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Research article

Note on a new class of operators between some spaces of holomorphic functions

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Abstract: The boundedness and compactness of a new class of linear operators from the weighted Bergman space to the weighted-type spaces on the unit ball are characterized.

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1. Introduction

By $\mathbb{B}$ we denote the open unit ball in $\mathbb{C}^n$, $\mathbb{S}$ is the unit sphere in $\mathbb{C}^n$, $B(z,r)$ is the open ball centered at z and with radius r, $d\sigma$ is the normalized rotation invariant measure on $\mathbb{S}$, $dV(z)$ is the Lebesgue measure, and $dV_\alpha(z) := c_{\alpha,n}(1-|z|^2)^\alpha dV(z)$, $\alpha > -1$, where $c_{\alpha,n}$ is the normalization constant such that $V_\alpha(\mathbb{B}) = 1$. The linear space of holomorphic functions on $\mathbb{B}$ we denote by $H(\mathbb{B})$, whereas $S(\mathbb{B})$ denotes the class of holomorphic self-maps of $\mathbb{B}$. The standard inner product between the vectors $z, w \in \mathbb{C}^n$ is denoted by $\langle z, w\rangle$, whereas $|z| = \sqrt{\langle z, z\rangle}$ is the Euclidean norm in $\mathbb{C}^n$. Many classical results on functions in $H(\mathbb{B})$ can be found in [1]. If $f \in C(\mathbb{B})$ is a positive function, then we call it a weight function, and the class of functions is denoted by $W(\mathbb{B})$. If $p, q \in \mathbb{N}_0$, $p \le q$, then the notation $j = \overline{p,q}$ is an abbreviation for the notation $j = p, p+1, \dots, q$. If X is a Banach space, then by B_X we denote the unit ball in X.

Each $\varphi \in S(\mathbb{B})$ induces the composition operator $C_\varphi f(z) = f(\varphi(z))$, whereas each $u \in H(\mathbb{B})$ induces the multiplication operator $M_u f(z) = u(z)f(z)$. The radial derivative of $f \in H(\mathbb{B})$ is defined by

$$\Re f(z) = \sum_{j=1}^{n} z_j D_j f(z),$$

where $D_j f(z) = \frac{\partial f}{\partial z_j}(z)$, $j = \overline{1,n}$ (if $n = 1$, then we regard $D_1 f := Df = f'$). There has been a huge interest in the operators and their products on subspaces of $H(\mathbb{B})$. The first investigations have been mostly devoted to the case $n = 1$. Beside the products of the operators C_φ and M_u, which have been studied a lot, there have been some investigations of the products of the operators D and C_φ. For some products of these and other concrete operators, see, for example, [2–25] and the related references therein. The boundedness and compactness [26, 27] of the operators have been predominately studied so far.

The weighted Bergman space $A^p_\alpha = A^p_\alpha(\mathbb{B})$, $p > 0$, $\alpha > -1$, consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{B}} |f(z)|^p dV_\alpha(z)\right)^{1/p} < +\infty,$$

which for $p \geq 1$ is a norm on A^p_α. With the norm the space is Banach. For some results on the space and operators on it, see, e.g., [4, 6, 14, 15, 22, 28–31].

If μ is a weight function, then the space of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H^\infty_\mu} = \sup_{z\in\mathbb{B}} \mu(z)|f(z)| < +\infty,$$

is called the weighted-type space and denoted by $H^\infty_\mu(\mathbb{B}) = H^\infty_\mu$, whereas the little weighted-type space is its closed subspace consisting of all $f \in H(\mathbb{B})$ such that $\lim_{|z|\to 1} \mu(z)|f(z)| = 0$, and is denoted by $H^\infty_{\mu,0}(\mathbb{B}) = H^\infty_{\mu,0}$. There has been a huge interest in investigating the spaces, their generalizations, and linear operators on them, especially in the boundedness and compactness [2, 11, 13, 19, 23, 31–34].

The product operator $\mathfrak{R}^m_{u,\varphi} = M_u C_\varphi \mathfrak{R}^m$ was introduced in [35]. For some investigations in the direction, see also [36]. Motivated, among others, by our investigations in [14–16, 35], I have introduced the operator

$$\mathfrak{S}^m_{\vec{u},\varphi} = \sum_{j=0}^{m} M_{u_j} C_\varphi \mathfrak{R}^j = \sum_{j=0}^{m} \mathfrak{R}^j_{u_j,\varphi}, \tag{1.1}$$

where $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, and $\varphi \in S(\mathbb{B})$, and studied it, for example, in [37]. For some related studies see also [2, 3].

This note continues some of our previous investigations (for example, the ones in [13–16, 35, 37]), by studying the boundedness and compactness of the operators $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ (or $H^\infty_{\mu,0}$), where $p \geq 1$ and $\alpha > -1$.

By C we denote some positive constants independent of essential variables and functions which may differ from line to line, whereas $a \lesssim b$ (resp. $a \gtrsim b$) means that there is $C > 0$ such that $a \leq Cb$ (resp. $a \geq Cb$). If $a \lesssim b$ and $b \lesssim a$, then we use the notation $a \asymp b$.

2. Auxiliary results

The first result is a standard Schwartz-type lemma [38].

Lemma 2.1. *Assume $p \geq 1$, $\alpha > -1$, $\mu \in W(\mathbb{B})$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $m \in \mathbb{N}$, $\varphi \in S(\mathbb{B})$, and that the operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ is bounded. Then, the operator is compact if and only if for every bounded sequence $(f_k)_{k\in\mathbb{N}} \subset A^p_\alpha$ uniformly converging to zero on compacts of $\mathbb{B}$, we have*

$$\lim_{k\to+\infty} \|\mathfrak{S}^m_{\vec{u},\varphi} f_k\|_{H^\infty_\mu} = 0.$$

The following lemma was essentially proved in [39], so we omit the proof.

Lemma 2.2. *A closed set K in $H^\infty_{\mu,0}$ is compact if and only if it is bounded and*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(z)|f(z)| = 0.$$

The following lemma is well known (see [29]; for a less precise version see also [1]).

Lemma 2.3. *Assume $p\in(0,\infty)$, $\alpha>-1$, and $f\in A^p_\alpha(\mathbb{B})$; Then,*

$$|f(z)| \le \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{n+\alpha+1}{p}}},\ z\in\mathbb{B}. \tag{2.1}$$

Lemma 2.4. *Assume $p\in(0,\infty)$, $\alpha>-1$, and $m\in\mathbb{N}$. Then,*

$$\left|\mathfrak{R}^m f(z)\right| \lesssim \frac{|z|}{(1-|z|^2)^{\frac{n+\alpha+1}{p}+m}}\|f\|_{A^p_\alpha}, \tag{2.2}$$

for every $f\in A^p_\alpha$ and $z\in\mathbb{B}$.

Proof. Note that it is enough to prove that for all $f\in A^p_\alpha$ and $z\in\mathbb{B}$,

$$|\mathfrak{R}^m f(z)| \lesssim \frac{|z|}{(1-|z|)^{\frac{n+\alpha+1}{p}+m}}\|f\|_{A^p_\alpha}. \tag{2.3}$$

Let $r\in(0,1)$ be fixed. Then, the Cauchy-Schwartz and Cauchy inequalities imply

$$|\mathfrak{R}f(z)| \lesssim |z|\frac{\sup_{w\in B(z,r(1-|z|))}|f(w)|}{1-|z|},\ z\in\mathbb{B},\ f\in H(\mathbb{B}). \tag{2.4}$$

Inequality (2.1) implies that

$$\sup_{w\in B(z,r(1-|z|))}|f(w)| \lesssim \frac{\|f\|_{A^p_\alpha}}{[(1-r)(1-|z|)]^{\frac{n+\alpha+1}{p}}}. \tag{2.5}$$

Since r is fixed, by (2.4) and (2.5) we get

$$|\mathfrak{R}f(z)| \lesssim \frac{|z|}{(1-|z|)^{\frac{n+\alpha+1}{p}+1}}\|f\|_{A^p_\alpha}, \tag{2.6}$$

that is, (2.3) holds when $m=1$.

Assume that for a $k\in\mathbb{N}\setminus\{1\}$ and all $f\in A^p_\alpha$ and $z\in\mathbb{B}$ holds,

$$|\mathfrak{R}^{k-1} f(z)| \lesssim \frac{|z|}{(1-|z|)^{\frac{n+\alpha+1}{p}+k-1}}\|f\|_{A^p_\alpha}. \tag{2.7}$$

Then, since for $w\in B(z,r(1-|z|))$ we have $(1-r)^{\frac{n+\alpha+1}{p}+k-1}(1-|z|)^{\frac{n+\alpha+1}{p}+k-1} \le (1-|w|)^{\frac{n+\alpha+1}{p}+k-1}$, from (2.7) we have

$$\sup_{w\in B(z,r(1-|z|))}|\mathfrak{R}^{k-1}f(w)| \lesssim \frac{1}{(1-|z|)^{\frac{n+\alpha+1}{p}+k-1}}\|f\|_{A^p_\alpha}. \tag{2.8}$$

If in (2.4) we replace f by $\mathfrak{R}^{k-1}f$, we get

$$|\mathfrak{R}^k f(z)| \lesssim |z| \frac{\sup_{w\in B(z,r(1-|z|))} |\mathfrak{R}^{k-1} f(w)|}{1-|z|}. \tag{2.9}$$

Combining (2.8) and (2.9), we have

$$|\mathfrak{R}^k f(z)| \lesssim \frac{|z|}{(1-|z|)^{\frac{n+\alpha+1}{p}+k}} \|f\|_{A^p_\alpha}.$$

Thus, (2.3) holds for each $m \in \mathbb{N}$, implying (2.2). □

The following lemma is well known.

Lemma 2.5. *Let $p \geq 1$ and $\alpha > -1$. Then, for any $t \geq 0$ and $w \in \mathbb{B}$,*

$$f_{w,t}(z) := \frac{(1-|w|^2)^{t+1}}{(1-\langle z,w\rangle)^{\frac{n+\alpha+1}{p}+t+1}}, \tag{2.10}$$

belongs to A^p_α and $\sup_{w\in\mathbb{B}} \|f_{w,t}\|_{A^p_\alpha} \lesssim 1$.

The following lemma is from [34] and [35].

Lemma 2.6. *Let $s \geq 0$, $w \in \mathbb{B}$ and $g_{w,s}(z) = (1-\langle z,w\rangle)^{-s}$. Then,*

$$\mathfrak{R}^k g_{w,s}(z) = s\frac{P_k(\langle z,w\rangle)}{(1-\langle z,w\rangle)^{s+k}}, \tag{2.11}$$

where $P_k(w) = s^{k-1}w^k + p^{(k)}_{k-1}(s)w^{k-1} + \cdots + p^{(k)}_2(s)w^2 + w$, and where $p^{(k)}_j(s)$, $j = \overline{2, k-1}$, are nonnegative polynomials for $s > 0$;

$$\mathfrak{R}^k g_{w,s}(z) = \sum_{t=1}^{k} a^{(k)}_t \Big(\prod_{j=0}^{t-1}(s+j)\Big) \frac{\langle z,w\rangle^t}{(1-\langle z,w\rangle)^{s+t}}, \tag{2.12}$$

where $(a^{(k)}_t)$, $t = \overline{1,k}$, $k \in \mathbb{N}$, are defined as

$$a^{(k)}_1 = a^{(k)}_k = 1, \ k \in \mathbb{N}; \tag{2.13}$$

and for $2 \leq t \leq k-1$, $k \geq 3$,

$$a^{(k)}_t = t a^{(k-1)}_t + a^{(k-1)}_{t-1}. \tag{2.14}$$

Lemma 2.7. *Assume $p \geq 1$, $\alpha > -1$, $m \in \mathbb{N}$, $w \in \mathbb{B}$, $f_{w,t}$ is defined in* (2.10)*, and $(a^{(k)}_t)_{t=\overline{1,k}}$, $k = \overline{1,m}$, are defined in* (2.13) *and* (2.14)*. Then,*

(a) for each $l \in \{1, \dots, m\}$, there is

$$h^{(l)}_w(z) = \sum_{k=0}^{m} c^{(l)}_k f_{w,k}(z), \tag{2.15}$$

where $c_k^{(l)}$, $k = \overline{0, m}$, *are numbers, such that*

$$\mathfrak{R}^j h_w^{(l)}(w) = 0,\ 0 \le j < l, \tag{2.16}$$

$$\mathfrak{R}^j h_w^{(l)}(w) = a_l^{(j)} \frac{|w|^{2l}}{(1-|w|^2)^{\frac{n+\alpha+1}{p}+l}},\ l \le j \le m, \tag{2.17}$$

hold. Moreover, we have $\sup_{w\in\mathbb{B}} \|h_w^{(l)}\|_{A_\alpha^p} < +\infty$;

(b) there is

$$h_w^{(0)}(z) = \sum_{k=0}^{m} c_k^{(0)} f_{w,k}(z), \tag{2.18}$$

where $c_k^{(0)}$, $k = \overline{0, m}$, *are numbers, such that*

$$h_w^{(0)}(w) = \frac{1}{(1-|w|^2)^{\frac{n+\alpha+1}{p}}},\ \mathfrak{R}^j h_w^{(0)}(w) = 0,\ j = \overline{1, m},$$

hold. Moreover, we have $\sup_{w\in\mathbb{B}} \|h_w^{(0)}\|_{A_\alpha^p} < +\infty$.

Proof. (a) Let $d_k = \frac{n+\alpha+1}{p} + k + 1$, $k \in \mathbb{N}_0$. Replace the constants $c_k^{(l)}$ in (2.15) by c_k. Then, from (2.12) we get

$$\begin{aligned}
h_w^{(l)}(w) &= \frac{c_0 + c_1 + \cdots + c_m}{(1-|w|^2)^{\frac{n+\alpha+1}{p}}},\\
\mathfrak{R} h_w^{(l)}(w) &= \frac{(d_0c_0 + d_1c_1 + \cdots + d_mc_m)|w|^2}{(1-|w|^2)^{\frac{n+\alpha+1}{p}+1}},\\
&\ \vdots\\
\mathfrak{R}^m h_w^{(l)}(w) &= a_1^{(m)} \frac{(d_0c_0 + d_1c_1 + \cdots + d_mc_m)|w|^2}{(1-|w|^2)^{\frac{n+\alpha+1}{p}+1}} + \cdots\\
&\quad + a_l^{(m)} \frac{(d_0\cdots d_{l-1}c_0 + d_1\cdots d_lc_1 + \cdots + d_m\cdots d_{m+l-1}c_m)|w|^{2l}}{(1-|w|^2)^{\frac{n+\alpha+1}{p}+l}} + \cdots\\
&\quad + a_m^{(m)} \frac{(d_0\cdots d_{m-1}c_0 + d_1\cdots d_mc_1 + \cdots + d_m\cdots d_{2m-1}c_m)|w|^{2m}}{(1-|w|^2)^{\frac{n+\alpha+1}{p}+m}}.
\end{aligned} \tag{2.19}$$

Lemma 2.5 in [11] shows that the determinant of the system,

$$\begin{bmatrix}
1 & 1 & \cdots & 1\\
d_0 & d_1 & \cdots & d_m\\
\vdots & \vdots & & \vdots\\
\prod_{k=0}^{l} d_k & \prod_{k=0}^{l} d_{k+1} & \cdots & \prod_{k=0}^{l} d_{m+k}\\
\vdots & \vdots & & \vdots\\
\prod_{k=0}^{m-1} d_k & \prod_{k=0}^{m-1} d_{k+1} & \cdots & \prod_{k=0}^{m-1} d_{m+k}
\end{bmatrix}
\begin{bmatrix} c_0\\ c_1\\ \\ \vdots\\ \\ c_m \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0 \end{bmatrix}, \tag{2.20}$$

is different from zero (on the right-hand side of (2.20), the unit is in the $(l+1)$th position). Thus, there is a unique solution $c_k = c_k^{(l)}$, $k = \overline{0,m}$, to (2.20). For these c_k-s, function (2.15) satisfies (2.16) and (2.17). By Lemma 2.5 we have $\sup_{w\in\mathbb{B}} \|h_w^{(l)}\|_{A_\alpha^p} < +\infty$.

(b) The proof is similar, so it is omitted. □

3. Main results

Our main results are formulated and proved in this section.

Theorem 3.1. *Let $p \ge 1$, $\alpha > -1$, $k \in \mathbb{N}$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operator $\mathfrak{R}_{u,\varphi}^k : A_\alpha^p \to H_\mu^\infty$ is bounded if and only if*

$$J_k := \sup_{z\in\mathbb{B}} \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+k}} < +\infty, \tag{3.1}$$

and if it is bounded, then we have

$$\|\mathfrak{R}_{u,\varphi}^k\|_{A_\alpha^p \to H_\mu^\infty} \asymp J_k. \tag{3.2}$$

Proof. Assume $\mathfrak{R}_{u,\varphi}^k : A_\alpha^p \to H_\mu^\infty$ is bounded. Let $g_w(z) = f_{\varphi(w),1}(z)$. By Lemma 2.6 the coefficients of the polynomial P_k therein are nonnegative, so we have

$$s\frac{\mu(w)|u(w)||\varphi(w)|^2}{(1-|\varphi(w)|^2)^{\frac{n+\alpha+1}{p}+k}} \le s\frac{\mu(w)|u(w)|P_k(|\varphi(w)|^2)}{(1-|\varphi(w)|^2)^{\frac{n+\alpha+1}{p}+k}} \le \|\mathfrak{R}_{u,\varphi}^k g_w\|_{H_\mu^\infty}. \tag{3.3}$$

The boundedness, (3.3) and the fact $\sup_{w\in\mathbb{B}} \|g_w\|_{A_\alpha^p} < +\infty$, imply

$$\sup_{|\varphi(z)|>1/2} \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+k}} \lesssim \|\mathfrak{R}_{u,\varphi}^k\|_{A_\alpha^p \to H_\mu^\infty}. \tag{3.4}$$

Further, the fact $f_j(z) = z_j \in A_\alpha^p$, $j = \overline{1,n}$, implies $\mathfrak{R}_{u,\varphi}^k f_j \in H_\mu^\infty$, $j = \overline{1,n}$, from which, together with $\mathfrak{R} f_j = f_j$, $j = \overline{1,n}$, we get

$$\sup_{z\in\mathbb{B}} \mu(z)|u(z)||\varphi_j(z)| = \|\mathfrak{R}_{u,\varphi}^k f_j\|_{H_\mu^\infty} \le \|\mathfrak{R}_{u,\varphi}^k\|_{A_\alpha^p \to H_\mu^\infty} \|z_j\|_{A_\alpha^p}, \ j = \overline{1,n},$$

from which we get

$$\sup_{z\in\mathbb{B}} \mu(z)|u(z)||\varphi(z)| \lesssim \|\mathfrak{R}_{u,\varphi}^k\|_{A_\alpha^p \to H_\mu^\infty}. \tag{3.5}$$

Inequality (3.5) together with

$$\sup_{|\varphi(z)|\le 1/2} \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+k}} \lesssim \sup_{|\varphi(z)|\le 1/2} \mu(z)|u(z)||\varphi(z)|,$$

implies

$$\sup_{|\varphi(z)|\le 1/2} \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+k}} \lesssim \|\mathfrak{R}_{u,\varphi}^k\|_{A_\alpha^p \to H_\mu^\infty}. \tag{3.6}$$

Combining (3.4) and (3.6), we get (3.1) and $J_k \lesssim \|\mathfrak{R}^k_{u,\varphi}\|_{A^p_\alpha \to H^\infty_\mu}$.

Assume (3.1) holds. Then, Lemma 2.4 implies that for any $f \in A^p_\alpha(\mathbb{B})$ and $z \in \mathbb{B}$,

$$\mu(z)\left|\mathfrak{R}^k_{u,\varphi} f(z)\right| \lesssim \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+k}}\|f\|_{A^p_\alpha}. \tag{3.7}$$

Taking the supremum in (3.7) over $B_{A^p_\alpha}$, and employing (3.1), the boundedness of $\mathfrak{R}^k_{u,\varphi} : A^p_\alpha \to H^\infty_\mu$ and the relation $\|\mathfrak{R}^k_{u,\varphi}\|_{A^p_\alpha \to H^\infty_\mu} \lesssim J_k$ follow, implying (3.2). □

The following result is known. For a more general result, see [31].

Theorem 3.2. *Let $p \geq 1$, $\alpha > -1$, $\mu \in W(\mathbb{B})$, $u \in H(\mathbb{B})$ and $\varphi \in S(\mathbb{B})$. Then, the operator $\mathfrak{R}^0_{u,\varphi} : A^p_\alpha \to H^\infty_\mu$ is bounded if and only if*

$$J_0 =: \sup_{z\in\mathbb{B}} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}}} < +\infty, \tag{3.8}$$

and if it is bounded, then $\|\mathfrak{R}^0_{u,\varphi}\|_{A^p_\alpha \to H^\infty_\mu} \asymp J_0$.

Theorem 3.3. *Let $p \geq 1$, $\alpha > -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operators $\mathfrak{R}^j_{u_j,\varphi} : A^p_\alpha \to H^\infty_\mu$, $j = \overline{0,m}$, are bounded if and only if $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ is bounded and*

$$\sup_{z\in\mathbb{B}} \mu(z)|u_j(z)||\varphi(z)| < +\infty, \ \ j = \overline{1,m}. \tag{3.9}$$

Proof. Assume $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ is bounded and (3.9) holds. We need to prove

$$I_j = \sup_{z\in\mathbb{B}} \frac{\mu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+j}} < +\infty, \ \ j = \overline{1,m}, \tag{3.10}$$

and

$$I_0 = \sup_{z\in\mathbb{B}} \frac{\mu(z)|u_0(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}}} < +\infty. \tag{3.11}$$

If $\varphi(w) \neq 0$, then there is $h^{(m)}_{\varphi(w)} \in A^p_\alpha$ such that

$$\mathfrak{R}^j h^{(m)}_{\varphi(w)}(\varphi(w)) = 0, \ 0 \leq j < m, \ \mathfrak{R}^m h^{(m)}_{\varphi(w)}(\varphi(w)) = \frac{|\varphi(w)|^{2m}}{(1-|\varphi(w)|^2)^{\frac{n+\alpha+1}{p}+m}},$$

and $\sup_{w\in\mathbb{B}} \|h^{(m)}_{\varphi(w)}\|_{A^p_\alpha} < +\infty$ (see Lemma 2.7 *(a)*). This, together with the boundedness of $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$, implies

$$\begin{aligned}\|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_\alpha \to H^\infty_\mu} &\gtrsim \|\mathfrak{S}^m_{\vec{u},\varphi} h^{(m)}_{\varphi(w)}\|_{H^\infty_\mu} \geq \mu(w)\left|\sum_{j=0}^m u_j(w)\mathfrak{R}^j h^{(m)}_{\varphi(w)}(\varphi(w))\right| \\ &= \frac{\mu(w)|u_m(w)||\varphi(w)|^{2m}}{(1-|\varphi(w)|^2)^{\frac{n+\alpha+1}{p}+m}},\end{aligned} \tag{3.12}$$

from which it follows that

$$\sup_{|\varphi(z)|>1/2} \frac{\mu(z)|u_m(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+m}} \lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_\alpha \to H^\infty_\mu},$$

and along with

$$\sup_{|\varphi(z)|\le 1/2} \frac{\mu(z)|u_m(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+m}} \lesssim \sup_{z\in\mathbb{B}} \mu(z)|u_m(z)||\varphi(z)| < +\infty,$$

implies $I_m < +\infty$.

Assume (3.10) holds for $j = \overline{s+1, m}$, for an $s \in \{1, 2, \ldots, m-1\}$. Let $h^{(s)}_{\varphi(w)}(z)$ be as in Lemma 2.7 *(a)*. Then, $\sup_{w\in\mathbb{B}} \|h^{(s)}_{\varphi(w)}\|_{A^p_\alpha} < +\infty$, and

$$\begin{aligned} \mu(w)\Big|\sum_{j=s}^{m} a^{(j)}_s u_j(w) \frac{|\varphi(w)|^{2s}}{(1-|\varphi(w)|^2)^{\frac{n+\alpha+1}{p}+s}}\Big| &\le \sup_{z\in\mathbb{B}} \mu(z)\Big|\sum_{j=0}^{m} u_j(z)\mathfrak{R}^j h^{(s)}_{\varphi(w)}(\varphi(z))\Big| \\ &\lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_\alpha \to H^\infty_\mu}, \end{aligned}$$

from which we easily get

$$\frac{\mu(w)|u_s(w)||\varphi(w)|^{2s}}{(1-|\varphi(w)|^2)^{\frac{n+\alpha+1}{p}+s}} \lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_\alpha \to H^\infty_\mu} + \sum_{j=s+1}^{m} \frac{\mu(w)|u_j(w)||\varphi(w)|^{2s}}{(1-|\varphi(w)|^2)^{\frac{n+\alpha+1}{p}+s}}. \tag{3.13}$$

From (3.13) and the fact $s \ge 1$, we have

$$\begin{aligned} \sup_{|\varphi(z)|>1/2} \frac{\mu(z)|u_s(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+s}} &\lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_\alpha \to H^\infty_\mu} + \sum_{j=s+1}^{m} \sup_{|\varphi(z)|>1/2} \frac{\mu(z)|u_j(z)||\varphi(z)|^{2s}}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+j}} \\ &\le \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_\alpha \to H^\infty_\mu} + \sum_{j=s+1}^{m} I_j. \end{aligned}$$

This, together with the fact

$$\sup_{|\varphi(z)|\le 1/2} \frac{\mu(z)|u_s(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+s}} \lesssim \sup_{z\in\mathbb{B}} \mu(z)|u_s(z)||\varphi(z)| < +\infty,$$

implies (3.10) for $j = s$. Thus, (3.10) holds for any $j \in \{1, \ldots, m\}$.

For any $w \in \mathbb{B}$, there is $h^{(0)}_{\varphi(w)} \in A^p_\alpha$ such that

$$h^{(0)}_{\varphi(w)}(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{\frac{n+\alpha+1}{p}}},\ \mathfrak{R}^j h^{(0)}_{\varphi(w)}(\varphi(w)) = 0,\ j = \overline{1, m},$$

and $\sup_{w\in\mathbb{B}} \|h^{(0)}_{\varphi(w)}\|_{A^p_\alpha} < +\infty$ (see Lemma 2.7 (b)).

This together with the boundedness of $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ implies

$$\frac{\mu(w)|u_0(w)|}{(1-|\varphi(w)|^2)^{\frac{n+\alpha+1}{p}}} \le \|\mathfrak{S}^m_{\vec{u},\varphi} h^{(0)}_{\varphi(w)}\|_{H^\infty_\mu} \lesssim \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_\alpha \to H^\infty_\mu}, \tag{3.14}$$

from which (3.11) follows, as claimed.

Assume $\mathfrak{R}^j_{u_j,\varphi} : A^p_\alpha \to H^\infty_\mu$, $j = \overline{0, m}$, are bounded. Then, $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ is also bounded. If u in (3.5) is replaced by u_j, we get (3.9). □

Theorem 3.4. *Let $p \geq 1$, $\alpha > -1$, $k \in \mathbb{N}$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operator $\mathfrak{R}^k_{u,\varphi} : A^p_\alpha \to H^\infty_\mu$ is compact if and only if it is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+k}} = 0. \tag{3.15}$$

Proof. If $\mathfrak{R}^k_{u,\varphi} : A^p_\alpha \to H^\infty_\mu$ is compact, it is also bounded. If $\|\varphi\|_\infty < 1$, (3.15) automatically/vacuously holds. If $\|\varphi\|_\infty = 1$ and $(z_j)_{j\in\mathbb{N}} \subset \mathbb{B}$ is such that $|\varphi(z_j)| \to 1$ as $j \to +\infty$, and $h_j(z) = f_{\varphi(z_j),t}(z)$, then $\sup_{j\in\mathbb{N}} \|h_j\|_{A^p_\alpha} < +\infty$. From $\lim_{j\to+\infty}(1-|\varphi(z_j)|^2)^{t+1} = 0$, we have $h_j \to 0$ as $j \to +\infty$, uniformly on compacta of $\mathbb{B}$. Using Lemma 2.1, it follows that $\lim_{j\to+\infty} \|\mathfrak{R}^k_{u,\varphi} h_j\|_{H^\infty_\mu} = 0$, from which, along with the consequence of (3.3),

$$\frac{\mu(z_j)|u(z_j)||\varphi(z_j)|}{(1-|\varphi(z_j)|^2)^{\frac{n+\alpha+1}{p}+k}} \leq C\|\mathfrak{R}^k_{u,\varphi} h_j\|_{H^\infty_\mu},$$

which holds for sufficiently large j, and we easily get (3.15).

If $\mathfrak{R}^k_{u,\varphi} : A^p_\alpha \to H^\infty_\mu$ is bounded and (3.15) holds, then Theorem 3.1 implies $\mu(z)|u(z)||\varphi(z)| \leq J_k < +\infty$, $z \in \mathbb{B}$, and (3.15) implies that for any $\varepsilon > 0$ there is $\delta \in (0,1)$ such that when $\delta < |\varphi(z)| < 1$,

$$\frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+k}} < \varepsilon. \tag{3.16}$$

Suppose $(f_j)_{j\in\mathbb{N}}$ is a bounded sequence in A^p_α converging to zero uniformly on compacts of $\mathbb{B}$. Let $s_\delta = \{z \in \mathbb{B} : |\varphi(z)| \leq \delta\}$. Then, Lemma 2.4, together with the fact $\sup_{z\in\mathbb{B}} \mu(z)|u(z)||\varphi(z)| < +\infty$, and (3.16), implies

$$\begin{aligned}\|\mathfrak{R}^k_{u,\varphi} f_j\|_{H^\infty_\mu} &\leq \sup_{z\in s_\delta} \mu(z)\left|u(z)\mathfrak{R}^k f_j(\varphi(z))\right| + \sup_{z\in\mathbb{B}\setminus s_\delta} \mu(z)\left|u(z)\mathfrak{R}^k f_j(\varphi(z))\right| \\ &\lesssim \sup_{z\in s_\delta} \mu(z)\left|u(z)\right|\left|\varphi(z)\right|\left|\nabla\mathfrak{R}^{k-1} f_j(\varphi(z))\right| + \sup_{z\in\mathbb{B}\setminus s_\delta} \frac{\mu(z)|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+k}} \\ &\lesssim \sup_{|w|\leq\delta} \left|\nabla\mathfrak{R}^{k-1} f_j(w)\right| + \varepsilon. \end{aligned} \tag{3.17}$$

The assumption $f_j \to 0$ on compacts along with Cauchy's estimate implies $\lim_{j\to+\infty} |\nabla\mathfrak{R}^{k-1} f_j| = 0$ uniformly on compacts of $\mathbb{B}$. The set $\{w : |w| \leq \delta\}$ is compact, so by letting $j \to +\infty$ in (3.17), it follows that $\limsup_{j\to+\infty} \|\mathfrak{R}^k_{u,\varphi} f_j\|_{H^\infty_\mu} \lesssim \varepsilon$, from which it follows that $\lim_{j\to+\infty} \|\mathfrak{R}^k_{u,\varphi} f_j\|_{H^\infty_\mu} = 0$. From this and Lemma 2.1, the compactness of $\mathfrak{R}^k_{u,\varphi} : A^p_\alpha \to H^\infty_\mu$ follows. □

The following theorem is known. For a more general result, see [31].

Theorem 3.5. *Let $p \geq 1$, $\alpha > -1$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operator $\mathfrak{R}^0_{u,\varphi} : A^p_\alpha \to H^\infty_\mu$ is compact if and only if it is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}}} = 0. \tag{3.18}$$

Theorem 3.6. *Let $p \geq 1$, $\alpha > -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ is compact and* (3.9) *holds if and only if the operators $\mathfrak{R}^j_{u_j,\varphi} : A^p_\alpha \to H^\infty_\mu$ are compact for $j = \overline{0,m}$.*

Proof. If $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ is compact and (3.9) holds, then the operator is bounded, from which, together with Theorem 3.3, the boundedness of $\mathfrak{R}^j_{u_j,\varphi} : A^p_\alpha \to H^\infty_\mu$, $j = \overline{0,m}$, follows. The previous two theorems show that it is enough to prove

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+j}} = 0, \;\; j = \overline{1,m}, \tag{3.19}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u_0(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}}} = 0. \tag{3.20}$$

If $\|\varphi\|_\infty < 1$, then (3.19) and (3.20) hold. Assume $\|\varphi\|_\infty = 1$. Let $(z_k)_{k\in\mathbb{N}} \subset \mathbb{B}$ be such that $\lim_{k\to+\infty} |\varphi(z_k)| = 1$, and $h^{(s)}_k(z) = h^{(s)}_{\varphi(z_k)}(z)$ for an $s \in \{1,\dots,m\}$ (see (2.15)). Then, $\sup_{k\in\mathbb{N}} \|h^{(s)}_k\|_{A^p_\alpha} < +\infty$. The fact $\lim_{k\to+\infty}(1-|\varphi(z_k)|^2)^{t+1} = 0$, implies $\lim_{k\to+\infty} h^{(s)}_k = 0$ uniformly on any compact of $\mathbb{B}$. So, Lemma 2.1 implies

$$\lim_{k\to+\infty} \|\mathfrak{S}^m_{\vec{u},\varphi} h^{(s)}_k\|_{H^\infty_\mu} = 0. \tag{3.21}$$

Relation (3.12) implies

$$\frac{\mu(z_k)|u_m(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+\alpha+1}{p}+m}} \lesssim \|\mathfrak{S}^m_{\vec{u},\varphi} h^{(m)}_k\|_{H^\infty_\mu}, \tag{3.22}$$

for sufficiently large k. From (3.22) and (3.21) with $s = m$, relation (3.19) with $j = m$ follows.

If (3.19) holds for $j = \overline{s+1,m}$, for a fixed $s \in \{1,\dots,m-1\}$, (3.13) implies

$$\frac{\mu(w)|u_s(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+\alpha+1}{p}+s}} \lesssim \|\mathfrak{S}^m_{\vec{u},\varphi} h^{(s)}_k\|_{A^p_\alpha\to H^\infty_\mu} + \sum_{j=s+1}^{m} \frac{\mu(w)|u_j(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+\alpha+1}{p}+j}},$$

for k large, from which, along with (3.21) and the hypothesis, the relation (3.19) with $j = s$ follows. Thus, (3.19) holds for any $s \in \{1,\dots,m\}$.

Let $h^{(0)}_k(z) = h^{(0)}_{\varphi(z_k)}(z)$ (see Lemma 2.7 *(b)*). Then, $\sup_{k\in\mathbb{N}} \|h^{(0)}_k\|_{A^p_\alpha} < +\infty$, and $\lim_{k\to+\infty} h^{(0)}_k(z) = 0$ uniformly on compacts of $\mathbb{B}$. From Lemma 2.1 we have that $\lim_{k\to+\infty} \|\mathfrak{S}^m_{\vec{u},\varphi} h^{(0)}_k\|_{H^\infty_\mu} = 0$, from which, along with the consequence of (3.14),

$$\frac{\mu(z_k)|u_0(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+\alpha+1}{p}}} \lesssim \|\mathfrak{S}^m_{\vec{u},\varphi} h^{(0)}_k\|_{H^\infty_\mu},$$

(3.20) follows.

Assume $\mathfrak{R}^j_{u_j,\varphi} : A^p_\alpha \to H^\infty_\mu$, $j = \overline{0,m}$, are compact. Then, $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ is also compact, and by Theorem 3.3 is obtained (3.9). □

Theorem 3.7. *Let $p \geq 1$, $\alpha > -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$ is bounded if and only if $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ is bounded and*

$$\lim_{|z|\to 1} \mu(z)\Big|\sum_{j=0}^{m} u_j(z) l^j\Big||\varphi(z)|^l = 0, \quad l \in \mathbb{N}_0. \tag{3.23}$$

Proof. If $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ is bounded and (3.23) holds, then since any polynomial p is represented as $p(z) = \sum_{l=0}^t p_l(z)$, where p_l, $l = \overline{0,t}$ are homogeneous polynomials of degree l, it follows that as $|z| \to 1$,

$$\mu(z)\Big|(\mathfrak{S}^m_{\vec{u},\varphi}p)(z)\Big| \le \sum_{l=0}^t \mu(z)\Big|\sum_{j=0}^m u_j(z)l^j\Big||p_l(\varphi(z))| \lesssim \sum_{l=0}^t \mu(z)\Big|\sum_{j=0}^m u_j(z)l^j\Big||\varphi(z)|^l \to 0.$$

Hence, $\mathfrak{S}^m_{\vec{u},\varphi}p \in H^\infty_{\mu,0}$. The density of the set of polynomials in A^p_α, implies that for any $f \in A^p_\alpha$ there are polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty} \|f - p_k\|_{A^p_\alpha} = 0$. From the boundedness of $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ we have

$$\|\mathfrak{S}^m_{\vec{u},\varphi}f - \mathfrak{S}^m_{\vec{u},\varphi}p_k\|_{H^\infty_\mu} \le \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_\alpha\to H^\infty_\mu}\|f - p_k\|_{A^p_\alpha} \to 0,$$

as $k \to +\infty$. So, $\mathfrak{S}^m_{\vec{u},\varphi}(A^p_\alpha) \subseteq H^\infty_{\mu,0}$, implying the boundedness of $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$.

If $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$ is bounded, then $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ is also bounded. The fact $f_{s,l}(z) = z_s^l \in A^p_\alpha$, $s = \overline{1,n}$, $l \in \mathbb{N}_0$, implies $\mathfrak{S}^m_{\vec{u},\varphi}f_{s,l} \in H^\infty_{\mu,0}$, $s = \overline{1,n}$, $l \in \mathbb{N}_0$. Hence, for $s = \overline{1,n}$, $l \in \mathbb{N}_0$, we have

$$\lim_{|z|\to1} \mu(z)|\mathfrak{S}^m_{\vec{u},\varphi}f_{s,l}(z)| = \lim_{|z|\to1} \mu(z)\Big|\sum_{j=0}^m u_j(z)l^j\Big||\varphi_s(z)|^l = 0,$$

from which, along with $|\varphi(z)|^l \lesssim \sum_{s=1}^n |\varphi_s(z)|^l$, (3.23) follows for each $l \in \mathbb{N}_0$. □

Theorem 3.8. *Let $p \ge 1$, $\alpha > -1$, $k \in \mathbb{N}$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operator $\mathfrak{R}^k_{u,\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to1} \frac{\mu(z)|u(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+k}} = 0. \tag{3.24}$$

Proof. Relation (3.24) implies (3.1). Taking the supremum in (3.7) over $\mathbb{B}$ and $B_{A^p_\alpha}$, and employing (3.1), it follows that

$$\sup_{f\in B_{A^p_\alpha}} \sup_{z\in\mathbb{B}} \mu(z)\Big|\mathfrak{R}^k_{u,\varphi}f(z)\Big| \lesssim \sup_{z\in\mathbb{B}} \frac{\mu(z)|u(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+k}} < +\infty. \tag{3.25}$$

Hence, the set $\mathcal{S} = \{\mathfrak{R}^k_{u,\varphi}f \in H^\infty_\mu : f \in B_{A^p_\alpha}\}$ is bounded in H^∞_μ. From (3.7) and (3.24) we easily get $\mathfrak{R}^k_{u,\varphi}f \in H^\infty_{\mu,0}$ for any $f \in B_{A^p_\alpha}$, i.e., $\mathcal{S} \subset H^\infty_{\mu,0}$. Taking the supremum in (3.7) over $B_{A^p_\alpha}$ and employing (3.24), it follows that

$$\lim_{|z|\to1} \sup_{f\in B_{A^p_\alpha}} \mu(z)\Big|\mathfrak{R}^k_{u,\varphi}f(z)\Big| = 0.$$

This fact and Lemma 2.2 imply the compactness of $\mathfrak{R}^k_{u,\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$.

If $\mathfrak{R}^k_{u,\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$ is compact, then $\mathfrak{R}^k_{u,\varphi} : A^p_\alpha \to H^\infty_\mu$ is also compact. From Theorem 3.4 we have that (3.15) and (3.16) hold. The fact $f_j(z) = z_j \in A^p_\alpha$, $j = \overline{1,n}$, implies $\mathfrak{R}^k_{u,\varphi}f_j \in H^\infty_{\mu,0}$, $j = \overline{1,n}$, from which we have $\lim_{|z|\to1} \mu(z)|u(z)||\varphi_j(z)| = 0$, $j = \overline{1,n}$. Hence,

$$\lim_{|z|\to1} \mu(z)|u(z)||\varphi(z)| = 0. \tag{3.26}$$

From (3.26) together with (3.16) we obtain (3.24) in a standard way. □

The following result is known. For a more general result, see [31].

Theorem 3.9. *Let $p \geq 1$, $\alpha > -1$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operator $\mathfrak{R}^0_{u,\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}}} = 0. \tag{3.27}$$

Theorem 3.10. *Let $p \geq 1$, $\alpha > -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$ is compact and*

$$\lim_{|z|\to 1} \mu(z)|u_j(z)||\varphi(z)| = 0, \ \ j = \overline{1,m}, \tag{3.28}$$

if and only if $\mathfrak{R}^j_{u_j,\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$ are compact for $j = \overline{0,m}$.

Proof. Suppose $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$ is compact and (3.28) holds. For the compactness of $\mathfrak{R}^j_{u_j,\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$, $j = \overline{0,m}$, it is enough to prove (see Theorems 3.8 and 3.9),

$$\lim_{|z|\to 1} \frac{\mu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+j}} = 0, \ \ j = \overline{1,m}, \tag{3.29}$$

and

$$\lim_{|z|\to 1} \frac{\mu(z)|u_0(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}}} = 0. \tag{3.30}$$

Note that $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_\mu$ is compact, whereas (3.9) follows from (3.28). The compactness of $\mathfrak{R}^j_{u_j,\varphi} : A^p_\alpha \to H^\infty_\mu$, $j = \overline{0,m}$, follows from Theorem 3.6. Hence, we have (3.19) and (3.20). Therefore, for every $\varepsilon > 0$ there is $\delta \in (0,1)$ such that for $\delta < |\varphi(z)| < 1$,

$$\frac{\mu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}+j}} < \varepsilon, \ \ j = \overline{1,m}, \ \ \textit{and} \ \ \frac{\mu(z)|u_0(z)|}{(1-|\varphi(z)|^2)^{\frac{n+\alpha+1}{p}}} < \varepsilon. \tag{3.31}$$

From (3.28) and (3.31), (3.29) easily follows. From the fact $f_0(z) \equiv 1 \in A^p_\alpha$ it follows that $\mathfrak{S}^m_{\vec{u},\varphi} 1 = u_0 \in H^\infty_{\mu,0}$, from which, together with (3.31), we similarly get (3.30).

If $\mathfrak{R}^j_{u_j,\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$, $j = \overline{0,m}$, are compact, then $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_\alpha \to H^\infty_{\mu,0}$ is also compact. Beside this (3.26) holds when u is replaced by u_j for each $j \in \{1,2,\ldots,m\}$, that is, (3.28) also holds. □

Remark 3.1. The quantities J_0 and J_k, $k \in \mathbb{N}$, in Theorems 3.1 and 3.2, are essentially obtained by using the point evaluations in (2.1) and (2.2), respectively. Since the numerator of the right-hand side in (2.1) does not contain the term $|z|$, the quantity J_0 does not contain the term $|\varphi(z)|$, unlike the quantities J_k, $k \in \mathbb{N}$. This is connected with the definition of the radial derivative operator.

4. Conclusions

Motivated, among others, by our investigations in [14–16, 35], in 2016 I came up with an idea of studying finite sums of the weighted differentiation composition operators and introduced several operators of this form acting on spaces of holomorphic functions on the unit disk or on the unit ball. One of them was the operator in (1.1). In [37] we have studied the operator from Hardy spaces to weighted-type spaces on the unit ball. Here we complement the main results therein by characterizing the boundedness and compactness of the operator from the weighted Bergman space to the weighted-type spaces on the unit ball. The methods, ideas and tricks presented here, with some modifications, can be used in some other settings, which should lead to some further investigations in the direction.

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Conflict of interest

The author declare no conflict of interest.

References

1. W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer, 1980.
2. Z. Guo, Y. Shu, On Stević-Sharma operators from Hardy spaces to Stević weighted spaces, *Math. Inequal. Appl.*, **23** (2020), 217–229. http://doi.org/10.7153/mia-2020-23-17
3. Z. T. Guo, L. L. Liu, Y. L. Shu, On Stević-Sharma operator from the mixed-norm spaces to Zygmund-type spaces, *Math. Inequal. Appl.*, **24** (2021), 445–461. http://doi.org/10.7153/mia-2021-24-31
4. Q. H. Hu, X. L. Zhu, Compact generalized weighted composition operators on the Bergman space, *Opuscula Math.*, **37** (2017), 303–312. http://doi.org/10.7494/OpMath.2017.37.2.303
5. S. X. Li, Volterra composition operators between weighted Bergman spaces and Bloch type spaces, *J. Korean Math. Soc.*, **45** (2008), 229–248. https://doi.org/10.4134/JKMS.2008.45.1.229
6. S. X. Li, Some new characterizations of weighted Bergman spaces, *Bull. Korean Math. Soc.*, **47** (2010), 1171–1180. https://doi.org/10.4134/BKMS.2010.47.6.1171
7. S. X. Li, On an integral-type operator from the Bloch space into the $Q_k(p,q)$ space, *Filomat*, **26** (2012), 331–339. http://doi.org/10.2298/FIL1202331L
8. S. X. Li, Differences of generalized composition operators on the Bloch space, *J. Math. Anal. Appl.*, **394** (2012), 706–711. https://doi.org/10.1016/j.jmaa.2012.04.009
9. S. X. Li, S. Stević, Integral-type operators from Bloch-type spaces to Zygmund-type spaces, *Appl. Math. Comput.*, **215** (2009), 464–473. https://doi.org/10.1016/j.amc.2009.05.011

10. S. Stević, Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces, *Siberian Math. J.*, **50** (2009), 726–736. https://doi.org/10.1007/S11202-009-0083-7

11. S. Stević, Composition followed by differentiation from H^{∞} and the Bloch space to nth weighted-type spaces on the unit disk, *Appl. Math. Comput.*, **216** (2010), 3450–3458. https://doi.org/10.1016/j.amc.2010.03.117

12. S. Stević, On operator P_{φ}^{g} from the logarithmic Bloch-type space to the mixed-norm space on unit ball, *Appl. Math. Comput.*, **215** (2010), 4248–4255. https://doi.org/10.1016/j.amc.2009.12.048

13. S. Stević, Weighted differentiation composition operators from the mixed-norm space to the nth weigthed-type space on the unit disk, *Abstr. Appl. Anal.*, **2010** (2010), 246287. https://doi.org/10.1155/2010/246287

14. S. Stević, A. K. Sharma, A. Bhat, Essential norm of products of multiplication composition and differentiation operators on weighted Bergman spaces, *Appl. Math. Comput.*, **218** (2011), 2386–2397. https://doi.org/10.1016/j.amc.2011.06.055

15. S. Stević, A. K. Sharma, A. Bhat, Products of multiplication composition and differentiation operators on weighted Bergman spaces, *Appl. Math. Comput.*, **217** (2011), 8115–8125. https://doi.org/10.1016/j.amc.2011.03.014

16. S. Stević, A. K. Sharma, R. Krishan, Boundedness and compactness of a new product-type operator from a general space to Bloch-type spaces, *J. Inequal. Appl.*, **2016** (2016), 219. https://doi.org/10.1186/s13660-016-1159-0

17. W. F. Yang, Products of composition and differentiation operators from $Q_k(p,q)$ spaces to Bloch-type spaces, *Abstr. Appl. Anal.*, **2009** (2009), 741920. https://doi.org/10.1155/2009/741920

18. W. F. Yang, Generalized weighted composition operators from the $F(p,q,s)$ space to the Bloch-type space, *Appl. Math. Comput.*, **218** (2012), 4967–4972. https://doi.org/10.1016/j.amc.2011.10.062

19. W. F. Yang, W. R. Yan, Generalized weighted composition operators from area Nevanlinna spaces to weighted-type spaces, *Bull. Korean Math. Soc.*, **48** (2011), 1195–1205. https://doi.org/10.4134/BKMS.2011.48.6.1195

20. W. F. Yang, X. L. Zhu, Generalized weighted composition operators from area Nevanlinna spaces to Bloch-type spaces, *Taiwanese J. Math.*, **16** (2012), 869–883. https://doi.org/10.11650/twjm/1500406662

21. X. L. Zhu, Multiplication followed by differentiation on Bloch-type spaces, *Bull. Allahbad Math. Soc.*, **23** (2008), 25–39.

22. X. L. Zhu, Generalized weighted composition operators on weighted Bergman spaces, *Numer. Funct. Anal. Optim.*, **30** (2009), 881–893. https://doi.org/10.1080/01630560903123163

23. X. L. Zhu, Generalized weighted composition operators from Bloch spaces into Bers-type spaces, *Filomat*, **26** (2012), 1163–1169. https://doi.org/10.2298/FIL1206163Z

24. X. L. Zhu, A new characterization of the generalized weighted composition operator from H^{∞} into the Zygmund space, *Math. Inequal. Appl.*, **18** (2015), 1135–1142. https://doi.org/10.7153/mia-18-87

25. X. L. Zhu, Essential norm and compactness of the product of differentiation and composition operators on Bloch type spaces, *Math. Inequal. Appl.*, **19** (2016), 325–334. https://doi.org/10.7153/mia-19-24

26. N. Dunford, J. T. Schwartz, *Linear operators I*, New York: Jon Willey and Sons, 1958.

27. W. Rudin, *Functional analysis*, New York: McGraw-Hill Book Campany, 1991.

28. K. L. Avetisyan, Integral representations in general weighted Bergman spaces, *Complex Var.*, **50** (2005), 1151–1161. http://doi.org/10.1080/02781070500327576

29. F. Beatrous, J. Burbea, Holomorphic Sobolev spaces on the ball, *Dissertationes Math.*, 1989.

30. G. Benke, D. C. Chang, A note on weighted Bergman spaces and the Cesáro operator, *Nagoya Math. J.*, **159** (2000), 25–43. https://doi.org/10.1017/S0027763000007406

31. S. Stević, Weighted composition operators from weighted Bergman spaces to weighted-type spaces on the unit ball, *Appl. Math. Comput.*, **212** (2009), 499–504. https://doi.org/10.1016/j.amc.2009.02.057

32. K. D. Bierstedt, W. H. Summers, Biduals of weighted Banach spaces of analytic functions, *J. Aust. Math. Soc.*, **54** (1993), 70–79. https://doi.org/10.1017/S1446788700036983

33. L. A. Rubel, A. L. Shields, The second duals of certain spaces of analytic functions, *J. Aust. Math. Soc.*, **11** (1970), 276–280. https://doi.org/10.1017/S1446788700006649

34. S. Stević, Weighted radial operator from the mixed-norm space to the nth weighted-type space on the unit ball, *Appl. Math. Comput.*, **218** (2012), 9241–9247. https://doi.org/10.1016/j.amc.2012.03.001

35. S. Stević, Weighted iterated radial composition operators between some spaces of holomorphic functions on the unit ball, *Abstr. Appl. Anal.*, **2010** (2010), 801264. https://doi.org/10.1155/2010/801264

36. S. Stević, Weighted iterated radial operators between different weighted Bergman spaces on the unit ball, *Appl. Math. Comput.*, **218** (2012), 8288–8294. https://doi.org/10.1016/j.amc.2012.01.052

37. S. Stević, C. S. Huang, Z. J. Jiang, Sum of some product-type operators from Hardy spaces to weighted-type spaces on the unit ball, *Math. Methods Appl. Sci.*, **45** (2022), 11581—11600. https://doi.org/10.1002/mma.8467

38. H. J. Schwartz, *Composition operators on H^p*, University of Toledo, 1969.

39. K. Madigan, A. Matheson, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.*, **347** (1995), 2679–2687. https://doi.org/10.1090/S0002-9947-1995-1273508-X