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Research article

Oscillation results for a fractional partial differential system with damping and forcing terms

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Abstract: In this paper, we study the forced oscillation of solutions of a fractional partial differential system with damping terms by using the Riemann-Liouville derivative and integral. We obtained some new oscillation results by using the integral averaging technique. The obtained results are illustrated by using some examples.

Keywords: fractional partial differential equation; forced oscillation; damping term
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1. Introduction

The theory of fractional calculus has played an important role in engineering and natural sciences. Currently, the concept of fractional calculus has been effectively used in many social, physical, signal, image processing, biological and engineering problems. Further, it has been realized that a fractional system provides a more accurate interpretation than the integer-order system in many real modeling problems. For more details, one can refer to [1–10].

Oscillation phenomena take part in different models of real world applications; see for instance the papers [11–17] and the papers cited therein. More precisely, we refer the reader to the papers [18, 19] on bio-mathematical models where oscillation and/or delay actions may be formulated by means of cross-diffusion terms. Recently and although it is rare, the study on the oscillation of fractional partial differential equations has attracted many researchers. In [20–23], the researchers have established the requirements of the oscillation for certain kinds of fractional partial differential equations.

In [24], Luo et al. studied the oscillatory behavior of the fractional partial differential equation of the form

$$\begin{aligned} &D_{+,t}^{1+\alpha}u(y,t) + p(t)D_{+,t}^{\alpha}u(y,t) + q(y,t)\int_0^t (t-\hbar)^{-\alpha}u(y,\hbar)d\hbar \\ &= a(t)\Delta u(y,t) + \sum_{i=1}^{m} a_i(t)\Delta u(y,t-\tau_i), \quad (y,t) \in Q \times \mathbb{R}_+ = H \end{aligned}$$

subject to either of the following boundary conditions

$$\begin{aligned} \frac{\partial u(y,t)}{\partial \nu} + \beta(y,t)u(y,t) = 0, \quad (y,t) \in \partial Q \times \mathbb{R}_+, \\ u(y,t) = 0, \quad (y,t) \in \partial Q \times \mathbb{R}_+. \end{aligned}$$

They have obtained some sufficient conditions for the oscillation of all solutions of this kind of fractional partial differential equations by using the integral averaging technique and Riccati transformations.

On other hand in [25], Xu and Meng considered a fractional partial differential equation of the form

$$\begin{aligned} &D_{+,t}^{\alpha}(r(t)D_{+,t}^{\alpha}u(y,t)) + p(t)D_{+,t}^{\alpha}u(y,t) + q(y,t)f(u(y,t)) \\ &= a(t)\Delta u(y,t) + \sum_{i=1}^{m} b_i(t)\Delta u(y,t-\tau_i), \quad (y,t) \in Q \times \mathbb{R}_+ = H \end{aligned}$$

with the Robin boundary condition

$$\frac{\partial u(y,t)}{\partial N} + g(y,t)u(y,t) = 0, \quad (y,t) \in \partial Q \times \mathbb{R}_+,$$

they obtained some oscillation criteria using the integral averaging technique and Riccati transformations.

Prakash et al. [26] considered the oscillation of the fractional differential equation

$$\frac{\partial}{\partial t}(r(t)D_{+,t}^{\alpha}u(y,t)) + q(y,t)f\left(\int_0^t (t-\nu)^{-\alpha}u(y,\nu)d\nu\right) = a(t)\Delta u(y,t), \quad (y,t) \in Q \times \mathbb{R}_+$$

with the Neumann boundary condition

$$\frac{\partial u(y,t)}{\partial N} = 0, \quad (y,t) \in \partial Q \times \mathbb{R}_+,$$

they obtained some oscillation criteria by using the integral averaging technique and Riccati transformations.

Furthermore in [27], Ma et al. considered the forced oscillation of the fractional partial differential equation with damping term of the form

$$\frac{\partial}{\partial t}(r(t)D_{+,t}^{\alpha}u(y,t)) + p(t)D_{+,t}^{\alpha}u(y,t) + q(y,t)f(u(y,t)) = a(t)\Delta u(y,t) + \tilde{g}(y,t), \quad (y,t) \in Q \times \mathbb{R}_+$$

with the boundary condition

$$\frac{\partial u(y,t)}{\partial N} + \beta(y,t)u(y,t) = 0, \quad (y,t) \in \partial Q \times \mathbb{R}_+ ,$$

they obtained some oscillation criteria by using the integral averaging technique.

From the above mentioned literature, one can notice that the Riccati transformation method has been incorporated into the proof of the oscillation results. Unlike previous results, however, we study in this paper the forced oscillation of the fractional partial differential equation with the damping term of the form

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(a(t)\frac{\partial}{\partial t}\left(r(t)g\left(D^{\alpha}_{+,t}u(y,t)\right)\right)\right) + p(t)\frac{\partial}{\partial t}\left(r(t)g(D^{\alpha}_{+,t}u(y,t))\right) \\
&= b(t)\Delta u(y,t) + \sum_{i=1}^{m} a_i(t)\Delta u(y,t-\tau_i) \\
&\quad - q(y,t)\int_0^t (t-\hbar)^{-\alpha}u(y,\hbar)d\hbar + f(y,t), \quad (y,t) \in Q \times \mathbb{R}_+ = H \qquad (1.1)
\end{aligned}$$

via the application of the integral averaging technique only. Equation (1.1) is presented under a high degree of generality providing a general platform for many particular cases. Here, $D^{\alpha}_{+,t}u(y,t)$ is the Riemann-Liouville fractional partial derivative of order α of u, $\alpha \in (0,1)$, Δ is the Laplacian in $\mathbb{R}^n$, i.e.,

$$\Delta u(y,t) = \sum_{r=1}^{n} \frac{\partial^2 u(y,t)}{\partial y_r^2},$$

Q is a bounded domain of $\mathbb{R}^n$ with the piecewise smooth boundary ∂Q and $\mathbb{R}_+ := (0,\infty)$.

Further, we assume the Robin and Dirichlet boundary conditions

$$\frac{\partial u(y,t)}{\partial N} + \gamma(y,t)u(y,t) = 0, \quad (y,t) \in \partial Q \times \mathbb{R}_+ \qquad (1.2)$$

and

$$u(y,t) = 0, \quad (y,t) \in \partial Q \times \mathbb{R}_+, \qquad (1.3)$$

where N is the unit outward normal to ∂Q and $\gamma(y,t) > 0$ is a continuous function on $\partial Q \times \mathbb{R}_+$. The following conditions are assumed throughout:

- **(H$_1$)** $a(t) \in C^1([t_0,\infty);\mathbb{R}_+)$ and $r(t) \in C^2([t_0,\infty);\mathbb{R}_+)$;
- **(H$_2$)** $g(t) \in C^2(\mathbb{R};\mathbb{R})$ is an increasing function and there exists a positive constant k such that $\frac{y}{g(y)} = k > 0$, $yg(y) \neq 0$ for $y \neq 0$;
- **(H$_3$)** $p(t) \in C([t_0,\infty);\mathbb{R})$ and $A(t) = \int_{t_0}^{t} \frac{p(\zeta)}{a(\zeta)}d\zeta$;
- **(H$_4$)** $b(t), a_i(t) \in C(\mathbb{R}_+;\mathbb{R}_+)$ and τ_i are non-negative constants, $i \in I_m = \{1,2,\ldots,m\}$;
- **(H$_5$)** $q(y,t) \in C(H;\mathbb{R}_+)$ and $q(t) = \min_{y\in Q} q(y,t)$;
- **(H$_6$)** $f(y,t) \in C(\bar{H};\mathbb{R})$.

By a solution of the problems (1.1) and (1.2) (or (1.1)–(1.3)), we mean a function $u(y,t) \in C^{2+\alpha}(\bar{Q}\times [0,\infty))$, which satisfies (1.1) on H and the boundary condition (1.2) (or (1.3)).

A solution $u(y,t)$ of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is *non-oscillatory*.

The rest of the paper is organized as follows. Some basic definitions and known lemmas are included in Section 2. In Sections 3 and 4, we study the oscillations of (1.1) and (1.2), and (1.1) and (1.3), respectively. Section 5 deals with some applications for the sake of showing the feasibility and effectiveness of our results. Lastly, we add a conclusion in Section 6.

2. Preliminaries

Before we start the main work, we present some basic lemmas and definitions which are applied in what follows.

Definition 1. [4] *The Riemann-Liouville fractional integral of order* $\alpha > 0$ *of a function* $y : \mathbb{R}_+ \to \mathbb{R}$ *on the half-axis* $\mathbb{R}_+$ *is defined by*

$$(I_+^{\alpha} y)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\vartheta)^{\alpha-1} y(\vartheta) d\vartheta, \quad t > 0$$

provided the right-hand side is pointwise defined on $\mathbb{R}_+$*, where* Γ *is the gamma function.*

Definition 2. [4] *The Riemann-Liouville fractional derivative of order* $\alpha > 0$ *of a function* $y : \mathbb{R}_+ \to \mathbb{R}$ *on the half-axis* $\mathbb{R}_+$ *is defined by*

$$(D_+^{\alpha} y)(t) := \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \left(I_+^{\lceil\alpha\rceil-\alpha} y\right)(t), \quad t > 0$$

provided the right-hand side is pointwise defined on $\mathbb{R}_+$*, where* $\lceil\alpha\rceil$ *is the ceiling function of* α.

Definition 3. [4] *The Riemann-Liouville fractional partial derivative of order* $0 < \alpha < 1$ *with respect to t of a function* $u(y,t)$ *is defined by*

$$(D_{+,t}^{\alpha} u)(y,t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\vartheta)^{-\alpha} u(y,\vartheta) d\vartheta$$

provided the right-hand side is pointwise defined on $\mathbb{R}_+$.

Lemma 1. [4] *Let y be a solution of* (1.1) *and*

$$L(t) := \int_0^t (t-\vartheta)^{-\alpha} y(\vartheta) d\vartheta$$

for $\alpha \in (0,1)$ *and* $t > 0$. *Then*

$$L'(t) = \Gamma(1-\alpha)(D_+^{\alpha} y)(t).$$

Lemma 2. [4] *Let* $\alpha \geq 0, m \in \mathbb{N}$ *and* $D = \frac{d}{dt}$. *If the fractional derivatives* $(D_{a+}^{\alpha} y)(t)$ *and* $(D_{a+}^{\alpha+m} y)(t)$ *exist, then*

$$\left(D^m D_{a+}^{\alpha} y\right)(t) = (D_{a+}^{\alpha+m} y)(t).$$

Lemma 3. [4] *If $\alpha \in (0,1)$, then*

$$\left(I_{a+}^{\alpha} D_{a+}^{\alpha} y\right)(t) = y(t) - \frac{y_{1-\alpha}(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1},$$

where $y_{1-\alpha}(t) = (I_{a+}^{1-\alpha} y)(t)$.

Lemma 4. [5] *The smallest eigenvalue β_0 of the Dirichlet problem*

$$\begin{aligned} \Delta\omega(y) + \beta\omega(y) &= 0 \text{ in } Q \\ \omega(y) &= 0 \text{ on } \partial Q \end{aligned}$$

is positive and the corresponding eigenfunction $\phi(y)$ is positive in Q.

3. Oscillation of (1.1) and (1.2)

In this section, we establish the oscillation criteria for (1.1) and (1.2).

Theorem 1. *If $(\mathbf{H_1}) - (\mathbf{H_6})$ are valid, $\lim_{t\to\infty} I_{+}^{1-\alpha} U(0) = C_0$ and if*

$$\liminf_{t\to\infty} \int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)} \left[C_2 + \int_{t_0}^{\hbar} \frac{C_1}{e^{A(\tau)}a(\tau)} d\tau + \int_{t_0}^{\hbar} \frac{1}{e^{A(\tau)}a(\tau)} \left(\int_{t_0}^{\tau} e^{A(\zeta)} F(\zeta) d\zeta\right) d\tau\right] d\hbar < 0 \tag{3.1}$$

and

$$\limsup_{t\to\infty} \int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)} \left[C_2 + \int_{t_0}^{\hbar} \frac{C_1}{e^{A(\tau)}a(\tau)} d\tau + \int_{t_0}^{\hbar} \frac{1}{e^{A(\tau)}a(\tau)} \left(\int_{t_0}^{\tau} e^{A(\zeta)} F(\zeta) d\zeta\right) d\tau\right] d\hbar > 0 \tag{3.2}$$

for some constants C_0, C_1 and C_2 with $F(t) = \int_Q f(y,t)dy$, then all solutions of (1.1) and (1.2) are oscillatory.

Proof. If $u(y,t)$ is a non-oscillatory solution of (1.1) and (1.2) then there exists a $t_0 \geq 0$ such that $u(y,t) > 0$ (or $u(y,t) < 0$), $t \geq t_0$.

Case 1. Let $u(y,t) > 0$ for $t \geq t_0$. Integrating (1.1) over Q, we get

$$\begin{aligned} &\frac{d}{dt}\left(a(t)\frac{d}{dt}\left(r(t)g(D_{+}^{\alpha}U(t))\right)\right) + p(t)\frac{d}{dt}\left(r(t)g(D_{+}^{\alpha}U(t))\right) \\ &= b(t)\int_Q \Delta u(y,t)dy + \sum_{i=1}^{m} a_i(t)\int_Q \Delta u(y,t-\tau_i)dy \\ &\quad - \int_Q \left(q(y,t)\int_0^t (t-\hbar)^{-\alpha} u(y,\hbar)d\hbar\right)dy + \int_Q f(y,t)dy, \end{aligned} \tag{3.3}$$

where $U(t) = \int_Q u(y,t)dy$ with $U(t) > 0$. By (1.2) and Green's formula, we have

$$\int_Q \Delta u(y,t)dy = \int_{\partial Q} \frac{\partial u}{\partial N} d\zeta = -\int_{\partial Q} \gamma(y,t)u(y,t)d\zeta < 0 \tag{3.4}$$

and

$$\int_Q \Delta u(y, t-\tau_i)dy < 0. \tag{3.5}$$

Also, by ($\mathbf{H_5}$), one can get

$$\begin{aligned}\int_Q \left(q(y,t)\int_0^t (t-\hbar)^{-\alpha}u(y,\hbar)d\hbar\right)dy &\geq q(t)\int_0^t (t-\hbar)^{-\alpha}\left(\int_Q u(y,\hbar)dy\right)d\hbar \\ &= q(t)L(t), \end{aligned} \tag{3.6}$$

where $L(t) = \int_0^t (t-\hbar)^{-\alpha}U(\hbar)d\hbar$. Because of the inequalities (3.4)–(3.6), (3.3) becomes

$$\frac{d}{dt}\left(a(t)\frac{d}{dt}\left(r(t)g(D_+^\alpha U(t))\right)\right) + p(t)\frac{d}{dt}\left(r(t)g(D_+^\alpha U(t))\right) \leq -q(t)L(t) + F(t) \leq F(t).$$

Thus, we get

$$\left(e^{A(t)}a(t)\left(r(t)g(D_+^\alpha U(t))\right)'\right)' = e^{A(t)}\left(\left(a(t)\left(r(t)g(D_+^\alpha U(t))\right)'\right)' + p(t)\left(r(t)g(D_+^\alpha U(t))\right)'\right) \leq e^{A(t)}F(t).$$

Integrating the above inequality over $[t_0, t]$, one can get

$$e^{A(t)}a(t)\left(r(t)g(D_+^\alpha U(t))\right)' \leq \int_{t_0}^t e^{A(\zeta)}F(\zeta)d\zeta + C_1,$$

where

$$C_1 = e^{A(t_0)}a(t_0)\left(r(t_0)g(D_+^\alpha U(t_0))\right)'.$$

Again integrating the above inequality over $[t_0, t]$, we get

$$r(t)g(D_+^\alpha U(t)) \leq C_2 + \int_{t_0}^t \frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \int_{t_0}^t \frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^\tau e^{A(\zeta)}F(\zeta)d\zeta\right)d\tau,$$

where $C_2 = r(t_0)g(D_+^\alpha U(t_0))$. Then using ($\mathbf{H_5}$), we obtain

$$\frac{D_+^\alpha U(t)}{k} \leq \frac{C_2}{r(t)} + \frac{1}{r(t)}\int_{t_0}^t \frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \frac{1}{r(t)}\int_{t_0}^t \frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^\tau e^{A(\zeta)}F(\zeta)d\zeta\right)d\tau.$$

Applying the Riemann-Liouville fractional integral operator of order α to the above inequality and using Lemma 3, we obtain

$$\begin{aligned}U(t) - \frac{I_0^{1-\alpha}U(0)}{\Gamma(\alpha)}t^{\alpha-1} \leq \frac{k}{\Gamma(\alpha)}\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\Bigg[C_2 &+ \int_{t_0}^\hbar \frac{C_1}{e^{A(\tau)}a(\tau)}d\tau \\ &+ \int_{t_0}^\hbar \frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^\tau e^{A(\zeta)}F(\zeta)d\zeta\right)d\tau\Bigg]d\hbar.\end{aligned}$$

Then

$$\liminf_{t\to\infty} U(t) \le \liminf_{t\to\infty} \frac{C_0}{\Gamma(\alpha)} t^{\alpha-1} + \liminf_{t\to\infty} \left\{ \frac{k}{\Gamma(\alpha)} \int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)} \left[C_2 + \int_{t_0}^{\hbar} \frac{C_1}{e^{A(\tau)}a(\tau)} d\tau + \int_{t_0}^{\hbar} \frac{1}{e^{A(\tau)}a(\tau)} \left(\int_{t_0}^{\tau} e^{A(\zeta)} F(\zeta) d\zeta \right) d\tau \right] d\hbar \right\}.$$

Therefore, by our hypothesis, as given by (3.1), we get $\liminf_{t\to\infty} U(t) \le 0$. This leads to a contradiction to $U(t) > 0$.

Case 2. Let $u(y,t) < 0$ for $t \ge t_0$. Just as in Case 1, we can obtain that (3.3) holds and $U(t) < 0$. By (1.2) and Green's formula, we get

$$\int_Q \Delta u(y,t) dy = \int_{\partial Q} \frac{\partial u}{\partial N} d\zeta = -\int_{\partial Q} \gamma(y,t) u(y,t) d\zeta > 0 \tag{3.7}$$

and

$$\int_Q \Delta u(y, t-\tau_i) dy > 0. \tag{3.8}$$

Also, by ($\mathbf{H_5}$), we have

$$\begin{aligned} \int_Q \left(q(y,t) \int_0^t (t-\hbar)^{-\alpha} u(y,\hbar) d\hbar \right) dy &\le q(t) \int_0^t (t-\hbar)^{-\alpha} \left(\int_Q u(y,\hbar) dy \right) d\hbar \\ &= q(t) L(t). \end{aligned} \tag{3.9}$$

Because of the inequalities (3.7)–(3.9), (3.3) becomes

$$\frac{d}{dt}\left(a(t) \frac{d}{dt} \left(r(t) g(D_+^\alpha U(t)) \right) \right) + p(t) \frac{d}{dt} \left(r(t) g(D_+^\alpha U(t)) \right) \ge -q(t)L(t) + F(t) \ge F(t), \tag{3.10}$$

that is,

$$\left(e^{A(t)} a(t) \left(r(t) g(D_+^\alpha U(t)) \right)' \right)' = e^{A(t)} \left(\left(a(t) \left(r(t) g(D_+^\alpha U(t)) \right)' \right)' + p(t) \left(r(t) g(D_+^\alpha U(t)) \right)' \right) \ge e^{A(t)} F(t).$$

Integrating the above inequality over $[t_0, t]$, we have

$$e^{A(t)} a(t) \left(r(t) g(D_+^\alpha U(t)) \right)' \ge \int_{t_0}^t e^{A(\zeta)} F(\zeta) d\zeta + C_1,$$

where

$$C_1 = e^{A(t_0)} a(t_0) \left(r(t_0) g(D_+^\alpha U(t_0)) \right)'.$$

Again integrating the above inequality over $[t_0, t]$, we obtain

$$r(t) g(D_+^\alpha U(t)) \ge C_2 + \int_{t_0}^t \frac{C_1}{e^{A(\tau)}a(\tau)} d\tau + \int_{t_0}^t \frac{1}{e^{A(\tau)}a(\tau)} \left(\int_{t_0}^{\tau} e^{A(\zeta)} F(\zeta) d\zeta \right) d\tau,$$

where $C_2 = r(t_0) g(D_+^\alpha U(t_0))$. Then using ($\mathbf{H_5}$), we obtain

$$\frac{D_+^\alpha U(t)}{k} \ge \frac{C_2}{r(t)} + \frac{1}{r(t)} \int_{t_0}^t \frac{C_1}{e^{A(\tau)}a(\tau)} d\tau + \frac{1}{r(t)} \int_{t_0}^t \frac{1}{e^{A(\tau)}a(\tau)} \left(\int_{t_0}^{\tau} e^{A(\zeta)} F(\zeta) d\zeta \right) d\tau.$$

Applying the Riemann-Liouville fractional integral operator of order α to the above inequality and using Lemma 3, we obtain

$$U(t) - \frac{I_0^{1-\alpha}U(0)}{\Gamma(\alpha)}t^{\alpha-1} \geq \frac{k}{\Gamma(\alpha)}\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\Big[C_2 + \int_{t_0}^{\hbar} \frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \int_{t_0}^{\hbar} \frac{1}{e^{A(\tau)}a(\tau)}\Big(\int_{t_0}^{\tau} e^{A(\zeta)}F(\zeta)d\zeta\Big)d\tau\Big]d\hbar.$$

Then

$$\limsup_{t\to\infty} U(t) \geq \limsup_{t\to\infty} \frac{C_0}{\Gamma(\alpha)}t^{\alpha-1} + \limsup_{t\to\infty}\Big\{\frac{k}{\Gamma(\alpha)}\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\Big[C_2 + \int_{t_0}^{\hbar} \frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \int_{t_0}^{\hbar} \frac{1}{e^{A(\tau)}a(\tau)}\Big(\int_{t_0}^{\tau} e^{A(\zeta)}F(\zeta)d\zeta\Big)d\tau\Big]d\hbar\Big\}.$$

Therefore, by our hypothesis given by (3.2), we get $\limsup_{t\to\infty} U(t) \geq 0$. This leads to a contradiction to $U(t) < 0$. □

4. Oscillation of (1.1) and (1.3)

In this section, we establish the oscillation criteria for (1.1) and (1.3).

Theorem 2. *If* $(\mathbf{H_1}) - (\mathbf{H_5})$ *are valid,* $\lim_{t\to\infty} I_+^{1-\alpha}U_1(0) = A_1$ *and if*

$$\liminf_{t\to\infty} \int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\Big[C_2 + \int_{t_0}^{\hbar} \frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \int_{t_0}^{\hbar} \frac{1}{e^{A(\tau)}a(\tau)}\Big(\int_{t_0}^{\tau} e^{A(\zeta)}F_1(\zeta)d\zeta\Big)d\tau\Big]d\hbar < 0 \tag{4.1}$$

and

$$\limsup_{t\to\infty} \int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\Big[C_2 + \int_{t_0}^{\hbar} \frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \int_{t_0}^{\hbar} \frac{1}{e^{A(\tau)}a(\tau)}\Big(\int_{t_0}^{\tau} e^{A(\zeta)}F_1(\zeta)d\zeta\Big)d\tau\Big]d\hbar > 0 \tag{4.2}$$

for some constants A_1, C_1 *and* C_2 *with*

$$F_1(t) = \int_Q f(y,t)\phi(y)dy \ \ \text{and} \ \ U_1(t) = \int_Q u(y,t)\phi(y)dy,$$

then all solutions of (1.1) *and* (1.3) *are oscillatory.*

Proof. If $u(y,t)$ is a non-oscillatory solution of (1.1) and (1.3) then there exists a $t_0 \geq 0$ such that $u(y,t) > 0$ (or $u(y,t) < 0$) for $t \geq t_0$.

Case 1. Let $u(y,t) > 0$ for $t \geq t_0$. Multiplying (1.1) by $\phi(y)$ and then integrating over Q, we get

$$\int_Q \frac{\partial}{\partial t}\Big(a(t)\frac{\partial}{\partial t}\Big(r(t)g(D_{+,t}^{\alpha}u(y,t))\Big)\Big)\phi(y)dy + \int_Q p(t)\frac{\partial}{\partial t}\Big(r(t)g(D_{+,t}^{\alpha}u(y,t))\Big)\phi(y)dy$$
$$= \int_Q b(t)\Delta u(y,t)\phi(y)dy + \int_Q \sum_{i=1}^{m} a_i(t)\Delta u(y,t-\tau_i)\phi(y)dy$$

$$-\int_Q\left(q(y,t)\int_0^t(t-\hbar)^{-\alpha}u(y,\hbar)d\hbar\right)\phi(y)dy+\int_Q f(y,t)\phi(y)dy. \tag{4.3}$$

By Lemma 4 and Green's formula, we have

$$\int_Q \Delta u(y,t)\phi(y)dy=\int_Q u(y,t)\Delta\phi(y)dy=-\beta_0\int_Q u(y,t)\phi(y)dy<0 \tag{4.4}$$

and

$$\int_Q \Delta u(y,t-\tau_i)\phi(y)dy<0. \tag{4.5}$$

Also, by $(\mathbf{H_5})$, we get

$$\begin{aligned}\int_Q\left(q(y,t)\int_0^t(t-\hbar)^{-\alpha}u(y,\hbar)\phi(y)d\hbar\right)dy &\geq q(t)\int_0^t(t-\hbar)^{-\alpha}\left(\int_Q u(y,\hbar)\phi(y)dy\right)d\hbar\\ &=q(t)L_1(t),\end{aligned} \tag{4.6}$$

where

$$L_1(t)=\int_0^t(t-\hbar)^{-\alpha}U_1(\hbar)d\hbar>0.$$

Because of the inequalities (4.4)–(4.6), (4.3) becomes

$$\frac{d}{dt}\left(a(t)\frac{d}{dt}\left(r(t)g(D_+^\alpha U_1(t))\right)\right)+p(t)\frac{d}{dt}\left(r(t)g(D_+^\alpha U_1(t))\right)\leq -q(t)L_1(t)+F_1(t)\leq F_1(t),$$

that is,

$$\left(e^{A(t)}a(t)\left(r(t)g(D_+^\alpha U_1(t))\right)'\right)'=e^{A(t)}\left[\left(a(t)\left(r(t)g(D_+^\alpha U_1(t))\right)'\right)'+p(t)\left(r(t)g(D_+^\alpha U_1(t))\right)'\right]\leq e^{A(t)}F_1(t).$$

Integrating the above inequality over $[t_0,t]$, we have

$$e^{A(t)}a(t)\left(r(t)g(D_+^\alpha U_1(t))\right)'\leq\int_{t_0}^t e^{A(\zeta)}F_1(\zeta)d\zeta+C_1,$$

where

$$C_1=e^{A(t_0)}a(t_0)\left(r(t_0)g(D_+^\alpha U_1(t_0))\right)'.$$

Again integrating the above inequality over $[t_0,t]$, we have

$$r(t)g(D_+^\alpha U_1(t))\leq C_2+\int_{t_0}^t\frac{C_1}{e^{A(\tau)}a(\tau)}d\tau+\int_{t_0}^t\frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^\tau e^{A(\zeta)}F_1(\zeta)d\zeta\right)d\tau,$$

where $C_2=r(t_0)g(D_+^\alpha U_1(t_0))$. Then using $(\mathbf{H_5})$, we obtain

$$\frac{D_+^\alpha U_1(t)}{k}\leq\frac{C_2}{r(t)}+\frac{1}{r(t)}\int_{t_0}^t\frac{C_1}{e^{A(\tau)}a(\tau)}d\tau+\frac{1}{r(t)}\int_{t_0}^t\frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^\tau e^{A(\zeta)}F_1(\zeta)d\zeta\right)d\tau.$$

Applying the Riemann-Liouville fractional integral operator of order α to the above inequality and using Lemma 3, we obtain

$$U_1(t) - \frac{I_0^{1-\alpha}U_1(0)}{\Gamma(\alpha)}t^{\alpha-1} \leq \frac{k}{\Gamma(\alpha)}\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\Big[C_2 + \int_{t_0}^{\hbar}\frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \int_{t_0}^{\hbar}\frac{1}{e^{A(\tau)}a(\tau)}\Big(\int_{t_0}^{\tau}e^{A(\zeta)}F_1(\zeta)d\zeta\Big)d\tau\Big]d\hbar.$$

Then

$$\liminf_{t\to\infty} U_1(t) \leq \liminf_{t\to\infty}\frac{A_1}{\Gamma(\alpha)}t^{\alpha-1} + \liminf_{t\to\infty}\Big\{\frac{k}{\Gamma(\alpha)}\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\Big[C_2 + \int_{t_0}^{\hbar}\frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \int_{t_0}^{\hbar}\frac{1}{e^{A(\tau)}a(\tau)}\Big(\int_{t_0}^{\tau}e^{A(\zeta)}F_1(\zeta)d\zeta\Big)d\tau\Big]d\hbar\Big\}.$$

Therefore, by our hypothesis given by (4.1), we get $\liminf_{t\to\infty} U_1(t) \leq 0$. This leads to a contradiction to $U_1(t) > 0$.

Case 2. Let $u(y,t) < 0$ for $t \geq t_0$. Multiplying (1.1) by $\phi(y)$ and then integrating over Q, one can get (4.3). Using Green's formula, we have

$$\int_Q \Delta u(y,t)\phi(y)dy = \int_Q u(y,t)\Delta\phi(y)dy = -\beta_0\int_Q u(y,t)\phi(y)dy > 0 \tag{4.7}$$

and

$$\int_Q \Delta u(y,t-\tau_i)\phi(y)dy > 0. \tag{4.8}$$

Also, by $(\mathbf{H_5})$, we have

$$\begin{aligned}\int_Q\Big(q(y,t)\int_0^t(t-\hbar)^{-\alpha}u(y,\hbar)d\hbar\Big)\phi(y)dy &\leq q(t)\int_0^t(t-\hbar)^{-\alpha}\Big(\int_Q u(y,\hbar)\phi(y)dy\Big)d\hbar\\ &= q(t)L_1(t),\end{aligned} \tag{4.9}$$

where

$$L_1(t) = \int_0^t (t-\hbar)^{-\alpha}U_1(\hbar)d\hbar < 0.$$

Because of the inequalities (4.7)–(4.9), (4.3) becomes

$$\frac{d}{dt}\Big(a(t)\frac{d}{dt}\Big(r(t)g(D_+^\alpha U_1(t))\Big)\Big)+p(t)\frac{d}{dt}\Big(r(t)g(D_+^\alpha U_1(t))\Big)\geq -q(t)L_1(t) + F_1(t) \geq F_1(t), \tag{4.10}$$

that is,

$$\Big(e^{A(t)}a(t)\Big(r(t)g(D_+^\alpha U_1(t))\Big)'\Big)' = e^{A(t)}\Big(\Big(a(t)\Big(r(t)g(D_+^\alpha U_1(t))\Big)'\Big)' + p(t)\Big(r(t)g(D_+^\alpha U_1(t))\Big)'\Big) \geq e^{A(t)}F_1(t).$$

Integrating the above inequality over $[t_0, t]$, we get

$$e^{A(t)}a(t)\left(r(t)g(D_+^\alpha U_1(t))\right)' \geq \int_{t_0}^t e^{A(\zeta)}F_1(\zeta)d\zeta + C_1,$$

where

$$C_1 = e^{A(t_0)}a(t_0)\left(r(t_0)g(D_+^\alpha U_1(t_0))\right)'.$$

Again integrating the above inequality over $[t_0, t]$, we get

$$r(t)g(D_+^\alpha U_1(t)) \ge C_2 + \int_{t_0}^t \frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \int_{t_0}^t \frac{1}{e^{A(\tau)}a(\tau)}\Big(\int_{t_0}^\tau e^{A(\zeta)}F_1(\zeta)d\zeta\Big)d\tau,$$

where $C_2 = r(t_0)g(D_+^\alpha U_1(t_0))$. Then using ($\mathbf{H_5}$), we obtain

$$\frac{D_+^\alpha U_1(t)}{k} \ge \frac{C_2}{r(t)} + \frac{1}{r(t)}\int_{t_0}^t \frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \frac{1}{r(t)}\int_{t_0}^t \frac{1}{e^{A(\tau)}a(\tau)}\Big(\int_{t_0}^\tau e^{A(\zeta)}F_1(\zeta)d\zeta\Big)d\tau.$$

Applying the Riemann-Liouville fractional integral operator of order α to the above inequality and using Lemma 3, we obtain

$$U_1(t) - \frac{I_0^{1-\alpha}U_1(0)}{\Gamma(\alpha)}t^{\alpha-1} \ge \frac{k}{\Gamma(\alpha)}\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\left[C_2 + \int_{t_0}^\hbar \frac{C_1}{e^{A(\tau)}a(\tau)}d\tau \right. \\ \left. + \int_{t_0}^\hbar \frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^\tau e^{A(\zeta)}F_1(\zeta)d\zeta\right)d\tau\right]d\hbar.$$

Then

$$\limsup_{t\to\infty} U_1(t) \ge \limsup_{t\to\infty} \frac{A_1}{\Gamma(\alpha)}t^{\alpha-1} + \limsup_{t\to\infty}\left\{\frac{k}{\Gamma(\alpha)}\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\left[C_2 + \int_{t_0}^\hbar \frac{C_1}{e^{A(\tau)}a(\tau)}d\tau \right.\right. \\ \left.\left. + \int_{t_0}^\hbar \frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^\tau e^{A(\zeta)}F_1(\zeta)d\zeta\right)d\tau\right]d\hbar\right\}.$$

Therefore, by our hypothesis given by (4.2), we get $\limsup_{t\to\infty} U_1(t) \ge 0$. This leads to a contradiction to $U_1(t) < 0$. □

5. Applications

In this section, we give two examples to illustrate our main results.

Example 1. Let us consider the fractional partial differential system

$$D_{+,t}^{\frac{5}{2}}u(y,t) = \frac{1}{\pi}\Delta u(y,t) + 2t\Delta u(y,t-1) \\ - \left(y^2 + \frac{1}{t^2}\right)\int_0^t (t-\hbar)^{-\frac{1}{2}}u(y,\hbar)d\hbar + e^{2t}\cos(t)\sin(y), \quad (y,t) \in (0,\pi)\times\mathbb{R}_+ \tag{5.1}$$

with the condition

$$u_y(0,t) = u_y(\pi,t) = 0. \tag{5.2}$$

In the above, $a(t) = 1, r(t) = 1, g(t) = t, \alpha = 1/2, p(t) = 0, b(t) = 1/\pi, m = 1, a_1(t) = 2t, \tau_1 = 1$, $q(y,t) = (y^2 + \frac{1}{t^2}), f(y,t) = e^{2t}\cos(t)\sin(y), Q = (0,\pi), q(t) = \min_{y\in(0,\pi)} q(y,t) = 1/t^2$ and $t_0 = 0$.

Since $F(t) = \int_0^\pi e^{2t}\cos(t)\sin(y)dy = 2e^{2t}\cos(t)$ and $A(t) = 0$, we have

$$\begin{aligned}
&\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\left[C_2 + \int_{t_0}^{\hbar}\frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \int_{t_0}^{\hbar}\frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^{\tau} e^{A(\zeta)}F(\zeta)d\zeta\right)d\tau\right]d\hbar \\
&= C_2\int_0^t (t-\hbar)^{-1/2}d\hbar + C_1\int_0^t (t-\hbar)^{-1/2}\left(\int_0^{\hbar} d\tau\right)d\hbar \\
&\quad + \frac{2}{5}\int_0^t (t-\hbar)^{-1/2}\left(2\int_0^{\hbar} e^{2\tau}\cos(\tau)d\tau + \int_0^{\hbar} e^{2\tau}\sin(\tau)d\tau - 2\int_0^{\hbar} d\tau\right)d\hbar \\
&= (C_2 - 6/25)\int_0^t (t-\hbar)^{-1/2}d\hbar + (C_1 - 4/5)\int_0^t \hbar(t-\hbar)^{-1/2}d\hbar \\
&\quad + 6/25\int_0^t e^{2\hbar}(t-\hbar)^{-1/2}\cos(\hbar)d\hbar + 8/25\int_0^t e^{2\hbar}(t-\hbar)^{-1/2}\sin(\hbar)d\hbar.
\end{aligned}$$

Fixing $y^2 = t - \hbar$, then

$$\begin{aligned}
&\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\left[C_2 + \int_{t_0}^{\hbar}\frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \int_{t_0}^{\hbar}\frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^{\tau} e^{A(\zeta)}F(\zeta)d\zeta\right)d\tau\right]d\hbar \\
&= 2(C_2 - 6/25)\sqrt{t} + 4/3(C_1 - 4/5)t^{3/2} \\
&\quad + 4/25e^{2t}\left(3\cos(t)\int_0^{\sqrt{t}} e^{-2y^2}\cos(y^2)dy + 3\sin(t)\int_0^{\sqrt{t}} e^{-2y^2}\sin(y^2)dy\right. \\
&\quad \left. +4\sin(t)\int_0^{\sqrt{t}} e^{-2y^2}\cos(y^2)dy - 4\cos(t)\int_0^{\sqrt{t}} e^{-2y^2}\sin(y^2)dy\right).
\end{aligned} \tag{5.3}$$

Pointing out that

$$|e^{-2y^2}\cos(y^2)| \le e^{-2y^2},\ |e^{-2y^2}\sin(y^2)| \le e^{-2y^2} \ \text{ and } \ \lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2y^2}dy = \frac{\sqrt{2\pi}}{4},$$

we can conclude that

$$\lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2y^2}\cos(y^2)dy \quad \text{and} \quad \lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2y^2}\sin(y^2)dy$$

are convergent. Thus, we have that

$$\begin{aligned}
\lim_{t\to\infty}&\left[\cos(t)\left(3\int_0^{\sqrt{t}} e^{-2y^2}\cos(y^2)dy - 4\int_0^{\sqrt{t}} e^{-2y^2}\sin(y^2)dy\right)\right. \\
&\left. + \sin t\left(3\int_0^{\sqrt{t}} e^{-2y^2}\sin(y^2)dy + 4\int_0^{\sqrt{t}} e^{-2y^2}\cos(y^2)dy\right)\right]
\end{aligned}$$

is convergent. Fixing

$$\lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2y^2}\cos(y^2)dy = P, \quad \lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2y^2}\sin(y^2)dy = Q$$

and considering the sequence

$$t_n = \frac{3\pi}{2} + 2n\pi - \arctan\left(\frac{3P-4Q}{3Q+4P}\right),$$

we get

$$\begin{aligned}
&\lim_{n\to\infty}\Bigg\{\cos(t_n)\left(3\int_0^{\sqrt{t_n}} e^{-2y^2}\cos(y^2)dy - 4\int_0^{\sqrt{t_n}} e^{-2y^2}\sin(y^2)dy\right)\\
&\quad + \sin(t_n)\left(3\int_0^{\sqrt{t_n}} e^{-2y^2}\sin(y^2)dy + 4\int_0^{\sqrt{t_n}} e^{-2y^2}\cos(y^2)dy\right)\Bigg\}\\
&= \sqrt{(3P-4Q)^2+(3Q+4P)^2}\sin\left(\frac{3\pi}{2} + 2n\pi - \arctan\left(\frac{3P-4Q}{3Q+4P}\right) + \arctan\left(\frac{3P-4Q}{3Q+4P}\right)\right)\\
&= 5\sqrt{P^2+Q^2}\sin\left(\frac{3\pi}{2} + 2n\pi\right)\\
&= -5\sqrt{P^2+Q^2}.
\end{aligned}$$

Since $\lim_{n\to\infty} t_n = \infty$, from (5.3), we have

$$\begin{aligned}
&\liminf_{t\to\infty}\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\left[C_2 + \int_{t_0}^{\hbar}\frac{C_1}{e^{A(\tau)}a(\tau)}d\tau + \int_{t_0}^{\hbar}\frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^{\tau} e^{A(\zeta)}F(\zeta)d\zeta\right)d\tau\right]d\hbar\\
&= \liminf_{n\to\infty}\Bigg\{2(C_2 - 6/25)\sqrt{t_n} + 4/3(e^{-1}C_1 - 4/5)t_n^{3/2} + 4/25e^{2t_n}\Bigg[\cos(t_n)\left(3\int_0^{\sqrt{t_n}} e^{-2y^2}\cos(y^2)dy\right.\\
&\quad \left.-4\int_0^{\sqrt{t_n}} e^{-2y^2}\sin(y^2)dy\right) + \sin(t_n)\left(3\int_0^{\sqrt{t_n}} e^{-2y^2}\sin(y^2)dy + 4\int_0^{\sqrt{t_n}} e^{-2y^2}\cos(y^2)dy\right)\Bigg]\Bigg\}\\
&= -\infty < 0.
\end{aligned}$$

Similarly, fixing

$$t_n = \frac{\pi}{2} + 2n\pi - \arctan\left(\frac{3P-4Q}{3Q+4P}\right),$$

we get

$$\begin{aligned}
&\lim_{n\to\infty}\Bigg[\cos(t_n)\left(3\int_0^{\sqrt{t_n}} e^{-2y^2}\cos(y^2)dy - 4\int_0^{\sqrt{t_n}} e^{-2y^2}\sin(y^2)dy\right)\\
&\quad + \sin(t_n)\left(3\int_0^{\sqrt{t_n}} e^{-2y^2}\sin(y^2)dy + 4\int_0^{\sqrt{t_n}} e^{-2y^2}\cos(y^2)dy\right)\Bigg]\\
&= 5\sqrt{P^2+Q^2}.
\end{aligned}$$

Thus, from (5.3), we can get

$$
\begin{aligned}
&\limsup_{t\to\infty}\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\left[C_2+\int_{t_0}^{\hbar}\frac{C_1}{e^{A(\tau)}a(\tau)}d\tau+\int_{t_0}^{\hbar}\frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^{\tau}e^{A(\zeta)}F(\zeta)d\zeta\right)d\tau\right]d\hbar\\
&=\limsup_{n\to\infty}\Bigg\{2(C_2-6/25)\sqrt{t_n}+4/3(e^{-1}C_1-4/5)t_n^{3/2}+4/25e^{2t_n}\Bigg[\cos(t_n)\left(3\int_0^{\sqrt{t_n}}e^{-2y^2}\cos(y^2)dy\right.\\
&\quad\left.-4\int_0^{\sqrt{t_n}}e^{-2y^2}\sin(y^2)dy\right)+\sin(t_n)\left(3\int_0^{\sqrt{t_n}}e^{-2y^2}\sin(y^2)dy+4\int_0^{\sqrt{t_n}}e^{-2y^2}\cos(y^2)dy\right)\Bigg]\Bigg\}\\
&=\infty>0.
\end{aligned}
$$

Therefore, by referring to Theorem 1, the solutions of (5.1) and (5.2) are oscillatory.

Example 2. Let us consider the fractional partial differential system

$$
\begin{aligned}
\frac{1}{1875\sqrt{\pi}}D_{+,t}^{\frac{5}{2}}u(y,t)=&\frac{1}{10^5t^{\frac{5}{2}}}\Delta u(y,t)+\left(\frac{16}{75\times10^2\pi t^3}-\frac{16e^{2t}\cos(t)}{5\pi t^3}\right)\int_0^t(t-\hbar)^{-\frac{1}{2}}u(y,\hbar)d\hbar\\
&+e^{2t}\cos(t)\cos(10y),\quad (y,t)\in(0,\pi)\times(0,1.5)
\end{aligned}
\tag{5.4}
$$

with the condition

$$
u(0,t)=u(\pi,t)=0. \tag{5.5}
$$

In the above, $a(t)=1, r(t)=\frac{1}{1875\sqrt{\pi}}$, $g(t)=t$, $\alpha=1/2$, $p(t)=0$, $b(t)=\frac{1}{2\times10^5t^{\frac{5}{2}}}$, $m=1$, $a_1(t)=\frac{1}{2\times10^5t^{\frac{5}{2}}}$, $\tau_1=0$, $q(y,t)=\frac{-16}{75\times10^2\pi t^3}+\frac{16e^{2t}\cos(t)}{5\pi t^3}$, $f(y,t)=e^{2t}\cos(t)\sin(y)$, $Q=(0,\pi)$, $q(t)=\min_{y\in(0,\pi)}q(y,t)=\frac{-16}{75\times10^2\pi t^3}+\frac{16e^{2t}\cos(t)}{5\pi t^3}$ and $t_0=0$. It is obvious that $\beta_0=1$ and $\phi(y)=\sin(y)$. Since $F_1(t)=\int_0^{\pi}e^{2t}\cos(t)\cos(10y)\sin(y)dy=\frac{-2}{99}e^{2t}\cos(t)$ and $A(t)=0$, we have

$$
\begin{aligned}
&\int_0^t\frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\left(C_2+\int_{t_0}^{\hbar}\frac{C_1}{e^{A(\tau)}a(\tau)}d\tau+\int_{t_0}^{\hbar}\frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^{\tau}e^{A(\zeta)}F(\zeta)d\zeta\right)d\tau\right)d\hbar\\
&=-\frac{3750\sqrt{\pi}}{99}\int_0^t(t-\hbar)^{\alpha-1}\left(\int_{t_0}^{\hbar}\left(\int_{t_0}^{\tau}e^{2\zeta}\cos(\zeta)d\zeta\right)d\tau\right)d\hbar\\
&=\frac{50\sqrt{\pi}}{11}\int_0^t(t-\hbar)^{-1/2}d\hbar+\frac{500\sqrt{\pi}}{33}\int_0^t\hbar(t-\hbar)^{-1/2}d\hbar\\
&\quad-\frac{50\sqrt{\pi}}{33}\left[3\int_0^te^{2\hbar}(t-\hbar)^{-1/2}\cos(\hbar)d\hbar+4\int_0^te^{2\hbar}(t-\hbar)^{-1/2}\sin(\hbar)d\hbar\right].
\end{aligned}
$$

Fixing $y^2=t-\hbar$, then

$$
\begin{aligned}
&\int_0^t\frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\left[C_2+\int_{t_0}^{\hbar}\frac{C_1}{e^{A(\tau)}a(\tau)}d\tau+\int_{t_0}^{\hbar}\frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^{\tau}e^{A(\zeta)}F(\zeta)d\zeta\right)d\tau\right]d\hbar\\
&=\frac{10^2\sqrt{\pi t}}{11}+\frac{2\times10^3}{99}t^{3/2}-\frac{10^2\sqrt{\pi}}{33}e^{2t}\Bigg\{3\cos(t)\int_0^{\sqrt{t}}e^{-2y^2}\cos(y^2)dy
\end{aligned}
$$

$$+3\sin(t)\int_0^{\sqrt{t}} e^{-2y^2}\sin(y^2)dy + 4\sin(t)\int_0^{\sqrt{t}} e^{-2y^2}\cos(y^2)dy - 4\cos(t)\int_0^{\sqrt{t}} e^{-2y^2}\sin(y^2)dy\Bigg\}. \quad (5.6)$$

Pointing out that

$$|e^{-2y^2}\cos(y^2)| \le e^{-2y^2},\ |e^{-2y^2}\sin(y^2)| \le e^{-2y^2} \text{ and } \lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2y^2}dy = \frac{\sqrt{2\pi}}{4},$$

we can conclude that

$$\lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2y^2}\cos(y^2)dy \quad \text{and} \quad \lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2y^2}\sin(y^2)dy$$

are convergent. Thus, we have that

$$\lim_{t\to\infty}\Bigg[\cos(t)\left(3\int_0^{\sqrt{t}} e^{-2y^2}\cos(y^2)dy - 4\int_0^{\sqrt{t}} e^{-2y^2}\sin(y^2)dy\right) + \sin t\left(3\int_0^{\sqrt{t}} e^{-2y^2}\sin(y^2)dy + 4\int_0^{\sqrt{t}} e^{-2y^2}\cos(y^2)dy\right)\Bigg]$$

is convergent. Fixing

$$\lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2y^2}\cos(y^2)dy = P \text{ and } \lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-2y^2}\sin(y^2)dy = Q$$

and considering the sequence

$$t_n = \frac{3\pi}{2} + 2n\pi - \arctan\left(\frac{3P-4Q}{3Q+4P}\right),$$

we get

$$\begin{aligned}
&\lim_{n\to\infty}\Bigg\{\cos(t_n)\left(3\int_0^{\sqrt{t_n}} e^{-2y^2}\cos(y^2)dy - 4\int_0^{\sqrt{t_n}} e^{-2y^2}\sin(y^2)dy\right) \\
&\qquad + \sin(t_n)\left(3\int_0^{\sqrt{t_n}} e^{-2y^2}\sin(y^2)dy + 4\int_0^{\sqrt{t_n}} e^{-2y^2}\cos(y^2)dy\right)\Bigg\} \\
&= \sqrt{(3P-4Q)^2+(3Q+4P)^2}\sin\left(\frac{3\pi}{2} + 2n\pi - \arctan\left(\frac{3P-4Q}{3Q+4P}\right) + \arctan\left(\frac{3P-4Q}{3Q+4P}\right)\right) \\
&= 5\sqrt{P^2+Q^2}\sin\left(\frac{3\pi}{2} + 2n\pi\right) \\
&= -5\sqrt{P^2+Q^2}.
\end{aligned}$$

Since $\lim_{n\to\infty} t_n = \infty$, from (5.6), we have

$$\limsup_{t\to\infty}\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\left[C_2+\int_{t_0}^{\hbar}\frac{C_1}{e^{A(\tau)}a(\tau)}d\tau+\int_{t_0}^{\hbar}\frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^{\tau}e^{A(\zeta)}F(\zeta)d\zeta\right)d\tau\right]d\hbar$$
$$=\limsup_{n\to\infty}\left\{\frac{10^2\sqrt{\pi t_n}}{11}\sqrt{t_n}+\frac{2\times10^3}{99}t_n^{3/2}-\frac{10^2\sqrt{\pi}}{33}e^{2t_n}\left[\cos(t_n)\left(3\int_0^{\sqrt{t_n}}e^{-2y^2}\cos(y^2)dy\right.\right.\right.$$
$$\left.\left.\left.-4\int_0^{\sqrt{t_n}}e^{-2y^2}\sin(y^2)dy\right)+\sin(t_n)\left(3\int_0^{\sqrt{t_n}}e^{-2y^2}\sin(y^2)dy+4\int_0^{\sqrt{t_n}}e^{-2y^2}\cos(y^2)dy\right)\right]\right\}$$
$$=\infty>0.$$

Similarly, fixing

$$t_n=\frac{\pi}{2}+2n\pi-\arctan\left(\frac{3P-4Q}{3Q+4P}\right),$$

we get

$$\lim_{n\to\infty}\left\{\cos(t_n)\left(3\int_0^{\sqrt{t_n}}e^{-2y^2}\cos(y^2)dy-4\int_0^{\sqrt{t_n}}e^{-2y^2}\sin(y^2)dy\right)\right.$$
$$\left.+\sin(t_n)\left(3\int_0^{\sqrt{t_n}}e^{-2y^2}\sin(y^2)dy+4\int_0^{\sqrt{t_n}}e^{-2y^2}\cos(y^2)dy\right)\right\}$$
$$=5\sqrt{P^2+Q^2}.$$

Thus, from (5.6), we get

$$\liminf_{t\to\infty}\int_0^t \frac{(t-\hbar)^{\alpha-1}}{r(\hbar)}\left[C_2+\int_{t_0}^{\hbar}\frac{C_1}{e^{A(\tau)}a(\tau)}d\tau+\int_{t_0}^{\hbar}\frac{1}{e^{A(\tau)}a(\tau)}\left(\int_{t_0}^{\tau}e^{A(\zeta)}F(\zeta)d\zeta\right)d\tau\right]d\hbar$$
$$=\liminf_{n\to\infty}\left\{\frac{10^2\sqrt{\pi t_n}}{11}\sqrt{t_n}+\frac{2\times10^3}{99}t_n^{3/2}-\frac{10^2\sqrt{\pi}}{33}e^{2t_n}\left[\cos(t_n)\left(3\int_0^{\sqrt{t_n}}e^{-2y^2}\cos(y^2)dy\right.\right.\right.$$
$$\left.\left.\left.-4\int_0^{\sqrt{t_n}}e^{-2y^2}\sin(y^2)dy\right)+\sin(t_n)\left(3\int_0^{\sqrt{t_n}}e^{-2y^2}\sin(y^2)dy+4\int_0^{\sqrt{t_n}}e^{-2y^2}\cos(y^2)dy\right)\right]\right\}$$
$$=-\infty<0.$$

Therefore, by referring to Theorem 2, the solutions of (5.4) and (5.5) are oscillatory. In fact, $u(y,t)=t^{5/2}\cos(10y)$ is a solution of (5.4) and (5.5) and its oscillatory behavior is demonstrated in Figure 1.

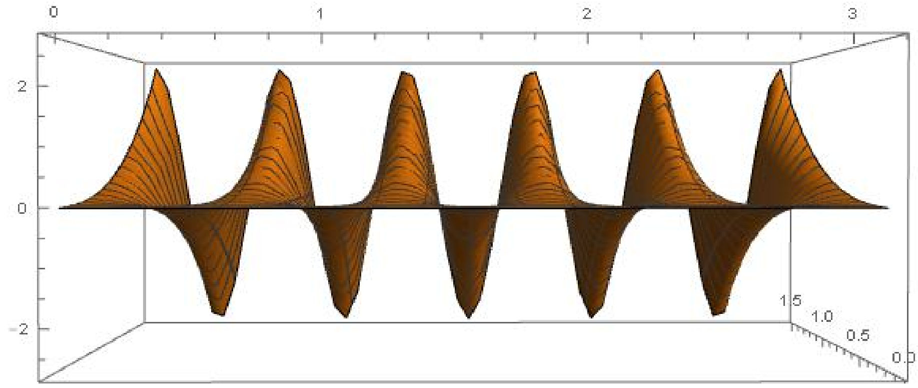

Figure 1. Oscillatory behavior of $u(y,t) = t^{5/2}\cos(10y)$.

6. Conclusions

In this paper, we have obtained some new oscillation results for the fractional partial differential equation with damping and forcing terms under Robin and Dirichlet boundary conditions. The main results are proved by using only the integral averaging technique and without implementing the Riccati approach. Further, the obtained results are justified by some examples which can not be commented upon by using the previous results. Our results have been obtained for the general equation which may cover other particular cases.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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