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Research article

On Pythagorean fuzzy ideals of a classical ring

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Abstract: The Pythagorean fuzzy set is an extension of the intuitionistic fuzzy set and is an effective approach of handling uncertain situations. Ring theory is a prominent branch of abstract algebra, vibrant in wide areas of current research in mathematics, computer science and mathematical/theoretical physics. In the theory of rings, the study of ideals is significant in many ways. Keeping in mind the importance of ring theory and Pythagorean fuzzy set, in the present article, we characterize the concept of Pythagorean fuzzy ideals in classical rings and study its numerous algebraic properties. We define the concept of Pythagorean fuzzy cosets of a Pythagorean fuzzy ideal and prove that the set of all Pythagorean fuzzy cosets of a Pythagorean fuzzy ideal forms a ring under certain binary operations. Furthermore, we present Pythagorean fuzzy version of the fundamental theorem of ring homomorphism. We also introduce the concept of Pythagorean fuzzy semi-prime ideals and give a detailed exposition of its different algebraic characteristics. In the end, we characterized regular rings by virtue of Pythagorean fuzzy ideals.

Keywords: Pythagorean fuzzy set; Pythagorean fuzzy ideal; Pythagorean fuzzy semi-prime ideal
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1. Introduction

Zadeh [1] introduced the notion of fuzzy sets (FSs) which has numerous applications in different branches of science and technology. A fuzzy subset $\mathfrak{F}$ on a universe K is denoted as $\{\alpha, \mu_{\mathfrak{F}}(\alpha) : \alpha \in K\}$, where $\mu_{\mathfrak{F}}$ is a function from K to $[0,1]$ and is called membership function.

Undoubtedly fuzzy set is a generalization of conventional set. In a conventional set A, the membership function is the characteristic function χ_A. The utilization of fuzzy set theory can be observed in almost every scientific field, particularly those involving set theory and mathematical logic. After the invention of fuzzy sets, many theories were put forward to deal with uncertainty and imprecision. Some of those are extensions of fuzzy sets, while others strive to deal with uncertainty in another suitable way. Later, it has been found that only the membership function is not sufficient to describe certain types of information. In this way, Atanassov [2] extended fuzzy sets to intuitionistic fuzzy sets (IFSs) to give a proper illustration of the information and allow a greater degree of freedom and flexibility in representing uncertainty. An IFS $\mathfrak{F}$ of a conventional set K is an object $\left\{\left(\alpha, \mu_{\mathfrak{F}}(\alpha), \nu_{\mathfrak{F}}(\alpha)\right): \alpha \in K\right\}$, where $\mu_{\mathfrak{F}}: K \to [0,1]$ and $\nu_{\mathfrak{F}}: K \to [0,1]$ are membership and non-membership functions respectively under the condition $\mu_{\mathcal{R}}(\alpha) + \nu_{\mathcal{R}}(\alpha) \le 1$ for all $\alpha \in K$. As compared to FS, the IFS handles uncertainty and vagueness in the field of decision-making [3,4] more effectively but even then, there is room for improvement. There exist many cases where IFSs unable to perform. For example: If a decision maker proposes $\mu_{\mathfrak{F}}(\alpha) = 0.7$ and $\nu_{\mathfrak{F}}(\alpha) = 0.4$ for some $\alpha \in K$. Then $\mu_{\mathcal{R}}(\alpha) + \nu_{\mathcal{R}}(\alpha) > 1$, therefore, such problems are beyond the limitations of IFS theory. To cope with these situations, Yager [5] generalized the notion of intuitionistic fuzzy sets by defining the Pythagorean fuzzy sets (PFSs). In [6], Zhang and Xu led the foundation of this novel concept. The Pythagorean fuzzy subset $\mathfrak{F}$ of K is denoted by $\left\{\left(\alpha, \mu_{\mathfrak{F}}(\alpha), \nu_{\mathfrak{F}}(\alpha)\right): \alpha \in K\right\}$, where $\mu_{\mathfrak{F}}: K \to [0,1]$ and $\nu_{\mathfrak{F}}: K \to [0,1]$ such that $\left(\mu_{\mathfrak{F}}(\alpha)\right)^2 + \left(\nu_{\mathfrak{F}}(\alpha)\right)^2 \le 1$ for all $\alpha \in K$. This idea is invented to transform vague and uncertain circumstances into mathematical form and to find an efficient solution [7,8].

1.1 Background and importance of ring theory

The theory of rings [9] is one of the important branches of mathematics. The idea of the ring was originally conceived to prove Fermat's Last Theorem, starting with Dedekind in the 1880s. After commitments from different fields (mainly number theory), the notion of the ring was summarized and firmly established in the decade of 1920 to 1930 by Noether and Krull. Present-day ring theory–an exceptionally dynamic numerical control–ponders rings in their very own right. To investigate rings, mathematicians have concocted different ideas to break rings into littler, better-reasonable pieces, for example, ideals, quotient rings and basic rings. In addition to these abstract attributes, ring theorists also make different qualifications between the theory of commutative rings and noncommutative rings. The commutative rings have a place in algebraic geometry and algebraic number theory. Noncommutative ring theory started with endeavors to extend complex numbers. The origins of commutative and non-commutative ring theories can be traced back to the early 19th century, while their maturity was achieved in the third decade of the 20th century. Over the past decade, ring theory has been applied to various branches of science, especially computer science, coding theory, and cryptography [10–12].

1.2 Literature review

In 1982, Liu [13] generalized the notion of the subring/ideal of a ring to the fuzzy subring/fuzzy ideal of a ring. In [14], different operations between fuzzy ideals of a ring have been defined. Mukherjee and Malik [15] published a fundamental paper on fuzzy ideals over Artinian rings. The author presented a characterization of Artinian rings with respect to fuzzy ideals. Hur et al. [16] defined the notion of intuitionistic fuzzy ideals of a ring. The notion of ω-fuzzy ideal of a ring is defined in [17]. Shabir et al. [18] investigated the approximation of bipolar fuzzy ideals of semirings. In [19], the notion of anti-fuzzy multi-ideals of the near ring is discussed. Madeline et al. [20] defined fuzzy multi-ideals of near rings and proved some fundamental theorems of this notion. A fuzzy version of Zorn's lemma is present in [21]. The author used it to demonstrate that every proper fuzzy ideal of a ring is contained in a maximal fuzzy ideal. Addis et al. [22] proved some important results regarding fuzzy homomorphism. The algebraic characteristics of (α,β)-Pythagorean fuzzy ideals of a ring are discussed in [23]. Hakim et al. presented a study [24] related to bipolar soft semiprime ideals over ordered semigroups. The concept of fuzzy ideals of LA rings is described in [25]. The authors presented a characterization of regular LA-rings with respect to fuzzy ideals.

1.3 Research gap in the current literature and motivation of the study

The above literature review highlights some research achievements in classical and intuitionistic fuzzy ring theory. Additionally, although certain results about (α,β)-Pythagorean fuzzy subrings of a ring and bipolar Pythagorean fuzzy subring of a ring have been demonstrated but some open questions remain to be answered.

1) In classical ring theory, the interstation of two subrings/ideals of a ring K is a subring/ideal of K. Therefore, a question arises, whether the intersection of two Pythagorean fuzzy subrings/ideals is a Pythagorean fuzzy subring/ideal of K? Moreover, if S is a subring of a ring K and I is an ideal of K, then $S\cap I$ is an ideal of S. The analogous version of this theorem in the Pythagorean fuzzy framework needs to be studied.

2) The characterization of classical/intuitionistic fuzzy subrings/ideals with respect to classical/intuitionistic fuzzy level sub-rings/ideals is given in the existing literature. Since a Pythagorean fuzzy ring theory is a generalization of intuitionistic fuzzy ring theory, therefore, it is important to understand the characterization of Pythagorean fuzzy subrings/ideals in terms of Pythagorean fuzzy level fuzzy subrings/ideals.

3) In classical ring theory, cosets of subring/ideal of a ring K is an important notion because it gives rise to quotient rings. It is well known that the set of all cosets of an ideal of K forms ring under a certain binary operation. A natural question comes into mind, does the set of all Pythagorean fuzzy cosets of Pythagorean fuzzy ideal of K, forms a ring under certain binary operations?

4) The fundamental theorem of ring homomorphism is one of the finest results in classical ring theory. Therefore, it is necessary to discuss this remarkable theorem in the context of Pythagorean fuzzy rings.

5) In the literature, various algebraic properties of fuzzy semi-prime ideals have been discussed. Furthermore, the characterization of regular rings by virtue of fuzzy ideals is presented. In the context of Pythagorean fuzzy theory, these studies have yet to be examined from a broader perspective.

Answering the above-mentioned open problems and bridging the knowledge gap in the existing literature is the ultimate aim of this research.

1.4 Comparative study and limitations of the current research

The results proved in this paper are valid for Pythagorean fuzzy ideals. Since every IFS is a PFS, therefore, the same hold for intuitionistic fuzzy ideals as well. Moreover, every fuzzy set is an IFS, so the present study can also be applied to fuzzy ideals. However, we cannot apply these results directly to q-rung orthopair fuzzy ideals, picture fuzzy ideals, neutrosophic fuzzy ideals, fuzzy soft ideals and fuzzy hypersoft ideals. Therefore, separate studies are recommended for these generalized structures. This is the main limitation of our research.

The rest of the paper is set up in this way: Section 2 contains basic definitions and concepts that are required to demonstrate our main results. In Section 3, the internal description of the Pythagorean fuzzy ideal along with its fundamental properties are discussed. The notions of Pythagorean fuzzy cosets of a Pythagorean fuzzy ideal is defined in Section 4. We prove that the set of all Pythagorean fuzzy cosets of a Pythagorean fuzzy ideal forms a ring under certain binary operations. Moreover, the Pythagorean fuzzy version of the fundamental theorem of ring homomorphisms has been proved. In Section 5, the concept of the Pythagorean fuzzy semi-prime ideal is defined. We investigate some important algebraic features of this newly defined notion. Furthermore, the characterization of regular rings by virtue of PFI is presented. The conclusion of this paper is presented in Section 6.

2. Preliminaries

This section contains some notions and concepts which are needed to prove our main theorems.

Definition 2.1. [13] A FS $\mathfrak{F} = \{(\alpha, \mu_{\mathfrak{F}}(\alpha)) : \alpha \in K\}$ of a ring K is called a fuzzy subring (FSR) of K if for all $\alpha_1, \alpha_2 \in K$, the following properties are satisfied:

i. $\mu_{\mathfrak{F}}(\alpha_1 - \alpha_2) \geq min\{\mu_{\mathfrak{F}}(\alpha_1), \mu_{\mathfrak{F}}(\alpha_2)\}$,

ii. $\mu_{\mathfrak{F}}(\alpha_1 \alpha_2) \geq min\{\mu_{\mathfrak{F}}(\alpha_1), \mu_{\mathfrak{F}}(\alpha_2)\}$.

The fuzzy ideal (FI) of K has the same definition with the only difference that in condition (ii) “min” is replaced by “max”.

Definition 2.2. [16] An IFS $\mathfrak{F} = \{(\alpha, \mu_{\mathfrak{F}}(\alpha), \nu_{\mathfrak{F}}(\alpha)) : \alpha \in K\}$ of K is called an intuitionistic fuzzy subring (IFSR) of K if for all $\alpha_1, \alpha_2 \in K$, the following requirements are fulfilled:

i. $\mu_{\mathfrak{F}}(\alpha_1 - \alpha_2) \geq min\{\mu_{\mathfrak{F}}(\alpha_1), \mu_{\mathfrak{F}}(\alpha_2)\}$ and $\nu_{\mathfrak{F}}(\alpha_1 - \alpha_2) \leq max\{\nu_{\mathfrak{F}}(\alpha_1), \nu_{\mathfrak{F}}(\alpha_2)\}$,

ii. $\mu_{\mathfrak{F}}(\alpha_1 \alpha_2) \geq min\{\mu_{\mathfrak{F}}(\alpha_1), \mu_{\mathfrak{F}}(\alpha_2)\}$ and $\nu_{\mathfrak{F}}(\alpha_1 \alpha_2) \leq max\{\nu_{\mathfrak{F}}(\alpha_1), \nu_{\mathfrak{F}}(\alpha_2)\}$.

By interchanging “min” and “max” by “max” and “min” respectively, in condition (ii), we obtain the definition of intuitionistic fuzzy ideal (IFI) of K.

Next, we define Pythagorean fuzzy subrings (PFSRs) and Pythagorean fuzzy ideals (PFIs) of a ring K.

Definition 2.3 A PFS $\mathfrak{F} = \{(\alpha, \mu_{\mathfrak{F}}(\alpha), \nu_{\mathfrak{F}}(\alpha)) : \alpha \in K\}$ of K is called a PFSR of K if for all $\alpha_1, \alpha_2 \in K$, the following requirements are fulfilled:

i. $\left(\mu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 \geq min\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$ and $\left(\nu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 \leq max\left\{\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$,

ii. $\left(\mu_{\mathfrak{F}}(\alpha_1 \alpha_2)\right)^2 \geq min\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$ and $\left(\nu_{\mathfrak{F}}(\alpha_1 \alpha_2)\right)^2 \leq max\left\{\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$.

The PFS $\mathfrak{F}$ is known to be PFI of K, if condition (ii) is replaced with $\left(\mu_{\mathfrak{F}}(\alpha_1 \alpha_2)\right)^2 \geq$

$max\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$ and $\left(\nu_{\mathfrak{F}}(\alpha_1\alpha_2)\right)^2 \leq min\left\{\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$.

It is evident from Definitions 2.2 and 2.3 that every IFR/IFI is a PFR/PFI. The following example demonstrates that the converse is not true.

Example 2.1. We know $Z_4 = \{0,1,2,3\}$ is a ring with respect to addition and multiplication modulo 4. It is easy to verify that $\mathfrak{F} = \{(0,0.9,0.3), (3,0.7,0.5), (2,0.8,0.4), (4,0.7,0.5)\}$ is a PFSR/PFI of $Z_4 = \{0,1,2,3\}$. Since the sum of membership and non-membership values is not less than or equal to one for all elements in Z_4, therefore, $\mathfrak{F}$ is not an IFS and hence not an IFR/IFI.

Definition 2.4. Let $\mathfrak{F} = \{(\alpha, \mu_{\mathfrak{F}}(\alpha), \nu_{\mathfrak{F}}(\alpha): \alpha \in K\}$ be a PFS of K. Then the set $\mathfrak{F}_{(\sigma,\rho)} = \left\{\alpha \in K: \left(\mu_{\mathfrak{F}}(\alpha)\right)^2 \geq \sigma, \left(\nu_{\mathfrak{F}}(\alpha)\right)^2 \leq \rho\right\}$ is known as Pythagorean fuzzy level set (PFLS) of $\mathfrak{F}$.

Definition 2.5. Let $\mathfrak{F}$ and $\mathfrak{A}$ be two PFS of K. The product $\mathfrak{F} \circ \mathfrak{A}$ of $\mathfrak{F}$ and $\mathfrak{A}$ is defined by

$$\mathfrak{F} \circ \mathfrak{A} = \{(\alpha, \mu_{\mathfrak{F}\circ\mathfrak{A}}(\alpha), \nu_{\mathfrak{F}\circ\mathfrak{A}}(\alpha): \alpha \in K\},$$

where $\left(\mu_{\mathfrak{F}\circ\mathfrak{A}}(\alpha)\right)^2 = \max\left\{\min\left(\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{A}}(\alpha_2)\right)^2\right): \alpha_1, \alpha_2 \in K, \alpha_1\alpha_2 = \alpha\right\}$ and $\left(\nu_{\mathfrak{F}\circ\mathfrak{A}}(\alpha)\right)^2 = \min\left\{\max\left(\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{A}}(\alpha_2)\right)^2\right): \alpha_1, \alpha_2 \in K, \alpha_1\alpha_2 = \alpha\right\}$.

3. Some fundamental results

This section contains some internal description of the Pythagorean fuzzy ideal and its fundamental properties.

Remark. 3.1. Every PFI of K is a PFSR of K.

In the following example, we see a PFSR of K which is not a PFI of K.

Example. 3.1. Consider $S = \{1,2,3\}$, then 2^S is the power set of S, that is,

$$2^S = \{\phi, S, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$$

forms ring under symmetric difference Δ and intersection $\cap$. Now, it is just a matter of simple calculation to conclude that

$$\mathfrak{F} = \left\{\begin{matrix} (\phi, 0.90,0.20), (S, 0.90,0.20), (\{1\}, 0.60,0.70), (\{2\}, 0.60,0.70), \\ (\{3\},0.90,0.20), (\{1,2\}, 0.90,0.20), (\{1,3\},0.60,0.70), (\{2,3\},0.60,0.70) \end{matrix}\right\}$$

is a PFSR of 2^S. But

$$\left(\mu_{\mathfrak{F}}(\{1,2\} \cap \{1\})\right)^2 = \left(\mu_{\mathfrak{F}}(\{1\})\right)^2 = (0.60)^2$$

and

$$max\left\{\left(\mu_{\mathfrak{F}}(\{1,2\})\right)^2, \left(\mu_{\mathfrak{F}}(\{1\})\right)^2\right\} = (0.90)^2$$

together imply that $\mathfrak{F}$ is not a PFI of R.

Theorem 3.1. Let $\mathfrak{F}$ be a PFI of a ring K, then,

i. $\left(\mu_{\mathfrak{F}}(0)\right)^2 \geq \left(\mu_{\mathfrak{F}}(\alpha)\right)^2$ and $\left(\nu_{\mathfrak{F}}(0)\right)^2 \leq \left(\nu_{\mathfrak{F}}(\alpha)\right)^2$ for all $\alpha \in K$.

ii. $\mu_{\mathfrak{F}}(\alpha_1) \neq \mu_{\mathfrak{F}}(\alpha_2)$ and $\nu_{\mathfrak{F}}(\alpha_1) \neq \nu_{\mathfrak{F}}(\alpha_2)$ for some $\alpha_1, \alpha_2 \in K$ inplies that $\left(\mu_{\mathfrak{F}}(\alpha_1-\alpha_2)\right)^2 = min\left[\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right]$ and $\left(\nu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 = max\left[\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right]$.

Proof. i. Let $\alpha \in K$, then $\left(\mu_{\mathfrak{F}}(0)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha - \alpha)\right)^2 \geq min\left\{\left(\mu_{\mathfrak{F}}(\alpha)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha)\right)^2\right\} = \left(\mu_{\mathfrak{F}}(\alpha)\right)^2$. Similarly, we can prove $\left(\nu_{\mathfrak{F}}(0)\right)^2 \leq \left(\nu_{\mathfrak{F}}(\alpha)\right)^2$ for all $\alpha \in K$.

ii. Assume that $\alpha_1,\alpha_2 \in K$ such that $\mu_{\mathfrak{F}}(\alpha_1) > \mu_{\mathfrak{F}}(\alpha_2)$, then obviously $\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2 > \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2$.

Consider

$$\left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha_1 - (\alpha_1 - \alpha_2))\right)^2 \geq min\left[\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2\right]. \tag{3.1}$$

Since $\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2 > \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2$, therefore, Eq (3.1) yields

$$\left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2 \geq \left(\mu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2. \tag{3.2}$$

Furthermore,

$$\left(\mu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 \geq min\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\} = \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2. \tag{3.3}$$

The Eqs (3.2) and (3.3) together imply that

$$\left(\mu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2 = min\left[\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right].$$

In the identical way, it can be shown that

$$\left(\nu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 = max\left[\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right].$$

Theorem 3.2. The intersection of two PFIs of K is a PFI of K.

Proof. Suppose that $\mathfrak{F}_1 = \{\alpha, \mu_{\mathfrak{F}_1}(\alpha), \nu_{\mathfrak{F}_1}(\alpha)\}$ and $\mathfrak{F}_2 = \{\alpha, \mu_{\mathfrak{F}_2}(\alpha), \nu_{\mathfrak{F}_2}(\alpha)\}$ are PFIs of K. Then for all $\alpha_1, \alpha_2 \in K$, we have

$$\begin{aligned}
\left(\mu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_1 - \alpha_2)\right)^2 &= min\left[\left(\mu_{\mathfrak{F}_1}(\alpha_1-\alpha_2)\right)^2, \left(\mu_{\mathfrak{F}_2}(\alpha_1-\alpha_2)\right)^2\right] \\
&\geq min\left[min\left(\left(\mu_{\mathfrak{F}_1}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}_1}(\alpha_2)\right)^2\right), min\left(\left(\mu_{\mathfrak{F}_2}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}_2}(\alpha_2)\right)^2\right)\right] \\
&= min\left[min\left(\left(\mu_{\mathfrak{F}_1}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}_2}(\alpha_1)\right)^2\right), min\left(\left(\mu_{\mathfrak{F}_1}(\alpha_2)\right)^2, \left(\mu_{\mathfrak{F}_2}(\alpha_2)\right)^2\right)\right] \\
&= min\left[\left(\mu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_2)\right)^2\right].
\end{aligned}$$

Therefore, $\left(\mu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_1-\alpha_2)\right)^2 \geq min\left[\left(\mu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_1)\right)^2,\left(\mu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_2)\right)^2\right]$. Similarly, $\left(\nu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_1-\alpha_2)\right)^2 \leq max\left[\left(\nu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_1)\right)^2,\left(\nu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_2)\right)^2\right]$. Next,

$$\begin{aligned}\left(\mu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_1\alpha_2)\right)^2 &= min\left[\left(\mu_{\mathfrak{F}_1}(\alpha_1\alpha_2)\right)^2,\left(\mu_{\mathfrak{F}_2}(\alpha_1\alpha_2)\right)^2\right]\\ &\geq min\left[\max\left(\left(\mu_{\mathfrak{F}_1}(\alpha_1)\right)^2,\left(\mu_{\mathfrak{F}_1}(\alpha_2)\right)^2\right),\max\left(\left(\mu_{\mathfrak{F}_2}(\alpha_1)\right)^2,\left(\mu_{\mathfrak{F}_2}(\alpha_2)\right)^2\right)\right]\\ &\geq max\left[min\left(\left(\mu_{\mathfrak{F}_1}(\alpha_1)\right)^2,\left(\mu_{\mathfrak{F}_2}(\alpha_1)\right)^2\right),min\left(\left(\mu_{\mathfrak{F}_1}(\alpha_2)\right)^2,\left(\mu_{\mathfrak{F}_2}(\alpha_2)\right)^2\right)\right]\\ &= max\left[\left(\mu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_1)\right)^2,\left(\mu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_2)\right)^2\right].\end{aligned}$$

That is, $\left(\mu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_1\alpha_2)\right)^2 \geq max\left[\left(\mu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_1)\right)^2,\left(\mu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_2)\right)^2\right]$. The utilization of the same arguments gives $\left(\nu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_1\alpha_2)\right)^2 \leq min\left[\left(\nu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_1)\right)^2,\left(\nu_{\mathfrak{F}_1\cap\mathfrak{F}_2}(\alpha_2)\right)^2\right]$. Thus, $\mathfrak{F}_1\cap\mathfrak{F}_2$ is a PFI of K.

Theorem 3.3. Let $\mathfrak{F}=\{\alpha,\mu_{\mathfrak{F}}(\alpha),\nu_{\mathfrak{F}}(\alpha)\}$ be a PFI of K. Then,

i. $\mathfrak{F}_{*+}=\left\{\alpha\in K:\left(\mu_{\mathfrak{F}}(\alpha)\right)^2=\left(\mu_{\mathfrak{F}}(0)\right)^2\right\}$ is an ideal of K.

ii. $\mathfrak{F}_{*-}=\left\{\alpha\in K:\left(\nu_{\mathfrak{F}}(\alpha)\right)^2=\left(\nu_{\mathfrak{F}}(0)\right)^2\right\}$ is an ideal of K.

iii. $\mathfrak{F}_{*}=\left\{\alpha\in K:\left(\mu_{\mathfrak{F}}(\alpha)\right)^2=\left(\mu_{\mathfrak{F}}(0)\right)^2 \text{ and } \left(\nu_{\mathfrak{F}}(\alpha)\right)^2=\left(\nu_{\mathfrak{F}}(0)\right)^2\right\}$ is an ideal of K.

Proof. i. By definition of $\mathfrak{F}_{*+}$, we have $0\in\mathfrak{F}_{*+}$. Therefore, $\mathfrak{F}_{*+}$ is non-empty subset of K.

Let $\alpha_1,\alpha_2\in\mathfrak{F}_{*+}$, then $\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2=\left(\mu_{\mathfrak{F}}(0)\right)^2=\left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2$. Consider

$$\begin{aligned}\left(\mu_{\mathfrak{F}}(\alpha_1-\alpha_2)\right)^2 &\geq min\left[\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2,\left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right]=min\left[\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2,\left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right]\\ &= min\left[\left(\mu_{\mathfrak{F}}(0)\right)^2,\left(\mu_{\mathfrak{F}}(0)\right)^2\right]=\left(\mu_{\mathfrak{F}}(0)\right)^2.\end{aligned}$$

Moreover, from Theorem 3.1, it can be obtained $\left(\mu_{\mathfrak{F}}(0)\right)^2\geq\left(\mu_{\mathfrak{F}}(\alpha_1-\alpha_2)\right)^2$. Therefore, $\left(\mu_{\mathfrak{F}}(\alpha_1-\alpha_2)\right)^2=\left(\mu_{\mathfrak{F}}(0)\right)^2$ implying that $\alpha_1-\alpha_2\in\mathfrak{F}_{*+}$.

Now, suppose that $\alpha\in\mathfrak{F}_{*+}$ and $\beta\in K$. Then,

$$\left(\mu_{\mathfrak{F}}(\alpha\beta)\right)^2\geq max\left[\left(\mu_{\mathfrak{F}}(\alpha)\right)^2,\left(\mu_{\mathfrak{F}}(\beta)\right)^2\right]=\left(\mu_{\mathfrak{F}}(0)\right)^2.$$

In view of Theorem 3.1, $\left(\mu_{\mathfrak{F}}(0)\right)^2\geq\left(\mu_{\mathfrak{F}}(\alpha\beta)\right)^2$. So, $\left(\mu_{\mathfrak{F}}(\alpha\beta)\right)^2=\left(\mu_{\mathfrak{F}}(0)\right)^2\Rightarrow\alpha\beta\in\mathfrak{F}_{*+}$. Similarly, it can be proved that $\beta\alpha\in\mathfrak{F}_{*+}$. Thus, $\mathfrak{F}_{*+}$ is an ideal of K.

ii. The proof is similar to that of (i).

iii. The proof is straightforward by using (i) and (ii).

Theorem 3.4. The intersection of a PFSR $\mathfrak{F}$ and a PFI $\mathfrak{A}$ of a ring K is a PFI of $\mathfrak{F}_*$.

Proof. Let $\alpha_1, \alpha_2 \in \mathfrak{F}_*$, then,

$$\left(\mu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_1-\alpha_2)\right)^2 = min\left[\left(\mu_{\mathfrak{F}}(\alpha_1-\alpha_2)\right)^2, \left(\mu_{\mathfrak{A}}(\alpha_1-\alpha_2)\right)^2\right]$$
$$\geq min\left[min\left(\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right), min\left(\left(\mu_{\mathfrak{A}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{A}}(\alpha_2)\right)^2\right)\right]$$
$$= min\left[min\left(\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{A}}(\alpha_1)\right)^2\right), min\left(\left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2, \left(\mu_{\mathfrak{A}}(\alpha_2)\right)^2\right)\right]$$
$$= min\left[\left(\mu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_2)\right)^2\right].$$

Therefore, $\left(\mu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_1-\alpha_2)\right)^2 \geq min\left[\left(\mu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_2)\right)^2\right]$. Similarly, $\left(\nu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_1-\alpha_2)\right)^2 \leq max\left[\left(\nu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_2)\right)^2\right]$. Next,

$$\left(\mu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_1\alpha_2)\right)^2 = min\left[\left(\mu_{\mathfrak{F}}(\alpha_1\alpha_2)\right)^2, \left(\mu_{\mathfrak{A}}(\alpha_1\alpha_2)\right)^2\right]$$
$$\geq min\left[min\left(\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right), max\left(\left(\mu_{\mathfrak{A}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{A}}(\alpha_2)\right)^2\right)\right]$$
$$= min\begin{bmatrix} max\left(\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right), \\ max\left(\left(\mu_{\mathfrak{A}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{A}}(\alpha_2)\right)^2\right)\end{bmatrix}, \text{ as } \left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2 = 0 = \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2$$
$$\geq max\left[min\left(\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{A}}(\alpha_1)\right)^2\right), min\left(\left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2, \left(\mu_{\mathfrak{A}}(\alpha_2)\right)^2\right)\right]$$
$$= max\left[\left(\mu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_2)\right)^2\right].$$

That is, $\left(\mu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_1\alpha_2)\right)^2 \geq max\left[\left(\mu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_2)\right)^2\right]$. By using the same arguments, it can be obtainable that $\left(\nu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_1\alpha_2)\right)^2 \leq \min\left[\left(\nu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}\cap\mathfrak{A}}(\alpha_2)\right)^2\right]$. Thus, $\mathfrak{F}\cap\mathfrak{A}$ is a PFI of $\mathfrak{F}_*$.

We present the following example to explain Theorem 3.4.

Example 3.2. Consider a PFSR $\mathfrak{F} = \left\{\left(a, \mu_{\mathfrak{F}}(a) = \begin{cases} 0.80, & if\ a \in \mathbb{Z}, \\ 0.70, & otherwise\end{cases}, \nu_{\mathfrak{F}}(a) = \begin{cases} 0.30, & if\ a \in \mathbb{Z}, \\ 0.40, & otherwise\end{cases}\right): a \in \mathbb{R}\right\}$ and a PFI $\mathfrak{A} = \left\{\left(a, \mu_{\mathfrak{A}}(a) = \begin{cases} 0.90, & if\ a = 0, \\ 0.75, & otherwise\end{cases}, \nu_{\mathfrak{A}}(a) = \begin{cases} 0.25, & if\ a = 0, \\ 0.50, & otherwise\end{cases}\right): a \in \mathbb{R}\right\}$ of the ring of real numbers $\mathbb{R}$. Then, $\mathfrak{F}_* = \mathbb{Z}$ and $\mathfrak{F}\cap\mathfrak{A} = \left\{\left(a, \mu_{\mathfrak{F}\cap\mathfrak{A}}(a) = \begin{cases} 0.80, & if\ a = 0, \\ 0.75, & if\ a \in \mathbb{Z}-\{0\}, \\ 0.70, & otherwise\end{cases}, \nu_{\mathfrak{F}\cap\mathfrak{A}}(a) = \begin{cases} 0.30, & if\ a = 0, \\ 0.50, & if\ a \in \mathbb{Z}-\{0\} \\ 0.50, & otherwise\end{cases}\right): a \in \mathbb{R}\right\}$. It can be easily validated that $\mathfrak{F}\cap\mathfrak{A}$ is a PFI of $\mathfrak{F}_* = \mathbb{Z}$.

Theorem 3.5. A PFS $\mathfrak{F}$ of a ring K is PFI of K if and only if $\mathfrak{F}_{(\sigma,\rho)}$ is an ideal of K for all $\sigma \in \left[0, \left(\mu_{\mathfrak{F}}(0)\right)^2\right]$ and $\rho \in \left[\left(\nu_{\mathfrak{F}}(0)\right)^2, 1\right]$.

Proof. Suppose that $\mathfrak{F} = \{(\alpha, \mu_{\mathfrak{F}}(\alpha), \nu_{\mathfrak{F}}(\alpha): \alpha \in K\}$ is PFI of K. We want to prove that $\mathfrak{F}_{(\sigma,\rho)}$ is an ideal of K for all $\sigma \in \left[0, \left(\mu_{\mathfrak{F}}(0)\right)^2\right]$ and $\rho \in \left[\left(\nu_{\mathfrak{F}}(0)\right)^2, 1\right]$.

For all such σ and ρ, clearly $\left(\mu_{\mathfrak{F}}(0)\right)^2 \geq \sigma$ and $\left(\nu_{\mathfrak{F}}(0)\right)^2 \leq \rho$. Therefore, $0 \in \mathfrak{F}_{(\sigma,\rho)}$ implying that $\mathfrak{F}_{(\sigma,\rho)}$ is a non-empty set.

Suppose $\alpha_1, \alpha_2 \in \mathfrak{F}_{(\sigma,\rho)}$, which means that $\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2 \geq \sigma$ and $\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2 \leq \rho$. Since $\mathfrak{F}$ is PFI of K, therefore,

$$\left(\mu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 \geq min\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\} \geq min\{\sigma, \sigma\} = \sigma$$

and

$$\left(\nu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 \leq max\left\{\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right\} \leq max\{\rho, \rho\} = \rho$$

together imply that $\alpha_1 - \alpha_2 \in \mathfrak{F}_{(\sigma,\rho)}$.

Again, assume that $\alpha \in \mathfrak{F}_{(\sigma,\rho)}$ and $\beta \in K$, then $\left(\mu_{\mathfrak{F}}(\alpha)\right)^2 \geq \sigma$ and $\left(\nu_{\mathfrak{F}}(\alpha)\right)^2 \leq \rho$. Since $\mathfrak{F}$ is PFI of K, therefore,

$$\left(\mu_{\mathfrak{F}}(\alpha\beta)\right)^2 \geq max\left\{\left(\mu_{\mathfrak{F}}(\alpha)\right)^2, \left(\mu_{\mathfrak{F}}(\beta)\right)^2\right\} \geq max\left\{\sigma, \left(\mu_{\mathfrak{F}}(\beta)\right)^2\right\} \geq \sigma$$

and

$$\left(\nu_{\mathfrak{F}}(\alpha\beta)\right)^2 \leq max\left\{\left(\nu_{\mathfrak{F}}(\alpha)\right)^2, \left(\nu_{\mathfrak{F}}(\beta)\right)^2\right\} \leq min\left\{\rho, \left(\nu_{\mathfrak{F}}(\beta)\right)^2\right\} \leq \rho$$

together imply that $\alpha\beta \in \mathfrak{F}_{(\sigma,\rho)}$. In a similar way, we can prove that $\beta\alpha \in \mathfrak{F}_{(\sigma,\rho)}$. Thus, $\mathfrak{F}_{(\sigma,\rho)}$ is an ideal of K.

Conversely, let $\mathfrak{F}_{(\sigma,\rho)}$ be an ideal of K for all $\sigma \in \left[0, \left(\mu_{\mathfrak{F}}(0)\right)^2\right]$ and $\rho \in \left[\left(\nu_{\mathfrak{F}}(0)\right)^2, 1\right]$. To show $\mathfrak{F}$ is a PFI of K, firstly suppose that $\alpha_1, \alpha_2 \in K$ and let $\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2 = \sigma_1$, $\left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2 = \sigma_2$, $\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2 = \rho_1$ and $\left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2 = \rho_2$. Then,

i. $\alpha_1, \alpha_2 \in \mathfrak{F}_{(min(\sigma_1,\sigma_2), max(\rho_1,\rho_2))}$, since $\mathfrak{F}_{(min(\sigma_1,\sigma_2), max(\rho_1,\rho_2))}$ is an ideal of K, therefore $\alpha_1 - \alpha_2 \in \mathfrak{F}_{(min(\sigma_1,\sigma_2), max(\rho_1,\rho_2))}$, which yields $\left(\mu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 \geq min(\sigma_1, \sigma_2) = min\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$ and $\left(\nu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 \leq max(\rho_1, \rho_2) = max\left\{\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$.

ii. Either $\alpha_1 \in \mathfrak{F}_{(max(\sigma_1,\sigma_2), min(\rho_1,\rho_2))}$ or $\alpha_2 \in \mathfrak{F}_{(max(\sigma_1,\sigma_2), min(\rho_1,\rho_2))}$. In both the case, we yield $\alpha_1\alpha_2 \in \mathfrak{F}_{(max(\sigma_1,\sigma_2), min(\rho_1,\rho_2))}$, since $\mathfrak{F}_{(\max(\sigma_1,\sigma_2), min(\rho_1,\rho_2))}$ is an ideal of K, which further implies that $\left(\mu_{\mathfrak{F}}(\alpha_1\alpha_2)\right)^2 \geq \max(\sigma_1, \sigma_2) = max\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$ and $\left(\nu_{\mathfrak{F}}(\alpha_1\alpha_2)\right)^2 \leq min(\rho_1, \rho_2) = \min\left\{\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$.

Thus, $\mathfrak{F}$ is a PFSR of K.

Theorem 3.6. Suppose that K is a division ring. Then, a PFS $\mathfrak{F}$ is a PFI of K if and only if $\left(\mu_{\mathfrak{F}}(\alpha)\right)^2 = \left(\mu_{\mathfrak{F}}(1)\right)^2 \leq \left(\mu_{\mathfrak{F}}(0)\right)^2$ and $\left(\nu_{\mathfrak{F}}(\alpha)\right)^2 = \left(\nu_{\mathfrak{F}}(1)\right)^2 \geq \left(\nu_{\mathfrak{F}}(0)\right)^2$ for all $\alpha \in R \setminus \{0\}$.

Proof. Let $\mathfrak{F}$ be a PFI of K. Then,

$$\left(\mu_{\mathfrak{F}}(\alpha)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha.1)\right)^2 \geq max\left\{\left(\mu_{\mathfrak{F}}(\alpha)\right)^2, \left(\mu_{\mathfrak{F}}(1)\right)^2\right\} \geq \left(\mu_{\mathfrak{F}}(1)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha\alpha^{-1})\right)^2$$
$$\geq max\left\{\left(\mu_{\mathfrak{F}}(\alpha)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha^{-1})\right)^2\right\} \geq \left(\mu_{\mathfrak{F}}(\alpha)\right)^2,$$

$\Rightarrow \left(\mu_{\mathfrak{F}}(\alpha)\right)^2 = \left(\mu_{\mathfrak{F}}(1)\right)^2$. Similarly, we can obtain that $\left(\nu_{\mathfrak{F}}(\alpha)\right)^2 = \left(\nu_{\mathfrak{F}}(1)\right)^2$. Finally, the application of Theorem 3.1 (i) gives the desired result.

Conversely, let $\left(\mu_{\mathfrak{F}}(\alpha)\right)^2 = \left(\mu_{\mathfrak{F}}(1)\right)^2 \leq \left(\mu_{\mathfrak{F}}(0)\right)^2$ and $\left(\nu_{\mathfrak{F}}(\alpha)\right)^2 = \left(\nu_{\mathfrak{F}}(1)\right)^2 \geq \left(\nu_{\mathfrak{F}}(0)\right)^2$ for all $\alpha \in R \setminus \{0\}$:

(i) For all $\alpha_1, \alpha_2 \in K$, if $\alpha_1 \neq \alpha_2$, then, $\left(\mu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 = \left(\mu_{\mathfrak{F}}(1)\right)^2 \geq min\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$ and $\left(\nu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 = \left(\nu_{\mathfrak{F}}(1)\right)^2 \leq max\left\{\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$, and if $\alpha_1 = \alpha_2$, then, $\left(\mu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 = \left(\mu_{\mathfrak{F}}(0)\right)^2 \geq min\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$ and $\left(\nu_{\mathfrak{F}}(\alpha_1 - \alpha_2)\right)^2 = \left(\nu_{\mathfrak{F}}(0)\right)^2 \leq max\left\{\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$.

(ii) For all $\alpha_1, \alpha_2 \in K$, if $\alpha_1 = 0$ or $\alpha_2 = 0$, then $\left(\mu_{\mathfrak{F}}(\alpha_1\alpha_2)\right)^2 \geq \max\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$ and $\left(\nu_{\mathfrak{F}}(\alpha_1\alpha_2)\right)^2 \leq min\left\{\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$ is obvious, and if $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, then $\left(\mu_{\mathfrak{F}}(\alpha_1\alpha_2)\right)^2 = \left(\mu_{\mathfrak{F}}(1)\right)^2 = max\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$ and $\left(\nu_{\mathfrak{F}}(\alpha_1\alpha_2)\right)^2 = \left(\mu_{\mathfrak{F}}(1)\right)^2 = min\left\{\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}$.

Thus, $\mathfrak{F}$ is a PFI of K.

4. Pythagorean fuzzy cosets of a Pythagorean fuzzy ideal

In this section, we define the notion of Pythagorean fuzzy cosets of a Pythagorean fuzzy ideal and prove that the set of all Pythagorean fuzzy cosets of a Pythagorean fuzzy ideal forms ring under certain binary operations. Furthermore, we prove Pythagorean fuzzy version of fundamental theorem of ring homomorphism.

We start this section with following theorem which provides basis to define Pythagorean fuzzy cosets (PFCs) of PFI in a ring K.

Theorem 4.1. Assume that $\mathfrak{F} = \left\{\left(\alpha, \mu_{\mathfrak{F}}(\alpha), \nu_{\mathfrak{F}}(\alpha)\right) : \alpha \in K\right\}$ is a PFI of a ring K and $\mu_{\Gamma_{\mathfrak{F}}}: K/\mathfrak{F}_* \longrightarrow [0,1]$ and $\nu_{\Gamma_{\mathfrak{F}}}: K/\mathfrak{F}_* \longrightarrow [0,1]$ are defined by $\mu_{\Gamma_{\mathfrak{F}}}(\alpha + \mathfrak{F}_*) = \mu_{\mathfrak{F}}(\alpha)$ and $\mu_{\Gamma_{\mathfrak{F}}}(\alpha + \mathfrak{F}_*) = \nu_{\mathfrak{F}}(\alpha)$ respectively. Then $\Gamma_{\mathfrak{F}} = \left\{\left(\alpha + \mathfrak{F}_*, \mu_{\Gamma_{\mathfrak{F}}}(\alpha + \mathfrak{F}_*), \nu_{\Gamma_{\mathfrak{F}}}(\alpha + \mathfrak{F}_*)\right) : \alpha + \mathfrak{F}_* \in K/\mathfrak{F}_*\right\}$ is a PFI of $K/\mathfrak{F}_*$.

Proof. Since $\mathfrak{F}$ is a PFI of K, therefore, $\mathfrak{F}_*$ is an ideal of K. Firstly, we show that $\mu_{\Gamma_{\mathfrak{F}}}$ and $\nu_{\Gamma_{\mathfrak{F}}}$, used to define PFS $\Gamma_{\mathfrak{F}}$, are well-defined. For this, let

$\alpha_1 + \mathfrak{F}_* = \alpha_2 + \mathfrak{F}_*$, where $\alpha_1, \alpha_2 \in K$,

$\Rightarrow \alpha_1 - \alpha_2 \in \mathfrak{F}_* \Rightarrow \alpha_1, \alpha_2 \in \mathfrak{F}_*$

$\Rightarrow \left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2$ and $\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2 = \left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2$

$\Rightarrow \left(\mu_{\Gamma_{\mathfrak{F}}}(\alpha_1 + \mathfrak{F}_*)\right)^2 = \left(\mu_{\Gamma_{\mathfrak{F}}}(\alpha_2 + \mathfrak{F}_*)\right)^2$ and $\left(\nu_{\Gamma_{\mathfrak{F}}}(\alpha_1 + \mathfrak{F}_*)\right)^2 = \left(\nu_{\Gamma_{\mathfrak{F}}}(\alpha_2 + \mathfrak{F}_*)\right)^2$.

Next, we prove that $\Gamma_{\mathfrak{F}}$ is a PFI of $K/_{\mathfrak{F}_*}$, so let $\alpha_1 + \mathfrak{F}_*, \alpha_2 + \mathfrak{F}_* \in K/_{\mathfrak{F}_*}$. Then,

$$\begin{aligned}
&\left(\mu_{\Gamma_{\mathfrak{F}}}\left((\alpha_1 + \mathfrak{F}_*) - (\alpha_2 + \mathfrak{F}_*)\right)\right)^2 \\
&= \left(\mu_{\Gamma_{\mathfrak{F}}}\left((\alpha_1 - \alpha_2) + \mathfrak{F}_*\right)\right)^2 = \left(\mu_{\Gamma_{\mathfrak{F}}}(\alpha_1 - \alpha_2)\right)^2 \\
&\geq min\left\{\left(\mu_{\Gamma_{\mathfrak{F}}}(\alpha_1)\right)^2, \left(\mu_{\Gamma_{\mathfrak{F}}}(\alpha_2)\right)^2\right\} \\
&= min\left\{\left(\mu_{\Gamma_{\mathfrak{F}}}(\alpha_1 + \mathfrak{F}_*)\right)^2, \left(\mu_{\Gamma_{\mathfrak{F}}}(\alpha_2 + \mathfrak{F}_*)\right)^2\right\}.
\end{aligned}$$

Similarly,

$$\left(\nu_{\Gamma_{\mathfrak{F}}}\left((\alpha_1 + \mathfrak{F}_*) - (\alpha_2 + \mathfrak{F}_*)\right)\right)^2 \leq max\left\{\left(\nu_{\Gamma_{\mathfrak{F}}}(\alpha_1 + \mathfrak{F}_*)\right)^2, \left(\nu_{\Gamma_{\mathfrak{F}}}(\alpha_2 + \mathfrak{F}_*)\right)^2\right\}.$$

Moreover,

$$\begin{aligned}
&\left(\mu_{\Gamma_{\mathfrak{F}}}\left((\alpha_1 + \mathfrak{F}_*)(\alpha_2 + \mathfrak{F}_*)\right)\right)^2 \\
&= \left(\mu_{\Gamma_{\mathfrak{F}}}(\alpha_1\alpha_2 + \mathfrak{F}_*)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha_1\alpha_2)\right)^2 \\
&\geq max\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\} \\
&= max\left\{\left(\mu_{\Gamma_{\mathfrak{F}}}(\alpha_1 + \mathfrak{F}_*)\right)^2, \left(\mu_{\Gamma_{\mathfrak{F}}}(\alpha_2 + \mathfrak{F}_*)\right)^2\right\}.
\end{aligned}$$

Similarly,

$$\left(\nu_{\Gamma_{\mathfrak{F}}}\left((\alpha_1 + \mathfrak{F}_*)(\alpha_2 + \mathfrak{F}_*)\right)\right)^2 \leq min\left\{\left(\nu_{\Gamma_{\mathfrak{F}}}(\alpha_1 + \mathfrak{F}_*)\right)^2, \left(\nu_{\Gamma_{\mathfrak{F}}}(\alpha_2 + \mathfrak{F}_*)\right)^2\right\}.$$

Thus, we conclude that $\Gamma_{\mathfrak{F}}$ is a PFI of $K/_{\mathfrak{F}_*}$.

The Example 4.1 describes the result proved in Theorem 4.1.

Example 4.1. Consider a PFI $\mathfrak{F}$ of Z_6, a ring of integers modulo 6, that is,

$$\mathfrak{F} = \begin{Bmatrix} (0,0.95,0.15), (1,0.70,0.40), (2,0.95,0.15), \\ (3,0.70,0.40), (4,0.95,0.15), (5,0.70,0.40) \end{Bmatrix}.$$

Then, we have $\mathfrak{F}_* = \{0,2,4\}$ and the quotient ring $Z_6/_{\mathfrak{F}_*} = \{\{0,2,4\}, \{1,3,5\}\}$. Now, following the technique described in Theorem 4.1, we construct a PFS $\Gamma_{\mathfrak{F}}$ of $Z_6/_{\mathfrak{F}_*}$ as follows:

$$\Gamma_{\mathfrak{F}} = \{(\{0,2,4\}, 0.95,0.15), (\{1,3,5\}, 0.70,0.40)\}.$$

As can be seen, the constructed PFS $\Gamma_{\mathfrak{F}}$ is a PFI of $Z_6/_{\mathfrak{F}_*}$.

Theorem 4.2. Suppose that I and $\mathfrak{O}$ are ideal and PFI of a ring K and K/I respectively. If $\mu_{\mathfrak{O}}(\alpha+I)=\mu_{\mathfrak{O}}(I)$ and $\nu_{\mathfrak{O}}(\alpha+I)=\nu_{\mathfrak{O}}(I)\Leftrightarrow \alpha\in I$, then there exists a PFI $\mathfrak{F}$ of K such that $\mathfrak{F}_*=I$.

Proof. Let us define a PFS $\mathfrak{F}$ of K in the following way:

$\mu_{\mathfrak{F}}(\alpha)=\mu_{\mathfrak{O}}(\alpha+I)$ and $\nu_{\mathfrak{F}}(\alpha)=\nu_{\mathfrak{O}}(\alpha+I)$, for all $\alpha\in K$,

$$\begin{aligned}\left(\mu_{\mathfrak{F}}(\alpha_1-\alpha_2)\right)^2&=\left(\mu_{\mathfrak{O}}\left((\alpha_1-\alpha_2)+I\right)\right)^2\\&=\left(\mu_{\mathfrak{O}}\left((\alpha_1+I)-(\alpha_1+I)\right)\right)^2\\&\geq min\left\{\left(\mu_{\mathfrak{O}}(\alpha_1+I)\right)^2,\left(\mu_{\mathfrak{O}}(\alpha_2+I)\right)^2\right\}\\&=min\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2,\left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}.\end{aligned}$$

Similarly,

$$\left(\mu_{\mathfrak{F}}(\alpha_1-\alpha_2)\right)^2\leq max\left\{\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2,\left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}.$$

Also,

$$\begin{aligned}\left(\mu_{\mathfrak{F}}(\alpha_1\alpha_2)\right)^2=\left(\mu_{\mathfrak{O}}(\alpha_1\alpha_2+I)\right)^2&=\left(\mu_{\mathfrak{O}}\left((\alpha_1+I)(\alpha_1+I)\right)\right)^2\\&\geq max\left\{\left(\mu_{\mathfrak{O}}(\alpha_1+I)\right)^2,\left(\mu_{\mathfrak{O}}(\alpha_2+I)\right)^2\right\}\\&=max\left\{\left(\mu_{\mathfrak{F}}(\alpha_1)\right)^2,\left(\mu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}.\end{aligned}$$

Similarly,

$$\left(\mu_{\mathfrak{F}}(\alpha_1\alpha_2)\right)^2\leq min\left\{\left(\nu_{\mathfrak{F}}(\alpha_1)\right)^2,\left(\nu_{\mathfrak{F}}(\alpha_2)\right)^2\right\}.$$

This implies that $\mathfrak{F}$ is a PFI of K.

Next, $\alpha\in\mathfrak{F}_*\Leftrightarrow\left(\mu_{\mathfrak{F}}(\alpha)\right)^2=\left(\mu_{\mathfrak{F}}(0)\right)^2$ and $\left(\nu_{\mathfrak{F}}(\alpha)\right)^2=\left(\nu_{\mathfrak{F}}(0)\right)^2\Leftrightarrow\left(\mu_{\mathfrak{O}}(\alpha+I)\right)^2=\left(\mu_{\mathfrak{F}}\left(\mu_{\mathfrak{O}}(I)\right)\right)^2$ and $\left(\nu_{\mathfrak{F}}\left(\mu_{\mathfrak{O}}(\alpha+I)\right)\right)^2=\left(\nu_{\mathfrak{F}}\left(\mu_{\mathfrak{O}}(I)\right)\right)^2\Leftrightarrow\alpha\in I$. Thus, $\mathfrak{F}_*=I$.

We verify the result proved in Theorem 4.2 in the following example.

Example 4.2. Consider an ideal $4\mathbb{Z}$ of $\mathbb{Z}$. Then, $\mathbb{Z}/4\mathbb{Z}=\{4\mathbb{Z},1+4\mathbb{Z},2+4\mathbb{Z},3+4\mathbb{Z}\}$. Keeping in mind the condition given in Theorem 4.2, we design a PFI $\mathfrak{O}$ of $\mathbb{Z}/4\mathbb{Z}$ as follows:

$$\mathfrak{O}=\{(4\mathbb{Z},0.90,0.20),(1+4\mathbb{Z},0.70,0.40),(2+4\mathbb{Z},0.80,0.35),(3+4\mathbb{Z},0.70,0.40)\}.$$

Again, we obtain a PFI $\mathfrak{F}=\left\{n,\mu_{\mathfrak{F}}(n)=\begin{cases}0.90, & if\ n\in 4\mathbb{Z},\\0.80, & if\ n\in 2+4\mathbb{Z}\\0.70, & otherwise,\end{cases},\nu_{\mathfrak{F}}(n)=\begin{cases}0.20, & if\ n\in 4\mathbb{Z},\\0.35, & if\ n\in 2+4\mathbb{Z},\\0.40, & otherwise,\end{cases}\right\}$ of $\mathbb{Z}$ by using the membership and non-membership functions defined in Theorem 4.2. It is easy to find $\mathfrak{F}_*=4\mathbb{Z}$, which affirms Theorem 4.2.

Definition 4.1. Let $\mathfrak{F}$ be a PFI of a ring K and $\beta\in K$. Then, the PFS $\mathfrak{F}_\beta=$

$\left\{\left(\alpha, \mu_{\mathfrak{F}_\beta}(\alpha), \nu_{\mathfrak{F}_\beta}(\alpha)\right): \alpha \in K\right\}$ of K, where $\left(\mu_{\mathfrak{F}_\beta}(\alpha)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha-\beta)\right)^2$ and $\left(\nu_{\mathfrak{F}_\beta}(\alpha)\right)^2 = \left(\nu_{\mathfrak{F}}(\alpha-\beta)\right)^2$, is called Pythagorean fuzzy coset (PFC) of PFI $\mathfrak{F}$ in a ring K associated with β.

The concept of PFC of a PFI $\mathfrak{F}$ in a ring K is explained in the example below.

Example 4.3. Consider a PFI $\mathfrak{F}$ of Z_6 as follows:

$$\mathfrak{F} = \begin{Bmatrix} (0,0.80,0.40), (1,0.70,0.50), (2,0.70,0.50), \\ (3,0.80,0.40), (4,0.70,0.50), (5,0.70,0.50) \end{Bmatrix}.$$

Next, we find PFCs $\mathfrak{F}_\beta$ of $\mathfrak{F}$ associated with all $\beta \in Z_6$.

(i) The PFC of J with respect to 0 is

$$\mathfrak{F}_0 = \begin{Bmatrix} \left(0, \mu_{\mathfrak{F}}(0-0), \nu_{\mathfrak{F}}(0-0)\right), \left(1, \mu_{\mathfrak{F}}(1-0), \nu_{\mathfrak{F}}(1-0)\right), \left(2, \mu_{\mathfrak{F}}(2-0), \nu_{\mathfrak{F}}(2-0)\right), \\ \left(3, \mu_{\mathfrak{F}}(3-0), \nu_{\mathfrak{F}}(3-0)\right), \left(4, \mu_{\mathfrak{F}}(4-0), \nu_{\mathfrak{F}}(4-0)\right), \left(5, \mu_{\mathfrak{F}}(5-0), \nu_{\mathfrak{F}}(5-0)\right) \end{Bmatrix}$$
$$= \begin{Bmatrix} (0,0.80,0.40), (1,0.70,0.50), (2,0.70,0.50), \\ (3,0.80,0.40), (4,0.70,0.50), (5,0.70,0.50) \end{Bmatrix}.$$

(ii) The PFC of J with respect to 1 is

$$\mathfrak{F}_1 = \begin{Bmatrix} \left(0, \mu_{\mathfrak{F}}(0-1), \nu_{\mathfrak{F}}(0-1)\right), \left(1, \mu_{\mathfrak{F}}(1-1), \nu_{\mathfrak{F}}(1-1)\right), \left(2, \mu_{\mathfrak{F}}(2-1), \nu_{\mathfrak{F}}(2-1)\right), \\ \left(3, \mu_{\mathfrak{F}}(3-1), \nu_{\mathfrak{F}}(3-1)\right), \left(4, \mu_{\mathfrak{F}}(4-1), \nu_{\mathfrak{F}}(4-1)\right), \left(5, \mu_{\mathfrak{F}}(5-1), \nu_{\mathfrak{F}}(5-1)\right) \end{Bmatrix}$$
$$= \begin{Bmatrix} \left(0, \mu_{\mathfrak{F}}(5), \nu_{\mathfrak{F}}(5)\right), \left(1, \mu_{\mathfrak{F}}(0), \nu_{\mathfrak{F}}(0)\right), \left(2, \mu_{\mathfrak{F}}(1), \nu_{\mathfrak{F}}(1)\right), \\ \left(3, \mu_{\mathfrak{F}}(2), \nu_{\mathfrak{F}}(2)\right), \left(4, \mu_{\mathfrak{F}}(3), \nu_{\mathfrak{F}}(3)\right), \left(5, \mu_{\mathfrak{F}}(4), \nu_{\mathfrak{F}}(4)\right) \end{Bmatrix}$$
$$= \begin{Bmatrix} (0,0.70,0.50), (1,0.80,0.40), (2,0.70,0.50), \\ (3,0.70,0.50), (4,0.80,0.40), (5,0.70,0.50) \end{Bmatrix}.$$

(iii) The PFC of J with respect to 2 is

$$\mathfrak{F}_2 = \begin{Bmatrix} \left(0, \mu_{\mathfrak{F}}(0-2), \nu_{\mathfrak{F}}(0-2)\right), \left(1, \mu_{\mathfrak{F}}(1-2), \nu_{\mathfrak{F}}(1-2)\right), \left(2, \mu_{\mathfrak{F}}(2-2), \nu_{\mathfrak{F}}(2-2)\right), \\ \left(3, \mu_{\mathfrak{F}}(3-2), \nu_{\mathfrak{F}}(3-2)\right), \left(4, \mu_{\mathfrak{F}}(4-2), \nu_{\mathfrak{F}}(4-2)\right), \left(5, \mu_{\mathfrak{F}}(5-2), \nu_{\mathfrak{F}}(5-2)\right) \end{Bmatrix}$$
$$= \begin{Bmatrix} \left(0, \mu_{\mathfrak{F}}(4), \nu_{\mathfrak{F}}(4)\right), \left(1, \mu_{\mathfrak{F}}(5), \nu_{\mathfrak{F}}(5)\right), \left(2, \mu_{\mathfrak{F}}(0), \nu_{\mathfrak{F}}(0)\right), \\ \left(3, \mu_{\mathfrak{F}}(1), \nu_{\mathfrak{F}}(1)\right), \left(4, \mu_{\mathfrak{F}}(2), \nu_{\mathfrak{F}}(2)\right), \left(5, \mu_{\mathfrak{F}}(3), \nu_{\mathfrak{F}}(3)\right) \end{Bmatrix}$$
$$= \begin{Bmatrix} (0,0.70,0.50), (1,0.70,0.50), (2,0.80,0.40), \\ (3,0.70,0.50), (4,0.70,0.50), (5,0.80,0.40) \end{Bmatrix}.$$

(iv) The PFC of J with respect to 3 is

$$\mathfrak{F}_3 = \begin{Bmatrix} \left(0, \mu_{\mathfrak{F}}(0-3), \nu_{\mathfrak{F}}(0-3)\right), \left(1, \mu_{\mathfrak{F}}(1-3), \nu_{\mathfrak{F}}(1-3)\right), \left(2, \mu_{\mathfrak{F}}(2-3), \nu_{\mathfrak{F}}(2-3)\right), \\ \left(3, \mu_{\mathfrak{F}}(3-3), \nu_{\mathfrak{F}}(3-3)\right), \left(4, \mu_{\mathfrak{F}}(4-3), \nu_{\mathfrak{F}}(4-3)\right), \left(5, \mu_{\mathfrak{F}}(5-3), \nu_{\mathfrak{F}}(5-3)\right) \end{Bmatrix}$$
$$= \begin{Bmatrix} \left(0, \mu_{\mathfrak{F}}(3), \nu_{\mathfrak{F}}(3)\right), \left(1, \mu_{\mathfrak{F}}(4), \nu_{\mathfrak{F}}(4)\right), \left(2, \mu_{\mathfrak{F}}(5), \nu_{\mathfrak{F}}(5)\right), \\ \left(3, \mu_{\mathfrak{F}}(0), \nu_{\mathfrak{F}}(0)\right), \left(4, \mu_{\mathfrak{F}}(1), \nu_{\mathfrak{F}}(1)\right), \left(5, \mu_{\mathfrak{F}}(2), \nu_{\mathfrak{F}}(2)\right) \end{Bmatrix}$$
$$= \begin{Bmatrix} (0,0.80,0.40), (1,0.70,0.50), (2,0.70,0.50), \\ (3,0.80,0.40), (4,0.70,0.50), (5,0.70,0.50) \end{Bmatrix}.$$

(v) The PFC of J with respect to 4 is

$$\mathfrak{F}_4=\left\{\begin{matrix}\big(0,\mu_{\mathfrak{F}}(0-4),\nu_{\mathfrak{F}}(0-4)\big),\big(1,\mu_{\mathfrak{F}}(1-4),\nu_{\mathfrak{F}}(1-4)\big),\big(2,\mu_{\mathfrak{F}}(2-4),\nu_{\mathfrak{F}}(2-4)\big),\\ \big(3,\mu_{\mathfrak{F}}(3-4),\nu_{\mathfrak{F}}(3-4)\big),\big(4,\mu_{\mathfrak{F}}(4-4),\nu_{\mathfrak{F}}(4-4)\big),\big(5,\mu_{\mathfrak{F}}(5-4),\nu_{\mathfrak{F}}(5-4)\big)\end{matrix}\right\}$$
$$=\left\{\begin{matrix}\big(0,\mu_{\mathfrak{F}}(2),\nu_{\mathfrak{F}}(2)\big),\big(1,\mu_{\mathfrak{F}}(3),\nu_{\mathfrak{F}}(3)\big),\big(2,\mu_{\mathfrak{F}}(4),\nu_{\mathfrak{F}}(4)\big),\\ \big(3,\mu_{\mathfrak{F}}(5),\nu_{\mathfrak{F}}(5)\big),\big(4,\mu_{\mathfrak{F}}(0),\nu_{\mathfrak{F}}(0)\big),\big(5,\mu_{\mathfrak{F}}(1),\nu_{\mathfrak{F}}(1)\big)\end{matrix}\right\}$$
$$=\left\{\begin{matrix}(0,0.70,0.50),(1,0.80,0.40),(2,0.70,0.50),\\ (3,0.70,0.50),(4,0.80,0.40),(5,0.70,0.50)\end{matrix}\right\}.$$

(vi) The PFC of J with respect to 5 is

$$\mathfrak{F}_5=\left\{\begin{matrix}\big(0,\mu_{\mathfrak{F}}(0-5),\nu_{\mathfrak{F}}(0-5)\big),\big(1,\mu_{\mathfrak{F}}(1-5),\nu_{\mathfrak{F}}(1-5)\big),\big(2,\mu_{\mathfrak{F}}(2-5),\nu_{\mathfrak{F}}(2-5)\big),\\ \big(3,\mu_{\mathfrak{F}}(3-5),\nu_{\mathfrak{F}}(3-5)\big),\big(4,\mu_{\mathfrak{F}}(4-5),\nu_{\mathfrak{F}}(4-5)\big),\big(5,\mu_{\mathfrak{F}}(5-5),\nu_{\mathfrak{F}}(5-5)\big)\end{matrix}\right\}$$
$$=\left\{\begin{matrix}\big(0,\mu_{\mathfrak{F}}(1),\nu_{\mathfrak{F}}(1)\big),\big(1,\mu_{\mathfrak{F}}(2),\nu_{\mathfrak{F}}(2)\big),\big(2,\mu_{\mathfrak{F}}(3),\nu_{\mathfrak{F}}(3)\big),\\ \big(3,\mu_{\mathfrak{F}}(4),\nu_{\mathfrak{F}}(4)\big),\big(4,\mu_{\mathfrak{F}}(5),\nu_{\mathfrak{F}}(5)\big),\big(5,\mu_{\mathfrak{F}}(0),\nu_{\mathfrak{F}}(0)\big)\end{matrix}\right\}$$
$$=\left\{\begin{matrix}(0,0.70,0.50),(1,0.70,0.50),(2,0.80,0.40),\\ (3,0.70,0.50),(4,0.70,0.50),(5,0.80,0.40)\end{matrix}\right\}.$$

Thus, there are three distinct PFCs of J in terms of all elements of Z_6, namely $\mathfrak{F}_0=\mathfrak{F}_3$, $\mathfrak{F}_1=\mathfrak{F}_4$ and $\mathfrak{F}_2=\mathfrak{F}_5$.

Theorem 4.3. Let $\mathfrak{F}$ be a PFI of K. Then $\mathfrak{R}_{\mathfrak{F}}$, the set of all PFCs of $\mathfrak{F}$ in K, forms ring with the following binary operations:

$$\mathfrak{F}_\beta+\mathfrak{F}_\gamma=\mathfrak{F}_{\beta+\gamma}\ \text{ and }\ \mathfrak{F}_\beta\mathfrak{F}_\gamma=\mathfrak{F}_{\beta\gamma}\ \text{ for all }\ \beta,\gamma\in K.$$

Proof. Firstly, we will prove that both the binary operations defined on $\mathfrak{R}_{\mathfrak{F}}$ are well-defined.

Suppose that $\beta,\gamma,\zeta,\eta\in K$ and $\mathfrak{F}_\beta=\mathfrak{F}_\gamma$ and $\mathfrak{F}_\zeta=\mathfrak{F}_\eta$. Then for all $\alpha\in K$,

$$\mu_{\mathfrak{F}_\beta}(\alpha)=\mu_{\mathfrak{F}_\gamma}(\alpha)\ \text{ and }\ \nu_{\mathfrak{F}_\beta}(\alpha)=\nu_{\mathfrak{F}_\gamma}(\alpha), \tag{4.1}$$

$$\mu_{\mathfrak{F}_\zeta}(\alpha)=\mu_{\mathfrak{F}_\eta}(\alpha)\ \text{ and }\ \nu_{\mathfrak{F}_\zeta}(\alpha)=\nu_{\mathfrak{F}_\eta}(\alpha). \tag{4.2}$$

So,

$$\mu_{\mathfrak{F}}(\alpha-\beta)=\mu_{\mathfrak{F}}(\alpha-\gamma)\ \text{ and }\ \nu_{\mathfrak{F}}(\alpha-\beta)=\nu_{\mathfrak{F}}(\alpha-\gamma), \tag{4.3}$$

$$\mu_{\mathfrak{F}}(\alpha-\zeta)=\mu_{\mathfrak{F}}(\alpha-\eta)\ \text{ and }\ \nu_{\mathfrak{F}}(\alpha-\zeta)=\nu_{\mathfrak{F}}(\alpha-\eta). \tag{4.4}$$

Putting $\alpha=\beta+\zeta-\eta$ in (4.3), $\alpha=\zeta$ in (4.4) and $\alpha=\beta$ in (4.3), we have

$$\mu_{\mathfrak{F}}(\zeta-\eta)=\mu_{\mathfrak{F}}(\beta+\zeta-\eta-\gamma)\ \text{ and }\ \nu_{\mathfrak{F}}(\zeta-\eta)=\nu_{\mathfrak{F}}(\beta+\zeta-\eta-\gamma), \tag{4.5}$$

$$\mu_{\mathfrak{F}}(0)=\mu_{\mathfrak{F}}(\zeta-\eta)\ \text{ and }\ \nu_{\mathfrak{F}}(0)=\nu_{\mathfrak{F}}(\zeta-\eta), \tag{4.6}$$

$$\mu_{\mathfrak{F}}(0)=\mu_{\mathfrak{F}}(\beta-\gamma)\ \text{ and }\ \nu_{\mathfrak{F}}(0)=\nu_{\mathfrak{F}}(\beta-\gamma). \tag{4.7}$$

Now,

$$\left(\mu_{\mathfrak{F}_\beta}(\alpha)\right)^2+\left(\mu_{\mathfrak{F}_\zeta}(\alpha)\right)^2=\left(\mu_{\mathfrak{F}_{\beta+\zeta}}(\alpha)\right)^2=\left(\mu_{\mathfrak{F}}(\alpha-\beta-\zeta)\right)^2$$
$$=\left(\mu_{\mathfrak{F}}((\alpha-\gamma-\eta)-(\beta-\gamma+\zeta-\eta))\right)^2$$
$$\geq min\left\{\left(\mu_{\mathfrak{F}}(\alpha-(\gamma+\eta))\right)^2,\left(\mu_{\mathfrak{F}}(\beta-\gamma+\zeta-\eta)\right)^2\right\}$$
$$\geq min\left\{\left(\mu_{\mathfrak{F}}(\alpha-(\gamma+\eta))\right)^2,\left(\mu_{\mathfrak{F}}(0)\right)^2\right\}\text{ (by using (4.6) and (4.7))}$$
$$=\left(\mu_{\mathfrak{F}}(\alpha-(\gamma+\eta))\right)^2=\left(\mu_{\mathfrak{F}_{\gamma+\eta}}(\alpha)\right)^2=\left(\mu_{\mathfrak{F}_\gamma}(\alpha)\right)^2+\left(\mu_{\mathfrak{F}_\eta}(\alpha)\right)^2.$$

So,

$$\left(\mu_{\mathfrak{F}_\beta}(\alpha)\right)^2+\left(\mu_{\mathfrak{F}_\zeta}(\alpha)\right)^2\geq\left(\mu_{\mathfrak{F}_\gamma}(\alpha)\right)^2+\left(\mu_{\mathfrak{F}_\eta}(\alpha)\right)^2. \tag{4.8}$$

Similarly, we can prove that

$$\left(\mu_{\mathfrak{F}_\gamma}(\alpha)\right)^2+\left(\mu_{\mathfrak{F}_\eta}(\alpha)\right)^2\geq\left(\mu_{\mathfrak{F}_\beta}(\alpha)\right)^2+\left(\mu_{\mathfrak{F}_\zeta}(\alpha)\right)^2. \tag{4.9}$$

The inequalities (4.8) and (4.9) yields

$$\left(\mu_{\mathfrak{F}_\beta}(\alpha)\right)^2+\left(\mu_{\mathfrak{F}_\zeta}(\alpha)\right)^2=\left(\mu_{\mathfrak{F}_\gamma}(\alpha)\right)^2+\left(\mu_{\mathfrak{F}_\eta}(\alpha)\right)^2. \tag{4.10}$$

By using (4.1) and (4.2), we obtain

$$\mu_{\mathfrak{F}_\beta}(\alpha)+\mu_{\mathfrak{F}_\zeta}(\alpha)=\mu_{\mathfrak{F}_\gamma}(\alpha)+\mu_{\mathfrak{F}_\eta}(\alpha). \tag{4.11}$$

The similar reasoning leads us to

$$\nu_{\mathfrak{F}_\beta}(\alpha)+\nu_{\mathfrak{F}_\zeta}(\alpha)=\nu_{\mathfrak{F}_\gamma}(\alpha)+\nu_{\mathfrak{F}_\eta}(\alpha). \tag{4.12}$$

Again, the utilization of the same method gives us

$$\mu_{\mathfrak{F}_\beta}(\alpha)\mu_{\mathfrak{F}_\zeta}(\alpha)=\mu_{\mathfrak{F}_\gamma}(\alpha)\mu_{\mathfrak{F}_\eta}(\alpha), \tag{4.13}$$

and

$$\nu_{\mathfrak{F}_\beta}(\alpha)\nu_{\mathfrak{F}_\zeta}(\alpha)=\nu_{\mathfrak{F}_\gamma}(\alpha)\nu_{\mathfrak{F}_\eta}(\alpha). \tag{4.14}$$

The Eqs (4.11)–(4.14) yield that

$$\mathfrak{F}_\beta+\mathfrak{F}_\zeta=\mathfrak{F}_\gamma+\mathfrak{F}_\eta\text{ and }\mathfrak{F}_\beta\mathfrak{F}_\zeta=\mathfrak{F}_\gamma\mathfrak{F}_\eta.$$

Hence, both addition and multiplication defined on $\mathfrak{R}_{\mathfrak{F}}$ are well-defined. Next, it is easy to verify that $\mathfrak{F}_0 = \mathfrak{F}$ serves as additive identity of $\mathfrak{R}_{\mathfrak{F}}$, and for each $\mathfrak{F}_\beta \in \mathfrak{R}_{\mathfrak{F}}$ there exists $\mathfrak{F}_{-\beta} \in \mathfrak{R}_{\mathfrak{F}}$ such that $\mathfrak{F}_\beta + \mathfrak{F}_{-\beta} = \mathfrak{F}_0 = \mathfrak{F}_{-\beta} + \mathfrak{F}_\beta$. The remaining properties are routine computations.

The following example illustrates the fact mentioned in Theorem 4.3.

Example 4.4. In Example 4.3, we find that the set of all PFCs of $\mathfrak{F} = \begin{Bmatrix} (0,0.80,0.40), (1,0.70,0.50), (2,0.70,0.50), \\ (3,0.80,0.40), (4,0.70,0.50), (5,0.70,0.50) \end{Bmatrix}$ in Z_6 is $\mathfrak{R}_{\mathfrak{F}} = \{\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2\}$. Consider the Cayley's tables (see Tables 1 and 2) of $\mathfrak{R}_{\mathfrak{F}}$ obtained by employing the operations defined in Theorem 4.3 as follows:

Table 1. Cayley's table of $(\mathfrak{R}_{\mathfrak{F}}, +)$.

$+$	$\boldsymbol{\mathfrak{F}_0}$	$\boldsymbol{\mathfrak{F}_1}$	$\boldsymbol{\mathfrak{F}_2}$
$\boldsymbol{\mathfrak{F}_0}$	$\mathfrak{F}_0$	$\mathfrak{F}_1$	$\mathfrak{F}_2$
$\boldsymbol{\mathfrak{F}_1}$	$\mathfrak{F}_1$	$\mathfrak{F}_2$	$\mathfrak{F}_0$
$\boldsymbol{\mathfrak{F}_2}$	$\mathfrak{F}_2$	$\mathfrak{F}_0$	$\mathfrak{F}_1$

Table 2. Cayley's table of $(\mathfrak{R}_{\mathfrak{F}}, .)$.

$\cdot$	$\boldsymbol{\mathfrak{F}_0}$	$\boldsymbol{\mathfrak{F}_1}$	$\boldsymbol{\mathfrak{F}_2}$
$\boldsymbol{\mathfrak{F}_0}$	$\mathfrak{F}_0$	$\mathfrak{F}_0$	$\mathfrak{F}_0$
$\boldsymbol{\mathfrak{F}_1}$	$\mathfrak{F}_0$	$\mathfrak{F}_1$	$\mathfrak{F}_2$
$\boldsymbol{\mathfrak{F}_2}$	$\mathfrak{F}_0$	$\mathfrak{F}_2$	$\mathfrak{F}_1$

From Tables 1 and 2, we see that $\mathfrak{R}_{\mathfrak{F}}$ is a ring under defined binary operations.

Remark 4.1. If $\mathfrak{F} = \{(\alpha, \mu_{\mathfrak{F}}(\alpha), \nu_{\mathfrak{F}}(\alpha) : \alpha \in K\}$ is a PFI of a ring K such that $\mu_{\mathfrak{F}}$ and $\nu_{\mathfrak{F}}$ are constant functions, then $\mathfrak{R}_{\mathfrak{F}} = \{\mathfrak{F}_0\}$.

Definition 4.2. Let $\mathfrak{F}$ be a PFI of a ring K, then the PFI $\mathfrak{F}'$ of $\mathfrak{R}_{\mathfrak{F}}$ defined by $\mu_{\mathfrak{F}'}(\mathfrak{F}_\beta) = \mu_{\mathfrak{F}}(\beta)$ and $\nu_{\mathfrak{F}'}(\mathfrak{F}_\beta) = \nu_{\mathfrak{F}}(\beta)$, for all $\beta \in K$, is called Pythagorean fuzzy quotient ideal (PFQI) associated with $\mathfrak{F}$.

Theorem 4.4. If $\mathfrak{F} = \{(\alpha, \mu_{\mathfrak{F}}(\alpha), \nu_{\mathfrak{F}}(\alpha) : \alpha \in K\}$ is a PFI of a ring K, then a mapping $\theta : K \to \mathfrak{R}_{\mathfrak{F}}$ defined by $\theta(\beta) = \mathfrak{F}_\beta$ for $\beta \in K$ is a ring homomorphism with kernel $\mathfrak{F}_*$.

Proof. Assume that $\beta, \gamma \in K$, then,

$$\theta(\beta + \gamma) = \mathfrak{F}_{\beta+\gamma} = \mathfrak{F}_\beta + \mathfrak{F}_\gamma = \theta(\beta) + \theta(\gamma)$$

and

$$\theta(\beta\gamma) = \mathfrak{F}_{\beta\gamma} = \mathfrak{F}_\beta \mathfrak{F}_\gamma = \theta(\beta)\theta(\gamma).$$

First, we prove that if $\left(\mu_{\mathfrak{F}}(\beta)\right)^2 = \left(\mu_{\mathfrak{F}}(0)\right)^2$ and $\left(\nu_{\mathfrak{F}}(\beta)\right)^2 = \left(\nu_{\mathfrak{F}}(0)\right)^2$ if and only if $\mathfrak{F}_\beta = \mathfrak{F}_0$.

Suppose that $\left(\mu_{\mathfrak{F}}(\beta)\right)^2 = \left(\mu_{\mathfrak{F}}(0)\right)^2$ and $\left(\nu_{\mathfrak{F}}(\beta)\right)^2 = \left(\nu_{\mathfrak{F}}(0)\right)^2$. Then for all $\alpha \in K$, we have $\left(\mu_{\mathfrak{F}}(\alpha)\right)^2 \le \left(\mu_{\mathfrak{F}}(\beta)\right)^2 = \left(\mu_{\mathfrak{F}}(0)\right)^2$ and $\left(\nu_{\mathfrak{F}}(\alpha)\right)^2 \ge \left(\nu_{\mathfrak{F}}(\beta)\right)^2 = \left(\nu_{\mathfrak{F}}(0)\right)^2$. If $\left(\mu_{\mathfrak{F}}(\alpha)\right)^2 < \left(\mu_{\mathfrak{F}}(\beta)\right)^2 \Rightarrow \left(\mu_{\mathfrak{F}}(\alpha - \beta)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha)\right)^2$ by Theorem 3.1 (ii). On the other hand, if $\left(\mu_{\mathfrak{F}}(\alpha)\right)^2 =$

$\left(\mu_{\mathfrak{F}}(\beta)\right)^2$, then,

$$\alpha, \beta \in \left\{\gamma \in K: \left(\mu_{\mathfrak{F}}(\gamma)\right)^2 = \left(\mu_{\mathfrak{F}}(0)\right)^2\right\} \Rightarrow \left(\mu_{\mathfrak{F}}(\alpha-\beta)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha)\right)^2 = \left(\mu_{\mathfrak{F}}(0)\right)^2.$$

So, in either case, we have

$$\left(\mu_{\mathfrak{F}}(\alpha-\beta)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha)\right)^2 \text{ for all } \alpha \in K.$$

In a similar way, we can prove

$$\left(\nu_{\mathfrak{F}}(\alpha-\beta)\right)^2 = \left(\nu_{\mathfrak{F}}(\alpha)\right)^2 \text{ for all } \alpha \in K.$$

Thus, $\mathfrak{F}_\beta = \mathfrak{F}_0$.

Conversely, suppose that $\mathfrak{F}_\beta = \mathfrak{F}_0$, so for all $\alpha \in K$, $\left(\mu_{\mathfrak{F}}(\alpha-\beta)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha-0)\right)^2$ and $\left(\nu_{\mathfrak{F}}(\alpha-\beta)\right)^2 = \left(\nu_{\mathfrak{F}}(\alpha-0)\right)^2$. Then,

$$\left(\mu_{\mathfrak{F}}(\beta)\right)^2 = \left(\mu_{\mathfrak{F}}(\alpha-(\alpha-\beta))\right)^2 \geq min\left\{\left(\mu_{\mathfrak{F}}(\alpha)\right)^2, \left(\mu_{\mathfrak{F}}(\alpha-\beta)\right)^2\right\} = \left(\mu_{\mathfrak{F}}(\alpha)\right)^2.$$

Thus, for all $\alpha \in K$, we obtain $\left(\mu_{\mathfrak{F}}(\beta)\right)^2 \geq \left(\mu_{\mathfrak{F}}(\alpha)\right)^2$, hence, $\left(\mu_{\mathfrak{F}}(\beta)\right)^2 = \left(\mu_{\mathfrak{F}}(0)\right)^2$.

The similar reasoning yields that $\left(\nu_{\mathfrak{F}}(\beta)\right)^2 = \left(\nu_{\mathfrak{F}}(0)\right)^2$. Now,

$$\begin{aligned}\text{Ker } \theta &= \{\beta \in K: \theta(\beta) = \mathfrak{F}_0\} = \{\beta \in K: \mathfrak{F}_\beta = \mathfrak{F}_0\} \\ &= \left\{\beta \in K: \left(\mu_{\mathfrak{F}}(\beta)\right)^2 = \left(\mu_{\mathfrak{F}}(0)\right)^2 \text{ and } \left(\nu_{\mathfrak{F}}(\beta)\right)^2 = \left(\nu_{\mathfrak{F}}(0)\right)^2\right\} = \mathfrak{F}_*.\end{aligned}$$

The following example explains the fact given in Theorem 4.4.

Example 4.5. In Example 4.2, we computed $\mathfrak{R}_{\mathfrak{F}} = \{\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2\}$ for PFI $\mathfrak{F} = \left\{\begin{matrix}(0,0.80,0.40),(1,0.70,0.50),(2,0.70,0.50), \\ (3,0.80,0.40),(4,0.70,0.50),(5,0.70,0.50)\end{matrix}\right\}$ of Z_6. By using the approach defined in Theorem 4.4, we define $\theta: Z_6 \to \mathfrak{R}_{\mathfrak{F}}$ as follows:

$$\theta(0) = \theta(3) = \mathfrak{F}_0,\ \theta(1) = \theta(4) = \mathfrak{F}_1 \text{ and } \theta(2) = \theta(5) = \mathfrak{F}_2.$$

Now, $\theta(1+2) = \theta(3) = \mathfrak{F}_0$ and $\theta(1) + \theta(2) = \mathfrak{F}_1 + \mathfrak{F}_2 = \mathfrak{F}_3 = \mathfrak{F}_0$ show that $\theta(1+2) = \theta(1) + \theta(2)$. Furthermore, $\theta\big((1)(2)\big) = \theta(2) = \mathfrak{F}_2$ and $\theta(1)\theta(2) = \mathfrak{F}_1\mathfrak{F}_2 = \mathfrak{F}_2$ together imply that $\theta\big((1)(2)\big) = \theta(1).\theta(2)$. Similarly, it is easy to verify that $\theta(n+m) = \theta(n) + \theta(m)$ and $\big((n)(m)\big) = \theta(n).\theta(m)$ for all $n, m \in Z_6$. It means that θ is a ring homomorphism. Also, since $\mathfrak{F}_0$ is zero of $\mathfrak{R}_{\mathfrak{F}}$, therefore, Ker $\theta = \{0,3\} = \mathfrak{F}_*$ satisfying Theorem 4.4.

Next, we present an analogue of Fundamental theorem of homomorphism.

Theorem 4.5. Let $\mathfrak{F}$ be a PFI of a ring K, then every PFI of $\mathfrak{R}_{\mathfrak{F}}$ corresponds in a natural way to a PFI of K.

Proof. Let $\mathfrak{F}'$ be a PFI of $\mathfrak{R}_{\mathfrak{F}}$. Define PFS $\mathfrak{P}$ of K in the following way:

$$\left(\mu_{\mathfrak{P}}(\beta)\right)^2 = \left(\mu_{\mathfrak{F}'}(\mathfrak{F}_\beta)\right)^2 \text{ and } \left(\nu_{\mathfrak{P}}(\beta)\right)^2 = \left(\nu_{\mathfrak{F}'}(\mathfrak{F}_\beta)\right)^2.$$

It is simple to verify that $\mathfrak{P}$ is a PFI of K.

5. Pythagorean fuzzy semi-prime ideals and characterization of regularity

In this section, the concept of Pythagorean fuzzy semi-prime ideals is defined. We investigate some important algebraic features of this newly defined notion. Furthermore, the characterization of regular rings by virtue of PFI is presented.

Definition 5.1. A PFI $\mathfrak{F}$ of a ring K is called Pythagorean fuzzy semi-prime ideal (PFSPI) of K if for any PFI $\mathfrak{P}$ of K, $\mathfrak{P}^n \subseteq \mathfrak{F} \Rightarrow \mathfrak{P} \subseteq \mathfrak{F}$ for all $n \in \mathbb{N}$.

Theorem 5.1. A PFI $\mathfrak{F}$ of a ring K is PFSPI if and only if $\mathfrak{F}_{(\sigma,\rho)}$ is a semi-prime ideal of K for all $\sigma \in \left[0, \left(\mu_{\mathfrak{F}}(0)\right)^2\right]$ and $\rho \in \left[\left(\nu_{\mathfrak{F}}(0)\right)^2, 1\right]$.

Proof. Suppose that $\mathfrak{F}$ is a PFSPI of K and $\beta \in K$ such that $\beta^n \in \mathfrak{F}_{(\sigma,\rho)}$. Let us define a PFI $\mathfrak{P}$ of K in the following way:

$$\left(\mu_{\mathfrak{P}}(\gamma)\right)^2 = \begin{cases} \sigma, & \text{if } \gamma \in \langle\beta\rangle, \\ 0, & \text{otherwise,} \end{cases} \text{ and } \left(\nu_{\mathfrak{P}}(\gamma)\right)^2 = \begin{cases} \rho, & \text{if } \gamma \in \langle\beta\rangle, \\ 1, & \text{otherwise.} \end{cases}$$

Suppose that $\left(\mu^n_{\mathfrak{P}}(\gamma)\right)^2 \neq 0$, then $\gamma = \gamma_1\gamma_2, \dots, \gamma_n$ such that $\left(\mu_{\mathfrak{P}}(\gamma_i)\right)^2 \neq 0$, for all $i = 1,2,\dots,n$, because otherwise, there exists γ_j in each representation $\gamma_1\gamma_2,\dots,\gamma_n$ of γ such that $\left(\mu_{\mathfrak{P}}(\gamma_j)\right)^2 = 0$, therefore,

$$\left(\mu^n_{\mathfrak{P}}(\gamma)\right)^2 = \left(\mu_{\mathfrak{P}}\,\mu^{n-1}_{\mathfrak{P}}(\gamma_j\gamma_1\gamma_2,\dots,\gamma_{j-1}\gamma_{j+1},\gamma_{j+2},\dots,\gamma_n)\right)^2 = 0,$$

which is a contradiction.

Similarly, we can prove that $\left(\nu^n_{\mathfrak{P}}(\gamma)\right)^2 \neq 1$ implies $\gamma = \gamma_1\gamma_2,\dots,\gamma_n$ such that $\left(\nu_{\mathfrak{P}}(\gamma_i)\right)^2 \neq 1$, for all $i = 1,2,\dots,n$. Thus,

$$\begin{aligned}
&\left(\mu_{\mathfrak{P}}(\gamma_i)\right)^2 = \sigma = \left(\mu^n_{\mathfrak{P}}(\gamma)\right)^2 \text{ and } \left(\nu_{\mathfrak{P}}(\gamma_i)\right)^2 = \rho = \left(\nu^n_{\mathfrak{P}}(\gamma)\right)^2 \\
&\quad \Rightarrow \gamma_i \in \langle\beta\rangle, \text{ by definition of } \mathfrak{P} \\
&\quad \Rightarrow \gamma = \gamma_1,\gamma_2,\dots,\gamma_n \in \langle\beta^n\rangle \subseteq \mathfrak{F}_{(\sigma,\rho)}, \text{ since } \beta^n \in \mathfrak{F}_{(\sigma,\rho)} \\
&\quad \Rightarrow \gamma \in \mathfrak{F}_{(\sigma,\rho)} \Rightarrow \left(\mu_{\mathfrak{F}}(\gamma)\right)^2 \geq \sigma \text{ and } \left(\nu_{\mathfrak{F}}(\gamma)\right)^2 \leq \rho \\
&\quad \Rightarrow \left(\mu_{\mathfrak{F}}(\gamma)\right)^2 \geq \left(\mu^n_{\mathfrak{P}}(\gamma)\right)^2 \text{ and } \left(\nu_{\mathfrak{F}}(\gamma)\right)^2 \leq \left(\nu^n_{\mathfrak{P}}(\gamma)\right)^2 \\
&\quad \Rightarrow \mathfrak{P}^n \subseteq \mathfrak{F} \\
&\quad \Rightarrow \mathfrak{P} \subseteq \mathfrak{F}, \text{ since } \mathfrak{F} \text{ is a PFSPI of } K \\
&\quad \Rightarrow \left(\mu_{\mathfrak{F}}(\beta)\right)^2 \geq \left(\mu_{\mathfrak{P}}(\beta)\right)^2 = \sigma \text{ and } \left(\nu_{\mathfrak{F}}(\beta)\right)^2 \leq \left(\nu_{\mathfrak{P}}(\beta)\right)^2 = \rho \\
&\quad \Rightarrow \beta \in \mathfrak{F}_{(\sigma,\rho)}.
\end{aligned}$$

Thus, $\mathfrak{F}_{(\sigma,\rho)}$ is a semi-prime ideal of K.

Conversely, suppose that $\mathfrak{F}_{(\sigma,\rho)}$ is a semi-prime ideal of K for all $\sigma \in \left[0,\left(\mu_{\mathfrak{F}}(0)\right)^2\right]$ and $\rho \in \left[\left(\nu_{\mathfrak{F}}(0)\right)^2,1\right]$. Assume that $\mathfrak{F}$ is not a PFSPI of K. It means that there exists a PFI $\mathfrak{P}$ of K such that $\mathfrak{P}^n \subseteq \mathfrak{F}$ but $\mathfrak{P} \nsubseteq \mathfrak{F}$. Therefore, for some $\beta \in K$, we have

$$\left(\mu_{\mathfrak{P}}(\beta)\right)^2 > \left(\mu_{\mathfrak{F}}(\beta)\right)^2 \text{ or } \left(\nu_{\mathfrak{P}}(\beta)\right)^2 < \left(\nu_{\mathfrak{F}}(\beta)\right)^2. \tag{5.1}$$

Let $\left(\mu_{\mathfrak{F}}(\beta)\right)^2 = \sigma$ and $\left(\nu_{\mathfrak{F}}(\beta)\right)^2 = \rho$, then $\beta \in \mathfrak{F}_{(\sigma,\rho)}$, so $\beta^n \in \mathfrak{F}_{(\sigma,\rho)}$. Moreover $\beta \notin \mathfrak{F}_{(\sigma',\rho')}$ for all $\sigma' > \sigma$ and $\rho' < \rho$. Since $\mathfrak{F}_{(\sigma',\rho')}$ is a semi-prime ideal of K, therefore, $\beta^n \notin \mathfrak{F}_{(\sigma',\rho')}$. Thus, $\left(\mu_{\mathfrak{F}}(\beta^n)\right)^2 = \sigma$ and $\left(\nu_{\mathfrak{F}}(\beta^n)\right)^2 = \rho$. Therefore,

$$\left(\mu_{\mathfrak{F}}(\beta^n)\right)^2 = \left(\mu_{\mathfrak{F}}(\beta)\right)^2 \text{ and } \left(\nu_{\mathfrak{F}}(\beta^n)\right)^2 = \left(\nu_{\mathfrak{F}}(\beta)\right)^2. \tag{5.2}$$

Next, it can be easily verified that

$$\left(\mu_{\mathfrak{P}}^{\,n}(\beta)\right)^2 \geq \left(\mu_{\mathfrak{P}}(\beta)\right)^2 \text{ and } \left(\nu_{\mathfrak{P}}^{\,n}(\beta)\right)^2 \leq \left(\nu_{\mathfrak{P}}(\beta)\right)^2. \tag{5.3}$$

Then, (5.1)–(5.3) imply that, either $\left(\mu_{\mathfrak{P}}^{\,n}(\beta)\right)^2 > \left(\mu_{\mathfrak{F}}(\beta^n)\right)^2$ or $\left(\nu_{\mathfrak{P}}^{\,n}(\beta)\right)^2 < \left(\nu_{\mathfrak{F}}(\beta^n)\right)^2$. This means that $\mathfrak{P}^n \nsubseteq \mathfrak{F}$, thus we reach at a contradiction.

Example 5.1. Consider a PFI $\mathfrak{F} = \left\{\begin{matrix}(0,0.90,0.35),(1,0.80,0.60),(2,0.80,0.60),(3,0.80,0.60),\\(4,0.90,0.35),(5,0.80,0.60),(6,0.80,0.60),(7,0.80,0.60),\\(8,0.90,0.35),(9,0.80,0.60),(10,0.80,0.60),(11,0.80,0.60)\end{matrix}\right\}$ of Z_{12}. The Pythagorean fuzzy level ideal $\mathfrak{F}_{(0.90^2,0.35^2)} = \{0,4,8\}$ is not a semi-prime ideal of Z_{12} since $(\{0,2,4,6,8,10\})^2 = \{0,4,8\} \subseteq \mathfrak{F}_{(0.90^2,0.35^2)}$ but $\{0,2,4,6,8,10\} \nsubseteq \mathfrak{F}_{(0.90^2,0.35^2)}$.

Also, $\mathfrak{F}$ is not a PFSPI ideal of Z_{12} because there exists a PFI $\mathfrak{P} = \left\{\begin{matrix}(0,0.90,0.35),(1,0.80,0.60),(2,0.90,0.35),(3,0.80,0.60),\\(4,0.90,0.35),(5,0.80,0.60),(6,0.90,0.35),(7,0.80,0.60),\\(8,0.90,0.35),(9,0.80,0.60),(10,0.90,0.35),(11,0.80,0.60)\end{matrix}\right\}$ of Z_{12} such that $\mathfrak{P}^2 = \mathfrak{P} \circ \mathfrak{P} = \mathfrak{F} \subseteq \mathfrak{F}$ but $\mathfrak{P} \nsubseteq \mathfrak{F}$. This satisfies the conditional statement "if $\mathfrak{F}$ is PFSPI of K, then $\mathfrak{F}_{(\sigma,\rho)}$ is a semi-prime ideal of K for all $\sigma \in \left[0,\left(\mu_{\mathfrak{F}}(0)\right)^2\right]$ and $\rho \in \left[\left(\nu_{\mathfrak{F}}(0)\right)^2,1\right]$" expressed in Theorem 5.1.

Example 5.2. Consider a PFI $\mathfrak{F} = \left\{\begin{matrix}(0,0.90,0.30),(1,0.60,0.50),(2,0.60,0.50),(3,0.60,0.50),\\(4,0.80,0.40),(5,0.60,0.50),(6,0.60,0.50),(7,0.60,0.50)\end{matrix}\right\}$ of Z_8. It is not a PFSPI ideal of Z_8 because there exists a PFI $\mathfrak{P} = \left\{\begin{matrix}(0,0.90,0.30),(1,0.60,0.50),(2,0.80,0.40),(3,0.60,0.50),\\(4,0.80,0.40),(5,0.60,0.50),(6,0.80,0.40),(7,0.60,0.50)\end{matrix}\right\}$ of Z_8 such that $\mathfrak{P}^2 = \mathfrak{P} \circ \mathfrak{P} = \mathfrak{F} \subseteq \mathfrak{F}$ but $\mathfrak{P} \nsubseteq \mathfrak{F}$.

Furthermore, for all $\sigma \in \left[0,\left(\mu_{\mathfrak{F}}(0)\right)^2\right]$ and $\rho \in \left[\left(\nu_{\mathfrak{F}}(0)\right)^2,1\right]$, we have the following three Pythagorean fuzzy level ideals $\mathfrak{F}_{(\sigma,\rho)}$:

i. $\mathfrak{F}_{(\sigma,\rho)} = \{0\} = I_1$ where $0.80 < \sigma \leq 0.90$ and $0.30 \leq \rho < 0.40$,

ii. $\mathfrak{F}_{(\sigma,\rho)} = \{0,4\} = I_2$ where $0.60 < \sigma \leq 0.80$ and $0.40 \leq \rho < 0.50$,

iii. $\mathfrak{F}_{(\sigma,\rho)} = Z_8 = I_3$ where $0 \le \sigma \le 0.60$ and $0.50 \le \rho \le 1$.

The Pythagorean fuzzy level ideal $I_2 = \{0,4\}$ is not a semi-prime ideal of Z_8 as $(\{0,2,4,6\})^2 = \{0,4\} \subseteq I_2$ but $\{0,2,4,6\} \nsubseteq I_2$. Thus, the conditional statement "if $\mathfrak{F}_{(\sigma,\rho)}$ is a semi-prime ideal of K for all $\sigma \in \left[0, \left(\mu_{\mathfrak{F}}(0)\right)^2\right]$ and $\rho \in \left[\left(\nu_{\mathfrak{F}}(0)\right)^2, 1\right]$, then $\mathfrak{F}$ is PFSPI of K" revealed in Theorem 5.1, is satisfied.

Theorem 5.2. If $\mathfrak{F}$ is a PFSPI of K, then $\mathfrak{R}_{\mathfrak{F}}$, the set of all PFCs of $\mathfrak{F}$ in K, has no non-zero nilpotent elements.

Proof. Suppose that $\mathfrak{F}$ is a PFSPI of a ring K, then by Theorem 5.1, it follows that $\mathfrak{F}_{(\sigma,\rho)}$ is a semi-prime ideal of K, where $\sigma = \left(\mu_{\mathfrak{F}}(0)\right)^2$ and $\rho = \left(\nu_{\mathfrak{F}}(0)\right)^2$. Moreover, in view of Theorem 4.3 it follows that $\mathfrak{R}_{\mathfrak{F}} \cong K/\mathfrak{F}_{(\sigma,\rho)}$.

Let $\alpha + \mathfrak{F}_{(\sigma,\rho)}$ be non-zero nilpotent element of $K/\mathfrak{F}_{(\sigma,\rho)}$. Therefore,

$$\left(\alpha + \mathfrak{F}_{(\sigma,\rho)}\right)^n = \mathfrak{F}_{(\sigma,\rho)}$$

$$\Rightarrow \alpha^n + \mathfrak{F}_{(\sigma,\rho)} = \mathfrak{F}_{(\sigma,\rho)}$$

$$\Rightarrow \alpha^n \in \mathfrak{F}_{(\sigma,\rho)} \Rightarrow \alpha \in \mathfrak{F}_{(\sigma,\rho)}$$
$$\Rightarrow \alpha + \mathfrak{F}_{(\sigma,\rho)} = \mathfrak{F}_{(\sigma,\rho)}.$$

Thus, we have a contradiction, thus, $K/\mathfrak{F}_{(\sigma,\rho)}$ has no non-zero nilpotent element. This together with $\mathfrak{R}_{\mathfrak{F}} \cong K/\mathfrak{F}_{(\sigma,\rho)}$ leads to the desired result.

Example 5.3. Consider a PFI $\mathfrak{F}$ of Z_6 as follows:

$$\mathfrak{F} = \begin{Bmatrix} (0,0.80,0.40), (1,0.70,0.50), (2,0.70,0.50), \\ (3,0.80,0.40), (4,0.70,0.50), (5,0.70,0.50) \end{Bmatrix}.$$

Clearly, it has two non-empty Pythagorean fuzzy level subsets namely {0,3} and Z_6 which are semi-prime ideals of Z_6. Therefore, by Theorem 5.1, $\mathfrak{F}$ is PFSPI of Z_6.

Also, the set of all PFCs of $\mathfrak{F}$ in Z_6 is $\mathfrak{R}_{\mathfrak{F}} = \{\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2\}$. One can see that both non-zero elements $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are not nilpotent. Thus, Theorem 5.2 is satisfied.

Definition 5.2. Let K be a ring and $U \subseteq K$. Suppose that $\chi_U: K \to [0,1]$ and $\chi_U{}^c: K \to [0,1]$ are defined by

$$\chi_U(\alpha) = \begin{cases} 1, & \text{if } \alpha \in U, \\ 0, & \text{if } \alpha \notin U, \end{cases} \text{ and } \chi_U{}^c(\alpha) = \begin{cases} 0, & \text{if } \alpha \in U, \\ 1, & \text{if } \alpha \notin U. \end{cases}$$

Then,

$$\Psi(U) = \left\{\left(\alpha, \chi_U(\alpha), \chi_U{}^c(\alpha)\right): \alpha \in K\right\}$$

is a PFS of K.

Lemma 5.1. $\Psi(U)$ is PFI of K if U is an ideal of K.

The proof involves simple computation.

Theorem 5.3. A ring K is regular if and only if $\mathfrak{F} \circ \mathfrak{D} = \mathfrak{F} \cap \mathfrak{D}$, where $\mathfrak{F}$ and $\mathfrak{D}$ are PFIs of K.

Proof. Suppose that K is a regular ring. We want to show that $\mathfrak{F} \circ \mathfrak{D} = \mathfrak{F} \cap \mathfrak{D}$. From routine computations, we get $\mathfrak{F} \circ \mathfrak{D} \subseteq \mathfrak{F} \cap \mathfrak{D}$. Let $\beta \in K$, the regularity of K ensures the existence of ζ

and η in K such that $\beta = \zeta\eta$. Now,

$$\left(\mu_{\mathfrak{F}\circ\mathfrak{D}}(\beta)\right)^2 = max\left\{min\left(\left(\mu_{\mathfrak{F}}(\zeta)\right)^2, \left(\mu_{\mathfrak{D}}(\eta)\right)^2\right)\right\}.$$

Since $\beta = \beta\gamma\beta$ for some $\gamma \in K$. Then,

$$\left(\mu_{\mathfrak{F}}(\beta)\right)^2 = \left(\mu_{\mathfrak{F}}(\beta\gamma\beta)\right)^2 \geq max\left\{\left(\mu_{\mathfrak{F}}(\beta\gamma)\right)^2, \left(\mu_{\mathfrak{F}}(\beta)\right)^2\right\} \geq \left(\mu_{\mathfrak{F}}(\beta\gamma)\right)^2$$
$$\geq max\left\{\left(\mu_{\mathfrak{F}}(\beta)\right)^2, \left(\mu_{\mathfrak{F}}(\gamma)\right)^2\right\} \geq \left(\mu_{\mathfrak{F}}(\beta)\right)^2.$$

In short, $\left(\mu_{\mathfrak{F}}(\beta)\right)^2 \geq \left(\mu_{\mathfrak{F}}(\beta\gamma)\right)^2 \geq \left(\mu_{\mathfrak{F}}(\beta)\right)^2$, therefore, $\left(\mu_{\mathfrak{F}}(\beta\gamma)\right)^2 = \left(\mu_{\mathfrak{F}}(\beta)\right)^2$. Then,

$$\left(\mu_{\mathfrak{F}\circ\mathfrak{D}}(\beta)\right)^2 = max\left\{min\left(\left(\mu_{\mathfrak{F}}(\zeta)\right)^2, \left(\mu_{\mathfrak{D}}(\eta)\right)^2\right)\right\}$$
$$\geq min\left(\left(\mu_{\mathfrak{F}}(\beta\gamma)\right)^2, \left(\mu_{\mathfrak{D}}(\beta)\right)^2\right), \text{taking } \zeta = \beta\gamma \text{ and } \eta = \beta$$
$$= min\left(\left(\mu_{\mathfrak{F}}(\beta)\right)^2, \left(\mu_{\mathfrak{D}}(\beta)\right)^2\right) = \left(\mu_{\mathfrak{F}\cap\mathfrak{D}}(\beta)\right)^2.$$

Similarly, we obtain $\left(\eta_{\mathfrak{F}\circ\mathfrak{D}}(\beta)\right)^2 \leq \left(\eta_{\mathfrak{F}\cap\mathfrak{D}}(\beta)\right)^2$, which gives $\mathfrak{F} \cap \mathfrak{D} \subseteq \mathfrak{F} \circ \mathfrak{D}$.

Conversely, suppose that $\mathfrak{F} \circ \mathfrak{D} = \mathfrak{F} \cap \mathfrak{D}$. Let Y and Z be two ideals of K. In view of Lemma 5.1, $\Psi(Y) = \{(\alpha, \chi_Y(\alpha), {\chi_Y}^c(\alpha)): \alpha \in K\}$ and $\Psi(Z) = \{(\alpha, \chi_Z(\alpha), {\chi_Z}^c(\alpha)): \alpha \in K\}$ are PFIs of K. Assume that $\beta \in Y \cap Z$, then $\left(\chi_Y(\beta)\right)^2 \cap \left(\chi_z(\beta)\right)^2 = 1$ and $\left({\chi_Y}^c(\beta)\right)^2 \cap \left({\chi_Z}^c(\beta)\right)^2 = 0$. Since $\Psi(Y) \cap \Psi(Z) = \Psi(Z) \circ \Psi(Z)$, therefore, $\left(\chi_Y(\beta)\right)^2 \cap \left(\chi_z(\beta)\right)^2 = \left(\chi_Y(\beta)\right)^2 \circ \left(\chi_z(\beta)\right)^2 = 1$ and $\left({\chi_Y}^c(\beta)\right)^2 \cap \left({\chi_Z}^c(\beta)\right)^2 = \left({\chi_Y}^c(\beta)\right)^2 \circ \left({\chi_Z}^c(\beta)\right)^2 = 0$, therefore,

$$max\left\{min\left(\left(\chi_Y(\beta_1)\right)^2, \left(\chi_z(\beta_2)\right)^2\right): \beta_1\beta_2 = \beta, \beta_1, \beta_2 \in K\right\} = 1 \text{ and}$$
$$min\left\{max\left(\left({\chi_Y}^c(\beta_1)\right)^2, \left({\chi_Z}^c(\beta_2)\right)^2\right): \beta_1\beta_2 = \beta, \beta_1, \beta_2 \in K\right\} = 0.$$

It means that there exists $\gamma_1, \gamma_2 \in K$ such that

$$\left(\chi_Y(\gamma_1)\right)^2 = 1 = \left(\chi_Z(\gamma_2)\right)^2 \text{ and } \left({\chi_Y}^c(\gamma_1)\right)^2 = 0 = \left({\chi_Z}^c(\gamma_2)\right)^2,$$

with $\beta = \gamma_1\gamma_2$. Thus, $\beta = \gamma_1\gamma_2 \in YZ$, which gives $Y \cap Z \subseteq Y.Z$. Furthermore, $Y.Z \subseteq Y \cap Z$ is obvious. So, $Y \cap Z = Y.Z$, then the regularity of K is directly followed by using the theorem on page 184 of [26].

Theorem 5.4. A ring K is regular if and only if every PFI of K is idempotent.

Proof. Let K be a regular ring and $\mathfrak{F}$ be a PFI of K. Then, in view of Theorem 5.3, it is straightforward to show that $\mathfrak{F}^2 = \mathfrak{F}$.

Conversely, let every PFI of K. Assume that $\mathfrak{F}$ and $\mathfrak{D}$ are PFIs of K. In view of Theorem 5.3, we require $\mathfrak{F} \circ \mathfrak{D} = \mathfrak{F} \cap \mathfrak{D}$ to prove the regularity of K. For this, we proceed as follows: be idempotent

$$\left(\mu_{\mathfrak{F}\cap\mathfrak{D}}(\beta)\right)^2 = \left({\mu_{\mathfrak{F}\cap\mathfrak{D}}}^2(\beta)\right)^2 = max\left\{min\left(\left(\mu_{\mathfrak{F}\cap\mathfrak{D}}(\zeta)\right)^2, \left(\mu_{\mathfrak{F}\cap\mathfrak{D}}(\eta)\right)^2\right): \beta = \zeta\eta\right\}$$
$$\leq max\left\{min\left(\left(\mu_{\mathfrak{F}}(\zeta)\right)^2, \left(\mu_{\mathfrak{D}}(\eta)\right)^2\right): \beta = \zeta\eta\right\} = \left(\mu_{\mathfrak{F}\circ\mathfrak{D}}(\beta)\right)^2.$$

The same reasoning leads us to $\left(\nu_{\mathfrak{F}\cap\mathfrak{D}}(\beta)\right)^2 \geq \left(\nu_{\mathfrak{F}\circ\mathfrak{D}}(\beta)\right)^2$. So, $\mathfrak{F}\cap\mathfrak{D} \subseteq \mathfrak{F}\circ\mathfrak{D}$. Furthermore $\mathfrak{F}\circ\mathfrak{D} \subseteq \mathfrak{F}\cap\mathfrak{D}$ is obvious. Thus, $\mathfrak{F}\circ\mathfrak{D} = \mathfrak{F}\cap\mathfrak{D}$.

Lemma 5.2. $\left(\Psi(\langle\beta\rangle)\right)^2 = \Psi(\langle\beta^2\rangle)$ for all $\beta \in K$.

The proof is simple.

Theorem 5.5. A commutative ring K is regular if and only if every PFI of K is PFSPI.

Proof. Suppose that K is a regular ring and $\mathfrak{F}$ is a PFI of K. Let $\mathfrak{P}$ be any PFI of K such that $\mathfrak{P}^n \subseteq \mathfrak{F}$. Since K is regular, therefore by Theorem 5.4, we obtain $\mathfrak{P}^n = \mathfrak{P}$. Thus, $\mathfrak{P} \subseteq \mathfrak{F}$. This shows that $\mathfrak{F}$ is a PFSPI of K.

Conversely, suppose that every PFI of K is PFSPI. In view of Lemma 5.2, we have $\left(\Psi(\langle\beta\rangle)\right)^2 = \Psi(\langle\beta^2\rangle)$ for all $\beta \in K$. Since $\Psi(\langle\beta^2\rangle)$ is PFSPI of K, therefore, $\Psi(\langle\beta\rangle) \subseteq \Psi(\langle\beta^2\rangle)$. Also, $\Psi(\langle\beta^2\rangle) \subseteq \Psi(\langle\beta\rangle)$ is obvious. Hence, $\Psi(\langle\beta\rangle) \subseteq \Psi(\langle\beta^2\rangle)$. It means that $\beta \in \Psi(\langle\beta^2\rangle)$, therefore, $\beta = \alpha\beta^2 = \beta\alpha\beta$ for some $\alpha \in K$. Thus, K is a regular ring.

Example 5.4. We know $\mathbb{Z}/_{5\mathbb{Z}}$ is a regular ring. We design a PFI $\mathfrak{F}$ of $\mathbb{Z}/_{5\mathbb{Z}}$ as follows:

$$\mathfrak{F} = \left\{\begin{matrix}(0+5\mathbb{Z}, 0.80, 0.40), (1+5\mathbb{Z}, 0.70, 0.50), (2+5\mathbb{Z}, 0.70, 0.50), \\ (3+5\mathbb{Z}, 0.70, 0.50), (4+5\mathbb{Z}, 0.70, 0.50)\end{matrix}\right\}.$$

It is easy to find that $\mathfrak{F}$ has two non-empty Pythagorean fuzzy level subsets namely $\{0+5\mathbb{Z}\}$ and $\mathbb{Z}/_{5\mathbb{Z}}$. Both of them are semi-prime ideals of $\mathbb{Z}/_{5\mathbb{Z}}$. Therefore, by Theorem 5.1, $\mathfrak{F}$ is PFSPI of $\mathbb{Z}/_{5\mathbb{Z}}$ satisfying the conditional statement "if a commutative ring K is regular then every PFI of K is PFSPI" expressed in Theorem 5.5.

Example 5.5. Consider a PFI $\mathfrak{F} = \left\{\begin{matrix}(0,0.90,0.35), (1,0.80,0.60), (2,0.80,0.60), (3,0.80,0.60), \\ (4,0.90,0.35), (5,0.80,0.60), (6,0.80,0.60), (7,0.80,0.60), \\ (8,0.90,0.35), (9,0.80,0.60), (10,0.80,0.60), (11,0.80,0.60)\end{matrix}\right\}$ of Z_{12}. Since Z_{12} contains zero divisors, therefore it not a regular ring. Moreover, in Example 4.6, we see that $\mathfrak{F}$ is not a PFSPI of Z_{12}. Thus, the conditional statement "if every PFI of a commutative ring K is PFSPI then K is regular" revealed in Theorem 5.5 is verified.

6. Conclusions

The basic purpose of this paper is to study the notion of the ideal of a classical ring under Pythagorean fuzzy environment. For this purpose, several notions of ring theory like cosets of an ideal, quotient ideal and semiprime ideal are converted into Pythagorean fuzzy format. We have proved that the intersection of two PFIs of a ring K is a PFI. We also show that the intersection of a PFSR $\mathfrak{F}$ and PFI $\mathfrak{A}$ of K is PFI of $\mathfrak{F}_*$. We define the concept of Pythagorean fuzzy cosets of a Pythagorean fuzzy ideal and prove that the set of all Pythagorean fuzzy cosets of a Pythagorean fuzzy ideal forms a ring under certain binary operations. Furthermore, we present a Pythagorean fuzzy version of the fundamental theorem of ring homomorphism. Next, we give the definition and related properties of Pythagorean fuzzy semi-prime ideals. Lastly, the characterization of regular rings by virtue of Pythagorean fuzzy ideals is presented.

By using the outcomes of present study, our future intention is to investigate the algebraic properties of prime, maximal and irreducible ideals in Pythagorean fuzzy context. Moreover, in future work, we will extend the present concepts under different extensions of the fuzzy sets such as q-rung orthopair fuzzy sets, fuzzy soft sets and fuzzy hypersoft sets etc.

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Conflict of interest

The authors declare no conflicts of interest.

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