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Research article

Proof of a conjecture on the ϵ-spectral radius of trees

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Abstract: The ϵ-spectral radius of a connected graph is the largest eigenvalue of its eccentricity matrix. In this paper, we identify the unique n-vertex tree with diameter 4 and matching number 5 that minimizes the ϵ-spectral radius, and thus resolve a conjecture proposed in [W. Wei, S. Li, L. Zhang, Characterizing the extremal graphs with respect to the eccentricity spectral radius, and beyond, Discrete Math. 345 (2022) 112686].

Keywords: eccentricity matrix; spectral radius; matching number
Mathematics Subject Classification: 05C50

1. Introduction

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The distance between vertices v_i and v_j in G, denoted by $d_G(v_i, v_j)$, is the length of the shortest path between v_i and v_j. The distance matrix of G, denoted by $D(G)$, is the $n \times n$ matrix whose (i, j)-entry is equal to $d_G(v_i, v_j)$. Let $A(G)$ be the adjacency matrix of G. The adjacency matrix and the distance matrix of a graph are well-studied matrix classes in the field of spectral graph theory. For more details about the study of classes of matrices associated with graphs, we refer to [2–4].

For $u \in V(G)$, $e_G(u)$ or $e(u)$ denotes the eccentricity of u in G, which is equal to the largest distance from u to other vertices in G. The diameter of a graph G, denoted by $diam(G)$, is defined to be the maximum of the eccentricity of all the vertices of G. A vertex v is said to be an eccentric vertex of the vertex u if $d_G(u, v) = e(u)$. A vertex $u \in V(G)$ is said to be a diametrical vertex of G if $e(u) = diam(G)$. If each vertex of G is a diametrical vertex and has a unique eccentric vertex, then G is called a diametrical graph.

A matching in G is a set of edges without common vertices. The maximum matching is a matching with the maximum size in G. The matching number is the size of a maximum matching in G.

The eccentricity matrix of the graph G, denoted by $\epsilon(G)$, is defined as [14]

$$\epsilon(G)_{u,v} = \begin{cases} d_G(u,v), & \text{if } d_G(u,v) = \min\{e_G(u), e_G(v)\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is also known as the DMAX-matrix defined by Randić in [10] as a tool for chemical graph theory. The eigenvalues of the eccentricity matrix of a graph G are called the eccentricity eigenvalues, or ϵ-eigenvalues, of G. Since $\epsilon(G)$ is symmetric, the ϵ-eigenvalues are all real, which are denoted by $\xi_1(G) \geq \xi_2(G) \geq \cdots \geq \xi_n(G)$. As usual, $\xi_1(G)$ is called the ϵ-spectral radius of G, denoted also by $\rho_\epsilon(G)$.

Recently, the ϵ-spectral radius has received much attention. Wang et al. [13] determined sharp lower and upper bounds for the ϵ-spectral radius of graphs and identified the corresponding extremal graphs. Wei et al. [16] determined the n-vertex trees with minimum ϵ-spectral radius. Furthermore, in [16], the authors identified all trees with given order and diameter having the minimum ϵ-spectral radius. He and Lu [5] identified the n-vertex trees with fixed odd diameter having the maximum ϵ-spectral radius. Wei, Li and Zhang [17] characterized the n-vertex trees having the second minimum ϵ-spectral radius and identified the n-vertex trees with small matching number having the minimum ϵ-spectral radii. For more advances on the ϵ-eigenvalues, one may be referred to [7–9, 12, 15].

Let $\widehat{T}_{n,5}^{a,b}$ be the n-vertex tree obtained from $P_5 = v_1v_2v_3v_4v_5v_6$ by attaching a pendent edges to v_3, b pendent edges and a path with length 2 to v_4, where $a, b \geq 1$ and $a + b = n - 8$, see Figure 1.

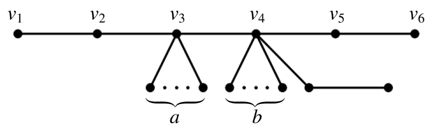

Figure 1. $\widehat{T}_{n,5}^{a,b}$.

Let $\mathscr{T}_{n,d}$ be the set of trees with order n and diameter d. Wei, Li and Zhang [17] proved in their Theorem 4.6 that, among the n-vertex trees with matching number 5 in $\bigcup_{d\neq 4} \mathscr{T}_{n,d}$, the tree $\widehat{T}_{n,5}^{1,n-9}$ has the minimum ϵ-spectral radius if $10 \leq n \leq 16$, and the trees $\widehat{T}_{n,5}^{\lfloor\gamma\rfloor,n-8-\lfloor\gamma\rfloor}$ or $\widehat{T}_{n,5}^{\lceil\gamma\rceil,n-8-\lceil\gamma\rceil}$ have the minimum ϵ-spectral radius if $n \geq 17$, where $\gamma = \frac{1}{144}(48n - 461 - 20\sqrt{6n-17})$. They commented that "since it is abnormal for the eccentricity matrices of trees of diameter 4", their Theorem 4.6 does not solve the problem for $d = 4$. So they proposed the following conjecture.

Conjecture 1.1. *Among the n-vertex trees with matching number* 5 *with* $n \geq 10$*, the tree* $\widehat{T}_{n,5}^{1,n-9}$ *has the minimum* ϵ*-spectral radius if* $10 \leq n \leq 16$*, and the trees* $\widehat{T}_{n,5}^{\lfloor\gamma\rfloor,n-8-\lfloor\gamma\rfloor}$ *or* $\widehat{T}_{n,5}^{\lceil\gamma\rceil,n-8-\lceil\gamma\rceil}$ *have the minimum* ϵ*-spectral radius if* $n \geq 17$*, where* $\gamma = \frac{1}{144}(48n - 461 - 20\sqrt{6n-17})$.

In this article, to avoid the earlier difficulty, we propose new graph transformations, characterize the certain structures of the extremal trees and determine the unique tree in $\mathscr{T}_{n,4}$ with matching number 5 having the minimum ϵ-spectral radius, so give an affirmative answer to the conjecture.

2. Preliminaries

In this section, we introduce some preliminary results which will be used to in our proofs. The following Lemma is a well-known result in the theory of nonnegative matrices, see [4, 6].

Lemma 2.1. *[4, 6] Let A and B be two nonnegative irreducible matrices with same order. If $A_{i,j} \leq B_{i,j}$ for each i, j, then $\rho(A) \leq \rho(B)$ with equality if and only if $A = B$, where $\rho(A)$ and $\rho(B)$ denote the spectral radius of A and B,respectively.*

Mahato et al. [8] obtained a lower bound for the ϵ-spectral radius of a graph with given diameter, which will be used in the proof of Lemma 3.1 and Lemma 3.2.

Lemma 2.2. *If G is a connected graph with diameter $d \geq 2$, then $\xi_1(G) \geq d$ with equality if and only if G is a diametrical graph.*

Lemma 2.3. *[14] The eccentricity matrix of a tree with order $n \geq 2$ is irreducible.*

Note that for a tree T with at least two vertices, $\epsilon(T)$ is an irreducible nonnegative matrix by Lemma 2.3. Thus by the Perron-Frobenius theorem, $\rho_\epsilon(T)$ is simple, and there is a positive unit eigenvector of $\epsilon(T)$, which is called Perron vector of $\epsilon(T)$, corresponding to $\rho_\epsilon(T)$.

Lemma 2.4. *[17] $\rho_\epsilon(\widehat{T}_{n,5}^{a,b})$ is the largest root of $f(\lambda) = 0$, where $f(\lambda) = \lambda^4 - 32a\lambda^2 - 16b\lambda^2 - 141\lambda^2 + 512ab + 1312a + 800b + 2050$.*

Lemma 2.5. *[17] Among $\bigcup_{d\neq 4} \mathscr{T}_{n,d}$ with matching number 5, $\widehat{T}_{n,5}^{1,n-9}$ is the tree with minimum ϵ-spectral radius for $10 \leq n \leq 16$, and $\widehat{T}_{n,5}^{\lfloor\gamma\rfloor,n-8-\lfloor\gamma\rfloor}$ or $\widehat{T}_{n,5}^{\lceil\gamma\rceil,n-8-\lceil\gamma\rceil}$ are trees with the minimum ϵ-spectral radius for $n \geq 17$, where $\gamma = \frac{1}{144}(48n - 461 - 20\sqrt{6n-17})$.*

3. Main results

Let $S_{n,\ell}(a_0, a_1, \ldots, a_\ell)$ be the tree obtained by adding an edge between the center v_0 of a star S_{a_0+1} and the center v_i of the star S_{a_i+1} for each $i = 1, 2, \ldots, \ell$, see Figure 2. Any tree with diameter 4 is of the form $S_{n,\ell}(a_0, a_1, \ldots, a_\ell)$ with $\ell \geq 2$ and $a_i \geq 1$ for each $i = 1, 2, \ldots, \ell$.

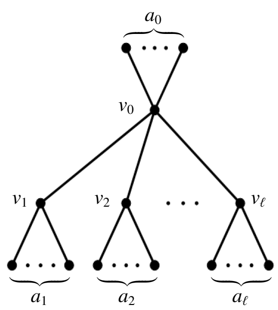

Figure 2. Tree $S_{n,\ell}(a_0, a_1, a_2, \ldots, a_\ell)$.

Lemma 3.1. *Let $T \cong S_{n,\ell}(a_0, a_1, \ldots, a_\ell)$ with $\ell \geq 2$ and $1 \leq a_1 \leq a_2 \leq \cdots \leq a_\ell$, where $n = l + 1 + \sum_{k=0}^{\ell} a_k$. If there exists k such that $1 \leq k \leq \ell - 1$ and $a_k \geq 2$, then*

$$\rho_\epsilon(S_{n,\ell}(a_0, a_1, \ldots, a_k - 1, \ldots, a_\ell + 1)) < \rho_\epsilon(T).$$

Proof. Let $U_i = V(S_{a_i+1})\backslash\{v_i\}$ for $0 \leq i \leq \ell$. We partition $V(T)$ into $U_0, \{v_0\}, \{v_1\}, ..., \{v_\ell\}, U_1, ..., U_\ell$, then $\epsilon(T)$ can be written as

$$\begin{pmatrix} 0_{a_0,a_0} & 0_{a_0,1} & 0_{a_0,1} & 0_{a_0,1} & \cdots & 0_{a_0,1} & 3J_{a_0,a_1} & 3J_{a_0,a_2} & \cdots & 3J_{a_0,a_l} \\ 0_{1,a_0} & 0 & 0 & 0 & \cdots & 0 & 2J_{1,a_1} & 2J_{1,a_2} & \cdots & 2J_{1,a_l} \\ 0_{1,a_0} & 0 & 0 & 0 & \cdots & 0 & 0_{1,a_1} & 3J_{1,a_2} & \cdots & 3J_{1,a_l} \\ 0_{1,a_0} & 0 & 0 & 0 & \cdots & 0 & 3J_{1,a_1} & 0_{1,a_2} & \cdots & 3J_{1,a_l} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{1,a_0} & 0 & 0 & 0 & \cdots & 0 & 3J_{1,a_1} & 3J_{1,a_2} & \cdots & 0_{1,a_l} \\ 3J_{a_1,a_0} & 2J_{a_1,1} & 0_{a_1,1} & 3J_{a_1,1} & \cdots & 3J_{a_1,1} & 0_{a_1,a_1} & 4J_{a_1,a_2} & \cdots & 4J_{a_1,a_l} \\ 3J_{a_2,a_0} & 2J_{a_2,1} & 3J_{a_2,1} & 0_{a_2,1} & \cdots & 3J_{a_2,1} & 4J_{a_2,a_1} & 0_{a_2,a_2} & \cdots & 4J_{a_2,a_l} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3J_{a_l,a_0} & 2J_{a_l,1} & 3J_{a_l,1} & 3J_{a_l,1} & \cdots & 0_{a_l,1} & 4J_{a_l,a_1} & 4J_{a_l,a_2} & \cdots & 0_{a_l,a_l} \end{pmatrix}.$$

Let x be a Perron eigenvector corresponding to $\rho := \rho_\epsilon(T)$, whose coordinate with respect to vertex v is x_v.

Since $\rho x_{u_0} = 3\sum_{j=1}^{\ell}\sum_{u_j \in U_j} x_{u_j}$ for any $u_0 \in U_0$, we have $x_{u_0} = x_{u_0'}$ for any $u_0, u_0' \in U_0$. Similarly, we have $x_{u_j} = x'_{u_j}$ for any $x_{u_j}, x'_{u_j} \in U_j$ with $j = 1, 2, ..., \ell$. Thus

$$\begin{aligned}
\rho x_{u_0} &= 3(a_1x_{u_1} + a_2x_{u_2} + \cdots + a_\ell x_{u_\ell});\\
\rho x_{v_0} &= 2(a_1x_{u_1} + a_2x_{u_2} + \cdots + a_\ell x_{u_\ell});\\
\rho x_{v_1} &= 3(0 + a_2x_{u_2} + \cdots + a_\ell x_{u_\ell});\\
\rho x_{v_2} &= 3(a_1x_{u_1} + 0 + \cdots + a_\ell x_{u_\ell});\\
&\cdots\\
\rho x_{v_\ell} &= 3(a_1x_{u_1} + \cdots + a_{\ell-1}x_{u_{\ell-1}} + 0);\\
\rho x_{u_1} &= 2x_{v_0} + 3(a_0x_{u_0} + 0 + x_{v_2} + \cdots + x_{v_\ell}) + 4(0 + a_2x_{u_2} + \cdots + a_\ell x_{u_\ell});\\
\rho x_{u_2} &= 2x_{v_0} + 3(a_0x_{u_0} + x_{v_1} + 0 + \cdots + x_{v_\ell}) + 4(a_1x_{u_1} + 0 + \cdots + a_\ell x_{u_\ell});\\
&\cdots\\
\rho x_{u_\ell} &= 2x_{v_0} + 3(a_0x_{u_0} + x_{v_1} + \cdots + x_{v_{\ell-1}} + 0) + 4(a_1x_{u_1} + \cdots + a_{\ell-1}x_{u_{\ell-1}} + 0).
\end{aligned}$$

By eliminating $x_{u_0}, x_{v_0}, \ldots x_{v_\ell}$ from the above system, we obtain

$$\begin{cases}
\rho^2 x_{u_1} = (9a_0 + 9\ell - 5)a_1x_{u_1} + (9a_0 + 9\ell - 14 + 4\rho)a_2x_{u_2} + \cdots + (9a_0 + 9\ell - 14 + 4\rho)a_\ell x_{u_\ell};\\
\rho^2 x_{u_2} = (9a_0 + 9\ell - 14 + 4\rho)a_1x_{u_1} + (9a_0 + 9\ell - 5)a_2x_{u_2} + \cdots + (9a_0 + 9\ell - 14 + 4\rho)a_\ell x_{u_\ell};\\
\cdots\\
\rho^2 x_{u_\ell} = (9a_0 + 9\ell - 14 + 4\rho)a_1x_{u_1} + (9a_0 + 9\ell - 14 + 4\rho)a_2x_{u_2} + \cdots + (9a_0 + 9\ell - 5)a_\ell x_{u_\ell}.
\end{cases}$$

Let $c = 9a_0 + 9\ell - 14$. Since $x_{u_i} > 0$ for all $1 \le i \le \ell$, ρ is the largest root of $f_{a_k,a_\ell}(\lambda) = 0$, where

$$f_{a_k,a_\ell}(\lambda) = \begin{vmatrix} \lambda^2-(c+9)a_1 & -(c+4\lambda)a_2 & \cdots & -(c+4\lambda)a_\ell \\ -(c+4\lambda)a_1 & \lambda^2-(c+9)a_2 & \cdots & -(c+4\lambda)a_\ell \\ \vdots & \vdots & \ddots & \vdots \\ -(c+4\lambda)a_1 & -(c+4\lambda)a_2 & \cdots & \lambda^2-(c+9)a_\ell \end{vmatrix}$$

$$= \begin{vmatrix} \lambda^2-(c+9)a_1 & -(c+4\lambda)a_2 & \cdots & -(c+4\lambda)a_\ell \\ -(\lambda^2+4\lambda a_1-9a_1) & \lambda^2+4\lambda a_2-9a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(\lambda^2+4\lambda a_1-9a_1) & 0 & \cdots & \lambda^2+4\lambda a_\ell-9a_\ell \end{vmatrix}.$$

$$= \Pi_{i=1}^{\ell}(\lambda^2+4a_i\lambda-9a_i)\begin{vmatrix} \frac{\lambda^2-(c+9)a_1}{\lambda^2+4a_1\lambda-9a_1} & \frac{-(c+4\lambda)a_2}{\lambda^2+4a_2\lambda-9a_2} & \cdots & \frac{-(c+4\lambda)a_\ell}{\lambda^2+4a_\ell\lambda-9a_\ell} \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{vmatrix}$$

$$= \left(1-\sum_{i=1}^{\ell}\frac{a_i(c+4\lambda)}{\lambda^2+4a_i\lambda-9a_i}\right)\Pi_{i=1}^{\ell}(\lambda^2+4a_i\lambda-9a_i).$$

By Lemma 2.2, $\rho \geq diam(T) = 4$. Then $\rho^2+4a_i\rho-9a_i > 0$ for all $1 \leq i \leq \ell$. Hence, ρ is the largest root of $g_{a_k,a_\ell}(\lambda) = 0$, where

$$g_{a_k,a_\ell}(\lambda) = 1-\sum_{i=1}^{\ell}\frac{a_i(c+4\lambda)}{\lambda^2+4a_i\lambda-9a_i}.$$

Since for $\lambda > 0$,

$$g'_{a_k,a_\ell}(\lambda) = \sum_{i=1}^{\ell}\frac{4a_i\lambda^2+2a_ic\lambda+36a_i^2+4a_i^2c}{(\lambda^2+4a_i\lambda-9a_i)^2} > 0,$$

we have $g_{a_k,a_\ell}(\lambda)$ is monotonically increasing for $\lambda > 0$. Thus it is sufficient to prove $g_{a_k-1,a_\ell+1}(\lambda) > g_{a_k,a_\ell}(\lambda)$ for $\lambda \geq \rho$.

By a direct calculation, one has

$$\begin{aligned} & g_{a_k-1,a_\ell+1}(\lambda) - g_{a_k,a_\ell}(\lambda) \\ = & -\frac{(a_k-1)(c+4\lambda)}{\lambda^2+4(a_k-1)\lambda-9(a_k-1)} - \frac{(a_\ell+1)(c+4\lambda)}{\lambda^2+4(a_\ell+1)\lambda-9(a_\ell+1)} + \frac{a_k(c+4\lambda)}{\lambda^2+4a_k\lambda-9a_k} + \frac{a_\ell(c+4\lambda)}{\lambda^2+4a_\ell\lambda-9a_\ell} \\ = & \frac{(c\lambda^2+4\lambda^3)(4\lambda-9)(a_\ell-a_k+1)\left((4\lambda-9)(a_k+a_\ell)+2\lambda^2\right)}{(\lambda^2+4a_k\lambda-9a_k)(\lambda^2+4(a_k-1)\lambda-9(a_k-1))(\lambda^2+4a_\ell\lambda-9a_\ell)(\lambda^2+4(a_\ell+1)\lambda-9(a_\ell+1))} \\ > & \, 0. \end{aligned}$$

Lemma 3.2. *Let $T_b = S_{n,4}(n-8-b,1,1,1,b)$ with $1 \leq b \leq n-9$. Then $\rho_\epsilon(T_b) \geq 6+\sqrt{36n-191}$ with equality if and only if $b = 1$.*

Proof. By the proof of Lemma 3.1, $\rho_\epsilon(T_b)$ is the largest root of

$$1-3\cdot\frac{9(n-8-b)+9\cdot 4-14+4\lambda}{\lambda^2+4\lambda-9}-\frac{b\,(9(n-8-b)+9\cdot 4-14+4\lambda)}{\lambda^2+4b\lambda-9b}=0,$$

i.e., $f_b(\lambda)=0$, where

$$f_b(\lambda)=\lambda^4-8\lambda^3+\left(9b^2+(20-9n)b-27n+141\right)\lambda^2+(144b^2+(872-144n)b)\lambda+(324(n-b)-1719)b.$$

Note that

$$\begin{aligned} f_1(\lambda)=&\lambda^4-8\lambda^3+(170-36n)\lambda^2+(1016-144n)\lambda+324n-2043\\ =&(\lambda^2+4\lambda-9)(\lambda^2-12\lambda+227-36n). \end{aligned}$$

By a direct calculation, $\rho_\epsilon(T_1)=6+\sqrt{36n-191}$.

Suppose now that $2\le b\le n-9$. Then it is sufficient to prove $f_1(\lambda)>f_b(\lambda)$ for $\lambda\ge\rho_\epsilon(T_b)$. By a direct calculation, one has

$$\begin{aligned} &f_1(\lambda)-f_b(\lambda)\\ =&(-9b^2+9nb-20b-9n+29)\lambda^2+(-144b^2+144nb-872b-144n+1016)\lambda\\ &+324b^2+1719b-324nb+324n-2043\\ =&(b-1)((9n-9b-29)\lambda^2+(144n-144b-1016)\lambda+9(36b-36n+227))\\ =&(b-1)g(\lambda), \end{aligned}$$

where $g(\lambda)=(9n-9b-29)\lambda^2+(144n-144b-1016)\lambda+9(36b-36n+227)$ with $2\le b\le n-9$. By Lemma 2.2, we get $\rho_\epsilon(T_b)\ge diam(T)=4$. Since $9n-9b-29=9(n-b)-29\ge 52>0$ and $-\frac{144n-144b-1016}{2(9n-9b-29)}=-\frac{144(n-b)-1016}{2(9n-9b-29)}<0$, we have $g(\lambda)$ is montoncially increasing for $\lambda>0$. Then $g(\lambda)\ge g(\rho_\epsilon(T_b))\ge g(4)=396(n-b-8)+683>0$. Thus, $f_1(\lambda)>f_b(\lambda)$ for $\lambda\ge\rho_\epsilon(T_b)$.

We note that a similar treatment has been used in studying the Estrada indices, see [1, 11].

Lemma 3.3. *Let* $T_1=S_{n,4}(n-9,1,1,1,1)$ *and* $T_1^*=S_{n,5}(0,1,1,1,1,n-10)$. *For* $n\ge 11$, *we have* $\rho_\epsilon(T_1^*)>\rho_\epsilon(T_1)$.

Proof. Let $N_{T_1}(v_0)\setminus\{v_1,v_2,v_3,v_4\}=\{w_0,w_1,\ldots,w_{n-10}\}$, $N_{T_1}(v_i)\setminus\{v_0\}=\{u_i\}$ for $i=1,2,3,4$. Then

$$T_1^*\cong T_1-\sum_{i=1}^{n-10}v_0w_i+\sum_{i=1}^{n-10}w_0w_i.$$

Let $V_1=\{u_1,u_2,u_3,u_4\}$, $V_2=\{v_1,v_2,v_3,v_4\}$ and $V_3=\{w_1,w_2,\ldots,w_{n-10}\}$. As we pass from T_1 to T_1^*, we have

$$\epsilon(T_1^*)_{u,v}-\epsilon(T_1)_{u,v}=\begin{cases}2 & \text{if } u\in\{v_0\}, v\in V_3,\\ 1 & \text{if } u\in V_1, v\in V_3,\\ 3 & \text{if } u\in V_2, v\in V_3,\\ 0 & \text{otherwise.}\end{cases}$$

Thus, $\epsilon(T_1^*)>\epsilon(T_1)$. Since T_1 and T_1^* are n-vertex trees, by Lemma 2.3, $\epsilon(T_1)$ and $\epsilon(T_1^*)$ are irreducible. By Lemma 2.1, $\rho_\epsilon(T_1^*)>\rho_\epsilon(T_1)$.

Theorem 3.1. *Let T be an n-vertex tree with diameter 4 and matching number 5. Then $\rho_\epsilon(T) \geq 6 + \sqrt{36n - 191}$ with equality if and only if $T \cong S_{n,4}(n-9, 1, 1, 1, 1)$.*

Proof. Let T be the tree having minimum ϵ-spectral radius among n-vertex trees with diameter 4 and matching number 5. Then $T \cong S_{n,\ell}(a_0, a_1, \ldots, a_\ell)$. Since the matching number of T is 5, $\ell = 4$ for $a_0 \geq 1$ and $\ell = 5$ for $a_0 = 0$. Thus, assume that $T \cong S_{n,4}(a_0, a_1, a_2, a_3, a_4)$ with $a_0 \geq 1$ and $1 \leq a_1 \leq a_2 \leq a_3 \leq a_4$ or $T \cong S_{n,5}(0, b_1, b_2, b_3, b_4, b_5)$ with $1 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq b_5$. In the former case, we have by Lemma 3.1 that $a_1 = a_2 = a_3 = 1$, and by Lemma 3.2 that $a_4 = 1$, so $a_0 = n - 9$, that is, $T \cong S_{n,4}(n-9, 1, 1, 1, 1)$. In the latter case, we have by Lemma 3.1 that $b_1 = b_2 = b_3 = b_4 = 1$, so $T \cong S_{n,5}(0, 1, 1, 1, 1, n-10)$. By Lemma 3.3, $T \cong S_{n,4}(n-9, 1, 1, 1, 1)$.

Theorem 3.2. *Among the n-vertex trees with matching number 5 with $n \geq 10$, $\widehat{T}_{n,5}^{1,n-9}$ is the tree with minimum ϵ-spectral radius for $10 \leq n \leq 16$, and $\widehat{T}_{n,5}^{\lfloor\gamma\rfloor,n-8-\lfloor\gamma\rfloor}$ or $\widehat{T}_{n,5}^{\lceil\gamma\rceil,n-8-\lceil\gamma\rceil}$ are trees with the minimum ϵ-spectral radius for $n \geq 17$, where $\gamma = \frac{1}{144}(48n - 461 - 20\sqrt{6n-17})$.*

Proof. Let T be the tree minimizing the ϵ-spectral radius in $\mathscr{T}_{n,d}$ with matching number 5. If $d = 4$, then $T \cong S_{n,4}(n-9, 1, 1, 1, 1)$ by Theorem 3.1. If $d \geq 5$, then $T \cong \widehat{T}_{n,5}^{a,b}$ by Lemma 2.5. Let ρ_ϵ be the spectral radius of $\widehat{T}_{n,5}^{a,b}$. Then by Lemma 2.4, ρ_ϵ is the largest root of $f(\lambda) = 0$, where

$$f(\lambda) = \lambda^4 - 32a\lambda^2 - 16b\lambda^2 - 141\lambda^2 + 512ab + 1312a + 800b + 2050.$$

Note that $a + b = n - 8$. Thus

$$f(\lambda) = \lambda^4 - (16n + 16a + 13)\lambda^2 + 512a(n - 7 - a) + 800n - 4350.$$

Since $n \geq 10$, then $512a(n - 7 - a) + 800n - 4350 > 0$. Thus

$$\begin{aligned}
\rho_\epsilon(\widehat{T}_{n,5}^{a,b}) &= \sqrt{\frac{16n + 16a + 13 + \sqrt{(16n + 16a + 13)^2 - 4(512a(n - 7 - a) + 800n - 435)}}{2}} \\
&< \sqrt{16n + 16a + 13} \\
&\leq \sqrt{16n + 16(n - 8) + 13} \\
&= \sqrt{32n - 115} \\
&< \sqrt{36n - 155 + 12\sqrt{36n - 191}} \\
&= 6 + \sqrt{36n - 191} \\
&= \rho_\epsilon(S_{n,4}(n - 9, 1, 1, 1, 1)).
\end{aligned}$$

Combining with Lemma 2.5, the result follows.

4. Conclusions

In this contribution, we characterize the unique tree among all trees with diameter 4 and matching number 5 that minimizes the ϵ-spectral radius. This confirms a conjecture in [17]. By combining the results from [17], the trees with minimum ϵ-spectral radius among all trees with matching number $r \leq 5$ have been characterized completely. For further study, one may try to determine the trees with the minimum ϵ-spectral radius among all trees with matching number $r \geq 6$ or even among trees with given fraction matching number.

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Conflict of interest

The authors declare that they have no competing interests.

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