Asymptotic behavior of projections of supercritical multi-type continuous state and continuous time branching processes with immigration

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Abstract

Under a fourth order moment condition on the branching and a second order moment condition on the immigration mechanisms, we show that an appropriately scaled projection of a supercritical and irreducible continuous state and continuous time branching process with immigration on certain left non-Perron eigenvectors of the branching mean matrix is asymptotically mixed normal. With an appropriate random scaling, under some conditional probability measure, we prove asymptotic normality as well. In case of a nontrivial process, under a first order moment condition on the immigration mechanism, we also prove the convergence of the relative frequencies of distinct types of individuals on a suitable event; for instance, if the immigration mechanism does not vanish, then this convergence holds almost surely.

1 Introduction

The asymptotic behavior of multi-type supercritical branching processes without or with immigration has been studied for a long time. Kesten and Stigum [20, Theorems 2.1, 2.2, 2.3, 2.4] investigated the limiting behaviors of the inner products $\langle \boldsymbol{a}, \boldsymbol{X}_n \rangle$ as $n \to \infty$, where $\boldsymbol{X}_n, n \in \{1, 2, \ldots\}$, is a supercritical, irreducible and positively regular *d*-type Galton–Watson

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branching process without immigration and $\boldsymbol{a} \in \mathbb{R}^d \setminus \{0\}$ is orthogonal to the left Perron eigenvector of the branching mean matrix $\boldsymbol{M} := (\mathbb{E}(\langle \boldsymbol{e}_j, \boldsymbol{X}_1 \rangle | \boldsymbol{X}_0 = \boldsymbol{e}_i))_{i,j \in \{1,...,d\}}$ of the process, where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ denotes the natural basis in \mathbb{R}^d . Of course, this can arise only if $d \in \{2, 3, \ldots\}$. It is enough to consider the case of $||\boldsymbol{a}|| = 1$, when $\langle \boldsymbol{a}, \boldsymbol{X}_n \rangle$ is the scalar projection of \boldsymbol{X}_n on \boldsymbol{a} . The appropriate scaling factor of $\langle \boldsymbol{a}, \boldsymbol{X}_n \rangle$, $n \in \{1, 2, \ldots\}$, depends not only on the Perron eigenvalue $r(\boldsymbol{M})$ (which is the spectral radius of \boldsymbol{M}) and on the left and right Perron eigenvectors of \boldsymbol{M} , but also on the full spectral representation of \boldsymbol{M} . Badalbaev and Mukhitdinov [4, Theorems 1 and 2] extended these results of Kesten and Stigum [20], namely, they described in a more explicit way the asymptotic behavior of $(\langle \boldsymbol{a}^{(1)}, \boldsymbol{X}_n \rangle, \ldots, \langle \boldsymbol{a}^{(d-1)}, \boldsymbol{X}_n \rangle)$ as $n \to \infty$, where $\{\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(d-1)}\}$ is a basis of the hyperplane in \mathbb{R}^d orthogonal to the left Perron eigenvector of \boldsymbol{M} . They also pointed out the necessity of considering the functionals above originated in statistical investigations for \boldsymbol{X}_n , $n \in \{1, 2, \ldots\}$.

Athreya [1, 2] investigated the limiting behavior of X_t and the inner products $\langle v, X_t \rangle$ as $t \to \infty$, where $(X_t)_{t \in [0,\infty)}$ is a supercritical, positively regular and non-singular d-type continuous time Galton–Watson branching process without immigration and $v \in \mathbb{C}^d$ is a right eigenvector corresponding to an eigenvalue $\lambda \in \mathbb{C}$ of the infinitesimal generator A of the branching mean matrix semigroup $\boldsymbol{M}(t) := (\mathbb{E}(\langle \boldsymbol{e}_j, \boldsymbol{X}_t \rangle | \boldsymbol{X}_0 = \boldsymbol{e}_i))_{i,j \in \{1,\dots,d\}} = e^{t\boldsymbol{A}}, t \in [0,\infty),$ of the process. Under a first order moment condition on the branching distributions, denoting by $s(\mathbf{A})$ the maximum of the real parts of the eigenvalues of \mathbf{A} , it was shown that there exists a non-negative random variable w_{u,X_0} such that $e^{-s(A)t}X_t$ converges to $w_{u,X_0}u$ almost surely as $t \to \infty$, where u denotes the left Perron eigenvector of the branching mean matrix M(1). Under a second order moment condition on the branching distributions, it was shown that if $\operatorname{Re}(\lambda) \in \left(\frac{1}{2}s(\boldsymbol{A}), s(\boldsymbol{A})\right]$, then $e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ converges almost surely and in L_2 to a (complex) random variable as $t \to \infty$, and if $\operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\boldsymbol{A})\right]$ and $\mathbb{P}(w_{\boldsymbol{u},\boldsymbol{X}_0} > 0) > 0$, then, under the conditional probability measure $\mathbb{P}(\cdot | w_{u,X_0} > 0)$, the limit distribution of $t^{-\theta} e^{-s(\mathbf{A})t/2} \langle \mathbf{v}, \mathbf{X}_t \rangle$ as $t \to \infty$ is mixed normal, where $\theta = \frac{1}{2}$ if $\operatorname{Re}(\lambda) = \frac{1}{2} s(\mathbf{A})$ and $\theta = 0$ if $\operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(A)\right)$. Further, in case of $\operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(A)\right]$, under the conditional probability measure $\mathbb{P}(\cdot | w_{u,X_0} > 0)$, with an appropriate random scaling, asymptotic normality has been derived as well with an advantage that the limit laws do not depend on the initial value X_0 . We also recall that Athreya [1] described the asymptotic behaviour of $\mathbb{E}(|\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle^2|)$ as $t \to \infty$ under a second order moment condition on the branching distributions. These results have been extended by Athreya [3] for the inner products $\langle a, X_t \rangle$, $t \in [0,\infty)$, with arbitrary $\boldsymbol{a} \in \mathbb{C}^d$. Janson [18, Theorem 3.1] gave a functional version of Athreya's above mentioned results in [1, 2]. Under some weaker conditions than Athreya [1, 2], Janson [18] described the asymptotic behaviour of $(\langle v, X_{t+s} \rangle)_{s \in [0,\infty)}$ as $t \to \infty$ by giving more explicit formulas for the asymptotic variances and covariances as well. For a more detailed comparison of Athreya's and Janson's results, see Janson [18, Section 6].

Kyprianou et al. [21] described the limit behavior of the inner product $\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle$ as $t \to \infty$ for supercritical and irreducible *d*-type continuous state and continuous time branching processes (without immigration), where \boldsymbol{u} denotes the left Perron vector of the branching mean

matrix of $(\mathbf{X}_t)_{t \in [0,\infty)}$. Barczy et al. [8] started to investigate the limiting behavior of the inner products $\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ as $t \to \infty$, where $(\boldsymbol{X}_t)_{t \in [0,\infty)}$ is a supercritical and irreducible d-type continuous state and continuous time branching process with immigration (CBI process) and $v \in \mathbb{C}^d$ is a left eigenvector corresponding to an eigenvalue $\lambda \in \mathbb{C}$ of the infinitesimal generator \tilde{B} of the branching mean matrix semigroup $e^{t\tilde{B}}$, $t \in [0, \infty)$, of the process. Note that for each $t \in [0,\infty)$ and $i, j \in \{1,\ldots,d\}$, we have $\langle \boldsymbol{e}_i, e^{t\tilde{\boldsymbol{B}}}\boldsymbol{e}_j \rangle = \mathbb{E}(\langle \boldsymbol{e}_i, \boldsymbol{Y}_t \rangle | \boldsymbol{Y}_0 = \boldsymbol{e}_j)$, where $(\boldsymbol{Y}_t)_{t\in[0,\infty)}$ is a multi-type continuous state and continuous time branching process without immigration and with the same branching mechanism as $(X_t)_{t \in [0,\infty)}$, so \tilde{B} plays the role of A^{\top} in Athreya [2], hence in our results the right and left eigenvectors are interchanged compared to Athreva [2]. Under first order moment conditions on the branching and immigration mechanisms, it was shown that there exists a non-negative random variable w_{u,X_0} such that $e^{-s(\widetilde{B})t} X_t$ converges to $w_{u,X_0} \widetilde{u}$ almost surely as $t \to \infty$, where \widetilde{u} is the right Perron vector of $e^{\hat{B}}$, see Barczy et al. [8, Theorem 3.3]. If v is a left non-Perron eigenvector of the branching mean matrix $e^{\hat{B}}$, then this result implies that $e^{-s(\hat{B})t}\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle \rightarrow w_{\boldsymbol{u}, \boldsymbol{X}_0} \langle \boldsymbol{v}, \tilde{\boldsymbol{u}} \rangle = 0$ almost surely as $t \to \infty$, since $\langle \boldsymbol{v}, \tilde{\boldsymbol{u}} \rangle = 0$ due to the so-called principle of biorthogonality (see, e.g., Horn and Johnson [14, Theorem 1.4.7(a)]), consequently, the scaling factor $e^{-s(B)t}$ is not appropriate for describing the asymptotic behavior of the projection $\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ as $t \to \infty$. Under suitable moment conditions on the branching and immigration mechanisms, it was shown that if $\operatorname{Re}(\lambda) \in \left(\frac{1}{2}s(\widetilde{\boldsymbol{B}}), s(\widetilde{\boldsymbol{B}})\right]$, then $e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ converges almost surely and in L_1 (in L_2) to a (complex) random variable as $t \to \infty$, see Barczy et al. [8, Theorems 3.1 and 3.4].

The aim of the present paper is to continue the investigations of Barczy et al. [8]. We will prove that under a fourth order moment condition on the branching mechanism and a second order moment condition on the immigration mechanism, if $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\widetilde{B})]$, then the limit distribution of $t^{-\theta}e^{-s(\widetilde{B})t/2}\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ as $t \to \infty$ is mixed normal, where $\theta = \frac{1}{2}$ if $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{B})$ and $\theta = 0$ if $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\widetilde{B}))$, see parts (ii) and (iii) of Theorem 3.1. If $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\widetilde{B})]$ and $(\boldsymbol{X}_t)_{t\in[0,\infty)}$ is non-trivial (equivalently, $\mathbb{P}(w_{u,\boldsymbol{X}_0} > 0) > 0$, see Lemma 3.3), then under the conditional probability measure $\mathbb{P}(\cdot | w_{u,\boldsymbol{X}_0} > 0)$, with an appropriate random scaling, we prove asymptotic normality as well with an advantage that the limit laws do not depend on the initial value \boldsymbol{X}_0 , see Theorem 3.4. For the asymptotic variances, explicit formulas are presented. In case of a non-trivial process, under a first order moment condition on the immigration mechanism, we also prove the convergence of the relative frequencies of distinct types of individuals on the event $\{w_{u,\boldsymbol{X}_0} > 0\}$ (see Proposition 3.6); for instance, if the immigration mechanism does not vanish, then this convergence holds almost surely (see Theorem 3.2).

Now, we summary the novelties of our paper. We point out that we investigate the asymptotic behavior of the projections of a multi-type CBI process on certain left non-Perron eigenvectors of its branching mean matrix. Our approach is based on a decomposition of the process $(e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle)_{t \in [0,\infty)}$ as the sum of a deterministic process and three square-integrable martingales, see the beginning of the proof of part (iii) of Theorem 3.1. For proving asymptotic normality of the martingales in question, we use a result due to Crimaldi and Pratelli [11, Theorem 2.2] (see also Theorem E.1), which provides a set of sufficient conditions for the asymptotic normality of multivariate martingales. These sufficient conditions are about the quadratic variation process and the jumps of the multivariate martingale in question. In the course of checking the conditions of Theorem E.1, we need to study the asymptotic behaviour of the expectation of the running supremum of the jumps of a compensated Poisson integral process having time dependent integrand over an interval [0, t] as $t \to \infty$. There is a new interest in this type of questions, see, e.g., the paper of He and Li [13] on the distributions of jumps of a single-type CBI process.

Next, we compare our methodology with the discrete-valued settings. Athreya [2] decomposed $e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ in three terms, where $(\boldsymbol{X}_t)_{t \in [0,\infty)}$ is a supercritical, positively regular and non-singular d-type continuous time Galton–Watson branching process without immigration, and he showed that two of them are small in probability and, using the central limit theorem, the third one converges to the desired normal distribution. Janson's proof [18, Theorem 3.1] for a functional extension of Athreya's results is based on a martingale convergence theorem (see [18, Proposition 9.1]) that relies on the convergence of the quadratic variation of an L_2 -locally bounded (see [18, condition (9.2)]) martingale sequence. Then, he needed to define a suitable martingale sequence, and estimate its quadratic variation. Observe that he asked for a finite second moment for the branching mechanism in order to have an L_2 -locally bounded martingale (see [18, assumption (A.2)]). In our case, where $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is a supercritical and irreducible d-type CBI process, the three martingales appearing in the previously mentioned decomposition of $(e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle)_{t \in [0,\infty)}$ turn out to be square-integrable under our moment assumptions of the branching and immigration mechanisms. One of the three martingales in question is an integral with respect to a standard Wiener process, and the other two are integrals with respect to compensated Poisson measures. The decomposition in question was derived using an SDE representation of $(X_t)_{t \in [0,\infty)}$ together with an application of the multidimensional Itô's formula, see Barczy et al. [7, Lemma 4.1]. Concerning our moment assumptions, in order to be able to check the conditions of the previously mentioned Theorem 2.2 in Crimaldi and Pratelli [11] (see also Theorem E.1) we need a fourth order moment condition on the branching mechanism and a second order moment condition on the immigration mechanism. So our proof technique can not be considered as an easy adaption of that of Athreya's [1, 2] or that of Janson [18, Theorem 3.1].

The paper is structured as follows. In Section 2, we recall the definition of multi-type CBI processes together with the notion of irreducibility, and we introduce a classification of multi-type CBI processes as well. Sections 3 and 4 contain our results and their proofs, respectively. We close the paper with five appendices. In Appendix A we recall a decomposition of multi-type CBI processes, Appendix B is devoted to a description of deterministic projections of multi-type CBI processes (i.e., projections that are deterministic). In Appendix C, based on Buraczewski et al. [10, Proposition 4.3.2], we recall some mild conditions under which the solution of a stochastic fixed point equation is atomless. Appendix D is devoted to the description of the asymptotic behaviour of the second moment of projections of multi-type CBI processes. In Appendix E we recall a result on the asymptotic behavior of multi-type CBI processes. In Appendix E we recall a result on the asymptotic behavior of multi-type CBI processes.

2 Preliminaries

Let \mathbb{Z}_+ , \mathbb{N} , \mathbb{R} , \mathbb{R}_+ , \mathbb{R}_{++} and \mathbb{C} denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers, positive real numbers and complex numbers, respectively. For $x, y \in \mathbb{R}$, we will use the notations $x \wedge y := \min\{x, y\}, x \vee y := \max\{x, y\}$ and $x^+ := \max\{0, x\}$. By $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{j=1}^d x_j \overline{y_j}$, we denote the Euclidean inner product of $\boldsymbol{x} = (x_1, \dots, x_d)^\top \in \mathbb{C}^d$ and $\boldsymbol{y} = (y_1, \dots, y_d)^\top \in \mathbb{C}^d$, and by $\|\boldsymbol{x}\|$ and $\|\boldsymbol{A}\|$, we denote the induced norm of $x \in \mathbb{C}^d$ and $A \in \mathbb{C}^{d \times d}$, respectively. By r(A), we denote the spectral radius of $A \in \mathbb{C}^{d \times d}$. The null vector and the null matrix will be denoted by **0**. Moreover, $I_d \in \mathbb{R}^{d \times d}$ denotes the identity matrix. If $A \in \mathbb{R}^{d \times d}$ is positive semidefinite, then $A^{1/2}$ denotes the unique positive semidefinite square root of A. If $A \in \mathbb{R}^{d \times d}$ is strictly positive definite, then $A^{1/2}$ is strictly positive definite and $A^{-1/2}$ denotes the inverse of $A^{1/2}$. The set of $d \times d$ matrices with non-negative off-diagonal entries (also called essentially non-negative matrices) is denoted by $\mathbb{R}^{d \times d}_{(+)}$. By $C^2_c(\mathbb{R}^d_+, \mathbb{R})$, we denote the set of twice continuously differentiable real-valued functions on \mathbb{R}^d_+ with compact support. By $B(\mathbb{R}^d_+,\mathbb{R})$, we denote the Banach space (endowed with the supremum norm) of real-valued bounded Borel functions on \mathbb{R}^d_+ . Convergence almost surely, in L_1 , in L_2 , in probability and in distribution will be denoted by $\xrightarrow{\text{a.s.}}$, $\xrightarrow{L_1}$, $\xrightarrow{L_2}$, $\xrightarrow{\mathbb{P}}$ and $\xrightarrow{\mathcal{D}}$, respectively. For an event A with $\mathbb{P}(A) > 0$, let $\mathbb{P}_A(\cdot) := \mathbb{P}(\cdot | A) = \mathbb{P}(\cdot \cap A) / \mathbb{P}(A)$ denote the conditional probability measure given A, and let $\xrightarrow{\mathcal{D}_A}$ denote convergence in distribution under the conditional probability measure \mathbb{P}_A . Almost sure equality and equality in distribution will be denoted by $\stackrel{\text{a.s.}}{=}$ and $\stackrel{\mathcal{D}}{=}$, respectively. If $V \in \mathbb{R}^{d \times d}$ is symmetric and positive semidefinite, then $\mathcal{N}_d(\mathbf{0}, \mathbf{V})$ denotes the *d*-dimensional normal distribution with zero mean and variance matrix V. Throughout this paper, we make the conventions $\int_a^b := \int_{(a,b]}$ and $\int_a^\infty := \int_{(a,\infty)}$ for any $a, b \in \mathbb{R}$ with a < b.

2.1 Definition. A tuple $(d, c, \beta, B, \nu, \mu)$ is called a set of admissible parameters if

- (i) $d \in \mathbb{N}$,
- (ii) $c = (c_i)_{i \in \{1,...,d\}} \in \mathbb{R}^d_+,$
- (iii) $\boldsymbol{\beta} = (\beta_i)_{i \in \{1, \dots, d\}} \in \mathbb{R}^d_+,$
- (iv) $\boldsymbol{B} = (b_{i,j})_{i,j \in \{1,...,d\}} \in \mathbb{R}^{d \times d}_{(+)},$
- (v) ν is a Borel measure on $\mathcal{U}_d := \mathbb{R}^d_+ \setminus \{\mathbf{0}\}$ satisfying $\int_{\mathcal{U}_d} (1 \wedge \|\boldsymbol{r}\|) \, \nu(\mathrm{d}\boldsymbol{r}) < \infty$,
- (vi) $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$, where, for each $i \in \{1, \ldots, d\}$, μ_i is a Borel measure on \mathcal{U}_d satisfying

$$\int_{\mathcal{U}_d} \bigg[\|\boldsymbol{z}\| \wedge \|\boldsymbol{z}\|^2 + \sum_{j \in \{1,...,d\} \setminus \{i\}} (1 \wedge z_j) \bigg] \mu_i(\mathrm{d}\boldsymbol{z}) < \infty$$

2.2 Theorem. Let $(d, c, \beta, B, \nu, \mu)$ be a set of admissible parameters. Then there exists a unique conservative transition semigroup $(P_t)_{t \in \mathbb{R}_+}$ acting on $B(\mathbb{R}^d_+, \mathbb{R})$ such that its Laplace transform has a representation

$$\int_{\mathbb{R}^d_+} e^{-\langle \boldsymbol{\lambda}, \boldsymbol{y} \rangle} P_t(\boldsymbol{x}, \mathrm{d}\boldsymbol{y}) = e^{-\langle \boldsymbol{x}, \boldsymbol{v}(t, \boldsymbol{\lambda}) \rangle - \int_0^t \psi(\boldsymbol{v}(s, \boldsymbol{\lambda})) \, \mathrm{d}s}, \qquad \boldsymbol{x} \in \mathbb{R}^d_+, \quad \boldsymbol{\lambda} \in \mathbb{R}^d_+, \quad t \in \mathbb{R}_+,$$

where, for any $\boldsymbol{\lambda} \in \mathbb{R}^d_+$, the continuously differentiable function $\mathbb{R}_+ \ni t \mapsto \boldsymbol{v}(t, \boldsymbol{\lambda}) = (v_1(t, \boldsymbol{\lambda}), \dots, v_d(t, \boldsymbol{\lambda}))^\top \in \mathbb{R}^d_+$ is the unique locally bounded solution to the system of differential equations

$$\partial_t v_i(t, \boldsymbol{\lambda}) = -\varphi_i(\boldsymbol{v}(t, \boldsymbol{\lambda})), \quad v_i(0, \boldsymbol{\lambda}) = \lambda_i, \quad i \in \{1, \dots, d\},$$

with

$$\varphi_i(\boldsymbol{\lambda}) := c_i \lambda_i^2 - \langle \boldsymbol{B} \boldsymbol{e}_i, \boldsymbol{\lambda} \rangle + \int_{\mathcal{U}_d} \left(e^{-\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle} - 1 + \lambda_i (1 \wedge z_i) \right) \mu_i(\mathrm{d}\boldsymbol{z})$$

for $\boldsymbol{\lambda} \in \mathbb{R}^d_+$, $i \in \{1, \dots, d\}$, and

$$\psi(\boldsymbol{\lambda}) := \langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle + \int_{\mathcal{U}_d} (1 - e^{-\langle \boldsymbol{\lambda}, \boldsymbol{r} \rangle}) \nu(\mathrm{d}\boldsymbol{r}), \qquad \boldsymbol{\lambda} \in \mathbb{R}^d_+.$$

Theorem 2.2 is a special case of Theorem 2.7 of Duffie et al. [12] with m = d, n = 0 and zero killing rate. For more details, see Remark 2.5 in Barczy et al. [6].

2.3 Definition. A conservative Markov process with state space \mathbb{R}^d_+ and with transition semigroup $(P_t)_{t\in\mathbb{R}_+}$ given in Theorem 2.2 is called a multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$. The function $\mathbb{R}^d_+ \ni \boldsymbol{\lambda} \mapsto (\varphi_1(\boldsymbol{\lambda}), \dots, \varphi_d(\boldsymbol{\lambda}))^\top \in \mathbb{R}^d$ is called its branching mechanism, and the function $\mathbb{R}^d_+ \ni \boldsymbol{\lambda} \mapsto \psi(\boldsymbol{\lambda}) \in \mathbb{R}_+$ is called its immigration mechanism. A multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ is called a CB process (a continuous state and continuous time branching process without immigration) if $\boldsymbol{\beta} = \mathbf{0}$ and $\nu = 0$ (equivalently, $\psi = 0$).

Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|) < \infty$ and the moment condition

(2.1)
$$\int_{\mathcal{U}_d} \|\boldsymbol{r}\| \mathbb{1}_{\{\|\boldsymbol{r}\| \ge 1\}} \nu(\mathrm{d}\boldsymbol{r}) < \infty$$

holds. Then, by formula (3.4) in Barczy et al. [6],

(2.2)
$$\mathbb{E}(\boldsymbol{X}_t \mid \boldsymbol{X}_0 = \boldsymbol{x}) = e^{t\widetilde{\boldsymbol{B}}} \boldsymbol{x} + \int_0^t e^{u\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} \, \mathrm{d}\boldsymbol{u}, \qquad \boldsymbol{x} \in \mathbb{R}^d_+, \quad t \in \mathbb{R}_+,$$

where

$$\widetilde{\boldsymbol{B}} := (\widetilde{b}_{i,j})_{i,j \in \{1,\dots,d\}}, \qquad \widetilde{b}_{i,j} := b_{i,j} + \int_{\mathcal{U}_d} (z_i - \delta_{i,j})^+ \, \mu_j(\mathrm{d}\boldsymbol{z}), \qquad \widetilde{\boldsymbol{\beta}} := \boldsymbol{\beta} + \int_{\mathcal{U}_d} \boldsymbol{r} \, \nu(\mathrm{d}\boldsymbol{r}),$$

with $\delta_{i,j} := 1$ if i = j, and $\delta_{i,j} := 0$ if $i \neq j$. Note that, for each $\boldsymbol{x} \in \mathbb{R}^d_+$, the function $\mathbb{R}_+ \ni t \mapsto \mathbb{E}(\boldsymbol{X}_t \mid \boldsymbol{X}_0 = \boldsymbol{x})$ is continuous, and $\widetilde{\boldsymbol{B}} \in \mathbb{R}^{d \times d}_{(+)}$ and $\widetilde{\boldsymbol{\beta}} \in \mathbb{R}^d_+$, since

$$\int_{\mathcal{U}_d} \|\boldsymbol{r}\| \, \nu(\mathrm{d}\boldsymbol{r}) < \infty, \qquad \int_{\mathcal{U}_d} (z_i - \delta_{i,j})^+ \, \mu_j(\mathrm{d}\boldsymbol{z}) < \infty, \quad i, j \in \{1, \ldots, d\},$$

see Barczy et al. [6, Section 2]. Further, $\mathbb{E}(X_t | X_0 = x)$, $x \in \mathbb{R}^d_+$, does not depend on the parameter c. One can give probabilistic interpretations of the modified parameters \widetilde{B} and $\widetilde{\beta}$, namely, for each $t \in \mathbb{R}_+$, we have $e^{t\widetilde{B}}e_j = \mathbb{E}(Y_t | Y_0 = e_j)$, $j \in \{1, \ldots, d\}$, and $t\widetilde{\beta} = \mathbb{E}(Z_t | Z_0 = 0)$, where $(Y_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+}$ are multi-type CBI processes with parameters $(d, c, 0, B, 0, \mu)$ and $(d, 0, \beta, 0, \nu, 0)$, respectively, see formula (2.2). The processes $(Y_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+}$ can be considered as pure branching (without immigration) and pure immigration (without branching) processes, respectively. Consequently, $e^{\widetilde{B}}$ and $\widetilde{\beta}$ may be called the branching mean matrix and the immigration mean vector, respectively. Note that the branching mechanism depends only on the parameters c, B and μ , while the immigration mechanism depends only on the parameters β and ν .

If $(d, c, \beta, B, \nu, \mu)$ is a set of admissible parameters, $\mathbb{E}(||X_0||) < \infty$ and the moment condition (2.1) holds, then the multi-type CBI process with parameters $(d, c, \beta, B, \nu, \mu)$ can be represented as a pathwise unique strong solution of the stochastic differential equation (SDE)

(2.3)
$$\boldsymbol{X}_{t} = \boldsymbol{X}_{0} + \int_{0}^{t} (\boldsymbol{\beta} + \widetilde{\boldsymbol{B}} \boldsymbol{X}_{u}) \, \mathrm{d}u + \sum_{\ell=1}^{d} \int_{0}^{t} \sqrt{2c_{\ell} \max\{0, X_{u,\ell}\}} \, \mathrm{d}W_{u,\ell} \, \boldsymbol{e}_{\ell}$$
$$+ \sum_{\ell=1}^{d} \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} \boldsymbol{z} \mathbb{1}_{\{w \leq X_{u-,\ell}\}} \, \widetilde{N}_{\ell}(\mathrm{d}u, \mathrm{d}\boldsymbol{z}, \mathrm{d}w) + \int_{0}^{t} \int_{\mathcal{U}_{d}} \boldsymbol{r} \, M(\mathrm{d}u, \mathrm{d}\boldsymbol{r})$$

for $t \in \mathbb{R}_+$, see, Theorem 4.6 and Section 5 in Barczy et al. [6], where (2.3) was proved only for $d \in \{1, 2\}$, but their method clearly works for all $d \in \mathbb{N}$. Here $X_{t,\ell}$, $\ell \in \{1, \ldots, d\}$, denotes the ℓ^{th} coordinate of \mathbf{X}_t , $\mathbb{P}(\mathbf{X}_0 \in \mathbb{R}^d_+) = 1$, $(W_{t,1})_{t \in \mathbb{R}_+}$, ..., $(W_{t,d})_{t \in \mathbb{R}_+}$ are standard Wiener processes, N_ℓ , $\ell \in \{1, \ldots, d\}$, and M are Poisson random measures on $\mathbb{R}_{++} \times \mathcal{U}_d \times \mathbb{R}_{++}$ and on $\mathbb{R}_{++} \times \mathcal{U}_d$ with intensity measures $du \, \mu_\ell(d\mathbf{z}) \, dw$, $\ell \in \{1, \ldots, d\}$, and $du \, \nu(d\mathbf{r})$, respectively, such that \mathbf{X}_0 , $(W_{t,1})_{t \in \mathbb{R}_+}$, ..., $(W_{t,d})_{t \in \mathbb{R}_+}$, N_1, \ldots, N_d and Mare independent, and $\widetilde{N}_\ell(du, d\mathbf{z}, dw) := N_\ell(du, d\mathbf{z}, dw) - du \, \mu_\ell(d\mathbf{z}) \, dw$, $\ell \in \{1, \ldots, d\}$.

Next we recall a classification of multi-type CBI processes. For a matrix $\mathbf{A} \in \mathbb{R}^{d \times d}$, $\sigma(\mathbf{A})$ will denote the spectrum of \mathbf{A} , that is, the set of all $\lambda \in \mathbb{C}$ that are eigenvalues of \mathbf{A} . Then $r(\mathbf{A}) = \max_{\lambda \in \sigma(\mathbf{A})} |\lambda|$ is the spectral radius of \mathbf{A} . Moreover, we will use the notation

$$s(\mathbf{A}) := \max_{\lambda \in \sigma(\mathbf{A})} \operatorname{Re}(\lambda).$$

A matrix $A \in \mathbb{R}^{d \times d}$ is called reducible if there exist a permutation matrix $P \in \mathbb{R}^{d \times d}$ and an integer r with $1 \leq r \leq d-1$ such that

$$oldsymbol{P}^ opoldsymbol{A} oldsymbol{P}^ opoldsymbol{A} oldsymbol{P}^ opoldsymbol{A} oldsymbol{A} = egin{pmatrix} oldsymbol{A}_1 & oldsymbol{A}_2 \ oldsymbol{0} & oldsymbol{A}_3 \end{pmatrix},$$

where $A_1 \in \mathbb{R}^{r \times r}$, $A_3 \in \mathbb{R}^{(d-r) \times (d-r)}$, $A_2 \in \mathbb{R}^{r \times (d-r)}$, and $\mathbf{0} \in \mathbb{R}^{(d-r) \times r}$ is a null matrix. A matrix $A \in \mathbb{R}^{d \times d}$ is called irreducible if it is not reducible, see, e.g., Horn and Johnson [14, Definitions 6.2.21 and 6.2.22]. We do emphasize that no 1-by-1 matrix is reducible.

2.4 Definition. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that the moment condition (2.1) holds. Then $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is called irreducible if $\widetilde{\mathbf{B}}$ is irreducible.

Recall that if $\widetilde{B} \in \mathbb{R}_{(+)}^{d \times d}$ is irreducible, then $e^{t\widetilde{B}} \in \mathbb{R}_{++}^{d \times d}$ for all $t \in \mathbb{R}_{++}$, and $s(\widetilde{B})$ is a real eigenvalue of \widetilde{B} , the algebraic and geometric multiplicities of $s(\widetilde{B})$ is 1, and the real parts of the other eigenvalues of \widetilde{B} are less than $s(\widetilde{B})$. Moreover, corresponding to the eigenvalue $s(\widetilde{B})$ there exists a unique (right) eigenvector $\widetilde{u} \in \mathbb{R}_{++}^d$ of \widetilde{B} such that the sum of its coordinates is 1 which is also the unique (right) eigenvector of $e^{\widetilde{B}}$, called the right Perron vector of $e^{\widetilde{B}}$, corresponding to the eigenvalue $r(e^{\widetilde{B}}) = e^{s(\widetilde{B})}$ of $e^{\widetilde{B}}$ such that the sum of its coordinates is 1. Further, there exists a unique left eigenvector $u \in \mathbb{R}_{++}^d$ of \widetilde{B} corresponding to the eigenvalue $s(\widetilde{B})$ with $\widetilde{u}^{\top}u = 1$, which is also the unique (left) eigenvector of $e^{\widetilde{B}}$, called the left Perron vector of $e^{\widetilde{B}}$, corresponding to the eigenvalue $r(e^{\widetilde{B}}) = e^{s(\widetilde{B})}$ of $e^{\widetilde{B}}$ such that the sum of its all the left Perron vector of $e^{\widetilde{B}}$, corresponding to the eigenvalue $r(e^{\widetilde{B}}) = e^{s(\widetilde{B})}$ of $e^{\widetilde{B}}$ such that $\widetilde{u}^{\top}u = 1$. Moreover, there exist $C_1, C_2, C_3, C_4 \in \mathbb{R}_{++}$ such that

(2.4)
$$\|e^{-s(\widetilde{\boldsymbol{B}})t}e^{t\widetilde{\boldsymbol{B}}} - \widetilde{\boldsymbol{u}}\boldsymbol{u}^{\top}\| \leq C_1 e^{-C_2 t}, \quad \|e^{t\widetilde{\boldsymbol{B}}}\| \leq C_3 e^{s(\widetilde{\boldsymbol{B}})t}, \quad t \in \mathbb{R}_+,$$

(2.5)
$$\mathbb{E}(\|\boldsymbol{X}_t\|) \leqslant C_4 \mathrm{e}^{s(\tilde{\boldsymbol{B}})t}, \quad t \in \mathbb{R}_+.$$

These Frobenius and Perron type results can be found, e.g., in Barczy and Pap [9, Appendix A] and Barczy et al. [8, (3.8)].

We will need the following dichotomy of the expectation of an irreducible multi-type CBI process.

2.5 Lemma. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be an irreducible multi-type CBI process with parameters $(d, c, \beta, B, \nu, \mu)$ such that $\mathbb{E}(||\mathbf{X}_0||) < \infty$ and the moment condition (2.1) holds. Then either $\mathbb{E}(\mathbf{X}_t) = \mathbf{0}$ for all $t \in \mathbb{R}_+$, or $\mathbb{E}(\mathbf{X}_t) \in \mathbb{R}_{++}^d$ for all $t \in \mathbb{R}_{++}$. Namely, if $\mathbb{P}(\mathbf{X}_0 = \mathbf{0}) = 1$, $\beta = \mathbf{0}$ and $\nu = 0$, then $\mathbb{E}(\mathbf{X}_t) = \mathbf{0}$ for all $t \in \mathbb{R}_+$, and hence $\mathbb{P}(\mathbf{X}_t = \mathbf{0}) = 1$ for all $t \in \mathbb{R}_+$, otherwise $\mathbb{E}(\mathbf{X}_t) \in \mathbb{R}_{++}^d$ for all $t \in \mathbb{R}_{++}$.

Proof. For each $t \in \mathbb{R}_+$, by (2.2), we have

$$\mathbb{E}(\boldsymbol{X}_t) = e^{t\widetilde{\boldsymbol{B}}} \mathbb{E}(\boldsymbol{X}_0) + \int_0^t e^{u\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} \, \mathrm{d} u, \qquad t \in \mathbb{R}^d_+.$$

Since $e^{u\widetilde{B}} \in \mathbb{R}^{d \times d}_{++}$ for all $u \in \mathbb{R}_{++}$, $\mathbb{E}(X_0) \in \mathbb{R}^d_+$ and $\widetilde{\beta} \in \mathbb{R}^d_+$, we obtain the assertions. \Box

2.6 Definition. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be an irreducible multi-type CBI process with parameters $(d, c, \beta, B, \nu, \mu)$. Then $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is called trivial if $\mathbb{P}(\mathbf{X}_0 = \mathbf{0}) = 1$, $\beta = \mathbf{0}$ and $\nu = 0$, equivalently, if $\mathbb{P}(\mathbf{X}_t = \mathbf{0}) = 1$ for all $t \in \mathbb{R}_+$. Otherwise $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is called non-trivial.

We do recall the attention that if $(\boldsymbol{X}_{t}^{(1)})_{t\in\mathbb{R}_{+}}$ and $(\boldsymbol{X}_{t}^{(2)})_{t\in\mathbb{R}_{+}}$ are multi-type CBI processes with parameters $(d, \boldsymbol{c}^{(1)}, \boldsymbol{\beta}, \boldsymbol{B}^{(1)}, \boldsymbol{\nu}, \boldsymbol{\mu}^{(1)})$ and $(d, \boldsymbol{c}^{(2)}, \boldsymbol{\beta}, \boldsymbol{B}^{(2)}, \boldsymbol{\nu}, \boldsymbol{\mu}^{(2)})$, respectively, $\boldsymbol{X}_{0}^{(1)} \stackrel{\text{a.s.}}{=} \boldsymbol{X}_{0}^{(2)}$ and $(\boldsymbol{X}_{t}^{(1)})_{t\in\mathbb{R}_{+}}$ is trivial, then $(\boldsymbol{X}_{t}^{(2)})_{t\in\mathbb{R}_{+}}$ is also trivial.

2.7 Definition. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be an irreducible multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|) < \infty$ and the moment condition (2.1) holds. Then $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is called

 $\begin{cases} subcritical & if \ s(\widetilde{B}) < 0, \\ critical & if \ s(\widetilde{B}) = 0, \\ supercritical & if \ s(\widetilde{B}) > 0. \end{cases}$

For motivations of Definitions 2.4 and 2.7, see Barczy and Pap [9, Section 3].

3 Results

Now we present the main result of this paper. Recall that $\boldsymbol{u} \in \mathbb{R}^d_{++}$ is the left Perron vector of $e^{\tilde{\boldsymbol{B}}}$ corresponding to the eigenvalue $e^{s(\tilde{\boldsymbol{B}})}$.

3.1 Theorem. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be a supercritical and irreducible multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|) < \infty$ and the moment condition (2.1) holds. Let $\lambda \in \sigma(\widetilde{\mathbf{B}})$ and let $\mathbf{v} \in \mathbb{C}^d$ be a left eigenvector of $\widetilde{\mathbf{B}}$ corresponding to the eigenvalue λ .

(i) If $\operatorname{Re}(\lambda) \in \left(\frac{1}{2}s(\widetilde{\boldsymbol{B}}), s(\widetilde{\boldsymbol{B}})\right]$ and the moment condition

(3.1)
$$\sum_{\ell=1}^{d} \int_{\mathcal{U}_{d}} g(\|\boldsymbol{z}\|) \mathbb{1}_{\{\|\boldsymbol{z}\| \ge 1\}} \mu_{\ell}(\mathrm{d}\boldsymbol{z}) < \infty$$

with

$$g(x) := \begin{cases} x^{\frac{s(\tilde{B})}{\operatorname{Re}(\lambda)}} & \text{if } \operatorname{Re}(\lambda) \in \left(\frac{1}{2}s(\tilde{B}), s(\tilde{B})\right), \\ x\log(x) & \text{if } \operatorname{Re}(\lambda) = s(\tilde{B}) \quad (\Longleftrightarrow \lambda = s(\tilde{B})), \end{cases} \quad x \in \mathbb{R}_{++}$$

holds, then there exists a complex random variable w_{v,X_0} with $\mathbb{E}(|w_{v,X_0}|) < \infty$ such that

(3.2) $e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle \to w_{\boldsymbol{v}, \boldsymbol{X}_0}$ as $t \to \infty$ in L_1 and almost surely.

(ii) If $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}})$ and the moment condition

(3.3)
$$\sum_{\ell=1}^{d} \int_{\mathcal{U}_{d}} \|\boldsymbol{z}\|^{4} \mathbb{1}_{\{\|\boldsymbol{z}\|\geq1\}} \mu_{\ell}(\mathrm{d}\boldsymbol{z}) < \infty, \qquad \int_{\mathcal{U}_{d}} \|\boldsymbol{r}\|^{2} \mathbb{1}_{\{\|\boldsymbol{r}\|\geq1\}} \nu(\mathrm{d}\boldsymbol{z}) < \infty$$

holds, then

(3.4)
$$t^{-1/2} e^{-s(\widetilde{\boldsymbol{B}})t/2} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \xrightarrow{\mathcal{D}} \sqrt{w_{\boldsymbol{u}, \boldsymbol{X}_0}} \boldsymbol{Z}_{\boldsymbol{v}} \quad as \ t \to \infty,$$

where Z_v is a 2-dimensional random vector such that $Z_v \stackrel{\mathcal{D}}{=} \mathcal{N}_2(\mathbf{0}, \Sigma_v)$ independent of w_{u, X_0} , where

(3.5)
$$\boldsymbol{\Sigma}_{\boldsymbol{v}} := \frac{1}{2} \sum_{\ell=1}^{d} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \left(C_{\boldsymbol{v},\ell} \boldsymbol{I}_{2} + \begin{pmatrix} \operatorname{Re}(\widetilde{C}_{\boldsymbol{v},\ell}) & \operatorname{Im}(\widetilde{C}_{\boldsymbol{v},\ell}) \\ \operatorname{Im}(\widetilde{C}_{\boldsymbol{v},\ell}) & -\operatorname{Re}(\widetilde{C}_{\boldsymbol{v},\ell}) \end{pmatrix} \mathbb{1}_{\{\operatorname{Im}(\lambda)=0\}} \right)$$

with

$$egin{aligned} C_{oldsymbol{v},\ell} &:= 2 |\langle oldsymbol{v}, oldsymbol{e}_\ell
angle|^2 c_\ell + \int_{\mathcal{U}_d} |\langle oldsymbol{v}, oldsymbol{z}
angle|^2 \mu_\ell(\mathrm{d}oldsymbol{z}), & \ell \in \{1,\dots,d\}, \ & \widetilde{C}_{oldsymbol{v},\ell} &:= 2 \langle oldsymbol{v}, oldsymbol{e}_\ell
angle^2 c_\ell + \int_{\mathcal{U}_d} \langle oldsymbol{v}, oldsymbol{z}
angle^2 \mu_\ell(\mathrm{d}oldsymbol{z}), & \ell \in \{1,\dots,d\}. \end{aligned}$$

(iii) If $\operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{B})\right)$ and the moment condition (3.3) holds, then

(3.6)
$$e^{-s(\tilde{\boldsymbol{B}})t/2} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \xrightarrow{\mathcal{D}} \sqrt{w_{\boldsymbol{u}, \boldsymbol{X}_0}} \boldsymbol{Z}_{\boldsymbol{v}} \quad as \ t \to \infty,$$

where Z_v is a 2-dimensional random vector such that $Z_v \stackrel{\mathcal{D}}{=} \mathcal{N}_2(0, \Sigma_v)$ independent of w_{u, X_0} , where

$$(3.7) \qquad \boldsymbol{\Sigma}_{\boldsymbol{v}} := \frac{1}{2} \sum_{\ell=1}^{d} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \left\{ \frac{C_{\boldsymbol{v},\ell}}{s(\widetilde{\boldsymbol{B}}) - 2\operatorname{Re}(\lambda)} \boldsymbol{I}_{2} + \begin{pmatrix} \operatorname{Re}\left(\frac{\widetilde{C}_{\boldsymbol{v},\ell}}{s(\widetilde{\boldsymbol{B}}) - 2\lambda}\right) & \operatorname{Im}\left(\frac{\widetilde{C}_{\boldsymbol{v},\ell}}{s(\widetilde{\boldsymbol{B}}) - 2\lambda}\right) \\ \operatorname{Im}\left(\frac{\widetilde{C}_{\boldsymbol{v},\ell}}{s(\widetilde{\boldsymbol{B}}) - 2\lambda}\right) & -\operatorname{Re}\left(\frac{\widetilde{C}_{\boldsymbol{v},\ell}}{s(\widetilde{\boldsymbol{B}}) - 2\lambda}\right) \end{pmatrix} \right\}$$
with $C = \ell \in [1, \dots, d]$, and $\widetilde{C} = \ell \in [1, \dots, d]$, defined in part (ii)

with $C_{\boldsymbol{v},\ell}$, $\ell \in \{1,\ldots,d\}$, and $\tilde{C}_{\boldsymbol{v},\ell}$, $\ell \in \{1,\ldots,d\}$, defined in part (ii).

First we have some remarks concerning the limit distributions in parts (ii) and (iii) of Theorem 3.1. Note that under the moment condition (3.3), the moment condition (3.1) holds for $\lambda = s(\tilde{B})$ and hence there exists a non-negative random variable w_{u,X_0} with $\mathbb{E}(w_{u,X_0}) < \infty$ such that $e^{-s(\tilde{B})t} \langle u, X_t \rangle \to w_{u,X_0}$ as $t \to \infty$ in L_1 and almost surely. Observe that if $(X_t)_{t \in \mathbb{R}_+}$ is not a trivial process (see Definition 2.6) and $\Sigma_v \neq 0$, then the scaling factors $t^{-1/2}e^{-s(\tilde{B})t/2}$ and $e^{-s(\tilde{B})t/2}$ in parts (ii) and (iii) of Theorem 3.1 are correct in the sense that the corresponding limits are non-degenerate random variables, since $\mathbb{P}(w_{u,X_0} = 0) < 1$ due to Theorem 3.1 in Barczy et al. [8] or to Lemma 3.3. The correctness of the scaling factor in part (i) of Theorem 3.1 will be studied later on, this motivates the forthcoming Theorem 3.2. Note also that Theorem 3.1 is valid even if Σ_v is not invertible. In Proposition D.3, necessary and sufficient conditions are given for the invertibility of Σ_{v} provided that $\mathbb{E}(||X_{0}||^{2}) < \infty$, $\operatorname{Im}(\lambda) \neq 0$, and the moment condition

(3.8)
$$\sum_{\ell=1}^{d} \int_{\mathcal{U}_{d}} \|\boldsymbol{z}\|^{2} \mathbb{1}_{\{\|\boldsymbol{z}\| \ge 1\}} \mu_{\ell}(\mathrm{d}\boldsymbol{z}) < \infty, \qquad \int_{\mathcal{U}_{d}} \|\boldsymbol{r}\|^{2} \mathbb{1}_{\{\|\boldsymbol{r}\| \ge 1\}} \nu(\mathrm{d}\boldsymbol{r}) < \infty$$

holds.

Moreover, in Proposition D.2 under the moment condition (3.8) together with $\mathbb{E}(||\mathbf{X}_0||^2) < \infty$ we describe the asymptotic behavior of the variance matrix of the real and imaginary parts of $\langle \boldsymbol{v}, \mathbf{X}_t \rangle$ as $t \to \infty$, which explains the phase transition at $\operatorname{Re}(\lambda) = \frac{1}{2}s(\tilde{B})$ in Theorem 3.1. This result can be considered as an extension of Proposition B.1 in Barczy et al. [8] (see also Proposition D.1), where the asymptotic behaviour of the second absolute moment $\mathbb{E}(|\langle \boldsymbol{v}, \mathbf{X}_t \rangle|^2)$ of $\langle \boldsymbol{v}, \mathbf{X}_t \rangle$ has been described as $t \to \infty$. The proof of Proposition D.2 is based on the decomposition of $e^{-\lambda t} \langle \boldsymbol{v}, \mathbf{X}_t \rangle$ mentioned in the Introduction (see the beginning of the proof of part (iii) of Theorem 3.1) yielding an appropriate decomposition of $\mathbb{E}(\langle \boldsymbol{v}, \mathbf{X}_t \rangle^2)$ containing $\mathbb{E}(\langle \boldsymbol{v}, \mathbf{X}_0 \rangle^2)$, $\mathbb{E}(\langle \boldsymbol{v}, \mathbf{X}_0 \rangle)$ and $\mathbb{E}(X_{u,\ell})$, $u \in [0, t]$, $\ell \in \{1, \ldots, d\}$. So the proof of Proposition D.2 can be finished by delicate estimations using the explicit form of $\mathbb{E}(\mathbf{X}_t | \mathbf{X}_0 = \mathbf{x})$, $\mathbf{x} \in \mathbb{R}_+$, given in (2.2).

In the next statement, sufficient conditions are derived for $\mathbb{P}(w_{\boldsymbol{v},\boldsymbol{X}_0}=0)=0$. Note that in case of $\mathbb{P}(w_{\boldsymbol{v},\boldsymbol{X}_0}=0)=0$, the scaling factor $e^{-\lambda t}$ is correct in part (i) of Theorem 3.1 in the sense that the limit is a non-degenerate random variable.

3.2 Theorem. Let $(\mathbf{X}_t)_{t\in\mathbb{R}_+}$ be a supercritical and irreducible multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|) < \infty$ and the moment conditions (2.1) and (3.8) hold. Let $\lambda \in \sigma(\widetilde{\mathbf{B}})$ be such that $\operatorname{Re}(\lambda) \in (\frac{1}{2}s(\widetilde{\mathbf{B}}), s(\widetilde{\mathbf{B}})]$, and let $\mathbf{v} \in \mathbb{C}^d$ be a left eigenvector of $\widetilde{\mathbf{B}}$ corresponding to the eigenvalue λ .

If the conditions

- (i) $\widetilde{\boldsymbol{\beta}} \neq \mathbf{0}$, *i.e.*, $\boldsymbol{\beta} \neq \mathbf{0}$ or $\nu \neq 0$,
- (ii) $\nu(\{\boldsymbol{r} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{r} \rangle \neq 0\}) > 0$, or there exists $\ell \in \{1, \dots, d\}$ such that $\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle c_\ell \neq 0$ or $\mu_\ell(\{\boldsymbol{z} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) > 0$

hold, then the law of $w_{\boldsymbol{v},\boldsymbol{X}_0}$ does not have atoms, where $w_{\boldsymbol{v},\boldsymbol{X}_0}$ is given in part (i) of Theorem 3.1. In particular, $\mathbb{P}(w_{\boldsymbol{v},\boldsymbol{X}_0}=0)=0.$

If the condition (ii) does not hold, then $\mathbb{P}(w_{\boldsymbol{v},\boldsymbol{X}_0} = \langle \boldsymbol{v}, \boldsymbol{X}_0 + \lambda^{-1} \widetilde{\boldsymbol{\beta}} \rangle) = 1$, and in particular, $\mathbb{P}(w_{\boldsymbol{v},\boldsymbol{X}_0} = 0) = \mathbb{P}(\langle \boldsymbol{v}, \boldsymbol{X}_0 + \lambda^{-1} \widetilde{\boldsymbol{\beta}} \rangle = 0).$

If $\lambda = s(\widetilde{\boldsymbol{B}})$, $\boldsymbol{v} = \boldsymbol{u}$ and the condition (i) holds, then $\mathbb{P}(w_{\boldsymbol{u},\boldsymbol{X}_0} = 0) = 0$.

If $\lambda = s(\widetilde{\boldsymbol{B}})$, $\boldsymbol{v} = \boldsymbol{u}$, and the conditions (i) and (ii) do not hold, then $\mathbb{P}(w_{\boldsymbol{u},\boldsymbol{X}_0} = 0) = \mathbb{P}(\boldsymbol{X}_0 = \boldsymbol{0})$.

Next, we show that with an appropriate random scaling in parts (ii) and (iii) in Theorem 3.1, $\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ is asymptotically normal as $t \to \infty$ under the conditional probability measure $\mathbb{P}(\cdot | w_{\boldsymbol{u}, \boldsymbol{X}_0} > 0)$, provided that $\mathbb{P}(w_{\boldsymbol{u}, \boldsymbol{X}_0} > 0) > 0$. Parts (ii) and (iii) of the forthcoming Theorem 3.4 are analogous to Theorems 1 and 2 and part 5 of Corollary 5 in Athreya [2]. First we give a necessary and sufficient condition for $w_{\boldsymbol{u}, \boldsymbol{X}_0} \stackrel{\text{a.s.}}{=} 0$.

3.3 Lemma. Suppose that $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is a supercritical and irreducible multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|) < \infty$, the moment condition (2.1) holds, and the moment condition (3.1) holds for $\lambda = s(\widetilde{\mathbf{B}})$. Then $w_{\mathbf{u},\mathbf{X}_0} \stackrel{\text{a.s.}}{=} 0$ if and only if $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is a trivial process (equivalently, $\mathbf{X}_0 \stackrel{\text{a.s.}}{=} \mathbf{0}$ and $\widetilde{\boldsymbol{\beta}} = \mathbf{0}$, see Lemma 2.5 and Definition 2.6).

3.4 Theorem. Suppose that $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is a supercritical, irreducible and non-trivial multitype CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|) < \infty$ and the moment condition (2.1) holds.

(i) If $\operatorname{Re}(\lambda) \in \left(\frac{1}{2}s(\widetilde{\boldsymbol{B}}), s(\widetilde{\boldsymbol{B}})\right]$ and the moment condition (3.1) holds, then

$$\mathbb{1}_{\{\boldsymbol{X}_t \neq \boldsymbol{0}\}} \frac{1}{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle^{\operatorname{Re}(\lambda)/s(\tilde{\boldsymbol{B}})}} \begin{pmatrix} \cos(\operatorname{Im}(\lambda)t) & \sin(\operatorname{Im}(\lambda)t) \\ -\sin(\operatorname{Im}(\lambda)t) & \cos(\operatorname{Im}(\lambda)t) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \\
\rightarrow \frac{1}{w_{\boldsymbol{u}, \boldsymbol{X}_0}^{\operatorname{Re}(\lambda)/s(\tilde{\boldsymbol{B}})}} \begin{pmatrix} \operatorname{Re}(w_{\boldsymbol{v}, \boldsymbol{X}_0}) \\ \operatorname{Im}(w_{\boldsymbol{v}, \boldsymbol{X}_0}) \end{pmatrix} \quad as \ t \to \infty$$

on the event $\{w_{u,X_0} > 0\}.$

(ii) If $\operatorname{Re}(\lambda) = \frac{1}{2}s(\tilde{\boldsymbol{B}})$ and the moment condition (3.3) holds, then, under the conditional probability measure $\mathbb{P}(\cdot | w_{\boldsymbol{u},\boldsymbol{X}_0} > 0)$, we have

$$\mathbb{1}_{\{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle > 1\}} \frac{1}{\sqrt{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle \log(\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle)}} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \stackrel{\mathcal{D}_{\{w_{\boldsymbol{u}, \boldsymbol{X}_0} > 0\}}}{\longrightarrow} \mathcal{N}_2\left(\boldsymbol{0}, \frac{1}{s(\widetilde{\boldsymbol{B}})} \boldsymbol{\Sigma}_{\boldsymbol{v}}\right)$$

as $t \to \infty$.

(iii) If $\operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\right)$ and the moment condition (3.3) holds, then, under the conditional probability measure $\mathbb{P}(\cdot | w_{\boldsymbol{u}, \boldsymbol{X}_0} > 0)$, we have

$$\mathbb{1}_{\{\boldsymbol{X}_t\neq\boldsymbol{0}\}}\frac{1}{\sqrt{\langle\boldsymbol{u},\boldsymbol{X}_t\rangle}}\begin{pmatrix}\operatorname{Re}(\langle\boldsymbol{v},\boldsymbol{X}_t\rangle)\\\operatorname{Im}(\langle\boldsymbol{v},\boldsymbol{X}_t\rangle)\end{pmatrix}\xrightarrow{\mathcal{D}_{\{w_{\boldsymbol{u},\boldsymbol{X}_0}>0\}}}\mathcal{N}_2(\boldsymbol{0},\boldsymbol{\Sigma}_{\boldsymbol{v}})\quad as \ t\to\infty.$$

3.5 Remark. The indicator function $\mathbb{1}_{\{X_t \neq 0\}}$ are needed in parts (i) and (iii) of Theorem 3.4, and the indicator function $\mathbb{1}_{\{(u,X_t)>1\}}$ is needed in part (ii) of Theorem 3.4, since it can happen that $\mathbb{P}(X_t = \mathbf{0}) > 0$, $t \in \mathbb{R}_{++}$, even if $\tilde{\boldsymbol{\beta}} \neq \mathbf{0}$. For example, if $(X_t)_{t \in \mathbb{R}_+}$ is a multi-type CBI process with parameters $(d, c, \mathbf{0}, B, \nu, \mathbf{0})$ such that $X_0 = \mathbf{0}$, B is irreducible

with $s(\boldsymbol{B}) > 0$ and $\nu \neq 0$ with $\int_{U_d} (1 \vee ||\boldsymbol{r}||) \nu(\mathrm{d}\boldsymbol{r}) < \infty$, then $\widetilde{\boldsymbol{B}} = \boldsymbol{B}$, thus $(\boldsymbol{X}_t)_{t \in \mathbb{R}_+}$ is irreducible and supercritical. One can choose, for instance, d = 2 and

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{R}^{2 \times 2}_{(+)},$$

yielding that $\sigma(\mathbf{B}) = \sigma(\widetilde{\mathbf{B}}) = \{0, 2\}$ and $s(\mathbf{B}) = s(\widetilde{\mathbf{B}}) = 2$, hence, by choosing $\lambda = 0 \in \sigma(\widetilde{\mathbf{B}})$, we have $\operatorname{Re}(\lambda) = 0 \in (-\infty, 1) = (-\infty, \frac{1}{2}s(\widetilde{\mathbf{B}}))$, and, by choosing $\lambda = 2 \in \sigma(\widetilde{\mathbf{B}})$, we have $\operatorname{Re}(\lambda) = 2 \in (1, 2] = (\frac{1}{2}s(\widetilde{\mathbf{B}}), s(\widetilde{\mathbf{B}})]$. If d = 2 and we choose

$$\boldsymbol{B} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \in \mathbb{R}^{2 \times 2}_{(+)},$$

then $\sigma(\boldsymbol{B}) = \sigma(\widetilde{\boldsymbol{B}}) = \{2, 4\}, \ s(\boldsymbol{B}) = s(\widetilde{\boldsymbol{B}}) = 4$, and with $\lambda = 2$ we have $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}})$. Further, using $\widetilde{\boldsymbol{\beta}} = \int_{\mathcal{U}_d} \boldsymbol{r} \, \nu(\mathrm{d}\boldsymbol{r})$, by Lemma 4.1 in Barczy et al. [7],

$$\begin{aligned} \boldsymbol{X}_{t} &= \int_{0}^{t} \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} \,\mathrm{d}u + \sum_{\ell=1}^{d} \int_{0}^{t} \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}} \boldsymbol{e}_{\ell} \sqrt{2c_{\ell}X_{u,\ell}} \,\mathrm{d}W_{u,\ell} + \int_{0}^{t} \int_{\mathcal{U}_{d}} \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}} \boldsymbol{r} \,\widetilde{\boldsymbol{M}}(\mathrm{d}u,\mathrm{d}\boldsymbol{r}) \\ &= \sum_{\ell=1}^{d} \int_{0}^{t} \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}} \boldsymbol{e}_{\ell} \sqrt{2c_{\ell}X_{u,\ell}} \,\mathrm{d}W_{u,\ell} + \int_{0}^{t} \int_{\mathcal{U}_{d}} \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}} \boldsymbol{r} \,\boldsymbol{M}(\mathrm{d}u,\mathrm{d}\boldsymbol{r}) \end{aligned}$$

for all $t \in \mathbb{R}_+$, where $\widetilde{M}(\mathrm{d} u, \mathrm{d} \boldsymbol{r}) := M(\mathrm{d} u, \mathrm{d} \boldsymbol{r}) - \mathrm{d} u \nu(\mathrm{d} \boldsymbol{r})$. Note that until the first jump of M in $\mathbb{R}_+ \times \mathcal{U}_d$, the pathwise unique solution of this SDE is the identically zero process. Hence, using that $\mathrm{e}^{(t-u)\widetilde{B}} \in \mathbb{R}^{d \times d}_{++}$ and $\mathrm{e}^{(t-u)\widetilde{B}}$ is invertible for all $t \in \mathbb{R}_{++}$ and $u \in [0, t]$, we have

$$\mathbb{P}(\boldsymbol{X}_{s} = \boldsymbol{0} \text{ for each } s \in [0, t]) \geq \mathbb{P}(\boldsymbol{M} \text{ has no point in } \{(u, \boldsymbol{r}) \in (0, t] \times \mathcal{U}_{d} : e^{(t-u)\tilde{\boldsymbol{B}}}\boldsymbol{r} \neq \boldsymbol{0}\})$$
$$= \mathbb{P}(\boldsymbol{M} \text{ has no point in } \{(u, \boldsymbol{r}) \in (0, t] \times \mathcal{U}_{d} : \boldsymbol{r} \neq \boldsymbol{0}\}) = e^{-\int_{0}^{t} \int_{\mathcal{U}_{d}} \mathbb{1}_{\{\boldsymbol{r} \neq \boldsymbol{0}\}} du \,\nu(d\boldsymbol{r})} = e^{-t\nu(\mathcal{U}_{d})}$$
for all $t \in \mathbb{R}_{++}$. Consequently, since $\nu(\mathcal{U}_{d}) < \infty$, we obtain $\mathbb{P}(\boldsymbol{X}_{t} = \boldsymbol{0}) > 0, \ t \in \mathbb{R}_{++}$.

Next we describe the asymptotic behavior of the relative frequencies of distinct types of individuals on the event $\{w_{u,X_0} > 0\}$. For different models, one can find similar results in Jagers [17, Corollary 1] and Yakovlev and Yanev [24, Theorem 2]. For critical and irreducible multi-type CBI processes, see Barczy and Pap [9, Corollary 4.1].

3.6 Proposition. If $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is a non-trivial, supercritical and irreducible multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|) < \infty$ and the moment condition (2.1) holds, then for each $i, j \in \{1, \ldots, d\}$, we have

$$\mathbb{1}_{\{\langle \boldsymbol{e}_j, \boldsymbol{X}_t \rangle \neq 0\}} \frac{\langle \boldsymbol{e}_i, \boldsymbol{X}_t \rangle}{\langle \boldsymbol{e}_j, \boldsymbol{X}_t \rangle} \to \frac{\langle \boldsymbol{e}_i, \widetilde{\boldsymbol{u}} \rangle}{\langle \boldsymbol{e}_j, \widetilde{\boldsymbol{u}} \rangle} \quad and \quad \mathbb{1}_{\{\boldsymbol{X}_t \neq \boldsymbol{0}\}} \frac{\langle \boldsymbol{e}_i, \boldsymbol{X}_t \rangle}{\sum_{k=1}^d \langle \boldsymbol{e}_k, \boldsymbol{X}_t \rangle} \to \langle \boldsymbol{e}_i, \widetilde{\boldsymbol{u}} \rangle \quad as \ t \to \infty$$

on the event $\{w_{u,X_0} > 0\}$.

3.7 Remark. The indicator functions $\mathbb{1}_{\{\boldsymbol{e}_j^\top \boldsymbol{X}_t \neq 0\}}$ and $\mathbb{1}_{\{\boldsymbol{X}_t \neq 0\}}$ are needed in Proposition 3.6, since it can happen that $\mathbb{P}(\boldsymbol{X}_t = \mathbf{0}) > 0, t \in \mathbb{R}_{++}$, see Remark 3.5.

3.8 Remark. If $\mathbb{P}(w_{u,X_0} = 0) = 0$, then the convergence in part (i) of Theorem 3.4 and in Proposition 3.6 holds almost surely, and the convergences in parts (ii) and (iii) hold under the unconditional probability measure \mathbb{P} .

4 Proofs

Proof of part (i) of Theorem 3.1. This statement has been proved in Barczy et al. [8, Theorem 3.1].

Proof of part (iii) of Theorem 3.1. The proof is divided into three main steps. First, we decompose the process $(e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle)_{t \in \mathbb{R}_+}$ as the sum of a deterministic process and three square-integrable martingales. We show that the deterministic process goes to zero as $t \to \infty$. For proving asymptotic normality of the martingales in question, we use Theorem E.1 due to Crimaldi and Pratelli [11, Theorem 2.2] which provides a set of sufficient conditions for the asymptotic normality of multivariate martingales. Then, the proof is complete as soon as we show that the conditions (E.1) and (E.2) of Theorem E.1 are satisfied. In the second and third steps, we prove that (E.1) and (E.2) are satisfied, respectively.

Step 1. For each $t \in \mathbb{R}_+$, we have the representation $e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle = Z_t^{(0,1)} + Z_t^{(2)} + Z_t^{(3,4)} + Z_t^{(5)}$ with

$$\begin{split} Z_t^{(0,1)} &:= \langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle + \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle \int_0^t e^{-\lambda u} \, \mathrm{d}u, \\ Z_t^{(2)} &:= \sum_{\ell=1}^d \langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle \int_0^t e^{-\lambda u} \sqrt{2c_\ell X_{u,\ell}} \, \mathrm{d}W_{u,\ell}, \\ Z_t^{(3,4)} &:= \sum_{\ell=1}^d \int_0^t \int_{\mathcal{U}_d} \int_{\mathcal{U}_1} e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle \mathbbm{1}_{\{w \leqslant X_{u-,\ell}\}} \, \widetilde{N}_\ell(\mathrm{d}u, \mathrm{d}\boldsymbol{z}, \mathrm{d}w), \\ Z_t^{(5)} &:= \int_0^t \int_{\mathcal{U}_d} e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle \, \widetilde{M}(\mathrm{d}u, \mathrm{d}\boldsymbol{r}), \end{split}$$

see Barczy et al. [7, Lemma 4.1]. Thus for each $t \in \mathbb{R}_+$, we have

$$e^{-s(\tilde{\boldsymbol{B}})t/2} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle = e^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t/2} \big(Z_t^{(0,1)} + Z_t^{(2)} + Z_t^{(3,4)} + Z_t^{(5)} \big).$$

First, we show

(4.1)
$$e^{-(s(\tilde{B})-2\lambda)t/2}Z_t^{(0,1)} \xrightarrow{\text{a.s.}} 0 \quad \text{as} \ t \to \infty.$$

Indeed, if $\lambda = 0$, then

$$e^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t/2}Z_t^{(0,1)} = e^{-s(\widetilde{\boldsymbol{B}})t/2}(\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle + \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle t) \xrightarrow{\text{a.s.}} 0 \qquad \text{as} \quad t \to \infty,$$

since $s(\widetilde{\boldsymbol{B}}) \in \mathbb{R}_{++}$. Otherwise, if $\operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\right)$ and $\lambda \neq 0$, then

$$e^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t/2}Z_t^{(0,1)} = e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t/2 + \operatorname{iIm}(\lambda)t} \left(\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle - \frac{\langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle}{\lambda} (e^{-\lambda t} - 1) \right)$$
$$= \left(\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle + \frac{\langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle}{\lambda} \right) e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t/2 + \operatorname{iIm}(\lambda)t} - \frac{\langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle}{\lambda} e^{-s(\tilde{\boldsymbol{B}})t/2} \xrightarrow{\text{a.s.}} 0$$

as $t \to \infty$.

For each $t \in \mathbb{R}_+$, we have

$$\begin{pmatrix} \operatorname{Re}\left(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t/2}\left(Z_{t}^{(2)}+Z_{t}^{(3,4)}+Z_{t}^{(5)}\right)\right)\\ \operatorname{Im}\left(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t/2}\left(Z_{t}^{(2)}+Z_{t}^{(3,4)}+Z_{t}^{(5)}\right)\right) \end{pmatrix} = \boldsymbol{Q}(t)\boldsymbol{M}_{t}$$

with

$$\boldsymbol{Q}(t) := \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t/2}) & -\operatorname{Im}(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t/2}) \\ \operatorname{Im}(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t/2}) & \operatorname{Re}(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t/2}) \end{pmatrix}, \quad t \in \mathbb{R}_+,$$

and

$$\boldsymbol{M}_{t} := \begin{pmatrix} \operatorname{Re}(Z_{t}^{(2)} + Z_{t}^{(3,4)} + Z_{t}^{(5)}) \\ \operatorname{Im}(Z_{t}^{(2)} + Z_{t}^{(3,4)} + Z_{t}^{(5)}) \end{pmatrix}, \qquad t \in \mathbb{R}_{+}$$

The assumption $\operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\right)$ implies

$$\boldsymbol{Q}(t) = e^{-(s(\tilde{\boldsymbol{B}}) - 2\operatorname{Re}(\lambda))t/2} \begin{pmatrix} \cos(\operatorname{Im}(\lambda)t) & -\sin(\operatorname{Im}(\lambda)t) \\ \sin(\operatorname{Im}(\lambda)t) & \cos(\operatorname{Im}(\lambda)t) \end{pmatrix} \to \boldsymbol{0} \quad \text{as} \ t \to \infty.$$

For each $t \in \mathbb{R}_+$, we can write $\boldsymbol{M}_t = \boldsymbol{M}_t^{(2)} + \boldsymbol{M}_t^{(3,4)} + \boldsymbol{M}_t^{(5)}$ with

$$\boldsymbol{M}_{t}^{(2)} := \begin{pmatrix} \operatorname{Re}(Z_{t}^{(2)}) \\ \operatorname{Im}(Z_{t}^{(2)}) \end{pmatrix}, \qquad \boldsymbol{M}_{t}^{(3,4)} := \begin{pmatrix} \operatorname{Re}(Z_{t}^{(3,4)}) \\ \operatorname{Im}(Z_{t}^{(3,4)}) \end{pmatrix}, \qquad \boldsymbol{M}_{t}^{(5)} := \begin{pmatrix} \operatorname{Re}(Z_{t}^{(5)}) \\ \operatorname{Im}(Z_{t}^{(5)}) \end{pmatrix}.$$

Note that under the moment condition (3.3), $(\boldsymbol{M}_{t}^{(2)})_{t\in\mathbb{R}_{+}}$, $(\boldsymbol{M}_{t}^{(3,4)})_{t\in\mathbb{R}_{+}}$ and $(\boldsymbol{M}_{t}^{(5)})_{t\in\mathbb{R}_{+}}$ are square-integrable martingales (see, e.g., Ikeda and Watanabe [15, pages 55 and 63]). One can also observe that, by the decomposition of $(e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_{t} \rangle)_{t\in\mathbb{R}_{+}}$ given at the beginning of this step, $(e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_{t} \rangle - \langle \boldsymbol{v}, \boldsymbol{\beta} \rangle \int_{0}^{t} e^{-\lambda u} du)_{t\in\mathbb{R}_{+}}$ is a martingale with respect to the filtration $\sigma(\boldsymbol{X}_{u}: u \in [0, t]), t \in \mathbb{R}_{+}$, which follows by Barczy et al. [8, Lemma 2.6] as well.

The aim of the following discussion is to apply Theorem E.1 for the 2-dimensional martingale $(\mathbf{M}_t)_{t\in\mathbb{R}_+}$ with the scaling $\mathbf{Q}(t), t\in\mathbb{R}_+$.

Step 2. Now we prove that condition (E.1) of Theorem E.1 holds for $(\mathbf{M}_t)_{t \in \mathbb{R}_+}$ with the scaling $\mathbf{Q}(t), t \in \mathbb{R}_+$. For each $t \in \mathbb{R}_+$, by Theorem I.4.52 in Jacod and Shiryaev [16], we

have

$$[\boldsymbol{M}^{(2)}]_{t} = \begin{pmatrix} [\operatorname{Re}(Z^{(2)}), \operatorname{Re}(Z^{(2)})]_{t} & [\operatorname{Re}(Z^{(2)}), \operatorname{Im}(Z^{(2)})]_{t} \\ [\operatorname{Im}(Z^{(2)}), \operatorname{Re}(Z^{(2)})]_{t} & [\operatorname{Im}(Z^{(2)}), \operatorname{Im}(Z^{(2)})]_{t} \end{pmatrix}$$
$$= \begin{pmatrix} \langle \operatorname{Re}(Z^{(2)}), \operatorname{Re}(Z^{(2)}) \rangle_{t} & \langle \operatorname{Re}(Z^{(2)}), \operatorname{Im}(Z^{(2)}) \rangle_{t} \\ \langle \operatorname{Im}(Z^{(2)}), \operatorname{Re}(Z^{(2)}) \rangle_{t} & \langle \operatorname{Im}(Z^{(2)}), \operatorname{Im}(Z^{(2)}) \rangle_{t} \end{pmatrix}$$
$$= 2\sum_{\ell=1}^{d} c_{\ell} \int_{0}^{t} \begin{pmatrix} \operatorname{Re}(e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle) \\ \operatorname{Im}(e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle) \\ \operatorname{Im}(e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle) \end{pmatrix}^{\top} X_{u,\ell} \, \mathrm{d}u,$$

since $(\boldsymbol{M}_{t}^{(2)})_{t\in\mathbb{R}_{+}}$ is continuous, where $([\boldsymbol{M}^{(2)}]_{t})_{t\in\mathbb{R}_{+}}$ and $(\langle \boldsymbol{M}^{(2)} \rangle_{t})_{t\in\mathbb{R}_{+}}$ denotes the quadratic variation process and the predictable quadratic variation process of $(\boldsymbol{M}_{t}^{(2)})_{t\in\mathbb{R}_{+}}$, respectively. Moreover, we have $\boldsymbol{M}_{t}^{(3,4)} = \sum_{\ell=1}^{d} \widetilde{\boldsymbol{Y}}_{t}^{(\ell)}$ with

$$\widetilde{\boldsymbol{Y}}_{t}^{(\ell)} := \begin{pmatrix} \operatorname{Re}(\widetilde{Y}_{t}^{(\ell)}) \\ \operatorname{Im}(\widetilde{Y}_{t}^{(\ell)}) \end{pmatrix}, \qquad \widetilde{Y}_{t}^{(\ell)} := \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} \operatorname{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle \mathbb{1}_{\{w \leq X_{u-,\ell}\}} \widetilde{N}_{\ell}(\mathrm{d}u, \mathrm{d}\boldsymbol{z}, \mathrm{d}w)$$

for $t \in \mathbb{R}_+$ and $\ell \in \{1, \ldots, d\}$. For each $t \in \mathbb{R}_+$ and $\ell \in \{1, \ldots, d\}$, $(\widetilde{\boldsymbol{Y}}_t^{(\ell)})_{t \in \mathbb{R}_+}$ is a square-integrable purely discontinuous martingale, see, e.g., Jacod and Shiryaev [16, Definition II.1.27 and Theorem II.1.33]). Hence, for each $t \in \mathbb{R}_+$ and $k, \ell \in \{1, \ldots, d\}$, by Lemma I.4.51 in Jacod and Shiryaev [16], we have

$$[\widetilde{\boldsymbol{Y}}^{(k)}, \widetilde{\boldsymbol{Y}}^{(\ell)}]_t = \sum_{s \in [0,t]} (\widetilde{\boldsymbol{Y}}^{(k)}_s - \widetilde{\boldsymbol{Y}}^{(k)}_{s-}) (\widetilde{\boldsymbol{Y}}^{(\ell)}_s - \widetilde{\boldsymbol{Y}}^{(\ell)}_{s-})^\top.$$

Further, by the proof of part (a) of Theorem II.1.33 in Jacod and Shiryaev [16], for each $t \in \mathbb{R}_+$ and $k \in \{1, \ldots, d\}$,

$$[\widetilde{\boldsymbol{Y}}^{(k)}]_{t} = \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} \left(\operatorname{Re}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \\ \operatorname{Im}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \right) \left(\operatorname{Re}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \right)^{\top} \mathbb{1}_{\{w \leq X_{u-,k}\}} N_{k}(\mathrm{d}u, \mathrm{d}\boldsymbol{z}, \mathrm{d}w).$$

The aim of the following discussion is to show that for each $t \in \mathbb{R}_+$ and $k, \ell \in \{1, \ldots, d\}$ with $k \neq \ell$, we have $[\widetilde{\boldsymbol{Y}}^{(k)}, \widetilde{\boldsymbol{Y}}^{(\ell)}]_t = \mathbf{0}$ almost surely. By the bilinearity of quadratic variation process, for all $\varepsilon \in \mathbb{R}_{++}$ and $t \in \mathbb{R}_+$, we have

(4.2)
$$[\widetilde{\boldsymbol{Y}}^{(k)}, \widetilde{\boldsymbol{Y}}^{(\ell)}]_{t} = [\widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}, \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t} + [\widetilde{\boldsymbol{Y}}^{(k)} - \widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}, \widetilde{\boldsymbol{Y}}^{(\ell)} - \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t} + [\widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}, \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}, \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t} + [\widetilde{\boldsymbol{Y}}^{(k)} - \widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}, \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t},$$

where, for all $\varepsilon \in \mathbb{R}_{++}, k \in \{1, \ldots, d\}$ and $t \in \mathbb{R}_+,$

$$\widetilde{\boldsymbol{Y}}_{t}^{(k,\varepsilon)} := \begin{pmatrix} \operatorname{Re}(\widetilde{Y}_{t}^{(k,\varepsilon)}) \\ \operatorname{Im}(\widetilde{Y}_{t}^{(k,\varepsilon)}) \end{pmatrix}, \qquad \widetilde{Y}_{t}^{(k,\varepsilon)} := \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle \mathbb{1}_{\{\|\boldsymbol{z}\| \ge \varepsilon\}} \mathbb{1}_{\{w \le X_{u-,k}\}} \widetilde{N}_{k}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{z}, \mathrm{d}\boldsymbol{w}),$$

which is well-defined and square-integrable, since, by (2.5) and (3.8),

$$\int_{0}^{t} \int_{\mathcal{U}_{d}} e^{-2\operatorname{Re}(\lambda)u} |\langle \boldsymbol{v}, \boldsymbol{z} \rangle|^{2} \mathbb{1}_{\{\|\boldsymbol{z}\| \ge \varepsilon\}} \mathbb{E}(X_{u,k}) \,\mathrm{d}u \,\mu_{k}(\mathrm{d}\boldsymbol{z})$$

$$\leq C_{4} \|\boldsymbol{v}\|^{2} \int_{0}^{t} e^{(s(\tilde{\boldsymbol{B}}) - 2\operatorname{Re}(\lambda))u} \,\mathrm{d}u \int_{\mathcal{U}_{d}} \|\boldsymbol{z}\|^{2} \mathbb{1}_{\{\|\boldsymbol{z}\| \ge \varepsilon\}} \,\mu_{k}(\mathrm{d}\boldsymbol{z}) < \infty$$

For each $\varepsilon \in \mathbb{R}_{++}$, $t \in \mathbb{R}_{+}$ and $k, \ell \in \{1, \ldots, d\}$, we have

$$[\widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}, \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t} = \sum_{s \in [0,t]} (\widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}_{s} - \widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}_{s-}) (\widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}_{s} - \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}_{s-})^{\top} = \sum_{s \in [0,t]} (\boldsymbol{Y}^{(k,\varepsilon)}_{s} - \boldsymbol{Y}^{(k,\varepsilon)}_{s-}) (\boldsymbol{Y}^{(\ell,\varepsilon)}_{s} - \boldsymbol{Y}^{(\ell,\varepsilon)}_{s-})^{\top}$$

with

$$\boldsymbol{Y}_{t}^{(k,\varepsilon)} := \begin{pmatrix} \operatorname{Re}(Y_{t}^{(k,\varepsilon)}) \\ \operatorname{Im}(Y_{t}^{(k,\varepsilon)}) \end{pmatrix}, \qquad Y_{t}^{(k,\varepsilon)} := \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle \mathbb{1}_{\{\|\boldsymbol{z}\| \ge \varepsilon\}} \mathbb{1}_{\{w \le X_{u-,k}\}} N_{k}(\mathrm{d} u, \mathrm{d} \boldsymbol{z}, \mathrm{d} w),$$

where the first equality follows by the proof of part (a) of Theorem II.1.33 in Jacod and Shiryaev [16], and the second equality, by (2.5), part (vi) of Definition 2.1 and (3.8), since

$$\int_{0}^{t} \int_{\mathcal{U}_{d}} e^{-\operatorname{Re}(\lambda)u} |\langle \boldsymbol{v}, \boldsymbol{z} \rangle| \mathbb{1}_{\{\|\boldsymbol{z}\| \ge \varepsilon\}} \mathbb{E}(X_{u,k}) \, \mathrm{d}u \, \mu_{k}(\mathrm{d}\boldsymbol{z})$$

$$\leq C_{4} \|\boldsymbol{v}\| \int_{0}^{t} e^{(s(\tilde{\boldsymbol{B}}) - \operatorname{Re}(\lambda))u} \, \mathrm{d}u \int_{\mathcal{U}_{d}} \|\boldsymbol{z}\| \mathbb{1}_{\{\|\boldsymbol{z}\| \ge \varepsilon\}} \, \mu_{k}(\mathrm{d}\boldsymbol{z})$$

$$\leq \frac{C_{4} \|\boldsymbol{v}\|}{\varepsilon} \int_{0}^{t} e^{(s(\tilde{\boldsymbol{B}}) - \operatorname{Re}(\lambda))u} \, \mathrm{d}u \int_{\mathcal{U}_{d}} \|\boldsymbol{z}\|^{2} \, \mu_{k}(\mathrm{d}\boldsymbol{z}) < \infty,$$

and hence we have

$$\widetilde{Y}_{t}^{(k,\varepsilon)} = Y_{t}^{(k,\varepsilon)} - \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle \mathbb{1}_{\{\|\boldsymbol{z}\| \ge \varepsilon\}} \mathbb{1}_{\{w \le X_{u-,k}\}} \, \mathrm{d}u \, \mu_{k}(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}w.$$

For each $\varepsilon \in \mathbb{R}_{++}$ and $k \in \{1, \ldots, d\}$, the jump times of $(\mathbf{Y}_t^{(k,\varepsilon)})_{t \in \mathbb{R}_+}$ is a subset of the jump times of the Poisson process $(N_k([0,t] \times \mathcal{U}_d \times \mathcal{U}_1))_{t \in \mathbb{R}_+}$. For each $k, \ell \in \{1, \ldots, d\}$ with $k \neq \ell$, the Poisson processes $(N_k([0,t] \times \mathcal{U}_d \times \mathcal{U}_1))_{t \in \mathbb{R}_+}$ and $(N_\ell([0,t] \times \mathcal{U}_d \times \mathcal{U}_1))_{t \in \mathbb{R}_+}$ are independent, hence they can jump simultaneously with probability zero, see, e.g., Revuz and Yor [22, Chapter XII, Proposition 1.5]. Consequently, for each $\varepsilon \in \mathbb{R}_{++}$, $t \in \mathbb{R}_+$ and $k, \ell \in \{1, \ldots, d\}$ with $k \neq \ell$, we have $[\widetilde{\mathbf{Y}}^{(k,\varepsilon)}, \widetilde{\mathbf{Y}}^{(\ell,\varepsilon)}]_t = \mathbf{0}$ almost surely.

Moreover, for each $t \in \mathbb{R}_+$, $\varepsilon \in \mathbb{R}_{++}$, $i, j \in \{1, 2\}$ and $k, \ell \in \{1, \ldots, d\}$ with $k \neq \ell$, by the Kunita–Watanabe inequality, we have

$$\begin{split} \left| \langle \boldsymbol{e}_{i}, [\widetilde{\boldsymbol{Y}}^{(k)} - \widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}, \widetilde{\boldsymbol{Y}}^{(\ell)} - \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t} \boldsymbol{e}_{j} \rangle \right| &= \left| [\langle \boldsymbol{e}_{i}, \widetilde{\boldsymbol{Y}}^{(k)} - \widetilde{\boldsymbol{Y}}^{(k,\varepsilon)} \rangle, \langle \boldsymbol{e}_{j}, \widetilde{\boldsymbol{Y}}^{(\ell)} - \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)} \rangle]_{t} \right| \\ &\leq [\langle \boldsymbol{e}_{i}, \widetilde{\boldsymbol{Y}}^{(k)} - \widetilde{\boldsymbol{Y}}^{(k,\varepsilon)} \rangle]_{t}^{1/2} [\langle \boldsymbol{e}_{j}, \widetilde{\boldsymbol{Y}}^{(k)} - \widetilde{\boldsymbol{Y}}^{(k,\varepsilon)} \rangle]_{t}^{1/2}, \\ &\left| \langle \boldsymbol{e}_{i}, [\widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}, \widetilde{\boldsymbol{Y}}^{(\ell)} - \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t} \boldsymbol{e}_{j} \rangle \right| \leq [\langle \boldsymbol{e}_{i}, \widetilde{\boldsymbol{Y}}^{(k,\varepsilon)} \rangle]_{t}^{1/2} [\langle \boldsymbol{e}_{j}, \widetilde{\boldsymbol{Y}}^{(\ell)} - \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)} \rangle]_{t}^{1/2}, \\ &\left| \langle \boldsymbol{e}_{i}, [\widetilde{\boldsymbol{Y}}^{(k)} - \widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}, \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t} \boldsymbol{e}_{j} \rangle \right| \leq [\langle \boldsymbol{e}_{i}, \widetilde{\boldsymbol{Y}}^{(k)} - \widetilde{\boldsymbol{Y}}^{(k,\varepsilon)} \rangle]_{t}^{1/2} [\langle \boldsymbol{e}_{j}, \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)} \rangle]_{t}^{1/2}. \end{split}$$

Hence it is enough to check that $[\langle e_j, \widetilde{Y}^{(\ell,\varepsilon)} \rangle]_t$ is stochastically bounded in $\varepsilon \in \mathbb{R}_{++}$ and

$$[\langle \boldsymbol{e}_j, \widetilde{\boldsymbol{Y}}^{(\ell)} - \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)} \rangle]_t \stackrel{L_1}{\longrightarrow} 0 \quad \text{as } \varepsilon \downarrow 0$$

for all $t \in \mathbb{R}_+$, $j \in \{1, 2\}$ and $\ell \in \{1, \ldots, d\}$. Indeed, in this case

$$\begin{split} \left| \langle \boldsymbol{e}_{i}, [\widetilde{\boldsymbol{Y}}^{(k)} - \widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}, \widetilde{\boldsymbol{Y}}^{(\ell)} - \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t}\boldsymbol{e}_{j} \rangle \right| \stackrel{\mathbb{P}}{\longrightarrow} 0 \quad \text{as } \varepsilon \downarrow 0, \\ \left| \langle \boldsymbol{e}_{i}, [\widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}, \widetilde{\boldsymbol{Y}}^{(\ell)} - \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t}\boldsymbol{e}_{j} \rangle \right| \stackrel{\mathbb{P}}{\longrightarrow} 0 \quad \text{as } \varepsilon \downarrow 0, \\ \left| \langle \boldsymbol{e}_{i}, [\widetilde{\boldsymbol{Y}}^{(k)} - \widetilde{\boldsymbol{Y}}^{(k,\varepsilon)}, \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t}\boldsymbol{e}_{j} \rangle \right| \stackrel{\mathbb{P}}{\longrightarrow} 0 \quad \text{as } \varepsilon \downarrow 0, \end{split}$$

and, by (4.2), for each $t \in \mathbb{R}_+$ and $k, \ell \in \{1, \ldots, d\}$ with $k \neq \ell$, we have $[\widetilde{\boldsymbol{Y}}^{(k)}, \widetilde{\boldsymbol{Y}}^{(\ell)}]_t = \mathbf{0}$ almost surely. By the proof of part (a) of Theorem II.1.33 in Jacod and Shiryaev [16],

$$[\widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t} = \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} \left(\operatorname{Re}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \right) \left(\operatorname{Re}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \right) \left(\operatorname{Re}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \right)^{\top} \mathbb{1}_{\{\|\boldsymbol{z}\| \ge \varepsilon\}} \mathbb{1}_{\{w \le X_{u-,\ell}\}} N_{\ell}(\mathrm{d}u, \mathrm{d}\boldsymbol{z}, \mathrm{d}w),$$

and

$$[\widetilde{\boldsymbol{Y}}^{(\ell)} - \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}]_{t} = \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} \left(\frac{\operatorname{Re}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle)}{\operatorname{Im}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle)} \right) \left(\frac{\operatorname{Re}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle)}{\operatorname{Im}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle)} \right)^{\top} \mathbb{1}_{\{\|\boldsymbol{z}\| < \varepsilon\}} \mathbb{1}_{\{w \leq X_{u-,\ell}\}} N_{\ell}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{z}, \mathrm{d}\boldsymbol{w}).$$

Consequently, using that $\|\boldsymbol{z}\boldsymbol{z}^{\top}\| \leqslant \|\boldsymbol{z}\|^2, \ \boldsymbol{z} \in \mathbb{R}^2$, we have

$$\left| \left[\langle \boldsymbol{e}_{j}, \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)} \rangle \right]_{t} \right| \leq \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} |\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle |^{2} \mathbb{1}_{\{w \leq X_{u-,\ell}\}} N_{\ell}(\mathrm{d}u, \mathrm{d}\boldsymbol{z}, \mathrm{d}w)$$

for all $\varepsilon \in \mathbb{R}_{++}$ and $j \in \{1, 2\}$, where the right-hand side is finite almost surely, since

$$\mathbb{E}\left(\int_{0}^{t}\int_{\mathcal{U}_{d}}\int_{\mathcal{U}_{1}}|\mathrm{e}^{-\lambda u}\langle\boldsymbol{v},\boldsymbol{z}\rangle|^{2}\mathbb{1}_{\{w\leqslant X_{u-,\ell}\}}N_{\ell}(\mathrm{d}\boldsymbol{u},\mathrm{d}\boldsymbol{z},\mathrm{d}\boldsymbol{w})\right) = \int_{0}^{t}\int_{\mathcal{U}_{d}}|\mathrm{e}^{-\lambda u}\langle\boldsymbol{v},\boldsymbol{z}\rangle|^{2}\mathbb{E}(X_{u,\ell})\,\mathrm{d}\boldsymbol{u}\,\mu_{\ell}(\mathrm{d}\boldsymbol{z})$$
$$\leqslant C_{4}\|\boldsymbol{v}\|^{2}\int_{0}^{t}\mathrm{e}^{(s(\tilde{\boldsymbol{B}})-2\mathrm{Re}(\lambda))u}\,\mathrm{d}\boldsymbol{u}\int_{\mathcal{U}_{d}}\|\boldsymbol{z}\|^{2}\,\mu_{\ell}(\mathrm{d}\boldsymbol{z})<\infty.$$

Further,

$$\mathbb{E}\left(\left|\left[\langle \boldsymbol{e}_{j}, \widetilde{\boldsymbol{Y}}^{(\ell)} - \widetilde{\boldsymbol{Y}}^{(\ell,\varepsilon)}\rangle\right]_{t}\right|\right) \leq \mathbb{E}\left(\int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} |\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z}\rangle|^{2} \mathbb{1}_{\{\|\boldsymbol{z}\| < \varepsilon\}} \mathbb{1}_{\{w \leq X_{u-,\ell}\}} N_{\ell}(\mathrm{d}u, \mathrm{d}\boldsymbol{z}, \mathrm{d}w)\right)$$

$$\leq \int_{0}^{t} \int_{\mathcal{U}_{d}} |\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z}\rangle|^{2} \mathbb{1}_{\{\|\boldsymbol{z}\| < \varepsilon\}} \mathbb{E}(X_{u,\ell}) \mathrm{d}u \,\mu_{\ell}(\mathrm{d}\boldsymbol{z})$$

$$\leq C_{4} \|\boldsymbol{v}\|^{2} \int_{0}^{t} \mathrm{e}^{(s(\tilde{B}) - 2\mathrm{Re}(\lambda))u} \mathrm{d}u \int_{\mathcal{U}_{d}} \|\boldsymbol{z}\|^{2} \mathbb{1}_{\{\|\boldsymbol{z}\| < \varepsilon\}} \,\mu_{\ell}(\mathrm{d}\boldsymbol{z}) \to 0$$

as $\varepsilon \downarrow 0$. Consequently, for each $t \in \mathbb{R}_+$ and $k, \ell \in \{1, \ldots, d\}$ with $k \neq \ell$, we have $[\widetilde{\boldsymbol{Y}}^{(k)}, \widetilde{\boldsymbol{Y}}^{(\ell)}]_t = \mathbf{0}$ almost surely.

In a similar way,

$$[\boldsymbol{M}^{(5)}]_{t} = \int_{0}^{t} \int_{\mathcal{U}_{d}} \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle) \\ \operatorname{Im}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle) \\ \operatorname{Im}(\mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle) \end{pmatrix}^{\top} M(\mathrm{d}u, \mathrm{d}\boldsymbol{r}), \qquad t \in \mathbb{R}_{+},$$

and $[\widetilde{\boldsymbol{Y}}^{(\ell)}, \boldsymbol{M}^{(5)}]_t = \boldsymbol{0}, \ \ell \in \{1, \dots, d\}$ almost surely. Consequently, for each $t \in \mathbb{R}_+$, we have $[\boldsymbol{M}^{(3,4)} + \boldsymbol{M}^{(5)}]_t = [\boldsymbol{M}^{(3,4)}]_t + [\boldsymbol{M}^{(5)}]_t$ with $[\boldsymbol{M}^{(3,4)}]_t = \sum_{\ell=1}^d [\widetilde{\boldsymbol{Y}}^{(\ell)}]_t$. Since $(\boldsymbol{M}^{(2)}_t)_{t \in \mathbb{R}_+}$ is a continuous martingale and $(\boldsymbol{M}^{(3,4)}_t + \boldsymbol{M}^{(5)}_t)_{t \in \mathbb{R}_+}$ is a purely discontinuous martingale, by Corollary I.4.55 in Jacod and Shiryaev [16], we have $[\boldsymbol{M}^{(2)}, \boldsymbol{M}^{(3,4)} + \boldsymbol{M}^{(5)}]_t = \boldsymbol{0}, \ t \in \mathbb{R}_+$. Consequently,

$$[\mathbf{M}]_t = [\mathbf{M}^{(2)}]_t + [\mathbf{M}^{(3,4)}]_t + [\mathbf{M}^{(5)}]_t, \qquad t \in \mathbb{R}_+.$$

For each $t \in \mathbb{R}_+$, we have

$$\boldsymbol{Q}(t)[\boldsymbol{M}^{(2)}]_{t}\boldsymbol{Q}(t)^{\top} = 2\sum_{\ell=1}^{d} c_{\ell} \int_{0}^{t} \boldsymbol{f}(t-\tau, \boldsymbol{e}_{\ell}) \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})\tau} X_{\tau,\ell} \,\mathrm{d}\tau$$

with

$$\boldsymbol{f}(w,\boldsymbol{z}) := \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)w/2}\langle \boldsymbol{v},\boldsymbol{z}\rangle) \\ \operatorname{Im}(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)w/2}\langle \boldsymbol{v},\boldsymbol{z}\rangle) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)w/2}\langle \boldsymbol{v},\boldsymbol{z}\rangle) \\ \operatorname{Im}(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)w/2}\langle \boldsymbol{v},\boldsymbol{z}\rangle) \end{pmatrix}^{\top}, \quad w \in \mathbb{R}_{+}, \quad \boldsymbol{z} \in \mathbb{R}^{d}.$$

First, we show

(4.3)
$$\boldsymbol{Q}(t)[\boldsymbol{M}^{(2)}]_{t}\boldsymbol{Q}(t)^{\top} - 2w_{\boldsymbol{u},\boldsymbol{X}_{0}}\sum_{\ell=1}^{d}c_{\ell}\langle\boldsymbol{e}_{\ell},\widetilde{\boldsymbol{u}}\rangle\int_{0}^{t}\boldsymbol{f}(w,\boldsymbol{e}_{\ell})\,\mathrm{d}w\xrightarrow{\mathrm{a.s.}}\boldsymbol{0} \quad \text{as } t\to\infty.$$

For each $t, T \in \mathbb{R}_+$, we have

$$\boldsymbol{Q}(t+T)[\boldsymbol{M}^{(2)}]_{t+T}\boldsymbol{Q}(t+T)^{\top} - 2w_{\boldsymbol{u},\boldsymbol{X}_{0}}\sum_{\ell=1}^{d}c_{\ell}\langle\boldsymbol{e}_{\ell},\widetilde{\boldsymbol{u}}\rangle\int_{0}^{t+T}\boldsymbol{f}(w,\boldsymbol{e}_{\ell})\,\mathrm{d}w = \boldsymbol{\Delta}_{t,T}^{(1)} + \boldsymbol{\Delta}_{t,T}^{(2)}$$

with

$$\boldsymbol{\Delta}_{t,T}^{(1)} := 2 \sum_{\ell=1}^{d} c_{\ell} \int_{0}^{T} \boldsymbol{f}(t+T-\tau, \boldsymbol{e}_{\ell}) (\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})\tau} X_{\tau,\ell} - w_{\boldsymbol{u},\boldsymbol{X}_{0}} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle) \,\mathrm{d}\tau,$$
$$\boldsymbol{\Delta}_{t,T}^{(2)} := 2 \sum_{\ell=1}^{d} c_{\ell} \int_{T}^{t+T} \boldsymbol{f}(t+T-\tau, \boldsymbol{e}_{\ell}) (\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})\tau} X_{\tau,\ell} - w_{\boldsymbol{u},\boldsymbol{X}_{0}} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle) \,\mathrm{d}\tau.$$

For each $t, T \in \mathbb{R}_+$, we have

$$\|\boldsymbol{\Delta}_{t,T}^{(1)}\| \leq 2 \left(\sup_{\tau \in [0,T]} \| \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})\tau} \boldsymbol{X}_{\tau} - w_{\boldsymbol{u},\boldsymbol{X}_{0}} \widetilde{\boldsymbol{u}} \| \right) \sum_{\ell=1}^{d} c_{\ell} \int_{0}^{T} \|\boldsymbol{f}(t+T-\tau,\boldsymbol{e}_{\ell})\| \,\mathrm{d}\tau,$$

where $\sup_{\tau \in [0,T]} \|e^{-s(\tilde{B})\tau} X_{\tau} - w_{u,X_0} \tilde{u}\| < \infty$ almost surely, since $(X_t)_{t \in \mathbb{R}_+}$ has càdlàg sample paths (due to Theorem 4.6 in Barczy et al. [5]). Then, using that $\|zz^{\top}\| \leq \|z\|^2$, $z \in \mathbb{R}^2$, we have

$$(4.4) \qquad \int_{0}^{T} \|\boldsymbol{f}(t+T-\tau,\boldsymbol{e}_{\ell})\| \,\mathrm{d}\tau = \int_{t}^{t+T} \|\boldsymbol{f}(w,\boldsymbol{e}_{\ell})\| \,\mathrm{d}w \leqslant \int_{t}^{t+T} |\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)w/2} \langle \boldsymbol{v},\boldsymbol{e}_{\ell} \rangle|^{2} \,\mathrm{d}w$$
$$(4.4) \qquad \leqslant \|\boldsymbol{v}\|^{2} \int_{t}^{t+T} \mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))w} \,\mathrm{d}w \leqslant \|\boldsymbol{v}\|^{2} \int_{t}^{\infty} \mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))w} \,\mathrm{d}w$$
$$= \frac{\|\boldsymbol{v}\|^{2}}{s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda)} \mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t} \to 0$$

as $t \to \infty$. Hence for each $T \in \mathbb{R}_+$, we obtain

$$\limsup_{t \to \infty} \|\boldsymbol{\Delta}_{t,T}^{(1)}\| = 0$$

almost surely. Moreover, for each $t, T \in \mathbb{R}_+$, we have

$$\|\boldsymbol{\Delta}_{t,T}^{(2)}\| \leq 2 \left(\sup_{\tau \in [T,\infty)} \| e^{-s(\widetilde{\boldsymbol{B}})\tau} \boldsymbol{X}_{\tau} - w_{\boldsymbol{u},\boldsymbol{X}_{0}} \widetilde{\boldsymbol{u}} \| \right) \sum_{\ell=1}^{d} c_{\ell} \int_{T}^{t+T} \|\boldsymbol{f}(t+T-\tau,\boldsymbol{e}_{\ell})\| \, \mathrm{d}\tau$$

almost surely, where

(4.5)
$$\int_{T}^{t+T} \|\boldsymbol{f}(t+T-\tau,\boldsymbol{e}_{\ell})\| \,\mathrm{d}\tau = \int_{0}^{t} \|\boldsymbol{f}(w,\boldsymbol{e}_{\ell})\| \,\mathrm{d}w$$
$$\leqslant \|\boldsymbol{v}\|^{2} \int_{0}^{\infty} \mathrm{e}^{-(s(\widetilde{\boldsymbol{B}})-2\mathrm{Re}(\lambda))w} \,\mathrm{d}w = \frac{\|\boldsymbol{v}\|^{2}}{s(\widetilde{\boldsymbol{B}})-2\mathrm{Re}(\lambda)}$$

Consequently, for each $T \in \mathbb{R}_+$, we obtain

$$\begin{split} & \limsup_{t \to \infty} \left\| \boldsymbol{Q}(t)[\boldsymbol{M}^{(2)}]_{t} \boldsymbol{Q}(t)^{\top} - 2w_{\boldsymbol{u},\boldsymbol{X}_{0}} \sum_{\ell=1}^{d} c_{\ell} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \int_{0}^{t} \boldsymbol{f}(w, \boldsymbol{e}_{\ell}) \, \mathrm{d}w \right\| \\ &= \limsup_{t \to \infty} \left\| \boldsymbol{Q}(t+T)[\boldsymbol{M}^{(2)}]_{t+T} \boldsymbol{Q}(t+T)^{\top} - 2w_{\boldsymbol{u},\boldsymbol{X}_{0}} \sum_{\ell=1}^{d} c_{\ell} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \int_{0}^{t+T} \boldsymbol{f}(w, \boldsymbol{e}_{\ell}) \, \mathrm{d}w \right\| \\ &\leqslant \limsup_{t \to \infty} \left\| \boldsymbol{\Delta}_{t,T}^{(1)} \right\| + \limsup_{t \to \infty} \left\| \boldsymbol{\Delta}_{t,T}^{(2)} \right\| \\ &\leqslant \frac{2 \| \boldsymbol{v} \|^{2}}{s(\widetilde{\boldsymbol{B}}) - 2 \mathrm{Re}(\lambda)} \left(\sup_{\tau \in [T,\infty)} \left\| \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})\tau} \boldsymbol{X}_{\tau} - w_{\boldsymbol{u},\boldsymbol{X}_{0}} \widetilde{\boldsymbol{u}} \right\| \right) \sum_{\ell=1}^{d} c_{\ell} \end{split}$$

almost surely. Letting $T \to \infty$, by Theorem 3.3 in Barczy et al. [8] (which can be used, since the moment condition (3.3) yields the moment condition (3.1) with $\lambda = s(\tilde{\boldsymbol{B}})$), we obtain (4.3). Moreover, $\int_0^t \boldsymbol{f}(w, \boldsymbol{e}_{\ell}) \, \mathrm{d}w \to \int_0^\infty \boldsymbol{f}(w, \boldsymbol{e}_{\ell}) \, \mathrm{d}w$ as $t \to \infty$, since we have

$$\int_0^\infty \|\boldsymbol{f}(w, \boldsymbol{e}_\ell)\| \, \mathrm{d}w \leqslant \|\boldsymbol{v}\|^2 \int_0^\infty \mathrm{e}^{-(s(\widetilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda))w} \, \mathrm{d}w = \frac{\|\boldsymbol{v}\|^2}{s(\widetilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda)} < \infty.$$

Consequently,

(4.6)
$$\boldsymbol{Q}(t)[\boldsymbol{M}^{(2)}]_{t}\boldsymbol{Q}(t)^{\top} \xrightarrow{\text{a.s.}} 2w_{\boldsymbol{u},\boldsymbol{X}_{0}} \sum_{\ell=1}^{d} c_{\ell} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \int_{0}^{\infty} \boldsymbol{f}(w, \boldsymbol{e}_{\ell}) \,\mathrm{d}w \quad \text{as} \quad t \to \infty.$$

Next, by Theorem 3.3 in Barczy et al. [8], we show that

$$\boldsymbol{Q}(t)[\boldsymbol{M}^{(3,4)}]_{t}\boldsymbol{Q}(t)^{\top} - w_{\boldsymbol{u},\boldsymbol{X}_{0}} \sum_{\ell=1}^{d} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \int_{0}^{t} \int_{\mathcal{U}_{d}} \boldsymbol{f}(w, \boldsymbol{z}) \, \mathrm{d}w \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \stackrel{L_{1}}{\longrightarrow} \boldsymbol{0}$$

as $t \to \infty$. Since

(4.7)
$$\boldsymbol{Q}(t)[\boldsymbol{M}^{(3,4)}]_{t}\boldsymbol{Q}(t)^{\top} = \sum_{\ell=1}^{d} \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} \boldsymbol{f}(t-u,\boldsymbol{z}) \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})u} \mathbb{1}_{\{w \leq X_{u-,\ell}\}} N_{\ell}(\mathrm{d}u,\mathrm{d}\boldsymbol{z},\mathrm{d}w),$$

it is enough to show that for each $\ell \in \{1, \ldots, d\}$ and $i, j \in \{1, 2\}$, we have

(4.8)
$$\int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} f_{i,j}(t-u, \boldsymbol{z}) \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})u} \mathbb{1}_{\{w \leq X_{u-,\ell}\}} N_{\ell}(\mathrm{d}u, \mathrm{d}\boldsymbol{z}, \mathrm{d}w) - w_{\boldsymbol{u}, \boldsymbol{X}_{0}} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \int_{0}^{t} \int_{\mathcal{U}_{d}} f_{i,j}(t-u, \boldsymbol{z}) \mathrm{d}u \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \xrightarrow{L_{1}} 0 \quad \text{as} \ t \to \infty,$$

where $f(w, z) =: (f_{i,j}(w, z))_{i,j \in \{1,2\}}, w \in \mathbb{R}_+, z \in \mathbb{R}^d$. For each $t \in \mathbb{R}_+, \ell \in \{1, ..., d\}$ and $i, j \in \{1, 2\}$, we have

where

$$I_{t,1} := \mathbb{E}\left(\left|\int_0^t \int_{\mathcal{U}_d} \int_{\mathcal{U}_1} f_{i,j}(t-u, \boldsymbol{z}) \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})u} \mathbb{1}_{\{w \leq X_{u-,\ell}\}} \widetilde{N}_{\ell}(\mathrm{d}u, \mathrm{d}\boldsymbol{z}, \mathrm{d}w)\right|\right)$$

and

$$I_{t,2} := \mathbb{E}\left(\left|\int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} f_{i,j}(t-u, \boldsymbol{z}) \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})u} \mathbb{1}_{\{w \leqslant X_{u,\ell}\}} \,\mathrm{d}u \,\mu_{\ell}(\mathrm{d}\boldsymbol{z}) \,\mathrm{d}w\right. \\ \left. - w_{\boldsymbol{u},\boldsymbol{X}_{0}} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \int_{0}^{t} \int_{\mathcal{U}_{d}} f_{i,j}(t-u, \boldsymbol{z}) \,\mathrm{d}u \,\mu_{\ell}(\mathrm{d}\boldsymbol{z}) \right|\right) \\ = \mathbb{E}\left(\left|\int_{0}^{t} \int_{\mathcal{U}_{d}} f_{i,j}(t-u, \boldsymbol{z}) (\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})u} X_{u,\ell} - w_{\boldsymbol{u},\boldsymbol{X}_{0}} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle) \,\mathrm{d}w \,\mu_{\ell}(\mathrm{d}\boldsymbol{z})\right|\right).$$

Here, for each $\ell \in \{1, \ldots, d\}$ and $i, j \in \{1, 2\}$, using Ikeda and Watanabe [15, page 63], (2.5) and that $|\operatorname{Re}(a)| \leq |a|$ and $|\operatorname{Im}(a)| \leq |a|$ for each $a \in \mathbb{C}$, we have

$$(4.10) \qquad I_{t,1}^{2} \leq \mathbb{E}\left(\left|\int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} f_{i,j}(t-u, \boldsymbol{z}) \mathrm{e}^{-s(\tilde{\boldsymbol{B}})u} \mathbb{1}_{\{w \leq X_{u-,\ell}\}} \widetilde{N}_{\ell}(\mathrm{d}u, \mathrm{d}\boldsymbol{z}, \mathrm{d}w)\right|^{2}\right) \\ = \int_{0}^{t} \int_{\mathcal{U}_{d}} |f_{i,j}(t-u, \boldsymbol{z})|^{2} \mathrm{e}^{-2s(\tilde{\boldsymbol{B}})u} \mathbb{E}(X_{u,\ell}) \mathrm{d}u \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \\ \leq \int_{0}^{t} \int_{\mathcal{U}_{d}} |\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)(t-u)/2} \langle \boldsymbol{v}, \boldsymbol{z} \rangle|^{4} \mathrm{e}^{-2s(\tilde{\boldsymbol{B}})u} \mathbb{E}(||\boldsymbol{X}_{u}||) \mathrm{d}u \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \\ \leq C_{4} ||\boldsymbol{v}||^{4} \int_{\mathcal{U}_{d}} ||\boldsymbol{z}||^{4} \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \, \mathrm{e}^{-2(s(\tilde{\boldsymbol{B}})-2\mathrm{Re}(\lambda))t} \int_{0}^{t} \mathrm{e}^{(s(\tilde{\boldsymbol{B}})-4\mathrm{Re}(\lambda))u} \, \mathrm{d}u \to 0$$

as $t \to \infty$. Indeed, if $s(\widetilde{B}) \neq 4 \operatorname{Re}(\lambda)$, using that $2\operatorname{Re}(\lambda) < s(\widetilde{B})$, we get

$$I_{t,1}^2 \leqslant C_4 \|\boldsymbol{v}\|^4 \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^4 \,\mu_\ell(\mathrm{d}\boldsymbol{z}) \frac{\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t} - \mathrm{e}^{-2(s(\widetilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda))t}}{s(\widetilde{\boldsymbol{B}}) - 4\mathrm{Re}(\lambda)} \to 0$$

as $t \to \infty$, since $\int_{\mathcal{U}_d} \|\boldsymbol{z}\|^4 \mu_\ell(\mathrm{d}\boldsymbol{z}) < \infty$. Otherwise, if $s(\widetilde{\boldsymbol{B}}) = 4\mathrm{Re}(\lambda)$, then we obtain

$$I_{t,1}^2 \leqslant C_4 \|\boldsymbol{v}\|^4 \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^4 \, \mu_\ell(\mathrm{d}\boldsymbol{z}) \, t\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t} \to 0 \qquad \mathrm{as} \ t \to \infty.$$

Further, for each $\ell \in \{1, \ldots, d\}$, $i, j \in \{1, 2\}$, and $t, T \in \mathbb{R}_+$, we have

(4.11)
$$I_{t+T,2} \leqslant J_{t,T}^{(1)} + J_{t,T}^{(2)}$$

with

$$J_{t,T}^{(1)} := \mathbb{E}\left(\left|\int_{0}^{T}\int_{\mathcal{U}_{d}}f_{i,j}(t+T-\tau,\boldsymbol{z})(\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})\tau}X_{\tau,\ell}-w_{\boldsymbol{u},\boldsymbol{X}_{0}}\langle\boldsymbol{e}_{\ell},\widetilde{\boldsymbol{u}}\rangle)\,\mathrm{d}\tau\,\mu_{\ell}(\mathrm{d}\boldsymbol{z})\right|\right),$$
$$J_{t,T}^{(2)} := \mathbb{E}\left(\left|\int_{T}^{t+T}\int_{\mathcal{U}_{d}}f_{i,j}(t+T-\tau,\boldsymbol{z})(\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})\tau}X_{\tau,\ell}-w_{\boldsymbol{u},\boldsymbol{X}_{0}}\langle\boldsymbol{e}_{\ell},\widetilde{\boldsymbol{u}}\rangle)\,\mathrm{d}\tau\,\mu_{\ell}(\mathrm{d}\boldsymbol{z})\right|\right).$$

By Theorem 3.3 in Barczy et al. [8], we have $K := \sup_{\tau \in \mathbb{R}_+} \mathbb{E}(\|e^{-s(\widetilde{\boldsymbol{B}})\tau} \boldsymbol{X}_{\tau} - w_{\boldsymbol{u},\boldsymbol{X}_0} \widetilde{\boldsymbol{u}}\|) < \infty$, and hence, similarly as in (4.4), for any $T \in \mathbb{R}_+$,

$$\begin{aligned} J_{t,T}^{(1)} &\leqslant K \int_0^T \int_{\mathcal{U}_d} |f_{i,j}(t+T-\tau, \boldsymbol{z})| \,\mathrm{d}\tau \,\mu_\ell(\mathrm{d}\boldsymbol{z}) = K \int_t^{t+T} \int_{\mathcal{U}_d} |f_{i,j}(w, \boldsymbol{z})| \,\mathrm{d}w \,\mu_\ell(\mathrm{d}\boldsymbol{z}) \\ &\leqslant K \|\boldsymbol{v}\|^2 \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^2 \,\mu_\ell(\mathrm{d}\boldsymbol{z}) \int_t^\infty \mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda))w} \,\mathrm{d}w \\ &= \frac{K \|\boldsymbol{v}\|^2}{s(\tilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda)} \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^2 \,\mu_\ell(\mathrm{d}\boldsymbol{z}) \,\mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda))t} \to 0 \end{aligned}$$

as $t \to \infty$. Further, similarly as in (4.5), for each $t, T \in \mathbb{R}_+$,

$$J_{t,T}^{(2)} \leq \sup_{\tau \in [T,\infty)} \mathbb{E}(|\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})\tau} X_{\tau,\ell} - w_{\boldsymbol{u},\boldsymbol{X}_0} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle|) \int_{T}^{t+T} \int_{\mathcal{U}_d} |f_{i,j}(t+T-\tau,\boldsymbol{z})| \,\mathrm{d}\tau \,\mu_{\ell}(\mathrm{d}\boldsymbol{z})$$
$$\leq \sup_{\tau \in [T,\infty)} \mathbb{E}(|\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})\tau} X_{\tau,\ell} - w_{\boldsymbol{u},\boldsymbol{X}_0} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle|) \frac{\|\boldsymbol{v}\|^2}{s(\widetilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda)} \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^2 \,\mu_{\ell}(\mathrm{d}\boldsymbol{z}).$$

Consequently, for each $T \in \mathbb{R}_+$, we obtain

$$\limsup_{t \to \infty} I_{t,2} = \limsup_{t \to \infty} I_{t+T,2} \leqslant \limsup_{t \to \infty} J_{t,T}^{(1)} + \limsup_{t \to \infty} J_{t,T}^{(2)}$$
$$\leqslant \sup_{\tau \in [T,\infty)} \mathbb{E}(|\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})\tau} X_{\tau,\ell} - w_{\boldsymbol{u},\boldsymbol{X}_0} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle|) \frac{\|\boldsymbol{v}\|^2}{s(\widetilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda)} \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^2 \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}).$$

Letting $T \to \infty$, by Theorem 3.3 in Barczy et al. [8], we have $\lim_{t\to\infty} I_{t,2} = 0$, as desired. All in all, $\lim_{t\to\infty} (I_{t,1} + I_{t,2}) = 0$, yielding (4.8). Moreover, for each $\ell \in \{1, \ldots, d\}$,

$$\int_0^t \int_{\mathcal{U}_d} \boldsymbol{f}(t-u,\boldsymbol{z}) \,\mathrm{d} u \,\mu_\ell(\mathrm{d}\boldsymbol{z}) = \int_0^t \int_{\mathcal{U}_d} \boldsymbol{f}(w,\boldsymbol{z}) \,\mathrm{d} w \,\mu_\ell(\mathrm{d}\boldsymbol{z}) \to \int_0^\infty \int_{\mathcal{U}_d} \boldsymbol{f}(w,\boldsymbol{z}) \,\mathrm{d} w \,\mu_\ell(\mathrm{d}\boldsymbol{z})$$

as $t \to \infty$, since we have

$$\int_0^\infty \int_{\mathcal{U}_d} \|\boldsymbol{f}(w, \boldsymbol{z})\| \,\mathrm{d}w \,\mu_\ell(\mathrm{d}\boldsymbol{z}) \leqslant \|\boldsymbol{v}\|^2 \int_0^\infty \int_{\mathcal{U}_d} \mathrm{e}^{-(s(\widetilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda))w} \|\boldsymbol{z}\|^2 \,\mathrm{d}w \,\mu_\ell(\mathrm{d}\boldsymbol{z})$$
$$= \frac{\|\boldsymbol{v}\|^2}{s(\widetilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda)} \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^2 \,\mu_\ell(\mathrm{d}\boldsymbol{z}) < \infty.$$

Consequently,

(4.12)
$$\boldsymbol{Q}(t)[\boldsymbol{M}^{(3,4)}]_t \boldsymbol{Q}(t)^\top \xrightarrow{L_1} w_{\boldsymbol{u},\boldsymbol{X}_0} \sum_{\ell=1}^d \langle \boldsymbol{e}_\ell, \widetilde{\boldsymbol{u}} \rangle \int_0^\infty \int_{\mathcal{U}_d} \boldsymbol{f}(w, \boldsymbol{z}) \,\mathrm{d}w \,\mu_\ell(\mathrm{d}\boldsymbol{z}) \quad \text{as } t \to \infty.$$

Further,

$$\boldsymbol{Q}(t)[\boldsymbol{M}^{(5)}]_{t}\boldsymbol{Q}(t)^{\top} = \int_{0}^{t} \int_{\mathcal{U}_{d}} \boldsymbol{f}(t-u,\boldsymbol{r}) \mathrm{e}^{-s(\tilde{\boldsymbol{B}})u} M(\mathrm{d}u,\mathrm{d}\boldsymbol{r}) \xrightarrow{L_{1}} \boldsymbol{0}$$

as $t \to \infty$, since if $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}}))$, then

$$\mathbb{E}(\|\boldsymbol{Q}(t)[\boldsymbol{M}^{(5)}]_{t}\boldsymbol{Q}(t)^{\top}\|) \leq \mathbb{E}\left(\int_{0}^{t}\int_{\mathcal{U}_{d}}\|\boldsymbol{f}(t-u,\boldsymbol{r})\mathrm{e}^{-s(\tilde{\boldsymbol{B}})u}\|M(\mathrm{d}u,\mathrm{d}\boldsymbol{r})\right)$$

$$=\int_{0}^{t}\int_{\mathcal{U}_{d}}\|\boldsymbol{f}(t-u,\boldsymbol{r})\mathrm{e}^{-s(\tilde{\boldsymbol{B}})u}\|\,\mathrm{d}u\,\nu(\mathrm{d}\boldsymbol{r})$$

$$\leq\int_{0}^{t}\int_{\mathcal{U}_{d}}|\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\lambda)(t-u)/2}\langle\boldsymbol{v},\boldsymbol{r}\rangle|^{2}\mathrm{e}^{-s(\tilde{\boldsymbol{B}})u}\,\mathrm{d}u\,\nu(\mathrm{d}\boldsymbol{r})$$

$$\leq\|\boldsymbol{v}\|^{2}\mathrm{e}^{-(s(\tilde{\boldsymbol{B}})-2\mathrm{Re}(\lambda))t}\int_{0}^{t}\mathrm{e}^{-2\mathrm{Re}(\lambda)u}\,\mathrm{d}u\int_{\mathcal{U}_{d}}\|\boldsymbol{r}\|^{2}\,\nu(\mathrm{d}\boldsymbol{r})\to 0$$

as $t \to \infty$. Indeed, if $\operatorname{Re}(\lambda) \neq 0$, then

$$e^{-(s(\widetilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t} \int_0^t e^{-2\operatorname{Re}(\lambda)u} \, \mathrm{d}u = \frac{1}{2\operatorname{Re}(\lambda)} \left(e^{-(s(\widetilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t} - e^{-s(\widetilde{\boldsymbol{B}})t} \right) \to 0$$

as $t \to \infty$, and if $\operatorname{Re}(\lambda) = 0$, then $e^{-(s(\tilde{B}) - 2\operatorname{Re}(\lambda))t} \int_0^t e^{-2\operatorname{Re}(\lambda)u} du = t e^{-s(\tilde{B})t} \to 0$ as $t \to \infty$. Consequently, by (4.6) and (4.12), we get

(4.14)

$$\boldsymbol{Q}(t)[\boldsymbol{M}]_{t}\boldsymbol{Q}(t)^{\top} \xrightarrow{\mathbb{P}} 2w_{\boldsymbol{u},\boldsymbol{X}_{0}} \sum_{\ell=1}^{d} c_{\ell} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \int_{0}^{\infty} \boldsymbol{f}(w, \boldsymbol{e}_{\ell}) \, \mathrm{d}w \\
+ w_{\boldsymbol{u},\boldsymbol{X}_{0}} \sum_{\ell=1}^{d} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \int_{0}^{\infty} \int_{\mathcal{U}_{d}} \boldsymbol{f}(w, \boldsymbol{z}) \, \mathrm{d}w \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) = w_{\boldsymbol{u},\boldsymbol{X}_{0}} \boldsymbol{\Sigma}_{\boldsymbol{v}}$$

as $t \to \infty$, hence the condition (E.1) of Theorem E.1 holds. Indeed, for each $a \in \mathbb{C}$, we have the identity

(4.15)
$$\begin{pmatrix} \operatorname{Re}(a) \\ \operatorname{Im}(a) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(a) \\ \operatorname{Im}(a) \end{pmatrix}^{\top} = \begin{pmatrix} \operatorname{Re}(a)^{2} & \operatorname{Re}(a)\operatorname{Im}(a) \\ \operatorname{Re}(a)\operatorname{Im}(a) & \operatorname{Im}(a)^{2} \end{pmatrix}$$
$$= \frac{1}{2}|a|^{2}\boldsymbol{I}_{2} + \frac{1}{2} \begin{pmatrix} \operatorname{Re}(a^{2}) & \operatorname{Im}(a^{2}) \\ \operatorname{Im}(a^{2}) & -\operatorname{Re}(a^{2}) \end{pmatrix}.$$

Hence, for each $\ell \in \{1, \ldots, d\}$, applying (4.15) with $a = e^{-(s(\tilde{B})-2\lambda)w/2} \langle v, e_{\ell} \rangle$, we have

$$\begin{split} &\int_{0}^{\infty} \boldsymbol{f}(\boldsymbol{w}, \boldsymbol{e}_{\ell}) \, \mathrm{d}\boldsymbol{w} = \frac{1}{2} \int_{0}^{\infty} \begin{pmatrix} \mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda))\boldsymbol{w}} |\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle|^{2} & 0 \\ 0 & \mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda))\boldsymbol{w}} |\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle|^{2} \end{pmatrix} \, \mathrm{d}\boldsymbol{w} \\ &+ \frac{1}{2} \int_{0}^{\infty} \begin{pmatrix} \mathrm{Re}(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)\boldsymbol{w}} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2}) & \mathrm{Im}(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)\boldsymbol{w}} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2}) \\ \mathrm{Im}(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)\boldsymbol{w}} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2}) & -\mathrm{Re}(\mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)\boldsymbol{w}} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2}) \end{pmatrix} \, \mathrm{d}\boldsymbol{w} \\ &= \frac{|\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle|^{2}}{2(s(\tilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda))} \boldsymbol{I}_{2} \\ &+ \frac{1}{2} \begin{pmatrix} \mathrm{Re}(\int_{0}^{\infty} \mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)\boldsymbol{w}} \, \mathrm{d}\boldsymbol{w} \, \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2}) & \mathrm{Im}(\int_{0}^{\infty} \mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)\boldsymbol{w}} \, \mathrm{d}\boldsymbol{w} \, \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2}) \\ \mathrm{Im}(\int_{0}^{\infty} \mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)\boldsymbol{w}} \, \mathrm{d}\boldsymbol{w} \, \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2}) & -\mathrm{Re}(\int_{0}^{\infty} \mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)\boldsymbol{w}} \, \mathrm{d}\boldsymbol{w} \, \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2}) \end{pmatrix} \\ &= \frac{|\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle|^{2}}{2(s(\tilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda))} \boldsymbol{I}_{2} + \frac{1}{2} \begin{pmatrix} \mathrm{Re}\left(\frac{\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2}}{s(\tilde{\boldsymbol{B}}) - 2\lambda}\right) & \mathrm{Im}\left(\frac{\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2}}{s(\tilde{\boldsymbol{B}}) - 2\lambda}\right) \\ \mathrm{Im}\left(\frac{\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2}}{s(\tilde{\boldsymbol{B}}) - 2\lambda}\right) & -\mathrm{Re}\left(\frac{\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2}}{s(\tilde{\boldsymbol{B}}) - 2\lambda}\right) \end{pmatrix}, \end{split}$$

and similarly

$$\int_0^\infty \int_{\mathcal{U}_d} \boldsymbol{f}(w, \boldsymbol{z}) \, \mathrm{d}w \, \mu_\ell(\mathrm{d}\boldsymbol{z}) =$$

$$= \frac{\int_{\mathcal{U}_d} |\langle \boldsymbol{v}, \boldsymbol{z} \rangle|^2 \,\mu_{\ell}(\mathrm{d}\boldsymbol{z})}{2(s(\tilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda))} \boldsymbol{I}_2 + \frac{1}{2} \begin{pmatrix} \mathrm{Re}\left(\frac{\int_{\mathcal{U}_d} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^2 \,\mu_{\ell}(\mathrm{d}\boldsymbol{z})}{s(\tilde{\boldsymbol{B}}) - 2\lambda}\right) & \mathrm{Im}\left(\frac{\int_{\mathcal{U}_d} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^2 \,\mu_{\ell}(\mathrm{d}\boldsymbol{z})}{s(\tilde{\boldsymbol{B}}) - 2\lambda}\right) \\ \mathrm{Im}\left(\frac{\int_{\mathcal{U}_d} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^2 \,\mu_{\ell}(\mathrm{d}\boldsymbol{z})}{s(\tilde{\boldsymbol{B}}) - 2\lambda}\right) & -\mathrm{Re}\left(\frac{\int_{\mathcal{U}_d} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^2 \,\mu_{\ell}(\mathrm{d}\boldsymbol{z})}{s(\tilde{\boldsymbol{B}}) - 2\lambda}\right) \end{pmatrix},$$

yielding (4.14). Note that $\Sigma_{\boldsymbol{v}}$ is non-negative definite irrespective of $\widetilde{\boldsymbol{\beta}} \neq \mathbf{0}$ or $\widetilde{\boldsymbol{\beta}} = \mathbf{0}$, since $\boldsymbol{c} \in \mathbb{R}^d_+$, $\widetilde{\boldsymbol{u}} \in \mathbb{R}^d_+$, and $\boldsymbol{f}(w, \boldsymbol{z})$ is non-negative definite for any $w \in \mathbb{R}_+$ and $\boldsymbol{z} \in \mathbb{R}^d$.

Step 3. Now we turn to prove that condition (E.2) of Theorem E.1 holds for $(\boldsymbol{M}_t)_{t \in \mathbb{R}_+}$ with the scaling $\boldsymbol{Q}(t), t \in \mathbb{R}_+$, namely,

$$\mathbb{E}\left(\sup_{u\in[0,t]} \|\boldsymbol{Q}(t)(\boldsymbol{M}_u - \boldsymbol{M}_{u-})\|\right) \to 0 \quad \text{as} \quad t \to \infty.$$

Since $(\boldsymbol{M}_t^{(2)})_{t \in \mathbb{R}_+}$ has continuous sample paths, we have for each $t \in \mathbb{R}_+$,

$$\sup_{u \in [0,t]} \|\boldsymbol{Q}(t)(\boldsymbol{M}_u - \boldsymbol{M}_{u-})\| = \sup_{u \in [0,t]} \|\boldsymbol{Q}(t)(\boldsymbol{M}_u^{(3,4)} - \boldsymbol{M}_{u-}^{(3,4)}) + \boldsymbol{Q}(t)(\boldsymbol{M}_u^{(5)} - \boldsymbol{M}_{u-}^{(5)})\|$$

$$\leq \|\boldsymbol{Q}(t)\| \sum_{\ell=1}^{d} \sup_{u \in [0,t]} \|\widetilde{\boldsymbol{Y}}_{u}^{(\ell)} - \widetilde{\boldsymbol{Y}}_{u-}^{(\ell)}\| + \|\boldsymbol{Q}(t)\| \sup_{u \in [0,t]} \|\boldsymbol{M}_{u}^{(5)} - \boldsymbol{M}_{u-}^{(5)}\|$$

almost surely. Since $\mathbf{Q}(t)\mathbf{Q}(t)^{\top} = e^{-(s(\tilde{B})-2\operatorname{Re}(\lambda))t}\mathbf{I}_2, \quad t \in \mathbb{R}_+$, we have $\|\mathbf{Q}(t)\| = e^{-(s(\tilde{B})-2\operatorname{Re}(\lambda))t/2}, \quad t \in \mathbb{R}_+$. Hence it is enough to show that

(4.17)
$$e^{-(s(\widetilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t/2} \mathbb{E}\left(\sup_{u\in[0,t]} \|\widetilde{\boldsymbol{Y}}_{u}^{(\ell)}-\widetilde{\boldsymbol{Y}}_{u-}^{(\ell)}\|\right) \to 0 \quad \text{as} \ t \to \infty$$

for every $\ell \in \{1, \ldots, d\}$, and

(4.18)
$$e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t/2} \mathbb{E}\left(\sup_{u\in[0,t]} \|\boldsymbol{M}_{u}^{(5)}-\boldsymbol{M}_{u-}^{(5)}\|\right) \to 0 \quad \text{as} \ t \to \infty$$

First, we prove (4.17) for each $\ell \in \{1, \ldots, d\}$. By Cauchy-Schwarz's inequality, for each $\varepsilon \in \mathbb{R}_{++}, t \in \mathbb{R}_+$ and $\ell \in \{1, \ldots, d\}$, we have

$$(4.19) e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t/2} \mathbb{E}\left(\sup_{u\in[0,t]} \|\tilde{\boldsymbol{Y}}_{u}^{(\ell)}-\tilde{\boldsymbol{Y}}_{u-}^{(\ell)}\|\right)$$

$$\leq e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t/2} \mathbb{E}\left(\sup_{u\in[0,t]} \|\tilde{\boldsymbol{Y}}_{u}^{(\ell,\varepsilon)}-\tilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)}\|\right)$$

$$+ e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t/2} \mathbb{E}\left(\sup_{u\in[0,t]} \|(\tilde{\boldsymbol{Y}}_{u}^{(\ell)}-\tilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)})-(\tilde{\boldsymbol{Y}}_{u-}^{(\ell)}-\tilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)})\|\right)$$

$$\leq e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t/2} \left(\mathbb{E}\left(\sup_{u\in[0,t]} \|\tilde{\boldsymbol{Y}}_{u}^{(\ell,\varepsilon)}-\tilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)}\|^{4}\right)\right)^{1/4}$$

$$+ e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t/2} \left(\mathbb{E}\left(\sup_{u\in[0,t]} \|(\tilde{\boldsymbol{Y}}_{u}^{(\ell)}-\tilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)})-(\tilde{\boldsymbol{Y}}_{u-}^{(\ell)}-\tilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)})\|^{2}\right)\right)^{1/2}.$$

Here, by (2.5), for each $\varepsilon \in \mathbb{R}_{++}$, $t \in \mathbb{R}_{+}$ and $\ell \in \{1, \ldots, d\}$, we have

$$\mathbb{E}\left(\sup_{u\in[0,t]}\|\widetilde{\boldsymbol{Y}}_{u}^{(\ell,\varepsilon)}-\widetilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)}\|^{4}\right) = \mathbb{E}\left(\sup_{u\in[0,t]}|\widetilde{Y}_{u}^{(\ell,\varepsilon)}-\widetilde{Y}_{u-}^{(\ell,\varepsilon)}|^{4}\right) = \mathbb{E}\left(\sup_{u\in[0,t]}|Y_{u}^{(\ell,\varepsilon)}-Y_{u-}^{(\ell,\varepsilon)}|^{4}\right) \\
\leq \mathbb{E}\left(\sum_{u\in[0,t]}|Y_{u}^{(\ell,\varepsilon)}-Y_{u-}^{(\ell,\varepsilon)}|^{4}\right) \\
(4.20) = \mathbb{E}\left(\int_{0}^{t}\int_{\mathcal{U}_{d}}\int_{\mathcal{U}_{1}}|e^{-\lambda u}\langle\boldsymbol{v},\boldsymbol{z}\rangle\mathbb{1}_{\{||\boldsymbol{z}||\geq\varepsilon\}}\mathbb{1}_{\{w\leq X_{u-,\ell}\}}|^{4}N_{\ell}(\mathrm{d}\boldsymbol{u},\mathrm{d}\boldsymbol{z},\mathrm{d}\boldsymbol{w})\right) \\
= \int_{0}^{t}\int_{\mathcal{U}_{d}}e^{-4\mathrm{Re}(\lambda)u}|\langle\boldsymbol{v},\boldsymbol{z}\rangle|^{4}\mathbb{1}_{\{||\boldsymbol{z}||\geq\varepsilon\}}\mathbb{E}(X_{u,\ell})\,\mathrm{d}\boldsymbol{u}\,\mu_{\ell}(\mathrm{d}\boldsymbol{z}) \\
\leq C_{4}\|\boldsymbol{v}\|^{4}\int_{0}^{t}e^{(s(\tilde{\boldsymbol{B}})-4\mathrm{Re}(\lambda))u}\,\mathrm{d}\boldsymbol{u}\,\int_{\mathcal{U}_{d}}\|\boldsymbol{z}\|^{4}\mathbb{1}_{\{||\boldsymbol{z}||\geq\varepsilon\}}\,\mu_{\ell}(\mathrm{d}\boldsymbol{z}).$$

Hence, by (4.10) and $2\operatorname{Re}(\lambda) < s(\widetilde{\boldsymbol{B}})$, for each $\varepsilon \in \mathbb{R}_{++}$ and $\ell \in \{1, \ldots, d\}$, we get

(4.21)
$$e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t/2} \left(\mathbb{E}\left(\sup_{u \in [0,t]} \| \tilde{\boldsymbol{Y}}_{u}^{(\ell,\varepsilon)} - \tilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)} \|^{4} \right) \right)^{1/4} \to 0 \quad \text{as} \ t \to \infty.$$

Further, since

$$\widetilde{\boldsymbol{Y}}_{t}^{(\ell)} - \widetilde{\boldsymbol{Y}}_{t}^{(\ell,\varepsilon)} = \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} \mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle \mathbb{1}_{\{\|\boldsymbol{z}\| < \varepsilon\}} \mathbb{1}_{\{w \leq X_{u-,\ell}\}} \widetilde{N}_{\ell}(\mathrm{d}\boldsymbol{u}, \mathrm{d}\boldsymbol{z}, \mathrm{d}\boldsymbol{w}), \qquad t \in \mathbb{R}_{+},$$

by the proof of part (a) of Theorem II.1.33 in Jacod and Shiryaev [16], we get

$$\begin{aligned}
& \mathbb{E}\left(\sup_{u\in[0,t]} \|(\widetilde{\boldsymbol{Y}}_{u}^{(\ell)}-\widetilde{\boldsymbol{Y}}_{u}^{(\ell,\varepsilon)})-(\widetilde{\boldsymbol{Y}}_{u-}^{(\ell)}-\widetilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)})\|^{2}\right) \\
&\leqslant \mathbb{E}\left(\sum_{u\in[0,t]} \|(\widetilde{\boldsymbol{Y}}_{u}^{(\ell)}-\widetilde{\boldsymbol{Y}}_{u}^{(\ell,\varepsilon)})-(\widetilde{\boldsymbol{Y}}_{u-}^{(\ell)}-\widetilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)})\|^{2}\right) \\
&= \mathbb{E}\left(\int_{0}^{t}\int_{\mathcal{U}_{d}}\int_{\mathcal{U}_{1}} |\mathrm{e}^{-\lambda u}\langle\boldsymbol{v},\boldsymbol{z}\rangle|^{2}\mathbb{1}_{\{\|\boldsymbol{z}\|<\varepsilon\}}\mathbb{1}_{\{w\leqslant X_{u,\ell}\}}N_{\ell}(\mathrm{d}\boldsymbol{u},\mathrm{d}\boldsymbol{z},\mathrm{d}\boldsymbol{w})\right) \\
&\leqslant C_{4}\|\boldsymbol{v}\|^{2}\int_{0}^{t}\int_{\mathcal{U}_{d}} \mathrm{e}^{(s(\widetilde{\boldsymbol{B}})-2\mathrm{Re}(\lambda))u}\|\boldsymbol{z}\|^{2}\mathbb{1}_{\{\|\boldsymbol{z}\|<\varepsilon\}}\mu_{\ell}(\mathrm{d}\boldsymbol{z}) \\
&\leqslant C_{4}\|\boldsymbol{v}\|^{2}\frac{\mathrm{e}^{(s(\widetilde{\boldsymbol{B}})-2\mathrm{Re}(\lambda))t}}{s(\widetilde{\boldsymbol{B}})-2\mathrm{Re}(\lambda)}\int_{\mathcal{U}_{d}}\|\boldsymbol{z}\|^{2}\mathbb{1}_{\{\|\boldsymbol{z}\|<\varepsilon\}}\mu_{\ell}(\mathrm{d}\boldsymbol{z}).
\end{aligned}$$

Hence, by (4.19) and (4.21), for all $\varepsilon \in \mathbb{R}_{++}$ and $\ell \in \{1, \ldots, d\}$, we have

$$\begin{split} \limsup_{t \to \infty} \mathrm{e}^{-(s(\widetilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda))t/2} \, \mathbb{E} \left(\sup_{u \in [0,t]} \| \widetilde{\boldsymbol{Y}}_{u}^{(\ell)} - \widetilde{\boldsymbol{Y}}_{u-}^{(\ell)} \| \right) \\ & \leqslant \left(\frac{C_{4} \| \boldsymbol{v} \|^{2}}{s(\widetilde{\boldsymbol{B}}) - 2\mathrm{Re}(\lambda)} \int_{\mathcal{U}_{d}} \| \boldsymbol{z} \|^{2} \mathbb{1}_{\{\| \boldsymbol{z} \| < \varepsilon\}} \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \right)^{1/2}, \end{split}$$

which tends to 0 as $\varepsilon \downarrow 0$ due to (3.8). Hence we conclude (4.17) for each $\ell \in \{1, \ldots, d\}$.

Next, we prove (4.18). By Cauchy-Schwarz's inequality, for each $t \in \mathbb{R}_+$, we have (4.23)

$$\mathbb{E}\left(\sup_{u\in[0,t]}\|\boldsymbol{M}_{u}^{(5)}-\boldsymbol{M}_{u-}^{(5)}\|\right) \leqslant \left(\mathbb{E}\left(\sup_{u\in[0,t]}\|\boldsymbol{M}_{u}^{(5)}-\boldsymbol{M}_{u-}^{(5)}\|^{2}\right)\right)^{1/2} = \left(\mathbb{E}\left(\sup_{u\in[0,t]}|Z_{u}^{(5)}-Z_{u-}^{(5)}|^{2}\right)\right)^{1/2},$$

hence it is enough to prove that

$$e^{-(s(\widetilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t} \mathbb{E}\left(\sup_{u\in[0,t]}|Z_u^{(5)}-Z_{u-}^{(5)}|^2\right) \to 0 \quad \text{as} \ t \to \infty$$

Since $\int_{\mathcal{U}_d} \|\boldsymbol{r}\| \, \nu(\mathrm{d}\boldsymbol{r}) < \infty$, for each $t \in \mathbb{R}_+$, we have $Z_t^{(5)} = Z_t^* - \int_0^t \int_{\mathcal{U}_d} \mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle \, \mathrm{d}u \, \nu(\mathrm{d}\boldsymbol{r})$ with $Z_t^* := \int_0^t \int_{\mathcal{U}_d} \mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle \, M(\mathrm{d}u, \mathrm{d}\boldsymbol{r})$, hence

$$\begin{aligned}
\mathbb{E}\left(\sup_{u\in[0,t]}|Z_{u}^{(5)}-Z_{u-}^{(5)}|^{2}\right) &= \mathbb{E}\left(\sup_{u\in[0,t]}|Z_{u}^{*}-Z_{u-}^{*}|^{2}\right) \leqslant \mathbb{E}\left(\sum_{u\in[0,t]}|Z_{u}^{*}-Z_{u-}^{*}|^{2}\right) \\
&= \mathbb{E}\left(\int_{0}^{t}\int_{\mathcal{U}_{d}}|\mathrm{e}^{-\lambda u}\langle\boldsymbol{v},\boldsymbol{r}\rangle|^{2}M(\mathrm{d}u,\mathrm{d}\boldsymbol{r})\right) &= \int_{0}^{t}\int_{\mathcal{U}_{d}}\mathrm{e}^{-2\mathrm{Re}(\lambda)u}|\langle\boldsymbol{v},\boldsymbol{r}\rangle|^{2}\mathrm{d}u\,\nu(\mathrm{d}\boldsymbol{r}) \\
&\leqslant \|\boldsymbol{v}\|^{2}\int_{0}^{t}\mathrm{e}^{-2\mathrm{Re}(\lambda)u}\,\mathrm{d}u\int_{\mathcal{U}_{d}}\|\boldsymbol{r}\|^{2}\,\nu(\mathrm{d}\boldsymbol{r})
\end{aligned}$$

hence, by (4.13), we conclude (4.18). Consequently, by Theorem E.1, we obtain

$$Q(t)M_t \xrightarrow{\mathcal{D}} (w_{u,X_0}\Sigma_v)^{1/2}N \quad \text{as} \quad t \to \infty,$$

where N is a 2-dimensional random vector with $N \stackrel{\mathcal{D}}{=} \mathcal{N}_2(\mathbf{0}, \mathbf{I}_2)$ independent of $w_{u, \mathbf{X}_0} \Sigma_{v}$. Clearly, $(w_{u, \mathbf{X}_0} \Sigma_{v})^{1/2} N = \sqrt{w_{u, \mathbf{X}_0}} \Sigma_{v}^{1/2} N \stackrel{\mathcal{D}}{=} \sqrt{w_{u, \mathbf{X}_0}} \mathbf{Z}_{v}$. By the decomposition

$$e^{-s(\tilde{\boldsymbol{B}})t/2} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} = \begin{pmatrix} \operatorname{Re}(e^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t/2}Z_t^{(0,1)}) \\ \operatorname{Im}(e^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t/2}Z_t^{(0,1)}) \end{pmatrix} + \boldsymbol{Q}(t)\boldsymbol{M}_t, \qquad t \in \mathbb{R}_+,$$

the convergence (4.1) and Slutsky's lemma (see, e.g., van der Vaart [23, Lemma 2.8]), we obtain (3.6).

Proof of part (ii) of Theorem 3.1. We use a similar approach as in the proof of part (iii) of Theorem 3.1. We divide the proof into three main steps.

Step 1. We use the same representation of $e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$, $t \in \mathbb{R}_+$ as in the proof of part (iii) of Theorem 3.1. We have

(4.25)
$$t^{-1/2} \mathrm{e}^{-(s(\widetilde{B}) - 2\lambda)t/2} Z_t^{(0,1)} \xrightarrow{\text{a.s.}} 0 \quad \text{as} \ t \to \infty,$$

since $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}}) > 0$ implies $\lambda \neq 0$, hence

$$t^{-1/2} \mathrm{e}^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t/2} Z_t^{(0,1)} = t^{-1/2} \mathrm{e}^{\mathrm{i}\mathrm{Im}(\lambda)t} \left(\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle - \frac{\langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle}{\lambda} (\mathrm{e}^{-\lambda t} - 1) \right)$$
$$= t^{-1/2} \mathrm{e}^{\mathrm{i}\mathrm{Im}(\lambda)t} \left(\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle + \frac{\langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle}{\lambda} \right) - t^{-1/2} \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t/2} \frac{\langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle}{\lambda} \xrightarrow{\text{a.s.}} 0$$

as $t \to \infty$.

For each $t \in \mathbb{R}_+$, with the notations of the proof of part (iii) of Theorem 3.1, we have

$$\begin{pmatrix} \operatorname{Re}(t^{-1/2}\mathrm{e}^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t/2}(Z_t^{(2)}+Z_t^{(3,4)}+Z_t^{(5)}))\\ \operatorname{Im}(t^{-1/2}\mathrm{e}^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t/2}(Z_t^{(2)}+Z_t^{(3,4)}+Z_t^{(5)})) \end{pmatrix} = t^{-1/2}\boldsymbol{Q}(t)\boldsymbol{M}_t,$$

where now

$$\boldsymbol{Q}(t) = \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{\mathrm{iIm}(\lambda)t}) & -\operatorname{Im}(\mathrm{e}^{\mathrm{iIm}(\lambda)t}) \\ \operatorname{Im}(\mathrm{e}^{\mathrm{iIm}(\lambda)t}) & \operatorname{Re}(\mathrm{e}^{\mathrm{iIm}(\lambda)t}) \end{pmatrix} = \begin{pmatrix} \cos(\operatorname{Im}(\lambda)t) & -\sin(\operatorname{Im}(\lambda)) \\ \sin(\operatorname{Im}(\lambda)t) & \cos(\operatorname{Im}(\lambda)t) \end{pmatrix}, \qquad t \in \mathbb{R}_+$$

We are again going to apply Theorem E.1 for the 2-dimensional martingale $(\boldsymbol{M}_t)_{t\in\mathbb{R}_+}$ now with the scaling $t^{-1/2}\boldsymbol{Q}(t), t\in\mathbb{R}_+$. We clearly have $t^{-1/2}\boldsymbol{Q}(t)\to\mathbf{0}$ as $t\to\infty$.

Step 2. Now we prove that condition (E.1) of Theorem E.1 holds. For each $t \in \mathbb{R}_+$, with the notations of the proof of part (iii) of Theorem 3.1, we have

$$t^{-1}\boldsymbol{Q}(t)[\boldsymbol{M}^{(2)}]_{t}\boldsymbol{Q}(t)^{\top} = \frac{2}{t}\sum_{\ell=1}^{d}c_{\ell}\int_{0}^{t}\boldsymbol{f}(t-\tau,\boldsymbol{e}_{\ell})\mathrm{e}^{-s(\tilde{\boldsymbol{B}})\tau}X_{\tau,\ell}\,\mathrm{d}\tau,$$

where now

$$\boldsymbol{f}(w, \boldsymbol{z}) = \begin{pmatrix} \operatorname{Re}(\operatorname{e}^{\operatorname{iIm}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \\ \operatorname{Im}(\operatorname{e}^{\operatorname{iIm}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(\operatorname{e}^{\operatorname{iIm}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \\ \operatorname{Im}(\operatorname{e}^{\operatorname{iIm}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \end{pmatrix}^{\top}, \qquad w \in \mathbb{R}_{+}, \quad \boldsymbol{z} \in \mathbb{R}^{d}.$$

First, we show

(4.26)
$$t^{-1}\boldsymbol{Q}(t)[\boldsymbol{M}^{(2)}]_{t}\boldsymbol{Q}(t)^{\top} - \frac{2w_{\boldsymbol{u},\boldsymbol{X}_{0}}}{t}\sum_{\ell=1}^{d}c_{\ell}\langle\boldsymbol{e}_{\ell},\widetilde{\boldsymbol{u}}\rangle\int_{0}^{t}\boldsymbol{f}(w,\boldsymbol{e}_{\ell})\,\mathrm{d}w\xrightarrow{\mathrm{a.s.}}\boldsymbol{0}$$
 as $t\to\infty$.

For each $t, T \in \mathbb{R}_{++}$, we have

$$(t+T)^{-1}\boldsymbol{Q}(t+T)[\boldsymbol{M}^{(2)}]_{t+T}\boldsymbol{Q}(t+T)^{\top} - \frac{2w_{\boldsymbol{u},\boldsymbol{X}_{0}}}{t+T}\sum_{\ell=1}^{d}c_{\ell}\langle\boldsymbol{e}_{\ell},\widetilde{\boldsymbol{u}}\rangle\int_{0}^{t+T}\boldsymbol{f}(w,\boldsymbol{e}_{\ell})\,\mathrm{d}w = \boldsymbol{\Delta}_{t,T}^{(1)} + \boldsymbol{\Delta}_{t,T}^{(2)}$$

with

$$\boldsymbol{\Delta}_{t,T}^{(1)} := \frac{2}{t+T} \sum_{\ell=1}^{d} c_{\ell} \int_{0}^{T} \boldsymbol{f}(t+T-\tau, \boldsymbol{e}_{\ell}) (\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})\tau} X_{\tau,\ell} - w_{\boldsymbol{u},\boldsymbol{X}_{0}} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle) \,\mathrm{d}\tau,$$
$$\boldsymbol{\Delta}_{t,T}^{(2)} := \frac{2}{t+T} \sum_{\ell=1}^{d} c_{\ell} \int_{T}^{t+T} \boldsymbol{f}(t+T-\tau, \boldsymbol{e}_{\ell}) (\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})\tau} X_{\tau,\ell} - w_{\boldsymbol{u},\boldsymbol{X}_{0}} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle) \,\mathrm{d}\tau.$$

For each $t, T \in \mathbb{R}_+$, we have

$$\|\boldsymbol{\Delta}_{t,T}^{(1)}\| \leq \frac{2}{t+T} \left(\sup_{\tau \in [0,T]} \|\mathbf{e}^{-s(\tilde{\boldsymbol{B}})\tau} \boldsymbol{X}_{\tau} - w_{\boldsymbol{u},\boldsymbol{X}_{0}} \tilde{\boldsymbol{u}}\| \right) \sum_{\ell=1}^{d} c_{\ell} \int_{0}^{T} \|\boldsymbol{f}(t+T-\tau,\boldsymbol{e}_{\ell})\| \,\mathrm{d}\tau,$$

where $\sup_{\tau \in [0,T]} \|e^{-s(\tilde{B})\tau} X_{\tau} - w_{\boldsymbol{u}, \boldsymbol{X}_0} \tilde{\boldsymbol{u}}\| < \infty$ almost surely since $(\boldsymbol{X}_t)_{t \in \mathbb{R}_+}$ has càdlàg sample paths, and using that $\|\boldsymbol{z}\boldsymbol{z}^{\top}\| \leq \|\boldsymbol{z}\|^2, \ \boldsymbol{z} \in \mathbb{R}^2$, we have

$$\int_0^T \|\boldsymbol{f}(t+T-\tau,\boldsymbol{e}_\ell)\| \,\mathrm{d}\tau = \int_t^{t+T} \|\boldsymbol{f}(w,\boldsymbol{e}_\ell)\| \,\mathrm{d}w \leqslant \int_t^{t+T} |\mathrm{e}^{\mathrm{i}\mathrm{Im}(\lambda)w} \langle \boldsymbol{v},\boldsymbol{e}_\ell \rangle|^2 \,\mathrm{d}w \leqslant \|\boldsymbol{v}\|^2 T.$$

Hence for each $T \in \mathbb{R}_+$, we obtain

$$\limsup_{t \to \infty} \|\boldsymbol{\Delta}_{t,T}^{(1)}\| = 0$$

almost surely. Moreover, for each $t, T \in \mathbb{R}_+$, we have

$$\|\boldsymbol{\Delta}_{t,T}^{(2)}\| \leq \frac{2}{t+T} \left(\sup_{\tau \in [T,\infty)} \|\mathbf{e}^{-s(\widetilde{\boldsymbol{B}})\tau} \boldsymbol{X}_{\tau} - w_{\boldsymbol{u},\boldsymbol{X}_{0}} \widetilde{\boldsymbol{u}}\| \right) \sum_{\ell=1}^{d} c_{\ell} \int_{T}^{t+T} \|\boldsymbol{f}(t+T-\tau,\boldsymbol{e}_{\ell})\| \,\mathrm{d}\tau$$

almost surely, where

$$\int_{T}^{t+T} \|\boldsymbol{f}(t+T-\tau,\boldsymbol{e}_{\ell})\| \,\mathrm{d}\tau = \int_{0}^{t} \|\boldsymbol{f}(w,\boldsymbol{e}_{\ell})\| \,\mathrm{d}w \leqslant \int_{0}^{t} |\mathrm{e}^{\mathrm{i}\mathrm{Im}(\lambda)w} \langle \boldsymbol{v},\boldsymbol{e}_{\ell} \rangle|^{2} \,\mathrm{d}w \leqslant \|\boldsymbol{v}\|^{2} t.$$

Consequently, for each $T \in \mathbb{R}_+$, we obtain

$$\begin{split} & \limsup_{t \to \infty} \left\| t^{-1} \boldsymbol{Q}(t) [\boldsymbol{M}^{(2)}]_t \boldsymbol{Q}(t)^\top - \frac{2w_{\boldsymbol{u}, \boldsymbol{X}_0}}{t} \sum_{\ell=1}^d c_\ell \langle \boldsymbol{e}_\ell, \widetilde{\boldsymbol{u}} \rangle \int_0^t \boldsymbol{f}(w, \boldsymbol{e}_\ell) \, \mathrm{d}w \right\| \\ &= \limsup_{t \to \infty} \left\| (t+T)^{-1} \boldsymbol{Q}(t+T) [\boldsymbol{M}^{(2)}]_{t+T} \boldsymbol{Q}(t+T)^\top - \frac{2w_{\boldsymbol{u}, \boldsymbol{X}_0}}{t+T} \sum_{\ell=1}^d c_\ell \langle \boldsymbol{e}_\ell, \widetilde{\boldsymbol{u}} \rangle \int_0^{t+T} \boldsymbol{f}(w, \boldsymbol{e}_\ell) \, \mathrm{d}w \right\| \\ &\leqslant \limsup_{t \to \infty} \left\| \boldsymbol{\Delta}_{t, T}^{(1)} \right\| + \limsup_{t \to \infty} \left\| \boldsymbol{\Delta}_{t, T}^{(2)} \right\| \\ &\leqslant 2 \| \boldsymbol{v} \|^2 \bigg(\sup_{\tau \in [T, \infty)} \left\| \mathrm{e}^{-s(\widetilde{B})\tau} \boldsymbol{X}_\tau - w_{\boldsymbol{u}, \boldsymbol{X}_0} \widetilde{\boldsymbol{u}} \right\| \bigg) \sum_{\ell=1}^d c_\ell \end{split}$$

almost surely. Letting $T \to \infty$, by Theorem 3.3 in Barczy et al. [8], we obtain (4.26). The aim of the following discussion is to show

(4.27)
$$\frac{1}{t} \int_0^t \boldsymbol{f}(w, \boldsymbol{e}_\ell) \, \mathrm{d}w \to \frac{1}{2} |\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle|^2 \boldsymbol{I}_2 + \frac{1}{2} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle^2) & \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle^2) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle^2) & -\operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle^2) \end{pmatrix} \mathbb{1}_{\{\operatorname{Im}(\lambda)=0\}}$$

as $t \to \infty$. Applying (4.15) for $a = e^{i \operatorname{Im}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle$, we obtain

$$\boldsymbol{f}(w, \boldsymbol{e}_{\ell}) = \frac{1}{2} |\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle|^{2} \boldsymbol{I}_{2} + \frac{1}{2} \begin{pmatrix} \operatorname{Re}((\mathrm{e}^{\mathrm{iIm}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle)^{2}) & \operatorname{Im}((\mathrm{e}^{\mathrm{iIm}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle)^{2}) \\ \operatorname{Im}((\mathrm{e}^{\mathrm{iIm}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle)^{2}) & -\operatorname{Re}((\mathrm{e}^{\mathrm{iIm}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle)^{2}) \end{pmatrix}.$$

Thus, if $Im(\lambda) = 0$, then we have

$$\frac{1}{t} \int_0^t \boldsymbol{f}(w, \boldsymbol{e}_\ell) \, \mathrm{d}w = \frac{1}{2} |\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle|^2 \boldsymbol{I}_2 + \frac{1}{2} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle)^2) & \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle)^2) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle)^2) & -\operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle)^2) \end{pmatrix}$$

for all $t \in \mathbb{R}_+$. If $\operatorname{Im}(\lambda) \neq 0$, then we have

$$\begin{aligned} \frac{1}{t} \int_0^t \operatorname{Re}((\mathrm{e}^{\mathrm{iIm}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle)^2) \, \mathrm{d}w &= \frac{1}{t} \operatorname{Re}\left(\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle^2 \int_0^t \mathrm{e}^{2\mathrm{iIm}(\lambda)w} \, \mathrm{d}w\right) \\ &= \frac{1}{t} \operatorname{Re}\left(\frac{\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle^2}{2\mathrm{iIm}(\lambda)} (\mathrm{e}^{2\mathrm{iIm}(\lambda)t} - 1)\right) \leqslant \frac{\|\boldsymbol{v}\|^2}{|\mathrm{Im}(\lambda)|t} \to 0 \end{aligned}$$

as $t \to \infty$, and, in a similar way, $\frac{1}{t} \int_0^t \operatorname{Im}((e^{\operatorname{iIm}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle)^2) dt \to 0$ as $t \to \infty$. Hence $\frac{1}{t} \int_0^t \boldsymbol{f}(w, \boldsymbol{e}_\ell) dw \to \frac{1}{2} |\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle|^2 \boldsymbol{I}_2$ as $t \to \infty$, and we conclude (4.27).

Next, using Theorem 3.3 in Barczy et al. [8], we show that

(4.28)
$$t^{-1}\boldsymbol{Q}(t)[\boldsymbol{M}^{(3,4)}]_{t}\boldsymbol{Q}(t)^{\top} - t^{-1}w_{\boldsymbol{u},\boldsymbol{X}_{0}}\sum_{\ell=1}^{d} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \int_{0}^{t} \int_{\mathcal{U}_{d}} \boldsymbol{f}(w,\boldsymbol{z}) \,\mathrm{d}w \,\mu_{\ell}(\mathrm{d}\boldsymbol{z}) \xrightarrow{L_{1}} \boldsymbol{0}$$

as $t \to \infty$. By the help of (4.7), it is enough to show that for each $\ell \in \{1, \ldots, d\}$ and $i, j \in \{1, 2\}$, we have

(4.29)
$$t^{-1} \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} f_{i,j}(t-u, \boldsymbol{z}) e^{-s(\tilde{\boldsymbol{B}})u} \mathbb{1}_{\{w \leq X_{u-,\ell}\}} N_{\ell}(\mathrm{d}u, \mathrm{d}\boldsymbol{z}, \mathrm{d}w) - t^{-1} w_{\boldsymbol{u}, \boldsymbol{X}_{0}} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \int_{0}^{t} \int_{\mathcal{U}_{d}} f_{i,j}(t-u, \boldsymbol{z}) \mathrm{d}u \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \xrightarrow{L_{1}} 0 \quad \text{as} \ t \to \infty.$$

For each $t \in \mathbb{R}_+$, $\ell \in \{1, \ldots, d\}$ and $i, j \in \{1, 2\}$, we use again the estimation (4.9). For each $\ell \in \{1, \ldots, d\}$ and $i, j \in \{1, 2\}$, as in (4.10), we have

$$((t+T)^{-1}I_{t,1})^{2} \leqslant (t+T)^{-2} \int_{0}^{t} \int_{\mathcal{U}_{d}} |\mathrm{e}^{\mathrm{i}\mathrm{Im}(\lambda)(t-u)} \langle \boldsymbol{v}, \boldsymbol{z} \rangle|^{4} \mathrm{e}^{-2s(\widetilde{\boldsymbol{B}})u} \mathbb{E}(||\boldsymbol{X}_{u}||) \,\mathrm{d}u \,\mu_{\ell}(\mathrm{d}\boldsymbol{z})$$
$$\leqslant C_{4} ||\boldsymbol{v}||^{4} (t+T)^{-2} \int_{\mathcal{U}_{d}} ||\boldsymbol{z}||^{4} \,\mu_{\ell}(\mathrm{d}\boldsymbol{z}) \int_{0}^{t} \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})u} \,\mathrm{d}u \to 0$$

as $t \to \infty$, since $\int_0^t e^{-s(\tilde{B})u} du \leq \int_0^\infty e^{-s(\tilde{B})u} du = \frac{1}{s(\tilde{B})}$ for every $t \in \mathbb{R}_+$, and $\int_{\mathcal{U}_d} \|\boldsymbol{z}\|^4 \mu_\ell(d\boldsymbol{z}) < \infty$. For each $t \in \mathbb{R}_+$ and $T \in \mathbb{R}_+$, we use again the decomposition (4.11). Similarly as in (4.4), for any $T \in \mathbb{R}_+$,

$$(t+T)^{-1}J_{t,T}^{(1)} \leq \frac{K}{t+T} \int_0^T \int_{\mathcal{U}_d} |f_{i,j}(t+T-\tau, \boldsymbol{z})| \,\mathrm{d}\tau \,\mu_\ell(\mathrm{d}\boldsymbol{z})$$
$$\leq \frac{K}{t+T} \int_0^T \int_{\mathcal{U}_d} |\langle \boldsymbol{v}, \boldsymbol{z} \rangle|^2 \,\mathrm{d}\tau \,\mu_\ell(\mathrm{d}\boldsymbol{z}) \leq \frac{K \|\boldsymbol{v}\|^2 T}{t+T} \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^2 \,\mu_\ell(\mathrm{d}\boldsymbol{z}) \to 0$$

as $t \to \infty$. Further, similarly as in (4.5), for each $t, T \in \mathbb{R}_+$,

$$\frac{J_{t,T}^{(2)}}{t+T} \leqslant \frac{1}{t+T} \sup_{\tau \in [T,\infty)} \mathbb{E}(|e^{-s(\widetilde{\boldsymbol{B}})\tau} X_{\tau,\ell} - w_{\boldsymbol{u},\boldsymbol{X}_0} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle|) \int_{T}^{t+T} \int_{\mathcal{U}_d} |f_{i,j}(t+T-\tau,\boldsymbol{z})| \,\mathrm{d}\tau \,\mu_{\ell}(\mathrm{d}\boldsymbol{z}) \\
\leqslant \frac{\|\boldsymbol{v}\|^2 t}{t+T} \sup_{\tau \in [T,\infty)} \mathbb{E}(|e^{-s(\widetilde{\boldsymbol{B}})\tau} X_{\tau,\ell} - w_{\boldsymbol{u},\boldsymbol{X}_0} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle|) \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^2 \,\mu_{\ell}(\mathrm{d}\boldsymbol{z}).$$

Consequently, for each $T \in \mathbb{R}_+$, we obtain

$$\limsup_{t \to \infty} t^{-1} I_{t,2} = \limsup_{t \to \infty} (t+T)^{-1} I_{t+T,2} \leq \limsup_{t \to \infty} (t+T)^{-1} J_{t,T}^{(1)} + \limsup_{t \to \infty} (t+T)^{-1} J_{t,T}^{(2)}$$
$$\leq \|\boldsymbol{v}\|^2 \sup_{\tau \in [T,\infty)} \mathbb{E}(|\mathrm{e}^{-s(\tilde{\boldsymbol{B}})\tau} X_{\tau,\ell} - w_{\boldsymbol{u},\boldsymbol{X}_0} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle|) \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^2 \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}).$$

Letting $T \to \infty$, by Theorem 3.3 in Barczy et al. [8], we have $\lim_{t\to\infty} t^{-1}I_{t,2} = 0$, as desired. All in all, $\lim_{t\to\infty} t^{-1}(I_{t,1} + I_{t,2}) = 0$, yielding (4.29). As in case of (4.27), one can derive

(4.30)

$$\frac{1}{t} \int_{0}^{t} \int_{\mathcal{U}_{d}} \boldsymbol{f}(w, \boldsymbol{z}) \, \mathrm{d}w \, \mu_{\ell}(\boldsymbol{z}) \to \frac{1}{2} \int_{\mathcal{U}_{d}} |\langle \boldsymbol{v}, \boldsymbol{z} \rangle|^{2} \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \boldsymbol{I}_{2} + \frac{1}{2} \begin{pmatrix} \operatorname{Re}(\int_{\mathcal{U}_{d}} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^{2} \, \mu_{\ell}(\mathrm{d}\boldsymbol{z})) & \operatorname{Im}(\int_{\mathcal{U}_{d}} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^{2} \, \mu_{\ell}(\mathrm{d}\boldsymbol{z})) \\ \operatorname{Im}(\int_{\mathcal{U}_{d}} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^{2} \, \mu_{\ell}(\mathrm{d}\boldsymbol{z})) & -\operatorname{Re}(\int_{\mathcal{U}_{d}} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^{2} \, \mu_{\ell}(\mathrm{d}\boldsymbol{z})) \end{pmatrix} \mathbb{1}_{\{\operatorname{Im}(\lambda)=0\}}$$

as $t \to \infty$. Indeed, we can apply (4.15) for $a = e^{i \operatorname{Im}(\lambda) w} \langle \boldsymbol{v}, \boldsymbol{z} \rangle$. In case of $\operatorname{Im}(\lambda) = 0$, we obtain

$$\begin{split} \frac{1}{t} \int_0^t \int_{\mathcal{U}_d} \boldsymbol{f}(w, \boldsymbol{z}) \, \mathrm{d}w \, \mu_\ell(\mathrm{d}\boldsymbol{z}) &= \frac{1}{2} \int_{\mathcal{U}_d} |\langle \boldsymbol{v}, \boldsymbol{z} \rangle|^2 \, \mu_\ell(\mathrm{d}\boldsymbol{z}) \boldsymbol{I}_2 \\ &+ \frac{1}{2} \begin{pmatrix} \mathrm{Re} \big(\int_{\mathcal{U}_d} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^2 \, \mu_\ell(\mathrm{d}\boldsymbol{z}) \big) & \mathrm{Im} \big(\int_{\mathcal{U}_d} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^2 \, \mu_\ell(\mathrm{d}\boldsymbol{z}) \big) \\ \mathrm{Im} \big(\int_{\mathcal{U}_d} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^2 \, \mu_\ell(\mathrm{d}\boldsymbol{z}) \big) & -\mathrm{Re} \big(\int_{\mathcal{U}_d} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^2 \, \mu_\ell(\mathrm{d}\boldsymbol{z}) \big) \end{pmatrix} \end{split}$$

for all $t \in \mathbb{R}_+$. If $\operatorname{Im}(\lambda) \neq 0$, then we have

$$\frac{1}{t} \int_0^t \int_{\mathcal{U}_d} \operatorname{Re}((\mathrm{e}^{\mathrm{iIm}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{z} \rangle)^2) \, \mathrm{d}w \, \mu_\ell(\mathrm{d}\boldsymbol{z}) = \frac{1}{t} \operatorname{Re}\left(\int_{\mathcal{U}_d} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^2 \, \mu_\ell(\mathrm{d}\boldsymbol{z}) \int_0^t \mathrm{e}^{2\mathrm{iIm}(\lambda)w} \, \mathrm{d}w\right)$$
$$= \frac{1}{t} \operatorname{Re}\left(\int_{\mathcal{U}_d} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^2 \, \mu_\ell(\mathrm{d}\boldsymbol{z}) \frac{\mathrm{e}^{2\mathrm{iIm}(\lambda)t} - 1}{2\mathrm{iIm}(\lambda)}\right) \leqslant \frac{\|\boldsymbol{v}\|^2}{|\mathrm{Im}(\lambda)|t} \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^2 \, \mu_\ell(\mathrm{d}\boldsymbol{z}) \to 0$$

as $t \to \infty$, and, in a similar way, $\frac{1}{t} \int_0^t \int_{\mathcal{U}_d} \operatorname{Im}((\mathrm{e}^{\mathrm{iIm}(\lambda)w} \langle \boldsymbol{v}, \boldsymbol{z} \rangle)^2) \, \mathrm{d}w \, \mu_\ell(\mathrm{d}\boldsymbol{z}) \to 0$ as $t \to \infty$. Hence $\frac{1}{t} \int_0^t \int_{\mathcal{U}_d} \boldsymbol{f}(w, \boldsymbol{z}) \, \mathrm{d}w \, \mu_\ell(\mathrm{d}\boldsymbol{z}) \to \frac{1}{2} \int_{\mathcal{U}_d} |\langle \boldsymbol{v}, \boldsymbol{z} \rangle|^2 \, \mu_\ell(\mathrm{d}\boldsymbol{z}) \boldsymbol{I}_2$ as $t \to \infty$, and we conclude (4.30).

Further,

$$t^{-1}\boldsymbol{Q}(t)[\boldsymbol{M}^{(5)}]_{t}\boldsymbol{Q}(t)^{\top} = t^{-1} \int_{0}^{t} \int_{\mathcal{U}_{d}} \boldsymbol{f}(t-u,\boldsymbol{r}) \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})u} M(\mathrm{d}u,\mathrm{d}\boldsymbol{r}) \xrightarrow{L_{1}} \boldsymbol{0}$$

as $t \to \infty$, since $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{B}) > 0$ implies $\operatorname{Re}(\lambda) \neq 0$, and hence

(4.31)

$$t^{-1} \mathbb{E}(\|\boldsymbol{Q}(t)[\boldsymbol{M}^{(5)}]_{t}\boldsymbol{Q}(t)^{\top}\|) \leq t^{-1} \mathbb{E}\left(\int_{0}^{t} \int_{\mathcal{U}_{d}} \|\boldsymbol{f}(t-u,\boldsymbol{r})e^{-s(\tilde{\boldsymbol{B}})u}\| M(\mathrm{d}u,\mathrm{d}\boldsymbol{r})\right)$$

$$= t^{-1} \int_{0}^{t} \int_{\mathcal{U}_{d}} \|\boldsymbol{f}(t-u,\boldsymbol{r})e^{-s(\tilde{\boldsymbol{B}})u}\| \mathrm{d}u\,\nu(\mathrm{d}\boldsymbol{r})$$

$$\leq t^{-1} \int_{0}^{t} \int_{\mathcal{U}_{d}} |\langle \boldsymbol{v},\boldsymbol{r}\rangle|^{2} e^{-s(\tilde{\boldsymbol{B}})u} \mathrm{d}u\,\nu(\mathrm{d}\boldsymbol{r})$$

$$\leq \|\boldsymbol{v}\|^{2} t^{-1} \int_{0}^{t} e^{-s(\tilde{\boldsymbol{B}})u} \mathrm{d}u \int_{\mathcal{U}_{d}} \|\boldsymbol{r}\|^{2}\,\nu(\mathrm{d}\boldsymbol{r}) \to 0$$

as $t \to \infty$. Consequently, by (4.26), (4.27), (4.28), (4.30) and (4.31), we get

$$t^{-1}\boldsymbol{Q}(t)[\boldsymbol{M}]_t\boldsymbol{Q}(t)^\top \xrightarrow{\mathbb{P}} w_{\boldsymbol{u},\boldsymbol{X}_0}\boldsymbol{\Sigma}_{\boldsymbol{v}} \quad \text{as} \ t \to \infty.$$

Step 3. Now we turn to prove that condition (E.2) of Theorem E.1 holds, namely,

$$\mathbb{E}\left(\sup_{u\in[0,t]}t^{-1/2}\|\boldsymbol{Q}(t)(\boldsymbol{M}_u-\boldsymbol{M}_{u-})\|\right)\to 0 \quad \text{as} \ t\to\infty.$$

By (4.16) and $\|\boldsymbol{Q}(t)\| = 1, t \in \mathbb{R}_+$, it is enough to show that

(4.32)
$$t^{-1/2} \mathbb{E}\left(\sup_{u \in [0,t]} \|\widetilde{\boldsymbol{Y}}_{u}^{(\ell)} - \widetilde{\boldsymbol{Y}}_{u-}^{(\ell)}\|\right) \to 0 \quad \text{as} \ t \to \infty$$

for every $\ell \in \{1, \ldots, d\}$, and

(4.33)
$$t^{-1/2} \mathbb{E}\left(\sup_{u \in [0,t]} \|\boldsymbol{M}_{u}^{(5)} - \boldsymbol{M}_{u-}^{(5)}\|\right) \to 0 \quad \text{as} \quad t \to \infty.$$

First, we prove (4.32) for each $\ell \in \{1, \ldots, d\}$. By (4.19), it is enough to prove that for all $\varepsilon \in \mathbb{R}_{++}$,

$$t^{-2} \mathbb{E}\left(\sup_{u \in [0,t]} |\widetilde{\boldsymbol{Y}}_{u}^{(\ell,\varepsilon)} - \widetilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)}|^{4}\right) \to 0 \quad \text{as} \quad t \to \infty$$

and

$$\limsup_{\varepsilon \downarrow 0} \limsup_{t \to \infty} t^{-1} \mathbb{E} \left(\sup_{u \in [0,t]} \| (\widetilde{\boldsymbol{Y}}_{u}^{(\ell)} - \widetilde{\boldsymbol{Y}}_{u}^{(\ell,\varepsilon)}) - (\widetilde{\boldsymbol{Y}}_{u-}^{(\ell)} - \widetilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)}) \|^{2} \right) = 0.$$

By (4.20), for all $\varepsilon \in \mathbb{R}_{++}$, we get

$$t^{-2} \mathbb{E} \left(\sup_{u \in [0,t]} |\widetilde{\boldsymbol{Y}}_{u}^{(\ell,\varepsilon)} - \widetilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)}|^{4} \right) = t^{-2} \mathbb{E} \left(\sup_{u \in [0,t]} |\widetilde{Y}_{u}^{(\ell,\varepsilon)} - \widetilde{Y}_{u-}^{(\ell,\varepsilon)}|^{4} \right)$$
$$\leq C_{4} \|\boldsymbol{v}\|^{4} t^{-2} \int_{0}^{t} e^{-s(\widetilde{\boldsymbol{B}})u} du \int_{\mathcal{U}_{d}} \|\boldsymbol{z}\|^{4} \mathbb{1}_{\{\|\boldsymbol{z}\| \ge \varepsilon\}} \mu_{\ell}(d\boldsymbol{z}) \to 0$$

as $t \to \infty$. Further, by (4.22), for all $t \in \mathbb{R}_{++}$,

$$t^{-1} \mathbb{E} \left(\sup_{u \in [0,t]} \| (\widetilde{\boldsymbol{Y}}_{u}^{(\ell)} - \widetilde{\boldsymbol{Y}}_{u}^{(\ell,\varepsilon)}) - (\widetilde{\boldsymbol{Y}}_{u-}^{(\ell)} - \widetilde{\boldsymbol{Y}}_{u-}^{(\ell,\varepsilon)}) \|^{2} \right) \leqslant C_{4} \| \boldsymbol{v} \|^{2} \int_{\mathcal{U}_{d}} \| \boldsymbol{z} \|^{2} \mathbb{1}_{\{\| \boldsymbol{z} \| < \varepsilon\}} \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \to 0$$

as $\varepsilon \downarrow 0$ due to (3.8). Hence we conclude (4.32) for each $\ell \in \{1, \ldots, d\}$.

Next, we prove (4.33). By (4.23), it is enough to prove that

$$t^{-1} \mathbb{E}\left(\sup_{u \in [0,t]} |Z_u^{(5)} - Z_{u-}^{(5)}|^2\right) \to 0 \quad \text{as} \quad t \to \infty.$$

By (4.24), we get

$$t^{-1} \mathbb{E}\left(\sup_{u \in [0,t]} |Z_u^{(5)} - Z_{u-}^{(5)}|^2\right) \leqslant t^{-1} \|\boldsymbol{v}\|^2 \int_0^t e^{-s(\tilde{\boldsymbol{B}})u} du \int_{\mathcal{U}_d} \|\boldsymbol{r}\|^2 \nu(d\boldsymbol{r}) \to 0$$

as $t \to \infty$, hence we conclude (4.33). Consequently, by Theorem E.1, we obtain

$$t^{-1/2} \boldsymbol{Q}(t) \boldsymbol{M}_t \xrightarrow{\mathcal{D}} (w_{\boldsymbol{u}, \boldsymbol{X}_0} \boldsymbol{\Sigma}_{\boldsymbol{v}})^{1/2} \boldsymbol{N} \quad \text{as} \ t \to \infty,$$

where N is a 2-dimensional random vector with $N \stackrel{\mathcal{D}}{=} \mathcal{N}_2(\mathbf{0}, \mathbf{I}_2)$ independent of $w_{u, \mathbf{X}_0} \Sigma_{v}$. Clearly, $(w_{u, \mathbf{X}_0} \Sigma_{v})^{1/2} N = \sqrt{w_{u, \mathbf{X}_0}} \Sigma_{v}^{1/2} N \stackrel{\mathcal{D}}{=} \sqrt{w_{u, \mathbf{X}_0}} \mathbf{Z}_{v}$. By the decomposition

$$t^{-1/2} \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t/2} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} = t^{-1/2} \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{\mathrm{iIm}(\lambda)t} Z_t^{(0,1)}) \\ \operatorname{Im}(\mathrm{e}^{\mathrm{iIm}(\lambda)t} Z_t^{(0,1)}) \end{pmatrix} + t^{-1/2} \boldsymbol{Q}(t) \boldsymbol{M}_t, \qquad t \in \mathbb{R}_+,$$

the convergence (4.25) and Slutsky's lemma, we obtain (3.4).

Proof of Theorem 3.2. First, suppose that the conditions (i) and (ii) hold. In the special case of $\mathbf{X}_0 \stackrel{\text{a.s.}}{=} \mathbf{0}$, applying Lemma A.1 with T = 1, we have $\mathbf{X}_{t+1} \stackrel{\mathcal{D}}{=} \mathbf{X}_t^{(1)} + \mathbf{X}_t^{(2,1)}$ for each $t \in \mathbb{R}_+$, where $(\mathbf{X}_s^{(1)})_{s \in \mathbb{R}_+}$ and $(\mathbf{X}_s^{(2,1)})_{s \in \mathbb{R}_+}$ are independent multi-type CBI processes with $\mathbf{X}_0^{(1)} \stackrel{\text{a.s.}}{=} \mathbf{0}$, $\mathbf{X}_0^{(2,1)} \stackrel{\mathcal{D}}{=} \mathbf{X}_1$, and with parameters $(d, c, \beta, B, \nu, \mu)$ and $(d, c, \mathbf{0}, B, 0, \mu)$, respectively. Without loss of generality, we may and do suppose that $(\mathbf{X}_s)_{s \in \mathbb{R}_+}$, $(\mathbf{X}_s^{(1)})_{s \in \mathbb{R}_+}$ are independent. Then, for each $t \in \mathbb{R}_+$, we have $e^{-\lambda(t+1)} \langle \mathbf{v}, \mathbf{X}_{t+1} \rangle \stackrel{\mathcal{D}}{=} e^{-\lambda} (e^{-\lambda t} \langle \mathbf{v}, \mathbf{X}_t^{(1)} \rangle) + e^{-\lambda} (e^{-\lambda t} \langle \mathbf{v}, \mathbf{X}_t^{(2,1)} \rangle)$. By (3.2), we obtain $w_{\mathbf{v},\mathbf{0}} \stackrel{\mathcal{D}}{=} e^{-\lambda} w_{\mathbf{v},\mathbf{0}}^{(1)} + e^{-\lambda} w_{\mathbf{v},\mathbf{0}}^{(2,1)}$, where $w_{\mathbf{v},\mathbf{0}}^{(1)}$ and $w_{\mathbf{v},\mathbf{0}_0^{(2,1)}}^{(2,1)}$ denote the almost sure limit of $e^{-\lambda t} \langle \mathbf{v}, \mathbf{X}_t^{(1)} \rangle$ and $e^{-\lambda t} \langle \mathbf{v}, \mathbf{X}_t^{(2,1)} \rangle$ as $t \to \infty$, respectively. Since, for each $t \in \mathbb{R}_+$, we have $\mathbf{X}_t^{(1)} \stackrel{\mathcal{D}}{=} \mathbf{X}_t$, we conclude $w_{\mathbf{v},\mathbf{0}}^{(1)} \stackrel{\mathcal{D}}{=} w_{\mathbf{v},\mathbf{0}}$.

The independence of $(\mathbf{X}_s)_{s \in \mathbb{R}_+}$ and $(\mathbf{X}_s^{(2,1)})_{s \in \mathbb{R}_+}$ implies the independence of $w_{v,0}$ and $w_{v,\mathbf{X}_0^{(2,1)}}^{(2,1)}$, hence $w_{v,0} \stackrel{\mathcal{D}}{=} e^{-\lambda} w_{v,0} + e^{-\lambda} w_{v,\mathbf{X}_0^{(2,1)}}^{(2,1)}$. Taking the real and imaginary parts, we get

$$\begin{pmatrix} \operatorname{Re}(w_{\boldsymbol{v},\boldsymbol{0}}) \\ \operatorname{Im}(w_{\boldsymbol{v},\boldsymbol{0}}) \end{pmatrix} \stackrel{\mathcal{D}}{=} \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{-\lambda}w_{\boldsymbol{v},\boldsymbol{0}}) \\ \operatorname{Im}(\mathrm{e}^{-\lambda}w_{\boldsymbol{v},\boldsymbol{0}}) \end{pmatrix} + \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{-\lambda}w_{\boldsymbol{v},\boldsymbol{0}}^{(2,1)}) \\ \operatorname{Im}(\mathrm{e}^{-\lambda}w_{\boldsymbol{v},\boldsymbol{0}}^{(2,1)}) \end{pmatrix} \\ = \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{-\lambda}) & -\operatorname{Im}(\mathrm{e}^{-\lambda}) \\ \operatorname{Im}(\mathrm{e}^{-\lambda}) & \operatorname{Re}(\mathrm{e}^{-\lambda}) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(w_{\boldsymbol{v},\boldsymbol{0}}) \\ \operatorname{Im}(w_{\boldsymbol{v},\boldsymbol{0}}) \end{pmatrix} + \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{-\lambda}) & -\operatorname{Im}(\mathrm{e}^{-\lambda}) \\ \operatorname{Im}(\mathrm{e}^{-\lambda}) & \operatorname{Re}(\mathrm{e}^{-\lambda}) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(w_{\boldsymbol{v},\boldsymbol{0}}) \\ \operatorname{Im}(w_{\boldsymbol{v},\boldsymbol{0}}) \end{pmatrix} \\ =: \boldsymbol{A} \begin{pmatrix} \operatorname{Re}(w_{\boldsymbol{v},\boldsymbol{0}}) \\ \operatorname{Im}(w_{\boldsymbol{v},\boldsymbol{0}}) \end{pmatrix} + \boldsymbol{AC}, \end{cases}$$

which is a 2-dimensional stochastic fixed point equation. We are going to apply Corollary C.2. We have $\det(\mathbf{A}) = (\operatorname{Re}(e^{-\lambda}))^2 + (\operatorname{Im}(e^{-\lambda}))^2 = |e^{-\lambda}|^2 = e^{-2\operatorname{Re}(\lambda)} \neq 0$. The eigenvalues of the matrix \boldsymbol{A} are $e^{-\lambda}$ and $e^{-\overline{\lambda}}$, hence the spectral radius of \boldsymbol{A} is $r(\boldsymbol{A}) = e^{-\operatorname{Re}(\lambda)} \in (0,1)$. Next we check that AC is not deterministic. Suppose that, on the contrary, AC is deterministic. Then $w_{\boldsymbol{v},\boldsymbol{X}_0^{(2,1)}}^{(2,1)}$ is deterministic, since \boldsymbol{A} is invertible. By Lemma 2.6 in Barczy et al. [8], the process $(e^{-s\tilde{B}}X_s^{(2,1)})_{s\in\mathbb{R}_+}$ is a *d*-dimensional martingale with respect to the filtration $\mathcal{F}_{s}^{\mathbf{X}^{(2,1)}} := \sigma(\mathbf{X}_{u}^{(2,1)} : u \in [0,s]), \ s \in \mathbb{R}_{+}, \ \text{hence} \ (e^{-\lambda s} \langle \boldsymbol{v}, \mathbf{X}_{s}^{(2,1)} \rangle)_{s \in \mathbb{R}_{+}} \ \text{is a complex}$ martingale with respect to the same filtration. By (3.2), we have $e^{-\lambda s} \langle \boldsymbol{v}, \mathbf{X}_{s}^{(2,1)} \rangle \rightarrow w_{\mathbf{v}, \mathbf{X}_{0}^{(2,1)}}^{(2,1)}$ as $s \to \infty$ in L_1 and almost surely, hence $\langle \boldsymbol{v}, \boldsymbol{X}_0^{(2,1)} \rangle = \mathbb{E}(w_{\boldsymbol{v}, \boldsymbol{X}_0^{(2,1)}}^{(2,1)} | \mathcal{F}_0^{\boldsymbol{X}^{(2,1)}}) = w_{\boldsymbol{v}, \boldsymbol{X}_0^{(2,1)}}^{(2,1)}$ almost surely, see, e.g., Karatzas and Shreve [19, Chapter I, Problem 3.20]. Thus $\langle \boldsymbol{v}, \boldsymbol{X}_{0}^{(2,1)} \rangle$ is deterministic as well. Then $\langle \boldsymbol{v}, \boldsymbol{X}_1 \rangle$ is also deterministic since $\boldsymbol{X}_1 \stackrel{\mathcal{D}}{=} \boldsymbol{X}_0^{(2,1)}$. However, applying Lemma B.1 for the process $(X_s)_{s \in \mathbb{R}_+}$, we obtain that $\langle v, X_1 \rangle$ is not deterministic, since the condition (i) of this theorem implies that the process $(X_s)_{s \in \mathbb{R}_+}$ is non-trivial, and the condition (ii) of this theorem yields that the condition (ii)/(b) of Lemma B.1 does not hold. Thus we get a contradiction, and we conclude that AC is not deterministic. Moreover, we have $\mathbb{E}(\|\boldsymbol{C}\|) = \mathbb{E}(|w_{\boldsymbol{v},\boldsymbol{X}_0^{(2,1)}}^{(2,1)}|) < \infty$, see (3.2). Applying Corollary C.2, we conclude that the distribution of $w_{v,0}$ does not have atoms. In particular, we obtain $\mathbb{P}(w_{v,0}=0)=0$.

If the conditions (i) and (ii) hold, but $\mathbf{X}_0 \stackrel{\text{a.s.}}{=} \mathbf{0}$ does not necessarily holds, then we apply Lemma A.1 with T = 0, and we obtain that $\mathbf{X}_t \stackrel{\mathcal{D}}{=} \mathbf{X}_t^{(1)} + \mathbf{X}_t^{(2,0)}$ for each $t \in \mathbb{R}_+$, where $(\mathbf{X}_s^{(1)})_{s \in \mathbb{R}_+}$ and $(\mathbf{X}_s^{(2,0)})_{s \in \mathbb{R}_+}$ are independent multi-type CBI processes with $\mathbf{X}_0^{(1)} \stackrel{\text{a.s.}}{=} \mathbf{0}$, $\mathbf{X}_0^{(2,0)} \stackrel{\mathcal{D}}{=} \mathbf{X}_0$, and with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ and $(d, \mathbf{c}, \mathbf{0}, \mathbf{B}, 0, \boldsymbol{\mu})$, respectively. Then, for each $t \in \mathbb{R}_+$, we have $e^{-\lambda t} \langle \mathbf{v}, \mathbf{X}_t \rangle \stackrel{\mathcal{D}}{=} e^{-\lambda t} \langle \mathbf{v}, \mathbf{X}_t^{(1)} \rangle + e^{-\lambda t} \langle \mathbf{v}, \mathbf{X}_t^{(2,0)} \rangle$. By (3.2), we obtain $w_{\mathbf{v},\mathbf{X}_0} \stackrel{\mathcal{D}}{=} w_{\mathbf{v},\mathbf{0}}^{(1)} + w_{\mathbf{v},\mathbf{X}_0^{(2,0)}}^{(2,0)}$, where $w_{\mathbf{v},\mathbf{0}}^{(1)}$ and $w_{\mathbf{v},\mathbf{X}_0^{(2,0)}}^{(2,0)}$ denotes the almost sure limit of $e^{-\lambda s} \langle \mathbf{v}, \mathbf{X}_s^{(1)} \rangle$ and of $e^{-\lambda s} \langle \mathbf{v}, \mathbf{X}_s^{(2,0)} \rangle$ as $s \to \infty$, respectively. The independence of $(\mathbf{X}_s^{(1)})_{s \in \mathbb{R}_+}$ and $(\mathbf{X}_s^{(2,0)})_{s \in \mathbb{R}_+}$ implies the independence of $w_{\mathbf{v},\mathbf{0}}^{(1)}$ and $w_{\mathbf{v},\mathbf{X}_0}^{(2,0)}$. We have already shown that $w_{\mathbf{v},\mathbf{0}}^{(1)} \stackrel{\mathcal{D}}{=} w_{\mathbf{v},\mathbf{0}}$ does not have atoms, yielding that $w_{\mathbf{v},\mathbf{X}_0}$ does not have atoms, since for each $z \in \mathbb{C}$, we have

$$\mathbb{P}(w_{\boldsymbol{v},\boldsymbol{X}_{0}}=z) = \mathbb{P}\left(w_{\boldsymbol{v},\boldsymbol{0}}^{(1)}=z-w_{\boldsymbol{v},\boldsymbol{X}_{0}^{(2,0)}}^{(2,0)}\right) = \mathbb{E}\left(\mathbb{P}\left(w_{\boldsymbol{v},\boldsymbol{0}}^{(1)}=z-w_{\boldsymbol{v},\boldsymbol{X}_{0}^{(2,0)}}^{(2,0)} \mid w_{\boldsymbol{v},\boldsymbol{X}_{0}^{(2,0)}}^{(2,0)}\right)\right) = \mathbb{E}(0) = 0.$$

In particular, we obtain $\mathbb{P}(w_{\boldsymbol{v},\boldsymbol{X}_0}=0)=0.$

If the condition (ii) does not hold, then, as in part (ii) \Longrightarrow (iii) of the proof of Lemma B.1, we obtain that in the representation (B.1) of $e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$, the terms $Z_t^{(2)}$, $Z_t^{(3,4)}$, and $Z_t^{(5)}$ are 0 almost surely, so $e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle = \langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle + \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle \int_0^t e^{-\lambda u} du$ for all $t \in \mathbb{R}_+$ almost surely, and hence, taking the limit $t \to \infty$, we have $w_{\boldsymbol{v}, \boldsymbol{X}_0} = \langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle + \lambda^{-1} \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle$ almost surely.

If $\lambda = s(\mathbf{B})$, $\mathbf{v} = \mathbf{u}$ and the conditions (i) and (ii) hold, then we have already derived $\mathbb{P}(w_{\mathbf{u},\mathbf{X}_0} = 0) = 0$.

If $\lambda = s(\widetilde{\boldsymbol{B}})$, $\boldsymbol{v} = \boldsymbol{u}$ and the condition (i) holds but the condition (ii) does not hold, then we have already derived $\mathbb{P}(w_{\boldsymbol{u},\boldsymbol{X}_0} = 0) = \mathbb{P}(\langle \boldsymbol{u},\boldsymbol{X}_0 \rangle + s(\widetilde{\boldsymbol{B}})^{-1} \langle \boldsymbol{u},\widetilde{\boldsymbol{\beta}} \rangle = 0)$, and this probability is 0, since $\boldsymbol{u} \in \mathbb{R}^d_{++}$, $\mathbb{P}(\boldsymbol{X}_0 \in \mathbb{R}^d_+) = 1$, $s(\widetilde{\boldsymbol{B}}) \in \mathbb{R}_{++}$ and $\widetilde{\boldsymbol{\beta}} \in \mathbb{R}^d_+ \setminus \{\mathbf{0}\}$ yielding that $\langle \boldsymbol{u},\widetilde{\boldsymbol{\beta}} \rangle > 0$.

If $\lambda = s(\widetilde{\boldsymbol{B}})$, $\boldsymbol{v} = \boldsymbol{u}$ and the conditions (i) and (ii) do not hold, then we have already derived $\mathbb{P}(w_{\boldsymbol{u},\boldsymbol{X}_0} = 0) = \mathbb{P}(\langle \boldsymbol{u},\boldsymbol{X}_0 \rangle + s(\widetilde{\boldsymbol{B}})^{-1} \langle \boldsymbol{u},\widetilde{\boldsymbol{\beta}} \rangle = 0) = \mathbb{P}(\langle \boldsymbol{u},\boldsymbol{X}_0 \rangle = 0)$, and this equals $\mathbb{P}(\boldsymbol{X}_0 = \boldsymbol{0})$, since $\boldsymbol{u} \in \mathbb{R}^d_{++}$ and $\mathbb{P}(\boldsymbol{X}_0 \in \mathbb{R}^d_+) = 1$.

Proof of Lemma 3.3. Note that $w_{\boldsymbol{u},\boldsymbol{X}_0} \stackrel{\text{a.s.}}{=} 0$ if and only if $\mathbb{E}(w_{\boldsymbol{u},\boldsymbol{X}_0}) = 0$. By (3.2), we have $e^{-s(\widetilde{\boldsymbol{B}})t}\langle \boldsymbol{u},\boldsymbol{X}_t\rangle \stackrel{L_1}{\longrightarrow} w_{\boldsymbol{u},\boldsymbol{X}_0}$ as $t \to \infty$. By (2.2), we obtain $\mathbb{E}(\boldsymbol{X}_t) = e^{t\widetilde{\boldsymbol{B}}} \mathbb{E}(\boldsymbol{X}_0) + \int_0^t e^{u\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} \, \mathrm{d}u$, $t \in \mathbb{R}_+$, hence

$$\mathbb{E}(\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t}\langle\boldsymbol{u},\boldsymbol{X}_t\rangle) = \langle\boldsymbol{u},\mathbb{E}(\boldsymbol{X}_0)\rangle + \langle\boldsymbol{u},\widetilde{\boldsymbol{\beta}}\rangle \int_0^t \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})(t-u)} \,\mathrm{d}\boldsymbol{u} \to \langle\boldsymbol{u},\mathbb{E}(\boldsymbol{X}_0)\rangle + \frac{\langle\boldsymbol{u},\widetilde{\boldsymbol{\beta}}\rangle}{s(\widetilde{\boldsymbol{B}})} = \mathbb{E}(w_{\boldsymbol{u},\boldsymbol{X}_0})$$

as $t \to \infty$, where $\boldsymbol{u} \in \mathbb{R}^d_{++}$ and $s(\widetilde{\boldsymbol{B}}) > 0$, thus $\mathbb{E}(w_{\boldsymbol{u},\boldsymbol{X}_0}) = 0$ if and only if $\boldsymbol{X}_0 \stackrel{\text{a.s.}}{=} \boldsymbol{0}$ and $\widetilde{\boldsymbol{\beta}} = \boldsymbol{0}$.

Proof of part (i) of Theorem 3.4. By Theorem 3.3 in Barczy et al. [8], we have $e^{-s(\tilde{B})t} X_t \xrightarrow{a.s.} w_{u,X_0} \tilde{u}$ as $t \to \infty$, hence $\mathbb{1}_{\{X_t \neq 0\}} = \mathbb{1}_{\{e^{-s(\tilde{B})t}X_t \neq 0\}} \to 1$ as $t \to \infty$ on the event $\{w_{u,X_0} > 0\}$, since $\tilde{u} \in \mathbb{R}^d_{++}$. By (3.2), we have $e^{-s(\tilde{B})t} \langle u, X_t \rangle \xrightarrow{a.s.} w_{u,X_0}$ and $e^{-\lambda t} \langle v, X_t \rangle \xrightarrow{a.s.} w_{v,X_0}$ as $t \to \infty$. Using that $\langle u, X_t \rangle \neq 0$ if and only if $X_t \neq 0$, we have

$$\begin{split} & \mathbb{1}_{\{\boldsymbol{X}_{t}\neq\boldsymbol{0}\}} \frac{1}{\langle \boldsymbol{u},\boldsymbol{X}_{t}\rangle^{\operatorname{Re}(\lambda)/s(\widetilde{\boldsymbol{B}})}} \begin{pmatrix} \cos(\operatorname{Im}(\lambda)t) & \sin(\operatorname{Im}(\lambda)t) \\ -\sin(\operatorname{Im}(\lambda)t) & \cos(\operatorname{Im}(\lambda)t) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v},\boldsymbol{X}_{t}\rangle) \\ \operatorname{Im}(\langle \boldsymbol{v},\boldsymbol{X}_{t}\rangle) \end{pmatrix} \\ & = \frac{\mathbb{1}_{\{\boldsymbol{X}_{t}\neq\boldsymbol{0}\}}}{(\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t}\langle \boldsymbol{u},\boldsymbol{X}_{t}\rangle)^{\operatorname{Re}(\lambda)/s(\widetilde{\boldsymbol{B}})} \mathrm{e}^{\operatorname{Re}(\lambda)t}} \begin{pmatrix} \cos(\operatorname{Im}(\lambda)t) & \sin(\operatorname{Im}(\lambda)t) \\ -\sin(\operatorname{Im}(\lambda)t) & \cos(\operatorname{Im}(\lambda)t) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{\lambda t}\mathrm{e}^{-\lambda t}\langle \boldsymbol{v},\boldsymbol{X}_{t}\rangle) \\ \operatorname{Im}(\mathrm{e}^{\lambda t}\mathrm{e}^{-\lambda t}\langle \boldsymbol{v},\boldsymbol{X}_{t}\rangle) \end{pmatrix} \\ & = \frac{\mathbb{1}_{\{\boldsymbol{X}_{t}\neq\boldsymbol{0}\}}}{(\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t}\langle \boldsymbol{u},\boldsymbol{X}_{t}\rangle)^{\operatorname{Re}(\lambda)/s(\widetilde{\boldsymbol{B}})}} \begin{pmatrix} \operatorname{Re}(\mathrm{e}^{-\lambda t}\langle \boldsymbol{v},\boldsymbol{X}_{t}\rangle) \\ \operatorname{Im}(\mathrm{e}^{-\lambda t}\langle \boldsymbol{v},\boldsymbol{X}_{t}\rangle) \end{pmatrix} \rightarrow \frac{1}{w_{\mathbf{u},\mathbf{X}_{0}}^{\operatorname{Re}(\lambda)/s(\widetilde{\boldsymbol{B}})}} \begin{pmatrix} \operatorname{Re}(w_{\mathbf{v},\mathbf{X}_{0}}) \\ \operatorname{Im}(w_{\mathbf{v},\mathbf{X}_{0}}) \end{pmatrix} \end{split}$$

as $t \to \infty$ on the event $\{w_{\boldsymbol{u},\boldsymbol{X}_0} > 0\}$, as desired.

Proof of part (iii) of Theorem 3.4. First, note that the moment condition (3.3) yields the moment condition (3.1) with $\lambda = s(\tilde{B})$, so, by Lemma 3.3, $w_{u,X_0} \stackrel{\text{a.s.}}{=} 0$ if and only if $(X_t)_{t \in \mathbb{R}_+}$ is trivial. For each $t \in \mathbb{R}_+$, we have the decomposition

$$\mathbb{1}_{\{\boldsymbol{X}_t\neq\boldsymbol{0}\}}\frac{1}{\sqrt{\langle\boldsymbol{u},\boldsymbol{X}_t\rangle}}\begin{pmatrix}\operatorname{Re}(\langle\boldsymbol{v},\boldsymbol{X}_t\rangle)\\\operatorname{Im}(\langle\boldsymbol{v},\boldsymbol{X}_t\rangle)\end{pmatrix} = \mathbb{1}_{\{\boldsymbol{X}_t\neq\boldsymbol{0}\}}\frac{\sqrt{w_{\boldsymbol{u},\boldsymbol{X}_0}}}{\sqrt{\mathrm{e}^{-s(\tilde{\boldsymbol{B}})t}\langle\boldsymbol{u},\boldsymbol{X}_t\rangle}}\frac{\mathrm{e}^{-s(\tilde{\boldsymbol{B}})t/2}}{\sqrt{w_{\boldsymbol{u},\boldsymbol{X}_0}}}\begin{pmatrix}\operatorname{Re}(\langle\boldsymbol{v},\boldsymbol{X}_t\rangle)\\\operatorname{Im}(\langle\boldsymbol{v},\boldsymbol{X}_t\rangle)\end{pmatrix}$$

on the event $\{w_{\boldsymbol{u},\boldsymbol{X}_0} > 0\}$. As we have seen in the proof of part (i) of Theorem 3.4, we have $\mathbb{1}_{\{\boldsymbol{X}_t \neq \boldsymbol{0}\}} \to 1$ as $t \to \infty$ on the event $\{w_{\boldsymbol{u},\boldsymbol{X}_0} > 0\}$, and $e^{-s(\tilde{\boldsymbol{B}})t} \langle \boldsymbol{u},\boldsymbol{X}_t \rangle \xrightarrow{\text{a.s.}} w_{\boldsymbol{u},\boldsymbol{X}_0}$ as $t \to \infty$. In case of $\boldsymbol{\Sigma}_{\boldsymbol{v}} = \boldsymbol{0}$, (3.6) yields

$$\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t/2} \begin{pmatrix} \mathrm{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \mathrm{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \xrightarrow{\mathbb{P}} \mathbf{0} \qquad \text{as} \ t \to \infty,$$

hence, using the above decomposition, by Slutsky's lemma, we obtain

$$\mathbb{1}_{\{\boldsymbol{X}_t\neq\boldsymbol{0}\}}\frac{1}{\sqrt{\langle\boldsymbol{u},\boldsymbol{X}_t\rangle}}\begin{pmatrix}\operatorname{Re}(\langle\boldsymbol{v},\boldsymbol{X}_t\rangle)\\\operatorname{Im}(\langle\boldsymbol{v},\boldsymbol{X}_t\rangle)\end{pmatrix}\overset{\mathcal{D}_{\{\boldsymbol{w}_{\boldsymbol{u},\boldsymbol{X}_0}>0\}}}{\longrightarrow}\boldsymbol{0}\qquad\text{as}\quad t\to\infty$$

In case of $\Sigma_{v} \neq 0$, as in the proof of (3.4), we may apply Theorem E.1 to obtain

$$\begin{pmatrix} e^{-s(\tilde{\boldsymbol{B}})t/2} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix}, \sqrt{w_{\boldsymbol{u}, \boldsymbol{X}_0}} \end{pmatrix} \xrightarrow{\mathcal{D}} \left((w_{\boldsymbol{u}, \boldsymbol{X}_0} \boldsymbol{\Sigma}_{\boldsymbol{v}})^{1/2} \boldsymbol{N}, \sqrt{w_{\boldsymbol{u}, \boldsymbol{X}_0}} \right) \quad \text{as} \ t \to \infty,$$

where N is a 2-dimensional random vector with $N \stackrel{\mathcal{D}}{=} \mathcal{N}_2(\mathbf{0}, \mathbf{I}_2)$ independent of $w_{u, \mathbf{X}_0} \Sigma_{v}$, and hence independent of w_{u, \mathbf{X}_0} because $\Sigma_{v} \neq \mathbf{0}$ and Σ_{v} is deterministic. Applying the continuous mapping theorem, we get

$$\frac{\mathrm{e}^{-s(\tilde{\boldsymbol{B}})t/2}}{\sqrt{w_{\boldsymbol{u},\boldsymbol{X}_0}}} \begin{pmatrix} \mathrm{Re}(\langle \boldsymbol{v},\boldsymbol{X}_t \rangle) \\ \mathrm{Im}(\langle \boldsymbol{v},\boldsymbol{X}_t \rangle) \end{pmatrix} \xrightarrow{\mathcal{D}_{\{w_{\boldsymbol{u},\boldsymbol{X}_0}>0\}}} \boldsymbol{\Sigma}_{\boldsymbol{v}}^{1/2} \boldsymbol{N} \quad \text{as} \ t \to \infty.$$

Hence, using again the above decomposition, by Slutsky's lemma and (3.2),

$$\mathbb{1}_{\{\boldsymbol{X}_t\neq\boldsymbol{0}\}}\frac{1}{\sqrt{\langle \boldsymbol{u},\boldsymbol{X}_t\rangle}} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v},\boldsymbol{X}_t\rangle)\\ \operatorname{Im}(\langle \boldsymbol{v},\boldsymbol{X}_t\rangle) \end{pmatrix} \stackrel{\mathcal{D}_{\{\boldsymbol{w}_{\boldsymbol{u},\boldsymbol{X}_0}>\boldsymbol{0}\}}}{\longrightarrow} \boldsymbol{\Sigma}_{\boldsymbol{v}}^{1/2}\boldsymbol{N} \quad \text{as} \ t \to \infty,$$

where $\boldsymbol{\Sigma}_{\boldsymbol{v}}^{1/2}\boldsymbol{N} \stackrel{\mathcal{D}}{=} \mathcal{N}_2(\boldsymbol{0},\boldsymbol{\Sigma}_{\boldsymbol{v}})$, as desired.

Proof of part (ii) of Theorem 3.4. First, note that the moment condition (3.3) yields the moment condition (3.1) with $\lambda = s(\tilde{B})$, so, by Lemma 3.3, $w_{u,X_0} \stackrel{\text{a.s.}}{=} 0$ if and only if $(X_t)_{t \in \mathbb{R}_+}$ is trivial. For each $t \in \mathbb{R}_+$, we have the decomposition

$$\mathbb{1}_{\{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle > 1\}} \frac{1}{\sqrt{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle \log(\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle)}} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \\
= \mathbb{1}_{\{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle > 1\}} \frac{\sqrt{w_{\boldsymbol{u}, \boldsymbol{X}_0}}}{\sqrt{\mathrm{e}^{-s(\tilde{\boldsymbol{B}})t} \langle \boldsymbol{u}, \boldsymbol{X}_t \rangle t^{-1} \log(\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle)}} \frac{t^{-1/2} \mathrm{e}^{-s(\tilde{\boldsymbol{B}})t/2}}{\sqrt{w_{\boldsymbol{u}, \boldsymbol{X}_0}}} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix}$$

on the event $\{w_{\boldsymbol{u},\boldsymbol{X}_0} > 0\}$. By Theorem 3.3 in Barczy et al. [8], we have $e^{-s(\tilde{\boldsymbol{B}})t}\boldsymbol{X}_t \xrightarrow{\text{a.s.}} w_{\boldsymbol{u},\boldsymbol{X}_0}\tilde{\boldsymbol{u}}$ as $t \to \infty$, hence $\mathbb{1}_{\{\langle \boldsymbol{u},\boldsymbol{X}_t \rangle > 1\}} = \mathbb{1}_{\{e^{-s(\tilde{\boldsymbol{B}})t}\langle \boldsymbol{u},\boldsymbol{X}_t \rangle - e^{-s(\tilde{\boldsymbol{B}})t} > 0\}} \to 1$ as $t \to \infty$ on the event $\{w_{\boldsymbol{u},\boldsymbol{X}_0} > 0\}$, since $e^{-s(\tilde{\boldsymbol{B}})t}\langle \boldsymbol{u},\boldsymbol{X}_t \rangle - e^{-s(\tilde{\boldsymbol{B}})t} \xrightarrow{\text{a.s.}} w_{\boldsymbol{u},\boldsymbol{X}_0}$. In case of $\boldsymbol{\Sigma}_{\boldsymbol{v}} = \mathbf{0}$, (3.4) yields

$$t^{-1/2} \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t/2} \begin{pmatrix} \mathrm{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \mathrm{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \stackrel{\mathbb{P}}{\longrightarrow} \mathbf{0} \qquad \mathrm{as} \ t \to \infty.$$

hence, using the above decomposition, by Slutsky's lemma, we obtain

$$\mathbb{1}_{\{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle > 1\}} \frac{1}{\sqrt{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle \log(\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle)}} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \stackrel{\mathcal{D}_{\{w_{\boldsymbol{u}, \boldsymbol{X}_0} > 0\}}}{\longrightarrow} \mathbf{0} \qquad \text{as} \ t \to \infty$$

since, by (3.2), we have $e^{-s(\tilde{\boldsymbol{B}})t}\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle \xrightarrow{\text{a.s.}} w_{\boldsymbol{u}, \boldsymbol{X}_0}$ as $t \to \infty$, which also implies

$$t^{-1}\log(\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle) = t^{-1}\log(e^{s(\widetilde{\boldsymbol{B}})t}) + t^{-1}\log(e^{-s(\widetilde{\boldsymbol{B}})t}\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle) \xrightarrow{\text{a.s.}} s(\widetilde{\boldsymbol{B}}) \in \mathbb{R}_{++} \quad \text{as} \ t \to \infty.$$

In case of $\Sigma_v \neq 0$, as in the proof of (3.4), we may apply Theorem E.1 to obtain

$$\begin{pmatrix} t^{-1/2} \mathrm{e}^{-s(\tilde{\boldsymbol{B}})t/2} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix}, \sqrt{w_{\boldsymbol{u}, \boldsymbol{X}_0}} \end{pmatrix} \overset{\mathcal{D}}{\longrightarrow} \begin{pmatrix} (w_{\boldsymbol{u}, \boldsymbol{X}_0} \boldsymbol{\Sigma}_{\boldsymbol{v}})^{1/2} \boldsymbol{N}, \sqrt{w_{\boldsymbol{u}, \boldsymbol{X}_0}} \end{pmatrix} \quad \text{as} \ t \to \infty,$$

where N is a 2-dimensional random vector with $N \stackrel{\mathcal{D}}{=} \mathcal{N}_2(\mathbf{0}, \mathbf{I}_2)$ independent of $w_{u, \mathbf{X}_0} \Sigma_{v}$, and hence independent of w_{u, \mathbf{X}_0} because $\Sigma_{v} \neq \mathbf{0}$ and Σ_{v} is deterministic. Applying the continuous mapping theorem, we get

$$\frac{t^{-1/2}\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t/2}}{\sqrt{w_{\boldsymbol{u},\boldsymbol{X}_0}}} \begin{pmatrix} \mathrm{Re}(\langle \boldsymbol{v},\boldsymbol{X}_t \rangle) \\ \mathrm{Im}(\langle \boldsymbol{v},\boldsymbol{X}_t \rangle) \end{pmatrix} \stackrel{\mathcal{D}_{\{w_{\boldsymbol{u},\boldsymbol{X}_0}>0\}}}{\longrightarrow} \boldsymbol{\Sigma}_{\boldsymbol{v}}^{1/2} \boldsymbol{N} \quad \text{ as } t \to \infty.$$

Hence, using again the above decomposition, by Slutsky's lemma and (3.2),

$$\mathbb{1}_{\{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle > 1\}} \frac{1}{\sqrt{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle \log(\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle)}} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \stackrel{\mathcal{D}_{\{\boldsymbol{w}_{\boldsymbol{u}, \boldsymbol{X}_0} > 0\}}}{\longrightarrow} \frac{1}{s(\widetilde{\boldsymbol{B}})^{1/2}} \Sigma_{\boldsymbol{v}}^{1/2} \boldsymbol{N} \quad \text{as} \ t \to \infty,$$

where $\frac{1}{s(\tilde{B})^{1/2}} \Sigma_{\boldsymbol{v}}^{1/2} \boldsymbol{N} \stackrel{\mathcal{D}}{=} \mathcal{N}_2(\boldsymbol{0}, \frac{1}{s(\tilde{B})} \Sigma_{\boldsymbol{v}})$, as desired.

Proof of Proposition 3.6. Theorem 3.3 in Barczy et al. [8] yields that $e^{-s(\tilde{B})t} \langle \boldsymbol{e}_i, \boldsymbol{X}_t \rangle \xrightarrow{\text{a.s.}} w_{\boldsymbol{u},\boldsymbol{X}_0} \langle \boldsymbol{e}_i, \tilde{\boldsymbol{u}} \rangle$ and $e^{-s(\tilde{B})t} \langle \boldsymbol{e}_j, \boldsymbol{X}_t \rangle \xrightarrow{\text{a.s.}} w_{\boldsymbol{u},\boldsymbol{X}_0} \langle \boldsymbol{e}_j, \tilde{\boldsymbol{u}} \rangle$ as $t \to \infty$. Consequently, since $\tilde{\boldsymbol{u}} \in \mathbb{R}_{++}$, we have $\mathbb{1}_{\{\langle \boldsymbol{e}_j, \boldsymbol{X}_t \rangle \neq 0\}} = \mathbb{1}_{\{e^{-s(\tilde{B})t} \langle \boldsymbol{e}_j, \boldsymbol{X}_t \rangle \neq 0\}} \to 1$ as $t \to \infty$ on the event $\{w_{\boldsymbol{u},\boldsymbol{X}_0} > 0\}$, and hence

$$\mathbb{1}_{\{\langle \boldsymbol{e}_j, \boldsymbol{X}_t \rangle \neq 0\}} \frac{\langle \boldsymbol{e}_i, \boldsymbol{X}_t \rangle}{\langle \boldsymbol{e}_j, \boldsymbol{X}_t \rangle} = \mathbb{1}_{\{\langle \boldsymbol{e}_j, \boldsymbol{X}_t \rangle \neq 0\}} \frac{\mathrm{e}^{-s(\boldsymbol{B})t} \langle \boldsymbol{e}_i, \boldsymbol{X}_t \rangle}{\mathrm{e}^{-s(\boldsymbol{\tilde{B}})t} \langle \boldsymbol{e}_j, \boldsymbol{X}_t \rangle} \to \frac{w_{\boldsymbol{u}, \boldsymbol{X}_0} \langle \boldsymbol{e}_i, \boldsymbol{\tilde{u}} \rangle}{w_{\boldsymbol{u}, \boldsymbol{X}_0} \langle \boldsymbol{e}_j, \boldsymbol{\tilde{u}} \rangle} = \frac{\langle \boldsymbol{e}_i, \boldsymbol{\tilde{u}} \rangle}{\langle \boldsymbol{e}_j, \boldsymbol{\tilde{u}} \rangle} \quad \text{as} \quad t \to \infty$$

on the event $\{w_{\boldsymbol{u},\boldsymbol{X}_0} > 0\}$, thus we obtain the first convergence. In a similar way, $\mathbb{1}_{\{\boldsymbol{X}_t \neq \boldsymbol{0}\}} \to 1$ as $t \to \infty$ on the event $\{w_{\boldsymbol{u},\boldsymbol{X}_0} > 0\}$, thus

$$\mathbb{1}_{\{\boldsymbol{X}_t\neq\boldsymbol{0}\}}\frac{\langle \boldsymbol{e}_i, \boldsymbol{X}_t \rangle}{\sum_{k=1}^d \langle \boldsymbol{e}_k, \boldsymbol{X}_t \rangle} = \mathbb{1}_{\{\boldsymbol{X}_t\neq\boldsymbol{0}\}}\frac{\mathrm{e}^{-s(\boldsymbol{B})t} \langle \boldsymbol{e}_i, \boldsymbol{X}_t \rangle}{\sum_{k=1}^d \mathrm{e}^{-s(\tilde{\boldsymbol{B}})t} \langle \boldsymbol{e}_k, \boldsymbol{X}_t \rangle} \to \frac{w_{\boldsymbol{u}, \boldsymbol{X}_0} \langle \boldsymbol{e}_i, \widetilde{\boldsymbol{u}} \rangle}{\sum_{k=1}^d w_{\boldsymbol{u}, \boldsymbol{X}_0} \langle \boldsymbol{e}_k, \widetilde{\boldsymbol{u}} \rangle} = \langle \boldsymbol{e}_i, \widetilde{\boldsymbol{u}} \rangle$$

as $t \to \infty$ on the event $\{w_{u,X_0} > 0\}$ since the sum of the coordinates of \tilde{u} is 1, hence we obtain the second convergence.

Appendix

A A decomposition of multi-type CBI processes

The following useful decomposition of a multi-type CBI process as an independent sum of a CBI process starting from **0** and a CB process has been derived in Barczy et al. [8, Lemma A.1].

A.1 Lemma. If $(\mathbf{X}_s)_{s \in \mathbb{R}_+}$ is a multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$, then for each $t, T \in \mathbb{R}_+$, we have $\mathbf{X}_{t+T} \stackrel{\mathcal{D}}{=} \mathbf{X}_t^{(1)} + \mathbf{X}_t^{(2,T)}$, where $(\mathbf{X}_s^{(1)})_{s \in \mathbb{R}_+}$ and $(\mathbf{X}_s^{(2,T)})_{s \in \mathbb{R}_+}$ are independent multi-type CBI processes with $\mathbb{P}(\mathbf{X}_0^{(1)} = \mathbf{0}) = 1$, $\mathbf{X}_0^{(2,T)} \stackrel{\mathcal{D}}{=} \mathbf{X}_T$, and with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ and $(d, \mathbf{c}, \mathbf{0}, \mathbf{B}, 0, \boldsymbol{\mu})$, respectively.

B On deterministic projections of multi-type CBI processes

B.1 Lemma. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be an irreducible multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|) < \infty$ and the moment conditions (2.1) and (3.8) hold. Let $\lambda \in \sigma(\widetilde{\mathbf{B}})$, and let $\mathbf{v} \in \mathbb{C}^d$ be a left eigenvector of $\widetilde{\mathbf{B}}$ corresponding to the eigenvalue λ . Then the following three assertions are equivalent:

- (i) There exists $t \in \mathbb{R}_{++}$ such that $\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ is deterministic.
- (ii) One of the following two conditions holds:
 - (a) $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is a trivial process (see Definition 2.6).
 - (b) $\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle$ is deterministic, $\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle c_\ell = 0$ and $\mu_\ell(\{\boldsymbol{z} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) = 0$ for every $\ell \in \{1, \ldots, d\}$, and $\nu(\{\boldsymbol{r} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{r} \rangle \neq 0\}) = 0$.
- (iii) For each $t \in \mathbb{R}_+$, $\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ is deterministic.

If
$$(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle)_{t \in \mathbb{R}_+}$$
 is deterministic, then $\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle \stackrel{\text{a.s.}}{=} e^{\lambda t} \langle \boldsymbol{v}, \mathbb{E}(\boldsymbol{X}_0) \rangle + \langle \boldsymbol{v}, \boldsymbol{\hat{\beta}} \rangle \int_0^t e^{\lambda u} du$, $t \in \mathbb{R}_+$.

Proof. (i) \implies (ii). We have the representation

(B.1)
$$e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle = Z_t^{(0)} + Z_t^{(1)} + Z_t^{(2)} + Z_t^{(3,4)} + Z_t^{(5)}$$

with

$$Z_{t}^{(0)} := \langle \boldsymbol{v}, \boldsymbol{X}_{0} \rangle,$$

$$Z_{t}^{(1)} := \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle \int_{0}^{t} e^{-\lambda u} du,$$

$$Z_{t}^{(2)} := \sum_{\ell=1}^{d} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle \int_{0}^{t} e^{-\lambda u} \sqrt{2c_{\ell}X_{u,\ell}} dW_{u,\ell},$$

$$Z_{t}^{(3,4)} := \sum_{\ell=1}^{d} \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle \mathbb{1}_{\{w \leq X_{u-,\ell}\}} \widetilde{N}_{\ell}(du, d\boldsymbol{z}, dw),$$

$$Z_{t}^{(5)} := \int_{0}^{t} \int_{\mathcal{U}_{d}} e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle \widetilde{M}(du, d\boldsymbol{r}),$$

see Barczy et al. [7, Lemma 4.1] or Barczy et al. [8, Lemma 2.7]. Note that under the moment condition (3.8), $(Z_t^{(2)})_{t\in\mathbb{R}_+}$, $(Z_t^{(3,4)})_{t\in\mathbb{R}_+}$ and $(Z_t^{(5)})_{t\in\mathbb{R}_+}$ are square-integrable martingales with initial values 0, hence $\mathbb{E}(Z_t^{(2)}) = \mathbb{E}(Z_t^{(3,4)}) = \mathbb{E}(Z_t^{(5)}) = 0$. Since $e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ and $Z_t^{(1)}$ are deterministic, we obtain $e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle = \mathbb{E}(e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) = \mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle) + Z_t^{(1)}$. Hence, by the representation (B.1), we get $0 = e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle - \mathbb{E}(e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) = \langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle - \mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle) + \sum_{j=2}^5 Z_t^{(j)}$ almost surely. Consequently,

$$\mathbb{E}(|\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle - \mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle) + Z_t^{(2)} + Z_t^{(3,4)} + Z_t^{(5)}|^2) = 0.$$

By the independence of \mathbf{X}_0 , $(W_{u,1})_{u \ge 0}$, ..., $(W_{u,d})_{u \ge 0}$, N_1 , ..., N_d and M, the random variables $\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle - \mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle)$, $Z_t^{(2)}$, $Z_t^{(3,4)}$, and $Z_t^{(5)}$ are conditionally independent with respect to $(\mathbf{X}_u)_{u \in [0,t]}$, thus

$$\begin{split} 0 &= \mathbb{E}\bigg(\bigg|\langle \boldsymbol{v}, \boldsymbol{X}_{0} \rangle - \mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_{0} \rangle) + Z_{t}^{(2)} + Z_{t}^{(3,4)} + Z_{t}^{(5)}\bigg|^{2}\bigg) \\ &= \mathbb{E}\bigg(\mathbb{E}\bigg(\bigg|\langle \boldsymbol{v}, \boldsymbol{X}_{0} \rangle - \mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_{0} \rangle) + Z_{t}^{(2)} + Z_{t}^{(3,4)} + Z_{t}^{(5)}\bigg|^{2}\bigg|(\boldsymbol{X}_{u})_{u\in[0,t]}\bigg)\bigg) \\ &= \mathbb{E}(\mathbb{E}(|\langle \boldsymbol{v}, \boldsymbol{X}_{0} \rangle - \mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_{0} \rangle)|^{2}|(\boldsymbol{X}_{u})_{u\in[0,t]})) + \mathbb{E}(\mathbb{E}(|Z_{t}^{(2)}|^{2}|(\boldsymbol{X}_{u})_{u\in[0,t]})) \\ &+ \mathbb{E}(\mathbb{E}(|Z_{t}^{(3,4)}|^{2}|(\boldsymbol{X}_{u})_{u\in[0,t]})) + \mathbb{E}(\mathbb{E}(|Z_{t}^{(5)}|^{2}|(\boldsymbol{X}_{u})_{u\in[0,t]})) \\ &= \mathbb{E}(|\langle \boldsymbol{v}, \boldsymbol{X}_{0} \rangle - \mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_{0} \rangle)|^{2}) + \mathbb{E}(|Z_{t}^{(2)}|^{2}) + \mathbb{E}(|Z_{t}^{(3,4)}|^{2}) + \mathbb{E}(|Z_{t}^{(5)}|^{2}), \end{split}$$

where we also used that $(Z_s^{(2)})_{s\in[0,t]}$, $(Z_s^{(3,4)})_{s\in[0,t]}$ and $(Z_s^{(5)})_{s\in[0,t]}$ are square-integrable martingales with initial values 0 conditionally on $(\mathbf{X}_u)_{u\in[0,t]}$. Consequently, $\mathbb{E}(|\langle \boldsymbol{v}, \mathbf{X}_0 \rangle - \mathbf{v})|_{u\in[0,t]}$

 $\mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle)|^2) = 0$ and $\mathbb{E}(|Z_t^{(2)}|^2) = \mathbb{E}(|Z_t^{(3,4)}|^2) = \mathbb{E}(|Z_t^{(5)}|^2) = 0.$ One can easily derive

$$\mathbb{E}(|Z_t^{(2)}|^2) = 2\sum_{\ell=1}^a |\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle|^2 c_\ell \int_0^t e^{-2\operatorname{Re}(\lambda)u} \mathbb{E}(X_{u,\ell}) \,\mathrm{d}u,$$

hence we conclude

$$|\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle|^{2} c_{\ell} \int_{0}^{t} \mathrm{e}^{-2\mathrm{Re}(\lambda)u} \mathbb{E}(X_{u,\ell}) \,\mathrm{d}u = 0, \qquad \ell \in \{1, \dots, d\}.$$

Consequently, for each $\ell \in \{1, \ldots, d\}$, we have $|\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle|^2 c_{\ell} = 0$ or $\int_0^t e^{-2\operatorname{Re}(\lambda)u} \mathbb{E}(X_{u,\ell}) du = 0$. In the first case we obtain $\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle c_{\ell} = 0$, which is in (ii)/(b). In the second case, using Lemma 2.5 and $e^{-2\operatorname{Re}(\lambda)u} \in \mathbb{R}_{++}$ for all $u \in \mathbb{R}_+$, we conclude (ii)/(a).

Since $\mathbb{E}(|Z_t^{(3,4)}|^2) = 0$, we have

$$\mathbb{E}(|Z_t^{(3,4)}|^2) = \sum_{\ell=1}^d \int_0^t \int_{\mathcal{U}_d} e^{-2\operatorname{Re}(\lambda)u} |\langle \boldsymbol{v}, \boldsymbol{z} \rangle|^2 \mathbb{E}(X_{u,\ell}) \,\mathrm{d}u \,\mu_\ell(\mathrm{d}\boldsymbol{z}) = 0.$$

Using $e^{-2\operatorname{Re}(\lambda)u} \in \mathbb{R}_{++}$ for all $u \in \mathbb{R}_{+}$, we conclude

$$\sum_{\ell=1}^{d} \int_{0}^{t} \int_{\mathcal{U}_{d}} \mathbb{1}_{\{\langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}} \mathbb{E}(X_{u,\ell}) \, \mathrm{d}u \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) = 0.$$

Then, using the non-negativity of the integrands, we obtain

$$\int_0^t \int_{\mathcal{U}_d} \mathbb{1}_{\{\langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}} \mathbb{E}(X_{u,\ell}) \, \mathrm{d}u \, \mu_\ell(\mathrm{d}\boldsymbol{z}) = 0, \qquad \ell \in \{1, \dots, d\}.$$

By Lemma 2.5, for each $\ell \in \{1, \ldots, d\}$, we have either (ii)/(a), or $\mathbb{E}(X_{u,\ell}) = \boldsymbol{e}_{\ell}^{\top} \mathbb{E}(\boldsymbol{X}_u) \in \mathbb{R}_{++}$ for all $u \in \mathbb{R}_{++}$. In the second case, we conclude

$$\int_0^t \int_{\mathcal{U}_d} \mathbb{1}_{\{\langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}} \, \mathrm{d}u \, \mu_\ell(\mathrm{d}\boldsymbol{z}) = t \mu_\ell(\{\boldsymbol{z} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) = 0,$$

and hence $\mu_{\ell}(\{\boldsymbol{z} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) = 0$, which is in (ii)/(b).

Since $\mathbb{E}(|Z_t^{(5)}|^2) = 0$, we have $Z_t^{(5)} = 0$ almost surely. Hence the random variable

$$\int_0^t \int_{\mathcal{U}_d} \mathrm{e}^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle \, M(\mathrm{d} u, \mathrm{d} \boldsymbol{r})$$

is deterministic, since $\int_0^t \int_{\mathcal{U}_d} e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle \, \mathrm{d}u \, \nu(\mathrm{d}\boldsymbol{r})$ is deterministic. We have $Z_s^{(5)} = \mathbb{E}(Z_t^{(5)} | \mathcal{F}_s^{Z^{(5)}}) = \mathbb{E}(0 | \mathcal{F}_s^{Z^{(5)}}) = 0$ for all $s \in [0, t]$ almost surely, where $\mathcal{F}_s^{Z^{(5)}} := \sigma(Z_u^{(5)} : u \in [0, s])$, since $(Z_s^{(5)})_{s \in \mathbb{R}_+}$ is a martingale. Thus $\mathbb{P}(A_t^{(M)}) = 1$, where $A_t^{(M)}$ is the event such that the Poisson random measure M has no point in the set H_t , where

$$H_t := \{(u, \boldsymbol{r}) \in (0, t] \times \mathcal{U}_d : e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle \neq 0\} = \{(u, \boldsymbol{r}) \in (0, t] \times \mathcal{U}_d : \mathbb{1}_{\{\langle \boldsymbol{v}, \boldsymbol{r} \rangle \neq 0\}} \neq 0\},\$$

since $e^{-\lambda u} \neq 0$ for all $u \in \mathbb{R}_+$. The number of the points of M in the set H_t has a Poisson distribution with parameter

$$\lambda_t := \int_0^t \int_{\mathcal{U}_d} \mathbb{1}_{\{\langle \boldsymbol{v}, \boldsymbol{r} \rangle \neq 0\}} \, \mathrm{d}u \, \nu(\mathrm{d}\boldsymbol{r}).$$

We have $1 = \mathbb{P}(A_t^{(M)}) = e^{-\lambda_t}$, yielding

$$\lambda_t = \int_0^t \int_{\mathcal{U}_d} \mathbb{1}_{\{\langle \boldsymbol{v}, \boldsymbol{r} \rangle \neq 0\}} \, \mathrm{d}u \, \nu(\mathrm{d}\boldsymbol{r}) = t\nu(\{\boldsymbol{r} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{r} \rangle \neq 0\}) = 0,$$

and hence $\nu(\{\boldsymbol{r} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{r} \rangle \neq 0\}) = 0$, which is in (ii)/(b).

(ii) \Longrightarrow (iii). If (ii)/(a) holds, then $\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle \stackrel{\text{a.s.}}{=} 0$ for all $t \in \mathbb{R}_+$. If (ii)/(b) holds, then we use again the representation (B.1) of $\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$. We have $\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle = \mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle) = \langle \boldsymbol{v}, \mathbb{E}(\boldsymbol{X}_0) \rangle$, since $\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle$ is deterministic. For each $t \in \mathbb{R}_+$, we have

$$Z_t^{(2)} = \sqrt{2} \sum_{\ell=1}^d \langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle \sqrt{c_\ell} \int_0^t e^{-\lambda u} \sqrt{X_{u,\ell}} \, \mathrm{d}W_{u,\ell} = 0,$$

since $\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle c_{\ell} = 0$ for every $\ell \in \{1, \dots, d\}$.

Further, for each $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, using the notation

$$f(u, \boldsymbol{z}, w) := \sum_{\ell=1}^{d} e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle \mathbb{1}_{\{w \leq X_{u-,\ell}\}} = e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle \sum_{\ell=1}^{d} \mathbb{1}_{\{w \leq X_{u-,\ell}\}}$$

for $u \in (0, t]$, $z \in \mathcal{U}_d$, and $w \in \mathcal{U}_1$, we have

$$\begin{split} &\int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} \mathbb{1}_{\{|f(u,\boldsymbol{z},w)| < n\}} \mathbb{1}_{\{||\boldsymbol{z}|| > 1/n\}} \mathbb{1}_{\{w < n\}} f(u,\boldsymbol{z},w) \, \widetilde{N}_{\ell}(\mathrm{d}u,\mathrm{d}\boldsymbol{z},\mathrm{d}w) \\ &= \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} \mathbb{1}_{\{|f(u,\boldsymbol{z},w)| < n\}} \mathbb{1}_{\{||\boldsymbol{z}|| > 1/n\}} \mathbb{1}_{\{w < n\}} f(u,\boldsymbol{z},w) \, N_{\ell}(\mathrm{d}u,\mathrm{d}\boldsymbol{z},\mathrm{d}w) \\ &- \int_{0}^{t} \int_{\mathcal{U}_{d}} \int_{\mathcal{U}_{1}} \mathbb{1}_{\{|f(u,\boldsymbol{z},w)| < n\}} \mathbb{1}_{\{||\boldsymbol{z}|| > 1/n\}} \mathbb{1}_{\{w < n\}} f(u,\boldsymbol{z},w) \, \mathrm{d}u \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}w = 0 \end{split}$$

almost surely, since $\int_{\mathcal{U}_d} \mathbb{1}_{\{\|\boldsymbol{z}\|>1/n\}} \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \leq n^2 \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^2 \mu_{\ell}(\mathrm{d}\boldsymbol{z}) < \infty$ due to part (vi) of Definition 2.1, (3.8) and

$$(\mathcal{L}_1 \otimes \mu_\ell \otimes \mathcal{L}_d)(\{(u, \boldsymbol{z}, w) \in (0, t] \times \mathcal{U}_d \times \mathcal{U}_1 : f(u, \boldsymbol{z}, w) \neq 0\}) = 0$$

for each $\ell \in \{1, \ldots, d\}$, where \mathcal{L}_1 and \mathcal{L}_d denote the Lebesgue measure on \mathbb{R} and on \mathbb{R}^d , respectively. Letting $n \to \infty$, by Ikeda and Watanabe [15, page 63], we conclude

$$Z_t^{(3,4)} = \sum_{\ell=1}^d \int_0^t \int_{\mathcal{U}_d} \int_{\mathcal{U}_1} e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{z} \rangle \mathbb{1}_{\{w \leq X_{u-,\ell}\}} \widetilde{N}_\ell(\mathrm{d} u, \mathrm{d} \boldsymbol{z}, \mathrm{d} w) = 0$$

almost surely.

Finally, for each $t \in \mathbb{R}_+$, we have

$$Z_t^{(5)} = \int_0^t \int_{\mathcal{U}_d} e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle M(\mathrm{d} u, \mathrm{d} \boldsymbol{r}) - \int_0^t \int_{\mathcal{U}_d} e^{-\lambda u} \langle \boldsymbol{v}, \boldsymbol{r} \rangle \,\mathrm{d} u \,\nu(\mathrm{d} \boldsymbol{r}) = 0$$

almost surely, since $\int_{\mathcal{U}_d} \|\boldsymbol{r}\| \, \nu(\mathrm{d}\boldsymbol{r}) < \infty$ (due to Definition 2.1 and (2.1)) and $\nu(\{\boldsymbol{r} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{r} \rangle \neq 0\}) = 0.$

(iii) \implies (i) is trivial.

If $(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle)_{t \in \mathbb{R}_+}$ is deterministic, then, by (2.2), for each $t \in \mathbb{R}_+$, we have $\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle = \mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) = \langle \boldsymbol{v}, \mathbb{E}(\boldsymbol{X}_t) \rangle = e^{\lambda t} \langle \boldsymbol{v}, \mathbb{E}(\boldsymbol{X}_0) \rangle + \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle \int_0^t e^{\lambda u} du$ almost surely. \Box

C A stochastic fixed point equation

Under some mild conditions, the solution of a stochastic fixed point equation is atomless, see, e.g., Buraczewski et al. [10, Proposition 4.3.2].

C.1 Theorem. Let (A, C) be a random element in $\mathbb{R}^{d \times d} \times \mathbb{R}^d$, where $d \in \mathbb{N}$. Assume that

- (i) \boldsymbol{A} is invertible almost surely,
- (ii) $\mathbb{P}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{C} = \boldsymbol{x}) < 1$ for every $\boldsymbol{x} \in \mathbb{R}^d$,
- (iii) the d-dimensional fixed point equation $X \stackrel{\mathcal{D}}{=} AX + C$, where (A, C) and X are independent, has a solution X, which is unique in distribution.

Then the distribution of X does not have atoms and is of pure type, i.e., it is either absolutely continuous or singular with respect to Lebesgue measure in \mathbb{R}^d .

C.2 Corollary. Let $A \in \mathbb{R}^{d \times d}$ with $\det(A) \neq 0$ and r(A) < 1. Let C be a d-dimensional non-deterministic random vector with $\mathbb{E}(||C||) < \infty$. Then the d-dimensional fixed point equation $X \stackrel{\mathcal{D}}{=} AX + C$, where X is independent of C, has a solution X which is unique in distribution, the distribution of X does not have atoms and is of pure type, i.e., it is either absolutely continuous or singular with respect to Lebesgue measure in \mathbb{R}^d .

Proof. The first condition of Theorem C.1 is trivially satisfied, since $\det(\mathbf{A}) \neq 0$. Since \mathbf{C} is not deterministic and for each $\mathbf{x} \in \mathbb{R}^d$, we have $\mathbb{P}(\mathbf{A}\mathbf{x} + \mathbf{C} = \mathbf{x}) = \mathbb{P}(\mathbf{C} = (\mathbf{I}_d - \mathbf{A})\mathbf{x})$, the second condition of Theorem C.1 is also satisfied. In order to check the third condition of Lemma C.1, first we suppose that \mathbf{X} is a solution of the stochastic fixed point equation $\mathbf{X} \stackrel{\mathcal{D}}{=} \mathbf{A}\mathbf{X} + \mathbf{C}$, where \mathbf{X} is a *d*-dimensional random vector independent of (\mathbf{A}, \mathbf{C}) , equivalently, independent of \mathbf{C} (since \mathbf{A} is deterministic and invertible). Then, iterating this equation, for each $n \in \mathbb{N}$, we obtain $\mathbf{X} \stackrel{\mathcal{D}}{=} \mathbf{A}^n \mathbf{X} + \sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{C}_k$, where $\mathbf{C}_k, \ k \in \mathbb{Z}_+$, are independent copies of \mathbf{C} . Since

 $r(\mathbf{A}) < 1$, we have $\mathbf{A}^n \to \mathbf{0}$ as $n \to \infty$, see, e.g., Horn and Johnson [14, Theorem 5.6.12]. Moreover, $\sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{C}_k \xrightarrow{L_1} \sum_{k=0}^{\infty} \mathbf{A}^k \mathbf{C}_k$ as $n \to \infty$, since $\sum_{k=n}^{\infty} \mathbf{A}^k \mathbf{C}_k \xrightarrow{L_1} \mathbf{0}$ as $n \to \infty$. Indeed, by the Gelfand formula, we have $r(\mathbf{A}) = \lim_{k\to\infty} \|\mathbf{A}^k\|^{1/k}$, see, e.g., Horn and Johnson [14, Corollary 5.6.14], hence there exists $k_0 \in \mathbb{N}$ such that $\|\mathbf{A}^k\|^{1/k} \leq (r(\mathbf{A}) + 1)/2 < 1$ for every $k \in \mathbb{N}$ with $k \ge k_0$. Thus, for each $n \in \mathbb{N}$ with $n \ge k_0$, we have

$$\mathbb{E}\left(\left\|\sum_{k=n}^{\infty} \mathbf{A}^{k} \mathbf{C}_{k}\right\|\right) \leqslant \sum_{k=n}^{\infty} \|\mathbf{A}^{k}\| \mathbb{E}(\|\mathbf{C}_{k}\|) \leqslant \mathbb{E}(\|\mathbf{C}\|) \sum_{k=n}^{\infty} \left(\frac{r(\mathbf{A})+1}{2}\right)^{k} \to 0$$

as $n \to \infty$, hence we obtain $\sum_{k=n}^{\infty} A^k C_k \xrightarrow{L_1} \mathbf{0}$ as $n \to \infty$, and hence $\sum_{k=0}^{n-1} A^k C_k \xrightarrow{L_1} \sum_{k=0}^{\infty} A^k C_k$ as $n \to \infty$. Consequently, if X is a solution of $X \stackrel{\mathcal{D}}{=} AX + C$, then, necessarily, $X \stackrel{\mathcal{D}}{=} \sum_{k=0}^{\infty} A^k C_k$. The *d*-dimensional random variable $\sum_{k=0}^{\infty} A^k C_k$ is a solution of $X \stackrel{\mathcal{D}}{=} AX + C$, then, $\sum_{k=0}^{\infty} A^k C_k$ and $\sum_{k=0}^{\infty} A^k C_{k+1}$ is independent of AC_0 (equivalently, of (A, AC_0)), hence the third condition of Lemma C.1 is also satisfied.

D On the second moment of projections of multi-type CBI processes

An explicit formula for the second absolute moment of the projection of a multi-type CBI process on the left eigenvectors of its branching mean matrix has been presented together with its asymptotic behavior in the supercritical and irreducible case in Barczy et al. [8, Proposition B.1].

D.1 Proposition. If $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is a multi-type CBI process with parameters $(d, c, \beta, B, \nu, \mu)$ such that $\mathbb{E}(||\mathbf{X}_0||^2) < \infty$ and the moment condition (3.8) holds, then for each left eigenvector $\mathbf{v} \in \mathbb{C}^d$ of $\widetilde{\mathbf{B}}$ corresponding to an arbitrary eigenvalue $\lambda \in \sigma(\widetilde{\mathbf{B}})$, we have

$$\mathbb{E}(|\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle|^2) = E_{\boldsymbol{v},\lambda}(t) + \sum_{\ell=1}^d C_{\boldsymbol{v},\ell} I_{\lambda,\ell}(t) + I_{\lambda}(t) \int_{\mathcal{U}_d} |\langle \boldsymbol{v}, \boldsymbol{r} \rangle|^2 \nu(\mathrm{d}\boldsymbol{r}), \qquad t \in \mathbb{R}_+,$$

where $C_{\boldsymbol{v},\ell}$, $\ell \in \{1, \ldots, d\}$, are defined in Theorem 3.1, and

$$E_{\boldsymbol{v},\lambda}(t) := \mathbb{E}\left(\left|e^{\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle + \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle \int_0^t e^{\lambda(t-u)} du\right|^2\right),$$
$$I_{\lambda,\ell}(t) := \int_0^t e^{2\operatorname{Re}(\lambda)(t-u)} \mathbb{E}(X_{u,\ell}) du, \qquad \ell \in \{1, \dots, d\},$$
$$I_{\lambda}(t) := \int_0^t e^{2\operatorname{Re}(\lambda)(t-u)} du.$$

If, in addition, $(X_t)_{t \in \mathbb{R}_+}$ is supercritical and irreducible, then we have

$$\lim_{t\to\infty} h(t) \mathbb{E}(|\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle|^2) = M_{\boldsymbol{v}}^{(2)},$$

where

$$h(t) := \begin{cases} e^{-s(\widetilde{\boldsymbol{B}})t} & \text{if } \operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\right), \\ t^{-1}e^{-s(\widetilde{\boldsymbol{B}})t} & \text{if } \operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}}), \\ e^{-2\operatorname{Re}(\lambda)t} & \text{if } \operatorname{Re}(\lambda) \in \left(\frac{1}{2}s(\widetilde{\boldsymbol{B}}), s(\widetilde{\boldsymbol{B}})\right], \end{cases}$$

and

$$M_{\boldsymbol{v}}^{(2)} := \begin{cases} \frac{1}{s(\widetilde{\boldsymbol{B}}) - 2\operatorname{Re}(\lambda)} \left(\langle \boldsymbol{u}, \mathbb{E}(\boldsymbol{X}_{0}) \rangle + \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}} \rangle}{s(\widetilde{\boldsymbol{B}})} \right) \sum_{\ell=1}^{d} C_{\boldsymbol{v},\ell} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle & \text{if } \operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\right), \\ \left(\langle \boldsymbol{u}, \mathbb{E}(\boldsymbol{X}_{0}) \rangle + \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}} \rangle}{s(\widetilde{\boldsymbol{B}})} \right) \sum_{\ell=1}^{d} C_{\boldsymbol{v},\ell} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle & \text{if } \operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}}), \\ \mathbb{E}\left(\left| \langle \boldsymbol{v}, \boldsymbol{X}_{0} \rangle + \frac{\langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle}{\lambda} \right|^{2} \right) + \frac{1}{2\operatorname{Re}(\lambda)} \int_{\mathcal{U}_{d}} |\langle \boldsymbol{v}, \boldsymbol{r} \rangle|^{2} \nu(\mathrm{d}\boldsymbol{r}) \\ + \sum_{\ell=1}^{d} C_{\boldsymbol{v},\ell} \boldsymbol{e}_{\ell}^{\top} (2\operatorname{Re}(\lambda)\boldsymbol{I}_{d} - \widetilde{\boldsymbol{B}})^{-1} \left(\mathbb{E}(\boldsymbol{X}_{0}) + \frac{\widetilde{\boldsymbol{\beta}}}{2\operatorname{Re}(\lambda)} \right) & \text{if } \operatorname{Re}(\lambda) \in \left(\frac{1}{2}s(\widetilde{\boldsymbol{B}}), s(\widetilde{\boldsymbol{B}}) \right]. \end{cases}$$

Based on Proposition D.1, we derive the asymptotic behavior of the variance matrix of the real and imaginary parts of the projection of a multi-type CBI process on certain left eigenvectors of its branching mean matrix $e^{\tilde{B}}$.

D.2 Proposition. If $(\mathbf{X}_t)_{t\in\mathbb{R}_+}$ is a supercritical and irreducible multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|^2) < \infty$ and the moment condition (3.8) holds, then for each left eigenvector $\mathbf{v} \in \mathbb{C}^d$ of $\widetilde{\mathbf{B}}$ corresponding to an arbitrary eigenvalue $\lambda \in \sigma(\widetilde{\mathbf{B}})$ with $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\widetilde{\mathbf{B}})]$ we have

$$\lim_{t\to\infty} h(t) \mathbb{E} \left(\begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix}^\top \right) = \left(\langle \boldsymbol{u}, \mathbb{E}(\boldsymbol{X}_0) \rangle + \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}} \rangle}{s(\widetilde{\boldsymbol{B}})} \right) \boldsymbol{\Sigma}_{\boldsymbol{v}},$$

where the scaling factor $h : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and the matrix Σ_{v} are defined in Proposition D.1 and in Theorem 3.1, respectively.

Proof. For each $t \in \mathbb{R}_+$, using the identity (4.15) for $a = \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle \in \mathbb{C}$, and then taking expectation, we obtain

(D.1)
$$\mathbb{E} \left(\begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix}^{\top} \right)$$
$$= \frac{1}{2} \mathbb{E}(|\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle|^2) \boldsymbol{I}_2 + \frac{1}{2} \begin{pmatrix} \operatorname{Re}(\mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle^2)) & \operatorname{Im}(\mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle^2)) \\ \operatorname{Im}(\mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle^2)) & -\operatorname{Re}(\mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle^2)) \end{pmatrix}$$

The asymptotic behavior of $\mathbb{E}(|\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle|^2)$ as $t \to \infty$ is described in Proposition D.1. The aim of the following discussion is to describe the asymptotic behavior of $\mathbb{E}((\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle)^2)$ as $t \to \infty$.

For each $t \in \mathbb{R}_+$, we use the representation of $e^{-\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ given at the beginning of the proof of part (iii) of Theorem 3.1. The independence of \boldsymbol{X}_0 , $(W_{u,1})_{u \in \mathbb{R}_+}$, ..., $(W_{u,d})_{u \in \mathbb{R}_+}$, N_1 , ..., N_d and \boldsymbol{M} implies the conditional independence of the random variables $Z_t^{(0,1)}$, $Z_t^{(2)}$, $Z_t^{(3,4)}$ and $Z_t^{(5)}$ with respect to $(\boldsymbol{X}_u)_{u \in [0,t]}$ for every $t \in \mathbb{R}_+$. Moreover, the conditional expectations of $Z_t^{(2)}$, $Z_t^{(3,4)}$ and $Z_t^{(5)}$ with respect to $(\boldsymbol{X}_u)_{u \in [0,t]}$ are 0, since the processes $(Z_t^{(2)})_{t \in \mathbb{R}_+}$, $(Z_t^{(3,4)})_{t \in \mathbb{R}_+}$ and $(Z_t^{(5)})_{t \in \mathbb{R}_+}$ are martingales with initial values 0. Consequently, for all $t \in \mathbb{R}_+$, we get

$$\mathbb{E}\left(\left(\mathrm{e}^{-\lambda t}\langle \boldsymbol{v}, \boldsymbol{X}_t\rangle\right)^2 \middle| (\boldsymbol{X}_u)_{u \in [0,t]}\right) = \mathbb{E}\left(\left(Z_t^{(0,1)}\right)^2 \middle| (\boldsymbol{X}_u)_{u \in [0,t]}\right) + \mathbb{E}\left(\left(Z_t^{(2)}\right)^2 \middle| (\boldsymbol{X}_u)_{u \in [0,t]}\right) \\ + \mathbb{E}\left(\left(Z_t^{(3,4)}\right)^2 \middle| (\boldsymbol{X}_u)_{u \in [0,t]}\right) + \mathbb{E}\left(\left(Z_t^{(5)}\right)^2 \middle| (\boldsymbol{X}_u)_{u \in [0,t]}\right)$$

almost surely. We have

$$\mathbb{E}\left(\left(Z_{t}^{(0,1)}\right)^{2} \mid (\boldsymbol{X}_{u})_{u \in [0,t]}\right) = \left(\langle \boldsymbol{v}, \boldsymbol{X}_{0} \rangle + \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle \int_{0}^{t} e^{-\lambda u} du\right)^{2},$$
$$\mathbb{E}\left(\left(Z_{t}^{(2)}\right)^{2} \mid (\boldsymbol{X}_{u})_{u \in [0,t]}\right) = 2\sum_{\ell=1}^{d} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle^{2} c_{\ell} \int_{0}^{t} e^{-2\lambda u} X_{u,\ell} du,$$
$$\mathbb{E}\left(\left(Z_{t}^{(3,4)}\right)^{2} \mid (\boldsymbol{X}_{u})_{u \in [0,t]}\right) = \sum_{\ell=1}^{d} \int_{0}^{t} e^{-2\lambda u} X_{u,\ell} du \int_{\mathcal{U}_{d}} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^{2} \mu_{\ell}(d\boldsymbol{z})$$
$$\mathbb{E}\left(\left(Z_{t}^{(5)}\right)^{2} \mid (\boldsymbol{X}_{u})_{u \in [0,t]}\right) = \int_{0}^{t} e^{-2\lambda u} du \int_{\mathcal{U}_{d}} \langle \boldsymbol{v}, \boldsymbol{r} \rangle^{2} \nu(d\boldsymbol{r})$$

almost surely. Taking the expectation and multiplying by $e^{2\lambda t}$, $t \in \mathbb{R}_+$, we obtain

$$\mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle^2) = \widetilde{E}_{\boldsymbol{v}, \lambda}(t) + \sum_{\ell=1}^d \widetilde{C}_{\boldsymbol{v}, \ell} \widetilde{I}_{\lambda, \ell}(t) + \widetilde{I}_{\lambda}(t) \int_{\mathcal{U}_d} \langle \boldsymbol{v}, \boldsymbol{r} \rangle^2 \nu(\mathrm{d}\boldsymbol{r})$$

with

$$\widetilde{E}_{\boldsymbol{v},\lambda}(t) := \mathbb{E}\bigg(\bigg(e^{\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle + \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle \int_0^t e^{\lambda(t-u)} du\bigg)^2\bigg),$$
$$\widetilde{I}_{\lambda,\ell}(t) := \int_0^t e^{2\lambda(t-u)} \mathbb{E}(X_{u,\ell}) du, \qquad \ell \in \{1, \dots, d\},$$
$$\widetilde{I}_{\lambda}(t) := \int_0^t e^{2\lambda(t-u)} du,$$

and $\widetilde{C}_{\boldsymbol{v},\ell}, \ \ell \in \{1,\ldots,d\}$ defined in Theorem 3.1. For each $t \in \mathbb{R}_+$, we have

$$|\widetilde{E}_{\boldsymbol{v},\lambda}(t)| \leq \mathbb{E}\left(\left|\mathrm{e}^{\lambda t} \langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle + \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle \int_0^t \mathrm{e}^{\lambda(t-u)} \,\mathrm{d}u\right|^2\right) = E_{\boldsymbol{v},\lambda}(t).$$

If $\operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\right]$, then $h(t)E_{\boldsymbol{v},\lambda}(t) \to 0$ as $t \to \infty$, see the proof of Proposition B.1 in Barczy et al. [8], hence

(D.2)
$$h(t)\widetilde{E}_{\boldsymbol{v},\lambda}(t) \to 0$$
 as $t \to \infty$.

Moreover, for each $t \in \mathbb{R}_+$ and $\ell \in \{1, \ldots, d\}$, by formula (2.2), we get

$$\widetilde{I}_{\lambda,\ell}(t) = \boldsymbol{e}_{\ell}^{\top} \widetilde{\boldsymbol{A}}_{\lambda,1}(t) \, \mathbb{E}(\boldsymbol{X}_0) + \boldsymbol{e}_{\ell}^{\top} \widetilde{\boldsymbol{A}}_{\lambda,2}(t) \widetilde{\boldsymbol{\beta}}$$

with

$$\widetilde{\boldsymbol{A}}_{\lambda,1}(t) := \int_0^t e^{2\lambda(t-u)} e^{u\widetilde{\boldsymbol{B}}} \, \mathrm{d}u, \qquad \widetilde{\boldsymbol{A}}_{\lambda,2}(t) := \int_0^t e^{2\lambda(t-u)} \left(\int_0^u e^{w\widetilde{\boldsymbol{B}}} \, \mathrm{d}w \right) \mathrm{d}u.$$

We have

$$\widetilde{\boldsymbol{A}}_{\lambda,1}(t) = e^{2\lambda t} \widetilde{\boldsymbol{A}}_{\lambda,1,1}(t) + e^{2\lambda t} \widetilde{\boldsymbol{A}}_{\lambda,1,2}(t), \qquad t \in \mathbb{R}_+,$$

with

$$\widetilde{\boldsymbol{A}}_{\lambda,1,1}(t) := \int_0^t e^{(s(\widetilde{\boldsymbol{B}}) - 2\lambda)u} \, \widetilde{\boldsymbol{u}} \boldsymbol{u}^\top du, \qquad \widetilde{\boldsymbol{A}}_{\lambda,1,2}(t) := \int_0^t e^{(s(\widetilde{\boldsymbol{B}}) - 2\lambda)u} (e^{-s(\widetilde{\boldsymbol{B}})u} e^{u\widetilde{\boldsymbol{B}}} - \widetilde{\boldsymbol{u}} \boldsymbol{u}^\top) \, du.$$

If $\operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\right)$, then we have

$$e^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t}\widetilde{\boldsymbol{A}}_{\lambda,1,1}(t) = e^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t} \frac{e^{(s(\widetilde{\boldsymbol{B}})-2\lambda)t}-1}{s(\widetilde{\boldsymbol{B}})-2\lambda} \widetilde{\boldsymbol{u}}\boldsymbol{u}^{\top} = \frac{1-e^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t}}{s(\widetilde{\boldsymbol{B}})-2\lambda} \widetilde{\boldsymbol{u}}\boldsymbol{u}^{\top} \to \frac{\widetilde{\boldsymbol{u}}\boldsymbol{u}^{\top}}{s(\widetilde{\boldsymbol{B}})-2\lambda}$$

as $t \to \infty$, and, by (2.4),

$$|e^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t}\tilde{\boldsymbol{A}}_{\lambda,1,2}(t)| \leqslant C_1 e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t} \int_0^t e^{(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))u} e^{-C_2u} du$$
$$\leqslant C_1 e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t} \int_0^t e^{(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda)-\tilde{C}_2)u} du$$
$$\leqslant C_1 e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t} \int_0^\infty e^{(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda)-\tilde{C}_2)u} du$$
$$= \frac{C_1}{s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda)-\tilde{C}_2} e^{-(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))t} \to 0$$

as $t \to \infty$, where $\widetilde{C}_2 \in (0, C_2 \land (s(\widetilde{\boldsymbol{B}}) - 2\operatorname{Re}(\lambda)))$. Hence, if $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}}))$, then

(D.3)
$$h(t)\widetilde{A}_{\lambda,1}(t) = e^{-s(\widetilde{B})t}\widetilde{A}_{\lambda,1}(t) \to \frac{\widetilde{u}u^{\top}}{s(\widetilde{B}) - 2\lambda} \quad \text{as} \ t \to \infty.$$

If $\lambda = 0$, then, by Fubini's theorem, we obtain

$$h(t)\widetilde{\boldsymbol{A}}_{\lambda,2}(t) = e^{-s(\widetilde{\boldsymbol{B}})t}\widetilde{\boldsymbol{A}}_{\lambda,2}(t) = e^{-s(\widetilde{\boldsymbol{B}})t} \int_0^t (t-w)e^{w\widetilde{\boldsymbol{B}}} \,\mathrm{d}w \to \frac{\widetilde{\boldsymbol{u}}\boldsymbol{u}^\top}{s(\widetilde{\boldsymbol{B}})^2} \qquad \text{as} \ t \to \infty,$$

see the proof of Proposition B.1 in Barczy et al. [8]. Hence, if $\lambda = 0$, then

(D.4)
$$h(t)\widetilde{I}_{\lambda,\ell}(t) = e^{-s(\widetilde{B})t}\widetilde{I}_{\lambda,\ell}(t) \to \frac{1}{s(\widetilde{B})}e_{\ell}^{\top}\widetilde{u}u^{\top}\left(\mathbb{E}(X_{0}) + \frac{\widetilde{\beta}}{s(\widetilde{B})}\right)$$
$$= \frac{\langle e_{\ell}, \widetilde{u} \rangle}{s(\widetilde{B})}\left(\langle u, \mathbb{E}(X_{0}) \rangle + \frac{\langle u, \widetilde{\beta} \rangle}{s(\widetilde{B})}\right)$$

as $t \to \infty$. If $\lambda \in \sigma(\widetilde{\boldsymbol{B}}) \setminus \{0\}$ with $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}}))$, then, by Fubini's theorem, we obtain

$$e^{-s(\widetilde{\boldsymbol{B}})t}\widetilde{\boldsymbol{A}}_{\lambda,2}(t) = e^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t} \int_{0}^{t} e^{-2\lambda u} \left(\int_{0}^{u} e^{w\widetilde{\boldsymbol{B}}} dw \right) du$$
$$= e^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t} \int_{0}^{t} \left(\int_{w}^{t} e^{-2\lambda u} du \right) e^{w\widetilde{\boldsymbol{B}}} dw$$
$$= \frac{1}{2\lambda} e^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t} \int_{0}^{t} \left(e^{-2\lambda w} - e^{-2\lambda t} \right) e^{w\widetilde{\boldsymbol{B}}} dw$$
$$= \frac{1}{2\lambda} \left(e^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t} \int_{0}^{t} e^{-2\lambda w} e^{w\widetilde{\boldsymbol{B}}} dw - e^{-s(\widetilde{\boldsymbol{B}})t} \int_{0}^{t} e^{w\widetilde{\boldsymbol{B}}} dw \right)$$
$$\to \frac{1}{2\lambda} \left(\frac{\widetilde{\boldsymbol{u}}\boldsymbol{u}^{\top}}{s(\widetilde{\boldsymbol{B}})-2\lambda} - \frac{\widetilde{\boldsymbol{u}}\boldsymbol{u}^{\top}}{s(\widetilde{\boldsymbol{B}})} \right) = \frac{\widetilde{\boldsymbol{u}}\boldsymbol{u}^{\top}}{(s(\widetilde{\boldsymbol{B}})-2\lambda)s(\widetilde{\boldsymbol{B}})}$$

as $t \to \infty$, since

$$e^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t} \int_0^t e^{-2\lambda w} e^{w\tilde{\boldsymbol{B}}} dw = e^{-s(\tilde{\boldsymbol{B}})t} \widetilde{\boldsymbol{A}}_{\lambda,1}(t) \to \frac{\widetilde{\boldsymbol{u}}\boldsymbol{u}^\top}{s(\tilde{\boldsymbol{B}})-2\lambda}, \qquad e^{-s(\tilde{\boldsymbol{B}})t} \int_0^t e^{w\tilde{\boldsymbol{B}}} dw \to \frac{\widetilde{\boldsymbol{u}}\boldsymbol{u}^\top}{s(\tilde{\boldsymbol{B}})}$$

as $t \to \infty$, by (D.3) and the proof of Proposition B.1 in Barczy et al. [9]. Hence, if $\lambda \in \sigma(\tilde{B}) \setminus \{0\}$ with $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\tilde{B}))$, then

(D.5)
$$h(t)\widetilde{I}_{\lambda,\ell}(t) = e^{-s(\widetilde{\boldsymbol{B}})t}\widetilde{I}_{\lambda,\ell}(t) \to \frac{1}{s(\widetilde{\boldsymbol{B}}) - 2\lambda}\boldsymbol{e}_{\ell}^{\top}\widetilde{\boldsymbol{u}}\boldsymbol{u}^{\top}\left(\mathbb{E}(\boldsymbol{X}_{0}) + \frac{\boldsymbol{\beta}}{s(\widetilde{\boldsymbol{B}})}\right)$$
$$= \frac{\langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle}{s(\widetilde{\boldsymbol{B}}) - 2\lambda} \left(\langle \boldsymbol{u}, \mathbb{E}(\boldsymbol{X}_{0}) \rangle + \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}} \rangle}{s(\widetilde{\boldsymbol{B}})}\right) \quad \text{as} \ t \to \infty.$$

If $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}})$ and $\operatorname{Im}(\lambda) = 0$, then we have

$$t^{-1}\mathrm{e}^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t}\widetilde{\boldsymbol{A}}_{\lambda,1,1}(t) = t^{-1}\int_0^t \widetilde{\boldsymbol{u}}\boldsymbol{u}^\top \mathrm{d}\boldsymbol{u} = \widetilde{\boldsymbol{u}}\boldsymbol{u}^\top, \qquad t \in \mathbb{R}_+$$

and, by (2.4),

$$|t^{-1}e^{-(s(\tilde{\boldsymbol{B}})-2\lambda)t}\tilde{\boldsymbol{A}}_{\lambda,1,2}(t)| = |t^{-1}\tilde{\boldsymbol{A}}_{\lambda,1,2}(t)| \leqslant C_1t^{-1}\int_0^t e^{(s(\tilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda))u}e^{-C_2u}\,\mathrm{d}u$$
$$\leqslant C_1t^{-1}\int_0^\infty e^{-C_2u}\,\mathrm{d}u = \frac{C_1}{C_2}t^{-1} \to 0 \qquad \text{as} \quad t \to \infty.$$

Hence, if $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}})$ and $\operatorname{Im}(\lambda) = 0$, then

$$h(t)\widetilde{\boldsymbol{A}}_{\lambda,1}(t) = t^{-1}\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t}\widetilde{\boldsymbol{A}}_{\lambda,1}(t) \to \widetilde{\boldsymbol{u}}\boldsymbol{u}^{\top} \quad \text{as} \ t \to \infty.$$

If $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}})$ and $\operatorname{Im}(\lambda) = 0$, then, by Fubini's theorem, we obtain

$$t^{-1} \mathrm{e}^{-s(\widetilde{B})t} \widetilde{A}_{\lambda,2}(t) = t^{-1} \int_0^t \mathrm{e}^{-s(\widetilde{B})u} \left(\int_0^u \mathrm{e}^{w\widetilde{B}} \, \mathrm{d}w \right) \mathrm{d}u = t^{-1} \int_0^t \left(\int_w^t \mathrm{e}^{-s(\widetilde{B})u} \, \mathrm{d}u \right) \mathrm{e}^{w\widetilde{B}} \, \mathrm{d}w$$
$$= \frac{1}{s(\widetilde{B})} t^{-1} \int_0^t \left(\mathrm{e}^{-s(\widetilde{B})w} - \mathrm{e}^{-s(\widetilde{B})t} \right) \mathrm{e}^{w\widetilde{B}} \, \mathrm{d}w$$
$$= \frac{1}{s(\widetilde{B})} t^{-1} \left(\int_0^t \mathrm{e}^{-s(\widetilde{B})w} \mathrm{e}^{w\widetilde{B}} \, \mathrm{d}w - \mathrm{e}^{-s(\widetilde{B})t} \int_0^t \mathrm{e}^{w\widetilde{B}} \, \mathrm{d}w \right) \to \frac{\widetilde{u}u^{\mathsf{T}}}{s(\widetilde{B})}$$

as $t \to \infty$, since

$$t^{-1} \int_0^t e^{-s(\widetilde{B})w} e^{w\widetilde{B}} dw \to \widetilde{u}u^\top, \quad e^{-s(\widetilde{B})t} \int_0^t e^{w\widetilde{B}} dw \to \frac{\widetilde{u}u^\top}{s(\widetilde{B})} \quad \text{as} \ t \to \infty,$$

see part (v) of Lemma A.2 and the proof of Proposition B.1 in Barczy et al. [9]. Consequently, if $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}})$ and $\operatorname{Im}(\lambda) = 0$, then

(D.6)
$$t^{-1} e^{-s(\widetilde{\boldsymbol{B}})t} \widetilde{I}_{\lambda,\ell}(t) \to \boldsymbol{e}_{\ell}^{\top} \widetilde{\boldsymbol{u}} \boldsymbol{u}^{\top} \left(\mathbb{E}(\boldsymbol{X}_0) + \frac{\widetilde{\boldsymbol{\beta}}}{s(\widetilde{\boldsymbol{B}})} \right) = \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \left(\langle \boldsymbol{u}, \mathbb{E}(\boldsymbol{X}_0) \rangle + \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}} \rangle}{s(\widetilde{\boldsymbol{B}})} \right)$$

as $t \to \infty$.

If $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}})$ and $\operatorname{Im}(\lambda) \neq 0$, then we have

$$t^{-1} e^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t} \widetilde{\boldsymbol{A}}_{\lambda,1,1}(t) = t^{-1} e^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t} \frac{e^{(s(\widetilde{\boldsymbol{B}})-2\lambda)t}-1}{s(\widetilde{\boldsymbol{B}})-2\lambda} \widetilde{\boldsymbol{u}} \boldsymbol{u}^{\top}$$
$$= \frac{1}{(s(\widetilde{\boldsymbol{B}})-2\lambda)t} (1-e^{2i\operatorname{Im}(\lambda)t}) \widetilde{\boldsymbol{u}} \boldsymbol{u}^{\top} \to \boldsymbol{0}$$

as $t \to \infty$ and

$$\begin{aligned} |t^{-1}\mathrm{e}^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t}\widetilde{\boldsymbol{A}}_{\lambda,1,2}(t)| &\leq C_1 t^{-1}\mathrm{e}^{-(s(\widetilde{\boldsymbol{B}})-2\mathrm{Re}(\lambda))t} \int_0^t \mathrm{e}^{(s(\widetilde{\boldsymbol{B}})-2\mathrm{Re}(\lambda))u}\mathrm{e}^{-C_2u}\,\mathrm{d}u \\ &\leq C_1 t^{-1} \int_0^\infty \mathrm{e}^{-C_2u}\,\mathrm{d}u = \frac{C_1}{C_2} t^{-1} \to 0 \qquad \text{as} \quad t \to \infty. \end{aligned}$$

Hence, if $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}})$ and $\operatorname{Im}(\lambda) \neq 0$, then

$$h(t)\widetilde{A}_{\lambda,1}(t) = t^{-1} e^{-s(\widetilde{B})t} \widetilde{A}_{\lambda,1}(t) \to \mathbf{0}$$
 as $t \to \infty$.

If $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}})$ and $\operatorname{Im}(\lambda) \neq 0$, then, by Fubini's theorem, as above, we obtain

$$t^{-1} \mathrm{e}^{-s(\widetilde{B})t} \widetilde{A}_{\lambda,2}(t) = t^{-1} \mathrm{e}^{-(s(\widetilde{B})-2\lambda)t} \int_0^t \left(\int_w^t \mathrm{e}^{-2\lambda u} \,\mathrm{d}u \right) \mathrm{e}^{w\widetilde{B}} \,\mathrm{d}w$$
$$= \frac{1}{2\lambda t} \mathrm{e}^{-(s(\widetilde{B})-2\lambda)t} \int_0^t (\mathrm{e}^{-2\lambda w} - \mathrm{e}^{-2\lambda t}) \mathrm{e}^{w\widetilde{B}} \,\mathrm{d}w$$
$$= \frac{1}{2\lambda t} \left(\mathrm{e}^{-(s(\widetilde{B})-2\lambda)t} \int_0^t \mathrm{e}^{-2\lambda w} \mathrm{e}^{w\widetilde{B}} \,\mathrm{d}w - \mathrm{e}^{-s(\widetilde{B})t} \int_0^t \mathrm{e}^{w\widetilde{B}} \,\mathrm{d}w \right) \to \mathbf{0}$$

as $t \to \infty$. Indeed, $e^{-s(\tilde{B})t} \int_0^t e^{w\tilde{B}} dw \to \frac{\tilde{u}u^{\top}}{s(\tilde{B})}$ as $t \to \infty$, and using that $\operatorname{Re}(\lambda) = \frac{1}{2}s(\tilde{B})$ and $\operatorname{Im}(\lambda) \neq 0$, for all $t \in \mathbb{R}_+$ we have

$$t^{-1} \mathrm{e}^{-(s(\widetilde{\boldsymbol{B}})-2\lambda)t} \int_{0}^{t} \mathrm{e}^{-2\lambda w} \mathrm{e}^{w\widetilde{\boldsymbol{B}}} \,\mathrm{d}w = t^{-1} \mathrm{e}^{2\mathrm{i}\mathrm{Im}(\lambda)t} \int_{0}^{t} \mathrm{e}^{-2\mathrm{i}\mathrm{Im}(\lambda)w} \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})w} \mathrm{e}^{w\widetilde{\boldsymbol{B}}} \,\mathrm{d}w$$
$$= t^{-1} \mathrm{e}^{2\mathrm{i}\mathrm{Im}(\lambda)t} \left(\int_{0}^{t} \mathrm{e}^{-2\mathrm{i}\mathrm{Im}(\lambda)w} \,\mathrm{d}w \right) \widetilde{\boldsymbol{u}} \boldsymbol{u}^{\top} + t^{-1} \mathrm{e}^{2\mathrm{i}\mathrm{Im}(\lambda)t} \int_{0}^{t} \mathrm{e}^{-2\mathrm{i}\mathrm{Im}(\lambda)w} (\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})w} \mathrm{e}^{w\widetilde{\boldsymbol{B}}} - \widetilde{\boldsymbol{u}} \boldsymbol{u}^{\top}) \,\mathrm{d}w,$$

where

$$\left|t^{-1}\mathrm{e}^{2\mathrm{i}\mathrm{Im}(\lambda)t}\int_{0}^{t}\mathrm{e}^{-2\mathrm{i}\mathrm{Im}(\lambda)w}\,\mathrm{d}w\right| = t^{-1}\left|\frac{\mathrm{e}^{-2\mathrm{i}\mathrm{Im}(\lambda)t}-1}{-2\mathrm{i}\mathrm{Im}(\lambda)}\right| \leqslant \frac{1}{t|\mathrm{Im}(\lambda)|} \to 0$$

as $t \to \infty$, and, by (2.4),

$$\left\| t^{-1} \mathrm{e}^{2\mathrm{i}\mathrm{Im}(\lambda)t} \int_0^t \mathrm{e}^{-2\mathrm{i}\mathrm{Im}(\lambda)w} (\mathrm{e}^{-s(\tilde{B})w} \mathrm{e}^{w\tilde{B}} - \tilde{u}u^\top) \,\mathrm{d}w \right\| \leq t^{-1} \int_0^t \|\mathrm{e}^{-s(\tilde{B})w} \mathrm{e}^{w\tilde{B}} - \tilde{u}u^\top\| \,\mathrm{d}w \leq t^{-1} C_1 \int_0^t \mathrm{e}^{-C_2w} \,\mathrm{d}w \leq t^{-1} C_1 \int_0^\infty \mathrm{e}^{-C_2w} \,\mathrm{d}w = \frac{C_1}{C_2t} \to 0 \qquad \text{as} \quad t \to \infty.$$

Consequently, if $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{B})$ and $\operatorname{Im}(\lambda) \neq 0$, then

(D.7)
$$h(t)\widetilde{I}_{\lambda,\ell}(t) = t^{-1}e^{-s(\widetilde{B})t}\widetilde{I}_{\lambda,\ell}(t) \to 0 \quad \text{as} \ t \to \infty.$$

By the help of (D.4), (D.5), (D.6) and (D.7), we have

$$(D.8) \lim_{t \to \infty} h(t) \widetilde{I}_{\lambda,\ell}(t) = \begin{cases} \frac{\langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle}{s(\widetilde{\boldsymbol{B}}) - 2\lambda} \left(\langle \boldsymbol{u}, \mathbb{E}(\boldsymbol{X}_0) \rangle + \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}} \rangle}{s(\widetilde{\boldsymbol{B}})} \right) & \text{if } \operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2} s(\widetilde{\boldsymbol{B}}) \right), \\ \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \left(\langle \boldsymbol{u}, \mathbb{E}(\boldsymbol{X}_0) \rangle + \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}} \rangle}{s(\widetilde{\boldsymbol{B}})} \right) & \text{if } \operatorname{Re}(\lambda) = \frac{1}{2} s(\widetilde{\boldsymbol{B}}) \text{ and } \operatorname{Im}(\lambda) = 0, \\ 0 & \text{if } \operatorname{Re}(\lambda) = \frac{1}{2} s(\widetilde{\boldsymbol{B}}) \text{ and } \operatorname{Im}(\lambda) \neq 0. \end{cases}$$

Further, we have

$$\widetilde{I}_{\lambda}(t) = \int_{0}^{t} e^{2\lambda w} dw = \begin{cases} t & \text{if } \lambda = 0, \\ \frac{1}{2\lambda} (e^{2\lambda t} - 1) & \text{if } \lambda \neq 0. \end{cases}$$

If $\lambda = 0$, then

$$e^{-s(\widetilde{B})t}\widetilde{I}_{\lambda}(t) = te^{-s(\widetilde{B})t} \to 0 \quad \text{as} \quad t \to \infty.$$

If $\lambda \in \sigma(\widetilde{\boldsymbol{B}}) \setminus \{0\}$ with $\operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\right)$, then

$$e^{-s(\widetilde{B})t}\widetilde{I}_{\lambda}(t) = \frac{1}{2\lambda} (e^{-(s(\widetilde{B})-2\lambda)t} - e^{-s(\widetilde{B})t}) \to 0 \quad \text{as} \quad t \to \infty.$$

If $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}})$, then

$$t^{-1} \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t} \widetilde{I}_{\lambda}(t) = \frac{1}{2\lambda t} (\mathrm{e}^{2\mathrm{i}\mathrm{Im}(\lambda)t} - \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t}) \to 0 \quad \text{as} \quad t \to \infty.$$

Consequently,

(D.9)
$$\lim_{t \to \infty} h(t) \widetilde{I}_{\lambda}(t) = 0.$$

Hence, by (D.2), (D.8) and (D.9), we have

$$\lim_{t \to \infty} h(t) \mathbb{E}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle^2) = \widetilde{M}_{\boldsymbol{v}}^{(2)}$$

with

$$\widetilde{M}_{\boldsymbol{v}}^{(2)} := \begin{cases} \sum_{\ell=1}^{d} \widetilde{C}_{\boldsymbol{v},\ell} \frac{\langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle}{s(\widetilde{\boldsymbol{B}}) - 2\lambda} \left(\langle \boldsymbol{u}, \mathbb{E}(\boldsymbol{X}_{0}) \rangle + \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}} \rangle}{s(\widetilde{\boldsymbol{B}})} \right) & \text{if } \operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\right), \\ \sum_{\ell=1}^{d} \widetilde{C}_{\boldsymbol{v},\ell} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \left(\langle \boldsymbol{u}, \mathbb{E}(\boldsymbol{X}_{0}) \rangle + \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}} \rangle}{s(\widetilde{\boldsymbol{B}})} \right) & \text{if } \operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}}) \text{ and } \operatorname{Im}(\lambda) = 0, \\ 0 & \text{if } \operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}}) \text{ and } \operatorname{Im}(\lambda) \neq 0. \end{cases}$$

Using the identity (D.1), then taking the limit as $t \to \infty$, and using Proposition D.1, we obtain

$$h(t) \mathbb{E} \left(\begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix}^{\mathsf{T}} \right) \rightarrow \frac{1}{2} M_{\boldsymbol{v}}^{(2)} \boldsymbol{I}_2 + \frac{1}{2} \begin{pmatrix} \operatorname{Re}(\widetilde{M}_{\boldsymbol{v}}^{(2)}) & \operatorname{Im}(\widetilde{M}_{\boldsymbol{v}}^{(2)}) \\ \operatorname{Im}(\widetilde{M}_{\boldsymbol{v}}^{(2)}) & -\operatorname{Re}(\widetilde{M}_{\boldsymbol{v}}^{(2)}) \end{pmatrix} \\ = \left(\langle \boldsymbol{u}, \mathbb{E}(\boldsymbol{X}_0) \rangle + \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}} \rangle}{s(\widetilde{\boldsymbol{B}})} \right) \boldsymbol{\Sigma}_{\boldsymbol{v}}$$

as $t \to \infty$, as desired.

D.3 Proposition. Let $(\mathbf{X}_t)_{t\in\mathbb{R}_+}$ be a supercritical and irreducible multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|^2) < \infty$ and the moment condition (3.8) holds. Let $\lambda \in \sigma(\widetilde{\mathbf{B}})$ with $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\widetilde{\mathbf{B}})]$ and $\mathbf{v} \in \mathbb{C}^d$ be a left-eigenvector of $\widetilde{\mathbf{B}}$ corresponding to the eigenvalue λ . Then $\mathbf{\Sigma}_{\mathbf{v}} = \mathbf{0}$ if and only if $c_{\ell}\langle \mathbf{v}, \mathbf{e}_{\ell} \rangle = 0$ and $\mu_{\ell}(\{\mathbf{z} \in \mathcal{U}_d : \langle \mathbf{v}, \mathbf{z} \rangle \neq 0\}) = 0$ for each $\ell \in \{1, \ldots, d\}$. If, in addition, $\operatorname{Im}(\lambda) \neq 0$, then $\mathbf{\Sigma}_{\mathbf{v}}$ is invertible if and only if there exists $\ell \in \{1, \ldots, d\}$ such that $c_{\ell}\langle \mathbf{v}, \mathbf{e}_{\ell} \rangle \neq 0$ or $\mu_{\ell}(\{\mathbf{z} \in \mathcal{U}_d : \langle \mathbf{v}, \mathbf{z} \rangle \neq 0\}) > 0$.

Proof. First, suppose that $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}}))$. If $c_{\ell}\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle = 0$ and $\mu_{\ell}(\{\boldsymbol{z} \in \mathcal{U}_{d} : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) = 0$ for each $\ell \in \{1, \ldots, d\}$, then we have $C_{\boldsymbol{v},\ell} = \widetilde{C}_{\boldsymbol{v},\ell} = 0, \ \ell \in \{1, \ldots, d\}$,

yielding that $\Sigma_{\boldsymbol{v}} = \boldsymbol{0}$. If there exists an $\ell \in \{1, \ldots, d\}$ such that $c_{\ell} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle \neq 0$ or $\mu_{\ell}(\{\boldsymbol{z} \in \mathcal{U}_{d} : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) > 0$, then we check that $\Sigma_{\boldsymbol{v}} \neq \boldsymbol{0}$, as requested. On the contrary, let us suppose that $\Sigma_{\boldsymbol{v}} = \boldsymbol{0}$. Due to the existence of such an ℓ , we have $C_{\boldsymbol{v},\ell} \in \mathbb{R}_{++}$ and hence $\sum_{k=1}^{d} \langle \boldsymbol{e}_{k}, \widetilde{\boldsymbol{u}} \rangle_{\overline{s(\widetilde{B})-2\operatorname{Re}(\lambda)}}^{C_{\boldsymbol{v},k}} \in \mathbb{R}_{++}$. However, using the notation $\Sigma_{\boldsymbol{v}} = ((\Sigma_{\boldsymbol{v}})_{i,j})_{i,j\in\{1,2\}}$, since

$$(\Sigma_{\boldsymbol{v}})_{1,1} = \frac{1}{2} \sum_{k=1}^{d} \langle \boldsymbol{e}_{k}, \widetilde{\boldsymbol{u}} \rangle \frac{C_{\boldsymbol{v},k}}{s(\widetilde{\boldsymbol{B}}) - 2\operatorname{Re}(\lambda)} + \frac{1}{2} \sum_{k=1}^{d} \langle \boldsymbol{e}_{k}, \widetilde{\boldsymbol{u}} \rangle \operatorname{Re}\left(\frac{\widetilde{C}_{\boldsymbol{v},k}}{s(\widetilde{\boldsymbol{B}}) - 2\lambda}\right) = 0,$$

$$(\Sigma_{\boldsymbol{v}})_{2,2} = \frac{1}{2} \sum_{k=1}^{d} \langle \boldsymbol{e}_{k}, \widetilde{\boldsymbol{u}} \rangle \frac{C_{\boldsymbol{v},k}}{s(\widetilde{\boldsymbol{B}}) - 2\operatorname{Re}(\lambda)} - \frac{1}{2} \sum_{k=1}^{d} \langle \boldsymbol{e}_{k}, \widetilde{\boldsymbol{u}} \rangle \operatorname{Re}\left(\frac{\widetilde{C}_{\boldsymbol{v},k}}{s(\widetilde{\boldsymbol{B}}) - 2\lambda}\right) = 0,$$

we have $\sum_{k=1}^{d} \langle \boldsymbol{e}_k, \widetilde{\boldsymbol{u}} \rangle_{\frac{C_{\boldsymbol{v},k}}{s(\widetilde{\boldsymbol{B}})-2\operatorname{Re}(\lambda)}} = 0$, yielding us to a contradiction.

Next, suppose that $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{B})$. If $c_{\ell}\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle = 0$ and $\mu_{\ell}(\{\boldsymbol{z} \in \mathcal{U}_{d} : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) = 0$ for each $\ell \in \{1, \ldots, d\}$, then, as in case of $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\widetilde{B}))$, we have $\Sigma_{\boldsymbol{v}} = \boldsymbol{0}$. If there exists an $\ell \in \{1, \ldots, d\}$ such that $c_{\ell}\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle \neq 0$ or $\mu_{\ell}(\{\boldsymbol{z} \in \mathcal{U}_{d} : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) > 0$, then we check that $\Sigma_{\boldsymbol{v}} \neq \boldsymbol{0}$, as requested. On the contrary, let us suppose that $\Sigma_{\boldsymbol{v}} = \boldsymbol{0}$. Similarly, as in case of $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\widetilde{B}))$, we have $\sum_{k=1}^{d} \langle \boldsymbol{e}_{k}, \widetilde{\boldsymbol{u}} \rangle C_{\boldsymbol{v},k} = 0$, yielding us to a contradiction.

Recall that, by (4.14),

$$\boldsymbol{\Sigma}_{\boldsymbol{v}} = 2\sum_{\ell=1}^{d} c_{\ell} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \int_{0}^{\infty} \boldsymbol{f}(w, \boldsymbol{e}_{\ell}) \, \mathrm{d}w + \sum_{\ell=1}^{d} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle \int_{0}^{\infty} \int_{\mathcal{U}_{d}} \boldsymbol{f}(w, \boldsymbol{z}) \, \mathrm{d}w \, \mu_{\ell}(\mathrm{d}\boldsymbol{z}) =: \boldsymbol{\Sigma}_{\boldsymbol{v}, 1} + \boldsymbol{\Sigma}_{\boldsymbol{v}, 2},$$

where both $\Sigma_{v,1}$ and $\Sigma_{v,2}$ (and consequently Σ_v) are symmetric and non-negative definite matrices, since $\boldsymbol{c} \in \mathbb{R}^d_+$, $\tilde{\boldsymbol{u}} \in \mathbb{R}^d_{++}$, and $\boldsymbol{f}(w, \boldsymbol{z})$ is symmetric and non-negative definite for any $w \in \mathbb{R}_+$ and $\boldsymbol{z} \in \mathcal{U}_d$.

In what follows, let us assume that $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\widetilde{B})]$ and $\operatorname{Im}(\lambda) \neq 0$. First, let us suppose that $\Sigma_{\boldsymbol{v}}$ is invertible, and, on the contrary, for each $\ell \in \{1, \ldots, d\}$, we have $c_{\ell} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle = 0$ and $\mu_{\ell}(\{\boldsymbol{z} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) = 0$. Then, for each $\ell \in \{1, \ldots, d\}$, we have $C_{\boldsymbol{v},\ell} = \widetilde{C}_{\boldsymbol{v},\ell} = 0$, and hence, by (3.5) and (3.7), $\Sigma_{\boldsymbol{v}} = \mathbf{0}$, yielding us to a contradiction.

Let us suppose now that there exists $\ell \in \{1, \ldots, d\}$ such that $c_{\ell} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle \neq 0$ or $\mu_{\ell} (\{\boldsymbol{z} \in \mathcal{U}_{d} : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) > 0$. Next we show that if $\ell \in \{1, \ldots, d\}$ is such that $c_{\ell} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle \neq 0$, then $\boldsymbol{\Sigma}_{\boldsymbol{v},1}$ is strictly positive definite, and that if $\ell \in \{1, \ldots, d\}$ is such that $\mu_{\ell} (\{\boldsymbol{z} \in \mathcal{U}_{d} : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) > 0$, then $\boldsymbol{\Sigma}_{\boldsymbol{v},2}$ is strictly positive definite, yielding that $\boldsymbol{\Sigma}_{\boldsymbol{v}}$ is strictly positive definite, and consequently is invertible. Here for all $\boldsymbol{w} \in \mathbb{R}_{+}, \boldsymbol{z} \in \mathcal{U}_{d}, \text{ and } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}$, we have

$$\begin{pmatrix} a \\ b \end{pmatrix}^{\top} \boldsymbol{f}(w, \boldsymbol{z}) \begin{pmatrix} a \\ b \end{pmatrix} = \left(a \operatorname{Re}(e^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)w/2} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) + b \operatorname{Im}(e^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)w/2} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \right)^{2}$$
$$= \left(\operatorname{Re}((a - \mathrm{i}b)e^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)w/2} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \right)^{2}.$$

Consequently, if $\ell \in \{1, \ldots, d\}$ is such that $\langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle \neq 0$, then for each $(a, b)^{\top} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, we have

$$\begin{pmatrix} a \\ b \end{pmatrix}^{\top} \int_{0}^{\infty} \boldsymbol{f}(w, \boldsymbol{e}_{\ell}) \, \mathrm{d}w \begin{pmatrix} a \\ b \end{pmatrix} = \int_{0}^{\infty} \left(\operatorname{Re}((a - \mathrm{i}b) \mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)w/2} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle) \right)^{2} \, \mathrm{d}w$$

is equal to 0 if and only if $\operatorname{Re}((a-\mathrm{i}b)e^{-(s(\tilde{B})-2\lambda)w/2}\langle v, e_{\ell}\rangle) = 0$ for every $w \in \mathbb{R}_+$ or equivalently $\operatorname{Re}(e^{\mathrm{iIm}(\lambda)w}(a-\mathrm{i}b)\langle v, e_{\ell}\rangle) = 0$ for every $w \in \mathbb{R}_+$. Since $(a-\mathrm{i}b)\langle v, e_{\ell}\rangle \neq 0$ and $\operatorname{Im}(\lambda) \neq 0$, there exists $w \in \mathbb{R}_+$ such that $\operatorname{Re}(e^{\mathrm{iIm}(\lambda)w}(a-\mathrm{i}b)\langle v, e_{\ell}\rangle) \neq 0$. Indeed, the multiplication by the complex number $e^{\mathrm{iIm}(\lambda)w}$ corresponds to a rotation by degree $\operatorname{Im}(\lambda)w$. Hence for each $(a, b)^{\top} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, we have

$$\binom{a}{b}^{+} \int_{0}^{\infty} \boldsymbol{f}(w, \boldsymbol{e}_{\ell}) \, \mathrm{d}w \, \binom{a}{b} \in \mathbb{R}_{++}.$$

This yields that if $\ell \in \{1, \ldots, d\}$ is such that $c_{\ell} \langle \boldsymbol{v}, \boldsymbol{e}_{\ell} \rangle \neq 0$, then for each $(a, b)^{\top} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$,

$$\begin{pmatrix} a \\ b \end{pmatrix}^{\top} \boldsymbol{\Sigma}_{\boldsymbol{v},1} \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}_{++},$$

implying that $\Sigma_{v,1}$ is strictly positive definite. Further, for each $(a, b)^{\top} \in \mathbb{R}^2$, we have

$$\begin{pmatrix} a \\ b \end{pmatrix}^{\top} \int_{0}^{\infty} \int_{\mathcal{U}_{d}} \boldsymbol{f}(w, \boldsymbol{z}) \, \mathrm{d}w \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \begin{pmatrix} a \\ b \end{pmatrix} = \int_{0}^{\infty} \int_{\mathcal{U}_{d}} \left(\mathrm{Re}((a - \mathrm{i}b)\mathrm{e}^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)w/2} \langle \boldsymbol{v}, \boldsymbol{z} \rangle) \right)^{2} \, \mathrm{d}w \mu_{\ell}(\mathrm{d}\boldsymbol{z}).$$

Since $\operatorname{Im}(\lambda) \neq 0$, for each $\boldsymbol{z} \in \mathcal{U}_d$ with $\langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0$ and $(a, b)^{\top} \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\}$, there exists an open subset $K_{\boldsymbol{z}}$ of \mathbb{R}_+ such that $\left(\operatorname{Re}((a - ib)e^{-(s(\tilde{\boldsymbol{B}}) - 2\lambda)w/2}\langle \boldsymbol{v}, \boldsymbol{z} \rangle)\right)^2 \in \mathbb{R}_{++}$ for all $w \in K_{\boldsymbol{z}}$. Consequently, if $\ell \in \{1, \ldots, d\}$ is such that $\mu_{\ell}(\{\boldsymbol{z} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) > 0$, then

$$\int_{0}^{\infty} \int_{\mathcal{U}_{d}} \left(\operatorname{Re}((a-\mathrm{i}b)\mathrm{e}^{-(s(\tilde{B})-2\lambda)w/2}\langle \boldsymbol{v},\boldsymbol{z}\rangle) \right)^{2} \mathrm{d}w \mu_{\ell}(\mathrm{d}\boldsymbol{z})$$

$$\geq \int_{\mathcal{U}_{d}} \mathbb{1}_{\{\langle \boldsymbol{v},\boldsymbol{z}\rangle \neq 0\}} \left(\int_{K_{\boldsymbol{z}}} \left(\operatorname{Re}((a-\mathrm{i}b)\mathrm{e}^{-(s(\tilde{B})-2\lambda)w/2}\langle \boldsymbol{v},\boldsymbol{z}\rangle) \right)^{2} \mathrm{d}w \right) \mu_{\ell}(\mathrm{d}\boldsymbol{z}) \in \mathbb{R}_{++}.$$

This yields that if $\ell \in \{1, \ldots, d\}$ is such that $\mu_{\ell}(\{\boldsymbol{z} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) > 0$, then for each $(a, b)^{\top} \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\}$, we have

$$\begin{pmatrix} a \\ b \end{pmatrix}^{\top} \Sigma_{v,2} \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}_{++},$$

implying that $\Sigma_{v,2}$ is strictly positive definite.

D.4 Remark. Under the conditions of Proposition D.3, if $\lambda \in \sigma(\tilde{B})$ with $\operatorname{Re}(\lambda) \in (-\infty, \frac{1}{2}s(\tilde{B})]$ and $\operatorname{Im}(\lambda) = 0$ and $v \in \mathbb{R}^d$ is a left eigenvector of \tilde{B} corresponding

to the eigenvalue λ , then Σ_{v} is singular. Indeed, in this case, by (3.7) and (3.5), we have

$$\boldsymbol{\Sigma}_{\boldsymbol{v}} = \begin{cases} \begin{pmatrix} \frac{1}{s(\widetilde{\boldsymbol{B}}) - 2\lambda} \sum_{\ell=1}^{d} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle C_{\boldsymbol{v},\ell} & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\right), \\ \begin{pmatrix} \sum_{\ell=1}^{d} \langle \boldsymbol{e}_{\ell}, \widetilde{\boldsymbol{u}} \rangle C_{\boldsymbol{v},\ell} & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}}). \end{cases}$$

Note that if $\boldsymbol{v} \in \mathbb{R}^d$ is a left eigenvector of $\widetilde{\boldsymbol{B}}$ corresponding to an eigenvalue λ of $\widetilde{\boldsymbol{B}}$, then $\lambda \in \mathbb{R}$ necessarily, and hence in case of $\lambda \in (-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})]$, we have $\boldsymbol{\Sigma}_{\boldsymbol{v}}$ is not invertible. However, if $\lambda \in \mathbb{R}$ is an eigenvalue of $\widetilde{\boldsymbol{B}}$ and $\boldsymbol{v} \in \mathbb{C}^d$ is a left eigenvector of $\widetilde{\boldsymbol{B}}$ corresponding to λ , then $\operatorname{Re}(\boldsymbol{v}) \in \mathbb{R}^d$ and $\operatorname{Im}(\boldsymbol{v}) \in \mathbb{R}^d$ are also left eigenvectors of $\widetilde{\boldsymbol{B}}$ or the zero vector. \Box

E A limit theorem for martingales

The next theorem is about the asymptotic behavior of multivariate martingales.

E.1 Theorem. (Crimaldi and Pratelli [11, Theorem 2.2]) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $(\mathbf{M}_t)_{t \in \mathbb{R}_+}$ be a d-dimensional martingale with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ such that it has càdlàg sample paths almost surely. Suppose that there exists a function $\mathbf{Q}: \mathbb{R}_+ \to \mathbb{R}^{d \times d}$ such that $\lim_{t \to \infty} \mathbf{Q}(t) = \mathbf{0}$,

(E.1)
$$\boldsymbol{Q}(t)[\boldsymbol{M}]_t \boldsymbol{Q}(t)^\top \stackrel{\mathbb{P}}{\longrightarrow} \boldsymbol{\eta} \quad as \ t \to \infty,$$

where $\boldsymbol{\eta}$ is a $d \times d$ random (necessarily positive semidefinite) matrix and $([\boldsymbol{M}]_t)_{t \in \mathbb{R}_+}$ denotes the quadratic variation (matrix-valued) process of $(\boldsymbol{M}_t)_{t \in \mathbb{R}_+}$, and

(E.2)
$$\mathbb{E}\left(\sup_{u\in[0,t]} \|\boldsymbol{Q}(t)(\boldsymbol{M}_u - \boldsymbol{M}_{u-})\|\right) \to 0 \quad as \ t \to \infty.$$

Then, for each $\mathbb{R}^{k \times \ell}$ -valued random matrix \mathbf{A} defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with some $k, \ell \in \mathbb{N}$, we have

$$(\boldsymbol{Q}(t)\boldsymbol{M}_t,\boldsymbol{A}) \stackrel{\mathcal{D}}{\longrightarrow} (\boldsymbol{\eta}^{1/2}\boldsymbol{Z},\boldsymbol{A}) \qquad as \ t \to \infty_{\mathcal{T}}$$

where Z is a d-dimensional random vector with $Z \stackrel{\mathcal{D}}{=} \mathcal{N}_d(\mathbf{0}, I_d)$ independent of $(\boldsymbol{\eta}, A)$.

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