이학박사 학위논문

Virtual pullbacks, cosection localization, and Donaldson-Thomas theory of Calabi-Yau 4-folds

(가상 당김, 여절단 국소화 및 4차원 칼라비-야우 다양체의 도널드슨-토마스 이론)

2022년 8월

서울대학교 대학원

수리과학부

박 현 준

Virtual pullbacks, cosection localization, and Donaldson-Thomas theory of Calabi-Yau 4-folds

(가상 당김, 여절단 국소화 및 4차원 칼라비-야우 다양체의 도널드슨-토마스 이론)

지도교수 김 영 훈

이 논문을 이학박사 학위논문으로 제출함

2022년 4월

서울대학교 대학원

수리과학부

박 현 준

박 현 준의 이학박사 학위논문을 인준함

2022년 6월

위 원 장	현 동 훈	(인)
부위원장	김 영 훈	(인)
위 원	최 진 원	(인)
위 원	Atanas Iliev	(인)
위 원	Martijn Kool	(인)

Virtual pullbacks, cosection localization, and Donaldson-Thomas theory of Calabi-Yau 4-folds

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

Park, Hyeonjun

Dissertation Director : Professor Young-Hoon Kiem

Department of Mathematical Sciences
Seoul National University

August 2022

Abstract

Virtual pullbacks, cosection localization, and Donaldson-Thomas theory of Calabi-Yau 4-folds

Park, Hyeonjun

Department of Mathematical Sciences
The Graduate School
Seoul National University

This dissertation is based on the four papers [KP1, KP2, Park1, AKLPR] and the two papers in progress [BKP, BP].

The main purpose is to generalize Manolache's virtual pullbacks and Kiem-Li's cosection localization to Donaldson-Thomas theory of Calabi-Yau 4-folds. The three main applications are Lefschetz principle, Pairs/Sheaves correspondence, and a foundation of surface counting theory.

A secondary purpose is to revisit virtual pullbacks and cosection localization via the Kimura sequence for Artin stacks, derived algebraic geometry, and algebraic cobordism. We also prove Graber-Pandaripande's torus localization formula in full generality.

Key words: Virtual pullbacks, cosection localization, Donaldson-Thomas theory of Calabi-Yau 4-folds
Student Number: 2018-20625

Contents

Introduction

Modern enumerative geometry studies invariants defined through *virtual cycles*. Moduli spaces are often singular and the fundamental cycles do not behave well. A remarkable idea of Kontsevich [Kon] is that these moduli spaces are actually truncations of quasi-smooth derived moduli spaces, and the fundamental cycles of these derived moduli spaces are well behaved. Conceptually, virtual cycles are the fundamental cycles of these derived enhancements. A rigorous mathematical foundation of virtual cycles was later established by Li-Tian [LT] and Behrend-Fantechi [BF] through the formalism of perfect obstruction theories.

These virtual enumerative invariants have been studied intensively during the last three decades and many interesting structures have been discovered. The main examples are Gromov-Witten theory [BM, Beh1] of counting curves and Donaldson-Thomas theory [DT, Tho] of counting sheaves.

There are two powerful tools handling virtual cycles developed to compute the virtual invariants.

A. Virtual pullbacks of Manolache [Man];

B. Cosection localization of Kiem-Li [KL1].

These two tools have vast applications in both the theoretical and computational aspects. In particular, other effective tools such as the torus localization formula [GP] and the degeneration formula [Li1, Li2, LW] can be shown as corollaries of the virtual pullback formula [Man].

Recently, a new type of virtual cycles was introduced for Donaldson-Thomas theory of Calabi-Yau 4-folds (in short DT4 theory) by Borisov-Joyce [BJ] and Oh-Thomas [OT]. Conceptually, these virtual cycles are the fundamental cycles of quasi-smooth derived Lagrangians of (-2)-shifted symplectic derived moduli spaces [PTVV]. There are already rich references on virtual invariants defined through these new virtual cycles, see [CL, CK1, CMT1, CMT2, CK2, CKM, CT19, CT20, CT21, Boj, COT1, COT2].

The main purpose of this dissertation is to develop analogs of the above two key tools in DT4 theory. In particular, this proves various conjectures in DT4 theory. Moreover, this opens a theory of counting *surfaces* on Calabi-Yau 4-folds. This is based on [Park1, KP2, BKP].

A secondary purpose of this dissertation is to revisit the above two key tools via recent developments in intersection theory of Artin stacks, derived algebraic geometry, and algebraic cobordism. This is based on [BP, AKLPR, KP1].

Background: Virtual intersection theory

Virtual cycles

Heuristically, virtual cycles are the fundamental cycles of quasi-smooth derived enhancements. The rigorous construction in [LT, BF] only uses a classical shadow of a derived enhancement, called a *perfect obstruction theory*.

We briefly summarize the construction. There are two key ingredients.

1. A perfect obstruction theory $\phi : \mathbb{F} \to \mathbb{L}_X$ for a scheme X induces a closed embedding
$$\iota : \mathfrak{C}_X \hookrightarrow \mathfrak{E}$$
of the intrinsic normal cone $\mathfrak{C}_X$ into a vector bundle stack $\mathfrak{E} := h^1/h^0(\mathbb{F}^\vee)$.

2. We have a Gysin pullback of the vector bundle stack $\mathfrak{E}$,
$$0^!_{\mathfrak{E}} : A_*(\mathfrak{E}) \to A_*(X),$$
given by the homotopy property of Chow groups [Kre2].

The *virtual cycle* is then defined as the cycle class
$$[X]^{\mathrm{vir}} := 0^!_{\mathfrak{E}}[\mathfrak{C}_X] \in A_*(X).$$

Virtual pullbacks

Manolache [Man] introduced the notion of virtual pullbacks as relative versions of virtual cycles. In the perspective of Fulton's intersection theory [Ful], virtual pullbacks are nothing but just the natural generalizations of the refined Gysin pullbacks for closed embeddings to arbitrary morphisms. This is achieved through replacing normal cones by intrinsic normal cones and vector bundles by vector bundle stacks.

More precisely, if $f : X \to Y$ is a morphism of schemes with a (relative) perfect obstruction theory $\phi : \mathbb{F} \to \mathbb{L}_{X/Y}$, then we have a closed embedding $\iota : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$ of the (relative) intrinsic normal cone $\mathfrak{C}_{X/Y}$ into a vector bundle stack $\mathfrak{E} := h^1/h^0(\mathbb{F}^\vee)$, and the *virtual pullback* is defined as the composition

$$f^! : A_*(Y) \xrightarrow{\mathrm{sp}_{X/Y}} A_*(\mathfrak{C}_{X/Y}) \xrightarrow{\iota_*} A_*(\mathfrak{E}) \xrightarrow{0^!_{\mathfrak{E}}} A_*(X)$$

where $\mathrm{sp} : A_*(Y) \to A_*(\mathfrak{C}_{X/Y})$ is the specialization map. Thus we may view virtual cycles/virtual pullbacks as generalizations of intersection theory for schemes to algebraic stacks.

The main property of virtual pullbacks is the functoriality. Indeed, if

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z. \end{tikzcd}$$

is a commutative diagram of schemes with a compatible triple of perfect obstruction theories

$$\begin{tikzcd} \mathbb{E}_{Y/Z}|_X \arrow[r] \arrow[d] & \mathbb{E}_{X/Z} \arrow[r] \arrow[d] & \mathbb{E}_{X/Y} \arrow[d] \\ \mathbb{L}_{Y/Z}|_X \arrow[r] & \mathbb{L}_{X/Z} \arrow[r] & \mathbb{L}_{X/Y} \end{tikzcd}$$

then we have

$$(g \circ f)^! = f^! \circ g^! : A_*(Z) \to A_*(X).$$

In particular, when $Z = \mathrm{Spec}(\mathbb{C})$, we have a *virtual pullback formula*

$$[X]^{\mathrm{vir}} = f^![Y]^{\mathrm{vir}} \in A_*(X).$$

It is desired to extend virtual cycles/virtual pullbacks to obstruction theories $\phi : \mathbb{E} \to L_{X/Y}$ of arbitrary tor-amplitude, where the above construction only works when $\mathbb{E}$ is of tor-amplitude $[-1, 0]$. We will consider three variants of virtual pullbacks in the following cases where $\mathbb{E}$ is of tor-amplitude $[-2, 0]$.

1. $\mathbb{E}$ is the cone of a map $O_X[1] \to \mathbb{F}$ where $\mathbb{F}$ is of tor-amplitude $[-1, 0]$;

2. $\mathbb{E}$ is a symmetric complex of tor-amplitude $[-2, 0]$;

3. $\mathbb{E}$ is a $\mathbb{G}_m$-equivariant complex where $\mathbb{E}^{\mathrm{fix}}$ is of tor-amplitude $[-1, 0]$.

Cosection localization

Kiem-Li [KL1] showed that a virtual cycle can be localized to a smaller locus when there is a cosection. We may view this cosection localization as a first variant of considering an obstruction theory of tor-amplitude $[-2, 0]$.

A precise statement for cosection localization can be divided into two parts, analogous to the construction of virtual cycles in the previous subsection.

1. A cosection $\sigma : \mathbb{E}^\vee[1] \to O_X$ for a scheme X with a perfect obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_X$ gives rise to a *cone reduction*,
$$(\mathfrak{C}_X)_{\mathrm{red}} \subseteq \mathfrak{K},$$
where $\mathfrak{K} := h^1/h^0(\mathbb{E}_\sigma^\vee)$ is the *kernel cone stack*, defined as the abelian cone stack associated to $\mathbb{E}_\sigma := \mathrm{cone}(\sigma^\vee[1] : O_X[1] \to \mathbb{E})$.

2. We have a *cosection-localized Gysin map*
$$0^!_{\mathfrak{E},\sigma} : A_*(\mathfrak{K}) \to A_*(X(\sigma))$$
where $X(\sigma)$ is the zero locus of $\overline{\sigma} := h^0(\sigma) : h^1(\mathbb{E}^\vee) \to O_X$ in X.

The main outcome is the *cosection-localized virtual cycle*, defined as
$$[X]^{\mathrm{loc}} := 0^!_{\mathfrak{E},\sigma}[\mathfrak{C}_X] \in A_*(X(\sigma)).$$

In particular, the virtual cycle vanishes when $\overline{\sigma}$ is surjective. In this case, the kernel cone stack $\mathfrak{K}$ is a vector bundle stack and we thus have an additional outcome, the *reduced virtual cycle*, defined as
$$[X]^{\mathrm{red}} := 0^!_{\mathfrak{K}}[\mathfrak{C}_X] \in A_{*+1}(X).$$

Donaldson-Thomas theory of Calabi-Yau 4-folds

Donaldson-Thomas invariants were first introduced by Thomas [Tho] as virtual counts of stable sheaves on Calabi-Yau 3-folds and Fano 3-folds. Many interesting structures have been discovered, e.g. connection to Gromov-Witten theory [MNOP1, MNOP2, PP] and rationality [PT1, PT2] for rank 1 invariants, reduction of higher rank invariants via rank 1 invariants [FT], motivic property [Beh2, JS], categorification [BBDJS, KL2, MT], and modularity for rank 0 invariants [GS, TT].

It was desired to extend the Donaldson-Thomas theory to higher-dimensional varieties. The main difficulty was that the natural obstruction theory on moduli spaces of sheaves on higher-dimensional varieties are no longer of tor-amplitude $[-1,0]$ and the standard method of constructing virtual cycles in [LT, BF] does not work. Thus a completely new method was required.

For moduli spaces of sheaves on Calabi-Yau 4-folds, Cao-Leung [CL] first defined virtual cycles in special cases and Borisov-Joyce later defined (topological) virtual cycles in the general case. However computation of DT4 invariants through the Borisov-Joyce virtual cycles was believed to be very difficult.

In the groundbreaking work [OT], Oh-Thomas constructed algebraic virtual cycles for Calabi-Yau 4-folds. This enabled us to extend the two key tools, virtual pullbacks and cosection localization, to DT4 theory [Park1, KP2].

Oh-Thomas virtual cycles

The crucial part of Oh-Thomas's construction is the following local model. Let E be a special orthogonal bundle on a scheme Y and $s \in \Gamma(Y, E)$ be an isotropic section. Let X be the zero locus of s in Y,

$$\begin{tikzcd} & E \arrow[d] \\ X \arrow[r, hook] & Y. \arrow[u, bend right, "s"'] \end{tikzcd}$$

Oh-Thomas constructed a localization

$$\sqrt{e}(E,s) : A_*(Y) \to A_*(X)$$

of the square root Euler class $\sqrt{e}(E) : A_*(Y) \to A_*(Y)$ of Edidin-Graham [EG1], using cosection localization of Kiem-Li [KL1].

The global construction is then given as follows. Let X be a moduli space of stable sheaves on a Calabi-Yau 4-fold. Then X carries a symmetric obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_X$ of tor-amplitude $[-2,0]$. If we choose a symmetric resolution $\mathbb{E} \cong [B \to E^\vee \to B^\vee]$, then the stupid truncation gives us a closed embedding $\mathfrak{C}_X \hookrightarrow [E/B]$. Form a fiber diagram

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & E \arrow[d] \\ \mathfrak{C}_X \arrow[r, hook] & {[E/B].} \end{tikzcd}$$

Then the zero section $0_C : X \hookrightarrow C$ is the zero locus of the tautological section $\tau \in \Gamma(C, E|_C)$,

$$\begin{tikzcd} & E|_C \arrow[d] \\ X \arrow[r, hook, "0_C"] & C. \arrow[u, bend right, "\tau"'] \end{tikzcd}$$

The *Oh-Thomas virtual cycle* is then defined as

$$[X]^{\mathrm{vir}} := \sqrt{e}(E|_C, \tau)[C] \in A_*(X).$$

The tautological section τ is isotropic by the derived Darboux theorem [BBJ, BBBJ, BG] and the orthogonal bundle E is orientable by [CGJ].

Square root virtual pullbacks

Oh-Thomas virtual cycles can be generalized to the relative setting, analogous to Manlache's virtual pullbacks.

Let $f : X \to Y$ be a morphism of schemes with an oriented symmetric obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_{X/Y}$ of tor-amplitude $[-2,0]$. Then there exists a canonical quadratic function $\mathfrak{q}_{\mathbb{E}} : \mathfrak{C}(\mathbb{E}) \to \mathbb{A}^1_X$ on the associated abelian cone stack $\mathfrak{C}(\mathbb{E}) := h^1/h^0(\mathbb{E}^\vee)$ induced by the symmetric form of $\mathbb{E}$.

1. If the intrinsic normal cone $\mathfrak{C}_X$ is *isotropic* with respect to $\mathfrak{q}_{\mathbb{E}}$, then we have a closed embedding
$$\iota : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{Q}(\mathbb{E})$$
into the *quadratic cone stack* $\mathfrak{Q}(\mathbb{E})$, defined as the zero locus of $\mathfrak{q}_{\mathbb{E}}$.

2. We have a *square root Gysin pullback*
$$\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}} : A_*(\mathfrak{Q}(\mathbb{E})) \to A_*(X)$$
for the quadratic cone stack $\mathfrak{Q}(\mathbb{E})$.

Definition A ([Park1])**.** Let $f : X \to Y$ be a morphism of schemes with an oriented symmetric obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_{X/Y}$ of tor-amplitude $[-2,0]$ satisfying the isotropic condition. We define the *square root virtual pullback* as the composition

$$\sqrt{f^!} : A_*(Y) \xrightarrow{\mathrm{sp}_{X/Y}} A_*(\mathfrak{C}_{X/Y}) \xrightarrow{\iota_*} A_*(\mathfrak{Q}(\mathbb{E})) \xrightarrow{\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}} A_*(X).$$

The square root virtual pullbacks are functorial in the following sense.

Theorem B ([Park1]). *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z. \end{tikzcd}$$

is a commutative diagram of quasi-projective schemes with symmetric obstruction theories $\phi_{X/Z} : \mathbb{E}_{X/Z} \to \mathbb{L}_{X/Z}$, $\phi_{Y/Z} : \mathbb{E}_{Y/Z} \to \mathbb{L}_{Y/Z}$ of tor-amplitude $[-2, 0]$ satisfying the isotropic condition, and a perfect obstruction theory $\phi_{X/Y} : \mathbb{E}_{X/Y} \to \mathbb{L}_{X/Y}$ of tor-amplitude $[-1, 0]$. Assume that there exists a commutative diagram

$$\begin{tikzcd} \mathbb{D}^\vee[2] \arrow[r, "\alpha^\vee"] \arrow[d, "\beta^\vee"] & \mathbb{E}_{X/Z} \arrow[r] \arrow[d, "\alpha"] & \mathbb{E}_{X/Y} \arrow[d, equal] \\ \mathbb{E}_{Y/Z}|_X \arrow[r, "\beta"] \arrow[d, "\phi_{Y/Z}"] & \mathbb{D} \arrow[r] \arrow[d, "\phi'_{X/Z}"] & \mathbb{E}_{X/Y} \arrow[d] \\ \mathbb{L}_{Y/Z}|_X \arrow[r] & \mathbb{L}_{X/Z} \arrow[r] & \mathbb{L}_{X/Y} \end{tikzcd}$$

for some $\mathbb{D}$, α, β, $\phi'_{X/Z}$ such that $\phi_{X/Z} = \phi'_{X/Z} \circ \alpha$ and the horizontal sequences are distinguished triangles. Then for each orientation of $\mathbb{E}_{Y/Z}$, there exists an induced orientation of $\mathbb{E}_{X/Z}$ such that we have

$$\sqrt{(g \circ f)^!} = f^! \circ \sqrt{g^!} : A_*(Z) \to A_*(X).$$

In particular, when $Z = \mathrm{Spec}(\mathbb{C})$, we have a virtual pullback formula

$$[X]^{\mathrm{vir}} = f^![Y]^{\mathrm{vir}} \in A_*(X).$$

The two main applications of the virtual pullback formula (Theorem B) are the *Lefschetz principle* (Corollary F) and the *Pairs/Sheaves correspondence* (Corollary G).

Cosection localization

Kiem-Li's cosection localization can be extended to Oh-Thomas virtual cycles.

In DT4 theory, there are two types of cone reduction. Let $\phi : \mathbb{E} \to \mathbb{L}_X$ be a symmetric obstruction theory on a scheme X satisfying the isotropic condition.

1. If $\Sigma : \mathbb{E}^\vee[1] \to K^\vee$ is an *isotropic* cosection, i.e. $\Sigma^2 = 0 : K \to K^\vee$, then we have
$$(\mathfrak{C}_X)_{\mathrm{red}} \subseteq \mathfrak{Q}(\mathbb{E}_{/\!/K})$$
where the symmetric complex $\mathbb{E}_{/\!/K}$ is given by the reduction diagram
$$\begin{tikzcd} \mathbb{D}^\vee[2] \arrow[r, "\alpha^\vee"] \arrow[d, "\beta^\vee"] & \mathbb{E} \arrow[r, "\Sigma"] \arrow[d, "\alpha"] & K^\vee[1] \arrow[d, equal] \\ \mathbb{E}_{/\!/K} \arrow[r, "\beta"] & \mathbb{D} \arrow[r] & K^\vee[1]. \end{tikzcd}$$

2. If $\Sigma : \mathbb{E}^\vee[1] \to F$ is a *non-degenerate* cosection, i.e. $\Sigma^2 : F^\vee \to F$ is an isomorphism, then we have
$$(\mathfrak{C}_X)_{\mathrm{red}} \subseteq \mathfrak{Q}(\mathbb{E}_{/F})$$
where the symmetric complex $\mathbb{E}_{/F}$ is given by the decomposition
$$\mathbb{E} = \mathbb{E}_{/F} \oplus F[1].$$

From the above cone reductions, we can define two types of reduced Oh-Thomas virtual cycles.

Definition C ([KP2])**.** Let X be a quasi-projective scheme with an oriented symmetric obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_X$ of tor-amplitude $[-2,0]$ satisfying the isotropic condition. Let $\Sigma : \mathbb{E}^\vee[1] \to K^\vee$ be an isotropic cosection such that $h^0(\Sigma) : h^1(\mathbb{E}^\vee) \to K^\vee$ is surjective. We define the *reduced Oh-Thomas virtual cycle* as
$$[X]^{\mathrm{red}} := \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_{/\!/K})}}[\mathfrak{C}_X] \in A_{*+k}(X)$$
where $k := \mathrm{rank}(K)$.

Definition D ([BKP])**.** Let X be a quasi-projective scheme with an oriented symmetric obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_X$ of tor-amplitude $[-2,0]$ satisfying the isotropic condition. Let $\Sigma : \mathbb{E}^\vee[1] \to F$ be a non-degenerate cosection. Choose an orientation of the orthogonal bundle (F, Σ^2). We define the *reduced Oh-Thomas virtual cycle* as
$$[X]^{\mathrm{red}} := \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_{/F})}}[\mathfrak{C}_X] \in A_{*+\frac{1}{2}f}(X)$$
where $f := \mathrm{rank}(F)$.

Definition C can be used to count curves on hyperkähler 4-folds [COT1, COT2] and Definition D can be used to count surfaces on Calabi-Yau 4-folds [BKP].

For isotropic cosections, we can define localized Oh-Thomas virtual cycles.

Theorem E ([KP2])**.** *Let X be a quasi-projective scheme with an oriented symmetric obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_X$ of tor-amplitude $[-2,0]$ satisfying the isotropic condition. Let $\sigma : \mathbb{E}^\vee[1] \to \mathcal{O}_X$ be an isotropic cosection. Then there exists a* cosection-localized Oh-Thomas virtual cycle

$$[X]^{\mathrm{loc}} \in A_*(X(\overline{\sigma}))$$

such that $i_*[X]^{\mathrm{loc}} = [X]^{\mathrm{vir}} \in A_*(X)$, where $X(\overline{\sigma})$ is the zero locus of $\overline{\sigma} := h^0(\sigma) : h^1(\mathbb{E}^\vee) \to \mathcal{O}_X$ and $i : X(\overline{\sigma}) \hookrightarrow X$ is the inclusion map.

The key idea is to localize Edidin-Graham's square root Euler classes by *two* isotropic section.

Three applications

Recall that the quantum Lefschetz principle of Kim-Kresch-Pantev [KKP] relates the genus zero Gromov-Witten invariants of an algebraic variety with the Gromov-Witten invariants of its divisor. The virtual pullback formula (Theorem B) provides an analogous formula in Donaldson-Thomas theory.

Corollary F ([Park1])**.** *Let X be a Calabi-Yau 4-fold and D be a smooth connected divisor of a line bundle L on X. Let $\beta \in H_2(X,\mathbb{Q})$ be a curve class and $n \in \mathbb{Z}$ be an integer. Consider the Hilbert schemes:*

$$I_{n,\beta}(X) := \{\text{closed subschemes } Z \subseteq X \text{ with } [Z] = \beta \text{ and } \chi(\mathcal{O}_Z) = n\}$$
$$I_{n,\beta}(D) := \{\text{closed subschemes } Z \subseteq D \text{ with } i_*[Z] = \beta \text{ and } \chi(\mathcal{O}_Z)=n\}$$

where $i : D \hookrightarrow X$ is the inclusion map. Assume that the tautological complex

$$\mathcal{L}_{n,\beta} := R\pi_*(\mathcal{O}_{\mathcal{Z}_X} \otimes L)$$

is a vector bundle concentrated in degree 0, where $\mathcal{Z}_X \subseteq I_{n,\beta}(X) \times X$ is the universal family and $\pi : I_{n,\beta}(X) \times X \to I_{n,\beta}(X)$ is the projection map. Then for any orientation on $I_{n,\beta}(X)$, there exists canonical signs $(-1)^{\sigma(e)}$ on the connected components $I_{n,\beta}(D)^e$ of $I_{n,\beta}(D)$ such that

$$\sum_e (-1)^{\sigma(e)} (j_e)_*[I_{n,\beta}(D)^e]^{\mathrm{vir}}_{BF} = e(\mathcal{L}_{n,\beta}) \cap [I_{n,\beta}(X)]^{\mathrm{vir}}_{OT}$$

where $j_e : I_{n,\beta}(D)^e \hookrightarrow I_{n,\beta}(D) \hookrightarrow I_{n,\beta}(X)$ are the inclusion maps.

If we apply Corollary F to points, we can prove the Cao-Kool conjecture [CK1] for line bundles with smooth divisors using the corresponding result for 3-folds [Li3, LP]. If we apply Corollary F to curves, we can prove the Cao-Kool-Monavari conjecture [CKM] on the DT/PT correspondence for line bundles with Calabi-Yau divisors, using the corresponding result for Calabi-Yau 3-folds [Toda, Bri]. Corollary F can also be generalized to surfaces [BKP].

The virtual pullback formula (Theorem B) also provides a correspondence between the moduli of stable pairs and the moduli of stable sheaves.

Corollary G ([Park1]). *Let X be a Calabi-Yau 4-fold with fixed very ample line bundle. Let $\beta \in H_2(X,\mathbb{Q})$ be a curve class and $n \in \mathbb{Z}$ be an integer. Consider the following moduli spaces:*

$$P_{n,\beta}(X) := \{\textit{stable pairs } (F,s) \textit{ on } X \textit{ with } \mathrm{ch}(F) = (0,0,0,\beta,n)\}$$
$$M_{n,\beta}(X) := \{\textit{stable sheaves } G \textit{ on } X \textit{ with } \mathrm{ch}(G) = (0,0,0,\beta,n)\}$$

Assume that β is irreducible and $M_{n,\beta}(X)$ has a universal family. Then we have a well-defined forgetful map

$$p : P_{n,\beta}(X) \to M_{n,\beta}(X) : (F,s) \mapsto F$$

which has a canonical perfect obstruction theory such that

$$[P_{n,\beta}(X)]^{\mathrm{vir}} = p^![M_{n,\beta}(X)]^{\mathrm{vir}} \in A_*(P_{n,\beta}(X))$$

for certain choice of orientations.

Since the forgetful map $p : P_{n,\beta}(X) \to M_{n,\beta}(X)$ is a virtual projective bundle, we also have a pushforward formula. In particular, this proves the Cao-Maulik-Toda conjecture [CMT1, CMT2] on the primary PT/Katz correspondence. Moreover, we also have a tautological PT/Katz correspondence.

One application of the cosection localization [KP2, BKP] is a foundation of a surface counting theory on Calabi-Yau 4-folds. Since a $(2,2)$-class does not remain a $(2,2)$-class under a deformation of a Calabi-Yau 4-fold in a generic situation, the Oh-Thomas virtual cycle usually vanishes. Hence we need to consider the *reduced* virtual cycles almost always.

Theorem H ([BKP]). *Let X be a Calabi-Yau 4-fold with nowhere vanishing Calabi-Yau 4-form $\omega \in H^0(X,\Omega^4_X)$. Let $v = (0,0,\gamma,\beta,n-\gamma\cdot\mathrm{td}_2(X)) \in H^*(X,\mathbb{Q})$ be* *a cohomology class. Let $I_v(X)$ be the Hilbert schemes of surfaces $S \subseteq X$ with $\mathrm{ch}(O_S) = v$. Then there exist a canonical reduced virtual cycle*

$$[I_v(X)]^{\mathrm{red}} \in A_{n-\frac{1}{2}\gamma^2+\frac{1}{2}\rho_\gamma}(I_v(X))$$

where ρ_γ is the rank of the symmetric bilinear form

$$\mathsf{B}_\gamma : H^1(X, T_X) \otimes H^1(X, T_X) \to \mathbb{C} : \xi_1 \otimes \xi_2 \mapsto \int_X (\iota_{\xi_1}\iota_{\xi_2}\gamma \cup \omega).$$

Moreover, the reduced virtual cycle $[I_v(X)]^{\mathrm{red}}$ is deformation invariant along the Hodge locus of (X, γ).

The Hodge conjecture predicts that for any smooth projective variety X, all rational (p, p)-classes on X are algebraic. In [Gro1], Grothendieck introduced a variant of the Hodge conjecture, called the *variational Hodge conjecture*: a deformation of an algebraic class is algebraic.

From the deformation invariance of reduced virtual cycles, we obtain a connection of DT4 theory with the variational Hodge conjecture.

Theorem I ([BKP])**.** *Let X be a Calabi-Yau 4-fold and let γ be a $(2,2)$-class on X. If for some $v \in H^*(X, \mathbb{Q})$ with $v_2 = \gamma$ and $q \in \{-1, 0, 1\}$*

$$[P_v^{(q)}(X)]^{\mathrm{red}} \neq 0 \in A_*(P_v^{(q)}(X))$$

then the variational Hodge conjecture holds for (X, γ).

There is a technical issue in deformation invariance of DT4 theory. We resolve this in [Park2]. Having a relative (-2)-shifted symplectic structure does not give a Darboux chart in general and an additional condition is required. Fortunately, this additional assumption is always satisfied if we choose sufficiently nice family of Calabi-Yau 4-folds. We refer to [Park2] for details.

Generalizations

Torus localization via equivariant virtual pullbacks

The torus localization formula of Graber-Pandharipande [GP] is an extremely useful tool for computing virtual invariants when there is a torus action. However, some technical assumptions were required in [GP]. This was significantly weakened by Chang-Kiem-Li [CKL], but there are still some cases where the full generality is desired (e.g. wall-crossing formula of Joyce [Joy, GJT]).

In [AKLPR], we prove the torus localization formula for Deligne-Mumford stacks, without any technical hypothesis. This inspired by Chang-Kiem-Li's approach [CKL].

Theorem J ([AKLPR]). *Let X be a separated Deligne-Mumford stack with a $\mathbf{T}$-action. Let $\phi : \mathbb{E} \to L_X$ be a $\mathbf{T}$-equivariant perfect obstruction theory. let $\phi_{X^{\mathbf{T}}}$ be the induced perfect obstruction theory on the fixed locus $X^{\mathbf{T}}$ (see Definition 7.2.1). Then we have*

$$[X]^{\mathrm{vir}} = i_*\left(\frac{[X^{\mathbf{T}}]^{\mathrm{vir}}}{e^{\mathbf{T}}(N^{\mathrm{vir}})}\right) \in A^{\mathbf{T}}_*(X)_{\mathbf{s}}$$

where $A^{\mathbf{T}}_*(X)_{\mathbf{s}} := A^{\mathbf{T}}_*(X) \otimes_{\mathbb{Q}[\mathbf{s}]} \mathbb{Q}[\mathbf{s}^{\pm 1}]$ denotes the localization by the Euler class $\mathbf{s}$ of the weight 1 representation of $\mathbf{T}$, $e^{\mathbf{T}}(N^{\mathrm{vir}})$ is the Euler class of the virtual normal bundle (see Definition 7.2.4) and $i : X^{\mathbf{T}} \hookrightarrow X$ is the inclusion map.

The key idea is to define an *equivariant virtual pullback*

$$i^!_{\mathbf{T}} : A^{\mathbf{T}}_*(X)_{\mathbf{s}} \to A^{\mathbf{T}}(X^{\mathbf{T}})_{\mathbf{s}}$$

for the inclusion map $i : X^{\mathbf{T}} \to X$. There is a canonical relative obstruction theory $\mathbb{E}|^{\mathrm{mov}}_{X^{\mathbf{T}}} \to \mathbb{L}_{X^{\mathbf{T}}/X}$ for $i : X^{\mathbf{T}} \to X$, but it is of tor-amplitude $[-2,-1]$ and thus we needed a new construction.

Consider the following general situation: Let $f : Y \to X$ be a $\mathbf{T}$-equivariant morphism of Deligne-Mumford stacks with $\mathbf{T}$-actions. Let $\phi : \mathbb{E} \to \mathbb{L}_{Y/X}$ be a $\mathbf{T}$-equivariant obstruction theory. Assume that the $\mathbf{T}$-action on Y is trivial, $\mathbb{E}^{\mathrm{fix}}$ is of tor-amplitude $[-1,0]$, and $\mathbb{E}^{\mathrm{mov}}$ is of tor-amplitude $[-2,-1]$. Then $\mathfrak{C}(\mathbb{E})$ is not necessarily a vector bundle stack, but we still have an *equivariant homotopy property*

$$A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{E}))_{\mathbf{s}} \cong A^{\mathbf{T}}_*(X)_{\mathbf{s}}.$$

Hence we can define the *equivariant Gysin pullback*

$$(0_{\mathfrak{C}(\mathbb{E})})^!_{\mathbf{T}} : A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{E}))_{\mathbf{s}} \to A^{\mathbf{T}}_*(Y)_{\mathbf{s}}.$$

We then define the $\mathbf{T}$-*equivariant virtual pullback* as the composition

$$f^!_{\mathbf{T}} : A^{\mathbf{T}}_*(Y)_{\mathbf{s}} \xrightarrow{\mathrm{sp}} A^{\mathbf{T}}_*(\mathfrak{C}_{X/Y})_{\mathbf{s}} \xrightarrow{\iota_*} A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{E}))_{\mathbf{s}} \xrightarrow{(0_{\mathfrak{C}(\mathbb{E})})^!_{\mathbf{T}}} A^{\mathbf{T}}_*(X)_{\mathbf{s}}.$$

Then the above torus localization formula follows from the functoriality of equivariant virtual pullbacks (Theorem 7.1.15).

Cosection localization via derived algebraic geometry

Kiem-Li's cosection localization can be reinterpreted using derived algebraic geometry. Note that in derived algebraic geometry, *quasi-smooth derived schemes* are natural analogs of schemes with perfect obstruction theories and (-1)-*shifted* 1-*forms* are natural analogs of cosections.

In [BKP, Appendix A], we prove *scheme-theoretical* cone reduction lemma when cosections can be enhanced to (-1)-shifted *closed* 1-forms. This extends Kiem-Li's original cone reduction lemma.

Proposition K ([BKP]). *Let $\mathbb{X}$ be a homotopically finitely presented derived scheme and α be a (-1)-shifted closed 1-form. Let $\phi : \mathbb{E} := \mathbb{L}_{\mathbb{X}}|_X \to \mathbb{L}_X$ be an induced obstruction theory on the classical truncation $X := \mathbb{X}_{\mathrm{cl}}$ and $\sigma := \alpha_0|_X^\vee : \mathbb{E}^\vee[1] \to \mathcal{O}_X$ be the induced cosection. Let $\mathbb{E}_\sigma := \mathrm{cone}(\sigma^\vee[1] : \mathcal{O}_X[1] \to \mathbb{E})$. Then we have a scheme-theoretical cone reduction*

$$\mathfrak{C}_X \subseteq \mathfrak{K}$$

as substacks of $\mathfrak{E} := h^1/h^0(\mathbb{E}^\vee)$, where $\mathfrak{K} := h^1/h^0(\mathbb{E}_\sigma^\vee)$ is the kernel cone stack. Equivalently, we have a reduced obstruction theory

$$\phi^{\mathrm{red}} : \mathbb{E}_\sigma \to \tau^{\geqslant -1}\mathbb{L}_X$$

that factors the original obstruction theory ϕ.

The main idea is to use the *derived Poincare lemma*, i.e., a (-1)-shifted closed 1-form is locally exact, which can be shown by the arguments in Brav-Bussi-Joyce [BBJ]. Then the proposition follows directly from the approach of Schurg-Toen-Vezzosi [STV] to construct reduced obstruction theories.

In the construction of Oh-Thomas virtual cycles, Kiem-Li's cosection localization was crucially used. We speculate the following converse.

Speculation L. *Let $\mathbb{X}$ be a quasi-smooth derived scheme and α be a (-1)-shifted closed 1-form. Let $\alpha_0 : \mathcal{O}_{\mathbb{X}} \to \mathbb{L}_{\mathbb{X}}[-1]$ be the underlying (-1)-shifted 1-form and let $\mathbb{X}(\alpha)$ be the derived zero locus. Then we have*

$$[X]^{\mathrm{loc}}_{KL} = [X(\sigma)]^{\mathrm{vir}}_{OT} \in A_*(X(\sigma))$$

where $[X]^{\mathrm{loc}}_{KL}$ is Kiem-Li's cosection-localized virtual cycle for the induced obstruction theory $\phi : \mathbb{E} := \mathbb{L}_{\mathbb{X}}|_X \to \mathbb{L}_X$ on the classical truncation $X := \mathbb{X}_{\mathrm{cl}}$ and the induced cosection $\sigma := \alpha_0|_X^\vee : \mathbb{E}^\vee[1] \to \mathcal{O}_X$, and $[X(\sigma)]^{\mathrm{vir}}_{OT}$ is the Oh-Thomas virtual cycle for the (-2)-shifted symplectic derived scheme $\mathbb{X}(\alpha)$.

At least in the local model case, Speculation L is essentially shown in [KP2]. We plan to prove Speculation L in the general case in [KP3].

The cosection-localized virtual cycles are mostly used in the areas that are not directly related to the DT4 theory (e.g. [ChLi]). Hence the above speculation provides an unexpected connection of those areas to the DT4 theory.

Virtual intersection theory in algebraic cobordism

In [LM], Levine-Morel introduced *algebraic coboridsm* Ω as the universal oriented Borel-Moore homology theory for schemes. In particular, we have natural maps

$$\begin{tikzcd} & \Omega_*(X) \arrow[dl] \arrow[dr] & \\ K_0(X)[\beta^{\pm 1}] & & A_*(X) \end{tikzcd}$$

to the algebraic K-theory of coherent sheaves and the Chow groups for a quasi-projective scheme X.

In [KP1], we extend the theory of virtual cycles (including the two key techniques) to algebraic cobordism. Since algebraic coboridsm is universal, this implies the same result in any other oriented Borel-Moore homology theory.

Theorem M ([KP1]). *Let $f : X \to Y$ be a morphism of quasi-projective schemes with a perfect obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_{X/Y}$.*

1. *Then there exists a virtual pullback*

$$f^! : \Omega_*(Y) \to \Omega_*(X)$$

which is bivariant and functorial (see Chapter 9 for the precise statements).

2. *Moreover, if $Y = \mathrm{Spec}(\mathbb{C})$ and $\sigma : \mathbb{E}^\vee[1] \to \mathcal{O}_X$ is a cosection, we have a* cosection-localized virtual cobordism class

$$[X]^{\mathrm{loc}} \in \Omega_*(X(\sigma))$$

such that $i_*[X]^{\mathrm{loc}} = [X]^{\mathrm{vir}}$ where $i : X(\sigma) \hookrightarrow X$ is the inclusion map.

3. *If $Y = \mathrm{Spec}(\mathbb{C})$, $\mathbf{T}$ acts on X, and ϕ is induced from a $\mathbf{T}$-equivariant perfect obstruction theory, then we have*

$$[X]^{\mathrm{vir}} = i_* \left(\frac{[X^{\mathbf{T}}]^{\mathrm{vir}}}{e^{\mathbf{T}}(N^{\mathrm{vir}})} \right) \in \Omega_*^{\mathbf{T}}(X)_{\mathrm{loc}}$$

after inverting the Euler class of the weight 1 representation of $\mathbf{T}$ *(see Chapter 9 for the precise statements).*

The key idea is to extend algebraic cobordism for schemes to Artin stacks via limit construction.

Definition N ([KP1])**.** Let X be an algebraic stack. We define the *limit algebraic coboridsm* as

$$\widehat{\Omega}_d(X) := \varprojlim_{T \to X} \Omega_{d+\dim(T/X)}(T)$$

where $T \to X$ are all smooth morphisms from quasi-projective schemes T.

Kimura sequence for Artin stacks

In [BP], we extend the Kimura sequence [Kim] to Kresch's Chow groups of Artin stacks [Kre2].

Proposition O ([BP])**.** *Let $p : Y \to X$ be a proper representable surjective morphism of algebraic stacks with affine stabilizers. Then we have a right exact sequence*

$$A_*(Y \times_X Y) \xrightarrow{(p_1)_* - (p_2)_*} A_*(Y) \xrightarrow{\quad p_* \quad} A_*(X) \longrightarrow 0$$

where $p_1, p_2 : Y \times_X Y \to Y$ are the projection maps. Here we used the proper pushforwards developed by Bae-Schmitt-Skowera [BS, Appendix B].

In many cases, algebraic stacks have proper covers by global quotient stacks. By the Kimura sequence, properties of Chow groups for these stacks can be reduced to those of global quotient stacks, which can be further reduced to those of quasi-projective scheme via Totaro's approximation [Tot]. This is technically useful in virtual intersection theory, especially when we want to remove the hypothesis on the resolution property. We plan to give various examples in [BP].

Notations and conventions

- All schemes and algebraic stacks are assumed to be of finite type over the field of complex numbers $\mathbb{C}$, unless stated otherwise.

- For any morphism $f : X \to Y$ of algebraic stacks, denote by $\mathbb{L}_{X/Y}$ the full cotangent complex [Ill] and $L_{X/Y} := \tau^{\geqslant -1}\mathbb{L}_{X/Y}$ the truncated cotangent complex [HT].
- A perfect obstruction theory is assumed to be of tor-amplitude $[-1, 0]$ and a symmetric obstruction theory is assumed to be of tor-amplitude $[-2, 0]$, unless stated otherwise.
- For any algebraic stack X, denote by $A_*(X)$ the Chow group of Kresch [Kre2] with $\mathbb{Q}$-coefficients, unless stated otherwise.

Part I

Virtual intersection theory

Chapter 1

Intersection theory

This chapter collects basics on intersection theory for scheme and algebraic stacks from [Ful, Vist, EG2, Kre2] and algebraic coboridsm from [LM, LP].

1.1 Intersection theory for schemes

In this section, we summarize basic properties of Chow groups for schemes and DM stacks, based on [Ful, Vist].

Definition 1.1.1 (Algebraic cycles)**.** Let X be a Deligne-Mumford stack.

1. The *cycle group of degree $d \in \mathbb{Z}_{\geqslant 0}$ on X* is defined as the $\mathbb{Q}$-vector space
$$Z_*(X) := \mathbb{Q}\langle [Z] \rangle$$
generated by integral closed substacks Z of X of dimension d.

2. The *cycle group on X* is defined as the graded $\mathbb{Q}$-vector space
$$Z_*(X) := \bigoplus_{d \in \mathbb{Z}} Z_d(X).$$

Definition 1.1.2 (Proper pushforward)**.** Let $f : X \to Y$ be a proper morphism of Deligne-Mumford stacks. We define the *proper pushforward* as

$$f_* : Z_*(X) \to Z_*(Y) : [Z] \mapsto \begin{cases} \deg(Z/f(Z)) \cdot [f(Z)] & \text{if } \dim(f(Z)) = \dim(Z) \\ 0 & \text{if } \dim(f(Z)) < \dim(Z) \end{cases}$$

where the *degree* $\deg(Z/f(Z))$ is given as follows:

1. If $Z \to f(Z)$ is representable, then by generic flatness $Z \cap f^{-1}(U) \to U$ is finite flat for some non-empty open $U \subseteq f(Z)$, and let $\deg(Z/f(Z))$ be the degree of the map $Z \cap f^{-1}(U) \to U$.

2. If $Z \to f(Z)$ is not representable, then choose a representable quasi-finite dominant morphism $V \to Z$ such that $V \to Z \to f(Z)$ is also representable, and let $\deg(Z/f(Z)) := \deg(V/f(Z))/\deg(V/Z)$.

Definition 1.1.3 (Flat pullback)**.** Let $f : X \to Y$ be a flat morphism of relative dimension d of Deligne-Mumford stacks. We define the *flat pullback* as

$$f^* : Z_*(Y) \to Z_{*+d}(X) : [Z] \mapsto \sum_{V \subseteq Y} \mathrm{mult}_V(f^{-1}(Z)) \cdot [V]$$

where the *multiplicity* $\mathrm{mult}_V(f^{-1}(Z))$ is given as follows:

1. If $f^{-1}(Z)$ is a scheme, then we let $\mathrm{mult}_V(f^{-1}(Z)) := \mathrm{length}(\mathcal{O}_{f^{-1}(Z),V})$.

2. If $f^{-1}(Z)$ is not a scheme, choose a smooth surjection $Z' \twoheadrightarrow f^{-1}(Z)$ and a connected component $V' \subseteq V \times_{f^{-1}(Z)} Z'$ and let $\mathrm{mult}_V(f^{-1}(Z)) := \mathrm{mult}_{V'}(Z')$.

Definition 1.1.4 (Rational functions)**.** Let V be an integral Deligne-Mumford stack. The *field of rational functions on V* is defined as the direct limit

$$k(V) := \varinjlim_{U \subseteq V} \Gamma(U, \mathcal{O}_U)$$

where the limit is taken over all open substacks $U \subseteq V$ and the transition maps are given by the restriction maps for all $U_1 \subseteq U_2$.

Definition 1.1.5 (Rational equivalence)**.** Let X be a Deligne-Mumford stack.

1. The *group of rational equivalences of degree d* is defined as the $\mathbb{Q}$-vector space
$$W_d(X) := \bigoplus_V k(V)^* \otimes_{\mathbb{Z}} \mathbb{Q}$$
where the direct sum is taken over all integral closed substacks V of dimension $d+1$ and $k(V)^*$ denotes the unit group of $k(V)$.

2. The *group of rational equivalences* is defined as the graded $\mathbb{Q}$-vector space
$$W_*(X) := \bigoplus_{d \in \mathbb{Z}} W_d(X).$$

Definition 1.1.6 (Boundary map)**.** Let X be a Deligne-Mumford stack. We define the *boundary map*

$$\partial : W_*(X) \to Z_*(X)$$

as follows:

1. *Case 1.* Assume that X is a scheme. Then we define

$$\partial : W_*(X) \to Z_*(X) : (f : U \subseteq V \to \mathbb{A}^1) \mapsto \sum_{Z \subseteq X} \mathrm{ord}_Z(f)[Z]$$

   where $\mathrm{ord}_Z(f) := \mathrm{length}_{O_{V,Z}}(O_{V,Z}/a) - \mathrm{length}_{O_{V,Z}}(O_{V,Z}/b)$ for $f = a/b$ and $a, b \in O_{V,Z}$.

2. *Case 2.* Let X be a Deligne-Mumford stack. Then both the group of algebraic cycles $Z_*(X)$ and the group of rational equivalences $W_*(X)$ satisfy the descent for the étale topology. We define the boundary map

$$\partial : W_*(X) \to Z_*(X)$$

   via descent.

Definition 1.1.7 (Chow group)**.** Let X be a Deligne-Mumford stack. We define the *Chow group* as the graded $\mathbb{Q}$-vector space

$$A_*(X) := \mathrm{coker}(\partial : W_*(X) \to Z_*(X)).$$

Proposition 1.1.8 (Basic properties)**.** *The proper pushforwards and flat pullbacks are well-defined, functorial, and commute with each others. More precisely, we have the followings:*

1. *Let $f : X \to Y$ be a proper morphism of Deligne-Mumford stacks. Then the proper pushforward in Definition 1.1.2 descends to the Chow groups*

$$f_* : A_*(X) \to A_*(Y).$$

   Moreover, if $g : Y \to Z$ is a proper morphism of Deligne-Mumford stacks, then we have

$$(g \circ f)_* = g_* \circ f_* : A_*(X) \to A_*(Z).$$

2. *Let $f : X \to Y$ be a flat morphism of relative dimension d of Deligne-Mumford stacks. Then the flat pullback in Definition 1.1.3 descends to the Chow groups*
$$f^* : A_*(Y) \to A_{*+d}(X).$$
Moreover, if $g : Y \to Z$ is a flat morphism of relative dimension e Deligne-Mumford stacks, then we have
$$(g \circ f)^* = f^* \circ g^* : A_*(Z) \to A_{*+d+e}(X).$$

3. *Let*
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
be a cartesian square of Deligne-Mumford stacks such that f is proper and g is equi-dimensional flat. Then we have
$$g^* \circ f_* = (f')_* \circ (g')^* : A_*(X) \to A_*(Y').$$

Proposition 1.1.9 (Localization sequence)**.** *Let X be a Deligne-Mumford stack. Let Z be a closed substack of X and U be the complement. Then we have a right exact sequence*
$$A_*(Z) \xrightarrow{i_*} A_*(X) \xrightarrow{j^*} A_*(U) \longrightarrow 0$$
where $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ are the inclusion maps.

Proposition 1.1.10 (Homotopy property)**.** *Let X be a Deligne-Mumford stack and E be a vector bundle on X of rank r. Then the smooth pullback*
$$\pi_E^* : A_*(X) \to A_{*+r}(E)$$
is an isomorphism where $\pi_E : E \to X$ is the projection map.

Proposition 1.1.11 (Chern classes)**.** *For each vector bundle E on a Deligne-Mumford stack Xand an integer $i \in \mathbb{Z}_{\geqslant 0}$, there exists unique maps*
$$c_i(E) : A_*(X) \to A_{*-i}(X)$$
satisfying the following properties:

1. *If $f : X \to Y$ is a proper morphism of Deligne-Mumford stacks, then we have*
$$f_* \circ c_i(f^*E) = c_i(E) \circ f_* : A_*(X) \to A_{*-i}(Y).$$

2. *If $f : X \to Y$ is a flat morphism of relative dimension d, then we have*
$$c_i(f^*E) \circ f^* = f^* \circ c_i(E) : A_*(Y) \to A_{*+e-i}(X).$$

3. *If D is an effective Cartier divisor of an integral Deligne-Mumford stack X of dimension d, then we have*
$$[D] = c_1(L)([X]) \in A_{d-1}(X).$$

4. *If L_1 and L_2 are line bundles on X, then we have*
$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) : A_*(X) \to A_{*-1}(X).$$

5. *If E and F are vector bundles of rank r and s on X, then we have*
$$c_i(E \oplus F) = \sum_{i=j+k} c_j(E) \circ c_k(F) : A_*(X) \to A_{*-r-s}(X).$$

Definition 1.1.12 (Intersection with an effective Cartier divisor)**.** Let X be a Deligne-Mumford stack and D be an effective Cartier divisor. We define the *intersection product map* as

$$D\cdot : A_*(X) \to A_{*-1}(D) : [Z] \mapsto \begin{cases} [Z \cap D] & \text{if } Z \not\subseteq D \\ c_1(\mathcal{O}_X(D)) \cap [Z] & \text{if } Z \subseteq D \end{cases}$$

Definition 1.1.13 (Normal cone)**.** Let $f : X \to Y$ be an unramified morphism of Deligne-Mumford stacks. We define the *normal cone* $C_{X/Y}$ as follows:

1. *Case 1.* Assume that $f : X \hookrightarrow Y$ is a closed embedding. We define
$$C_{X/Y} := \mathrm{Spec}(\bigoplus_{n\geqslant 0} \mathcal{I}^n_{X/Y}/\mathcal{I}^{n+1}_{X/Y})$$
where $\mathcal{I}_{X/Y}$ is the ideal sheaf of X in Y.

2. *Case 2.* Assume that $f : X \to Y$ is an unramified morphism. Then we can find a fiber diagram of Deligne-Mumford stacks

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook, "\widetilde{f}"] \arrow[d, two heads] & \widetilde{Y} \arrow[d, two heads] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

   where vertical arrows are étale surjective and $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is a closed embedding. We define

$$C_{X/Y} := [\, C_{\widetilde{X}\times_X\widetilde{X}/\widetilde{Y}\times_Y\widetilde{Y}} \rightrightarrows C_{\widetilde{X}/\widetilde{Y}} \,]$$

   where the induced map $\widetilde{X} \times_X \widetilde{X} \to \widetilde{Y} \times_Y \widetilde{Y}$ is a closed embedding.

Definition 1.1.14 (Deformation space)**.** Let $f : X \to Y$ be an unramified morphism of Deligne-Mumford stacks. We define the *deformation space* $M^{\circ}_{X/Y}$ as follows:

1. *Case 1.* Assume that $f : X \hookrightarrow Y$ is a closed embedding. We define

$$M^{\circ}_{X/Y} := M_{X/Y} \backslash \widetilde{Y}$$

   where $M_{X/Y} := \mathrm{Bl}_{X\times\{0\}}(Y \times \mathbb{P}^1)$ and $\widetilde{Y} := \mathrm{Bl}_{X\times\{0\}}(Y \times \{0\})$.

2. *Case 2.* Assume that $f : X \to Y$ is an unramified morphism. Then we can find a fiber diagram of Deligne-Mumford stacks

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook, "\widetilde{f}"] \arrow[d, two heads] & \widetilde{Y} \arrow[d, two heads] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

   where vertical arrows are étale surjective and $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is a closed embedding. We define

$$M^{\circ}_{X/Y} := [\, M^{\circ}_{\widetilde{X}\times_X\widetilde{X}/\widetilde{Y}\times_Y\widetilde{Y}} \rightrightarrows M^{\circ}_{\widetilde{X}/\widetilde{Y}} \,].$$

Definition 1.1.15 (Specialization map)**.** Let $f : X \to Y$ be an unramified morphism of Deligne-Mumford stacks. We define the *specialization map*

$$\mathrm{sp}_{X/Y} : A_*(Y) \longrightarrow A_*(C_{X/Y})$$

as the unique map that fits into the commutative diagram

$$\begin{tikzcd} A_*(C_{X/Y}) \arrow[r] \arrow[dr, "0"'] & A_*(M^\circ_{X/Y}) \arrow[r] \arrow[d, "D_0\cdot"] \arrow[dr, "D_\zeta\cdot"] & A_*(Y \times \mathbb{A}^1) \arrow[r] \arrow[d, "\cong"] & 0 \\ & A_*(C_{X/Y}) & A_*(Y) \arrow[l, dotted, "\mathrm{sp}_{X/Y}"] & \end{tikzcd}$$

where $D_0 = C_{X/Y} \subseteq M^\circ_{X/Y}$ and $D_\zeta = Y \subseteq M^\circ_{X/Y}$ are the effective Cartier divisors given by the fibers of $M^\circ_{X/Y} \to \mathbb{P}^1$ over $0 \in \mathbb{P}^1$ and $\zeta \in \mathbb{P}^1$.

Definition 1.1.16 (Lci pullback)**.** Let

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a cartesian square of Deligne-Mumford stacks such that $f : X \to Y$ be a local complete intersection morphism of codimension c. We define the *lci pullback* as

$$f^! : A_*(Y') \xrightarrow{\mathrm{sp}_{X'/Y'}} A_*(C_{X'/Y'}) \xrightarrow{i_*} A_*(N_{X/Y}|_{X'}) \cong A_{*-c}(X')$$

where $i : C_{X'/Y'} \hookrightarrow N_{X/Y}|_{X'}$ is the inclusion map.

We refer to [Ful, Vist] for the proofs of the above propositions.

Remark 1.1.17 (Integral coefficient)**.** For schemes, everything in this section also works with $\mathbb{Z}$*-coefficients*. However, for Deligne-Mumford stacks, the Chow groups (with $\mathbb{Z}$-coefficients) in this section do not give us a *correct* theory since they do not the homotopy property.

Remark 1.1.18 (Naive Chow groups for Artin stacks)**.** For an arbitrary *Artin* stack X, we can still define a graded $\mathbb{Q}$-vector space $A_*(X)$ as in this section. This is what Kresch in [Kre2] calls the *naive* Chow group of X. These Chow groups do not have the homotopy property as Remark 1.1.17. We will consider the *correct* Chow theory, introduced by Kresch [Kre2], in the next section.

1.2 Intersection theory for Artin stacks

In this section, we summarize basic properties of Chow groups for *Artin stacks*, based on [EG2, Kre2].

1.2.1 Equivariant Chow groups

Definition 1.2.1 (Totaro's approximation)**.** Let G be a linear algebraic group. We say that that maps

$$EG_i/G \to BG$$

for $i \in \mathbb{Z}_{\geqslant 0}$ are *Totaro's approximations* of the classifying stack BG if there exist G-representations V_i and G-invariant closed subschemes $Z_i \subseteq V_i$ of codimension $\geqslant i$ such that $EG_i = V_i \backslash Z_i$ and the quotient stack $[EG_i/G]$ is a quasi-projective scheme.

The existence of Totaro's approximation is shown in [Tot].

Definition 1.2.2 (Equivariant Chow group)**.** Let X be an algebraic space with an action of a linear algebraic group G.

1. We define the *equivariant Chow group* of degree d as the $\mathbb{Q}$-vector space

$$A_d^G(X) := A_{d+\dim(EG_i)-\dim(G)}(X \times_G EG_i)$$

for big enough i, where $EG_i/G \to BG$ are Totaro's approximations.

2. We define the *equivariant Chow group* as the graded $\mathbb{Q}$-vector space

$$A_*^G(X) := \bigoplus_{d \in \mathbb{Z}} A_d^G(X).$$

It is easy to show that the equivariant Chow group is independent of the choice of Totaro's approximation.

Proposition 1.2.3 (Reductive to torus)**.** *Let X be an algebraic space with an action of a connected reductive group G. Let $\mathbf{T}$ be a maximal torus of G. Then there is an action of the Weyl group W on $A_*^{\mathbf{T}}(X)$ such that*

$$A_*^G(X) = A_*^{\mathbf{T}}(X)^W.$$

Proof. We refer to [EG2, Prop. 6] for the proof. □

1.2.2 Kresch's Chow groups

Definition 1.2.4 (Kresch's Chow group)**.** We define the Chow group of algebraic stacks as follows:

1. Firstly, for any algebraic stack X, we define the *naive Chow group* of degree d as
$$A_d^\circ(X) := \operatorname{coker}(\partial : W_d(X) \to Z_d(X))$$
where the group of algebraic cycles $Z_d(X)$, the group of rational equivalence $W_d(X)$, and the boundary map $\partial : W_d(X) \to Z_d(X)$ are defined as in Definition 1.1.1, Definition 1.1.5, and Definition 1.1.6.

2. (a) For a connected algebraic stack X, we define
$$\widehat{A}_d(X) := \varinjlim_{E \to X} A^\circ_{d+\operatorname{rank}(E)}(E)$$
where E are vector bundles on X and the transition maps are given by the smooth pullbacks between surjections $E_1 \twoheadrightarrow E_2$ of vector bundles.

   (b) For an algebraic stack X, we define
$$\widehat{A}_d(X) := \bigoplus_i \widehat{A}_d(X_i)$$
where $X = \bigsqcup_i X_i$ is the decomposition of the connected components.

3. (a) For a projective morphism $f : X \to Y$ of algebraic stacks such that X is connected, we define
$$\widehat{A}_d^f(X) := \varinjlim_{E \to Y} A^\circ_{d+\operatorname{rank}(E)}(E|_X)$$
where E are vector bundles on Y and the transition maps are given by the smooth pullbacks for $E_1|_X \to E_2|_X$ for all surjections $E_1 \to E_2$.

   (b) For a projective morphism $f : X \to Y$ of algebraic stacks , we define
$$\widehat{A}_d^f(X) := \bigoplus_i \widehat{A}_d^f(X_i)$$
where $X = \bigsqcup_i X_i$ is the decomposition of the connected components.

4. (a) For projective morphisms $p_1, p_2 : T \to X$ of algebraic stacks, we define

$$\widehat{B}_d^{p_1,p_2}(X) := \mathrm{im}(\ker(\widehat{A}_d^{p_1}(T) \oplus \widehat{A}_d^{p_2}(T) \to \widehat{A}_d(T)) \to \widehat{A}_d(X)) \subseteq \widehat{A}_d(X).$$

   (b) For any morphism $f : X \to Y$ of algebraic stacks, we define

$$\widehat{B}_d^f(X) := \sum_{p_1,p_2:T\to X} \widehat{B}_d^{p_1,p_2}(X) \subseteq \widehat{A}_d(X)$$

   where $p_1, p_2 : T \to X$ are morphisms from algebraic stacks T such that $f \circ p_1 \cong f \circ p_2$.

5. (a) For any algebraic stack X, we define the *Chow group* of degree d as

$$A_d(X) := \varinjlim_{f:Y\to X} (\widehat{A}_d(Y)/\widehat{B}_d^f(Y))$$

   where $f : Y \to X$ are projective morphisms from algebraic stacks Y and the transition maps are given by the pushforwards for open and closed embeddings $Y_1 \hookrightarrow Y_2$ over X.

   (b) For any algebraic stack X, we define the *Chow group* as

$$A_*(X) := \bigoplus_{d\in\mathbb{Z}} A_d(X).$$

Proposition 1.2.5 (Compatibility)**.** *Let X be an algebraic stack.*

1. *If X is a Deligne-Mumford stack, then Kresch's Chow group in Definition 1.2.4 equals to the Chow group in Definition 1.1.7,*

$$A_*^\circ(X) = A_*(X).$$

2. *If $X = [P/G]$ for an algebraic space P with an action of a linear algebraic group G, then Kresch's Chow group in Definition 1.2.4 equals to the equivariant Chow group in Definition 1.2.2,*

$$A_*(X) = A^G_{*+\dim(G)}(P).$$

Proposition 1.2.6 (Basic operations)**.** *We have the following operations in Kresch's Chow groups.*

1. *For any projection morphism $f : X \to Y$ of algebraic stacks, there exists a pushforward*
$$f_* : A_*(X) \to A_*(Y).$$
Moreover, the projective pushforwards are functorial.

2. *For any flat morphism $f : X \to Y$ of relative dimension e, there exists a pullback*
$$f^* : A_*(Y) \to A_{*+e}(X).$$
Moreover, the flat pullbacks are functorial and commute with projective pushforwards.

3. *For any vector bundle E on an algebraic stack X and an integer i, there exists a Chern class*
$$c_i(E) : A_*(X) \to A_{*-i}(X).$$
Moreover, Chern classes commute with projective pushforwards, flat pullbacks, other Chern classes, and satisfies the Whitney sum formula.

4. *For any fiber diagram of algebraic stacks*
$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, hook, "f"] & Y \end{tikzcd}$$
with a regular closed embedding $f : X \hookrightarrow Y$ of codimension c, there exists a refined Gysin pullback
$$f^! : A_*(Y') \to A_{*-c}(X').$$
Moreover, the refined Gysin pullbacks are functorial and commute with projective pushforward, flat pullbacks, Chern classes, and other refined Gysin pullbacks.

Proposition 1.2.7 (Localization sequence)**.** *Let X be an algebraic stack. Let Z be a closed substack of X and U be the complement. Then we have a right exact sequence*
$$A_*(Z) \xrightarrow{i_*} A_*(X) \xrightarrow{j^*} A_*(U) \longrightarrow 0$$
where $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ are the inclusion maps.

We also present one additional ingredient that was not developed in [Kre2]. The pushforwards for proper DM morphisms were later developed in [BS, Appendix B].

Proposition 1.2.8 (Skowera's proper pushforward)**.** *For any proper DM morphism $f : X \to Y$ of algebraic stacks, there exists a pushforward*

$$f_* : A_*(X) \to A_*(Y).$$

Moreover, the proper DM pushforwards are functorial and commute with flat pullbacks, Chern classes, and the refined Gysin pullbacks.

Remark 1.2.9 (Integral coefficients)**.** Everything in this section (except Proposition 1.2.5) work in $\mathbb{Z}$-coefficients.

1.3 Algebraic cobordism

In this section, we review definition and basic properties of algebraic cobordism of Levine-Morel [LM]. Basically, algebraic cobordism is an algebraic analog of Quillen's complex cobordism [Quil]

Levine-Morel introduced the notion of oriented Borel-Moore homology theory as follows.

Definition 1.3.1 (Oriented Borel-Moore homology theory)**.** An *oriented Borel-Moore homology theory H* for schemes consists of the following data:

(D1) For each quasi-projective scheme X, we have a $\mathbb{Z}$-graded abelian group

$$H_*(X).$$

(D2) For each projective morphism $f : X \to Y$ of quasi-projective schemes, we have a morphism of graded abelian groups

$$f_* : H_*(X) \to H_*(Y).$$

(D3) For each local complete intersection morphism $f : X \to Y$ of quasi-projective schemes of codimension c, we have a morphism of graded abelian groups

$$f^! : H_*(Y) \to H_{*-c}(X).$$

We denote $f^!$ by f^* when f is smooth.

For any line bundle L on a quasi-projective scheme X, we denote by $c_1(L) := (0_L)^! \circ (0_L)_* : H_*(X) \to H_{*-1}(X)$ where $0_L : X \hookrightarrow L$ is the zero section.

(D4) For each quasi-projective schemes X and Y, we have a morphism of graded abelian groups

$$\times : H_*(X) \otimes_{\mathbb{Z}} H_*(Y) \to H_*(X \times Y).$$

These data are assume to satisfy the following assumptions:

(A1) Projective pushforwards in (D2) are *functorial*, i.e.

(a) For any quasi-projective scheme X, we have

$$(\mathrm{id}_X)_* = \mathrm{id}_{H_*(X)} : H_*(X) \to H_*(X).$$

(b) For projective morphisms $f : X \to Y$ and $g : Y \to Z$ of quasi-projective schemes, we have

$$(g \circ f)_* = g_* \circ f_* : H_*(X) \to H_*(Z).$$

(A2) Lci pullbacks in (D3) are *functorial*, i.e.

(a) For any quasi-projective scheme X, we have

$$(\mathrm{id}_X)^* = \mathrm{id}_{H_*(X)} : H_*(X) \to H_*(X).$$

(b) For local complete intersection morphisms $f : X \to Y$ and $g : Y \to Z$ of quasi-projective schemes of codimension c and d, we have

$$(g \circ f)^! = f^! \circ g^! : H_*(Z) \to H_{*-c-d}(X).$$

(A3) External products in (D4) are unital, associative, and commutative, i.e.

(a) There exists an element $1 \in H_0(\mathrm{Spec}(\mathbb{C}))$ such that for any quasi-projective scheme X and $\alpha \in H_*(X)$, we have

$$1 \times \alpha = \alpha \in H_*(X).$$

(b) For any quasi-projective schemes X, Y, Z, and $\alpha \in H_*(X)$, $\beta \in H_*(Y)$, $\gamma \in H_*(Z)$, we have

$$(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma) \in H_*(X \times Y \times Z).$$

(c) For any quasi-projective schemes X, Y and $\alpha \in H_*(X), \beta \in H_*(Y)$, we have

$$\alpha \times \beta = \beta \times \alpha \in H_*(X \times Y).$$

(A4) Projective pushforwards, lci pullbacks, and external products commute with each others, i.e.

(a) For any cartesian square of quasi-projective schemes

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

which is tor-independent, if f is projective and g is lci of codimension c, then we have

$$g^! \circ f_* = (f')_* \circ (g')^! : H_*(X) \to H_{*-c}(Y').$$

(b) For any projective morphisms $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ of quasi-projective schemes, and $\alpha_1 \in H_*(X_1)$, $\alpha_2 \in H_*(X_2)$, we have

$$(f_1)_*(\alpha_1) \times (f_2)_*(\alpha_2) = (f_1 \times f_2)_*(\alpha_1 \times \alpha_2) \in H_*(Y_1 \times Y_2).$$

(c) For any local complete intersection morphisms $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ of quasi-projective schemes, and $\alpha_1 \in H_*(Y_1)$, $\alpha_2 \in H_*(Y_2)$, we have

$$(f_1)^!(\alpha_1) \times (f_2)^!(\alpha_2) = (f_1 \times f_2)!(\alpha_1 \times \alpha_2) \in H_*(X_1 \times X_2).$$

(A5) We have a projective bundle formula, i.e. if $\pi : \mathbb{P}(E) \to X$ is a projective bundle associated to a vector bundle E of rank $n+1$ over a quasi-projective scheme X, then the map

$$\sum_{i=0}^{n} c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^i \circ \pi^* : \bigoplus_{i=0}^{n} H_{*+i}(X) \to H_{*+n}(\mathbb{P}(E))$$

is an isomorphism.

(A6) We have an extended homotopy property, i.e. if $\pi : A \to X$ is a torsor of a vector bundle of rank n over a quasi-projective scheme X, then the smooth pullback

$$\pi^* : H_*(X) \to H_{*+n}(A)$$

is an isomorphism.

(A7) We have a localization sequence, i.e. if $i : Z \hookrightarrow X$ is a closed embedding of quasi-projective schemes and $j : U := X \backslash Z \hookrightarrow X$ is the complement, then the sequence

$$H_*(Z) \xrightarrow{i_*} H_*(X) \xrightarrow{j^*} H_*(U) \longrightarrow 0$$

is exact.

Remark 1.3.2. Compared to [LM, Def. 5.1.3], there are two minor differences:

1. We use the category of quasi-projective schemes, while [LM] uses general "admissible" subcategories of the category of schemes.
2. In (A7), we assumed the localization sequence, while [LM] uses a weaker axiom called (CD).

Since our main objects, algebraic cobordism, Chow groups, and algebraic K-theory are defined over all quasi-projective schemes and satisfy the axiom (A7), this simpler convention does not affect anything in this paper.

Fulton's Chow theory [Ful] and Grothendieck's algebraic K-theory are the basic examples.

Example 1.3.3 (Chow groups)**.** The Chow groups $A_*(X)$ of rational equivalence classes of algebraic cycles for quasi-projective schemes X in section 1.1 form an oriented Borel-Moore homology theory.

Example 1.3.4 (Algebraic K-theory)**.** The algebraic K-theory $K_0(X)[\beta^{\pm 1}]$ of coherent sheaves for quasi-projective schemes X form an oriented Borel-Moore homology theory. More precisely, we define

$$K_0(X) := K_0(\mathsf{Coh}(X))$$

to be the Grothendieck group of coherent sheaves. Then we define

$$K_0(X)[\beta^{\pm 1}] = \bigoplus_{d \in \mathbb{Z}} K_0(X) \cdot \beta^d$$

for a formal parameter β of degree 1.

Algebraic coborism is defined as the universal oriented Borel-Moore homology theory.

Proposition 1.3.5 (Algebraic cobordism)**.** *There exists a universal oriented Borel-Moore homology theory Ω, called algebraic cobordism. More precisely, for any oriented Borel-Moore homology theory H, there exists a unique map*

$$\theta^H(X) : \Omega_*(X) \to H_*(X)$$

for each quasi-projective scheme X such that θ^H commutes with projective pushforwards, lci pullbacks, and external products.

Algebraic cobordism has a geometric description via *double point relations*, which was discovered by Levine-Pandharipande [LP].

Proposition 1.3.6 (Double point cobordism)**.** *Let X be a quasi-projective scheme. Then there exists an isomorphism*

$$\Omega_*(X) \cong \frac{Z_*^{\Omega}(X)}{D_*^{\Omega}(X)}$$

that commutes with projective pushforwards and smooth pullbacks. Here the group of cobordism cycles $Z_*^{\Omega}(X)$ and the group of double points relations $D_*^{\Omega}(X)$ are defined as follows:

1. *A* cobordism cycle *of degree d is a projective morphism*

   $$f : Z \to X$$

   from a smooth quasi-projective scheme Z of dimension d. We let

   $$Z_*^{\Omega}(X) := \mathbb{Z}\langle [f : Z \to X] \rangle$$

   be the free abelian group generated by all cobordism cycles $f : Z \to X$, where the grading is given by the dimension of Z.

2. *Let $h : W \to X \times \mathbb{P}^1$ be a projective morphism from a smooth quasi-projective scheme W such that the fiber W_∞ over $\infty \in \mathbb{P}^1$ is smooth and the fiber W_0 over $0 \in \mathbb{P}^1$ is the sum of two smooth divisors $A, B \subseteq W$ such that $A \cap B$ is smooth of codimension 2. The* double point relation *associated to W is*

   $$[A \to X] + [B \to X] - [\mathbb{P}(N_{A\cap B/A} \oplus O_{A\cap B}) \to X] - [W_\infty \to X] \in Z_*^{\Omega}(X).$$

We let

$$D_*^{\Omega}(X) \subseteq Z_*^{\Omega}(X)$$

be the subgroup generated by all double point relations.

Algebraic cobordism recovers Chow groups and algebraic K-theory.

Proposition 1.3.7. *For any quasi-projective scheme X, we have isomorphisms*

$$A_*(X) \cong \Omega_*(X) \otimes_{\Omega_*(\mathrm{Spec}(\mathbb{C}))} A_*(\mathrm{Spec}(\mathbb{C}))$$
$$K_0(X)[\beta^{\pm 1}] \cong \Omega_*(X) \otimes_{\Omega_*(\mathrm{Spec}(\mathbb{C}))} K_0(\mathrm{Spec}(\mathbb{C}))[\beta^{\pm 1}]$$

which commute with projective pushforwards, lci pullbacks, and external products.

Proof. The first isomorphism is shown in [LM, Thm. 4.5.1]. The second isomorphism for smooth X is shown in [LM, Cor. 4.2.12] and the general case is shown in [Dai]. □

Algebraic coboridsm with *rational* coefficients has more concrete descriptions.

Proposition 1.3.8 (Rational cobordism ring)**.** *The rational algebraic cobordism ring of the point* $\mathrm{Spec}(\mathbb{C})$ *is the polynomial ring, freely generated by the projective spaces,*

$$\Omega_*(\mathrm{Spec}(\mathbb{C}))_{\mathbb{Q}} = \mathbb{Q}[\mathbb{P}^1, \mathbb{P}^2, \cdots].$$

Proof. It follows from [LM, Thm. 4.3.7]. □

Proposition 1.3.9. *For any quasi-projective scheme X, we have an isomorphism of graded $\mathbb{Q}$-vector spaces*

$$\Omega_*(X)_{\mathbb{Q}} \cong A_*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[\mathbb{P}^1, \mathbb{P}^2, \cdots],$$

which commutes with projective pushforwards (but not necessarily with the lci pullbacks).

Proof. This is shown in [LM, Thm. 4.1.28]. □

Remark 1.3.10 (Proper pushforwards)**.** In [GK1], Gonzalez-Karu constructed proper pushforwards in algebraic cobordism.

Remark 1.3.11 (Flat pullbacks)**.** In [Lev1], Levine showed that flat pullbacks do not exist in algebraic cobordism.

Chapter 2

Virtual pullbacks

This chapter reviews the notions of *virtual cycles* of Behrend-Fantechi [BF] and *virtual pullbacks* of Manolache [Man].

Summary A remarkable idea of Fulton [Ful] on intersection theory is to define a *refine Gysin pullback* via the deformation to normal cone instead of using the moving lemma. More precisely, given a fiber square

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, hook, "f"] & Y \end{tikzcd}$$

with a regular closed embedding f, Fulton defined the refined Gysin pullback

$$f^! : A_*(Y') \xrightarrow{\text{sp}} A_*(C_{X'/Y'}) \to A_*(N_{X/Y}|_{X'}) \xrightarrow{\cong} A_*(X')$$

where

1. the first map is given by the deformation to the normal cone,
2. the second map is given by the inclusion map $C_{X'/Y'} \hookrightarrow N_{X/Y}|_{X'}$, and
3. the third map is given by the homotopy property of vector bundles.

The virtual cycles/virtual pullbacks are natural generalization of Fulton's refined Gysin pullback to algebraic stacks. As the above paragraph, we need three ingredients to do this:

1. The *intrinsic normal cone* $\mathfrak{C}_{X/Y}$ for an arbitrary morphism $f : X \to Y$ and a deformation $Y \rightsquigarrow \mathfrak{C}_{X/Y}$.

2. A closed embedding $\mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$ of the intrinsinc normal cone into a vector bundle stack. This data is equivalent to a *perfect obstruction theory*.

3. The homotopy property $A_*(\mathfrak{E}) \cong A_*(X)$ for vector bundle stacks, which gives us a *Gysin pullback* $0^!_{\mathfrak{E}} : A_*(\mathfrak{E}) \to A_*(X)$.

Based on the above three ingredients, we can define the *virtual pullback* for a DM morphism $f : X \to Y$ with a perfect obstruction theory $\iota : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$ as

$$f^! : A_*(Y) \xrightarrow{\mathrm{sp}} A_*(\mathfrak{C}_{X/Y}) \xrightarrow{\iota_*} A_*(\mathfrak{E}) \xrightarrow{0^!_{\mathfrak{E}}} A_*(X).$$

In particular, the *virtual cycle* for a Deligne-Mumford stack X with a perfect obstruction theory can be defined as

$$[X]^{\mathrm{vir}} := 0^!_{\mathfrak{E}}[\mathfrak{C}_X] \in A_*(X).$$

The most important property of virtual cycles is the *deformation invariance*. This is a special case of *functoriality* of virtual pullbacks: given a commutative diagram of DM morphisms of algebraic stacks

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z, \end{tikzcd}$$

and a compatible triple of perfect obstruction theories, we have

$$(g \circ f)^! = f^! \circ g^! : A_*(Z) \to A_*(X).$$

The key idea for proving the functoriality is to use the *double deformation space* of Kim-Kresch-Pantev [KKP].

2.1 Intrinsic normal cones

In this section, we review the concept of *intrinsic normal cones* introduced by Behrend-Fantechi in [BF]. These intrinsic normal cones are stacky generalizations of the normal cones for closed embeddings to arbitrary morphisms.

2.1.1 Abelian cone stacks

Recall that the *cones* are main objects in intersection theory [Ful, Vist]. In virtual intersection theory [BF, Man], the *cone stacks* are the analogous main objects. Roughly speaking, cone stacks are algebraic stacks with $\mathbb{A}^1$-actions and zero sections. For the precise definition, we refer to [BF, Def. 1.8].

Definition 2.1.1 (Abelian cone stack)**.** Let X be an algebraic stack and let $\mathbb{F} \in \mathsf{D}^{(-\infty,0]}_{\mathrm{coh}}(X)$. An *abelian cone stack* associated to $\mathbb{F}$ is the cone stack

$$\mathfrak{C}(\mathbb{F}) := h^1/h^0(\mathbb{F}^\vee)$$

defined in [BF, Prop. 2.4].

An abelian cone stack has an explicit description as a quotient stack when there is a *global resolution*. The general case can be regarded as a gluing of this special case.

Example 2.1.2 (Global presentation)**.** Let X be an algebraic stack and let $\mathbb{F} \in \mathsf{D}^{(-\infty,0]}_{\mathrm{coh}}(X)$. If $\mathbb{F} \cong [F \to E]$ for a coherent sheaf F and a vector bundle E, then we have

$$\mathfrak{C}(\mathbb{F}) \cong [C(F)/E^\vee]$$

where $C(F) := \mathrm{Spec}(\mathrm{Sym}(F))$ is the abelian cone associated to F.

An abelian cone stack also has a *derived* interpretation. This allow us to view an abelian cone stack as a natural generalization of an abelian cone.

Remark 2.1.3 (Derived enhancement)**.** Let X be an algebraic stack and let $\mathbb{F} \in \mathsf{D}^{(-\infty,0]}_{\mathrm{coh}}(X)$. Consider the *derived linear stack* defined as the ∞-functor

$$\mathrm{Spec}(\mathrm{Sym}(\mathbb{F}[-1])) : \mathsf{sAlg}_{/X} \to \mathsf{sSet} : (\mathrm{Spec}(A) \xrightarrow{s} X) \mapsto \mathrm{Map}_{\mathrm{Mod}_A}(x^*\mathbb{F}[-1], A)$$

such that $\mathbb{L}_{\mathrm{Spec}(\mathrm{Sym}(\mathbb{F}[-1]))/X} = \mathbb{F}[-1]$ by [AG]. Then the abelian cone stack $\mathfrak{C}(\mathbb{F})$ is the classical truncation of the derived linear stack,

$$\mathfrak{C}(\mathbb{F}) = \mathrm{Spec}(\mathrm{Sym}(\mathbb{F}[-1]))_{\mathrm{cl}}.$$

In particular, if $\mathbb{F}$ is a perfect complex, then the abelian cone stack $\mathfrak{C}(\mathbb{F})$ is the classical truncation of the total space of $\mathbb{F}^\vee[1]$,

$$\mathfrak{C}(\mathbb{F}) = \mathrm{Tot}(\mathbb{F}^\vee[1])_{\mathrm{cl}}.$$

Recall that the (contravariant) functor

$$C : \mathrm{Coh}(X) \xrightarrow{\cong} \{\text{abelian cones on } X\} : F \mapsto C(F) := \mathrm{Spec}(\mathrm{Sym}^\bullet(F))$$

is an equivalence of categories (cf. [Sie]). There is a similar equivalence for abelian cone stacks.

Proposition 2.1.4 (Equivalence)**.** *Let X be an algebraic stack. Then the 2-functor*

$$\mathfrak{C} : \mathsf{D}^{[-1,0]}_{\mathrm{coh}}(X) \xrightarrow{\cong} \{\textit{abelian cone stacks on } X\} : \mathbb{F} \mapsto \mathfrak{C}(\mathbb{F}) := h^1/h^0(\mathbb{F}^\vee)$$

is an equivalence of 2-categories. Moreover, the 2-functor

$$\{\textit{abelian cone stacks on } X\} \to \mathsf{D}^{[-1,0]}_{\mathrm{coh}}(X) : \mathfrak{A} \mapsto L_{X/\mathfrak{A}} := \tau^{\geqslant -1}\mathbb{L}_{X/\mathfrak{A}}$$

is the inverse of $\mathfrak{C}$.

For complexes with global resolutions, the equivalence in Proposition 2.1.4 can be described explicitly as Remark 2.1.5 below. The general case can be shown by descent (in the ∞-categorical sense). We omit the proof here.

Remark 2.1.5. Let X be an algebraic stack and let $\mathbb{F} \in \mathsf{D}^{(-\infty,0]}_{\mathrm{coh}}(X)$. Assume that there is a global resolution

$$\tau^{\geqslant -1}\mathbb{F} \cong [F \to E]$$

by a coherent sheaf F and a vector bundle E. Then the zero section of the abelian cone stack $\mathfrak{C}(\mathbb{F})$ can be factored as

$$\begin{tikzcd} & C(F) \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[ur, hook, "0_{C(F)}"] \arrow[r, "0_{\mathfrak{C}(\mathbb{F})}"'] & \mathfrak{C}(\mathbb{F}) \arrow[r] & B(E^\vee). \end{tikzcd}$$

Hence we can simply obtain the following

$$\tau^{\geqslant -1}\mathbb{L}_{X/\mathfrak{C}(\mathbb{F})} \cong \left[\mathcal{I}_{X/C(F)}/\mathcal{I}^2_{X/C(F)} \to \Omega_{C(F)/\mathfrak{C}(\mathbb{F})}|_X\right] \cong [F \to E] \cong \tau^{\geqslant -1}\mathbb{F}$$

as desired.

2.1.2 Intrinsic normal cones

We will work with morphisms that are relatively of Deligne-Mumford type.

Definition 2.1.6 (DM morphism)**.** We say that a morphism $f : X \to Y$ of algebraic stacks is a *DM morphism* if one of the following equivalent conditions is satisfied:

1. The diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is unramified.
2. The fibers $X_T := X \times_Y T$ are DM stacks for all morphisms $T \to Y$ from DM stacks T.
3. We have $\mathbb{L}_{X/Y} \in D^{(-\infty,0]}_{\mathrm{coh}}(X)$, i.e., $h^1(\mathbb{L}_{X/Y}) = 0$.

It is easy to show that the above three conditions are indeed equivalent.

Definition 2.1.7 (Intrinsic normal sheaf)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks. We define the *intrinsic normal sheaf* as

$$\mathfrak{N}_{X/Y} := \mathfrak{C}(\mathbb{L}_{X/Y}),$$

the abelian cone stack associated to the cotangent complex $\mathbb{L}_{X/Y}$.

We define the main object in this section.

Definition 2.1.8 (Intrinsic normal cone)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks. We define the *intrinsic normal cone*

$$\mathfrak{C}_{X/Y} \subseteq \mathfrak{N}_{X/Y}$$

to be the unique subcone stack satisfying the following property: for any commutative square

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook, "\widetilde{f}"] \arrow[d] & \widetilde{Y} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with smooth vertical arrows, and a closed embedding $\widetilde{f}$, we have a cartesian square

$$\begin{tikzcd} C_{\widetilde{X}/\widetilde{Y}} \arrow[r, hook] \arrow[d, dotted] & N_{\widetilde{X}/\widetilde{Y}} \arrow[d] \\ \mathfrak{C}_{X/\widetilde{Y}} \arrow[r, hook] & \mathfrak{N}_{X/Y} \end{tikzcd}$$

for some dotted arrow. Here the map $\mathfrak{N}_{\widetilde{X}/\widetilde{Y}} \to \mathfrak{N}_{X/Y}$ is induced by the canonical map $\mathbb{L}_{X/Y}|_{\widetilde{X}} \to \mathbb{L}_{\widetilde{X}/\widetilde{Y}}$ of cotangent complexes.

Lemma 2.1.9 (Well-definedness)**.** *The intrinsic normal cone in Definition 2.1.8 exists.*

Proof. We refer to [BF, Kre2, Man] for the proof. $\square$

The intrinsic normal cone has a simple presentation as a quotient stack when there is a *global factorization*.

Example 2.1.10 (Global presentation)**.** Let $f : X \to Y$ be a DM morphism between algebraic stacks. If there exists a factorization

$$\begin{tikzcd} & \widetilde{Y} \arrow[d, "\overline{f}"] \\ X \arrow[ur, hook, "\widetilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd}$$

by a closed embedding $X \hookrightarrow \widetilde{Y}$ and a smooth morphism $\widetilde{Y} \to Y$, then we have

$$\mathfrak{C}_{X/Y} \cong \left[C_{X/\widetilde{Y}}/T_{\widetilde{Y}/Y}|_X\right] \subseteq \mathfrak{N}_{X/Y} \cong \left[N_{X/\widetilde{Y}}/T_{\widetilde{Y}/Y}|_X\right].$$

We provide one lemma which is technically quite useful.

Lemma 2.1.11. *Let $f : X \to Y$ be a DM morphism of algebraic stacks. Then we have canonical isomorphisms of the truncated cotangent complexes*

$$L_{X/\mathfrak{C}_{X/Y}} \cong L_{X/\mathfrak{N}_{X/Y}} \cong L_{X/Y}.$$

Proof. Since the cotangent complexes satisfy the étale descent (in the ∞-categorical sense), we may assume that there is a global factorization

$$\begin{tikzcd} & \widetilde{Y} \arrow[d, "\overline{f}"] \\ X \arrow[ur, hook, "\widetilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd}$$

by a closed embedding $X \hookrightarrow \widetilde{Y}$ and a smooth morphism $\widetilde{Y} \to Y$. Then we have induced factorizations

$$\begin{tikzcd} & N_{X/\widetilde{Y}} \arrow[d] \\ X \arrow[ur, hook] \arrow[r] & \mathfrak{N}_{X/Y} \end{tikzcd} \qquad \begin{tikzcd} & C_{X/\widetilde{Y}} \arrow[d] \\ X \arrow[ur, hook] \arrow[r] & \mathfrak{C}_{X/Y} \end{tikzcd}$$

of the zero sections of the intrinsic normal sheaf and the intrinsic normal cone. Moreover, by the definitions of normal cones and normal sheaves, we have

$$C_{X/\widetilde{Y}} = \mathrm{Spec}(\bigoplus_{n\geqslant 0} \mathcal{I}^n_{X/\widetilde{Y}}/\mathcal{I}^{n+1}_{X/\widetilde{Y}}), \qquad N_{X/\widetilde{Y}} = \mathrm{Spec}(\bigoplus_{n\geqslant 0} \mathrm{Sym}^n(\mathcal{I}_{X/\widetilde{Y}}/\mathcal{I}^2_{X/\widetilde{Y}})).$$

Hence the three truncated cotangent complexes $L_{X/\mathfrak{C}_{X/Y}}$, $L_{X/\mathfrak{N}_{X/Y}}$, $L_{X/Y}$ are all isomorphic to the complex

$$\left[\mathcal{I}_{X/\widetilde{Y}}/\mathcal{I}^2_{X/\widetilde{Y}} \to \Omega_{\widetilde{Y}/Y}|_X\right].$$

It completes the proof. □

We provide a heuristic explanation why we call $\mathfrak{C}_X$ an intrinsic normal cone.

Remark 2.1.12 (Heuristic description)**.** For a Deligne-Mumford stack X, the intrinsic normal cone $\mathfrak{C}_X$ is an intrinsic object, which is *homotopically equivalent* to the normal cone $C_{X/Y}$ whenever we have a closed embedding $X \hookrightarrow Y$ to a smooth Deligne-Mumford stack Y. Here we say $C_{X/Y}$ is homotopically equivalent to $\mathfrak{C}_X$ since it is a vector bundle torsor.

2.1.3 Deformation to the normal cone

Definition 2.1.13 (Deformation space)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks. The *deformation space* of $f : X \to Y$ is a flat family

$$M^\circ_{X/Y} \to \mathbb{P}^1$$

defined as follows:

1. *Case 1.* Assume that $f : X \hookrightarrow Y$ is a closed embedding. Then we define

$$M^\circ_{X/Y} := M_{X/Y} \backslash \widetilde{Y}$$

where $M_{X/Y} := \mathrm{Bl}_{X\times\{0\}}(Y \times \mathbb{P}^1)$ and $\widetilde{Y} := \mathrm{Bl}_{X\times\{0\}}(Y \times \{0\})$.

2. *Case 2.* Assume that $f : X \to Y$ is an unramified morphism. Then we can find a fiber diagram of algebraic stacks

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook, "\widetilde{f}"] \arrow[d, twoheadrightarrow] & \widetilde{Y} \arrow[d, twoheadrightarrow] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2.1.1}$$

where vertical arrows are étale surjective and $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is a closed embedding. We define

$$M^{\circ}_{X/Y} := [\, M^{\circ}_{\widetilde{X}\times_X\widetilde{X}/\widetilde{Y}\times_Y\widetilde{Y}} \rightrightarrows M^{\circ}_{\widetilde{X}/\widetilde{Y}} \,]$$

where the induced map $\widetilde{X} \times_X \widetilde{X} \to \widetilde{Y} \times_Y \widetilde{Y}$ is a closed embedding.

3. *Case 3.* Assume that $f : X \to Y$ is a DM morphism. Then we can find a fiber diagram of algebraic stacks

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook, "\widetilde{f}"] \arrow[d, two heads] & \widetilde{Y} \arrow[d, two heads] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2.1.2}$$

where vertical arrows are smooth surjective and $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is a closed embedding. We define

$$M^{\circ}_{X/Y} := [\, M^{\circ}_{\widetilde{X}\times_X\widetilde{X}/\widetilde{Y}\times_Y\widetilde{Y}} \rightrightarrows M^{\circ}_{\widetilde{X}/\widetilde{Y}} \,]$$

where the induced map $\widetilde{X} \times_X \widetilde{X} \to \widetilde{Y} \times_Y \widetilde{Y}$ is unramified.

We note that the deformation space in Definition 2.1.13 is well-defined.

Lemma 2.1.14 (Well-definedness)**.** *In the situation of Case 2 in Definition 2.1.13, $M^{\circ}_{X/Y}$ is independent of the choice of the fiber diagram* (2.1.1)*. Also, in the situation of Case 3 in Definition 2.1.13, $M^{\circ}_{X/Y}$ is independent of the choice of the fiber diagram* (2.1.2)*.*

Proof. We refer to [Kre2, Man] for the proof. □

Remark 2.1.15. The diagonal of the deformation space $M^{\circ}_{X/Y}$ may not be separated, see [Kre2].

We recall the basic properties of the deformations spaces from [Ful, Kre2].

Proposition 2.1.16 (Fibers)**.** *Let $f : X \to Y$ be a DM morphism of algebraic stacks. Then there exists a canonical map*

$$m : X \times \mathbb{P}^1 \to M^{\circ}_{X/Y}$$

such that the fibers over $\zeta \in \mathbb{P}^1$ are given as follows:

1. *The fiber of the above map m over $\zeta \neq 0 \in \mathbb{P}^1$ is the given map*

$$f : X \to Y.$$

2. *The fiber of the above map m over $0 \in \mathbb{P}^1$ is the zero section*

$$0_{\mathfrak{C}_{X/Y}} : X \to \mathfrak{C}_{X/Y},$$

Proof. If $f : X \to Y$ is a closed embedding of schemes, then this is shown in [Ful]. The general case follows by descent. □

Proposition 2.1.17 (Base change)**.** *Let*

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{Y} \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a fiber diagram of algebraic stacks such that $f : X \to Y$ is a DM morphism. Then the canonical map

$$M^{\circ}_{\widetilde{X}/\widetilde{Y}} \to M^{\circ}_{X/Y} \times_Y \widetilde{Y} \tag{2.1.3}$$

is a closed embedding. Moreover, if $g : \widetilde{Y} \to Y$ is flat, then the above canonical map (2.1.3) *is an isomorphism.*

Proof. We refer to [Kre2] for the proof. □

Heuristically, we may view $M^{\circ}_{X/Y}$ as the space of a deformation from Y to the normal cone $\mathfrak{C}_{X/Y}$,

$$M^{\circ}_{X/Y} : Y \rightsquigarrow \mathfrak{C}_{X/Y}.$$

Rigorously, this does not give us a genuine map of algebraic stacks. However, we indeed have a map between the Chow groups, called the *specialization map*.

Definition 2.1.18 (Specialization map)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks. We define the *specialiation map*

$$\mathrm{sp}_{X/Y} : A_*(Y) \longrightarrow A_*(\mathfrak{C}_{X/Y})$$

as the unique map that fits into the commutative diagram

$$\begin{tikzcd}
A_*(\mathfrak{C}_{X/Y}) \arrow[r] \arrow[dr, "0"'] & A_*(M^\circ_{X/Y}) \arrow[r] \arrow[d, "\iota_0^!"] \arrow[dr, "\iota_\zeta^!"] & A_*(Y \times \mathbb{A}^1) \arrow[r] \arrow[d, "\cong"] & 0 \\
& A_*(\mathfrak{C}_{X/Y}) & A_*(Y) \arrow[l, dotted, "\mathrm{sp}_{X/Y}"] &
\end{tikzcd}$$

where $\iota_0 : \{0\} \hookrightarrow \mathbb{P}^1$ and $\iota_\zeta : \{\zeta\} \hookrightarrow \mathbb{P}^1$ are the inclusion maps and $\zeta \neq 0$.

The specialization maps are *bivariant* classes.

Proposition 2.1.19 (Bivariance)**.** *Let*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

be a fiber diagram of algebraic stacks such that f is a DM morphism. Consider the induced commutative diagram

$$\begin{tikzcd}
\mathfrak{C}_{X'/Y'} \arrow[r, hook, "j"] \arrow[dr, "g'''"'] & \mathfrak{C}_{X/Y}|_{X'} \arrow[r] \arrow[d, "g''"] & X' \arrow[d, "g'"] \\
& \mathfrak{C}_{X/Y} \arrow[r] & X
\end{tikzcd}$$

where the square is cartesian.

1. *If g is a proper DM morphism, then we have*
$$\mathrm{sp}_{X/Y} \circ g_* = (g''')_* \circ \mathrm{sp}_{X'/Y'} : A_*(Y') \to A_*(\mathfrak{C}_{X'/Y'}).$$

2. *If g is an equi-dimensional flat morphism, then we have*
$$\mathrm{sp}_{X'/Y'} \circ g^* = g^* \circ \mathrm{sp}_{X/Y} : A_*(Y) \to A_*(\mathfrak{C}_{X/Y}|_{X'} = \mathfrak{C}_{X'/Y'}).$$

3. *If g is a local complete intersection morphism and Y' have affine stabilizers, then we have*
$$j_* \circ \mathrm{sp}_{X'/Y'} \circ g^! = g^! \circ \mathrm{sp}_{X/Y} : A_*(Y) \to A_*(\mathfrak{C}_{X/Y}|_{X'}).$$

Proof. If follows directly from Proposition 2.1.17, Definition 2.1.18, and Proposition 2.1.22 below. □

Remark 2.1.20. Proposition 2.1.19 is slightly general than the corresponding statements in [Man] since we use proper DM pushforwards, instead of projective pushforwards. This is based on the development of proper DM pushforwards of Bae-Schmitt-Skowera [BS, Appendix B].

The proof of Proposition 2.1.19.2 also works for a *commutative* square.

Lemma 2.1.21. *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a commutative diagram of algebraic stacks (not necessarily cartesian) such that g *,* g' *are smooth morphisms and* f *is a DM morphism. Then the canonical map*

$$g'' : \mathfrak{C}_{X'/Y'} \to \mathfrak{C}_{X/Y}$$

is smooth and we have

$$\mathrm{sp}_{X'/Y'} \circ g^* = g^* \circ \mathrm{sp}_{X/Y} : A_*(Y) \to A_*(\mathfrak{C}_{X'/Y'}).$$

We rephrase Vistoli's rational equivalence [Vist, Lem. 3.16] as follows.

Proposition 2.1.22 (Vistoli's rational equivalence)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a fiber diagram of algebraic stacks such that f *and* g *are DM morphisms. Consider the induced fiber diagram*

$$\begin{tikzcd} \mathfrak{C}_{X/Y}|_{X'} \times_{X'} \mathfrak{C}_{Y'/Y}|_{X'} \arrow[r] \arrow[d] & \mathfrak{C}_{Y'/Y}|_{X'} \arrow[r, "f''"] \arrow[d] & \mathfrak{C}_{Y'/Y} \arrow[d] \\ \mathfrak{C}_{X/Y}|_{X'} \arrow[r] \arrow[d, "g''"] & X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ \mathfrak{C}_{X/Y} \arrow[r] & X \arrow[r, "f"] & Y \end{tikzcd}$$

and the canonical closed embeddings

$$\mathfrak{C}_{\mathfrak{C}_{X/Y}|_{X'}/\mathfrak{C}_{X/Y}} \xhookrightarrow{a} \mathfrak{C}_{X/Y}|_{X'} \times_{X'} \mathfrak{C}_{Y'/Y}|_{X'} \xhookleftarrow{b} \mathfrak{C}_{\mathfrak{C}_{Y'/Y}|_{X'}/\mathfrak{C}_{Y'/Y}}.$$

Then we have

$$a_* \circ \mathrm{sp}_{\mathfrak{C}_{X/Y|_{X'}}/\mathfrak{C}_{X/Y}} \circ \mathrm{sp}_{X/Y} = b_* \circ \mathrm{sp}_{\mathfrak{C}_{Y'/Y}|_{X'}/\mathfrak{C}_{Y'/Y}} \circ \mathrm{sp}_{Y'/Y}.$$

Proof. We follow the argument in [Kre1]. Form a commutative diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & M_g^\circ \arrow[d] & \mathfrak{C}_g \arrow[l] \arrow[d] & \{1\} \arrow[d, "i_1"] \\
M_f^\circ \arrow[r] & M_f^\circ \times_Y M_g^\circ & M_f^\circ \times_Y \mathfrak{C}_g \arrow[l] & \mathbb{A}^1 \\
\mathfrak{C}_f \arrow[r] \arrow[u] & \mathfrak{C}_f \times_Y M_g^\circ \arrow[u] & \mathfrak{C}_f \times_Y \mathfrak{C}_g \arrow[l] \arrow[u] & \{0\} \arrow[u, "i_0"]
\end{tikzcd}$$

$$\{1\} \xrightarrow{j_1} \mathbb{A}^1 \xleftarrow{j_0} \{0\}.$$

Choose an element $\alpha \in A_*(Y)$. Then there exists a cycle class

$$\widetilde{\alpha} \in A_*(M_f^\circ \times_Y M_g^\circ)$$

such that

$$i_1^! \circ j_1^!(\widetilde{\alpha}) = j_1^! \circ i_1^!(\widetilde{\alpha}) = \alpha.$$

By the definition of the specialization map in Definition 2.1.18, we have

$$\mathrm{sp}_f(\alpha) = i_0^! \circ j_1^!(\widetilde{\alpha}) = j_1^! \circ i_0^!(\widetilde{\alpha}), \qquad \mathrm{sp}_g(\alpha) = j_0^! \circ i_1^!(\widetilde{\alpha}) = j_1^! \circ j_0^!(\widetilde{\alpha}).$$

Consider a commutative diagram

$$\begin{tikzcd}
\mathfrak{C}_f \arrow[r] \arrow[d, equal] & M_{g''}^\circ \arrow[d, hook] & \mathfrak{C}_{g''} \arrow[l] \arrow[d, hook, "a"] \\
\mathfrak{C}_f \arrow[r] & \mathfrak{C}_f \times_Y M_g^\circ & \mathfrak{C}_f \times_Y \mathfrak{C}_g \arrow[l]
\end{tikzcd}$$

$$\{1\} \xrightarrow{j_1} \mathbb{A}^1 \xleftarrow{j_0} \{0\}$$

induced by Proposition 2.1.16. From the above diagram, we can easily show that

$$a_* \circ \mathrm{sp}_{g''}(\mathrm{sp}_f(\alpha)) = j_0^!(i_0^!(\widetilde{\alpha})).$$

Analogously, we can also deduce

$$b_* \circ \mathrm{sp}_{f''}(\mathrm{sp}_g(\alpha)) = i_0^!(j_0^!(\widetilde{\alpha})).$$

Since $i_0^! \circ j_0^! = j_0^! \circ i_0^!$, we have

$$a_* \circ \mathrm{sp}_{g''}(\mathrm{sp}_f(\alpha)) = b_* \circ \mathrm{sp}_{f''}(\mathrm{sp}_g(\alpha)),$$

as desired. □

We recall *MacPherson's graph construction* from [Ful, Rem. 5.1.1].

Remark 2.1.23 (MacPherson's graph construction)**.** Consider a diagram

$$\begin{tikzcd} & E \arrow[d] \\ X \arrow[r, hook] & Y \arrow[u, bend right, "s"'] \end{tikzcd}$$

where Y is a smooth scheme, E is a vector bundle on Y, s is a section of E, and X is the zero locus of s. By [Ful, Rem. 5.1.1], the deformation space $M^\circ_{X/Y}$ is the closure of the embedding

$$Y \times \mathbb{A}^1 \to E \times \mathbb{P}^1 : (y, \zeta) \mapsto (\zeta \cdot y, [\zeta : 1])$$

In particular, the normal cone is the flat limit

$$C_{X/Y} = \varinjlim_{\zeta \to 0} \Gamma_{\zeta \cdot s},$$

where $\Gamma_{\zeta \cdot s} \subseteq E$ is the image of the embedding $\zeta \cdot s : Y \to E$.

2.2 Perfect obstruction theories

In this section, we recall the notion of *perfect obstruction theories* introduced by Behrend-Fantechi in [BF]. These perfect obstruction theories are the necessary additional data to define the virtual cycles.

2.2.1 Vector bundle stacks

Definition 2.2.1 (Vector bundle stack)**.** Let X be an algebraic stack and $\mathbb{F}$ be a perfect complex of tor-amplitude $[-1,0]$. We define the *vector bundle stack* associated to $\mathbb{F}$ as the abelian cone stack

$$\mathfrak{E}(\mathbb{F}) := \mathfrak{C}(\mathbb{F}).$$

Lemma 2.2.2 (Smoothness)**.** *Let X be an algebraic stack. A cone stack on X is a vector bundle stack if and only if it is smooth over X.*

Proof. If follows from [BF, Lem. 1.1] via descent. □

Proposition 2.2.3 (Cotangent complex)**.** *Let X be an algebraic stack and let* $\mathbb{F}$ *be a perfect complex on X of tor-amplitude* $[-1,0]$*. Then we have a canonical isomorphism*

$$\mathbb{L}_{\mathfrak{E}(\mathbb{F})/X} \cong \pi^*_{\mathfrak{E}(\mathbb{F})}(\mathbb{F})[-1],$$

where $\pi_{\mathfrak{E}(\mathbb{F})} : \mathfrak{E}(\mathbb{F}) \to X$ *denotes the projection map.*

Proof. It follows from the Remark 2.2.4 below. □

Remark 2.2.4 (Derived interpretation)**.** Let $\mathbb{F}$ be a perfect complex of tor-amplitude $[-1,0]$ on an algebraic stack X. Then the associated vector bundle stack $\mathfrak{E}(\mathbb{F})$ is the total space of the perfect complex $\mathbb{F}^\vee[1]$,

$$\mathfrak{E}(\mathbb{F}) = \mathrm{Tot}_X(\mathbb{F}^\vee[1]).$$

Indeed, this follows from Remark 2.1.3 since $\mathrm{Tot}_X(\mathbb{F}^\vee[1])$ is smooth.

2.2.2 Perfect obstruction theories

Definition 2.2.5 (Obstruction theory)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks. An *obstruction theory* for $f : X \to Y$ is a morphism

$$\phi : \mathbb{F} \to L_{X/Y} := \tau^{\geqslant -1}\mathbb{L}_{X/Y}$$

in $\mathsf{D}^{(-\infty,0]}_{\mathrm{coh}}(X)$ such that

1. $h^0(\phi)$ is bijective, $h^{-1}(\phi)$ is surjective, and
2. $\mathbb{F}$ is a perfect complex of tor-amplitude $[-d,0]$ for some $d \in \mathbb{Z}_{\geqslant 0}$.

Definition 2.2.6 (Perfect obstruction theory)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks. A *perfect obstruction theory* for $f : X \to Y$ is an obstruction theory $\phi : \mathbb{F} \to L_{X/Y}$ such that $\mathbb{F}$ is a perfect complex of tor-amplitude $[-1, 0]$.

Remark 2.2.7. The notion of *perfect* obstruction theory is quite misleading. It would be more natural to call an obstruction theory $\phi : \mathbb{F} \to L_{X/Y}$ perfect when $\mathbb{F}$ is a perfect complex. However, since the terminology is already standard in the literatures, we will also follow this tradition in this paper.

We note that the classical truncation of a derived scheme has a canonical obstruction theory. Most of the practical examples of the obstruction theories arise from derived structures.

Example 2.2.8 (Derived enhancement)**.** Let $\mathbb{X}$ be a homotopically finitely presented derived scheme. Let $X := \mathbb{X}_{\mathrm{cl}}$ be the classical truncation.

1. The canonical map
$$\mathbb{L}_{\mathbb{X}}|_{X_{\mathrm{cl}}} \to \mathbb{L}_{X_{\mathrm{cl}}} \to L_{X_{\mathrm{cl}}}$$
is an obstruction theory by [STV, Prop. 1.2].

2. If $\mathbb{X}$ is *quasi-smooth*, i.e. $\mathbb{L}_{\mathbb{X}}$ has tor-amplitude $[-1, 0]$, then the above induced obstruction theory $\mathbb{L}_{\mathbb{X}}|_{X_{\mathrm{cl}}} \to L_{X_{\mathrm{cl}}}$ is a perfect obstruction theory.

A perfect obstruction theory is equivalent to a closed embedding of the intrinsic normal cone into a vector bundle stack.

Proposition 2.2.9 (Equivalence)**.** *Let $f : X \to Y$ be a DM morphism of algebraic stacks.*

1. *If $\phi : \mathbb{F} \to L_{X/Y}$ is perfect obstruction theory, then the composition*
$$\mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{N}_{X/Y} = \mathfrak{C}(L_{X/Y}) \xrightarrow{\mathfrak{C}(\phi)} \mathfrak{E}(\mathbb{F})$$
is a closed embedding of cone stacks.

2. *Conversely, if $\mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$ is a closed embedding of cone stacks for some vector bundle stack $\mathfrak{E}$, then the composition*
$$\mathbb{L}_{X/\mathfrak{E}} \to \mathbb{L}_{X/\mathfrak{C}_{X/Y}} \to L_{X/\mathfrak{C}_{X/Y}} \cong L_{X/Y}$$
is a perfect obstruction theory.

Moreover, the above two operations are inverse to each others.

Proof. It follows from Proposition 2.1.4, Lemma 2.1.11 and [BF, Prop. 2.6]. □

Remark 2.2.10. Proposition 2.2.9 is a folklore, but there are two technical issues that are often ignored in the literatures:

1. We need to use the *truncated* cotangent complex $L_{X/Y} := \tau^{\geqslant -1}\mathbb{L}_{X/Y}$.

2. The closed embedding $\mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$ should be $\mathbb{A}^1$*-equivariant*.

These are necessary to have the equivalence in Proposition 2.2.9.

In various literatures, the full cotangent complex $\mathbb{L}_{X/Y}$ is used in the definition of obstruction theories, instead of the truncated cotangent complex $L_{X/Y} := \tau^{\geqslant -1}\mathbb{L}_{X/Y}$. Practically, this difference of definitions was not regarded seriously since most of the examples have perfect obstruction theories in the stronger version. However, there are some technical examples that only the existence of perfect obstruction theories in the weaker version is known. In general, these two versions of perfect obstruction theories are not equivalent, see Example 2.2.12 below. Thus Proposition 2.2.9 does not hold for the full cotangent complex version.[1]

If we have a closed embedding $\mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$ which is *not* $\mathbb{A}^1$-equivariant, then we still have an induced perfect obstruction theory $\mathbb{L}_{X/\mathfrak{E}} \to L_{X/Y}$, but the associated closed embedding $\mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$ may differ with the given embedding. Thus we need to consider a *closed embedding of cone stacks*, i.e., an $\mathbb{A}^1$-equivariant closed embedding, as in [BF, Def. 1.8].

Remark 2.2.11. In this paper, we use the truncated version of perfect obstruction theories. Then we have Proposition 2.2.9 as a technical advantage. On the other hand, there is one technical disadvantage. We need to be careful when dealing with distinguished triangles of truncated cotangent complexes. Indeed, consider a commutative diagram of DM morphisms of algebraic stacks

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z. \end{tikzcd}$$

Then the maps between the truncated cotangent complexes

$$f^*L_{Y/Z} \longrightarrow L_{X/Z} \longrightarrow L_{X/Y}$$

[1]It is stated in [Man, Prop. 3.11] (and [Qu, §1.5]) that Proposition 2.2.9 holds for the full cotangent complex version. However, the author expects that this is a mistake and actually it is meant for the truncated cotangent complex version.

do not form a distinguished triangle in general.[2]

Here we provide an example of a perfect obstruction theory $\phi : \mathbb{F} \to L_{X/Y}$ that does not lift to a map $\mathbb{F} \to \mathbb{L}_{X/Y}$.

Example 2.2.12. Let $X = \mathbb{P}^2$ and let $x_0, x_1, x_2 \in \Gamma(X, \mathcal{O}_{\mathbb{P}}(1))$ be the coordinate sections so that

$$\Gamma(X, \mathcal{O}_{\mathbb{P}^2}(1)) = \bigoplus_{0\leqslant i\leqslant 2} \mathbb{C}\cdot x_i.$$

Let $Z = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3})$ be a vector bundle on X. Let π_Z denote the projection map and 0_Z denote the zero section. Let $t_0, t_1, t_2 \in \Gamma(Z, \pi_Z^*\mathcal{O}_{\mathbb{P}^2}(1))$ be the three tautological sections so that

$$\Gamma(Z, \pi_Z^*\mathcal{O}_{\mathbb{P}^2}(1)) \cong \Gamma(X, \mathcal{O}_{\mathbb{P}^2}(1)) \oplus \bigoplus_{0\leqslant i\leqslant 2} \Gamma(X, \mathcal{O}_{\mathbb{P}^2})\cdot t_i.$$

Then $\Gamma(Z, \pi_Z^*\mathcal{O}_{\mathbb{P}^2}(2))$ can be expressed as

$$\Gamma(X, \mathcal{O}_{\mathbb{P}^2}(2)) \oplus \bigoplus_{0\leqslant i\leqslant 2} \Gamma(X, \mathcal{O}_{\mathbb{P}^2}(1))\cdot t_i \oplus \bigoplus_{0\leqslant i,j\leqslant 2} \Gamma(X, \mathcal{O}_{\mathbb{P}^2})\cdot t_i t_j.$$

Let $E = \mathrm{Tot}(\pi_Z^*\mathcal{O}_{\mathbb{P}^2}(2)^{\oplus 3})$ be a vector bundle on Z. Consider a diagram

$$\begin{tikzcd}
 & & E \arrow[d] \\
X \arrow[r, hook] \arrow[rr, bend right, "0_Z"'] & Y \arrow[r, hook] & Z \arrow[u, bend right, "s"']
\end{tikzcd}$$

where Y is the zero locus of the section

$$s = \begin{pmatrix} \pi_Z^*(x_2)\cdot t_3 - \pi_Z^*(x_3)\cdot t_2 \\ \pi_Z^*(x_3)\cdot t_1 - \pi_Z^*(x_1)\cdot t_3 \\ \pi_Z^*(x_1)\cdot t_2 - \pi_Z^*(x_2)\cdot t_1 + t_3^2 \end{pmatrix} \in \Gamma(Z, E).$$

A simple local computation show that s is a regular section. Hence from the canonical distinguished triangle

$$\mathbb{L}_{Y/Z}|_X \longrightarrow \mathbb{L}_{X/Z} \longrightarrow \mathbb{L}_{X/Y}$$

[2]It is stated in [KP1, Thm. 4.4(2)] that the truncated cotangent complexes form a distinguished triangle, but this is not true in general. We explain how to fix this in §2.3.3.

we can deduce that

$$\mathbb{L}_{X/Y} = \left[\mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \to 0\right]$$

where

$$M = \begin{pmatrix} 0 & x_1 & -x_2 \\ x_2 & 0 & -x_0 \\ x_0 & -x_1 & 0 \end{pmatrix}.$$

Consider a perfect obstruction theory

$$\phi := 1_{\mathcal{O}_{\mathbb{P}^2}[1]} : \mathcal{O}_{\mathbb{P}^2}[1] \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2}[1] = L_{X/Y},$$

for the inclusion map $X \hookrightarrow Y$. Then it does not lift to a map $\mathcal{O}_{\mathbb{P}^2} \to \mathbb{L}_{X/Y}$. More precisely, there is no map that fits into the commutative diagram

$$\begin{tikzcd} & \mathcal{O}_{\mathbb{P}^2}[1] \arrow[dl, dotted, "\nexists"'] \arrow[d, "\phi"] \arrow[dr, "\neq 0"] & \\ \mathbb{L}_{X/Y} \arrow[r] & L_{X/Y} \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-3)[3] \end{tikzcd}$$

as the dotted arrow since the map $\mathcal{O}_{\mathbb{P}^2}[1] \to \mathcal{O}_{\mathbb{P}^2}(-3)[3]$ is non-zero.

2.3 Virtual pullbacks and virtual cycles

In this section, we recall the definitions and main properties of *virtual cycles* and *virtual pullbacks* associated to perfect obstruction theories from [BF, Man].

2.3.1 Gysin pullbacks

We begin with a special case. Note that the zero section

$$0_{\mathfrak{E}} : X \to \mathfrak{E}$$

of a vector bundle stack $\mathfrak{E}$ on X has a canonical perfect obstruction theory since it is a local complete intersection morphism. We construct the Gysin pullback $0^!_{\mathfrak{E}}$ via the *homotopy property* of vector bundle stacks.

Proposition 2.3.1 (Homotopy property)**.** *Let X be an algebraic stack with affine stabilizers. Let $\mathfrak{E}$ be a vector bundle stack on X. Then the smooth pullback*

$$\pi_{\mathfrak{E}}^* : A_*(X) \xrightarrow{\cong} A_*(\mathfrak{E})$$

is an isomorphism, where $\pi_{\mathfrak{E}} : \mathfrak{E} \to X$ is the projection map.

Proof. We refer to [Kre2, Prop. 4.3.2] for the proof. □

Consequently, we have Kresch's *Gysin pullbacks* [Kre2] for the zero sections of vector bundle stacks.

Definition 2.3.2 (Gysin pullback)**.** Let X be an algebraic stack with affine stabilizers. Let $\mathfrak{E}$ be a vector bundle stack on X. Let $\pi_{\mathfrak{E}} : \mathfrak{E} \to X$ denote the projection map and let $0_{\mathfrak{E}} : X \to \mathfrak{E}$ denote the zero section. We define the *Gysin pullback* as

$$0_{\mathfrak{E}}^! := (\pi_{\mathfrak{E}}^*)^{-1} : A_*(\mathfrak{E}) \to A_*(X)$$

where the smooth pullback $\pi_{\mathfrak{E}}^*$ is an isomorphism by Proposition 2.3.1.

2.3.2 Virtual pullbacks

We then consider the general case. We define Manolache's virtual pullbacks [Man] by reducing the situation to the special case in the previous subsection via deformation to normal cone.

Definition 2.3.3 (Virtual pullback)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks and let $\phi : \mathbb{F} \to L_{X/Y}$ be a perfect obstruction theory. Assume that X has affine stabilizers. We define the *virtual pullback* as the composition

$$f^! : A_*(Y) \xrightarrow{\mathrm{sp}_{X/Y}} A_*(\mathfrak{C}_{X/Y}) \xrightarrow{\iota_*} A_*(\mathfrak{E}(\mathbb{F})) \xrightarrow{0^!_{\mathfrak{E}(\mathbb{F})}} A_*(X)$$

where $\iota : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}(\mathbb{F})$ denotes the closed embedding induced by the obstruction theory ϕ. Here $\mathrm{sp}_{X/Y}$ is the specialization map in Definition 2.1.18 and $0^!_{\mathfrak{E}(\mathbb{F})}$ is the Gysin pullback in Definition 2.3.2.

We explain three special cases of virtual pullbacks. Firstly, when $Y = \mathrm{Spec}(\mathbb{C})$, we obtain Behrend-Fantechi's virtual cycle [BF].

Definition 2.3.4 (Virtual cycle)**.** Let X be a Deligne-Mumford stack endowed with a perfect obstruction theory $\psi : \mathbb{F} \to L_X$. We define the *virtual cycle* as

$$[X]^{\mathrm{vir}} := p^![\mathrm{Spec}(\mathbb{C})] = 0^!_{\mathfrak{E}(\mathbb{F})}[\mathfrak{C}_X] \in A_*(X)$$

where $p : X \to \mathrm{Spec}(\mathbb{C})$ denotes the projection map.

Secondly, when f is a closed embedding, we obtain Fulton's refined Gysin pullback [Ful].

Remark 2.3.5 (Refined Gysin pullbacks as virtual pullbacks)**.** Let $f : X \to Y$ be a closed embedding of schemes. Then a perfect obstruction theory is equivalent to a closed embedding $C_{X/Y} \hookrightarrow N$ of the normal cone into a vector bundle N. Then the virtual pullback is the *refined Gysin pullback*

$$f^! := 0^!_N \circ \mathrm{sp}_{X/Y} : A_*(Y) \to A_*(X).$$

Thirdly, when f is the zero section of a vector bundle stack, then we obtain the Gysin pullback of the vector bundle stack in Definition 2.3.2.

Remark 2.3.6 (Gysin pullbacks as virtual pullbacks)**.** Let X be an algebraic stack with affine stabilizers and $\mathfrak{E}$ be a vector bundle stack. Then the zero section $0_{\mathfrak{E}} : X \to \mathfrak{E}$ has a canonical perfect obstruction theory by Proposition 2.2.9, and the associated virtual pullback is the Gysin pullback in Definition 2.3.2.

The virtual pullbacks are bivariant classes. This can be shown directly from the bivariance of the specialization maps.

Proposition 2.3.7 (Bivariance)**.** *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of algebraic stacks. Assume that the two maps f and f' are Deligne-Mumford morphisms, and the two algebraic stacks X and X' have affine stabilizers. Let $\phi : \mathbb{F} \to L_{X/Y}$ be a perfect obstruction theory. Then the composition

$$\phi' : (g')^*\mathbb{E} \xrightarrow{(g')^*(\phi)} (g')^*L_{X/Y} \to L_{X'/Y'}$$

is a perfect obstruction theory satisfying the following properties:

1. *If g is a proper Deligne-Mumford morphism, then we have*

$$f^! \circ g_* = g'_* \circ (f')^! : A_*(Y') \to A_*(X).$$

2. *If g is an equi-dimensional flat morphism, then we have*

$$(f')^! \circ g^* = (g')^* \circ f^! : A_*(Y) \to A_*(X').$$

3. *If g is a local complete intersection morphism and Y' has affine stabilizers, then we have*

$$(f')^! \circ g^! = (g')^! \circ f^! : A_*(Y) \to A_*(X').$$

Proof. It follows immediately from Proposition 2.1.19. □

Remark 2.3.8. In [Man, Thm. 4.1] (see also [Man, Rem. 4.2]), only the projective morphisms are considered instead of the proper DM morphisms. Based on the development of proper DM pushforwards in [BS, Appendix B], we can generalize the result in [Man] to proper DM morphisms as in Proposition 2.3.7.

Remark 2.3.9 (Generalization)**.** In Definition 2.3.3, we need two technical assumptions for defining virtual pullbacks:

1. $f : X \to Y$ is a DM morphism;

2. X has affine stabilizers.

These assumptions are required due to the foundational issues in Chow groups for Artin stacks. The assumption 2 can be removed whenever we have homotopy property for vector bundle stacks. The assumption 1 can be removed when we can extend the Chow groups to higher Artin stacks. In particular, if we use Khan's motivic Borel-Moore homology theory [Khan, KR] and Aranha-Pstragowski's intrinsic normal cone for Artin morphisms [AP], we can remove the above two technical assumptions for defining virtual pullbacks.

2.3.3 Functoriality

We now prove the *functoriality* of virtual pullbacks, following the arguments in [KKP, Man].

Notation 2.3.10 (Distinguished triangle of truncated cotangent complexes)**.** Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z. \end{tikzcd}$$

be a commutative diagram of DM morphisms of algebraic stacks. The canonical distinguished triangle of cotangent complexes

$$f^*(\mathbb{L}_{Y/Z}) \longrightarrow \mathbb{L}_{X/Z} \longrightarrow \mathbb{L}_{X/Y} \longrightarrow$$

induces a distinguished triangle of truncated cotangent complexes

$$\tau^{\geqslant -1} f^*(L_{Y/Z}) \xrightarrow{a} L_{X/Z} \longrightarrow L'_{X/Y} \longrightarrow$$

where $L'_{X/Y} := \mathrm{cone}(a)$. Let

$$r : L'_{X/Y} \to \tau^{\geqslant -1}(L'_{X/Y}) \cong L_{X/Y}$$

denote the canonical map.

Definition 2.3.11 (Compatible triple of obstruction theories)**.** Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

be a commutative diagram of *DM* morphism of algebraic stacks. We say that the triple $(\phi_{X/Y}, \phi_{Y/Z}, \phi_{X/Z})$ of obstruction theories $\phi_{X/Y} : \mathbb{F}_{X/Y} \to L_{X/Y}$, $\phi_{Y/Z} : \mathbb{F}_{Y/Z} \to L_{Y/Z}$, and $\phi_{X/Z} : \mathbb{F}_{X/Z} \to L_{X/Z}$ is *compatible* if there exists a morphism of distinguished triangles

$$\begin{tikzcd} f^*(\mathbb{F}_{Y/Z}) \arrow[r] \arrow[d, "f^*(\phi_{Y/Z})"] & \mathbb{F}_{X/Z} \arrow[r] \arrow[d, "\phi_{X/Z}"] & \mathbb{F}_{X/Y} \arrow[r] \arrow[d, dashed, "\phi'_{X/Y}"] & {} \\ \tau^{\geqslant -1} f^*(L_{Y/Z}) \arrow[r] & L_{X/Z} \arrow[r] & L'_{X/Y} \arrow[r] & {} \end{tikzcd}$$

for some $\phi'_{X/Y}$ such that $\phi_{X/Y} = r \circ \phi'_{X/Y}$.

Theorem 2.3.12 (Functoriality)**.** *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

be a commutative diagram of DM morphisms of algebraic stacks. Assume that X and Y have affine stabilizers. Given a compatible triple of perfect obstruction theories, we have

$$(g \circ f)^! = f^! \circ g^! : A_*(Z) \to A_*(X).$$

We have a *virtual pullback formula* as an immediate corollary of the functoriality of virtual pullbacks in Theorem 2.3.12.

Corollary 2.3.13 (Virtual pullback formula)**.** *Let $f : X \to Y$ be a morphism of DM stacks. Given a compatible triple of perfect obstruction theories for*

$$X \xrightarrow{f} Y \longrightarrow \mathrm{Spec}(\mathbb{C}),$$

we have a virtual pullback formula

$$[X]^{\mathrm{vir}} = f^![Y]^{\mathrm{vir}} \in A_*(X).$$

The notion of a compatible triple of perfect obstrucion theories in Definition 2.3.11 is slightly general than the standard one in [Man, Def. 4.5] since we are considering the truncated version of perfect obstruction theories.

Remark 2.3.14. Let us recall the notion of compatible triple in [Man, Def. 4.5]. Consider a commutative diagram of algebraic stacks

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

where f and g are DM morphisms. Let $\phi_{X/Y} : \mathbb{F}_{X/Y} \to \mathbb{L}_{X/Y}$, $\phi_{Y/Z} : \mathbb{F}_{Y/Z} \to \mathbb{L}_{Y/Z}$, and $\phi_{X/Z} : \mathbb{F}_{X/Z} \to \mathbb{L}_{X/Z}$ be maps that induce perfect obstruction theories. The triple $(\phi_{X/Y}, \phi_{Y/Z}, \phi_{X/Z})$ is said to be *compatible* if there is a morphism of distinguished triangles

$$\begin{tikzcd} f^*(\mathbb{F}_{Y/Z}) \arrow[r] \arrow[d, "f^*(\phi_{Y/Z})"] & \mathbb{F}_{X/Z} \arrow[r] \arrow[d, "\phi_{X/Z}"] & \mathbb{F}_{X/Y} \arrow[r] \arrow[d, "\phi_{X/Y}"] & {} \\ f^*(\mathbb{L}_{Y/Z}) \arrow[r] & \mathbb{L}_{X/Z} \arrow[r] & \mathbb{L}_{X/Y} \arrow[r] & {} \end{tikzcd}$$

where the lower triangle is the canonical one. If we apply the truncation functor, we obtain the compatibility diagram in Definition 2.3.11 .

Note that a natural source of compatible triples of perfect obstruction theories is quasi-smooth morphisms of derived schemes.

Example 2.3.15 (Derived enhancement)**.** Consider a commutative diagram of derived Artin stacks

$$\begin{tikzcd} \mathbb{X} \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & \mathbb{Y} \arrow[r, "g"] & \mathbb{Z} \end{tikzcd}$$

where f and g are quasi-smooth DM morphisms. Then we have a canonical homotopy cofiber sequence of cotangent complexes

$$f^*(\mathbb{L}_{\mathbb{Y}/\mathbb{Z}}) \longrightarrow \mathbb{L}_{\mathbb{X}/\mathbb{Z}} \longrightarrow \mathbb{L}_{\mathbb{X}/\mathbb{Y}} .$$

By considering the classical truncations, we obtain a compatible triple of perfect obstruction theories in the sense of [Man, Def. 4.5], see Remark 2.3.14. Hence by applying the truncation functor, we obtain a compatible triple in the sense of Definition 2.3.11.

An elementary example of compatible triple is a modification of a relative perfect obstruction theory for a smooth base to an absolute perfect obstruction theory.

Example 2.3.16 (Relative to absolute)**.** Let $f : X \to Y$ be a morphism of Deligne-Mumford stacks. Let $\phi_{X/Y} : \mathbb{F}_{X/Y} \to L_{X/Y}$ be a perfect obstruction theory. Assume that Y is smooth. Then we can form a morphism of distinguished triangles

$$\begin{tikzcd} f^*(\Omega_Y) \arrow[r] \arrow[d, equal] & \mathbb{F}_X \arrow[r] \arrow[d, dotted, "\phi_X"] & \mathbb{F}_{X/Y} \arrow[r] \arrow[d, "\phi_{X/Y}"] & f^*(\Omega_Y)[1] \arrow[d, equal] \\ f^*(\Omega_Y) \arrow[r] & L_X \arrow[r] & L_{X/Y} \arrow[r] & f^*(\Omega_Y)[1] \end{tikzcd}$$

where $\mathbb{F}_X := \mathrm{cone}(\mathbb{F}_{X/Y} \xrightarrow{\phi_{X/Y}} L_{X/Y} \to \Omega_Y[1])[-1]$. Then $\phi_X : \mathbb{F}_X \to L_X$ is a perfect obstruction theory and we have

$$[X]^{\mathrm{vir}} = f^![Y] \in A_*(X)$$

by Theorem 2.3.12.

We note that this approach can be generalized to the case when Y is a smooth *Artin* stack.

We often want to lift a perfect obstruction theory by a smooth morphism. However there is an obstruction for such lifting in general.

Remark 2.3.17 (Lift along smooth morphism)**.** Let $f : X \to Y$ be a smooth morphism of DM stacks. Let $\phi_Y : \mathbb{F}_Y \to L_Y$ be a perfect obstruction theory. We want to find a perfect obstruction theory $\phi_X : \mathbb{F}_X \to L_X$ such that $[X]^{\mathrm{vir}} = f^*[Y]^{\mathrm{vir}}$. This can be achieved if there exists a commutative diagram

$$\begin{tikzcd} \Omega_{X/Y} \arrow[r, dotted] \arrow[d, equal] & f^*(\mathbb{F}_Y)[1] \arrow[d, "f^*(\phi_Y)"] \\ \Omega_{X/Y} \arrow[r] & \tau^{\geqslant -1} f^*(L_Y)[1] \end{tikzcd}$$

for some dotted arrow. This is possible for the following two cases:

1. If X is an affine scheme.
2. If $f : X \to Y$ can be enhanced to a smooth morphism $\mathbb{X} \to \mathbb{Y}$ of quasi-smooth DM stacks.

A possible alternative approach for this situation is to use the Siebert formula [Sie].

We now prove Theorem 2.3.12 through 3 steps.

Step 1: Special case via homotopy property We first consider the special case

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "0_{\mathfrak{C}} \circ f"'] & Y \arrow[r, "0_{\mathfrak{C}}"] & \mathfrak{C} \end{tikzcd}$$

where $\mathfrak{C}$ is a cone stack over Y. The functoriality for this case can be shown easily from the homotopy property of vector bundle stacks.

Lemma 2.3.18 (Cone stack case)**.** *Consider a commutative diagram of algebraic stacks*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "0_{\mathfrak{C}} \circ f"'] & Y \arrow[r, "0_{\mathfrak{C}}"] & \mathfrak{C} \end{tikzcd}$$

where f is a DM morphism, $\mathfrak{C}$ is a cone stack over Y, and $0_{\mathfrak{C}} : Y \to \mathfrak{C}$ is the zero section. Let $\phi_{X/Y} : \mathbb{F}_{X/Y} \to L_{X/Y}$ and $\phi : \mathbb{F} \to L_{Y/\mathfrak{C}}$ be perfect obstruction theories. Form a compatible triple of perfect obstruction theories as

$$\begin{tikzcd}
f^*(\mathbb{F}) \arrow[r] \arrow[d, "f^*(\phi)"] & f^*(\mathbb{F}) \oplus \mathbb{F}_{X/Y} \arrow[r] \arrow[d, "f^*(\phi)\oplus\phi_{X/Y}"] & \mathbb{F}_{X/Y} \arrow[d, "\phi_{X/Y}"] \\
\tau^{\geqslant -1} f^*(L_{Y/\mathfrak{C}}) \arrow[r] & L_{X/\mathfrak{C}} \arrow[r] & L_{X/Y}
\end{tikzcd}$$

where $\mathbb{L}_{X/\mathfrak{C}} = f^*(\mathbb{L}_{Y/\mathfrak{C}}) \oplus \mathbb{L}_{X/Y}$. Assume that X and Y have affine stabilizers. Then we have

$$(0_{\mathfrak{C}} \circ f)^! = f^! \circ 0_{\mathfrak{C}}^! : A_*(\mathfrak{C}) \to A_*(X).$$

Proof. We first reduce the situation to the vector bundle stack case,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend right, "0\circ f"'] & Y \arrow[r, "0"] & \mathfrak{C}
\end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend right, "0\circ f"'] & Y \arrow[r, "0"] & \mathfrak{E}(\mathbb{F})
\end{tikzcd}.$$

Indeed, we can form a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rr, bend left, "0_{\mathfrak{C}}\circ f"] \arrow[d, equal] & Y \arrow[r, "0_{\mathfrak{C}}"'] \arrow[d, equal] & \mathfrak{C} \arrow[d, hook] \\
X \arrow[r, "f"] \arrow[rr, bend right, "0_{\mathfrak{E}(\mathbb{F})}\circ f"'] & Y \arrow[r, "0_{\mathfrak{E}(\mathbb{F})}"] & \mathfrak{E}(\mathbb{F}).
\end{tikzcd}$$

Since virtual pullbacks commute with proper pushforwards by Proposition 2.3.7, replacing $\mathfrak{C}$ by $\mathfrak{E}(\mathbb{F})$, we may assume that $\mathfrak{C}$ is a vector bundle and $\phi : \mathbb{F} \to L_{Y/\mathfrak{C}}$ is an isomorphism. Let $\mathfrak{E} := \mathfrak{C}$.

By the homotopy property of the vector bundle stack $\mathfrak{E}$, if suffices to show the functoriality for

$$\begin{tikzcd}
X \arrow[r, "0_{\mathfrak{E}}\circ f"] \arrow[rr, bend right, "f"'] & \mathfrak{E} \arrow[r, "\pi_{\mathfrak{E}}"] & Y
\end{tikzcd}$$

where $\pi_{\mathfrak{E}} : \mathfrak{E} \to Y$ denotes the projection map. Indeed, we have

$$f^! = (0_{\mathfrak{E}} \circ f)^! \circ \pi_{\mathfrak{E}}^* \implies f^! \circ 0_{\mathfrak{E}}^! = (0_{\mathfrak{E}} \circ f)^!$$

since $0^!_{\mathfrak{E}} = (\pi^*_{\mathfrak{E}})^{-1}$.

Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "0_{\mathfrak{E}}\circ f"] \arrow[d, equal] & \mathfrak{E} \arrow[d, "\pi_{\mathfrak{E}}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where the vertical arrows are smooth. Hence by Lemma 2.1.21, we have

$$(\mathrm{id}_{\mathfrak{E}} \times \pi_{\mathfrak{E}}|_X)^* \circ \mathrm{sp}_{X/Y} = \mathrm{sp}_{X/\mathfrak{E}} \circ \pi^*_{\mathfrak{E}} : A_*(Y) \to A_*(\mathfrak{C}_{X/\mathfrak{E}}) \tag{2.3.4}$$

where $\mathfrak{C}_{X/\mathfrak{E}} = \mathfrak{C}_{X/Y} \times \mathfrak{E}|_X$. Applying the Gysin pullback $0^!_{\mathfrak{E}(\mathbb{F}_{X/Y})\times\mathfrak{E}|_X}$ to (2.3.4), we obtain the desired identity. □

Step 2: Deformation to the normal cone We then consider the general case. The main idea is to reduce the situation to the special case in the previous subsection via deformation to the normal cone,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "0\circ f"'] & Y \arrow[r, "0"] & \mathfrak{C}_{Y/Z} \end{tikzcd}. \tag{2.3.5}$$

Indeed, consider the composition

$$h : X \times \mathbb{A}^1 \xrightarrow{f\times\mathrm{id}} Y \times \mathbb{A}^1 \to M^\circ_{Y/Z}$$

where the second arrow is the canonical map. Then the generic fiber of the map h over $\zeta \neq 0 \in \mathbb{A}^1$ is the formal diagram in (2.3.5) and the special fiber of the map h over $0 \in \mathbb{A}^1$ is the latter diagram in (2.3.5). In other words, we have a fiber diagram

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d] \arrow[rr, bend left, "g\circ f"] & Y \arrow[r, "g"'] \arrow[d] & Z \arrow[r] \arrow[d] & \{\zeta\} \arrow[d, "\iota_\zeta"'] \\ X \times \mathbb{A}^1 \arrow[r, "f\times\mathrm{id}"] & Y \times \mathbb{A}^1 \arrow[r] & M^\circ_{Y/Z} \arrow[r] & \mathbb{A}^1 \\ X \arrow[r, "f"] \arrow[u] \arrow[rr, bend right, "0\circ f"'] & Y \arrow[r, "0"] \arrow[u] & \mathfrak{C}_{Y/Z} \arrow[r] \arrow[u] & \{0\}. \arrow[u, "\iota_0"'] \end{tikzcd}$$

We will construct a perfect obstruction theory for $h : X \times \mathbb{A}^1 \to M^\circ_{Y/Z}$.

Lemma 2.3.19 (Deformation to normal cone)**.** *Consider a commutative diagram of algebraic stacks*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

where f and g are DM morphisms. Consider a compatible triple $(\phi_{X/Y}, \phi_{Y/Z}, \phi_{X/Z})$ of perfect obstruction theories. Then the composition

$$h : X \times \mathbb{A}^1 \to Y \times \mathbb{A}^1 \to M^\circ_{Y/Z}$$

has a perfect obstruction theory

$$\phi : \mathbb{F} \to L_{X \times \mathbb{A}^1 / M^\circ_{Y/Z}}$$

satisfying the following properties:

1. *The fiber of ϕ at $\zeta \neq 0 \in \mathbb{A}^1$ is*

$$\phi_\zeta = \phi_{X/Z} : \mathbb{F}_{X/Z} \to L_{X/Z}$$

2. *The fiber of ϕ at $0 \in \mathbb{A}^1$ is*

$$\phi_0 = \begin{pmatrix} \phi_{Y/Z} & \xi \\ 0 & \eta \end{pmatrix} : f^*(\mathbb{F}_{Y/Z}) \oplus \mathbb{F}_{X/Y} \to \tau^{\geqslant -1} f^*(L_{Y/Z}) \oplus L_{X/Y}$$

such that the diagram

$$\begin{tikzcd} f^*(\mathbb{F}_{Y/Z}) \arrow[r] \arrow[d, "f^*(\phi_{Y/Z})"] & \mathbb{F}_{X/Z} \arrow[r] \arrow[d, "\phi_{X/Z}"] & \mathbb{F}_{X/Y} \arrow[r] \arrow[d, "\phi'_{X/Y}"'] \arrow[d, dotted, shift left=2, "\eta'"] & f^*(\mathbb{F}_{Y/Z})[1] \arrow[d] \\ \tau^{\geqslant -1} f^*(L_{Y/Z}) \arrow[r, "a"] & L_{X/Z} \arrow[r] & L'_{X/Y} \arrow[r] & \tau^{\geqslant -1} f^*(L_{Y/Z})[1] \end{tikzcd}$$

commutes for some η' with $\eta = r \circ \eta'$.

Before we proof Lemma 2.3.19, we recall the following key result of Kim-Kresch-Pantev in [KKP].

Lemma 2.3.20 ([KKP]). *Consider a commutative diagram of algebraic stacks*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

where f and g are DM morphisms. Form a distinguished triangle on $X \times \mathbb{A}^1$

$$\tau^{\geqslant -1} f^*(L_{Y/Z}) \boxtimes \mathcal{O}_{\mathbb{A}^1} \xrightarrow{(T,a)} (\tau^{\geqslant -1} f^*(L_{Y/Z}) \oplus L_{X/Z}) \boxtimes \mathcal{O}_{\mathbb{A}^1} \longrightarrow L'_{X\times\mathbb{A}^1/M^\circ_{Y/Z}}$$

for some $L'_{X\times\mathbb{A}^1/M^\circ_{Y/Z}}$. Then we have a canonical isomorphism

$$\tau^{\geqslant -1}(L'_{X\times\mathbb{A}^1/M^\circ_{Y/Z}}) \cong L_{X\times\mathbb{A}^1/M^\circ_{Y/Z}}.$$

Proof of Lemma 2.3.19. Form a morphism of distinguished triangles

$$\begin{tikzcd} f^*(\mathbb{F}_{Y/Z}) \boxtimes \mathcal{O}_{\mathbb{A}^1} \arrow[r, "(T,a)"] \arrow[d, "\phi_{Y/Z}"] & (f^*(\mathbb{F}_{Y/Z}) \oplus \mathbb{F}_{X/Z}) \boxtimes \mathcal{O}_{\mathbb{A}^1} \arrow[r] \arrow[d, "(\phi_{Y/Z},\phi_{X/Z})"] & \mathbb{F} \arrow[d, dotted, "\phi'"] \\ \tau^{\geqslant -1} f^*(L_{Y/Z}) \boxtimes \mathcal{O}_{\mathbb{A}^1} \arrow[r, "(T,a)"] & (\tau^{\geqslant -1} f^*(L_{Y/Z}) \oplus L_{X/Z}) \boxtimes \mathcal{O}_{\mathbb{A}^1} \arrow[r] & L'_{X\times\mathbb{A}^1/M^\circ_{Y/Z}} \end{tikzcd} \tag{2.3.6}$$

for some perfect complex $\mathbb{F}$ and a map $\phi' : \mathbb{F} \to L'_{X\times\mathbb{A}^1/M^\circ_{Y/Z}}$, where the lower distinguished triangle is given by Lemma 2.3.20. Let

$$\phi : \mathbb{F} \xrightarrow{\phi'} L'_{X\times\mathbb{A}^1/M^\circ_{Y/Z}} \to L_{X\times\mathbb{A}^1/M^\circ_{Y/Z}}$$

be the composition. The long exact sequence associated to (2.3.6) assures that ϕ is a perfect obstruction theory. Then fibers of ϕ over $\zeta \in \mathbb{A}^1$ have the desired properties. ◻

Step 3 By combining the deformation result in Lemma 2.3.19 with the special case in Lemma 2.3.18, we can now show the functoriality in Theorem 2.3.12.

Proof of Theorem 2.3.12. By Lemma 2.3.19, we have

$$(g\circ f)^! = (0_{\mathfrak{C}_{Y/Z}} \circ f)^!_{\phi_0} \circ \mathrm{sp}_g \tag{2.3.7}$$

where

$$\phi_0 = \begin{pmatrix} \phi_{Y/Z} & \xi \\ 0 & \eta \end{pmatrix} : f^*(\mathbb{F}_{Y/Z}) \oplus \mathbb{F}_{X/Y} \to \tau^{\geqslant -1} f^*(L_{Y/Z}) \oplus L_{X/Y} = L_{X/\mathfrak{C}_{Y/Z}}.$$

Note that

$$\begin{pmatrix} \phi_{Y/Z} & t_1 \cdot \xi \\ 0 & (1-t_2)\cdot \eta + t_2 \cdot \phi_{X/Y} \end{pmatrix} : f^*(\mathbb{F}_{Y/Z}) \oplus \mathbb{F}_{X/Y} \to \tau^{\geqslant -1} f^*(L_{Y/Z}) \oplus L_{X/Y}$$

are perfect obstruction theories for all $t_1 \in \mathbb{A}^1$ and $t_2 \in \mathbb{A}^1$. By a deformation argument, we have

$$(0_{\mathfrak{C}_{Y/Z}} \circ f)^!_{\phi_0} = (0_{\mathfrak{C}_{Y/Z}} \circ f)^! \tag{2.3.8}$$

where the second virtual pullback is given by

$$f^*(\phi_{Y/Z}) \oplus \phi_{X/Y} : f^*(\mathbb{F}_{Y/Z}) \oplus \mathbb{F}_{X/Y} \to \tau^{\geqslant -1} f^*(L_{Y/Z}) \oplus L_{X/Y}.$$

By Lemma 2.3.18, we have

$$(0_{\mathfrak{C}_{Y/Z}} \circ f)^! = f^! \circ 0^!_{\mathfrak{C}_{Y/Z}}. \tag{2.3.9}$$

By combining the three equations (2.3.7), (2.3.8), and (2.3.9), we obtain the desired formula. □

We explain one technical difference in the proof of functoriality given here and the standard references [KKP, Man].

Remark 2.3.21. There is one technical issue in the proof of functoriality that was ignored in the standard references [KKP, Man]. In the construction of the perfect obstruction theory $\phi : \mathbb{E} \to L_{X\times\mathbb{A}^1/M^\circ_{Y/Z}}$, it is not clear that the special fiber ϕ_0 over $0 \in \mathbb{A}^1$ is $f^*(\phi_{Y/Z}) \oplus \phi_{X/Y}$.[3] Hence here we provided additional deformation argument (that was not given in [KKP, Man]) to compared the two perfect obstruction theories ϕ_0 and $f^*(\phi_{Y/Z}) \oplus \phi_{X/Y}$. This issue was considered in [Park1], in the context of DT4 theory.

[3] The author expects that this issue was not regarded seriously in the classical references [KKP, Man] since the Siebert formula [Sie] assures that the virtual cycle only depends on the K-theory class $[\mathbb{F}] \in K^0(X)$, but not on the map $\phi : \mathbb{F} \to L_X$ (for quasi-projective schemes). However the author does not know whether the Siebert type formula exists for arbitrary DM morphism of algebraic stacks, or in other homology theories.

Chapter 3

Cosection localization

This chapter reviews the *cosection localization* technique of Kiem-Li [KL1].

Summary Recall from Chapter 2 that the virtual cycle of a Deligne-Mumford stack X with a perfect obstruction theory $\mathfrak{C}_X \hookrightarrow \mathfrak{E}$ is defined as

$$[X]^{\mathrm{vir}} = 0^!_{\mathfrak{E}}[\mathfrak{C}_X] \in A_{\mathrm{vd}}(X).$$

Kiem-Li showed the followings for a *cosection* $\sigma : \mathfrak{E} \to \mathbb{A}^1_X$.

1. There is a *cone reduction*, i.e., a smaller closed embedding

$$(\mathfrak{C}_X)_{\mathrm{red}} \hookrightarrow \mathfrak{K}(\mathfrak{E}, \sigma)$$

into the *kernel cone stack* $\mathfrak{K}(\mathfrak{E}, \sigma) := \mathfrak{E} \times_{\sigma, \mathbb{A}^1_X, 0} X \subseteq \mathfrak{E}$.

2. There exists a *localized Gysin pullback*

$$0^!_{\mathfrak{E},\sigma} : A_*(\mathfrak{K}(\mathfrak{E}, \sigma)) \to A_*(X(\sigma))$$

to the zero locus $X(\sigma)$ of the cosection σ in X.

The two main outcomes are the followings:

1. We have a *localized virtual cycle*

$$[X]^{\mathrm{vir}}_{\mathrm{loc}} := 0^!_{\mathfrak{E},\sigma}[\mathfrak{C}_X] \in A_{\mathrm{vd}}(X(\sigma))$$

that localizes the ordinary virtual cycle $[X]^{\mathrm{vir}} \in A_{\mathrm{vd}}(X)$.

2. If the cosection σ is nowhere vanishing, then the ordinary virtual cycle vanishes
$$[X]^{\mathrm{vir}} = 0 \in A_{\mathrm{vd}}(X).$$
Moreover the kernel cone stack $\mathfrak{K}(\mathfrak{E},\sigma)$ is a vector bundle stack and we have a canonical *reduced virtual cycle*
$$[X]^{\mathrm{vir}}_{\mathrm{loc}} := 0^!_{\mathfrak{K}(\mathfrak{E},\sigma)}[\mathfrak{C}_X] \in A_{\mathrm{vd}+1}(X).$$

Usually the zero locus $X(\sigma)$ is much more smaller than the original space X and the cosection-localized virtual cycles are very useful for computation in this case (e.g. Gromov-Witten/Poincare invariants for surfaces with holomorphic 2-forms [KL1, CK]).

The reducing is required in many cases (e.g. GW/PT invariants for $K3$ surfaces [MPT, KT1, KT2]). The resulting reduced invariants in this cases have turned out have rich structures.

Moreover, the cosection localization has a deep connection to the algebraic foundation of Donaldson-Thomas theory of Calabi-Yau 4-folds [OT].

3.1 Cone reduction

In this section, we provide a basic framework for the theory of cosection localization.

3.1.1 Kernel cone stacks

In the theory of cosection localization, the *kernel cone stacks* associated to *cosections* play the role of vector bundle stacks in Chapter 2.

We first fix the notion of *cosections*.

Definition 3.1.1 (Cosection)**.** Let $\mathbb{F}$ be a perfect complex on an algebraic stack X. A *cosection* of $\mathbb{F}$ is a map
$$\sigma : \mathbb{F}^\vee[1] \to \mathcal{O}_X$$
in the derived category of X.

We observe that a cosection induces the *canonical linear function* on the associated abelian cone stack.

Definition 3.1.2 (Canonical linear function)**.** Let $\mathbb{F}$ be a perfect complex on an algebraic stack X and let $\mathfrak{C}(\mathbb{F}) := h^1/h^0(\mathbb{F}^\vee)$ denote the associated abelian cone stack. Let $\sigma : \mathbb{F}^\vee[1] \to \mathcal{O}_X$ be a cosection of $\mathbb{F}$. We define the *canonical linear function* on as

$$\mathfrak{l}_\sigma := \mathfrak{C}(\sigma^\vee[1]) : \mathfrak{C}(\mathbb{F}) \to \mathbb{A}^1_X.$$

We now define the *kernel cone stacks*.

Definition 3.1.3 (Kernel cone stack)**.** Let $\mathbb{F}$ be a perfect complex on an algebraic stack X and $\sigma : \mathbb{F}^\vee[1] \to \mathcal{O}_X$ be a cosection of $\mathbb{F}$. We define the *kernel cone stack* as the abelian cone stack

$$\mathfrak{K}(\mathbb{F},\sigma) := \mathfrak{C}(\mathbb{F}_\sigma) = h^1/h^0(\mathbb{F}_\sigma^\vee)$$

where $\mathbb{F}_\sigma := \mathrm{cone}(\sigma^\vee[1] : \mathcal{O}_X[1] \to \mathbb{F})$.

Remark 3.1.4 (Base change)**.** The construction of the kernel cone stack $\mathfrak{K}(\mathbb{F},\sigma)$ is stable under the base change of X.

The following lemma justifies the terminology *kernel cone stacks*.

Lemma 3.1.5. *Let $\mathbb{F}$ be a perfect complex on an algebraic stack X and σ : $\mathbb{F}^\vee[1] \to \mathcal{O}_X$ be a cosection of $\mathbb{F}$. Then we have a canonical cartesian square*

$$\begin{tikzcd}
\mathfrak{K}(\mathbb{F},\sigma) \arrow[r] \arrow[d, hook] & X \arrow[d, hook, "0"] \\
\mathfrak{C}(\mathbb{F}) \arrow[r, "\mathfrak{l}_\sigma"] & \mathbb{A}^1_X
\end{tikzcd}$$

where $\mathfrak{l}_\sigma : \mathfrak{C}(\mathbb{F}) \to \mathbb{A}^1_X$ is the canonical linear function.

We compare the notions of cosections and kernel cone stacks given here with those in the original paper of Kiem-Li [KL1].

Remark 3.1.6 (Comparison to Kiem-Li)**.** Let $\mathbb{F}$ be a perfect complex of tor-amplitude $[-1,0]$ on a Deligne-Mumford stack X.

1. A cosection $\sigma : \mathbb{F}^\vee[1] \to \mathcal{O}_X$ is equivalent to a map of coherent sheaves

$$h^1(\mathbb{F}^\vee) \to \mathcal{O}_X,$$

which is the definition of a cosection in [KL1].

2. Let $\sigma : \mathbb{F}^\vee[1] \to O_X$ be a cosection. Let $X(\sigma)$ denote the zero locus of the induced map $\overline{\sigma} : h^1(\mathbb{F}^\vee) \to O_X$ and let $U := X \backslash X(\sigma)$ be the complement. Then we have a fiber diagram

$$\begin{tikzcd}
\mathfrak{E}(\mathbb{F}|_{X(\sigma)}) \arrow[r, hook] \arrow[d] & \mathfrak{K}(\mathbb{F}, \sigma) \arrow[d] & \mathfrak{E}(\mathbb{F}_\sigma|_U) \arrow[l, hook'] \arrow[d] \\
X(\sigma) \arrow[r, hook] & X & U \arrow[l, hook']
\end{tikzcd}$$

where both $\mathfrak{E}(\mathbb{F}|_{X(\sigma)}) := \mathfrak{C}(\mathbb{F}|_{X(\sigma)})$ and $\mathfrak{E}(\mathbb{F}_\sigma|_U) := \mathfrak{C}(\mathbb{F}_\sigma|_U)$ are vector bundle stacks (of different ranks). Thus *set-theoretically*, we have

$$\mathfrak{K}(\mathbb{F}, \sigma) = \mathfrak{E}(\mathbb{F}|_{X(\sigma)}) \cup \mathfrak{E}(\mathbb{F}_\sigma|_U)$$

which is the definition of the kernel cone stack in [KL1].[1]

3.1.2 Cone reduction

The *cone reduction property* is the crucial ingredient in the theory of cosection localization.

Definition 3.1.7 (Cone reduction property)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks. We say that an obstruction theory $\phi : \mathbb{F} \to L_{X/Y}$ satisfies the *cone reduction property* with respect to a cosection $\sigma : \mathbb{F}^\vee[1] \to O_X$ if the composition

$$(\mathfrak{C}_{X/Y})_{\mathrm{red}} \hookrightarrow \mathfrak{C}_{X/Y} \xrightarrow{\iota} \mathfrak{C}(\mathbb{F}) \xrightarrow{\mathfrak{l}_\sigma} \mathbb{A}^1_X$$

is zero. Here $(\mathfrak{C}_{X/Y})_{\mathrm{red}} \subseteq \mathfrak{C}_{X/Y}$ is the reduced closed substack of the intrinsic normal cone, $\iota : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{C}(\mathbb{F})$ is the closed embedding induced by ϕ, and $\mathfrak{l}_\sigma : \mathfrak{C}(\mathbb{F}) \to \mathbb{A}^1_X$ is the canonical linear function in Definition 3.1.2.

We observe that an obstruction theory satisfying the cone reduction property is equivalent to a closed embedding of the intrinsic normal cone into a kernel cone stack.

Proposition 3.1.8 (Equivalence)**.** *Let $f : X \to Y$ be a DM morphism of algebraic stacks.*

[1]The author learned the *scheme-theoretical* description of the kernel cone stacks from Jeongseok Oh.

1. *If $\phi : \mathbb{F} \to L_{X/Y}$ is an obstruction theory satisfying the cone reduction property, then there exists a unique closed embedding*

$$\iota^{\sigma,\mathrm{red}} : (\mathfrak{C}_{X/Y})_{\mathrm{red}} \hookrightarrow \mathfrak{K}(\mathbb{F},\sigma)$$

that fits into the commutative diagram

$$\begin{tikzcd}
 & & \mathfrak{K}(\mathbb{F},\sigma) \arrow[r] \arrow[d, hook] & X \arrow[d, hook, "0"] \\
(\mathfrak{C}_{X/Y})_{\mathrm{red}} \arrow[r, hook] \arrow[urr, dotted, "\iota^{\sigma,\mathrm{red}}"] & \mathfrak{C}_{X/Y} \arrow[r, hook, "\iota"] & \mathfrak{C}(\mathbb{F}) \arrow[r, "\mathfrak{l}_\sigma"] & \mathbb{A}^1_X
\end{tikzcd}$$

as the dotted arrow.

2. *If $\mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{K}(\mathbb{F},\sigma)$ is a closed embedding of cone stacks for some kernel cone stack $\mathfrak{K}(\mathbb{F},\sigma)$ associated to a perfect complex $\mathbb{F}$ and a cosection $\sigma : \mathbb{F}^\vee[1] \to \mathcal{O}_X$, then the composition*

$$\mathbb{F} \to \tau^{\geqslant -1}\mathbb{F} \cong L_{X/\mathfrak{C}(\mathbb{F})} \to L_{X/\mathfrak{K}(\mathbb{F},\sigma)} \to L_{X/\mathfrak{C}_{X/Y}} \cong L_{X/Y}$$

is an obstruction theory satisfying the cone reduction property.

Moreover, the above two operations are inverse to each others.

Proof. We omit the proof, see Proposition 2.2.9. □

There are two sources of the cone reduction property:

1. cone reduction lemma of Kiem-Li [KL1];
2. Reductions via (-1)-shifted 1-forms (see Chapter 8).

Firstly, we recall Kiem-Li's cone reduction lemma.

Proposition 3.1.9 (Kiem-Li's cone reduction lemma)**.** *Let X be a Deligne-Mumford stack, $\phi : \mathbb{F} \to L_X$ be an obstruction theory, and $\sigma : \mathbb{F}^\vee[1] \to \mathcal{O}_X$ be a cosection. Then ϕ satisfies the cone reduction property.*

Proof. For perfect obstruction theories, this is shown in [KL1, Prop. 4.3]. The general case of arbitrary tor-amplitude can be reduced to the perfect obstruction theory case (cf. [BKP, Lem. 4.18]). □

In Kiem-Li's cone reduction lemma, it is crucial to take the reduced closed substack $(\mathfrak{C}_X)_{\mathrm{red}}$ of the intrinsic normal cone.

Remark 3.1.10 (Maulik-Pandharipande-Thomas's counterexample)**.** Consider a commutative diagram

$$\begin{tikzcd} & E \arrow[r, "t"] \arrow[d, "\pi"'] & \mathbb{A}^1_U \\ X \arrow[r, hook, "i"] & U \arrow[u, bend right, "s"'] & \end{tikzcd}$$

where $U = \mathbb{A}^1$, $E = \mathcal{O}_U^{\oplus 2}$, $s = (T^2, 0)$, $t = (T, 1)$, and X is the zero locus of s in U. Then we have an induced perfect obstruction theory and a cosection

$$\begin{tikzcd} \mathcal{O}_X[1] \arrow[r, "\sigma^\vee[1]"] & \mathbb{E} \arrow[d, "\phi"] \\ & L_X \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_X \arrow[r, "t|_X"] & E|_X^\vee \arrow[r, "ds"] \arrow[d, "s"] & \Omega_U|_X \arrow[d, equal] \\ & \mathcal{I}_{X/U}/\mathcal{I}_{X/U}^2 \arrow[r, "d"] & \Omega_U|_X. \end{tikzcd}$$

Since the composition

$$ds \circ t|_X = (2T, 0) \circ (T, 1) = 0 : \mathcal{O}_X \to \Omega_U|_X \cong \mathcal{O}_X$$

is zero, σ given above is indeed a cosection. On the other hand, the composition

$$s \circ t|_X = (T^2, 0) \circ (T, 1) = (T^3) \neq 0 : \mathcal{O}_X \to \mathcal{I}_{X/U}/\mathcal{I}_{X/U}^2 = (T^2)/(T^4)$$

is *not* zero, we have

$$N_{X/U} \not\subseteq \ker(\sigma|_X : E|_X \to \mathcal{O}_X).$$

Consequently, we also have

$$C_{X/U} \not\subseteq \ker(\sigma|_X : E|_X \to \mathcal{O}_X)$$

and the cone reduction property does not hold *scheme-theoretically*.

We also note that Kiem-Li's cone reduction lemma does not hold for the relative setting in general.

Remark 3.1.11. Let $f : X \hookrightarrow Y$ be a regular closed embedding of schemes. Assume that there is a surjection $N_{X/Y} \twoheadrightarrow \mathcal{O}_X$ of coherent sheaves. Then the cone reduction property does not hold for the canonical perfect obstruction theory $\mathbb{F} := \mathbb{L}_{X/Y} \xrightarrow{\cong} L_{X/Y}$.

However, if the base is smooth, then Kiem-Li's cone reduction lemma holds under an additional assumption.

Example 3.1.12 (Relative cone reduction for smooth base)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks, let $\phi : \mathbb{F} \to L_{X/Y}$ be an obstruction theory, and $\sigma : \mathbb{F}^\vee[1] \to \mathcal{O}_X$ be a cosection. Assume that Y is smooth. If the composition

$$\mathcal{O}_X \xrightarrow{\sigma^\vee} \mathbb{F}[-1] \xrightarrow{\phi[-1]} L_{X/Y}[-1] \xrightarrow{\mathsf{KS}_{X/Y}} \Omega_Y|_X$$

vanishes, then $\phi : \mathbb{F} \to L_{X/Y}$ satisfies the cone reduction property. Indeed, this can be shown by modifying the relative obstruction theory $\phi : \mathbb{F} \to L_{X/Y}$ into an absolute obstruction theory as in Example 2.3.16.

Secondly, if a cosection can be enhanced to a (-1)-shifted *closed* 1-form [PTVV], then the cone reduction property holds.

Example 3.1.13 (Reduction by (-1)-shifted 1-form)**.** Let $\mathbb{X}$ be a homotopically finitely presented derived Deligne-Mumford stack over an affine scheme Y and α be a (-1)-shifted closed 1-form. Then the induced obstruction theory

$$\phi : \mathbb{F} := \mathbb{L}_{\mathbb{X}/Y}|_X \to \mathbb{L}_{X/Y} \to L_{X/Y}$$

on the classical truncation $X := \mathbb{X}_{\mathrm{cl}}$ satisfies the *scheme-theoretical* cone reduction property with respect to the cosection

$$\alpha_0|_X^\vee : \mathbb{F}^\vee[1] \to \mathcal{O}_X$$

induced by the underlying (-1)-shifted 1-form $\alpha_0 : \mathcal{O}_{\mathbb{X}} \to \mathbb{L}_{\mathbb{X}}[-1]$ of α. In particular, the cone reduction property in Definition 3.1.7 is satisfied. We refer to §**??** for details.

Finally, we provide a straightforward generalization of Kiem-Li's cone reduction lemma to *multiple* cosections.

Remark 3.1.14 (Multiple cosections)**.** Let X be a Deligne-Mumford stack and $\phi : \mathbb{F} \to L_X$ be an obstruction theory. Consider a *generalized cosection*, i.e., a map

$$\Sigma : \mathbb{F}^\vee[1] \to F$$

in the derived category of X for some vector bundle F. Then we have a *generalized cone reduction property*, i.e., there is a commutative diagram

$$\begin{tikzcd}
 & & \mathfrak{K}(\mathbb{F},\Sigma) \arrow[r] \arrow[d, hook] & X \arrow[d, hook, "0_F"] \\
(\mathfrak{C}_X)_{\mathrm{red}} \arrow[r, hook] \arrow[urr, dotted, "\iota^{\Sigma,\mathrm{red}}"] & \mathfrak{C}_X \arrow[r, hook, "\iota"] & \mathfrak{C}(\mathbb{F}) \arrow[r, "\mathfrak{l}_\Sigma"] & F
\end{tikzcd}$$

for some dotted arrow, where $\mathfrak{l}_\Sigma := \mathfrak{C}(\Sigma^\vee[1]) : \mathfrak{C}(\mathbb{F}) \to \mathbb{A}^1_X$ is the canonical linear function associated to Σ and the square is cartesian.

3.2 Reduced virtual cycles

In this section, we construct *reduced virtual pullbacks* for surjective cosections.

Definition 3.2.1 (Reduced virtual pullback)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks and $\phi : \mathbb{F} \to L_{X/Y}$ be a perfect obstruction theory satisfying the cone reduction property with respect to a cosection $\sigma : \mathbb{F}^\vee[1] \to O_X$. Assume that $h^0(\sigma) : h^1(\mathbb{F}^\vee) \to O_X$ is surjective so that the kernel cone stack $\mathfrak{K}(\mathbb{F},\sigma)$ is a vector bundle stack. We define the *reduced virtual pullback*

$$f^!_{\sigma,\mathrm{red}} : A_*(Y) \to A_*(X)$$

as the composition

$$A_*(Y) \xrightarrow{\mathrm{sp}_{X/Y}} A_*(\mathfrak{C}_{X/Y}) \cong A_*((\mathfrak{C}_{X/Y})_{\mathrm{red}}) \xrightarrow{\iota^{\sigma,\mathrm{red}}_*} A_*(\mathfrak{E}(\mathbb{F}_\sigma)) \xrightarrow{0^!_{\mathfrak{E}(\mathbb{F}_\sigma)}} A_*(X)$$

where $\mathrm{sp}_{X/Y}$ is the specialization map in Definition 2.1.18, $\iota^{\sigma,\mathrm{red}} : (\mathfrak{C}_X)_{\mathrm{red}} \hookrightarrow \mathfrak{K}(\mathbb{E},\sigma)$ is the closed embedding in Proposition 3.1.8.1, and $0^!_{\mathfrak{E}(\mathbb{F}_\sigma)}$ is the Gysin pullback of the vector bundle stack $\mathfrak{E}(\mathbb{F}_\sigma) = \mathfrak{K}(\mathbb{F},\sigma)$ in Definition 2.3.2.

As a special case, we define the *reduced virtual cycles*. In this case, Kiem-Li's cone reduction lemma (Proposition 3.1.9) assures the cone reduction property.

Definition 3.2.2 (Reduced virtual cycle)**.** Let X be a Deligne-Mumford stack equipped with a perfect obstruction theory $\phi : \mathbb{F} \to L_X$ and a cosection $\sigma : \mathbb{F}^\vee[1] \to O_X$. Assume that $h^0(\sigma) : h^1(\mathbb{F}^\vee) \to O_X$ is surjective so that the kernel cone stack $\mathfrak{K}(\mathbb{E},\sigma) = \mathfrak{E}(\mathbb{E}_\sigma)$ is a vector bundle stack. We define the *reduced virtual cycles* as

$$[X]^{\mathrm{red}} := 0^!_{\mathfrak{E}(\mathbb{F}_\sigma)}[\mathfrak{C}_X] \in A_{\mathrm{vd}+1}(X)$$

where $\mathrm{vd} = \mathrm{rank}(\mathbb{E}^\vee)$.

The reduced virtual cycles are *deformation invariant* under an additional assumption.

Proposition 3.2.3 (Deformation invariance)**.** *Let $f : \mathcal{X} \to \mathcal{B}$ be a morphism of Deligne-Mumford stacks. Assume that $\mathcal{B}$ is smooth. Form a fiber diagram*

$$\begin{tikzcd} \mathcal{X}_b \arrow[r, hook] \arrow[d, "f_b"] & \mathcal{X} \arrow[d, "f"] \\ \{b\} \arrow[r, hook, "i_b"] & \mathcal{B} \end{tikzcd}$$

where $b \in \mathcal{B}$. Let $\phi : \mathbb{E} \to L_{\mathcal{X}/\mathcal{B}}$ be a perfect obstruction theory and $\sigma : \mathbb{E}^\vee[1] \to \mathcal{O}_{\mathcal{X}}$ be a cosection. Assume that the composition

$$\mathcal{O}_X \xrightarrow{\sigma^\vee} \mathbb{E}[-1] \xrightarrow{\phi} L_{X/B}[-1] \xrightarrow{\mathsf{KS}} \Omega_{\mathcal{B}}|_X$$

vanishes. Then there exists a cycle class $[\mathcal{X}]^{\mathrm{red}} \in A_*(\mathcal{X})$ such that

$$[\mathcal{X}_b]^{\mathrm{red}} = i_b^![\mathcal{X}]^{\mathrm{red}} \in A_*(\mathcal{X}_b)$$

for all $b \in \mathcal{B}$, where $i_b^! : A_*(\mathcal{X}) \to A_*(\mathcal{X}_b)$ denotes the refined Gysin pullback.

Proof. As in Example 2.3.16, we can modify the relative perfect obstruction theory $\phi : \mathbb{E} \to L_{\mathcal{X}/\mathcal{B}}$ for $f : \mathcal{X} \to \mathcal{B}$ into an absolute perfect obstruction theory for $\mathcal{X}$. As explained in Example 3.1.12, the vanishing condition of the composition

$$\mathcal{O}_X \xrightarrow{\sigma^\vee} \mathbb{E}[-1] \xrightarrow{\phi} L_{X/B}[-1] \xrightarrow{\mathsf{KS}} \Omega_{\mathcal{B}}|_X$$

assures that the relative cosection $\sigma : \mathbb{E}^\vee[1] \to \mathcal{O}_{\mathcal{X}}$ lifts to an absolute cosection and hence we have a cone reduction property for $f : \mathcal{X} \to \mathcal{B}$. Then we can define a reduced virtual pullback

$$f^!_{\sigma,\mathrm{red}} : A_*(\mathcal{B}) \to A_*(\mathcal{X}).$$

By Vistoli's rational equivalence in Proposition 2.1.22, we have

$$[\mathcal{X}_b]^{\mathrm{red}} := (f_b)^!_{\sigma_b,\mathrm{red}}([\mathrm{Spec}(\mathbb{C})]) = (f_b)^!_{\sigma_b,\mathrm{red}} \circ i_b^!([\mathcal{B}]) = i_b^! \circ f^!_{\sigma,\mathrm{red}}([\mathcal{B}]).$$

Then the cycle class $[\mathcal{X}]^{\mathrm{red}} := f^!_{\sigma,\mathrm{red}}[\mathcal{B}]$ satisfies the desired property. ◻

We can easily generalize the reduced virtual cycles to *multiple* cosections.

Definition 3.2.4 (Reduced virtual cycle for multiple cosections)**.** Let X be a Deligne-Mumford stack equipped with a perfect obstruction theory $\phi : \mathbb{F} \to L_X$ and a map $\Sigma : \mathbb{F}^\vee[1] \to F$ to a vector bundle F. Assume that $h^0(\Sigma) : h^1(\mathbb{F}^\vee) \to F$ is surjective. Let $\iota^{\Sigma,\mathrm{red}} : (\mathfrak{C}_X)_{\mathrm{red}} \hookrightarrow \mathfrak{R}(\mathbb{F}, \Sigma)$ be the induced closed embedding in Remark 3.1.14. We define the *reduced virtual cycle* as

$$[X]^{\mathrm{red}} := 0^!_{\mathfrak{C}(\mathbb{F}_\Sigma)}[\mathfrak{C}_X] \in A_{\mathrm{vd}+\mathrm{rank}(F)}(X)$$

where $\mathfrak{R}(\mathbb{E}, \Sigma) := \mathfrak{C}(\mathbb{F}_\sigma)$ is a vector bundle stack since $\mathbb{F}_\Sigma := \mathrm{cone}(\Sigma^\vee[1] : F^\vee[1] \to \mathbb{F})$ is of tor-amplitude $[-1, 0]$.

Remark 3.2.5 (Compatibility)**.** In the situation of Definition 3.2.2, we have

$$[X]^{\mathrm{vir}} = e(F) \cap [X]^{\mathrm{red}} \in A_*(X).$$

In particular, the virtual cycle $[X]^{\mathrm{vir}}$ vanishes when $e(F) = 0$.

We note that the *reduced obstruction theories* may not exist in general.

Remark 3.2.6 (Reduced obstruction theory)**.** Let X be a Deligne-Mumford stack with a perfect obstruction theory $\phi : \mathbb{F} \to L_X$ and a cosection $\sigma : \mathbb{F}^\vee[1] \to O_X$. We would like to know whether the *reduced obstruction theory* exists. More precisely, we would like to find a commutative diagram

$$\begin{tikzcd}
O_X[1] \arrow[r, "\sigma^\vee"] & \mathbb{F} \arrow[r] \arrow[d, "\phi"] & \mathbb{F}_\sigma \arrow[dl, dotted, "\phi^{\mathrm{red}}"] \\
 & L_X &
\end{tikzcd}$$

for some dotted arrow. This is equivalent to the *scheme-theoretical* cone reduction property, i.e., there exists a commutative diagram

$$\begin{tikzcd}
 & \mathfrak{R}(\mathbb{F}, \sigma) \arrow[r] \arrow[d, hook] & X \arrow[d, hook, "0"] \\
\mathfrak{C}_{X/Y} \arrow[ur, dotted] \arrow[r, hook, "\iota"] & \mathfrak{C}(\mathbb{F}) \arrow[r, "\mathrm{l}_\sigma"] & \mathbb{A}^1_X
\end{tikzcd}$$

for some dotted arrow. By [MPT], this property may not hold in general. However, we will see in Chapter 8 that this property is satisfied when the cosection σ can be enhanced to a (-1)-*shifted closed* 1-*form*.

3.3 Cosection-localized virtual pullbacks

In this section, we construct *cosection-localized virtual pullbacks* for perfect obstruction theories with cosections satisfying the cone reduction property.

3.3.1 Cosection-localized Gysin pullbacks

As a special case, we first construct the *cosection-localized Gysin map* for kernel cone stacks.

Notation 3.3.1 (Blowup diagram)**.** Let X be a Deligne-Mumford stack, $\mathbb{F}$ be a perfect complex of tor-amplitude $[-1,0]$ on X, and $\sigma : \mathbb{F}^\vee[1] \to \mathcal{O}_X$ be a cosection of $\mathbb{F}$. Form a fiber diagram

$$\begin{tikzcd} \mathfrak{K}(\mathbb{F},\sigma) \arrow[r] \arrow[d, hook, "k"] & X \arrow[d, hook, "0"] \\ \mathfrak{E}(\mathbb{F}) \arrow[r, "\mathfrak{l}_\sigma"] & \mathbb{A}^1_X \end{tikzcd}$$

where $\mathbb{F}_\sigma := \mathrm{cone}(\sigma^\vee[1] : \mathcal{O}_X[1] \to \mathbb{F})$. Let $X(\sigma) \subseteq X$ be the zero locus of the induced map $\overline{\sigma} : \mathrm{Ob} := h^1(\mathbb{F}^\vee) \to \mathcal{O}_X$. Let $\widetilde{X} := \mathrm{Bl}_{X(\sigma)}X$ be the blowup of X along $X(\sigma)$. Form a fiber diagram

$$\begin{tikzcd} D \arrow[r, hook, "j"] \arrow[d, "q"] & \widetilde{X} \arrow[d, "p"] \\ X(\sigma) \arrow[r, hook, "i"] & X \end{tikzcd}$$

where D is the exceptional divisor. Note that $\sigma|_{\widetilde{X}}$ factors as

$$\begin{tikzcd} \mathbb{F}^\vee[1]|_{\widetilde{X}} \arrow[r, dotted, "\widetilde{\sigma}"] \arrow[dr, "\sigma|_{\widetilde{X}}"'] & \mathcal{O}_{\widetilde{X}}(-D) \arrow[d, hook] \\ & \mathcal{O}_{\widetilde{X}} \end{tikzcd}$$

where $\mathcal{I}_{D/\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-D)$. Then $\mathbb{K} := \mathrm{cone}(\widetilde{\sigma}^\vee[1] : \mathcal{O}_{\widetilde{X}}(D)[1] \to \mathbb{F}|_{\widetilde{X}})$ is of tor-amplitude $[-1,0]$. Hence we have an abstract blowup square

$$\begin{tikzcd} \mathfrak{E}(\mathbb{K}|_D) \arrow[r, hook, "a"] \arrow[d, "b"] & \mathfrak{E}(\mathbb{K}) \arrow[d, "d"] \\ \mathfrak{E}(\mathbb{F}|_{X(\sigma)}) \arrow[r, hook, "c"] & \mathfrak{K}(\mathbb{F},\sigma) \end{tikzcd}$$

where $\mathfrak{E}(\mathbb{K}|_D) := \mathfrak{C}(\mathbb{K}|_D)$, $\mathfrak{E}(\mathbb{K}) := \mathfrak{C}(\mathbb{K})$, and $\mathfrak{E}(\mathbb{F}|_{X(\sigma)}) := \mathfrak{C}(\mathbb{F}|_{X(\sigma)})$ are vector bundle stacks.

Definition 3.3.2 (Cosection-localized Gysin map)**.** Let X be a Deligne-Mumford stack, $\mathbb{F}$ be a perfect complex of tor-amplitude $[-1,0]$, and $\sigma : \mathbb{F}^\vee[1] \to \mathcal{O}_X$ be a cosection. We use the notations in Notation 3.3.1. We define the *cosection-localized Gysin map*

$$0^!_{\mathfrak{E}(\mathbb{F}),\sigma} : A_*(\mathfrak{R}(\mathbb{F},\sigma)) \to A_*(X(\sigma))$$

as the unique map that fits into the commutative diagram

$$\begin{tikzcd}
A_*(\mathfrak{E}(\mathbb{K}|_D)) \arrow[r, "{(-a_*,b_*)}"] \arrow[drr, "0"'] & A_*(\mathfrak{E}(\mathbb{K})) \oplus A_*(\mathfrak{E}(\mathbb{F}|_{X(\sigma)})) \arrow[r, "{(c_*,d_*)}"] \arrow[d, "{(u,v)}"] & A_*(\mathfrak{R}(\mathbb{F},\sigma)) \arrow[r] \arrow[dl, dotted, "0^!_{\mathfrak{E}(\mathbb{F}),\sigma}"'] \arrow[d, "0^!_{\mathfrak{E}(\mathbb{F})}\circ k_*"] & 0 \\
& A_*(X(\sigma)) \arrow[r, "i_*"] & A_*(X) &
\end{tikzcd}$$

where the top horizontal sequence is the abstract blowup sequence (cf. Corollary A.2.7) and the two maps u and v are given as follows:

$$u : A_*(\mathfrak{E}(\mathbb{K})) \xrightarrow{0^!_{\mathfrak{E}(\mathbb{K})}} A_*(\widetilde{X}) \xrightarrow{-j^!} A_*(D) \xrightarrow{q_*} A_*(X(\sigma))$$
$$v : A_*(\mathfrak{E}(\mathbb{F}|_{X(\sigma)}) \xrightarrow{0^!_{\mathfrak{E}(\mathbb{F}|_{X(\sigma)})}} A_*(X(\sigma)).$$

We first show that the cosection-localized Gysin map is well-defined.

Lemma 3.3.3 (Well-definedness)**.** *In this situation of Definition 3.3.2, we have*

$$u \circ a_* = v \circ b_*, \qquad i_* \circ u = 0^!_{\mathfrak{E}(\mathbb{F})} \circ k_* \circ c_*, \qquad i_* \circ v = 0^!_{\mathfrak{E}(\mathbb{F})} \circ k_* \circ d_*.$$

Proof. Firstly, we have

$$\begin{aligned}
u \circ a_* &= q_* \circ j^! \circ 0^!_{\mathfrak{E}(\mathbb{K})} \circ a_* \\
&= -q_* \circ j^! \circ j_* \circ 0^!_{\mathfrak{E}(\mathbb{K}|_D)} \\
&= -q_* \circ c_1(\mathcal{O}_{\widetilde{X}}(D)) \circ 0^!_{\mathfrak{E}(\mathbb{K}|_D)} \\
&= q_* \circ 0^!_{\mathfrak{E}(\mathbb{F}|_D)} \circ e_* \\
&= 0^!_{\mathfrak{E}(\mathbb{F}|_{X(\sigma)})} \circ b_* = v \circ b_*
\end{aligned}$$

where $e : \mathfrak{E}(\mathbb{K}|_D) \hookrightarrow \mathfrak{E}(\mathbb{F}|_D)$ is the canonical inclusion map.

Secondly, we have

$$\begin{aligned} i_* \circ u &= -i_* \circ q_* \circ j^! \circ 0^!_{\mathfrak{E}(\mathbb{K})} \\ &= -p_* \circ j_* \circ j^! \circ 0^!_{\mathfrak{E}(\mathbb{K})} \\ &= -p_* \circ c_1(\mathcal{O}_{\widetilde{X}}(D)) \circ 0^!_{\mathfrak{E}(\mathbb{K})} \\ &= p_* \circ 0^!_{\mathfrak{E}(\mathbb{F}|_{\widetilde{X}})} \circ f_* \\ &= 0^!_{\mathfrak{E}(\mathbb{F})} \circ k_* \circ c_* \end{aligned}$$

where $f : \mathfrak{E}(\mathbb{K}) \hookrightarrow \mathfrak{E}(\mathbb{F}|_{\widetilde{X}})$ is the canonical inclusion map.

Finally, we have

$$i_* \circ v = i_* \circ 0^!_{\mathfrak{E}(\mathbb{F}|_{X(\sigma)})} = 0^!_{\mathfrak{E}(\mathbb{F})} \circ k_* \circ d_*.$$

It completes the proof. □

We note that the cosection-localized Gysin maps are *bivariant classes*.

Proposition 3.3.4 (Bivariance)**.** *Let $f : Y \to X$ be a morphism of Deligne-Mumford stacks. Let $\mathbb{F}$ be a perfect complex of tor-amplitude $[-1,0]$ on X and $\sigma : \mathbb{F}^\vee[1] \to \mathcal{O}_X$ be a cosection. Form a fiber diagram*

$$\begin{tikzcd} \mathfrak{R}(\mathbb{F}|_Y, \sigma|_Y) \arrow[r, "\widehat{f}"] \arrow[d] & \mathfrak{R}(\mathbb{F}, \sigma) \arrow[d] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

1. *If $f : Y \to X$ is a proper morphism, then we have*

$$f_* \circ 0^!_{\mathfrak{E}(\mathbb{F}|_Y), \sigma|_Y} = 0^!_{\mathfrak{E}(\mathbb{F}), \sigma} \circ \widehat{f}_* : A_*(\mathfrak{R}(\mathbb{F}|_Y, \sigma|_Y)) \to A_*(X).$$

2. *If $f : Y \to X$ is a equi-dimensional flat morphism, then we have*

$$f^* \circ 0^!_{\mathfrak{E}(\mathbb{F}|_Y, \sigma_Y)} = 0^!_{\mathfrak{E}(\mathbb{F}), \sigma} \circ \widehat{f}^* : A_*(\mathfrak{R}(\mathbb{E}, \sigma)) \to A_*(Y).$$

3. *If $f : Y \to X$ is a local complete intersection morphism, then we have*

$$f^! \circ 0^!_{\mathfrak{E}(\mathbb{F}|_Y, \sigma_Y)} = 0^!_{\mathfrak{E}(\mathbb{F}), \sigma} \circ f^! : A_*(\mathfrak{R}(\mathbb{E}, \sigma)) \to A_*(Y).$$

Proof. By the universal property of blowup, we can form a commutative diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[r, dotted, "\widetilde{f}"] \arrow[d] & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

for some $\widetilde{f}$. Then we can form a commutative diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "f''"'] \arrow[rr, bend left, "\widetilde{f}"] \arrow[dr] & Y' \arrow[r, "f'"'] \arrow[d] & \widetilde{X} \arrow[d, "p"] \\ & Y \arrow[r, "f"] & X \end{tikzcd}$$

where the square is cartesian.

Then Proposition 3.3.4.1 follows immediately since $\widetilde{f}$, f'' are proper and all the operations in Definition 3.3.2 commute with projective proper pushforwards.

Similarly, Proposition 3.3.4.2 follows immediately since the flatness of f implies $\widetilde{Y} = Y'$ and all the operations in Definition 3.3.2 commute with flat pullbacks.

Finally, Proposition 3.3.4.3 follows from Lemma 3.3.5 below and Proposition 3.3.4.1 since all the operations in Definition 3.3.2 commute with lci pullbacks. $\square$

Lemma 3.3.5. *Let X be a Deligne-Mumford stack, $\mathbb{F}$ be a perfect complex of tor-amplitude $[-1,0]$, and $\sigma : \mathbb{F}^\vee[1] \to O_X$ be a cosection. Assume that there exists a factorization*

$$\begin{tikzcd} & L^\vee \arrow[d] \\ \mathbb{F}^\vee[1] \arrow[ur, dotted, "\tau"] \arrow[r, "\sigma"'] & O_X \end{tikzcd}$$

for some line bundle L and a map τ. Let $\mathbb{F}_\tau :=$ $\mathrm{cone}(\tau^\vee[1] : L[1] \to \mathbb{F})$. Then $\mathfrak{K} := \mathfrak{C}(\mathbb{F}_\tau)$ is a vector bundle stack and we have

$$0^!_{\mathfrak{E},\sigma} \circ a_* = -j^! \circ 0^!_{\mathfrak{K}} : A_*(\mathfrak{K}) \to A_*(X(\sigma))$$

where $a : \mathfrak{K} \hookrightarrow \mathfrak{K}(\mathbb{E}, \sigma)$ is the inclusion map.

Proof. By the blowup sequence, we may reduce the situation to the case when $X(\sigma)$ is a divisor of X and $L = O_X(X(\sigma))$. Then the statement follows directly from the definition. $\square$

Remark 3.3.6 (Uniqueness)**.** The cosection-localized Gysin pullbacks $0^!_{\mathfrak{E},\sigma}$ in Definition 3.3.2 are uniquely determined by the bivariance in Proposition 3.3.4, the compatibility $0^!_{\mathfrak{E},0} = 0^!_{\mathfrak{E}}$, and the special case in Lemma 3.3.5.

Remark 3.3.7 (Generalization)**.** We may want to generalize the cosection-localized Gysin pullbacks $0^!_{\mathfrak{E},\sigma}$ in Definition 3.3.2 to *Artin* stacks X. As long as we have a generalized blowup sequence in Corollary A.2.7 for arbitrary Artin stacks, then everything in this subsection generalize to Artin stacks immediately.

3.3.2 Cosection-localized virtual pullbacks

We now consider the general case. We construct *cosection-localized virtual pullbacks* for perfect obstruction theories with cosections

Definition 3.3.8 (Cosection-localized virtual pullback)**.** Let $f : X \to Y$ be a morphism from a DM stack X to an algebraic stack Y. Let $\phi : \mathbb{F} \to L_{X/Y}$ be a perfect obstruction theory satisfying the cone reduction property with respect to a cosection $\sigma : \mathbb{F}^\vee[1] \to L_{X/Y}$. We define the *cosection-localized virtual pullback*

$$f^!_\sigma : A_*(Y) \to A_*(X(\sigma))$$

as the composition

$$A_*(Y) \xrightarrow{\mathrm{sp}_{X/Y}} A_*(\mathfrak{C}_{X/Y}) \cong A_*((\mathfrak{C}_{X/Y})_{\mathrm{red}}) \xrightarrow{\iota^{\sigma,\mathrm{red}}} A_*(\mathfrak{K}(\mathbb{F},\sigma)) \xrightarrow{0^!_{\mathfrak{E}(\mathbb{F}),\sigma}} A_*(X(\sigma))$$

where $\mathrm{sp}_{X/Y}$ is the specialization map, $\iota^{\sigma,\mathrm{red}} : (\mathfrak{C}_{X/Y})_{\mathrm{red}} \hookrightarrow \mathfrak{K}(\mathbb{F},\sigma)$ is the closed embedding to the kernel cone stack, and $0^!_{\mathfrak{E}(\mathbb{F}),\sigma}$ is the cosection-localized Gysin pullback in Definition 3.3.2

We now define the *cosection-localized virtual cycle* as a special case of the cosection-localized virtual pullback.

Definition 3.3.9 (Cosection-localized virtual cycle)**.** Let X be a Deligne-Mumford stack equipped with a perfect obstruction theory $\phi : \mathbb{F} \to L_X$ and a cosection $\sigma : \mathbb{F}^\vee[1] \to O_X$. We define the *cosection-localized virtual cycle* as

$$[X]^{\mathrm{vir}}_{\mathrm{loc}} := f^!_\sigma([\mathrm{Spec}(\mathbb{C})]) = 0^!_{\mathfrak{E}(\mathbb{F}),\sigma}[\mathfrak{C}_X] \in A_*(X(\sigma))$$

where $f^!_\sigma : A_*(\mathrm{Spec}(\mathbb{C})) \to A_*(X(\sigma))$ is the cosection-localized virtual pullback in Definition 3.3.8.

The cosection-localized virtual pullbacks are *bivariant* classes.

Proposition 3.3.10 (Bivariance)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a cartesian square of algebraic stacks where X and X' are Deligne-Mumford stacks. Let $\phi : \mathbb{F} \to L_{X/Y}$ be a perfect obstruction theory satisfying the cone reduction property with respect to a cosection $\sigma : \mathbb{F}^\vee[1] \to O_X$. Let

$$\phi' : \mathbb{F}' := (g')^*(\mathbb{F}) \xrightarrow{(g')^*(\phi)} (g')^*(L_{X/Y}) \to L_{X'/Y'}$$

be the induced perfect obstruction theory. Then ϕ' also satisfies the cone reduction property with respect to the induced cosection

$$\sigma' := (g')^*(\sigma) : (\mathbb{F}')^\vee[1] \to O_{X'}$$

and we have the following properties:

1. *If g is a proper DM morphism, then we have*
$$f^!_\sigma \circ g_* = g'_* \circ (f')^!_\sigma : A_*(Y') \to A_*(X(\sigma)).$$

2. *If g is a equi-dimensional flat morphism, then we have*
$$(f')^!_{\sigma'} \circ g^* = (g')^* \circ f^!_\sigma : A_*(Y) \to A_*(X'(\sigma')).$$

3. *If g is a local complete intersection morphism and Y' has affine stabilizers, then we have*
$$(f')^!_{\sigma'} \circ g^! = (g')^! \circ f^!_\sigma : A_*(Y) \to A_*(X'(\sigma')).$$

Proof. If follows directly from Proposition 2.1.19 and Proposition 3.3.4. □

3.3.3 Functoriality

In this subsection, we collect *functorility* results of cosection-localized virtual cycles shown in Chang-Kiem-Li [CKL].

Theorem 3.3.11 (Functoriality I)**.** *Let $f : X \to Y$ be a morphism of Deligne-Mumford stacks. Let $(\phi_f : \mathbb{F}_f \to L_f, \phi_Y : \mathbb{F}_Y \to L_Y, \phi_X : \mathbb{F}_X \to L_X)$ be a compatible triple of perfect obstruction theories in the sense of Definition 2.3.11. Consider a commutative triangle*

$$\begin{tikzcd} \mathbb{F}_X^\vee[1] \arrow[d] \arrow[dr, "\sigma_X"] & \\ \mathbb{F}_Y|_X^\vee[1] \arrow[r, "\sigma_Y|_X"'] & \mathcal{O}_X \end{tikzcd}$$

for some cosections $\sigma_X : \mathbb{E}_X|^\vee[1] \to \mathcal{O}_X$ and $\sigma_Y : \mathbb{E}_Y^\vee[1] \to \mathcal{O}_Y$. Then we have a fiber diagram

$$\begin{tikzcd} X(\sigma_X) \arrow[r] \arrow[d, hook] & Y(\sigma_Y) \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and a virtual pullback formula

$$[X]^{\mathrm{vir}}_{\mathrm{loc}} = f^![Y]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(X(\sigma_X)).$$

Most of the arguments of Theorem 3.3.11 are straightforward generalizations of Theorem 2.3.12. The crucial additional ingredient is [CKL, Lem. 2.7]: the induced perfect obstruction theory

$$\mathbb{E}_h \to L_h$$

for $h : X \times \mathbb{A}^1 \to M^\circ_{Y/Z}$ in Lemma 2.3.19 satisfies the cone reduction property with respect to a cosection

$$\sigma_h : \mathbb{E}_h^\vee[1] \to \mathcal{O}_{X\times\mathbb{A}^1}$$

that fits into a morphism of distinguished triangles

$$\begin{tikzcd} \mathbb{E}_h^\vee[1] \arrow[r] \arrow[d, dotted, "\sigma_h"] & (f^*\mathbb{E}_Y^\vee[1] \oplus \mathbb{E}_X^\vee[1]) \boxtimes \mathcal{O}_{\mathbb{A}^1} \arrow[r] \arrow[d, "f^*\sigma_Y \oplus \sigma_X"] & f^*\mathbb{E}_Y^\vee[1] \boxtimes \mathcal{O}_{\mathbb{A}^1} \arrow[d, "\sigma_Y"] \\ \mathcal{O}_{X\times\mathbb{A}^1} \arrow[r] & \mathcal{O}_{X\times\mathbb{A}^1} \oplus \mathcal{O}_{X\times\mathbb{A}^1} \arrow[r] & \mathcal{O}_{X\times\mathbb{A}^1}. \end{tikzcd}$$

We refer to [CKL, Thm. 2.6] for details.

Theorem 3.3.12 (Functoriality II)**.** *Let $f : X \to Y$ be a morphism of Deligne-Mumford stacks. Let $(\phi_f : \mathbb{F}_f \to L_f, \phi_Y : \mathbb{F}_Y \to L_Y, \phi_X : \mathbb{F}_X \to L_X)$ be a compatible triple of perfect obstruction theories in the sense of Definition 2.3.11. Consider a commutative triangle*

$$\begin{tikzcd} \mathbb{F}_f^\vee[1] \arrow[d] \arrow[dr, "\sigma_f"] & \\ \mathbb{F}_X^\vee[1] \arrow[r, "\sigma_X"'] & O_X \end{tikzcd}$$

for some cosections $\sigma_X : \mathbb{E}_X^\vee[1] \to O_X$ and $\sigma_f : \mathbb{E}_f^\vee[1] \to O_X$. Then ϕ_f satisfies the cone reduction property with respect to the cosection σ_f and we have

$$c_*[X]^{\mathrm{vir}}_{\mathrm{loc}} = f^!_\sigma[Y]^{\mathrm{vir}} \in A_*(X(\sigma_f))$$

where $c : X(\sigma_X) \hookrightarrow X(\sigma_f)$ is the inclusion map.

The proof of Theorem 3.3.12 is relatively easier than that of Theorem 3.3.11 since the cone reduction property for ϕ_h follows immediately from the cone reduction property for ϕ_X. We refer to [CKL, Thm. 2.10] for details.

Part II

Donaldson-Thomas theory of Calabi-Yau 4-folds

Chapter 4

Virtual pullbacks in Donaldson-Thomas theory of Calabi-Yau 4-folds

In this chapter, we generalize Manolache's virtual pullbacks [Man] to Donaldson-Thomas theory of Calabi-Yau 4-folds. This is based on [Park1].

Summary We first recall the local model of Behrend-Fantechi virtual cycles. Let U be a smooth scheme, E be a vector bundle on U, s be a global section of E, and X be the zero locus of s in U,

$$\begin{tikzcd} & E \arrow[d] \\ X \arrow[r, hook] & U. \arrow[u, bend right, "s"'] \end{tikzcd}$$

The Behrend-Fantechi virtual cycle is defined as the localized Euler class,

$$[X]^{\mathrm{vir}}_{BF} := e(E,s)[U] \in A_*(X).$$

Analogously, the local model of Oh-Thomas virtual cycles is given by the following replacements:

$$\begin{array}{rcl} \text{vector bundle } E & \rightsquigarrow & \text{special orthogonal bundle } E \\ \text{section } s \text{ of } E & \rightsquigarrow & \text{isotropic section } s \text{ of } E \\ \text{localized Euler class } e(E,s) & \rightsquigarrow & \text{localized square root Euler class } \sqrt{e}(E,s). \end{array}$$

where the construction of $\sqrt{e}(E, s)$ is the crucial part.

A general philosophy in DT4 theory is to replace everything in the virtual intersection theory to the *symmetric* versions of them. In particular, we consider the following global replacement:

$$\text{perfect obstruction theories} \quad \rightsquigarrow \quad \text{symmetric obstruction theories.}$$

A new feature is that we need two additional ingredients: (1) *isotropic condition* and (2) an *orientation*.

We briefly summarize the global construction of virtual cycles/virtual pullbacks in DT4 theory. Let $f : X \to Y$ be a morphism of schemes and let $\phi : \mathbb{E} \to L_{X/Y}$ be a symmetric obstruction theory. Then there is a *canonical quadratic function*

$$\mathfrak{q}_{\mathbb{E}} : \mathfrak{C}(\mathbb{E}) \to \mathbb{A}^1_X$$

induced by the symmetric form of $\mathbb{E}$. Let $\mathfrak{Q}(\mathbb{E})$ be the *quadratic cone stack*, defined as the zero locus of the canonical quadratic function $\mathfrak{q}_{\mathbb{E}}$ in $\mathfrak{C}(\mathbb{E})$.

1. We say that ϕ satisfies the *isotropic condition* if

$$\mathfrak{C}_X \subseteq \mathfrak{Q}(\mathbb{E})$$

as substacks of $\mathfrak{C}(\mathbb{E})$.

2. We will construct the *square root Gysin pullback*

$$\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}} : A_*(\mathfrak{Q}(\mathbb{E})) \to A_*(X)$$

for the quadratic cone stack $\mathfrak{Q}(\mathbb{E})$, which depends on a choice of an *orientation* $o : O_X \xrightarrow{\cong} \det(\mathbb{E})$.

We define the *square root virtual pullback* as the composition

$$\sqrt{f^!} : A_*(Y) \xrightarrow{\mathrm{sp}} A_*(\mathfrak{C}_{X/Y}) \to A_*(\mathfrak{Q}(\mathbb{E})) \xrightarrow{0^!_{\mathfrak{Q}(\mathbb{E})}} A_*(X).$$

The *Oh-Thomas virtual cycle* is then defined as a special case for $Y = \mathrm{Spec}(\mathbb{C})$,

$$[X]^{\mathrm{vir}}_{OT} := \sqrt{f^!}[\mathrm{Spec}(\mathbb{C})] = \sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}[\mathfrak{C}_X] \in A_*(X).$$

As virtual pullbacks, the most important property of square root virtual pullbacks is a *functoriality*. If

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z, \end{tikzcd}$$

is a commutative diagram of schemes such that g, $g \circ f$ are equipped with symmetric obstruction theories and f is equipped with a perfect obstruction theory, then we have

$$\sqrt{(g\circ f)^!} = f^! \circ \sqrt{g^!} : A_*(Z) \to A_*(X)$$

for a natural compatibility condition. In particular, when $Z = \mathrm{Spec}(\mathbb{C})$, we have a virtual pullback formula

$$[X]^{\mathrm{vir}}_{OT} = f^![Y]^{\mathrm{vir}}_{OT} \in A_*(X).$$

We present various applications of the virtual pullback formula in Chapter 6.

4.1 Local models

In this section, we review the *square root Euler classes* $\sqrt{e}(E)$ for special orthogonal bundles E of Edidin-Graham [EG1] and its localization $\sqrt{e}(E,s)$ by isotropic sections s of Oh-Thomas [OT]. Roughly speaking, $\sqrt{e}(E,s)$ are the local models of the square root virtual pullbacks.

4.1.1 Orthogonal bundles

In this subsection, we fix the notions of *orthogonal bundles* and *special orthogonal bundles* on algebraic stacks. We also present three basic operations of them.

Definition 4.1.1 (Orthogonal bundle)**.** Let X be an algebraic stack. An *orthogonal bundle* on X is a pair (E,θ) where

1. E is a vector bundle on X, and
2. θ is a *non-degenerate symmetric bilinear form* on E, i.e., a map

$$\theta : E\otimes E \to O_X$$

satisfying the following properties:

(a) $\theta \circ \sigma = \theta$ for the transition map $\sigma : E \otimes E \to E \otimes E$;
(b) the induced map $E^\vee \to E$ is an isomorphism.

By abuse of notation, we say that E is an orthogonal bundle on X.

We present some elementary facts on orthogonal bundles.

Remark 4.1.2. Let X be an algebraic stack. Then an orthogonal bundle of rank n on X is equivalent to a map
$$X \to B\mathrm{O}(n)$$
to the classifying stack of the orthogonal group $\mathrm{O}(n)$.

Remark 4.1.3. Let X be a scheme. Then an orthogonal bundle on X is étale-locally trivial, but not necessarily Zariski-locally trivial. A counterexample is provided in [EG1]. In other words, the orthogonal group $\mathrm{O}(n)$ is not a special group.

Remark 4.1.4 (Canonical quadratic function)**.** Let E be an orthogonal bundle on an algebraic stack X. Then there exists a *canonical quadratic function*
$$\mathfrak{q}_E : E \to \mathbb{A}^1_X$$
defined by the symmetric form $\theta \in \mathrm{Sym}^2(E^\vee)$ where $E = \mathrm{Spec}(\mathrm{Sym}(E^\vee))$.

There are three basic operations for orthogonal bundles.

Example 4.1.5 (Three operations)**.** Let X be an algebraic stack.

1. If E_1 and E_2 are orthogonal bundles, then we have a direct sum $E_1 \oplus E_2$ as an orthogonal bundle.

2. If F is a *non-degenerate subbundle* of an orthogonal bundle E, i.e., a subbundle such that $\theta|_F : F \otimes F \to \mathcal{O}_X$ is non-degenerate, then we have an *orthgonal complement* $F^\perp$ as an orthogonal bundle. Moreover, we have a canonical direct sum decomposition
$$E = F \oplus F^\perp.$$
We sometimes denote the orthogonal complement $F^\perp$ by $E_{/F}$.

3. If K is an *isotropic subbundle* of an orthogonal bundle E, i.e., a subbundle such that $\theta|_K : K \otimes K \to \mathcal{O}_X$ is zero, then we have a *reduction* $K^\perp/K$ as an orthogonal bundle. Moreover, we have a canonical commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, hook] \arrow[d, equal] & K^\perp \arrow[r, two heads] \arrow[d, hook] & K^\perp/K \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & K \arrow[r, hook] \arrow[d] & E \arrow[r, two heads] \arrow[d] & E/K \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & K^\vee \arrow[r, equal] & K^\vee \arrow[r] & 0 \end{tikzcd}$$

where the rows and columns are exact. We sometimes denote the reduction $K^\perp/K$ by $E_{/\!/K}$.

Definition 4.1.6 (Orientation)**.** Let E be an orthogonal bundle on an algebraic stack X. An *orientation* of E is an isomorphism of line bundles

$$o : \mathcal{O}_X \xrightarrow{\cong} \det(E)$$

such that the square

$$o^2 : \mathcal{O}_X \xrightarrow{o} \det(E) \cong \det(E^\vee) \xrightarrow{o^\vee} \mathcal{O}_X$$

is the identity $1 \in \Gamma(X, \mathcal{O}_X)$, where the second isomorphism $\det(E) \cong \det(E^\vee)$ is given by the symmetric form of E.

Remark 4.1.7 (Orientation bundle)**.** Let E be an orthogonal bundle on an algebraic stack X. We define the *orientation bundle* of E as the functor

$$\mathsf{Or}(E) : \mathsf{Sch}^{\mathrm{op}}_{/X} \to \mathsf{Set} : (T \to X) \mapsto \{\text{orientations of } E|_T\}.$$

Consider the canonical short exact sequence of algebraic groups

$$\begin{tikzcd} 0 \arrow[r] & SO(n) \arrow[r, hook] & O(n) \arrow[r, two heads, "\det"] & \mu_2 \arrow[r] & 0 . \end{tikzcd}$$

Then the orientation bundle $\mathsf{Or}(E)$ fits into the fiber diagram

$$\begin{tikzcd} \mathsf{Or}(E) \arrow[r] \arrow[d] & BSO(n) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{C}) \arrow[d] \\ X \arrow[r, "E"] & BO(n) \arrow[r, "\det"] & B\mu_2. \end{tikzcd}$$

Hence the orientation bundle $\mathsf{Or}(E)$ is a principal μ_2-bundle over X. In particular, E is étale-locally orientable since the pullback $E|_{\mathsf{Or}(E)}$ has a canonical orientation.

There are canonical induced orientations for the three operations of orthogonal bundles in Example 4.1.5.

Example 4.1.8 (Induced orientations)**.** Let X be an algebraic stack.

1. Let $E = E_1 \oplus E_2$ be the direct sum of two orthogonal bundles. Then we have a canonical isomorphism of line bundles
$$\det(E) \cong \det(E_1) \otimes \det(E_2).$$
Hence orientations of E_1 and E_2 induce an orientation of E.

2. Let $F^\perp$ be the orthogonal complement of a non-degenerate subbundle F of an orthogonal bundle E. Then we have a canonical isomorphism of line bundles
$$\det(F^\perp) \cong \det(E) \otimes \det(F)^\vee.$$
Hence orientations of E and F induce an orientation of $F^\perp$.

3. Let $K^\perp/K$ be the reduction of an isotropic subbundle K of an orthogonal bundle E. Then we have a canonical isomorphism of line bundles
$$\begin{aligned}\det(K^\perp/K) &\cong \det(K^\perp) \otimes \det(K)^\vee \\ &\cong \det(E) \otimes \det(K^\vee)^\vee \otimes \det(K)^\vee \cong \det(E).\end{aligned}$$
Hence an orientation of E induces an orientation of $K^\perp/K$.

Definition 4.1.9 (Special orthogonal bundles)**.** Let X be an algebraic stack. A *special orthogonal bundle* on X is a triple (E, θ, o) where

1. (E, θ) is an orthogonal bundle on X, and
2. $o : \mathcal{O}_X \to \det(E)$ is an orientation of E.

By abuse of notation, we say that E is a special orthogonal bundle on X.

Remark 4.1.10 (Maximal isotropic subbundles)**.** Let E be a special orthogonal bundle of rank $2n$ on an algebraic stack X.

1. We say that an isotropic subbundle M of E is *maximal* if $\mathrm{rank}(M) = n$.
2. For a maximal isotropic subbundle M of E, the reduction $M^\perp/M$ of E by M is zero. Thus an orientation of the orthogonal bundle $M^\perp/M = 0$ is equivalent to a *sign*. We say that a maximal isotropic subbundle M of E is *positive* if the induced orientation on $M^\perp/M$ is 1. Otherwise, if the induced orientation is -1, then we say that M is a *negative* maximal isotropic subbundle.

4.1.2 Edidin-Graham classes

In this subsection, we construct *square root Euler classes* of special orthogonal bundles, introduced by Edidin-Graham [EG1].

We first recall the notion of *isotropic flag bundles*.

Definition 4.1.11 (Isotropic flag bundle)**.** Let E be an orthogonal bundle of rank $2n$ on an algebraic stack X. We define the *isotropic flag bundle* of E as the functor

$$\mathsf{Flag}(E) : \mathsf{Sch}^{\mathrm{op}}_{/X} \to \mathsf{Set} : T \mapsto \{(K_1 \subseteq K_2 \subseteq \cdots \subseteq K_{n-1} \subseteq E|_T)\}$$

where K_i are isotropic subbundle of $E|_T$ of rank i.

Proposition 4.1.12. *Let E be an orthogonal bundle of even rank on an algebraic stack X. Let $\mathsf{Flag}(E)$ be the isotropic flag bundle of E.*

1. *The canonical map*
$$p : \mathsf{Flag}(E) \to X$$
is a smooth, projective, surjective morphism of algebraic stacks.

2. *Given an orientation of E, there exists a canonical positive maximal isotropic subbundle Λ of $E|_{\mathsf{Flag}(E)}$.*

Proof. It follows from the results in [EG1] for schemes via descent. ◻

We define the *square root Euler classes* through the isotropic flag bundles.

Definition 4.1.13 (Square root Euler class)**.** Let E be a special orthogonal bundle of rank $2n$ on a Deligne-Mumford stack X. Let $p : \mathsf{F} := \mathsf{Flag}(E) \to X$ denote the isotropic flag bundle of E. We define the *square root Euler class* of E as the unique map

$$\sqrt{e}(E) : A_*(X) \to A_*(X)$$

that fits into the commutative diagram

$$\begin{tikzcd}
A_*(\mathsf{F} \times_X \mathsf{F}) \arrow[r, "(p_1)_* - (p_2)_*"] \arrow[d, "e(p_1^*\Lambda) = e(p_2^*\Lambda)"] & A_*(\mathsf{F}) \arrow[r, "p_*"] \arrow[d, "e(\Lambda)"] & A_*(X) \arrow[r] \arrow[d, dotted, "\sqrt{e}(E)"] & 0 \\
A_*(\mathsf{F} \times_X \mathsf{F}) \arrow[r, "(p_1)_* - (p_2)_*"'] & A_*(\mathsf{F}) \arrow[r, "p_*"'] & A_*(X) \arrow[r] & 0
\end{tikzcd}$$

as the dotted arrow. Here $\Lambda \subseteq E|_{\mathsf{F}}$ is the canonical positive maximal isotropic subbundle in Proposition 4.1.12.2, the rows are exact by the Kimura sequence (see Theorem A.1.1), and $e(p_1^*\Lambda) = e(p_2^*\Lambda)$ by Fulton's conjecture in Lemma 4.1.14 below.

We need the following version of *Fulton's conjecture* to assure that the square root Euler class in Definition 4.1.13 is well-defined.

Lemma 4.1.14 (Fulton's conjecture)**.** *Let E be a special orthogonal bundle on a Deligne-Mumford stack X. If M_1 and M_2 are positive maximal isotropic subbundles of E, then we have*

$$e(M_1) = e(M_2) : A_*(X) \to A_*(X).$$

Proof. By [EHKV, Thm. 2.7], there exists a finite surjective map $p : \mathsf{F} \to X$ from a scheme F. Since the pushforward

$$f_* : A_*(\mathsf{F}) \to A_*(X)$$

is surjective by the Kimura sequence in Theorem A.1.1, the result for schemes in [EG1, Thm. 1] completes the proof. □

We note that the square root Euler classes are bivariant classes.

Proposition 4.1.15 (Bivariance)**.** *Let $f : Y \to X$ be a morphism of Deligne-Mumford stacks and E be a special orthogonal bundle on X.*

1. *If $f : Y \to X$ is a proper morphism, then we have*

$$\sqrt{e}(E) \circ f_* = f_* \circ \sqrt{e}(E).$$

2. *If $f : Y \to X$ is an equi-dimensional flat morphism, then we have*

$$\sqrt{e}(E) \circ f^* = f^* \circ \sqrt{e}(E).$$

3. *If $f : Y \to X$ is a local complete intersection morphism, then we have*

$$\sqrt{e}(E) \circ f^! = f^! \circ \sqrt{e}(E).$$

By abuse of notation, we denoted $\sqrt{e}(f^*E)$ by $\sqrt{e}(E)$.

We describe how the square root Euler classes are related to the basic operations of special orthogonal bundles in Example 4.1.5 (and Example 4.1.8).

Proposition 4.1.16 (Whitney sum formula)**.** *Let E_1 and E_2 be special orthogonal bundles on a Deligne-Mumford stack X. Then we have*

$$\sqrt{e}(E_1 \oplus E_2) = \sqrt{e}(E_1) \circ \sqrt{e}(E_2) : A_*(X) \to A_*(X).$$

Proposition 4.1.17 (Reduction formula)**.** *Let E be a special orthogonal bundle on a Deligne-Mumford stack X and K be an isotropic subbundle. Then we have*

$$\sqrt{e}(E) = e(K) \circ \sqrt{e}(E_{//K}) : A_*(X) \to A_*(X)$$

where $E_{//K} := K^{\perp}/K$ is the reduction of E by K.

We omit the proofs of Proposition 4.1.15, Proposition 4.1.16, and Proposition 4.1.17 since they follow immediately from the definition.

Remark 4.1.18 (Uniqueness)**.** Note that Fulton's conjecture in Lemma 4.1.14 is a special case of the reduction formula in Proposition 4.1.17. Thus we can observe that the square root Euler classes are uniquely determined by the bivariance in Proposition 4.1.15 and the reduction formula in Proposition 4.1.17.

Remark 4.1.19 (Integral coefficients)**.** Square root Euler classes can be defined in the Chow groups with $\mathbb{Z}[1/2]$-coefficients. However, Totaro [**?**] showed that square root Euler class does not exists with $\mathbb{Z}$-coefficients.

The square root Euler classes can be generalized to a certain class of Artin stacks.

Remark 4.1.20 (Generalization)**.** Let E be a special orthogonal bundle on an algebraic stack X. Assume that X admits a proper cover by a quotient stack (in the sense of Definition A.2.1). Then we can define a square root Euler

$$\sqrt{e}(E) : A_*(X) \to A_*(X)$$

as in Definition 4.1.13, since Fulton's conjecture holds for the isotropic flag bundle $\mathsf{F} := \mathsf{Flag}(E)$ and $\mathsf{F} \times_X \mathsf{F}$ by Lemma 4.1.21 below.

In particular, we have square root Euler classes for algebraic stacks with reductive stabilizers and affine diagonals (see Proposition **??**).

Lemma 4.1.21. *Let E be a special orthogonal bundle on an algebraic stack X. Let M_1 and M_2 be two positive maximal isotropic subbundles of E. Assume that X admits a proper cover by a quotient stack (in the sense of Definition A.2.1). Then we have*

$$e(M_1) = e(M_2) : A_*(X) \to A_*(X).$$

Proof. Let $p : \widetilde{X} \to X$ be a proper representable surjection from the quotient stack $\widetilde{X} = [P/G]$ of a separated Deligne-Mumford stack P by an action of a linear algebraic group G. By the Kimura sequence in Theorem A.1.1, the pushforward

$$f_* : A_*(\widetilde{X}) \to A_*(X)$$

is surjective. Let $EG_i/G \to BG$ be Totaro's algebraic approximations [Tot]. By the homotopy property, the pullback

$$A_d([P/G]) \to A_{d+\dim(EG_i)}([P \times EG_i/G])$$

for each $d \in \mathbb{Z}$, is an isomorphism for sufficiently large i. Hence Lemma 4.1.14 completes the proof since $[P \times EG_i/G]$ is a Deligne-Mumford stack. □

4.1.3 Oh-Thomas classes

In this subsection, we construct *localized square root Euler classes* for special orthogonal bundles with isotropic sections, introduced by Oh-Thomas in [OT]. Instead of following the construction in [OT] directly, we use the blowup construction introduced in [KP2]. This construction is inspired by the cosection localization [KL1].

Notation 4.1.22 (Blowup diagram)**.** Let E be a special orthogonal bundle on an algebraic stack X. Let s be an isotropic section of E. Let $X(s)$ denote the zero locus of s in X. Let $\widetilde{X} := \mathrm{Bl}_{X(s)}X$ denote the blowup of X along $X(s)$. Form a fiber diagram

$$\begin{tikzcd} D \arrow[r, hook, "j"] \arrow[d, "q"] & \widetilde{X} \arrow[d, "p"] \\ X(s) \arrow[r, hook, "i"] & X \end{tikzcd}$$

where D is the exceptional divisor. Then the section s defines a surjection

$$E|_{\widetilde{X}} \twoheadrightarrow \mathcal{O}_{\widetilde{X}}(-D) \subseteq \mathcal{O}_{\widetilde{X}}.$$

Since s is an isotropic section, $L := \mathcal{O}_{\widetilde{X}}(D)$ is an isotropic subbundle of $E|_{\widetilde{X}}$. Let $L^{\perp}/L$ be the reduction of $E|_{\widetilde{X}}$ by L.

Definition 4.1.23 (Localized square root Euler class)**.** Let X be a Deligne-Mumford stack, E be a special orthogonal bundle on X, and s be an isotropic section of E.

We use the notations in Notation 5.3.1. We define the *localized square root Euler class* of E by s as the unique map

$$\sqrt{e}(E,s) : A_*(X) \to A_*(X(s))$$

that fits into the commutative diagram

$$\begin{tikzcd}
A_*(D) \arrow[r, "{(-j_*,q_*)}"] & A_*(\widetilde{X}) \oplus A_*(X(s)) \arrow[r, "{(p_*,i_*)}"] \arrow[d, "{(u,v)}"'] & A_*(X) \arrow[r] \arrow[d, "\sqrt{e}(E)"] \arrow[dl, dotted, "\sqrt{e}(E,s)"'] & 0 \\
& A_*(X(s)) \arrow[r, "i_*"'] & A_*(X) &
\end{tikzcd}$$

where the middle vertical arrow is given by the two maps

$$u : A_*(\widetilde{X}) \xrightarrow{j^!} A_*(D) \xrightarrow{\sqrt{e}(L^\perp/L)} A_*(D) \xrightarrow{q_*} A_*(X(s))$$
$$v : A_*(X(s)) \xrightarrow{\sqrt{e}(E|_{X(s)})} A_*(X(s))$$

and the top horizontal right exact sequence is the abstract blowup sequence in Corollary A.2.7.

To show that the localized square root Euler class $\sqrt{e}(E,s)$ in Definition 4.1.23 is well-defined, we need the following identities.

Lemma 4.1.24 (Well-definedness)**.** *In the situation of Definition 4.1.23, we have the identities*

$$u \circ j_* = v \circ q_*, \quad i_* \circ u = \sqrt{e}(E) \circ p_*, \quad i_* \circ v = \sqrt{e}(E) \circ i_*.$$

Proof. Note that $\sqrt{e}(E)$ is a bivariant class by Proposition 4.1.15 and we have

$$\sqrt{e}(E|_{\widetilde{X}}) = \sqrt{e}(L^\perp/L) \circ e(L) = e(L) \circ \sqrt{e}(L^\perp/L) : A_*(\widetilde{X}) \to A_*(\widetilde{X})$$

by the reduction formula in Proposition 4.1.17. The first identity follows from

$$q_* \circ \sqrt{e}(L^\perp/L) \circ j^! \circ j_* = q_* \circ \sqrt{e}(L^\perp/L) \circ e(L) = q_* \circ \sqrt{e}(E) = \sqrt{e}(E) \circ q_*,$$

the second identity follows from

$$\begin{aligned} i_* \circ q_* \circ \sqrt{e}(L^\perp/L) \circ j^! &= p_* \circ j_* \circ \sqrt{e}(L^\perp/L) \circ j^! = p_* \circ j_* \circ j^! \circ \sqrt{e}(L^\perp/L) \\ &= p_* \circ e(L) \circ \sqrt{e}(L^\perp/L) = p_* \circ \sqrt{e}(E) = \sqrt{e}(E) \circ p_*, \end{aligned}$$

and the third identity follows from the bivariance of $\sqrt{e}(E)$. □

The localized square root Euler classes are bivariant classes. This localizes the results in Proposition 4.1.15.

Proposition 4.1.25 (Bivariance). *Let $f : X \to Y$ be a morphism of Deligne-Mumford stacks. Let E be a special orthogonal bundle on X and s be an isotropic section of E. Form a fiber diagram*

$$\begin{tikzcd} Y(s) \arrow[r, hook] \arrow[d, "f(s)"'] & Y \arrow[d, "f"] \\ X(s) \arrow[r, hook, "i"] & X. \end{tikzcd}$$

1. *If $f : Y \to X$ is a proper morphism, then*
$$\sqrt{e}(E,s) \circ f_* = f(s)_* \circ \sqrt{e}(E,s) : A_*(Y) \to A_*(X(s)).$$

2. *If $f : Y' \to Y$ is an equi-dimensional flat morphism, then*
$$(f(s))^* \circ \sqrt{e}(E,s) = \sqrt{e}(E,s) \circ f^* : A_*(X) \to A_*(Y(s)).$$

3. *If $f : Y \to X$ is a local complete intersection morphism, then*
$$f^! \circ \sqrt{e}(E,s) = \sqrt{e}(E,s) \circ f^! : A_*(X) \to A_*(Y(s)).$$

By abuse of notation, we denoted $\sqrt{e}(f^*E, f^*s)$ by $\sqrt{e}(E,s)$.

We provide localized versions of the *Whitney sum formula* in Proposition 4.1.16 and the *reduction formula* in Proposition 4.1.17.

Proposition 4.1.26 (Whitney sum formula). *Let E_1 and E_2 be special orthogonal bundles on a Deligne-Mumford stack X. Let s_1 and s_2 be isotropic sections of E_1 and E_2, respectively. Then we have*

$$\sqrt{e}((E_1 \oplus E_2),(s_1,s_2)) = \sqrt{e}(E_1,s_1) \circ \sqrt{e}(E_2,s_2) : A_*(X) \to A_*(X(s_1,s_2))$$

where $X(s_1,s_2) := X(s_1) \cap X(s_2)$ is the common zero locus of s_1 and s_2.

Proposition 4.1.27 (Reduction formula). *Let E be a special orthogonal bundle on a Deligne-Mumford stack X and K be an isotropic subbundle. Let s be an isotropic section of E such that $s \cdot K = 0$. Let $s_1 \in \Gamma(X, K^\perp)$ and $s_2 \in \Gamma(X(s_1), K|_{X(s_1)})$ be the induced sections. Then we have*

$$\sqrt{e}(E,s) = e(K|_{X(s_1)}, s_2) \circ \sqrt{e}(E_{/\!/K}, s_1) : A_*(X) \to A_*(X(s))$$

where $E_{/\!/K} := K^\perp/K$ is the reduction of E by K.

We have the following corollary which will be use frequently in §4.3.

Corollary 4.1.28. *Let X be a Deligne-Mumford stack, E be a special orthogonal bundle on X, and K be an isotropic subbundle. Let C be an isotropic subcone of the reduction $E_{/\!/K} := K^{\perp}/K$. Form a commutative diagram*

$$\begin{tikzcd} & \widetilde{C} \arrow[r, hook] \arrow[d, "r"] & K^{\perp} \arrow[r, hook] \arrow[d] & E \\ X \arrow[ur, hook, "0_{\widetilde{C}}"] \arrow[r, hook, "0_C"'] & C \arrow[r, hook] & E_{/\!/K} & \end{tikzcd}$$

where the square is cartesian. Then we have

$$\sqrt{e}(E_{/\!/K}|_C, \tau) = \sqrt{e}(E|_{\widetilde{C}}, \widetilde{\tau}) \circ r^* : A_*(C) \to A_*(X)$$

where $\tau \in \Gamma(C, E_{/\!/K}|_C)$ and $\widetilde{\tau} \in \Gamma(\widetilde{C}, E|_{\widetilde{C}})$ are the tautological sections.

The proofs of Proposition 4.1.25, Proposition 4.1.26, Proposition 4.1.27, and Corollary 4.1.28 are straightforward. We refer to [KP2] for the details.

Remark 4.1.29 (Uniqueness)**.** As the ordinary square root Euler classes $\sqrt{e}(E)$, the localized square root Euler classes $\sqrt{e}(E,s)$ are uniquely determined by the bivariance in Proposition 4.1.25, the reduction formula in Proposition 4.1.27, and the compatibility $i_* \circ \sqrt{e}(E,s) = \sqrt{e}(E)$.

Remark 4.1.30. Following Remark 4.1.20, everything in this subsection can be generalized to algebraic stacks which admit proper covers by quotient stacks (in the sense of Definition A.2.1).

We briefly review the original construction of $\sqrt{e}(E,s)$ in [OT].

Remark 4.1.31 (Oh-Thomas construction)**.** Let E be a special orthogonal bundle on a Deligne-Mumford stack X and s be an isotropic section of E.

- *Case 1.* Assume that E has a positive maximal isotropic subbundle M. Then we have a short exact sequence

  $$0 \longrightarrow M \hookrightarrow E \cong E^{\vee} \twoheadrightarrow M^{\vee} \longrightarrow 0.$$

  Let $s_1 \in \Gamma(X, M^{\vee})$ and $s_2 \in \Gamma(X_1, M|_{X_1})$ be the induced sections. Let $X_1 \subseteq X$ be the zero locus of s_1 in X. Then the zero locus $X(s)$ of s in X is the zero locus of s_2 in X_1. Since $s \in \Gamma(X,E)$ is isotropic, we have

  $$C_{X_1/X} \subseteq M|^{\vee}_{X_1}(s_2^{\vee})$$

as subcones of $M|^{\vee}_{X_1}$ where $M|^{\vee}_{X_1}(s_2^{\vee}) := M|^{\vee}_{X_1} \times_{s_2^{\vee},\mathbb{A}^1,0} X$ is the kernel cone. We define

$$\sqrt{e}(E, s, M)^{OT} : A_*(X) \to A_*(X(s))$$

as the composition

$$A_*(X) \xrightarrow{\mathrm{sp}_{X_1/X}} A_*(C_{X_1/X}) \to A_*(M|^{\vee}_{X_1}(s_2^{\vee})) \xrightarrow{0^!_{M|^{\vee}_{X_1},s_2^{\vee}}} A_*(X_1(s_2))$$

where $0^!_{M|^{\vee}_{X_1},s_2^{\vee}}$ is the cosection-localized Gysin map.

- *Case 2.* Consider the general case. Let $\pi : \mathsf{F} := \mathsf{Flag}(E) \to X$ be the isotropic flag bundle of E. Then there exists a canonical operational class $h \in A^*(\mathsf{F})$ such that

  $$\alpha = \pi_*(h \cap \pi^*\alpha) \in A_*(X)$$

  for all $\alpha \in A_*(X)$. Let Λ be the canonical positive maximal isotropic subbundle of $E|_{\mathsf{F}}$. We define

  $$\sqrt{e}(E, s)^{OT} : A_*(X) \to A_*(X(s))$$

  as the composition

  $$A_*(X) \xrightarrow{\pi^*} A_*(\mathsf{F}) \xrightarrow{\sqrt{e}(\pi^*E,\pi^*s,\Lambda)^{OT}} A_*(\mathsf{F}(\pi^*s)) \xrightarrow{\pi(s)_*} A_*(X(s))$$

  where $\pi(s) : \mathsf{F}(\pi^*s) \to X(s)$ is the restriction of $\pi : \mathsf{F} \to X$.

The two definitions of localized square root Euler class coincide,

$$\sqrt{e}(E, s) = \sqrt{e}(E, s)^{OT}.$$

We refer to [KP2, Thm. 5.2] for the proof of the comparison.

4.2 Symmetric obstruction theories

In this section, we fix the notion of *symmetric obstruction theories*. They are the necessary additional data to define Oh-Thomas virtual cycles (or more generally, square root virtual pullbacks). Compared to the ordinary virtual cycles (or virtual pullbacks) associated to perfect obstruction theories in Chapter 2, a new feature is that we now need an additional data, an *orientation* and an additional property, the *isotropic condition*.

4.2.1 Symmetric complexes

Definition 4.2.1 (Symmetric complex)**.** Let X be an algebraic stack. We say that a pair $(\mathbb{E}, \theta)$ is a *symmetric complex* on X if

1. $\mathbb{E}$ is a perfect complex on X of tor-amplitude $[-2, 0]$, and

2. θ is a (-2)-shifted non-degenerate symmetric form, i.e., a morphism

$$\theta : O_X \to (\mathbb{E} \otimes \mathbb{E})[-2]$$

satisfying the following properties:

(a) $\sigma[-2] \circ \theta = \theta$ for the transition map $\sigma : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E} \otimes \mathbb{E}$;

(b) the induced map $\mathbb{E}^\vee[2] \to \mathbb{E}$ is an isomorphism.

By abuse of notation, we say that $\mathbb{E}$ is a symmetric complex on X.

Remark 4.2.2. The notion of symmetric complexes in Definition 4.2.1 can be generalized to *d-shifted symmetric complexes of tor-amplitude* $[a, b]$ in a straightforward manner.

In this paper, we will always assume that symmetric complexes are (-2)-shifted symmetric and of tor-amplitude $[-2, 0]$, unless stated otherwise.

Three operations We note that there are three basic operations for symmetric complexes:

1. The *direct sum* $\mathbb{E}_1 \oplus \mathbb{E}_2$ of two symmetric complexes $\mathbb{E}_1$ and $\mathbb{E}_2$;

2. The *orthogonal complement* $\mathbb{E}_{/\mathbb{F}}$ of a non-degenerate subcomplex $\mathbb{F}$ of a symmetric complex $\mathbb{E}$;

3. The *reduction* $\mathbb{E}_{/\!/\mathbb{K}}$ of a symmetric complex $\mathbb{E}$ by an isotropic subcomplex $\mathbb{K}$.

These operations are analogous to the basic operations of orthogonal bundles in Example 4.1.5. We now explain how to define these operations. The direct sum operation is obvious. To define the other two operations, we introduce the notions of *non-degenerate subcomplexes* and *isotropic subcomplexes*.

Definition 4.2.3 (Non-degenerate subcomplex)**.** Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. We say that a perfect complex $\mathbb{F}$ on X is a *non-degenerate subcomplex* of $\mathbb{E}$ with respect to $\epsilon : \mathbb{E} \to \mathbb{F}$ if the square

$$\epsilon^2 : \mathbb{F}^\vee[2] \xrightarrow{\epsilon^\vee[2]} \mathbb{E}^\vee[2] \xrightarrow{\theta} \mathbb{E} \xrightarrow{\epsilon} \mathbb{F}$$

is an isomorphism.

Definition 4.2.4 (Isotropic subcomplex)**.** Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. We say that a perfect complex $\mathbb{K}$ on X is an *isotropic subcomplex* of $\mathbb{E}$ with respect to $\delta : \mathbb{E} \to \mathbb{K}$ if the square

$$\delta^2 : \mathbb{K}^\vee[2] \xrightarrow{\delta^\vee[2]} \mathbb{E}^\vee[2] \xrightarrow{\theta} \mathbb{E} \xrightarrow{\delta} \mathbb{K}$$

is zero and $\mathbb{K}$ is of tor-amplitude $[-1,0]$.

We define the notions of *orthogonal complements* $\mathbb{E}_{/\mathbb{F}}$ and *reductions* $\mathbb{E}_{/\!/\mathbb{K}}$ via the following propositions.

Proposition 4.2.5 (Orthogonal complement)**.** *Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. Let $\mathbb{F}$ be a non-degenerated subcomplex of $\mathbb{E}$ with respect to $\epsilon : \mathbb{E} \to \mathbb{F}$. Note that $\mathbb{F}$ is a symmetric complex with the induced symmetric form*

$$\epsilon_*(\theta) : O_X \xrightarrow{\theta} (\mathbb{E} \otimes \mathbb{E})[-2] \xrightarrow{(\epsilon\otimes\epsilon)[-2]} (\mathbb{F} \otimes \mathbb{F})[-2].$$

Then there exists a unique symmetric complex $\mathbb{E}_{/\mathbb{F}}$ that fits into an isomorphism of symmetric complexes

$$\mathbb{E} \cong (\mathbb{F} \oplus \mathbb{E}_{/\mathbb{F}})$$

where $\epsilon : \mathbb{E} \to \mathbb{F}$ corresponds to the projection $(1,0) : (\mathbb{F} \oplus \mathbb{E}_{/\mathbb{F}}) \to \mathbb{F}$.

We omit the proof of Proposition 4.2.5 since it is straightforward.

Proposition 4.2.6 (Reduction)**.** *Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. Let $\mathbb{K}$ be an isotropic subcomplex of $\mathbb{E}$ with respect to $\delta : \mathbb{E} \to \mathbb{K}$. Then there exists a unique symmetric complex $\mathbb{E}_{/\!/\mathbb{K}}$ that fits into a morphism of distinguished triangles*

$$\begin{tikzcd}
\mathbb{D}^\vee[2] \arrow[r, "\alpha^\vee"] \arrow[d, "\beta^\vee"] & \mathbb{E} \arrow[r, "\delta"] \arrow[d, "\alpha"] & \mathbb{K} \arrow[r] \arrow[d, equal] & {} \\
\mathbb{E}_{/\!/\mathbb{K}} \arrow[r, "\beta"] & \mathbb{D} \arrow[r] & \mathbb{K} \arrow[r] & {}
\end{tikzcd}$$

for some $\mathbb{D}$, α, β, where $\alpha^\vee$, $\beta^\vee$ are the duals of α, β with respect to the identifications $\mathbb{E}^\vee[2] \cong \mathbb{E}$ and $(\mathbb{E}_{/\!/\mathbb{K}})^\vee[2] \cong \mathbb{E}_{/\!/\mathbb{K}}$.

Unlike Proposition 4.2.5, proving Proposition 4.2.6 is quite difficult than it seems. We refer to [Park1, Appendix C] for the proof.

Definition 4.2.7 (Symmetric resolution)**.** Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. A *symmetric resolution* of $\mathbb{E}$ is an isomorphism

$$[B \to E^\vee \to B^\vee] \xrightarrow{\cong} \mathbb{E} \tag{4.2.1}$$

for some orthogonal bundle E and a vector bundle B such that the symmetric form of $\mathbb{E}$ is represented by the chain map

$$\begin{tikzcd} \mathbb{E}^\vee \arrow[d, "\theta"] & B \arrow[r, "d"] \arrow[d, equal] & E \arrow[r, "d^\vee \circ q"] \arrow[d, "q"] & B^\vee \arrow[d, equal] \\ \mathbb{E}[-2] & B \arrow[r, "q\circ d"] & E^\vee \arrow[r, "d^\vee"] & B^\vee \end{tikzcd}$$

where $q : E \to E^\vee$ is the symmetric form of E.

We observe that the symmetric resolutions are special cases of the *reduction* operation in Proposition 4.2.6.

Lemma 4.2.8. *Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X and let $[B \xrightarrow{d} E^\vee \xrightarrow{d^\vee} B^\vee] \cong \mathbb{E}$ be a symmetric resolution. Then the symmetric complex $\mathbb{E}$ is the reduction of the symmetric complex $E^\vee[1]$ by the isotropic subcomplex $B^\vee[1]$ with respect to $d^\vee[1] : E^\vee[1] \to B^\vee[1]$,*

$$\mathbb{E} \cong E^\vee[1]_{/\!/B^\vee[1]}.$$

Proof. This is immediate from the definitions. □

Proposition 4.2.9. *Let X be an algebraic stack with the resolution property. Then every symmetric complex on X has a symmetric resolution.*

Sketch of the proof. Let $\mathbb{E}$ be a symmetric complex on X. By the resolution property, there exists a resolution

$$\mathbb{E}^\vee \cong [A^{-2} \to A^{-1} \to A^0]$$

for some vector bundles A^{-2}, A^{-1}, and A^0 such that the symmetric form of $\mathbb{E}$ is represented by a self-dual map

$$\begin{tikzcd} \mathbb{E}^\vee[2] \arrow[d, "\theta"] & A^{-2} \arrow[r] \arrow[d] & A^{-1} \arrow[r] \arrow[d] & A^0 \arrow[d] \\ \mathbb{E} & (A^0)^\vee \arrow[r] & (A^{-1})^\vee \arrow[r] & (A^{-2})^\vee. \end{tikzcd}$$

Consider the induced orthogonal bundle

$$E := \operatorname{coker}(A^{-2} \to A^{-1} \oplus (A^0)^\vee) \cong \ker((A^{-1})^\vee \oplus A^0 \to (A^{-2})^\vee) = E^\vee.$$

Then we have a symmetric resolution

$$\mathbb{E} \cong [(A^0)^\vee \to E \cong E^\vee \to A^0]$$

as desired. We refer to [OT, Prop. 4.1] for the details. □

Lemma 4.2.10. *Let X be an algebraic stack with the resolution property. Let $\mathbb{E}$ be a symmetric complex on X and $\delta : \mathbb{E} \to \mathbb{K}$ be a map to a perfect complex $\mathbb{K}$ such that $h^0(\delta)$ is surjective. Then there exists a symmetric resolution*

$$[B \to E^\vee \to B^\vee] \xrightarrow{\cong} \mathbb{E}$$

for some orthogonal bundle E and a vector bundle B and a resolution

$$[K^\vee \to D^\vee] \xrightarrow{\cong} \mathbb{K}$$

for some vector bundles K and D such that the map δ is represented by a surjective chain map

$$\begin{array}{ccc} \mathbb{E} & & B \longrightarrow E^\vee \longrightarrow B^\vee \\ \big\downarrow \delta & & \big\downarrow \qquad \big\downarrow \qquad \big\downarrow \\ \mathbb{K} & & 0 \longrightarrow K^\vee \longrightarrow D^\vee. \end{array}$$

Proof. It is easy to show the statement from the proof of Proposition 4.2.9. As Proposition 4.2.9, we refer to [OT, Prop. 4.1] for the details. □

Corollary 4.2.11. *Let $X = [P/G]$ be the quotient stack of a quasi-projective scheme P by a linear action of a linear algebraic group G. Then any symmetric complex on X has a symmetric resolution.*

Proof. The quotient stack $X = [P/G]$ has the resolution property by [Tho1, Lem. 2.6]. Hence Proposition 4.2.9 completes the proof. □

Definition 4.2.12 (Orientation)**.** Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. An *orientation* of $\mathbb{E}$ is an isomorphism of line bundles

$$o : \mathcal{O}_X \xrightarrow{\cong} \det(\mathbb{E})$$

such that

$$o^2 = 1 : \mathcal{O}_X \xrightarrow{o} \det(\mathbb{E}) \cong \det(\mathbb{E}^\vee) \xrightarrow{o^\vee} \mathcal{O}_X$$

where the second isomorphism is given by the symmetric form of $\mathbb{E}$.

As in Remark 4.1.7, we have an *orientation bundle* of a symmetric complex as a μ_2-torsor.

Remark 4.2.13 (Orientation bundle)**.** Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. We define the *orientation bundle* of $\mathbb{E}$ as the functor

$$\mathsf{Or}(\mathbb{E})\mathsf{Sch}_{/X}^{\mathrm{op}} \to \mathsf{Set} : (T \to X) \mapsto \{\text{orientations of } \mathbb{E}|_T\}.$$

Then the orientation bundle $\mathsf{Or}(\mathbb{E})$ fits into the fiber diagram

$$\begin{tikzcd} \mathsf{Or}(\mathbb{E}) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{C}) \arrow[d] \\ X \arrow[r, "\det(\mathbb{E})"] & B\mu_2 \end{tikzcd}$$

where the map $X \to B\mu_2$ is given by the line bundle $\det(\mathbb{E})$ and the isomorphism $\mathcal{O}_X \cong \det(\mathbb{E})^{\otimes 2}$ induced by the symmetric form of $\mathbb{E}$. Hence the orientation bundle $\mathsf{Or}(\mathbb{E}) \to X$ is a principal μ_2-bundle. In particular, the symmetric complex $\mathbb{E}$ is étale-locally orientable since the pullback $\mathbb{E}|_{\mathsf{Or}(\mathbb{E})}$ has a canonical orientation.

Remark 4.2.14 (Induced orientations)**.** Let X be an algebraic stack.

1. Let $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$ be the direct sum of two symmetric complexes. Then we have a canonical isomorphism of line bundles

   $$\det(\mathbb{E}) \cong \det(\mathbb{E}_1) \otimes \det(\mathbb{E}_2).$$

   Hence orientations of $\mathbb{E}_1$ and $\mathbb{E}_2$ induce an orientation of $\mathbb{E}$.

2. Let $\mathbb{E}_{/\mathbb{F}}$ be the orthogonal complement of a non-degenerate subcomplex $\mathbb{F}$ of a symmetric complex $\mathbb{E}$ (in the sense of Proposition 4.2.5). Then we have a canonical isomorphism of line bundles

   $$\det(\mathbb{E}_{/\mathbb{F}}) \cong \det(\mathbb{E}) \otimes \det(\mathbb{F})^{\vee}.$$

   Hence orientations of $\mathbb{E}$ and $\mathbb{F}$ induce an orientation of $\mathbb{E}_{/\mathbb{F}}$.

3. Let $\mathbb{E}_{/\!/\mathbb{K}}$ be the reduction of an isotropic subcomplex $\mathbb{K}$ of a symmetric complex $\mathbb{E}$ (in the sense of Proposition 4.2.6). Based on the notations in Proposition 4.2.6, we have a canonical isomorphism of line bundles

   $$\begin{aligned} \det(\mathbb{E}_{/\!/\mathbb{K}}) &\cong \det(\mathbb{D}) \otimes \det(\mathbb{K})^{\vee} \\ &\cong \det(\mathbb{E}) \otimes \det(\mathbb{K}^{\vee}[2])^{\vee} \otimes \det(\mathbb{K})^{\vee} \cong \det(\mathbb{E}). \end{aligned}$$

   Hence an orientation of $\mathbb{E}$ induces an orientation of $\mathbb{E}_{/\!/\mathbb{K}}$.

4.2.2 Quadratic cone stacks

In the DT4 theory, the *quadratic cone stacks* associated to symmetric complexes play the role of the vector bundle stacks in virtual intersection theory.

We first define the *canonical quadratic function* on the abelian cone stack associated to a symmetric complex.

Proposition 4.2.15 (Canonical quadratic function). *For each symmetric complex $\mathbb{E}$ on an algebraic stack X, there exists a canonical function*

$$\mathfrak{q}_{\mathbb{E}} : \mathfrak{C}(\mathbb{E}) \to \mathbb{A}^1_X$$

on the associated abelian cone stack $\mathfrak{C}(\mathbb{E})$ satisfying the following properties:

1. *If $\mathbb{E} = E[1]$ for an orthogonal bundle E, then*

$$\mathfrak{q}_{\mathbb{E}} = \mathfrak{q}_E : \mathfrak{C}(\mathbb{E}) = E \to \mathbb{A}^1_X,$$

is the cannonical quadratic function on the orthogonal bundle E.

2. *For any morphism $f : Y \to X$ of algebraic stacks, we have*

$$f^*\mathfrak{q}_{\mathbb{E}} = \mathfrak{q}_{f^*\mathbb{E}} : f^*\mathfrak{C}(\mathbb{E}) = \mathfrak{C}(f^*\mathbb{E}) \to \mathbb{A}^1_Y.$$

3. *If $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$ is the direct sum of two symmetric complexes $\mathbb{E}_1$ and $\mathbb{E}_2$, then we have*

$$\mathfrak{q}_{\mathbb{E}} = \mathfrak{q}_{\mathbb{E}_1} \circ p_1 + \mathfrak{q}_{\mathbb{E}_2} \circ p_2 : \mathfrak{C}(\mathbb{E}) = \mathfrak{C}(\mathbb{E}_1) \times \mathfrak{C}(\mathbb{E}_2) \to \mathbb{A}^1_X$$

where p_1 and p_2 denote the projection maps.

4. *If $\mathbb{E}_{/\!/\mathbb{K}}$ is the reduction of $\mathbb{E}$ by an isotropic subcomplex $\mathbb{K}$, then the diagram*

$$\begin{tikzcd}
\mathfrak{C}(\mathbb{D}) \arrow[r, "\mathfrak{C}_\alpha"] \arrow[d, "\mathfrak{C}_\beta"'] & \mathfrak{C}(\mathbb{E}) \arrow[d, "\mathfrak{q}_{\mathbb{E}}"] \\
\mathfrak{C}(\mathbb{E}_{/\!/\mathbb{K}}) \arrow[r, "\mathfrak{q}_{\mathbb{E}_{/\!/\mathbb{K}}}"'] & \mathbb{A}^1_X
\end{tikzcd}$$

commutes, where α, β, $\mathbb{D}$ are given as in Proposition 4.2.6 above.

Moreover, the functions $\mathfrak{q}_{\mathbb{E}}$ are uniquely determined by the above properties.

We can explicitly describe the canonical quadratic function $\mathfrak{q}_{\mathbb{E}}$ as Example 4.2.16 below, when the symmetric complex $\mathbb{E}$ has a symmetric resolution. We refer to [Park1, Prop. 1.7] for the proof of Proposition 4.2.15 in the general case.

Example 4.2.16. Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. If there exists a symmetric resolution $\mathbb{E} \cong [B \to E \to B^\vee]$, then we have $\mathfrak{C}(\mathbb{E}) = [C(D)/B]$, where $D := \mathrm{coker}(B \to E)$. In this case, the restriction

$$\mathfrak{q}_E|_{C(D)} : C(D) \hookrightarrow E \to \mathbb{A}^1_X$$

is B-invariant and it descends to the canonical quadratic function

$$\mathfrak{q}_{\mathbb{E}} : \mathfrak{C}(\mathbb{E}) = [C(D)/B] \to \mathbb{A}^1_X.$$

There is a simple description of the canonical quadratic function $\mathfrak{q}_{\mathbb{E}}$ using derived algebraic geometry.

Remark 4.2.17 (Derived interpretation)**.** Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. In Remark 2.1.3, we observed that

$$\mathfrak{C}(\mathbb{E}) = \mathrm{Tot}(\mathbb{E}^\vee[1]) := \mathrm{Spec}(\mathrm{Sym}^\bullet(\mathbb{E}[-1]))_{\mathrm{cl}}.$$

Then the symmetric form $\theta \in \mathrm{Sym}^2(\mathbb{E}[-1])$ defines a map

$$\mathrm{Tot}(\mathbb{E}^\vee[1])) \to \mathbb{A}^1_X.$$

The canonical quadratic function $\mathfrak{q}_{\mathbb{E}}$ in Proposition 4.2.15 is the restriction of the above function to the classical truncation $\mathfrak{C}(\mathbb{E})$.

Remark 4.2.18. We note that the canonical quadratic function $\mathfrak{q}_{\mathbb{E}}$ can be generalized to perfect complexes $\mathbb{E}$ with *degenerate* symmetric bilinear forms in a straightforward manner.

We finally define our main object in this subsection, the *quadratic cone stacks*, as follows.

Definition 4.2.19 (Quadratic cone stack)**.** Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. We define the *quadratic cone stack* associated to $\mathbb{E}$ as the subcone stack

$$\mathfrak{Q}(\mathbb{E}) \subseteq \mathfrak{C}(\mathbb{E})$$

defined as the zero locus of the canonical quadratic function $\mathfrak{q}_{\mathbb{E}}$ in Proposition 4.2.15. Equivalently, we have a fiber diagram

$$\begin{tikzcd} \mathfrak{Q}(\mathbb{E}) \arrow[r] \arrow[d, hook] & X \arrow[d, hook, "0"] \\ \mathfrak{C}(\mathbb{E}) \arrow[r, "\mathfrak{q}_{\mathbb{E}}"] & \mathbb{A}^1_X \end{tikzcd}$$

of cone stacks.

4.2.3 Symmetric obstruction theories

In DT4 theory, the *symmetric obstruction theories* satisfying the *isotropic condition* play the role of perfect obstruction theories in virtual intersection theory.

Definition 4.2.20 (Symmetric obstruction theory)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks. We say that $\phi : \mathbb{E} \to L_{X/Y}$ is a *symmetric obstruction theory* for $f : X \to Y$ if

1. $\mathbb{E}$ is a symmetric complex on X, and
2. $\phi : \mathbb{E} \to L_{X/Y}$ is an obstruction theory for $f : X \to Y$.

An important new ingredient in DT4 theory is the *isotropic condition*.

Definition 4.2.21 (Isotropic condition)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks. We say that a symmetric obstruction theory $\phi : \mathbb{E} \to L_{X/Y}$ satisfies the *isotropic condition* if the composition

$$\mathfrak{C}_{X/Y} \xrightarrow{\iota} \mathfrak{C}(\mathbb{E}) \xrightarrow{\mathfrak{q}_{\mathbb{E}}} \mathbb{A}^1_X$$

vanishes, where $\iota : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{C}(\mathbb{E})$ is the closed embedding induced by ϕ and $\mathfrak{q}_{\mathbb{E}} : \mathfrak{C}(\mathbb{E}) \to \mathbb{A}^1_X$ is the canonical quadratic function.

We provide a technical generalization of the isotropic condition.

Remark 4.2.22 (Weak isotropic condition)**.** In the situation of Definition 4.2.21, we say that the symmetric obstruction theory $\phi : \mathbb{E} \to L_{X/Y}$ satisfies the *weak isotropic condition* if the composition

$$(\mathfrak{C}_{X/Y})_{\mathrm{red}} \hookrightarrow \mathfrak{C}_{X/Y} \xrightarrow{\iota} \mathfrak{C}(\mathbb{E}) \xrightarrow{\mathfrak{q}_{\mathbb{E}}} \mathbb{A}^1_X$$

vanishes, where $(\mathfrak{C}_{X/Y})_{\mathrm{red}} \subseteq \mathfrak{C}_{X/Y}$ is the reduced closed substack of the intrinsic normal cone $\mathfrak{C}_{X/Y}$. This weak isotropic condition is sufficient to define the square root virtual pullbacks in the next section.

We observe that a symmetric obstruction theory satisfying the isotropic condition is equivalent to a closed embedding of the intrinsic normal cones into a quadratic cone stack. This is a DT4 analog of Proposition 2.2.9.

Proposition 4.2.23. *Let $f : X \to Y$ be a DM morphism of algebraic stacks.*

1. *If $\phi : \mathbb{E} \to L_{X/Y}$ is a symmetric obstruction theory satisfying the isotropic condition, then there is a unique closed embedding*

$$\mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{Q}(\mathbb{E})$$

of cone stacks that fits into the commutative diagram

$$\begin{tikzcd} & \mathfrak{Q}(\mathbb{E}) \arrow[r] \arrow[d, hook] & X \arrow[d, hook, "0"] \\ \mathfrak{C}_{X/Y} \arrow[r, hook, "\iota"] \arrow[ur, dotted] & \mathfrak{C}(\mathbb{E}) \arrow[r, "\mathfrak{q}_{\mathbb{E}}"] & \mathbb{A}^1_X \end{tikzcd}$$

as the dotted arrow.

2. *If $\mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{Q}(\mathbb{E})$ is a closed embedding of cone stacks for some quadratic cone stack associated to a symmetric complex $\mathbb{E}$, then the composition*

$$\mathbb{E} \to \tau^{\geqslant -1}\mathbb{E} \cong L_{X/\mathfrak{C}(\mathbb{E})} \to L_{X/\mathfrak{Q}(\mathbb{E})} \to L_{X/\mathfrak{C}_{X/Y}} \cong L_{X/Y}$$

is a symmetric obstruction theory satisfying the isotropic condition.

Moreover, the above two operations are inverse to each others.

Proof. If follows immediately from Proposition 2.2.9 and Definition 4.2.21. □

Proposition 4.2.24 (Criterion for isotropic condition)**.** *Let $f : X \to Y$ be a DM morphism of algebraic stacks and let $\phi : \mathbb{E} \to L_{X/Y}$ be a symmetric obstruction theory. Assume that there is a factorization of $f : X \to Y$ as*

$$\begin{tikzcd} & \widetilde{Y} \arrow[d, "\overline{f}"] \\ X \arrow[ur, hook, "\widetilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd}$$

with a closed embedding $\widetilde{f}$ and a smooth morphism $\overline{f}$, and a symmetric resolution of $\mathbb{E}$ such that $\phi : \mathbb{E} \to L_f$ is represented by a surjective chain map

$$\begin{tikzcd} \mathbb{E} \arrow[d, "\phi"] & B \arrow[r, "q\circ d"] \arrow[d] & E^\vee \arrow[r, "d^\vee"] \arrow[d, "\widetilde{\phi}"] & B^\vee \arrow[d] \\ L_f & 0 \arrow[r] & \mathcal{I}/\mathcal{I}^2 \arrow[r] & \Omega_{\overline{f}}|_X \end{tikzcd}$$

where $\mathcal{I} := \mathcal{I}_{X/\widetilde{Y}}$ is the ideal sheaf. Then ϕ satisfies the isotropic condition if and only if the induced symmetric obstruction theory

$$\widetilde{\phi} : E^\vee[1] \longrightarrow \mathcal{I}/\mathcal{I}^2[1] \cong L_{X/\widetilde{Y}}$$

satisfies the isotropic condition.

Proof. Consider the commutative diagram

$$\begin{tikzcd} C_{\widetilde{f}} \arrow[r] \arrow[dr, "r"'] & C \arrow[r, hook] \arrow[d] & C(Q) \arrow[r, hook] \arrow[d] & E \arrow[d] \\ & \mathfrak{C}_f \arrow[r, hook] & \mathfrak{C}(\mathbb{E}) \arrow[r, hook] & {[E/B]}. \end{tikzcd}$$

where the squares are cartesian, the horizontal arrows are closed embeddings, and the vertical arrows are smooth morphisms. Here $Q := \mathrm{coker}(B \to E)$. By Lemma 4.2.8, the symmetric complex $\mathbb{E}$ is the reduction of the symmetric complex $E^\vee[1]$ by an isotropic subcomplex $B^\vee[1]$. Hence by Proposition 4.2.15.4, we have

$$r^*(\mathfrak{q}_{\mathbb{E}}|_{\mathfrak{C}_f}) = \mathfrak{q}_E|_{C_{\widetilde{f}}}.$$

Since the projection map $r : C_{\widetilde{f}} \to \mathfrak{C}_f$ is smooth and surjective, the two isotropic conditions are equivalent. $\square$

Example 4.2.25 ((-2)-shifted symplectic derived schemes)**.** Let $\mathbb{X}$ be a derived DM stack with a (-2)-shifted symplectic structure ω. Let $X := \mathbb{X}_{\mathrm{cl}}$ be the classical truncation. Then the canonical map

$$\phi : \mathbb{L}_{\mathbb{X}}|_X \to \mathbb{L}_X \to L_X$$

is a symmetric obstruction theory by [STV, Prop 1.2]. Here the symmetric form of $\mathbb{L}_{\mathbb{X}}|_X$ is induced by the underlying (-2)-shifted 2-form $\omega_0 : \mathcal{O}_{\mathbb{X}} \to \wedge^2\mathbb{L}_{\mathbb{X}}[-2]$.

The isotropic condition follows by the Darboux theorem [BBJ, BG]. Indeed, since the isotropic condition can be shown locally, we may assume that X is the zero locus of an isotropic section s of an orthogonal bundle E over a smooth scheme U. Moreover, the symmetric obstruction theory ϕ can be written as

$$\begin{tikzcd}
\mathbb{L}_{\mathbb{X}}|_X \arrow[d, "\phi"] & T_U|_X \arrow[r, "ds"] \arrow[d] & E|_X^\vee \arrow[r, "ds"] \arrow[d, "s"] & \Omega_U|_X \arrow[d, equal] \\
L_X & 0 \arrow[r] & \mathcal{I}_{X/U}/\mathcal{I}_{X/U}^2 \arrow[r, "d"] & \Omega_U|_X.
\end{tikzcd}$$

By the criterion in Proposition 4.2.24, it suffices to show that the normal cone

$$C_{X/U} \subseteq E|_X$$

is isotropic. By MacPherson's graph construction [Ful, Rem. 5.1.1], we have

$$C_{X/U} = \lim_{t\to\infty} \Gamma_{t\cdot s}$$

where $\Gamma_{t\cdot s} \subseteq E$ is the image of the section $t\cdot s : U \hookrightarrow E$. Since the section s is isotropic, the cone $C_{X/U}$ is also isotropic.

4.3 Square root virtual pullbacks

In this section, we construct *square root virtual pullbacks* for symmetric obstruction theories, based on [Park1]. The Oh-Thomas virtual cycles [OT] will be defined defined as a special case of square root virtual pullbacks.

4.3.1 Square root Gysin pullbacks

Definition 4.3.1 (Square root Gysin pullback)**.** Let X be a separated Deligne-Mumford stack. Let $\mathfrak{Q}(\mathbb{E})$ be the quadratic cone stack associated to a symmetric complex $\mathbb{E}$ on X. Choose an orientation $o : \mathcal{O}_X \xrightarrow{\cong} \det(\mathbb{E})$. We define the *square root Gysin pullback*

$$\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}} : A_*(\mathfrak{Q}(\mathbb{E})) \to A_*(X)$$

as follows:

Case 1. Assume that X is a quasi-projecitve scheme. By Proposition 4.2.9, we have a symmetric resolution $\mathbb{E} \cong [B \to E^\vee \to B^\vee]$ for some special orthogonal bundle E and a vector bundle B. Form a fiber diagram

$$\begin{tikzcd} Q \arrow[rr, hook] \arrow[d, "r"] & & E \arrow[d] \\ \mathfrak{Q}(\mathbb{E}) \arrow[r, hook] & \mathfrak{C}(\mathbb{E}) \arrow[r, hook] & {[E/B]} \end{tikzcd}$$

where the closed embedding $\mathfrak{C}(\mathbb{E}) \hookrightarrow [E/B]$ is given by the stupid truncation

$$[0 \to E^\vee \to B^\vee] \to [B \to E^\vee \to B^\vee] \cong \mathbb{E}$$

Consider the factorization of the zero section $0_{\mathfrak{Q}(\mathbb{E})}$ as

$$\begin{tikzcd} & E|_Q \arrow[d, shift right] \\ X \arrow[r, hook, "0_Q"] \arrow[dr, "0_{\mathfrak{Q}(\mathbb{E})}"'] & Q \arrow[u, bend right, "\tau"'] \arrow[d, "r"] \\ & \mathfrak{Q}(\mathbb{E}) \end{tikzcd}$$

where 0_Q is the zero section of Q and τ is the tautological section. We define the square root Gysin pullback as the composition

$$\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}} : A_*(\mathfrak{Q}(\mathbb{E})) \xrightarrow{r^*} A_*(Q) \xrightarrow{\sqrt{e}(E|_Q,\tau)} A_*(X)$$

where $\sqrt{e}(E|_Q, \tau)$ is the localized square root Euler class in Definition 4.1.23.

Case 2. Assume that X is a separated Deligne-Mumford stack. By the Chow lemma [LMB, Cor. 16.6.1], there exists a projective surjective map $p : \widetilde{X} \to X$ from a quasi-projective scheme X. We define $\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}$ as the unique map that fits into the commutative diagram

$$\begin{tikzcd} A_*(\mathfrak{Q}(\mathbb{E}|_{\widetilde{X}\times_X\widetilde{X}})) \arrow[r] \arrow[d, "\sqrt{0^!_{\mathfrak{Q}(\mathbb{E}|_{\widetilde{X}\times_X\widetilde{X}})}}"] & A_*(\mathfrak{Q}(\mathbb{E}|_{\widetilde{X}})) \arrow[r] \arrow[d, "\sqrt{0^!_{\mathfrak{Q}(\mathbb{E}|_{\widetilde{X}})}}"] & A_*(\mathfrak{Q}(\mathbb{E})) \arrow[r] \arrow[d, dotted, "\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}"] & 0 \\ A_*(\widetilde{X}\times_X\widetilde{X}) \arrow[r] & A_*(\widetilde{X}) \arrow[r, "p_*"] & A_*(X) \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact by the Kimura sequence in Theorem A.1.1.

Lemma 4.3.2 (Well-definedness)**.** *In the situation of Definition 4.3.1, the square root Gysin pullback* $\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}$ *is well-defined, i.e.,*

1. *In Case 1,* $\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}$ *is independent of the choice of a symmetric resolution.*
2. *In Case 2,* $\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}$ *is independent of the choice of a projective cover.*

Proof. 1. We will only sketch the proof and refer to [OT] for details. Let

$$[B_1 \to E_1^\vee \to B_1^\vee] \cong \mathbb{E} \cong [B_2 \to E_2^\vee \to B_2^\vee]$$

be two symmetric resolutions. By a deformation argument, it suffices to consider the following special case: there exists a surjective chain map

$$\begin{tikzcd} B_1 \arrow[r] & E_1^\vee \arrow[r] \arrow[d] & B_1^\vee \arrow[d] \\ & K^\vee \arrow[r, equal] & K^\vee \end{tikzcd}$$

for some vector bundles such that there is an isomorphism of chain complexes

$$[(B_1/K) \to (K^\perp/K)^\vee \to (B_1/K)^\vee] \cong [B_2 \to E_2^\vee \to B_2^\vee].$$

Then we can form a commutative diagram

$$\begin{tikzcd} & Q_1 \arrow[r, hook] \arrow[d, two heads, "s"] & K^\perp \arrow[r, hook] \arrow[d, two heads] & E \arrow[dd] \\ X \arrow[ur, hook, "0_{Q_1}"] \arrow[r, hook, "0_{Q_2}"] \arrow[dr, "0_{\mathfrak{Q}(\mathbb{E})}"'] & Q_2 \arrow[r, hook] \arrow[d, "r_2"] & K^\perp/K \arrow[d] & \\ & \mathfrak{Q}(\mathbb{E}) \arrow[r, hook] & {[(K^\perp/K)/(B/K)]} \arrow[r, hook] & {[E/B]} \end{tikzcd}$$

where the squares are cartesian, the horizontal arrows are closed embeddings, and the vertical arrows are smooth morphisms. By Corollary 4.1.28, we have

$$\sqrt{e}(K^\perp/K|_{Q_2}, \tau_2) = \sqrt{e}(E|_{Q_1}, \tau_1) \circ s^* : A_*(Q_2) \to A_*(X)$$

where $\tau_1 \in \Gamma(Q_1, E|_{Q_1})$ and $\tau_2 \in \Gamma(Q_2, K^\perp/K|_{Q_2})$ are the tautological section. Consequently, we have the desired equality

$$\sqrt{e}(K^\perp/K|_{Q_2}, \tau_2) \circ r_2^* = \sqrt{e}(E|_{Q_1}, \tau_1) \circ r_1^* : A_*(\mathfrak{Q}(\mathbb{E})) \to A_*(X)$$

where $r_1 := r_2 \circ s$.

2. Let $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ be two projective surjective maps from quasi-projective schemes $\widetilde{X}_1$ and $\widetilde{X}_2$. By replacing $\widetilde{X}_1$ by $\widetilde{X}_1 \times_X \widetilde{X}_2$, we may assume that there exists a factorization

$$\begin{tikzcd}
\widetilde{X}_1 \arrow[r, dotted, "q"] \arrow[dr, "p_1"'] & \widetilde{X}_2 \arrow[d, "p_2"] \\
& X
\end{tikzcd}$$

for some dotted arrow. Then the commutative square

$$\begin{tikzcd}
A_*(\mathfrak{Q}(\mathbb{E}|_{\widetilde{X}_1})) \arrow[r] \arrow[d, "\sqrt{0^!_{\mathfrak{Q}(\mathbb{E}|_{\widetilde{X}_1})}}"] & A_*(\mathfrak{Q}(\mathbb{E}|_{\widetilde{X}_2})) \arrow[d, dotted, "\sqrt{0^!_{\mathfrak{Q}(\mathbb{E}|_{\widetilde{X}_2})}}"] \\
A_*(\widetilde{X}_1) \arrow[r, "p_*"] & A_*(\widetilde{X}_2)
\end{tikzcd}$$

completes the proof since $A_*(\mathfrak{Q}(\mathbb{E}|_{\widetilde{X}_1}) \to A_*(\mathfrak{Q}(\mathbb{E}))$ is surjective. $\square$

Proposition 4.3.3 (Bivariance). *Let $f : Y \to X$ be a morphism of separated Deligne-Mumford stacks. Let $\mathbb{E}$ be a symmetric complex on X with an orientation $o : O_X \cong \det(\mathbb{E})$. Let*

$$\begin{tikzcd}
\mathfrak{Q}(\mathbb{E}|_Y) \arrow[r, "\widetilde{f}"] \arrow[d] & \mathfrak{Q}(\mathbb{E}) \arrow[d] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

be a fiber diagram.

1. *If $f : Y \to X$ is a proper morphism, then we have*

$$f_* \circ \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}|_Y)}} = \sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}} \circ \widetilde{f}_* : A_*(\mathfrak{Q}(\mathbb{E}|_Y)) \to A_*(X).$$

2. *If $f : Y \to X$ is an equi-dimensional flat morphism, then we have*

$$f^* \circ \sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}} = \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}|_Y)}} \circ \widetilde{f}^* : A_*(\mathfrak{Q}(\mathbb{E})) \to A_*(Y).$$

3. *If $f : Y \to X$ is a local complete intersection morphism, then we have*

$$f^! \circ \sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}} = \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}|_Y)}} \circ f^! : A_*(\mathfrak{Q}(\mathbb{E})) \to A_*(Y).$$

Proof. It follows immediately from Proposition 4.1.25. □

Proposition 4.3.4 (Whitney sum formula)**.** *Let $\mathbb{E}_1$ be a symmetric complex on a separated Deligne-Mumford stack X with an orientation and E_2 be a special orthogonal bundle on X. Note that we have a canonical closed embedding*

$$c : \mathfrak{Q}(\mathbb{E}_1) \times \mathfrak{Q}(E_2) \hookrightarrow \mathfrak{Q}(\mathbb{E}_1 \oplus E_2[1])$$

by Proposition 4.2.15.3, where $\mathfrak{Q}(E_2) := \mathfrak{Q}(E_2[1])$. Then we have

$$\sqrt{0^!_{\mathfrak{Q}(E_2)}} \circ \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_1|_{\mathfrak{Q}(E_2)})}} = \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_1 \oplus E_2[1])}} \circ c_* : A_*(\mathfrak{Q}(\mathbb{E}_1) \times \mathfrak{Q}(E_2)) \to A_*(X).$$

Proof. It follows immediately from the Whitney sum formula of localized square root Euler classes in Proposition 4.1.26. □

Proposition 4.3.5 (Reduction formula)**.** *Let $\mathbb{E}$ be a symmetric complex on a separated Deligne-Mumford stack X and $\mathbb{K}$ be an isotropic subcomplex of $\mathbb{E}$ with respect to $\delta : \mathbb{E} \to \mathbb{K}$. We use the notations in Proposition 4.2.6. Consider the canonical diagram of cone stacks*

$$\begin{tikzcd}
\mathfrak{E}(\mathbb{K}) \arrow[d] & \\
\mathfrak{Q}(\mathbb{D}) \arrow[r, hook, "a"] \arrow[d, "b"] & \mathfrak{Q}(\mathbb{E}) \\
\mathfrak{Q}(\mathbb{E}_{/\!/\mathbb{K}}) &
\end{tikzcd}$$

where $\mathfrak{Q}(\mathbb{D})$ is the zero locus of $\mathfrak{q}_{\mathbb{E}}|_{\mathfrak{C}(\mathbb{D})} = \mathfrak{q}_{\mathbb{E}_{/\!/\mathbb{K}}}|_{\mathfrak{C}(\mathbb{D})}$, $a : \mathfrak{Q}(\mathbb{D}) \hookrightarrow \mathfrak{Q}(\mathbb{E})$ is a closed embedding, and $b : \mathfrak{Q}(\mathbb{D}) \twoheadrightarrow \mathfrak{Q}(\mathbb{E}_{/\!/\mathbb{K}})$ is a torsor of the vector bundle stack $\mathfrak{E}(\mathbb{K}) := \mathfrak{C}(\mathbb{K})$. Then we have

$$\sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_{/\!/\mathbb{K}})}} = \sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}} \circ a_* \circ b^* : A_*(\mathfrak{Q}(\mathbb{E}_{/\!/\mathbb{K}})) \to A_*(X).$$

Proof. By the Kimura sequence in Theorem A.1.1, we may assume that X is a quasi-projective scheme. By Proposition 4.2.9 (and Lemma 4.2.10), we have a symmetric resolution $\mathbb{E} \cong [B \to E^\vee \to B^\vee]$ and a resolution $\mathbb{K} \cong [K^\vee \to D^\vee]$ such that δ is represented by a surjective chain map. Here E is a special orthogonal bundle, K is an isotropic subbundle of E, B is a vector bundle, and D is a subbundle of B. Then the maps $\mathfrak{C}(\alpha) : \mathfrak{C}(\mathbb{D}) \hookrightarrow \mathfrak{C}(\mathbb{E})$ and $\mathfrak{C}(\beta) : \mathfrak{C}(\mathbb{D}) \to \mathfrak{C}(\mathbb{E})$ can be expressed as

$$\begin{tikzcd} \left[\dfrac{C(\mathrm{coker}(B/D \to E/K))}{B}\right] \arrow[r, hook, "\mathfrak{C}(\alpha)"] \arrow[d, "\mathfrak{C}(\beta)"] & \left[\dfrac{C(\mathrm{coker}(B \to E))}{B}\right] \\ \left[\dfrac{C(\mathrm{coker}(B/D \to K^{\perp}/K))}{B/D}\right]. & \end{tikzcd}$$

Hence the desired equality follows from Corollary 4.1.28. ◻

Remark 4.3.6 (Uniqueness)**.** The square root Gysin pullbacks are uniquely determined by the bivariance in Proposition 4.3.3, the reduction formula in Proposition 4.3.5, and the *compatibility formula*: if E is a special orthogonal bundle on a quasi-projective scheme X, then

$$\sqrt{0^!_{\mathfrak{Q}(E)}} = \sqrt{e}(E|_{\mathfrak{Q}(E)}, \tau) : A_*(\mathfrak{Q}(E)) \to A_*(X)$$

where $\mathfrak{Q}(E) := \mathfrak{Q}(E[1])$ and $\tau \in \Gamma(\mathfrak{Q}(E), E|_{\mathfrak{Q}(E)})$ is the tautological section.

Remark 4.3.7. As in Remark 4.1.20 and Remark 4.1.30, everything in this subsection can be generalized to algebraic stacks which admit proper covers by quotient stacks (in the sense of Definition A.2.1).

Based on this generalization, the Whitney sum formula in Proposition 4.3.4 can be generalized as follows: Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be symmetric complexes with orientations on a separated Deligne-Mumford stack X. Then the quadratic cone stack $\mathfrak{Q}(\mathbb{E}_1)$ admits a proper cover by a quotient stack. Moreover, we have a *Whitney sum formula*

$$\sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_2)}} \circ \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_1|_{\mathfrak{Q}(\mathbb{E}_2)})}} = \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_1 \oplus \mathbb{E}_2)}} \circ c_* : A_*(\mathfrak{Q}(\mathbb{E}_1) \times \mathfrak{Q}(\mathbb{E}_2)) \to A_*(X)$$

where $c : \mathfrak{Q}(\mathbb{E}_1) \times \mathfrak{Q}(\mathbb{E}_2) \hookrightarrow \mathfrak{Q}(\mathbb{E}_1 \oplus \mathbb{E}_2)$ is the inclusion map.

4.3.2 Square root virtual pullbacks

Definition 4.3.8 (Square root virtual pullback)**.** Let $f : X \to Y$ be a morphism from a Deligne-Mumford stack X to an algebraic stack Y. Let $\phi : \mathbb{E} \to L_{X/Y}$ be a symmetric obstruction theory satisfying the isotropic condition. Let $o : O_X \cong \det(\mathbb{E})$ be an orientation. Then we define the *square root virtual pullback*

$$\sqrt{f^!} : A_*(Y) \to A_*(X)$$

as the composition

$$A_*(Y) \xrightarrow{\mathrm{sp}_{X/Y}} A_*(\mathfrak{C}_{X/Y}) \to A_*(\mathfrak{Q}(\mathbb{E})) \xrightarrow{\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}} A_*(X)$$

where $\mathrm{sp}_{X/Y}$ is the specialization map for f in Definition 2.1.18, the second map is the pushforward for the closed embedding $\mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{Q}(\mathbb{E})$ in Proposition 4.2.23, and $\sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}$ is the square root Gysin pullack of the quadratic cone stack $\mathfrak{Q}(\mathbb{E})$ in Definition 4.3.1.

Definition 4.3.9 (Oh-Thomas virtual cycle)**.** Let X be a Deligne-Mumford stacks, $\phi : \mathbb{E} \to L_X$ be a symmetric obstruction theory satisfying the isotropic condition, and $o : O_X \cong \det(\mathbb{E})$ be an orientation. We define the *Oh-Thomas virtual cycle* as

$$[X]^{\mathrm{vir}} := \sqrt{p^!}[\mathrm{Spec}(\mathbb{C})] = \sqrt{0^!_{\mathfrak{Q}(\mathbb{E})}}[\mathfrak{C}_X] \in A_*(X)$$

where $p : X \to \mathrm{Spec}(\mathbb{C})$ is the projection map.

Proposition 4.3.10 (Bivariance)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a cartesian square of algebraic stacks where X and X' are Deligne-Mumford stacks. Let $\phi : \mathbb{E} \to L_{X/Y}$ be a symmetric obstruction theory satisfying the isotropic condition. Let

$$\phi' : (g')^*\mathbb{E} \xrightarrow{(g')^*(\phi)} (g')^*(L_{X/Y}) \to L_{X'/Y'}$$

be the induced symmetric obstruction theory. Then ϕ' also satisfies the isotropic condition and we have the following properties:

1. *If g is a proper DM morphism, then we have*

$$\sqrt{f^!} \circ g_* = g'_* \circ \sqrt{(f')^!} : A_*(Y') \to A_*(X).$$

2. *If g is a equi-dimensional flat morphism, then we have*

$$\sqrt{(f')^!} \circ g^* = (g')^* \circ \sqrt{f^!} : A_*(Y) \to A_*(X').$$

3. *If g is a local complete intersection morphism and Y' has affine stabilizers, then we have*

$$\sqrt{(f')^!} \circ g^! = (g')^! \circ \sqrt{f^!} : A_*(Y) \to A_*(X').$$

Proposition 4.3.11 (Commutativity)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a cartesian square of Deligne-Mumford stacks. Let $\phi_{X/Y} : \mathbb{E}_{X/Y} \to L_{X/Y}$ and $\phi_{Y'/Y} : \mathbb{E}_{Y'/Y} \to L_{Y'/Y}$ be symmetric obstruction theories with orientations satisfying the isotropic condition. Then we have

$$\sqrt{f^!} \circ \sqrt{g^!} = \sqrt{g^!} \circ \sqrt{f^!} : A_*(Y) \to A_*(X').$$

Proof. If follows directly from Proposition 2.1.22 and Remark 4.3.7. ◻

Proposition 4.3.12 (Reduction formula)**.** *Let $f : X \to Y$ be a morphism from a Deligne-Mumford stack X to an algebraic stack Y. Let*

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\delta"] \arrow[dr, "\phi"'] & \mathbb{K} \arrow[d, "\psi"] \\ & L_{X/Y} \end{tikzcd}$$

be a commutative diagram such that $\phi : \mathbb{E} \to L_{X/Y}$ is a symmetric obstruction theory, $\psi : \mathbb{K} \to L_{X/Y}$ is a perfect obstruction theory, and $\mathbb{K}$ is an isotropic subcomplex of $\mathbb{E}$ with respect to δ (see Definition 4.2.4). Let $o : O_X \cong \det(\mathbb{E})$ be an orientation. Then the symmetric obstruction theory ϕ satisfies the isotropic condition and we have

$$\sqrt{f^!_\phi} = \sqrt{e}(G) \circ f^!_\psi : A_*(Y) \to A_*(X)$$

where $G[1]$ is the reduction $\mathbb{E}_{/\!/\mathbb{K}}$ of $\mathbb{E}$ by $\mathbb{K}$.

Corollary 4.3.13 (Local complete intersection)**.** *Let $f : X \to Y$ be a morphism from a Deligne-Mumford stack X to an algebraic stack Y. Let $\phi : \mathbb{E} \to L_{X/Y}$ be a symmetric obstruction theory satisfying the isotropic condition. Assume that $f : X \to Y$ is a local complete intersection morphism. Then and we have*

$$\sqrt{f^!} = \sqrt{e}(G) \circ f^! : A_*(Y) \to A_*(X)$$

where $G[1]$ is the reduction of $\mathbb{E}$ by $\mathbb{L}_{X/Y} = L_{X/Y}$.

4.3.3 Functoriality

Definition 4.3.14 (Compatible triple of obstruction theories)**.** Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

be a commutative diagram of DM morphisms of algebraic stacks We use the notations in Notation 2.3.10. We say that a triple $(\psi_{X/Y}, \phi_{Y/Z}, \phi_{X/Z})$ of symmetric obstruction theories $\phi_{Y/Z} : \mathbb{E}_{Y/Z} \to L_{Y/Z}$, $\phi_{X/Z} : \mathbb{E}_{X/Z} \to L_{X/Z}$, and a perfect obstruction theory $\psi_f : \mathbb{K}_{X/Y} \to L_{X/Y}$ is *compatible* if there exist two morphisms of distinguished triangles

$$\begin{tikzcd} \mathbb{D}^\vee[2] \arrow[r, "\alpha^\vee"] \arrow[d, "\beta^\vee"] & \mathbb{E}_{X/Z} \arrow[r, "\delta"] \arrow[d, "\alpha"] & \mathbb{E}_{X/Y} \arrow[r] \arrow[d, equal] & {} \\ f^*\mathbb{E}_{Y/Z} \arrow[r, "\beta"] \arrow[d, "f^*\phi_{Y/Z}"] & \mathbb{D} \arrow[r, "\gamma"] \arrow[d, "\phi'_{X/Z}"] & \mathbb{E}_{X/Y} \arrow[r] \arrow[d, "\phi'_{X/Y}"] & {} \\ \tau^{\geqslant -1} f^* L_{Y/Z} \arrow[r, "a"] & L_{X/Z} \arrow[r, "b"] & L'_{X/Y} \arrow[r] & {} \end{tikzcd}$$

for some $\mathbb{D}, \alpha, \beta, \gamma, \delta, \phi'_{X/Y}, \phi'_{X/Z}$ such that $\phi_{X/Z} = \phi'_{X/Z} \circ \alpha$ and $\psi_{X/Y} = r \circ \phi'_{X/Y}$.

Theorem 4.3.15 (Functoriality)**.** *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

be a commutative diagram of algebraic stacks. Assume that X, Y are Deligne-Mumford stacks and X has the resolution property. Let $(\psi_{X/Y} : \mathbb{E}_{X/Y} \to L_{X/Y}, \phi_{Y/Z} :$

$\mathbb{E}_{Y/Z} \to L_{Y/Z}, \phi_{X/Z} : \mathbb{E}_{X/Z} \to L_{X/Z})$ *be a compatible triple of obstruction theories in the sense of Definition 4.3.14. Assume that* $\phi_{X/Z}$ *and* $\phi_{Y/Z}$ *satisfy the isotropic condition. Then for each orientation* $o_{Y/Z} : O_Y \cong \det(\mathbb{E}_{Y/Z})$*, there exists a canonical orientation* $o_{X/Z} : O_X \cong \det(\mathbb{E}_{X/Z})$ *such that we have*

$$\sqrt{(g \circ f)^!} = f^! \circ \sqrt{g^!} : A_*(Z) \to A_*(X).$$

Proof. We refer to [Park1] for the proof. □

The isotropic condition for $\phi_{g \circ f}$ is redundant in Theorem **??**.

Lemma 4.3.16 (Redundancy of isotropic condition)**.** *Given a compatible triple* $(\psi_{X/Y}, \phi_{Y/Z}, \phi_{X/Z}$ *of obstruction theories in the sense of Definition 4.3.14, the isotropic condition for* $\phi_{Y/Z}$ *implies the isotropic condition for* $\phi_{X/Z}$*.*

Proof. By Proposition 4.2.15.4, the diagram

$$\begin{tikzcd}
\mathfrak{C}_{X/Z} \arrow[r] \arrow[d] & \mathfrak{C}(\mathbb{D}) \arrow[r] \arrow[d] & \mathfrak{C}(\mathbb{E}_{X/Z}) \arrow[d, "\mathfrak{q}(\mathbb{E}_{X/Z})"] \\
\mathfrak{C}_{Y/Z}|_X \arrow[r] & \mathfrak{C}(\mathbb{E}_{Y/Z}|_X) \arrow[r, "\mathfrak{q}(\mathbb{E}_{Y/Z}|_X)"] & \mathbb{A}^1_X
\end{tikzcd}$$

commutes. Hence the isotropic condition for $\phi_{Y/Z}$ implies the isotropic condition for $\phi_{X/Z}$. □

Corollary 4.3.17 (Virtual pullback formula)**.** *Let* $f : X \to Y$ *be a morphism of Deligne-Mumford stacks. Assume that X has the resolution property. Let* $(\psi_{X/Y} : \mathbb{E}_{X/Y} \to L_{X/Y}, \phi_Y : \mathbb{E}_Y \to L_Y, \phi_X : \mathbb{E}_X \to L_X)$ *be a compatible triple of obstruction theories in the sense of Definition 4.3.14. Assume that* ϕ_X *and* ϕ_Y *satisfy the isotropic condition. Then for each orientation* $o_Y : O_Y \cong \det(\mathbb{E}_Y)$*, there exists a canonical orientation* $o_X : O_X \cong \det(\mathbb{E}_X)$ *such that we have*

$$[X]^{\mathrm{vir}} = f^![Y]^{\mathrm{vir}} \in A_*(X).$$

Remark 4.3.18 (Generalization)**.** In the current version of the proof of Theorem 4.3.15 in [Park1, Thm. 2.2], the resolution property for X is necessary. However, there is an alternative proof that does not use the resolution property. The details will appear in [BP].

Chapter 5

Cosection localization in Donaldson-Thomas theory of Calabi-Yau 4-folds

In this chapter, we generalize Kiem-Li's *cosection localization* [KL1] to Donaldson-Thomas theory of Calabi-Yau 4-folds. This is based on [KP2, BKP].

Summary We introduce *reduced virtual cycles* for two types of cosections:

1. *isotropic* cosections;

2. *non-degenerate* cosections.

The first one is studied in [KP2] and the second one is studied in [BKP]. Both of them are constructed by generalizing Kiem-Li's *cone reduction* lemma [KL1] to these cosections.

This cosection localization approach to the reduced theory become more important in DT4 theory. In the classical cases of surfaces and threefolds, the standard approach to the reduced theory is to use the algebraic twistor family of Kool-Thomas [KT1]. However, this algebraic twistor approach does not give us a reduced virtual cycle in DT4 theory. Thus we really need the cosection localization approach to obtain a reduced virtual cycle.

We also introduce *cosection-localized virtual cycles* for isotropic cosections. This is achieved by localizing the Edidin-Graham classes [EG1] by *two* isotropic sections.

5.1 Cone reductions

Recall that there are two special types of subbundles of an orthogonal bundle:

1. a *non-degenerate* subbundle F of an orthogonal bundle E.
2. an *isotropic* subbundle K of an orthogonal bundle E;

In the first case, we can form an *orthogonal complement* $F^{\perp}$ as an induced orthogonal bundle. In the second case, we can form a *reduction* $K^{\perp}/K$ as an induced orthogonal bundle. We thus provide two versions of the *cone reduction lemma* in this section.

We will work with *generalized* cosections.

Definition 5.1.1 (Generalized cosections)**.** Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. A *generalized cosection* is a map

$$\Sigma : \mathbb{E}^{\vee}[1] \to F$$

in the derived category of X for some vector bundle F.

By abuse of notation, we sometimes drop the letter "generalized" and simply call the map $\Sigma : \mathbb{E}^{\vee}[1] \to F$ a cosection.

We now fix the notions of *non-degenerate/isotropic* cosections.

Definition 5.1.2 (Non-degenerate cosections)**.** Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. We say that a cosection

$$\Sigma : \mathbb{E}^{\vee}[1] \to F$$

is *non-degenerate* if the square

$$\Sigma^2 : F^{\vee} \xrightarrow{\Sigma^{\vee}} \mathbb{E}[-1] \cong \mathbb{E}^{\vee}[1] \xrightarrow{\Sigma} F$$

is an isomorphism.

Definition 5.1.3 (Isotropic cosections)**.** Let $\mathbb{E}$ be a symmetric complex on an algebraic stack X. We say that a cosection

$$\Sigma : \mathbb{E}^{\vee}[1] \to K^{\vee}$$

is *isotropic* if the square

$$\Sigma^2 : K \xrightarrow{\Sigma^{\vee}} \mathbb{E}[-1] \cong \mathbb{E}^{\vee}[1] \xrightarrow{\Sigma} K^{\vee}$$

is zero.

Recall the followings from subsection 4.2.1:

1. Given a non-degenerate cosection $\Sigma : \mathbb{E}^\vee[1] \to F$, the perfect complex $F[1]$ is a non-degenerate subcomplex of $\mathbb{E}$ with respect to Σ in the sense of Definition 4.2.3. We then have an *orthogonal complement* $\mathbb{E}_{/\Sigma}$ in the sense of Proposition 4.2.5.

2. Given an isotropic cosection $\Sigma : \mathbb{E}^\vee[1] \to K^\vee$, the perfect complex $K^\vee[1]$ is an isotropic subcomplex of $\mathbb{E}$ with respect to Σ in the sense of Definition 4.2.4. We then have a *reduction* $\mathbb{E}_{//\Sigma}$ in the sense of Proposition 4.2.6.

The following two versions of the *cone reduction lemma* is the main result in this section.

Proposition 5.1.4 (Cone reduction for non-degenerate cosections)**.** *Let X be a Deligne-Mumford stack and $\phi : \mathbb{E} \to L_X$ be a symmetric obstruction theory satisfying the isotropic condition. Let $\Sigma : \mathbb{E}^\vee[1] \to F$ be a non-degenerate cosection. Then we have a closed embedding*

$$(\mathfrak{C}_X)_{\mathrm{red}} \hookrightarrow \mathfrak{Q}(\mathbb{E}_{/\Sigma})$$

that fits into the commutative diagram

$$\begin{tikzcd} & \mathfrak{Q}(\mathbb{E}_{/\Sigma}) \arrow[d, hook] \\ (\mathfrak{C}_X)_{\mathrm{red}} \arrow[r, hook] \arrow[ur, dotted] & \mathfrak{Q}(\mathbb{E}) \end{tikzcd}$$

as the dotted arrow. Here the closed embedding $(\mathfrak{C}_X)_{\mathrm{red}} \hookrightarrow \mathfrak{Q}(\mathbb{E})$ is induced by the obstruction theory ϕ as in Proposition 4.2.23.

Proof. Form a commutative diagram of cartesian squares

$$\begin{tikzcd} & \mathfrak{Q}(\mathbb{E}_{/\Sigma}) \arrow[r, hook] \arrow[d, hook] & \mathfrak{C}(\mathbb{E}_{/\Sigma}) \arrow[r] \arrow[d, hook] & X \arrow[d, hook, "0_F"] \\ (\mathfrak{C}_X)_{\mathrm{red}} \arrow[r, hook] \arrow[ur, dotted] & \mathfrak{Q}(\mathbb{E}) \arrow[r, hook] \arrow[d] & \mathfrak{C}(\mathbb{E}) \arrow[r, "\mathfrak{l}_\Sigma"] \arrow[d, "\mathfrak{q}_\mathbb{E}"] & F \\ & X \arrow[r, hook, "0"] & \mathbb{A}^1_X. & \end{tikzcd}$$

Indeed, the middle bottom square is cartesian by the definition of $\mathfrak{Q}(\mathbb{E})$, the right upper square is cartesian since $\mathbb{E}_{/\Sigma} := \mathrm{cone}(\Sigma^\vee[1] : F^\vee[1] \to \mathbb{E})$, and the middle upper square is cartesian by Proposition 4.2.15.3 since

$$\mathbb{E} = \mathbb{E}_{/\Sigma} \oplus F[1]$$

as symmetric complexes. Then Kiem-Li's cone reduction lemma (see Proposition 3.1.9) gives us the desried dotted arrow. ◻

Proposition 5.1.5 (Cone reduction for isotropic cosections)**.** *Let X be a Deligne-Mumford stack and $\phi : \mathbb{E} \to L_X$ be a symmetric obstruction theory satisfying the isotropic condition. Let $\Sigma : \mathbb{E}^\vee[1] \to K^\vee$ be an isotropic cosection such that $h^0(\Sigma) : h^1(\mathbb{E}^\vee) \to K^\vee$ is surjective. Then we have a closed embedding*

$$(\mathfrak{C}_X)_{\mathrm{red}} \hookrightarrow \mathfrak{Q}(\mathbb{E}_{//\Sigma})$$

that fits into the commutative diagram

$$\begin{tikzcd}
(\mathfrak{C}_X)_{\mathrm{red}} \arrow[dr] \arrow[drr, bend left] \arrow[ddr, dotted, bend right] & & \\
& \mathfrak{Q}(\mathbb{D}) \arrow[r, hook] \arrow[d] & \mathfrak{Q}(\mathbb{E}) \\
& \mathfrak{Q}(\mathbb{E}_{//\Sigma}) &
\end{tikzcd}$$

as the dotted arrow for some closed embedding $(\mathfrak{C}_X)_{\mathrm{red}} \hookrightarrow \mathfrak{Q}(\mathbb{D})$. Here the perfect complex $\mathbb{D}$ is given as in Proposition 4.2.6, $\mathfrak{Q}(\mathbb{D})$ is the zero locus of $\mathfrak{q}_{\mathbb{E}}|_{\mathfrak{C}(\mathbb{D})} = \mathfrak{q}_{\mathbb{E}_{//\Sigma}}|_{\mathfrak{C}(\mathbb{D})}$, and the closed embedding $(\mathfrak{C}_X)_{\mathrm{red}} \hookrightarrow \mathfrak{Q}(\mathbb{E})$ is induced by the obstruction theory ϕ as in Proposition 4.2.23.

Proof. Since the statement is local, we may assume that X is an affine scheme. Since X is affine, the surjection $h^{-1}(\mathbb{E}) \twoheadrightarrow K^\vee$ has a right inverse

$$s : K^\vee \to h^{-1}(\mathbb{E})$$

Consider the morphism of distinguished triangles

$$\begin{tikzcd}
h^{-2}(\mathbb{E})[2] \arrow[r] \arrow[d, equal] & \tau^{\leqslant -1}(\mathbb{E}) \arrow[r] \arrow[d] & h^{-1}(\mathbb{E}) \arrow[r] \arrow[d] & h^{-2}(\mathbb{E})[3] \arrow[d, equal] \\
h^{-2}(\mathbb{E})[2] \arrow[r] & \mathbb{E} \arrow[r] & \tau^{\geqslant -1}(\mathbb{E}) \arrow[r] & h^{-2}(\mathbb{E})[3].
\end{tikzcd}$$

Since X is affine, we have

$$\mathrm{Hom}_X(K^\vee[1], h^{-2}(\mathbb{E})[2]) = \mathrm{Hom}_X(K^\vee[1], h^{-2}(\mathbb{E})[3]) = 0.$$

Hence the map s gives us a right inverse of the composition

$$\tau^{\geqslant -1}\mathbb{E} \to \mathbb{E} \cong \mathbb{E}^\vee[2] \xrightarrow{\Sigma[1]} K^\vee[1].$$

Therefore we also have a right inverse

$$r : K^\vee[1] \to \mathbb{E}$$

of the map $\mathbb{E} \to \mathbb{E} \cong \mathbb{E}^\vee[2] \xrightarrow{\Sigma[1]} K^\vee[1]$. Then the map

$$\Sigma' := (r^\vee[1], \Sigma) : \mathbb{E}^\vee[1] \to (K \oplus K^\vee)$$

is a non-degenerate cosection such that $\mathbb{E}_{/\Sigma'} = \mathbb{E}_{/\!/\Sigma}$. Hence the cone reduction for non-degenerate cosection in Proposition 5.1.4 completes the proof. □

5.2 Reduced virtual cycles

In this section, we define *reduced virtual cycles* using the cone reduction lemmas in the previous section.

Definition 5.2.1 (Reduced virtual cycle for non-degenerate cosection)**.** Let X be a Deligne-Mumford stack, $\phi : \mathbb{E} \to L_X$ be a symmetric obstruction theory satisfying the isotropic condition, and $o : O_X \to \det(\mathbb{E})$ be an orientation. Let $\Sigma : \mathbb{E}^\vee[1] \to F$ be a non-degenerate cosection. Then we have a closed embedding $(\mathfrak{C}_X)_{\mathrm{red}} \hookrightarrow \mathfrak{Q}(\mathbb{E}_{/\Sigma})$ by Proposition 5.1.4. Let $o_2 : O_X \to \det(F)$ be an orientation of the orthogonal bundle (F, Σ). We define the *reduced virtual cycle* as

$$[X]^{\mathrm{red}}_{/\Sigma} := \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_{/\Sigma})}}[\mathfrak{C}_X] \in A_*(X)$$

where $\sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_{/\Sigma})}}$ is the square root Gysin pullback in Definition 4.3.1.

By abuse of notation, we drop the subscript $_{/\Sigma}$ if it is clear from the context.

Definition 5.2.2 (Reduced virtual cycle for isotropic cosection)**.** Let X be a Deligne-Mumford stack, $\phi : \mathbb{E} \to L_X$ be a symmetric obstruction theory satisfying the isotropic condition, and $o : O_X \to \det(\mathbb{E})$ be an orientation. Let $\Sigma : \mathbb{E}^\vee[1] \to K^\vee$

be an isotropic cosection such that $h^0(\Sigma)$ is surjective. Then we have a closed embedding $(\mathfrak{C}_X)_{\mathrm{red}} \hookrightarrow \mathfrak{Q}(\mathbb{E}_{//\Sigma})$ by Proposition 5.1.5. We define the *reduced virtual cycle* as

$$[X]^{\mathrm{red}}_{//\Sigma} := \sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_{//\Sigma})}}[\mathfrak{C}_X] \in A_*(X)$$

where $\sqrt{0^!_{\mathfrak{Q}(\mathbb{E}_{//\Sigma})}}$ is the square root Gysin pullback in Definition 4.3.1.

By abuse of notation, we drop the subscript $_{//\Sigma}$ if it is clear from the context.

We have the following compatibility results.

Proposition 5.2.3 (Compatibility)**.** *Let X be a Deligne-Mumford stack, $\phi : \mathbb{E} \to L_X$ be a symmetric obstruction theory satisfying the isotropic condition, and $o : O_X \to \det(\mathbb{E})$ be an orientation.*

1. *Let $\Sigma : \mathbb{E}^\vee[1] \to F$ be a non-degenerate cosection. Then we have*
$$[X]^{\mathrm{vir}} = \sqrt{e}(F) \cap [X]^{\mathrm{red}}_{/\Sigma}.$$
2. *Let $\Sigma : \mathbb{E}^\vee[1] \to K^\vee$ be an isotropic cosection such that $h^0(\Sigma)$ is surjective. Then we have*
$$[X]^{\mathrm{vir}} = e(K) \cap [X]^{\mathrm{red}}_{//\Sigma}.$$
3. *Let $\Sigma : \mathbb{E}^\vee[1] \to F$ be a non-degenerate cosection. If M is a positive maximal isotropic subbundle of F, then we have*
$$[X]^{\mathrm{red}}_{//\Sigma_M} = [X]^{\mathrm{red}}_{/\Sigma}$$
where $\Sigma_M : \mathbb{E}^\vee[1] \xrightarrow{\Sigma} F \cong F^\vee \twoheadrightarrow M^\vee$ is the composition.

Proof. 1. Since we have
$$\mathbb{E} = \mathbb{E}_{/\Sigma} \oplus F[1]$$
as symmetric complexes, Proposition 4.3.4 proves the desired formula.

2. The desired formula follows from Proposition 4.3.5 since
$$\mathfrak{Q}(\mathbb{D}) \to \mathfrak{Q}(\mathbb{E}_{//\Sigma})$$
is a K-torsor.

3. The desired formula follows directly from the canonical isomorphism
$$\mathbb{E}_{/\Sigma} \cong \mathbb{E}_{//\Sigma_M}$$
of symmetric complexes. □

In particular, we have the following vanishing result.

Corollary 5.2.4 (Vanishing)**.** *Let X be a Deligne-Mumford stack, $\phi : \mathbb{E} \to L_X$ be a symmetric obstruction theory satisfying the isotropic condition, $o : \mathcal{O}_X \to$ $\det(\mathbb{E})$ be an orientation, and $\Sigma : \mathbb{E}^\vee[1] \to \mathcal{O}_X$ be a cosection. Assume one of the following conditions:*

1. *$\Sigma^2 \in \Gamma(X, \mathcal{O}_X)$ is nowhere vanishing.*
2. *$\Sigma^2 = 0$ is isotropic and $h^0(\Sigma) : h^1(\mathbb{E}^\vee) \to \mathcal{O}_X$ is surjective.*

Then we have

$$[X]^{\mathrm{vir}} = 0 \in A_*(X).$$

The reduced virtual cycles are deformation invariant under an additional assumption.

Proposition 5.2.5 (Deformation invariance)**.** *Let $f : \mathcal{X} \to \mathcal{B}$ be a morphism of Deligne-Mumford stacks. Assume that $\mathcal{B}$ is smooth. Form a fiber diagram*

$$\begin{tikzcd} \mathcal{X}_b \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \{b\} \arrow[r, hook, "i_b"] & \mathcal{B} \end{tikzcd}$$

where $b \in \mathcal{B}$. Let $\phi : \mathbb{E} \to L_{\mathcal{X}/\mathcal{B}}$ be a symmetric obstruction theory satisfying the isotropic condition and $o : \mathcal{O}_X \to \det(\mathbb{E})$ be an orientation.

1. *Let $\Sigma : \mathbb{E}^\vee[1] \to F^\vee$ be a non-degenerate cosection. Let $o_2 : \mathcal{O}_X \to \det(F)$ be an orientation. Assume that the composition*

   $$F \xrightarrow{\Sigma^\vee} \mathbb{E}[-1] \xrightarrow{\phi} L_{X/B}[-1] \xrightarrow{\mathsf{KS}} \Omega_{\mathcal{B}}|_X$$

   vanishes. Then there exists a cycle class $[\mathcal{X}]^{\mathrm{red}} \in A_*(\mathcal{X})$ such that

   $$[\mathcal{X}_b]^{\mathrm{red}}_{/\Sigma_b} = i_b^![\mathcal{X}] \in A_*(\mathcal{X}_b)$$

   for all $b \in \mathcal{B}$, where $\Sigma_b : \mathbb{E}|_{\mathcal{X}_b}^\vee[1] \to F|_{\mathcal{X}_b}^\vee$ is the induced cosection.

2. *Let $\Sigma : \mathbb{E}^\vee[1] \to K^\vee$ be an isotropic cosection such that $h^0(\Sigma)$ is surjective. Assume the followings:*

(a) The composition

$$K \xrightarrow{\Sigma^\vee} \mathbb{E}[-1] \xrightarrow{\phi} L_{X/B}[-1] \xrightarrow{\mathsf{KS}} \Omega_{\mathcal{B}}|_X$$

vanishes.

(b) The induced map $\mathrm{coker}(\Omega_{\mathcal{B}}|_X \to h^1(\mathbb{E}^\vee)) \to K^\vee$ *is surjective.*

Then there exists a cycle class $[\mathcal{X}]^{\mathrm{red}} \in A_*(\mathcal{X})$ *such that*

$$[\mathcal{X}_b]^{\mathrm{red}}_{/\!/\Sigma_b} = i_b^![\mathcal{X}] \in A_*(\mathcal{X}_b)$$

for all $b \in \mathcal{B}$*, where* $\Sigma_b : \mathbb{E}|^\vee_{\mathcal{X}_b}[1] \to K|^\vee_{\mathcal{X}_b}$ *is the induced cosection.*

Proof. We refer to [KP2, Lem. 8.5] for the proof of the first case[1] and [BKP, Thm. 5.1] for the proof of the second case. □

5.3 Cosection-localized virtual cycles

5.3.1 Local model

Notation 5.3.1 (Blowup diagram)**.** Let E be a special orthogonal bundle on an algebraic stack X. Let s and t be an isotropic section of E such that $s \cdot t = 0$. Let $X(s)$ denote the zero locus of s in X. Let $\widetilde{X} := \mathrm{Bl}_{X(s)}X$ denote the blowup of X along $X(s)$ and D be the exceptional divisor. Then $L := \mathcal{O}_{\widetilde{X}}(D)$ is an isotropic subbundle of $E|_{\widetilde{X}}$. Let $E_{/\!/L}$ be the reduction of $E|_{\widetilde{X}}$ by L. Since $s \cdot t = 0$, we have an induced isotropic section t_L of $E_{/\!/L}$. Let

$$X(s,t)^\# := p(D(t_L)) \cup X(s,t)$$

where $D(t_L)$ is the zero locus of t_L in D and $X(s,t)$ is the common zero locus of s and t in X. Form a commutative diagram

$$\begin{tikzcd}
 & D(t_L) \arrow[r, hook] \arrow[d, "r"] & D \arrow[r, hook, "j"] \arrow[d, "q"] & \widetilde{X} \arrow[d, "p"] \\
X(s,t) \arrow[r, hook, "l"] & X(s,t)^\# \arrow[r, hook, "k"] & X(s) \arrow[r, hook, "i"] & X.
\end{tikzcd}$$

[1]It is written for the localized virtual cycles but the same proof work for the reduced virtual cycles.

Definition 5.3.2 (Localized square root Euler class for two cosections)**.** Let X be a Deligne-Mumford stack. Let E be a special orthogonal bundle on X. Let s and t be an isotropic section of E such that $s \cdot t = 0$. We use the notations in Notation 5.3.1. We define the *localized square root Euler class*

$$\sqrt{e}(E, s, t) : A_*(X) \to A_*(X(s,t)^\#)$$

as the unique map that fits into the commutative diagram

$$\begin{tikzcd}
A_*(D) \arrow[r, "(-j_*,q_*)"] & A_*(\widetilde{X}) \oplus A_*(X(s)) \arrow[r, "(p_*,i_*)"] \arrow[d, "(u,v)"'] & A_*(X) \arrow[r] \arrow[d, "\sqrt{e}(E,s)"] \arrow[dll, dotted, "\sqrt{e}(E,s,t)"'] & 0 \\
& A_*(X(s,t)^\#) \arrow[r, "k_*"'] & A_*(X(s)) &
\end{tikzcd}$$

where the middle vertical arrow is given by the two maps

$$u : A_*(\widetilde{X}) \xrightarrow{j^!} A_*(D) \xrightarrow{\sqrt{e}(E_{/\!/L},t_L)} A_*(D(t_L)) \xrightarrow{r_*} A_*(X(s,t)^\#)$$
$$v : A_*(X(s)) \xrightarrow{\sqrt{e}(E|_{X(s)},t)} A_*(X(s,t)) \xrightarrow{l_*} A_*(X(s,t)^\#)$$

and the top horizontal right exact sequence is the abstract blowup sequence in Corollary A.2.7.

To show that the localized square root Euler class $\sqrt{e}(E,s,t)$ in Definition 5.3.2 is well-defined, we need the following identities.

Lemma 5.3.3 (Well-definedness)**.** *In the situation of Definition 5.3.2, we have the identities*

$$u \circ j_* = v \circ q_*, \quad k_* \circ u = \sqrt{e}(E,s) \circ p_*, \quad k_* \circ v = \sqrt{e}(E,s) \circ i_*.$$

Proof. The first identity follows from the reduction formula in Proposition 4.1.17 and the bivariance of $\sqrt{e}(E,s)$ in Proposition 4.1.15. The second identity follows from the definition of $\sqrt{e}(E,s)$ in Definition 4.1.23, the compatibility $\sqrt{e}(E,0) = \sqrt{e}(E)$, and the bivariance of $\sqrt{e}(E,s)$. The third identity follows from the definition of $\sqrt{e}(E,s)$. □

We now state some basic properties of $\sqrt{e}(E,s,t)$. We omit the proofs since they follow from standard arguments.

Proposition 5.3.4 (Bivariance)**.** *Let $f : Y \to X$ be a morphism of Deligne-Mumford stacks. Let E be a special orthogonal bundle on X and s, t be isotropic sections of E such that $s \cdot t = 0$. Then we have*

$$Y(s,t)^{\#} \subseteq f^{-1}(X(s,t)^{\#})$$

as substacks of Y. Form a fiber diagram

$$\begin{tikzcd} Y(s,t)^{\#} \arrow[r, hook, "a"] \arrow[dr, "f(s,t)^{\#}"'] & f^{-1}(X(s,t)^{\#}) \arrow[r, hook] \arrow[d] & Y(s) \arrow[r, hook] \arrow[d, "f(s)"] & Y \arrow[d, "f"] \\ & X(s,t)^{\#} \arrow[r, hook] & X(s) \arrow[r, hook] & X. \end{tikzcd}$$

1. *If $f : Y \to X$ is a proper morphism, then*

$$\sqrt{e}(E, s, t) \circ f_* = f(s,t)^{\#}_* \circ \sqrt{e}(E, s, t) : A_*(Y) \to A_*(X(s,t)^{\#}).$$

2. *If $f : Y' \to Y$ is an equi-dimensional flat morphism, then a is an isomorphism and*

$$(f(s,t)^{\#})^* \circ \sqrt{e}(E, s, t) = \sqrt{e}(E, s, t) \circ f^* : A_*(X) \to A_*(Y(s,t)^{\#}).$$

3. *If $f : Y \to X$ is a local complete intersection morphism, then*

$$f^! \circ \sqrt{e}(E, s, t) = a_* \circ \sqrt{e}(E, s, t) \circ f^! : A_*(X) \to A_*(f^{-1}(X(s,t)^{\#})).$$

By abuse of notation, we denoted $Y(f^*s)$, $Y(f^*s, f^*t)$, and $\sqrt{e}(f^*E, f^*s, f^*t)$ by $Y(s)$, $Y(s,t)$, and $\sqrt{e}(E, s, t)$, respectively.

Proposition 5.3.5 (Reduction formula)**.** *Let E be a special orthogonal bundle on a Deligne-Mumford stack X and K be an isotropic subbundle. Let s and t be isotropic sections of E such that $s \cdot t = 0$ and $s \cdot K = t \cdot K = 0$. Let s_1 and t_1 be the isotropic sections of the reduction $K^{\perp}/K$. Let s_2 be the induced section of $K|_{X(s_1)}$. Then for any cycle class $\alpha \in A_*(X)$, we have*

$$\sqrt{e}(E, s, t)(\alpha) = e(K, s_2) \circ \sqrt{e}(K^{\perp}/K, s_1, t_1)(\alpha)$$

in $A_*(X(s,t)^{\#} \cup X(s_1, t_1)^{\#}(s_2))$.

Corollary 5.3.6. *Let X be a separated Deligne-Mumford stack, E be a special orthogonal bundle on X, and K be an isotropic subbundle. Let C be an isotropic subcone of the reduction $E_{/\!/K} := K^{\perp}/K$. Form a commutative diagram*

$$\begin{tikzcd}
 & \widetilde{C} \arrow[r, hook] \arrow[d, "r"] & K^{\perp} \arrow[r, hook] \arrow[d] & E \\
X \arrow[ur, hook, "0_{\widetilde{C}}"] \arrow[r, hook, "0_C"'] & C \arrow[r, hook] & E_{/\!/K} &
\end{tikzcd}$$

where the square is cartesian. Let $\tau \in \Gamma(C, E_{/\!/K}|_C)$ and $\widetilde{\tau} \in \Gamma(\widetilde{C}, E|_{\widetilde{C}})$ be the tautological sections. Let t be an vanishing isotropic section of E such that $t\cdot K = 0$ and $t_{\widetilde{C}} \cdot \widetilde{\tau} = 0$. Let t_1 be the induced isotropic section on $E|_{/\!/K}$. Then for any cycle class $\alpha \in A_*(C)$, we have

$$\sqrt{e}(E_{/\!/K}|_C, \tau, t_1)(\alpha) = \sqrt{e}(E|_{\widetilde{C}}, \widetilde{\tau}, t) \circ r^*(\alpha)$$

in $A_*(\widetilde{C}(\widetilde{\tau}, t)^{\#} \cup C(\tau, t_1)^{\#})$.

5.3.2 Global construction

Definition 5.3.7 (Cosection-localized virtual cycle)**.** Let X be a DM stack which has the resolution property. Let $\phi : \mathbb{E} \to L_X$ be a symmetric obstruction theory satisfying the isotropic condition. Let $o : O_X \to \det(\mathbb{E})$ be an orientation. Let $\sigma : \mathbb{E}^{\vee}[1] \to O_X$ be an isotropic section.

Choose a symmetric resolution of $\mathbb{E}$ such that the cosection σ is represented by a surjective chain map

$$\begin{tikzcd}
\mathbb{E}^{\vee}[1] \arrow[d, "\sigma"] & B \arrow[r, "d"] \arrow[d] & E^{\vee} \arrow[r, "d^{\vee}"] \arrow[d, "\widetilde{\sigma}"] & B^{\vee} \arrow[d] \\
O_X & 0 \arrow[r] & O_X \arrow[r] & 0.
\end{tikzcd} \tag{5.3.1}$$

Form a fiber diagram

$$\begin{tikzcd}
C \arrow[rr, hook] \arrow[d] & & E \arrow[d] \\
\mathfrak{C}_X \arrow[r, hook] & \mathfrak{C}(\mathbb{E}) \arrow[r, hook] & {[E/B]}.
\end{tikzcd}$$

where the closed embedding $\mathfrak{C}_X \hookrightarrow \mathfrak{C}(\mathbb{E})$ is induced by ϕ. Consider a diagram

$$\begin{tikzcd} & E|_D \arrow[d] \\ X_{\mathrm{red}} \arrow[r, hook, "0"] & D \arrow[u, bend left, "\tau"] \arrow[u, bend right, "\widetilde{\sigma}|_D^\vee"'] \end{tikzcd}$$

where $D := C_{\mathrm{red}}$ and $\tau \in \Gamma(D, E|_D)$ is the tautological section. Then $D(\tau) = X_{\mathrm{red}}$ and $\tau \cdot \widetilde{\sigma}|_D^\vee = 0$ by Proposition 5.1.5.

We define the *cosection-localized virtual cycle* as

$$[X]^{\mathrm{vir}}_{\mathrm{loc}} := \sqrt{e}(E|_D, \tau, \widetilde{\sigma}|_D^\vee)[C] \in A_*(X(\overline{\sigma}))$$

where $X(\overline{\sigma})$ is the zero locus of $\overline{\sigma} := h^0(\sigma) : h^1(\mathbb{E}^\vee) \to \mathcal{O}_X$ and

$$D(\tau, \widetilde{\sigma}|_D^\vee)^\# \subseteq X(\overline{\sigma})$$

by Lemma 5.3.8 below.

Lemma 5.3.8 (Well-definedness)**.** *In the situation of Definition 5.3.7, we have the followings.*

1. $D(\tau, \widetilde{\sigma}|_D^\vee)^\# \subseteq X(\overline{\sigma})$.
2. $[X]^{\mathrm{vir}}_{\mathrm{loc}}$ *is independent of the choice* (5.3.1).

Proof. 1. Replacing X by $X \backslash X(\overline{\sigma})$, we may assume that $\overline{\sigma} : h^1(\mathbb{E}^\vee) \to \mathcal{O}_X$ is surjective. Then it suffices to prove that

$$D(\tau, \widetilde{\sigma}|_D^\vee)^\# = \varnothing.$$

Since $\widetilde{\sigma} : E \to \mathcal{O}_X$ is surjective, $D(\tau)(\widetilde{\sigma}|_D^\vee) = \varnothing$. Since D is a cone over X_{red}, the projective cone $\mathbb{P}(D)$ over X_{red} is the exceptional divisor of the blowup of D along $D(\tau) = X_{\mathrm{red}}$. Hence it remains to show that

$$\mathbb{P}(D)(\widetilde{\sigma}_L) = \varnothing,$$

where $L := \mathcal{O}_{\mathbb{P}(D)}(-1)$ and $\widetilde{\sigma}_L \in \Gamma(\mathbb{P}(D), L^\perp/L)$ is the induced isotropic section. Since the composition

$$D \hookrightarrow E \twoheadrightarrow E/\langle \widetilde{\sigma}^\vee \rangle$$

is a closed immersion by Proposition 5.1.5, the composition

$$L \hookrightarrow E|_{\mathbb{P}(D)} \twoheadrightarrow (E/\langle\widetilde{\sigma}^\vee\rangle)|_{\mathbb{P}(D)}$$

is nowhere vanishing. Hence $\widetilde{\sigma}_L$ is also a nowhere vanishing section, which completes the proof.

2. It follows from a deformation argument and Corollary 5.3.6. □

Remark 5.3.9. In [OT2], it is shown that the Oh-Thomas virtual cycles [OT] map to the Borisov-Joyce virtual cycles [BJ] under the cycle class map. It is desirable to know whether the cosection-localization Oh-Thomas virtual cycles in [KP2] map to the cosection-localized Borisov-Joyce virtual cycles of Savvas [Sav].

Chapter 6

Applications to enumerative geometry

In this chapter, we apply the tools in Chapter 4 and Chapter 5 to the moduli spaces of sheaves on Calabi-Yau 4-folds. This is based on [Park1, KP2, BKP].

6.1 Moduli spaces, virtual cycles, and invariants

6.1.1 Moduli spaces

In this paper, we consider two types of moduli spaces:

1. moduli spaces of *pairs*;
2. moduli spaces of *sheaves*.

We first define the *moduli stacks* of all pairs/sheaves.

Definition 6.1.1 (Moduli stack of pairs)**.** Let X be a smooth projective variety. We define the *moduli stack of pairs* on X as the 2-functor

$$\underline{\mathrm{Pair}}(X) : \mathsf{Sch}^{\mathrm{op}}_{/\mathbb{C}} \to \mathsf{Groupoid} : T \mapsto \left\{ \begin{matrix} \text{pairs } (F, s) \text{ of a coherent sheaf } F \text{ on } X \times T \\ \text{flat over } T \text{ and a section } s \in \Gamma(X \times T, F) \end{matrix} \right\}.$$

Definition 6.1.2 (Moduli stack of sheaves)**.** Let X be a smooth projective variety. We define the *moduli stack of sheaves* on X as the 2-functor

$$\underline{\mathrm{Coh}}(X) : \mathsf{Sch}^{\mathrm{op}}_{/\mathbb{C}} \to \mathsf{Groupoid} : T \mapsto \{\text{coherent sheaves } F \text{ on } X \times T \text{ flat over } T\}.$$

Proposition 6.1.3 (Representability)**.** *Let X be a smooth projective variety. Then the moduli stacks $\underline{\mathrm{Pair}}(X)$ and $\underline{\mathrm{Coh}}(X)$ are representable by algebraic stacks of locally of finite type.*

Proof. We refer to [LMB, Thm. 4.6.2.1] for the representability of $\underline{\mathrm{Coh}}(X)$. The representability of $\underline{\mathrm{Pair}}(X)$ follows from the fact that $\underline{\mathrm{Pair}}(X)$ is an abelian cone over $\underline{\mathrm{Coh}}(X)$ (see for example [Bri, Lem. 2.4]). □

Remark 6.1.4 (Derived enhancement)**.** Let X be a smooth projective variety. Then the *derived moduli stack of pairs* and the *derived moduli stack of sheaves* defines as the ∞-functors

$$\begin{aligned}\mathbb{R}\underline{\mathrm{Pair}}(X) : \mathsf{dSch}^{\mathrm{op}}_{/\mathbb{C}} &\to \infty\text{–}\mathsf{Groupoid}\\ T &\mapsto \left\{\begin{matrix}\text{pairs } (F,s) \text{ of a perfect complex } F \text{ on } X\times T\\ \text{and a map } \mathcal{O}_{X\times T}\to F \text{ such that}\\ \text{the fibers } F_t \text{ are coherent sheaves for all } t\in T(\mathbb{C})\end{matrix}\right\}\end{aligned}$$

$$\begin{aligned}\mathbb{R}\underline{\mathrm{Coh}}(X) : \mathsf{dSch}^{\mathrm{op}}_{/\mathbb{C}} &\to \infty\text{–}\mathsf{Groupoid}\\ T &\mapsto \left\{\begin{matrix}\text{perfect complexes } F \text{ on } X\times T \text{ such that}\\ \text{the fibers } F_t \text{ are coherent sheaves for all } t\in T(\mathbb{C})\end{matrix}\right\}\end{aligned}$$

are represeentable by derived Artin stacks.

The derived moduli stack of pairs is the total space of the derived moduli stack of sheaves,

$$\mathbb{R}\underline{\mathrm{Pair}}(X) = \mathrm{Tot}_{\mathbb{R}\underline{\mathrm{Coh}}(X)}(\mathbb{R}\pi_*(\mathbb{F})),$$

where $\mathbb{F}$ is the universal complex of $\mathbb{R}\underline{\mathrm{Coh}}(X)\times X$ and $\pi : \mathbb{R}\underline{\mathrm{Coh}}(X)\times X \to \mathbb{R}\underline{\mathrm{Coh}}(X)$ is the projection map.

The cotangent complexes of the derived moduli stacks can be expressed as

$$\begin{aligned}\mathbb{L}_{\mathbb{R}\underline{\mathrm{Pair}}(X)} &= (\mathbb{R}\mathcal{H}om_\pi(\mathbb{I},\mathbb{F}))^\vee\\ \mathbb{L}_{\mathbb{R}\underline{\mathrm{Coh}}(X)} &= (\mathbb{R}\mathcal{H}om_\pi(\mathbb{F},\mathbb{F})[1])\end{aligned}$$

where $\mathbb{I}\to\mathcal{O}_{\mathbb{R}\underline{\mathrm{Pair}}(X)\times X}\to\mathbb{F}$ is the universal homotopy cofiber sequence on $\mathbb{R}\underline{\mathrm{Pair}}(X)\times X$ and the projection map $\mathbb{R}\underline{\mathrm{Pair}}(X)\times X\to\mathbb{R}\underline{\mathrm{Pair}}(X)$ is denoted by the same letter π. This can be shown by the derived loop stacks.

Clearly, the moduli stacks in Definition 6.1.1/Definition 6.1.2 are the classical truncations of the above derived moduli stacks.

Proposition 6.1.5 (Obstruction theories)**.** *Let X be a smooth projective variety of dimension n.*

1. *The map*
$$\mathbb{R}\mathcal{H}om_\pi(\mathbb{F}, \mathbb{I} \otimes \omega_X)[n] \xrightarrow{\mathrm{At}(\mathbb{F},s)} \mathbb{L}_{\underline{\mathrm{Pair}}(X)}$$
is an obstruction theory, where $\mathrm{At}(\mathbb{F}, s)$ *is the Atiyah class of the universal pair* $(\mathbb{F}, s)$ *(see Remark 6.1.6 below),* $\mathbb{I} = [\mathcal{O}_{\underline{\mathrm{Pair}}(X)\times X} \xrightarrow{s} \mathbb{F}$*, and* $\pi : \underline{\mathrm{Pair}}(X) \times X \to \underline{\mathrm{Pair}}(X)$ *is the projection map.*

2. *The map*
$$\mathbb{R}\mathcal{H}om_\pi(\mathbb{F}, \mathbb{F} \otimes \omega_X)[n-1] \xrightarrow{\mathrm{At}(\mathbb{F})} \mathbb{L}_{\underline{\mathrm{Coh}}(X)}$$
is an obstruction theory, where $\mathrm{At}(\mathbb{F})$ *is the Atiyah class of the universal sheaf* $\mathbb{F}$ *and* $\pi : \underline{\mathrm{Coh}}(X) \times X \to \underline{\mathrm{Coh}}(X)$ *is the projection map.*

Proof. It follows directly from Remark 6.1.4. □

Remark 6.1.6 (Atiyah class of pair)**.** Let X be a scheme. Let F be a perfect complex on X and $s : \mathcal{O}_X \to F$ be a map. Let

$$I \longrightarrow \mathcal{O}_X \xrightarrow{s} F$$

be a distinguished triangle. We define the *Atiyah class of the pair* (F, s)

$$\mathrm{At}_X(F, s) : F \to I \otimes \mathbb{L}_X$$

as the unique dotted arrow given by the homotopy square in the diagram in the stable ∞-category

$$\begin{tikzcd}
\mathbb{R}\mathcal{H}om_X(F, \mathcal{O}_X) \arrow[r, "s"] \arrow[d, "s"] & \mathbb{R}\mathcal{H}om_X(F, F) \arrow[r] \arrow[d, "\mathrm{At}_X(F)"] & \mathbb{R}\mathcal{H}om_X(F, I) \arrow[dl, dotted, "\mathrm{At}_X(F,s)"] \\
\mathbb{R}\mathcal{H}om_X(\mathcal{O}_X, \mathcal{O}_X) \arrow[r, "\mathrm{At}_X(\mathcal{O}_X)\cong 0"] & \mathbb{L}_X. &
\end{tikzcd}$$

Then the differential of the map $(F, s) : X \to \mathbb{R}\underline{\mathrm{Pair}}$

$$\mathbb{L}_{\mathbb{R}\underline{\mathrm{Pair}}}|_X = \mathbb{R}\mathcal{H}om_X(F, I) \to \mathbb{L}_X$$

can be identified to $\mathrm{At}_X(F, s)$.

Since the moduli stacks $\underline{\mathrm{Pair}}(X)$ and $\underline{\mathrm{Coh}}(X)$ are not *bounded* in general, we need *stability conditions*. We first consider the PT_q*-stability condition* on pairs, introduced in [BKP].

Definition 6.1.7 (PT_q-stability condition)**.** Let X be a smooth projective variety and $q \geqslant -1$ be an integer. We say that the pair (F, s) of a coherent sheaf F on X and a section $s \in \Gamma(X, F)$ is PT_q*-stable* if

1. $F \in \mathrm{Coh}_{\geqslant q+1}(X)$, and
2. $Q := \mathrm{coker}(s : \mathcal{O}_X \to F) \in \mathrm{Coh}_{\leqslant q}(X)$.

By abbreviation, we also refer to PT_q-stable pairs as PT_q *pairs*.

The two extremes of PT_q-stability are the well-known DT/PT-stability.

Example 6.1.8 (DT/PT-stability condition)**.** Let X be a smooth projective variety. Let F be a coherent sheaf on X of dimension d and $s \in \Gamma(X, F)$ be a section.

DT) The pair (F, s) is PT_{-1}-stable if and only if $s : \mathcal{O}_X \to F$ is surjective. Hence the PT_{-1} pairs on X correspond to the closed subschemes of X. Thus we refer to PT_{-1}-stability as DT-stability.

PT) The pair (F, s) is PT_{d-1}-stable if and only if F is pure and $\dim(Q) < d$. Hence PT_{d-1} pairs are exactly the stable pairs in the sense of Le Potier [Pot1, Def. 4.2] that are natural generalization of Pandharipande-Thomas stable pairs. Thus we sometimes refer to PT_{d-1}-stability as PT-stability.

Heuristically, d-dimensional PT_q-stable pairs are intermediate notions

$$\mathrm{DT} := \mathrm{PT}_{-1} \rightsquigarrow \mathrm{PT}_0 \rightsquigarrow \cdots \rightsquigarrow \mathrm{PT}_{d-2} \rightsquigarrow \mathrm{PT}_{d-1} =: \mathrm{PT}$$

between DT-stable pairs and PT-stable pairs.

Theorem 6.1.9 (Moduli space of PT_q-stable pairs)**.** *Let X be a smooth projective variety of dimension n, and $v \in H^*(X, \mathbb{Q})$ be a cohomology class such that $v_{\leqslant n-3} = 0$, and $q \in \{-1, 0, 1\}$ be an integer. Then the open locus of* PT_q*-stable pairs*

$$P_v^{(q)}(X) := \{\mathrm{PT}_q \textit{ pairs } (F, s) \textit{ on } X \textit{ with } \mathrm{ch}(F) = v\} \subseteq \underline{\mathrm{Pair}}(X)$$

is a projective scheme.

Proof. When $q = -1$, then this is shown in [Gro2]. When $v_{n-2} = 0$ and $q = 0$, or $q = 1$, then this is shown in [Pot1, Pot2]. When $v_{n-2} \neq 0$ and $q = 0$, this is shown in [BKP]. □

We will consider the following six moduli spaces of pairs:

1. $X^{[n]}$: Hilbert scheme of n-points [Gro2];
2. $I_{n,\beta}$: Hilbert scheme of curves [Gro2];
3. $P_{n,\beta}$: moduli space (1-dimensional) PT stable pairs [PT1, Pot1, Pot2];
4. $I_{n,\beta,\gamma}$: Hilbert scheme of surfaces [Gro2];
5. $P^{(0)}_{n,\beta,\gamma}$: moduli space of (2-dimensional) PT_0-stable pairs [BKP];
6. $P^{(1)}_{n,\beta,\gamma}$: moduli space of (2-dimensional) PT_1-stable pairs [Pot1, Pot2].

Secondly, we consider the *Gieseker stability* on sheaves.

Definition 6.1.10 (Gieseker stability)**.** Let X be a smooth projective variety and H be an ample line bundle. We say that a coherent sheaf F is *H-stable* (resp. *H-semi-stable*) if

1. F is pure sheaf of dimension d;
2. for any subsheaf F', we have
$$p_{F'}(t) < p_F(t) \quad (\text{resp. } P_{F'}(t) \leqslant P_F(t))$$
where $p_F(t)$ is the reduced Hilbert polynomial of F with respect to H.

For any coherent sheaf F, the automophism group $\underline{\mathrm{Aut}}(F)$ contains $\mathbb{G}_m$. Hence we will consider the *rigidified* moduli stack of coherent sheaves
$$\underline{\mathrm{Coh}}(X)/B\mathbb{G}_m,$$
defined as the quotient stack of the natural $B\mathbb{G}_m$-action on $\underline{\mathrm{Coh}}(X)$.

Theorem 6.1.11 (Moduli space of Gieseker-stable sheaves)**.** *Let X be a smooth projective variety, $v \in H^*(X, \mathbb{Q})$ be a cohomology class, and H be an ample line bundle. Then the open locus of H-stable sheaves*
$$M^H_v(X) := \{H\text{-stable sheaves } F \text{ with } \mathrm{ch}(F) = v\} \subseteq \underline{\mathrm{Coh}}(X)/B\mathbb{G}_m$$
is a quasi-projective scheme. Moreover, if there are no strictly semi-stable sheaves of Chern character v, then $M^H_v(X)$ is a projective scheme.

Proof. We refer to [HL] for the proof. □

6.1.2 Virtual cycles and invariants

Let X be a Calabi-Yau 4-fold, i.e., a smooth projective variety of dimension 4 with trivial canonical line bundle. Let $v \in H^*(X, \mathbb{Q})$ be a cohomology class. Let $\underline{\mathrm{Perf}}(X, v)^{\mathrm{spl}}$ be the moduli stack of simple perfect complexes F on X with $\mathrm{ch}(F) = v$ [Ina, Lie] (cf. [ToVa, STV]). Then $\underline{\mathrm{Perf}}(X, v)^{\mathrm{spl}}$ is always an *Artin* stack. Thus we consider the two variants:

1. $\underline{\mathrm{Perf}}(X, v)^{\mathrm{spl}}_L$: the moduli stack of simple perfect complexes on X with fixed determinant L and Chern character v. More precisely, $\underline{\mathrm{Perf}}(X)^{\mathrm{spl}}_L$ is defined as the fiber product
$$\begin{tikzcd} \underline{\mathrm{Perf}}(X,v)^{\mathrm{spl}}_L \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{C}) \arrow[d, "L"] \\ \underline{\mathrm{Perf}}(X,v)^{\mathrm{spl}} \arrow[r, "\det"] & \underline{\mathrm{Pic}}(X) \end{tikzcd}$$
where $\underline{\mathrm{Pic}}(X)$ is the Picard stack of line bundle on X and $\det : \underline{\mathrm{Perf}}(X) \to \underline{\mathrm{Pic}}(X)$ is the determinant map.

2. $\underline{\mathrm{Perf}}(X, v)^{\mathrm{spl}}/B\mathbb{G}_m$: the moduli space of simple perfect complexes on X with Chern character v, rigidified by the action of $B\mathbb{G}_m$.

Then the two moduli stacks $\underline{\mathrm{Perf}}(X, v)^{\mathrm{spl}}_L$ (for $v_0 \neq 0$) and $\underline{\mathrm{Perf}}(X, v)^{\mathrm{spl}}/B\mathbb{G}_m$ are Deligne-Mumford stacks.

Theorem 6.1.12. *Let X be a smooth projective variety, $v \in H^*(X, \mathbb{Q})$ be a cohomology class such that $v_{\leqslant n-3} = 0$, and $q \in \{-1, 0, 1\}$ be an integer. Then the canonical map*
$$P^{(q)}_v(X) \to \underline{\mathrm{Perf}}(X)^{\mathrm{spl}}_{O_X} : (F, s) \mapsto I := [O_X \xrightarrow{s} F]$$
is an open embedding.

Corollary 6.1.13. *Let X be a smooth projective variety of dimension n. Let $v \in H^*(X, \mathbb{Q})$ be a cohomology class such that $v_{\leqslant n-3} = 0$, and $q \in \{-1, 0, 1\}$ be an integer. Then the canonical map*
$$\phi : \mathbb{E} := \mathbb{R}\mathcal{H}om_\pi(\mathbb{I}, \mathbb{I} \otimes \omega_X)_0[n-1] \xrightarrow{\mathrm{At}(\mathbb{I})} \mathbb{L}_{P^{(q)}_v(X)}$$
is a symmetric obstruction theory satisfying the isotropic condition, where $\mathbb{I} := [O_{P^{(q)}_v(X) \times X} \xrightarrow{s} \mathbb{F}]$ is the universal pair and $\pi : P^{(q)}_v(X) \times X \to P^{(q)}_v(X)$ is the projection map. Moreover, the symmetric complex $\mathbb{E}$ is orientable.

Remark 6.1.14. The obstruction theory in Corollary 6.1.13 is *different* with that in Proposition 6.1.5.1.

Proposition 6.1.15. *Let X be a smooth projective variety, $v \in H^*(X, \mathbb{Q})$ be a cohomology class, and H be an ample line bundle. Then the canonical map*

$$M_v^H(X) \to \underline{\mathrm{Perf}}(X)^{\mathrm{spl}}/B\mathbb{G}_m$$

is an open embedding.

Corollary 6.1.16. *Let X be a smooth projective variety of dimension $n \geqslant 4$, $v \in H^*(X, \mathbb{Q})$ be a cohomology class, and H be an ample line bundle. Then the canonical map*

$$\phi : \mathbb{E} := \tau^{[-2,0]}\mathbb{R}\mathcal{H}om_\pi(\mathbb{F}, \mathbb{F})[3] \xrightarrow{\mathrm{At}(\mathbb{F})} \mathbb{L}_{M_v^H(X)}$$

is a symmetric obstruction theory satisfying the isotropic condition, where $\mathbb{F}$ is the universal sheaf and $\pi : M_v^H(X) \times X \to M_v^H(X)$ is the projection map. Moreover, the symmetric complex $\mathbb{E}$ is orientable.

Definition 6.1.17 (Virtual cycle for moduli space of stable pairs)**.** Let X be a Calabi-Yau 4-fold, $v \in H^*(X, \mathbb{Q})$ be a cohomology class such that $v_{\leqslant n-3} = 0$, and $q \in \{-1, 0, 1\}$ be an integer. We define the *virtual cycle*

$$[P_v^{(q)}(X)]_o^{\mathrm{vir}} \in A_{\mathrm{vd}}(P_v^{(q)}(X)), \qquad \mathrm{vd} = v_4 + \mathrm{td}_2(X) \cdot v_2 - \tfrac{1}{2}v_2^2$$

as the Oh-Thomas virtual cycle (Definition 4.3.9) associated to the symmetric obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_{P_v^{(q)}(X)}$ in Corollary 6.1.13 for an orientation $o : \mathcal{O}_{P_v^{(q)}(X)} \cong \det(\mathbb{E})$.

Definition 6.1.18 (Virtual cycle for moduli space of stable sheaves)**.** Let X be a Calabi-Yau 4-fold, $v \in H^*(X, \mathbb{Q})$ be a cohomology class, and H be an ample line bundle. We define the *virtual cycle*

$$[M_v^H(X)]_o^{\mathrm{vir}} \in A_{\mathrm{vd}}(M_v^H(X)), \qquad \mathrm{vd} = 1 - \tfrac{1}{2}\langle v, v\rangle$$

as the Oh-Thomas virtual cycle (Definition 4.3.9) associated to the symmetric obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_{M_v^H(X)}$ in Corollary 6.1.16 for an orientation $o : \mathcal{O}_{M_v^H(X)} \cong \det(\mathbb{E})$.

Remark 6.1.19 (Generalization)**.** The constructions of Oh-Thomas virtual cycles in Definition 6.1.17 and Definition 6.1.18 can be generalized to any open substack of $\underline{\mathrm{Perf}}(X, v)_L^{\mathrm{spl}}$ (for $v_0 \neq 0$) or $\underline{\mathrm{Perf}}(X, v)^{\mathrm{spl}}/B\mathbb{G}_m$, which is a separated Deligne-Mumford stack.

6.1.3 Invariants and conjectures

Definition 6.1.20 (Tautological complex)**.** Let X be a Calabi-Yau 4-fold, $v \in H^*(X, \mathbb{Q})$ be a cohomology class, $q \in \{-1, 0, 1\}$ be an integer, and H be an ample line bundle. Write $P_v(k) = \sum_{i \geqslant 0} a_i \binom{k+i-1}{i}$ for integers a_i.

1. Assume that $v_0 = v_1 = 0$. For any perfect complex E on X, we define the associated *tautological complex* on $P_v^{(q)}(X)(X)$ as

   $$\Phi_{\mathbb{F}}(E) := \mathbb{R}\pi_*(\mathbb{F} \otimes q^* E)$$

   where $(\mathbb{F}, s)$ is the universal pair and $\pi : P_v^{(q)}(X) \times X \to P_v^{(q)}(X)$, $q : P_v^{(q)}(X) \times X \to X$ are the projection maps.

   We sometimes omit the subscript $\mathbb{F}$ in $\Phi_{\mathbb{F}}(E)$ and just write $\Phi(E)$.

2. Assume that g.c.d$(a_i) = 1$. Fix a universal family $\mathbb{G}$ of $M_v^H(X)$. For any perfect complex E on X, we define the associated *tautological complex* on $M_v^H(X)$ as

   $$\Phi_{\mathbb{G}}(E) := \mathbb{R}\pi_*(\mathbb{G} \otimes q^* E)$$

   where $\pi : M_v^H(X) \times X \to M_v^H(X)$, $q : M_v^H(X) \times X \to X$ are the projection maps.

   The tautological complex $\Phi_{\mathbb{G}}(E)$ depends on the choice of $\mathbb{G}$.

Definition 6.1.21 (Primary insertions)**.** Let X be a Calabi-Yau 4-fold, $v \in H^*(X, \mathbb{Q})$ be a cohomology class, $q \in \{-1, 0, 1\}$ be an integer, and H be an ample line bundle. Write $P_v(k) = \sum_{i \geqslant 0} a_i \binom{k+i-1}{i}$ for integers a_i.

1. Assume that $v_0 = v_1 = 0$. For any cohomology class $\delta \in H^*(X, \mathbb{Q})$, we define the *primary insertion* as

   $$\Phi_0(\delta) := \pi_*(\mathrm{ch}_2(\mathbb{F}) \cup q^* \delta) \in H^*(P_v^{(q)}(X), \mathbb{Q})$$

   where $(\mathbb{F}, s)$ is the universal pair and $\pi : P_v^{(q)}(X) \times X \to P_v^{(q)}(X)$, $q : P_v^{(q)}(X) \times X \to X$ are the projection maps.

2. Assume that g.c.d$(a_i) = 1$. We define the *primary insertion* as

   $$\Phi_0(\delta) := \mathbb{R}\pi_*(\mathbb{G} \otimes q^* E) \in H^*(M_v^H(X), \mathbb{Q})$$

   where $\mathbb{G}$ is a universal sheaf and $\pi : M_v^H(X) \times X \to M_v^H(X)$, $q : M_v^H(X) \times X \to X$ are the projection maps.

   The primary insertion $\Phi_0(\delta)$ is independent of the choice of $\mathbb{G}$.

In [CK1], Cao-Kool conjectured that the tautoglogical Hilbert scheme invariants can be expressed by the MacMahon function as follows.

Conjecture 6.1.22 (Tautological Hilbert scheme invariants)**.** *Let X be a Calabi-Yau 4-fold. Let L be a line bundle on X. Then there exist orientations such that*

$$\sum_{n\geqslant 0}\int_{[X^{[n]}]} e(L^{[n]})\cdot q^n = M(-q)^{\int_X c_3(T_X)c_1(L)}$$

where $L^{[n]} := \Phi(L)$ is the tautological bundle and $M(q) := \prod_{n\geqslant 1}(1-q^n)^{-n}$ is the MacMahon function.

In [CK2, Conj. 0.3], Cao-Kool conjectured (1-dimensional) DT/PT correspondence for primary insertions.

Conjecture 6.1.23 (Primary DT/PT correspondence)**.** *Let X be a Calabi-Yau 4-fold, $\beta \in H_2(X,\mathbb{Q})$ be a curve class, and $n \in Z$ be an integer. Let $\gamma_i \in H^*(X,\mathbb{Q})$ be cohomology classes. Then there exist orientations such that*

$$\int_{[I_{n,\beta}(X)]^{\mathrm{vir}}} \Phi_0(\gamma_1)\cup\cdots\cup\Phi_0(\gamma_k) = \int_{[P_{n,\beta}(X)]^{\mathrm{vir}}} \Phi_0(\gamma_1)\cup\cdots\cup\Phi_0(\gamma_k).$$

In [CKM, Conj. 0.13], Cao-Kool-Monavari conjectured (1-dimensional) DT/PT correspondence for tautological insertions.

Conjecture 6.1.24 (Tautological DT/PT correspondence)**.** *Let X be a Calabi-Yau 4-fold and $\beta \in H_2(X,\mathbb{Q})$ be a curve class. Let L be a line bundle on X. Then there exist orientations such that*

$$\frac{\sum_{n\geqslant 0}\int_{[I_{n,\beta}(X)]^{\mathrm{vir}}} e(\Phi(L))\cdot q^n}{\sum_{n\geqslant 0}\int_{[X^{[n]}]^{\mathrm{vir}}} e(L^{[n]})\cdot q^n} = \sum_{n\geqslant 0}\int_{[P_{n,\beta}(X)]^{\mathrm{vir}}} e(\Phi(L))\cdot q^n\,.$$

In [CMT1, CMT2], Cao-Maulik-Toda conjectured (1-dimensional) PT/Katz corrspondence for primary insertions.

Conjecture 6.1.25 (Primary PT/Katz correspondence)**.** *Let X be a Calabi-Yau 4-fold, $\beta \in H_2(X,\mathbb{Q})$ be a curve class, and $n \in Z$ be an integer. Let $\gamma \in H^4(X,\mathbb{Q})$ be cohomology classes. Then there exist orientations such that*

$$\int_{[P_{n,\beta}(X)]^{\mathrm{vir}}} \Phi_0(\gamma)^n = \sum_{\sum_{i=0}^n \beta_i=\beta}\left(\int_{[P_{0,\beta_0}(X)]^{\mathrm{vir}}} 1\cdot\prod_{i=1}^{n}\int_{[M_{1,\beta}(X)]^{\mathrm{vir}}}\Phi_0(\gamma)\right).$$

In [BKP], a (2-dimensional) $\mathrm{DT}/\mathrm{PT}_0$ correspondence for tautological insertions were introduced.

Conjecture 6.1.26 (Tautological $\mathrm{DT}/\mathrm{PT}_0$ correspondence)**.** *Let X be a Calabi-Yau 4-fold, $\gamma \in H_4(X,\mathbb{Q})$ be a surface class, and $\beta \in H_2(X,\mathbb{Q})$ be a curve class. Let L be a line bundle on X. Then there exist orientations such that*

$$\frac{\sum_{n\geqslant 0}\int_{[I_{n,\beta,\gamma}(X)]^{\mathrm{vir}}} e(\Phi(L))\cdot q^n}{\sum_{n\geqslant 0}\int_{[X^{[n]}]^{\mathrm{vir}}} e(L^{[n]})\cdot q^n} = \sum_{n\geqslant 0}\int_{[P^{(0)}_{n,\beta,\gamma}(X)]^{\mathrm{vir}}} e(\Phi(L))\cdot q^n .$$

Remark 6.1.27 (Descendent insertions)**.** In the situation of Definition 6.1.21, we can define the *descendent insertion* as

$$\Phi_i(\delta) := \pi_*(\mathrm{ch}_{2+i}(\mathbb{F}) \cup q^*\delta) \in H^*(P^{(q)}_v(X),\mathbb{Q})$$

for $i > 0$.

6.2 Lefschetz principle

Recall [KKP] that the quantum Lefschetz principle relates the Gromov-Witten invariants of an algebraic variety with the Gromov-Witten invariants of its divisor. The virtual pullback formula in Theorem 4.3.17 provides an analogous formula in Donaldson-Thomas theory. This section is based on [Park1].

Theorem 6.2.1 (Lefschetz principle)**.** *Let X be a Calabi-Yau 4-fold and D be a smooth connected divisor of a line bundle L on X. Let $v \in H^*(X,\mathbb{Q})$ be a cohomology class such that $v_0 = v_1 = 0$ and $q \in \{-1,0,1\}$ be an integer. Consider the following moduli spaces:*

$$P(X) := \{\mathrm{PT}_q \textit{ pairs } (F,s) \textit{ on } X \textit{ with } \mathrm{ch}(F) = v\}$$
$$P(D) := \{\mathrm{PT}_q \textit{ pairs } (F,s) \textit{ on } D \textit{ with } \mathrm{ch}(i_*F) = v\}$$

where $i : D \hookrightarrow X$ is the inclusion map. Assume the followings:

A1) The tautological complex $\Phi(L)$ is a vector bundle.

A2) The canonical map $R\mathcal{H}om_\pi(\mathbb{I}_D,\mathbb{I}_D\otimes L)_0[2] \xrightarrow{\mathrm{At}(\mathbb{I}_D)} \mathbb{L}_{P(D)}$ is a perfect obstruction theory.

Then for any orientation on $P(X)$, there exists canonical signs $(-1)^{\sigma(e)}$ on the connected components $P(D)^e$ of $P(D)$ such that

$$\sum_e (-1)^{\sigma(e)} (j_e)_*[P(D)^e]^{\mathrm{vir}}_{BF} = e(\Phi(L)) \cap [P(X)]^{\mathrm{vir}}_{OT}$$

where $j_e : P(D)^e \hookrightarrow P(D) \hookrightarrow P(X)$ are the inclusion maps.

Sketch of the proof. □

Corollary 6.2.2 (Tautological Hilbert scheme invariants)**.** *Let X be a Calabi-Yau 4-fold. Let L be a line bundle on X. Assume that L has a smooth connected divisor. Then Conjecture 6.1.22 holds for X and L.*

Proof. Since $D^{[n]}$ is connected, the Lefschetz principle gives us

$$\int_{[X^{[n]}]^{\mathrm{vir}}} e(L^{[n]}) = \int_{[D^{[n]}]^{\mathrm{vir}}} 1.$$

By [LP, Li3], the generating series of the degree zero MNOP invariants [MNOP1, MNOP2] of a smooth projective 3-fold D can be expressed as

$$\sum_{n \geqslant 0} \int_{[D^{[n]}]^{\mathrm{vir}}} 1 \cdot q^n = M(-q)^{\int_D c_3(T_D \otimes K_D)}.$$

By an elementary argument, we can deduce

$$\int_D c_3(T_D \otimes K_D) = \int_X c_3(T_X) c_1(L)$$

(cf. [CK1, (2.5)]). It completes the proof. □

Corollary 6.2.3 (Tautological DT/PT correspondence)**.** *Let X be a Calabi-Yau 4-fold and $\beta \in H_2(X, \mathbb{Q})$ be a curve class. Let L be a line bundle. Assume that there is a smooth connected divisor D of L such that the following conditions are satisfied:*

A1) D is a Calabi-Yau 3-fold.

A2) For all pure 1-dimensional closed subschemes C of X with $[C] = \beta$, we have $H^1(C, L) = 0$.

A3) For all n, the inclusion maps $I_{n,\beta}(D) \hookrightarrow I_{n,\beta}(X)$ and $P_{n,\beta}(D) \hookrightarrow P_{n,\beta}(X)$ induce injective maps between the sets of connected components.

Then Conjecture 6.1.24 holds for X, β, and L.

Proof. Applying the Lefschetz principle to the three moduli space $I_{n,\beta}(X)$, $P_{n,\beta}(X)$, $X^{[n]}$, the 3-fold DT/PT correspondence [Bri, Toda]

$$\frac{\sum_{n\geqslant 0}\int_{[I_{n,\beta}(D)]^{\mathrm{vir}}} 1\cdot q^n}{\sum_{n\geqslant 0}\int_{[D^{[n]}]^{\mathrm{vir}}} 1\cdot q^n} = \sum_{n\geqslant 0}\int_{[P_{n,\beta}(D)]^{\mathrm{vir}}} 1\cdot q^n$$

completes the proof. □

Corollary 6.2.4 (Tautological DT/PT$_0$ correspondence)**.** *Let X be a Calabi-Yau 4-fold, $\gamma \in H_4(X,\mathbb{Q})$ be a surface class, and $\beta \in H_2(X,\mathbb{Q})$ be a curve class. Let L be a line bundle. Assume that there is a smooth connected divisor D of L such that the following conditions are satisfied:*

A1) D is a Calabi-Yau 3-fold.

A2) For all 2-dimensional closed subschemes S of X with $\mathrm{ch}_2(\mathcal{O}_S) = \gamma$ and $\mathrm{ch}_3(\mathcal{O}_S) = \beta$, we have $H^1(S,L) = H^2(S,L) = 0$.

A3) For all n, the inclusion maps $I_{n,\beta,\gamma}(D) \hookrightarrow I_{n,\beta,\gamma}(X)$ and $P^{(0)}_{n,\beta,\gamma}(D) \hookrightarrow P^{(0)}_{n,\beta,\gamma}(X)$ induce injective maps between the sets of connected components.

Then Conjecture 6.1.26 holds for X, β, γ, and L.

Proof. Applying the Lefschetz principle to the three moduli space $I_{n,\beta,\gamma}(X)$, $P^{(0)}_{n,\beta,\gamma}(X)$, $X^{[n]}$, the 3-fold (1-dimensional) DT/PT correspondence [Bri, Toda]

$$\frac{\sum_{n\geqslant 0}\int_{[I_{n,\beta}(D)]^{\mathrm{vir}}} 1\cdot q^n}{\sum_{n\geqslant 0}\int_{[D^{[n]}]^{\mathrm{vir}}} 1\cdot q^n} = \sum_{n\geqslant 0}\int_{[P_{n,\beta}(D)]^{\mathrm{vir}}} 1\cdot q^n$$

completes the proof. □

6.3 Pairs/Sheaves correspondence

In many cases, maps between moduli spaces of sheaves or complexes can be realized as *virtual projective bundles*. Since there is a general pushforward formula for virtual projective bundles, a virtual pullback formula for these cases is practically effective for computing invariants. We provide a correspondence between the moduli of stable pairs and the moduli of stable sheaves as an example. This section is based on [Park1].

6.3.1 Virtual projective bundles

We first fix the notion of *virtual projective bundles*.

Definition 6.3.1 (Virtual projective bundle)**.** Let X be a scheme and $\mathbb{K}$ be a perfect complex of tor-amplitude $[0,1]$. We define the *virtual projective bundle* as the projective cone
$$p : \mathbb{P}(\mathbb{K}) := \mathrm{Proj}(\mathrm{Sym}(h^0(\mathbb{K}^\vee))) \to X.$$

The virtual projective bundles are classical truncations of *derived projective bundles*.

Remark 6.3.2 (Derived enhancement)**.** Let X be a scheme and $\mathbb{K}$ be a perfect complex of tor-amplitude $[0,1]$. Then we have
$$\mathbb{P}(\mathbb{K}) = [(\mathrm{Tot}_X(\mathbb{K})\backslash 0)/\mathbb{G}_m]_{\mathrm{cl}}$$
where $0 : X \to \mathrm{Tot}_X(\mathbb{K})$ is the zero section.

The quasi-smooth derived enhancements on the virtual projective bundles induce *perfect obstruction theories*.

Proposition 6.3.3 (Obstruction theory)**.** *Let X be a scheme and $\mathbb{K}$ be a perfect complex of tor-amplitude $[0,1]$. Then the virtual projective cone*
$$p : \mathbb{P}(\mathbb{K}) := \mathrm{Proj}(\mathrm{Sym}(h^0(\mathbb{K}^\vee))) \to X$$
has a natural perfect obstruction theory
$$\mathbb{E} := \mathrm{cone}(\mathcal{O}_{\mathbb{P}(\mathbb{K})} \to p^*\mathbb{K}(1))^\vee \to \mathbb{L}_{\mathbb{P}(\mathbb{K})/X}.$$

There is a *pushforward formula* for virtual projective bundles.

Proposition 6.3.4 (Pushforward formula)**.** *Let $p : \mathbb{P}(\mathbb{K}) \to \mathcal{X}$ be a virtual projective bundle over a quasi-projective scheme $\mathcal{X}$. For any cycle class $\alpha \in A_*(\mathcal{X})$ and a K-theory class $\xi \in K^0(\mathcal{X})$, we have*
$$p_*(c_m(p^*\xi(1)) \cap p^!\alpha) = \sum_{0\leqslant i\leqslant m} \binom{s-i}{m-i} \cdot c_i(\xi) \cap c_{m-i+1-r}(-\mathbb{K}) \cap \alpha$$
where r is the rank of $\mathbb{K}$ and s is the rank of ξ.

Proof. Fix a global resolution $\mathbb{K} \cong [K_0 \to K_1]$ and consider the factorization

$$\begin{tikzcd} & K_1(1)|_{\mathbb{P}(K_0)} \arrow[d] \\ \mathbb{P}(\mathbb{K}) \arrow[r, hook, "i"] \arrow[dr, "p"'] & \mathbb{P}(K_0) \arrow[d, "q"] \arrow[u, bend right, "t"'] \\ & X \end{tikzcd}$$

where $\mathbb{P}(\mathbb{K})$ is the zero locus of the tautological section t. Then

$$\begin{tikzcd} \mathbb{E} \arrow[d] & = & (K_1(1))|^{\vee}_{\mathbb{P}(\mathbb{K})} \arrow[r, "dt"] \arrow[d, "t"] & \Omega_{\mathbb{P}(K_0)/\mathcal{X}}|_{\mathbb{P}(\mathbb{K})} \arrow[d, equal] \\ \mathbb{L}_{\mathbb{P}(\mathbb{K})/\mathcal{X}} & = & \mathcal{I}_{\mathbb{P}(\mathbb{K})/\mathbb{P}(K_0)}/\mathcal{I}^2_{\mathbb{P}(\mathbb{K})/\mathbb{P}(K_0)} \arrow[r, "d"] & \Omega_{\mathbb{P}(K_0)/\mathcal{X}}|_{\mathbb{P}(\mathbb{K})} \end{tikzcd}$$

is the perfect obstruction theory. Manolache's virtual pullback formula $p^! = i^! \circ q^*$ implies

$$p_*(c_m(\xi(1)) \cap p^!\alpha) = q_*(c_m(\xi(1)) \cap c_{r_1}(K_1(1)) \cap q^*\alpha)$$

where r_0 and r_1 are the ranks of K_0 and K_1, respectively. Note that

$$c_m(\xi(1)) = \sum_{0 \leqslant i \leqslant m} \binom{s-i}{m-i} c_i(\xi) c_1(\mathcal{O}(1))^{m-i}$$

by [Ful, Example 3.2.2]. Therefore, we have

$$\begin{aligned}
&p_*(c_m(\xi(1)) \cap p^!\alpha) \\
&= \sum_{0 \leqslant i \leqslant m} \sum_{0 \leqslant j \leqslant r_1} \binom{s-i}{m-i} \cdot c_i(\xi) \cap c_j(K_1) \cap q_*(c_1(\mathcal{O}(1))^{m+r_1-i-j} \cap q^*\alpha) \\
&= \sum_{0 \leqslant i \leqslant m} \sum_{0 \leqslant j \leqslant r_1} \binom{s-i}{m-i} \cdot c_i(\xi) \cap c_j(K_1) \cap s_{m+r_1-i-j-r_0+1}(K_0) \cap \alpha \\
&= \sum_{0 \leqslant i \leqslant m} \binom{s-i}{m-i} \cdot c_i(\xi) \cap c_{m-i+1-r}(-\mathbb{K}) \cap \alpha
\end{aligned}$$

where $s_\bullet(K_0)$ denotes the Segre class of K_0. ◻

6.3.2 Pairs/Sheaves correspondence

Theorem 6.3.5 (Pairs/Sheaves correspondence)**.** *Let X be a Calabi-Yau 4-fold, $v \in H^*(X, \mathbb{Q})$ be a cohomology class such that $v_0 = v_1 = 0$, and H be an ample line bundle. Consider the following moduli spaces:*

$$P(X) := \{\mathrm{PT}\text{-}\textit{stable pairs } (F, s) \textit{ on } X \textit{ with } \mathrm{ch}(F) = v\}$$
$$M(X) := \{H\text{-}\textit{stable sheaves } G \textit{ on } X \textit{ with } \mathrm{ch}(F) = v\}$$

Assume the followings:

A1) *(a)* *$v_2 \neq 0$ is an irreducible surface class, or*

(b) *$v_2 = 0$ and $v_3 \neq 0$ is an irreducible curve class.*

A2) *There exists a universal family $\mathbb{G}$ on $M(X) \times X$ and the tautological complex $O_X^{M(X)} := R\pi_*\mathbb{G}$ is of tor-amplitude $[0, 1]$.*

Then the forgetful map

$$p : P(X) \to M(X) : (F, s) \mapsto F$$

is the virtual projective bundle of $O_X^{M(X)}$. Moreover, for any orientation on $M(X)$, there exists a canonical orientation on $P(X)$ such that

$$[P(X)]^{\mathrm{vir}} = p^![M(X)]^{\mathrm{vir}} \in A_*(P(X)).$$

Corollary 6.3.6 (Pushforward formula)**.** *In the situation of Theorem 6.3.5, for any perfect complex E of rank N on X, we have*

$$p_*(c_{n-1}(\Phi(E)) \cap [P(X)]^{\mathrm{vir}}) = N \cdot [M(X)]^{\mathrm{vir}} \in A_*(M(X)).$$

Corollary 6.3.7 (Primary PT/Katz correspondence)**.** *Let X be a Calabi-Yau* 4-*fold, $\beta \in H_2(X, \mathbb{Q})$ be a curve class, and $n \in Z$ be an integer. Let $\gamma \in H^4(X, \mathbb{Q})$ be cohomology classes. Assume that β is irreducible. Then Conjecture 6.1.25 holds for X, β, n, and γ.*

Corollary 6.3.8 (Tautological PT/Katz correspondence)**.** *Let X be a Calabi-Yau* 4-*fold, $\beta \in H_2(X, \mathbb{Q})$ be a curve class, $n \in Z$ be an integer, and H be an ample* *line bundle. Let E be a perfect complex on X. Assume that β is irreducible and* $\mathrm{g.c.d}(\beta \cdot H, n) = 1$. *Then there exist orientations such that*

$$\int_{[P_{n,\beta}(X)]^{\mathrm{vir}}} c_n(\Phi(E))$$
$$= \begin{cases} -\binom{N}{n} \cdot \int_{[M_{n,\beta}(X)]^{\mathrm{vir}}} c_1(\Phi_{\mathbb{G}}(\mathcal{O}_X)) & \text{if } n = 0 \\ \binom{N-1}{n-1} \cdot \int_{[M_{n,\beta}(X)]^{\mathrm{vir}}} c_1(\Phi_{\mathbb{G}}(\mathcal{O}_X)) - \binom{N}{n} \cdot \int_{[M_{n,\beta}(X)]^{\mathrm{vir}}} c_1(\Phi_{\mathbb{G}}(\mathcal{O}_X)) & \text{if } n \geqslant 1 \end{cases}$$

where $\mathbb{G}$ is the universal sheaf of $M^H_{n,\beta}(X)$ and $N = n \cdot \mathrm{rank}(E) + \int_\beta c_1(E)$.

6.4 Counting surfaces on Calabi-Yau 4-folds

This section is based on [BKP].

Theorem 6.4.1 (Reduced virtual cycle)**.** *Let X be a Calabi-Yau 4-fold with nowhere vanishing Calabi-Yau 4-form $\omega \in H^0(X, \Omega_X^4)$. Let $v = (0, 0, \gamma, \beta, n - \gamma \cdot \mathrm{td}_2(X)) \in H^*(X, \mathbb{Q})$ be a cohomology class, $q \in \{-1, 0, 1\}$ be an integer, and H be an ample line bundle. Then there exist canonical reduced virtual cycles*

$$[P_v^{(q)}(X)]^{\mathrm{red}} \in A_{n-\frac{1}{2}\gamma^2+\frac{1}{2}\rho_\gamma}(P_v^{(q)}(X))$$
$$[M_v^H(X)]^{\mathrm{red}} \in A_{1-\frac{1}{2}\gamma^2+\frac{1}{2}\rho_\gamma}(M_v(X))$$

where ρ_γ is the rank of the symmetric bilinear form

$$\mathsf{B}_\gamma : H^1(X, T_X) \otimes H^1(X, T_X) \to \mathbb{C} : \xi_1 \otimes \xi_2 \mapsto \int_X (\iota_{\xi_1}\iota_{\xi_2}\gamma \cup \omega).$$

The Hodge conjecture predicts that for any smooth projective variety X, all rational (p, p)-classes on X are algebraic. In [Gro1] Grothendieck introduced a variant of the Hodge conjecture.

Conjecture 6.4.2 (variational Hodge conjecture)**.** *Let X be a smooth projective variety and γ be an algebraic (p, p)-class on X. For any smooth projective morphism $f : \mathcal{X} \to \mathcal{B}$ to a smooth connected scheme $\mathcal{B}$ and a horizontal section $\widetilde{\nu}_p$ of $F^p\mathcal{H}^{2p}_{DR}(\mathcal{X}/\mathcal{B})$ such that $\mathcal{X}_0 \cong X$ and $(\widetilde{\nu}_p)_0 = \gamma$ for some closed point $0 \in \mathcal{B}$, the cohomology classes $(\widetilde{\nu}_p)_b$ are algebraic for all closed points $b \in \mathcal{B}$.*

Theorem 6.4.3. *Let X be a Calabi-Yau 4-fold and let γ be a $(2,2)$-class on X. If for some $v \in H^*(X,\mathbb{Q})$ with $v_2 = \gamma$ and $q \in \{-1,0,1\}$*

$$[P_v^{(q)}(X)]^{\mathrm{red}} \neq 0 \in A_*(P_v^{(q)}(X))$$

then Conjecture 6.4.2 holds for X and γ.

This recovers the results of Buchweitz-Flenner [BuFl] (cf. Bloch [Blo]) for Calabi-Yau 4-folds since the reduced virtual cycle equals to the fundamental cycle near the semi-regular point.

Part III

Generalizations

Chapter 7

Torus localization via equivariant virtual pullbacks

In this chapter, we prove Graber-Pandharipande's torus localization formula [GP] via *equivariant virtual pullbacks*. This chapter is based on [AKLPR]

Summary The torus localization formula is an extremely useful tool for computing virtual enumerative invariants when there is a torus action. However, there were some necessary technical assumptions in the original proof of [GP]. These assumptions were significantly weakened by Chang-Kiem-Li [CKL] by using Manolache's virtual pullbacks [Man]. However, it was still desired to fully remove the assumptions.

Inspired by Chang-Kiem-Li's work, we fully remove the technical assumptions by developing equivariant virtual pullbacks for obstruction theories of tor-amplitude $[-2,0]$, when the fixed part is of tor-amplitude $[-1,0]$. In this case, the associated abelian cone stack is not necessarily a vector bundle stack, but we still have the equivariant homotopy property in the localized Chow groups, which allows us to define the equivariant virtual pullback.

Instead of using motivic Borel-Moore homology spectra as in [AKLPR], we use Kresch's Chow groups for simplicity.

7.1 Equivariant virtual pullbacks

In this section, we construct *equivariant virtual pullbacks* for *good obstruction theories*, which are not necessary of tor-amplitude $[-1,0]$.

7.1.1 Equivariant Chow groups

In this subection, we recall basic facts on equivariant Chow groups. We first fix some notations.

Notations

- Let $\mathbf{T} := \mathbb{G}_m$ be the 1-dimensional torus.
- Let $\mathbf{t} \in \mathrm{Pic}^{\mathbf{T}}(\mathrm{Spec}(\mathbb{C}))$ be the 1-dimensional weight 1 representation.
- For an algebraic stack X with a $\mathbf{T}$-action, we define the *equivariant Chow group* as
$$A_*^{\mathbf{T}}(X) := A_*([X/\mathbf{T}]).$$
- Let $\mathbf{s} := c_1(\mathbf{t})$ be the first Chern class. Then $A_*^{\mathbf{T}}(\mathrm{Spec}(\mathbb{C})) = \mathbb{Q}[\mathbf{s}]$.
- For an algebraic stack X with a $\mathbf{T}$-action, we define
$$A^{\mathbf{T}}(X)_{\mathbf{s}} := A^{\mathbf{T}}(X) \otimes_{\mathbb{Q}[\mathbf{s}]} \mathbb{Q}[\mathbf{s}^{\pm 1}].$$
- For any vector bundle E on $X \times B\mathbf{T}$, we have a weight decomposition
$$E = \bigoplus_{w \in \mathbb{Z}} E(w).$$
We let $E^{\mathrm{fix}} := E(0)$ and $E^{\mathrm{mov}} := \bigoplus_{w \neq 0} E(w)$.

We define the *fixed locus* as in [AHR].

Definition 7.1.1 (Fixed locus)**.** Let X be a Deligne-Mumford stack with a $\mathbf{T}$-action. We define the *fixed locus* as

$$X^{\mathbf{T}} := \varinjlim_{\mathbf{T}' \to \mathbf{T}} \underline{\mathrm{Map}}^{\mathbf{T}'}(\mathrm{Spec}(\mathbb{C}), X)$$

where $\underline{\mathrm{Map}}^{\mathbf{T}'}(-,-)$ denotes the equivariant mapping stack and the direct limit is taken for all finite surjection $\mathbf{T}' \to \mathbf{T}$ of tori.

By [AHR], the fixed locus is a closed substack.

Proposition 7.1.2. *Let X be an Deligne-Mumford stack with a $\mathbf{T}$-action. There exists a finite surjection $\mathbf{T}' \to \mathbf{T}$ of tori such that*

$$X^{\mathbf{T}} = \underline{\mathrm{Map}}^{\mathbf{T}'}(\mathrm{Spec}(\mathbb{C}), X).$$

Moreover, the canonical map $X^{\mathbf{T}} \to X$ is a closed embedding.

We will use the following localization theorem of Kresch in [Kre2, Thm. 5.3.5].

Proposition 7.1.3 (Localization of Chow groups)**.** *Let X be a Deligne-Mumford stack with a $\mathbf{T}$-action. Let $X^{\mathbf{T}}$ be the fixed locus and $i : X^{\mathbf{T}} \hookrightarrow X$ be the inclusion map. Then the pushforward*

$$i_*^{\mathbf{T}} : A_*^{\mathbf{T}}(X^{\mathbf{T}})_{\mathbf{s}} \to A_*^{\mathbf{T}}(X)_{\mathbf{s}}$$

is an isomorphism.[1]

We note that the reparametrization $\mathbf{T}' \to \mathbf{T}$ does not affect the Chow groups.

Lemma 7.1.4. *Let X be a Deligne-Mumford stack with a $\mathbf{T}$-action. For any finite surjection $\mathbf{T}' \to \mathbf{T}$ of tori, the smooth pullback*

$$A_*^{\mathbf{T}}(X) \to A_*^{\mathbf{T}'}(X)$$

is an isomorphism.

Proof. Let $E\mathbf{T}_i := \mathbf{t}^{\oplus i} \backslash \{0\}$. Then it suffices to show that the smooth pullback

$$A_*([X \times E\mathbf{T}_i/\mathbf{T}]) \to A_*([X \times E\mathbf{T}_i/\mathbf{T}'])$$

is an isomorphism. Since both $[X \times E\mathbf{T}_i/\mathbf{T}]$ and $[X \times E\mathbf{T}_i/\mathbf{T}']$ are DM stacks with the same coarse moduli space, [Vist, Prop. 6.1] completes the proof. □

It is easy to show that the Euler class of vector bundle of non-zero weights is invertible, directly from the definitions. Here we observe that this can also be deduced as a corollary of Proposition 7.1.3.

Corollary 7.1.5. *Let X be a Deligne-Mumford stack with a trivial $\mathbf{T}$-action. Let E be a $\mathbf{T}$-equivariant vector bundle on X. Assume that $E^{\mathrm{fix}} = 0$. Then the equivariant Euler class*

$$e^{\mathbf{T}}(E) : A_*^{\mathbf{T}}(X)_{\mathbf{s}} \to A_*^{\mathbf{T}}(X)_{\mathbf{s}}$$

is an isomorphism.

[1] The reduced substack $X^{\mathbf{T}}_{\mathrm{red}}$ is $\mathbf{T}$-invariant substack of X.

Proof. Consider the zero section

$$0_E : X \hookrightarrow E.$$

Since $E^{\mathrm{fix}} = 0$, the fixed locus $E^{\mathbf{T}}$ of E is the zero section $0_E : X \hookrightarrow E$. Hence by Proposition 7.1.3, the pushforward

$$(0_E)_* : A^{\mathbf{T}}_*(X)_{\mathbf{s}} \to A^{\mathbf{T}}_*(E)_{\mathbf{s}}$$

is an isomorphism. Therefore the Euler class

$$e^{\mathbf{T}}(E) = 0^!_E \circ (0_E)_* : A^{\mathbf{T}}_*(X)_{\mathbf{s}} \to A^{\mathbf{T}}_*(X)_{\mathbf{s}}$$

is also an isomorphism. □

7.1.2 Equivariant Gysin pullbacks

In this subsection, we introduce **T**-*good cone stacks*, which are analogues of vector bundle stacks in **T**-equivariant geometry. In particular, we will define *equivariant Gysin pullbacks* for **T**-good cone stacks.

We first generalized the equivariant Euler classes of vector bundles to perfect complexes.

Definition 7.1.6 (Equivariant Euler class)**.** Let X be a separated DM stack with a trivial **T**-action. Let $\mathbb{K}$ be a perfect complex of tor-amplitude $[0,1]$ such that $\mathbb{K}^{\mathrm{fix}} = 0$. Then we define the *$T$-equivariant Euler class*

$$e^{\mathbf{T}}(\mathbb{K}) : A^{\mathbf{T}}(X)_{\mathbf{s}} \to A^{\mathbf{T}}(X)_{\mathbf{s}}$$

as follows:

1. *Case 1)* Assume that X is a quasi-projective scheme. Then there is a **T**-equivariant resolution $\mathbb{K} \cong [K_0 \to K_1]$ for some vector bundles K_0 and K_1 on X of non-zero weights. We define the T-equivariant Euler class as

$$e^{\mathbf{T}}(\mathbb{K}) := \frac{e^{\mathbf{T}}(K_0)}{e^{\mathbf{T}}(K_1)} : A^{\mathbf{T}}(X)_{\mathbf{s}} \to A^{\mathbf{T}}(X)_{\mathbf{s}}$$

where $e^{\mathbf{T}}(K_1)$ is invertible by Corollary 7.1.5.

2. *Case 2)* Assume that X is a separated DM stack. Then by the Chow lemma [LMB, Cor. 16.6.1], there is a projective surjective map $p : \widetilde{X} \to X$ from a quasi-projective scheme $\widetilde{X}$. We define the T-equivariant Euler class as

$$\begin{tikzcd} A^{\mathbf{T}}_*(\widetilde{X} \times_X \widetilde{X})_{\mathbf{s}} \arrow[r] \arrow[d, "e^T(\mathbb{K}|_{\widetilde{X}\times_X\widetilde{X}})"] & A^{\mathbf{T}}_*(\widetilde{X})_{\mathbf{s}} \arrow[r] \arrow[d, "e^T(\mathbb{K}|_{\widetilde{X}})"] & A^{\mathbf{T}}(X)_{\mathbf{s}} \arrow[r] \arrow[d, dotted, "e^T(\mathbb{K})"] & 0 \\ A^{\mathbf{T}}_*(\widetilde{X} \times_X \widetilde{X})_{\mathbf{s}} \arrow[r] & A^{\mathbf{T}}_*(\widetilde{X})_{\mathbf{s}} \arrow[r] & A^{\mathbf{T}}(X)_{\mathbf{s}} \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact by the Kimura sequence in Theorem A.1.1.

Remark 7.1.7. In the situation of Definition 7.1.6, it is easy to show that $e^{\mathbf{T}}(\mathbb{K})$ is independent of the choices of a resolution $\mathbb{K} \cong [K_0 \to K_1]$ and a projective cover $\widetilde{X} \to X$.

We will consider the following class of cone stacks.

Definition 7.1.8 (Good cone stacks)**.** Let X be a separated DM stack with a trivial $\mathbf{T}$-action. Let $\mathbb{F}$ be a $\mathbf{T}$-equivariant perfect complex on X. Let $\mathfrak{C}(\mathbb{F})$ denote the associated abelian cone stack. We say that $\mathfrak{C}(\mathbb{F})$ is a $\mathbf{T}$-*good cone stack* if

1. $\mathbb{F}^{\text{fix}}$ has tor-amplitude $[-1, 0]$, and
2. $\mathbb{F}^{\text{mov}}$ has tor-amplitude $[-2, -1]$.

We can simply define the *equivariant Gysin pullbacks* for $\mathbf{T}$-good cone stacks via the localization of Chow groups in Proposition 7.1.3.

Definition 7.1.9 (Equivariant Gysin pullback)**.** Let X be a separated DM stack with a trivial $\mathbf{T}$-action. Let $\mathfrak{C}(\mathbb{F})$ be a $\mathbf{T}$-good cone stack on X associated to a $\mathbf{T}$-equivariant perfect complex $\mathbb{F}$ on X. We define the *T-equivariant Gysin pullback*

$$(0_{\mathfrak{C}(\mathbb{F})})^!_{\mathbf{T}} : A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{F}))_{\mathbf{s}} \to A^{\mathbf{T}}_*(X)_{\mathbf{s}}$$

of the zero section $0_{\mathfrak{C}(\mathbb{F})} : X \to \mathfrak{C}(\mathbb{F})$ as follows:

1. *Case 1)* Assume that $\mathbb{F}^{\text{fix}} = 0$. Then $\mathfrak{C}(\mathbb{F})$ is a cone and $0_{\mathfrak{C}(\mathbb{F})} : X \to \mathfrak{C}(\mathbb{F})$ is a closed embedding We define the T-equivariant Gysin pullback as

$$(0_{\mathfrak{C}(\mathbb{F})})^!_{\mathbf{T}} := e^{\mathbf{T}}(\mathbb{F}^\vee[1]) \circ (0_{\mathfrak{C}(\mathbb{F})})^{-1}_* : A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{F}))_{\mathbf{s}} \to A^T_*(X)_{\mathbf{s}}$$

where the pushforward $(0_{\mathfrak{C}(\mathbb{F})})_* : A^{\mathbf{T}}_*(X)_t \to A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{F}))_{\mathbf{s}}$ is an isomorphism by Proposition 7.1.3 and $e^{\mathbf{T}}(\mathbb{F}^\vee[1])$ is the $\mathbf{T}$-equivariant Euler class in Definition 7.1.6.

2. *Case 2)* Consider the general case. Note that $\mathbb{F} = \mathbb{F}^{\text{fix}} \oplus \mathbb{F}^{\text{mov}}$ and

$$\mathfrak{C}(\mathbb{F}) = \mathfrak{C}(\mathbb{F}^{\text{fix}}) \times \mathfrak{C}(\mathbb{F}^{\text{mov}})$$

where $\mathfrak{C}(\mathbb{F}^{\text{fix}})$ is a vector bundle stack. We define the **T**-equivariant Gysin pullback as the composition

$$(0_{\mathfrak{C}(\mathbb{F})})^!_{\mathbf{T}} : A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{F}))_{\mathbf{s}} \xrightarrow{0^!_{\mathfrak{C}(\mathbb{F}^{\text{fix}})}} A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{F}^{\text{mov}}))_{\mathbf{s}} \xrightarrow{(0_{\mathfrak{C}(\mathbb{F}^{\text{mov}})})^!_{\mathbf{T}}} A^{\mathbf{T}}_*(X)_{\mathbf{s}}$$

where the first map is the Gysin pullback of the vector bundle stack

$$\mathfrak{C}(\mathbb{F}) = \mathfrak{C}(\mathbb{F}^{\text{fix}}) \times \mathfrak{C}(\mathbb{F}^{\text{mov}}) = \mathfrak{C}(\mathbb{F}^{\text{fix}}|_{\mathfrak{C}(\mathbb{F}^{\text{mov}})}) \to \mathfrak{C}(\mathbb{F}^{\text{mov}})$$

and the second map is given by Case 1.

Remark 7.1.10 (Equivariant homotopy property)**.** In **T**-equivariant geometry, the **T**-good cone stacks are the natural generalizations of vector bundle stacks since we have **T**-*equivariant homotopy property*: for any **T**-good cone stack $\mathfrak{C}(\mathbb{F})$ on a separated DM stack X, the **T**-equivariant Gysin pullback gives us an isomorphism

$$(0_{\mathfrak{C}(\mathbb{F})})^!_{\mathbf{T}} : A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{F}))_{\mathbf{s}} \cong A^{\mathbf{T}}_*(X)_{\mathbf{s}}.$$

Proposition 7.1.11 (Whitney sum formula)**.** *Let X be a separated DM stack with trivial* **T**-*action. Let $\mathfrak{C}(\mathbb{F}_1)$ and $\mathfrak{C}(\mathbb{F}_2)$ be two* **T**-*good cone stacks on X. Assume that $\mathbb{F}_2^{\text{fix}} = 0$. Then we have*

$$(0_{\mathfrak{C}(\mathbb{F}_1 \oplus \mathbb{F}_2)})^!_{\mathbf{T}} = (0_{\mathfrak{C}(\mathbb{F}_2)})^!_{\mathbf{T}} \circ (0_{\mathfrak{C}(\mathbb{F}_1|_{\mathfrak{C}(\mathbb{F}_2)})})^!_{\mathbf{T}} : A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{F}_1 \oplus \mathbb{F}_2))_{\mathbf{s}} \to A^{\mathbf{T}}_*(X)_{\mathbf{s}}.$$

7.1.3 Equivariant virtual pullbacks

Definition 7.1.12 (Good obstruction theories)**.** Let $f : X \to Y$ be a **T**-equivariant morphism of algebraic stacks with **T**-actions. Assume that X is a separated DM stack and the **T**-action on X is trivial. Let $\phi : \mathbb{F} \to L_{X/Y}$ be an obstruction theory. We say that $\phi : \mathbb{F} \to L_{X/Y}$ is a **T**-*good obstruction theory* if

1. $\mathbb{F}^{\text{fix}}$ has tor-amplitude $[-1, 0]$, and

2. $\mathbb{F}^{\text{mov}}$ has tor-amplitude $[-2, -1]$.

Definition 7.1.13 (Equivariant virtual pullback)**.** Let $f : X \to Y$ be a $\mathbf{T}$-equivariant morphism of algebraic stacks with $\mathbf{T}$-actions. Let $\phi : \mathbb{F} \to L_{X/Y}$ be a $\mathbf{T}$-good obstruction theory. Assume that X is a separated DM stack and the $\mathbf{T}$-action on X is trivial. We define the $\mathbf{T}$-*equivariant pullback* as the composition

$$f^!_{\mathbf{T}} : A^{\mathbf{T}}_*(Y)_{\mathbf{s}} \xrightarrow{\mathrm{sp}^{\mathbf{T}}_{X/Y}} A^{\mathbf{T}}_*(\mathfrak{C}_{X/Y})_{\mathbf{s}} \xrightarrow{\iota_*} A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{F}))_{\mathbf{s}} \xrightarrow{(0_{\mathfrak{C}(\mathbb{F})})^!_{\mathbf{T}}} A^{\mathbf{T}}_*(X)_{\mathbf{s}}$$

where $\mathrm{sp}^{\mathbf{T}}_{X/Y}$ is the specialization map for the induced map $[X/\mathbf{T}] \to [Y/\mathbf{T}]$, $\iota : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{C}(\mathbb{F})$ is the closed embedding associated to the obstruction theory ϕ, and $(0_{\mathfrak{C}(\mathbb{F})})^!_{\mathbf{T}}$ is the $\mathbf{T}$-equivariant Gysin pullback for the $\mathbf{T}$-good cone stack $\mathfrak{C}(\mathbb{F})$ in Definition 7.1.9.

Proposition 7.1.14 (Bivariance)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a cartesian square of $\mathbf{T}$-*equivariant morphisms of algebraic stacks with* $\mathbf{T}$-*actions. Assume that* X *and* X' *are separated Deligne-Mumford stacks and the* $\mathbf{T}$-*actions on* X *and* X' *are trivial. Let* $\phi : \mathbb{F} \to L_{X/Y}$ *be a* $\mathbf{T}$-*good obstruction theory. Let*

$$\phi' : (g')^*\mathbb{F} \xrightarrow{(g')^*(\phi)} (g')^*(L_{X/Y}) \to L_{X'/Y'}$$

be the induced $\mathbf{T}$-*good obstruction theory. Then we have the following properties:*

1. *If* g *is a proper DM morphism, then we have*

$$f^!_{\mathbf{T}} \circ g_* = g'_* \circ (f')^!_{\mathbf{T}} : A_*(Y') \to A_*(X).$$

2. *If* g *is a equi-dimensional flat morphism, then we have*

$$(f')^!_{\mathbf{T}} \circ g^* = (g')^* \circ f^!_{\mathbf{T}} : A_*(Y) \to A_*(X').$$

3. *If* g *is a local complete intersection morphism and* Y' *has affine stabilizers, then we have*

$$(f')^!_{\mathbf{T}} \circ g^! = (g')^! \circ f^!_{\mathbf{T}} : A_*(Y) \to A_*(X').$$

Proof. It follows immediately from Proposition 2.1.19. □

7.1.4 Funtoriality

Theorem 7.1.15 (Functoriality). *Consider a commutative diagram of algebraic stacks with* $\mathbf{T}$*-actions*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

where f and g are $\mathbf{T}$*-equivariant DM morphisms. Assume that X is a separated DM stack and the* $\mathbf{T}$*-action on X is trivial. Assume that Y has affine stabilizers. Let $\phi_{X/Y} : \mathbb{F}_{X/Y} \to L_{X/Y}$, $\phi_{X/Z} : \mathbb{F}_{X/Z} \to L_{X/Z}$ be* $\mathbf{T}$*-good obstruction theories and $\phi_{Y/Z} : \mathbb{F}_{Y/Z} \to L_{Y/Z}$ be a* $\mathbf{T}$*-equivariant perfect obstruction theory. Assume that there exists a morphism of distinguished triangles*

$$\begin{tikzcd} f^*(\mathbb{F}_{Y/Z}) \arrow[r] \arrow[d, "f^*(\phi_{Y/Z})"] & \mathbb{F}_{X/Z} \arrow[r] \arrow[d, "\phi_{X/Z}"] & \mathbb{F}_{X/Y} \arrow[r] \arrow[d, dotted, "\phi'_{X/Y}"] & {} \\ \tau^{\geqslant -1} f^*(L_{Y/Z}) \arrow[r] & L_{X/Z} \arrow[r] & L'_{X/Y} \arrow[r] & {} \end{tikzcd}$$

for some $\phi'_{X/Y}$ such that $\phi_{X/Y} = r \circ \phi'_{X/Y}$. Then we have

$$(g\circ f)^!_{\mathbf{T}} = f^!_{\mathbf{T}} \circ g^! : A^{\mathbf{T}}_*(Z)_{\mathbf{s}} \to A^{\mathbf{T}}_*(X)_{\mathbf{s}}.$$

The proof is similar to the functoriality of ordinary virtual pullbacks in Theorem 2.3.12. As in Lemma 2.3.18, we begin with a special case

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "0_{\mathfrak{C}(\mathbb{F})}\circ f"'] & Y \arrow[r, "0_{\mathfrak{C}(\mathbb{F})}"] & \mathfrak{C}(\mathbb{F}) \end{tikzcd}$$

where $Z = \mathfrak{C}(\mathbb{F})$ is a $\mathbf{T}$-good cone stack over Y and $g = 0_{\mathfrak{C}(\mathbb{F})}$ is the zero section.

Lemma 7.1.16. *Let $f : X \to Y$ be a* $\mathbf{T}$*-equivariant morphism of algebraic stacks with* $\mathbf{T}$*-actions. Assume that X is a separated DM stack and the* $\mathbf{T}$*-action on X is trivial. Let $\phi_{X/Y} : \mathbb{F}_{X/Y} \to L_{X/Y}$ be a perfect obstruction theory for $f : X \to Y$. Let $\mathfrak{C}(\mathbb{F})$ be a* $\mathbf{T}$*-good cone stack for some* $\mathbf{T}$*-equivariant perfect complex $\mathbb{F}$. Let*

$$\phi : \mathbb{F} \to L_{Y/\mathfrak{C}(\mathbb{F})}$$

be the canonical $\mathbf{T}$*-good obstruction theory of the zero section $0_{\mathfrak{C}(\mathbb{F})} : Y \to \mathfrak{C}(\mathbb{F})$. Then*

$$f^*(\phi) \oplus \phi_{X/Y} : f^*(\mathbb{F}) \oplus \mathbb{F}_{X/Y} \to L_{X/\mathfrak{C}(\mathbb{F})} = \tau^{\geqslant -1} f^*(L_{Y/\mathfrak{C}(\mathbb{F})}) \oplus L_{X/Y}$$

is a $\mathbf{T}$*-good obstruction theory for the composition*

$$X \xrightarrow{f} Y \xrightarrow{0_{\mathfrak{C}(\mathbb{F})}} \mathfrak{C}(\mathbb{F})$$

and we have

$$(0_{\mathfrak{C}(\mathbb{F})} \circ f)^!_{\mathbf{T}} = f^! \circ (0_{\mathfrak{C}(\mathbb{F})})^!_{\mathbf{T}} : A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{F}))_{\mathbf{s}} \to A^{\mathbf{T}}_*(X)_{\mathbf{s}}.$$

Proof. Note that

$$\mathfrak{C}(\mathbb{F}) = \mathfrak{E}(\mathbb{F}^{\mathrm{fix}}) \times_Y \mathfrak{C}(\mathbb{F}^{\mathrm{mov}})$$

and $\mathfrak{E}(\mathbb{F}^{\mathrm{fix}}) := \mathfrak{C}(\mathbb{F}^{\mathrm{fix}})$ is a vector bundle stack. Consider the commutative diagram

$$\begin{tikzcd}
 & & \mathfrak{C}(\mathbb{F}) \arrow[d, "\pi"] \\
X \arrow[r, "f"] \arrow[urr, bend left, "0_{\mathfrak{C}(\mathbb{F})}\circ f"] \arrow[rr, bend right, "0_{\mathfrak{C}(\mathbb{F}^{\mathrm{mov}})}\circ f"'] & Y \arrow[r, "0_{\mathfrak{C}(\mathbb{F}^{\mathrm{mov}})}"] \arrow[ur, "0_{\mathfrak{C}(\mathbb{F})}"] & \mathfrak{C}(\mathbb{F}^{\mathrm{mov}}).
\end{tikzcd}$$

Then we can form a diagram

$$\begin{tikzcd}
\mathfrak{C}(\mathbb{F}) \arrow[r, squiggly, "\mathrm{sp}"] \arrow[d, "\pi"] & \mathfrak{C}_{X/\mathfrak{C}(\mathbb{F})} \arrow[r, hook] \arrow[d, "r"] & \mathfrak{C}_f \times \mathfrak{C}(\mathbb{F}) \arrow[r, hook] \arrow[d, "\mathrm{id}\times\pi"] & \mathfrak{E}(\mathbb{F}_{X/Y}) \times \mathfrak{C}(\mathbb{F}) \arrow[d, "\mathrm{id}\times\pi"] \\
\mathfrak{C}(\mathbb{F}^{\mathrm{mov}}) \arrow[r, squiggly, "\mathrm{sp}"] & \mathfrak{C}_{X/\mathfrak{C}(\mathbb{F}^{\mathrm{mov}})} \arrow[r, hook] & \mathfrak{C}_f \times \mathfrak{C}(\mathbb{F}^{\mathrm{mov}}) \arrow[r, hook] & \mathfrak{E}(\mathbb{F}_{X/Y}) \times \mathfrak{C}(\mathbb{F}^{\mathrm{mov}})
\end{tikzcd}$$

where two right two squares are cartesian and the vertical arrows are $\mathfrak{E}(\mathbb{F}^{\mathrm{fix}})$ torsors (curly arrows are not genuine morphisms). By Lemma 2.1.21, we have

$$r^* \circ \mathrm{sp}_{X/\mathfrak{C}(\mathbb{F}^{\mathrm{mov}})} = \mathrm{sp}_{X/\mathfrak{C}(\mathbb{F})} \circ \pi^*.$$

Since the smooth pullback

$$\pi^* : A_*(\mathfrak{C}(\mathbb{F}^{\mathrm{mov}})) \to A_*(\mathfrak{C}(\mathbb{F}))$$

is an isomorphism, it suffices to show the statement for

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend right, "0_{\mathfrak{C}(\mathbb{F}^{\mathrm{mov}})}\circ f"'] & Y \arrow[r, "0_{\mathfrak{C}(\mathbb{F}^{\mathrm{mov}})}"] & \mathfrak{C}(\mathbb{F}^{\mathrm{mov}}).
\end{tikzcd}$$

Hence we may assume that $\mathbb{F}^{\text{fix}} = 0$ and $\mathbb{F} = \mathbb{F}^{\text{mov}}$.

Note that $\mathfrak{C}(\mathbb{F})$ is a cone and $0_{\mathfrak{C}(\mathbb{F}} : Y \hookrightarrow \mathfrak{C}(\mathbb{F})$ is a closed embedding. Form a cartesian diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, hook, "0_{\mathfrak{C}(\mathbb{F})}"] \\ X \arrow[r, "f"] & Y \arrow[r, hook, "0_{\mathfrak{C}(\mathbb{F})}"] & \mathfrak{C}(\mathbb{F}). \end{tikzcd}$$

Since the pushforward

$$(0_{\mathfrak{C}(\mathbb{F})})_* : A^{\mathbf{T}}_*(Y)_{\mathbf{s}} \to A^{\mathbf{T}}_*(\mathfrak{C}(\mathbb{F}))_{\mathbf{s}}$$

is an isomorphism, if suffices to show that

$$(0_{\mathfrak{C}(\mathbb{F}|_X)\times\mathfrak{E}(\mathbb{F}_{X/Y})})^!_{\mathbf{T}} \circ a_* \circ \mathrm{sp}_{X/Y} = f^! \circ (0_{\mathfrak{C}(\mathbb{F})})^!_{\mathbf{T}} \circ (0_{\mathfrak{C}(\mathbb{F})})_*$$

where $b : \mathfrak{C}_{X/Y} \xrightarrow{\iota} \mathfrak{E}(\mathbb{F}_{X/Y}) \xrightarrow{(0,1)} \mathfrak{C}(\mathbb{F}|_X) \times \mathfrak{E}(\mathbb{F}_{X/Y})$ is the inclusion map, and $\iota : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}(\mathbb{F}_{X/Y})$ is the closed embedding induced by ϕ. By Proposition 7.1.11, this is equivalent to

$$0^!_{\mathfrak{E}(\mathbb{F}_{X/Y})} \circ e^{\mathbf{T}}(\mathbb{F}^\vee[1]) \circ \iota \circ \mathrm{sp}_{X/Y} = f^! \circ e^{\mathbf{T}}(\mathbb{F}^\vee[1]).$$

Since the equivariant Euler class $e^{\mathbf{T}}(\mathbb{F}^\vee[1])$ commutes with Gysin pullbacks and virtual pullbacks, we have the desired equality. □

As in Lemma 2.3.19, we use the double deformation space of [KKP] to reduce the general case to the special case in Lemma 7.1.16.

Proof of Theorem 7.1.15. Let

$$h : X \times \mathbb{A}^1 \to Y \times \mathbb{A}^1 \to M^\circ_{Y/Z}$$

be the composition. Form a morphism of distinguished triangles

$$\begin{tikzcd} f^*(\mathbb{F}_{Y/Z}) \boxtimes \mathcal{O}_{\mathbb{A}^1} \arrow[r, "{(T,a)}"] \arrow[d, "f^*\phi_{Y/Z}"] & (f^*(\mathbb{F}_{Y/Z}) \oplus \mathbb{F}_{X/Z}) \boxtimes \mathcal{O}_{\mathbb{A}^1} \arrow[r] \arrow[d, "{(f^*\phi_{Y/Z},\phi_{X/Z})}"] & \mathbb{F}_h \arrow[d, dotted, "\phi'_h"] \\ \tau^{\geqslant -1} f^*(L_{Y/Z}) \boxtimes \mathcal{O}_{\mathbb{A}^1} \arrow[r, "{(T,a)}"] & (\tau^{\geqslant -1} f^*(L_{Y/Z}) \oplus L_{X/Z}) \boxtimes \mathcal{O}_{\mathbb{A}^1} \arrow[r] & L'_h \end{tikzcd}$$

for some perfect complex $\mathbb{F}_h$ and a map ϕ'_h, where the lower distinguished triangle is given as in Lemma 2.3.20. Then the composition

$$\phi_h : \mathbb{F}_h \xrightarrow{\phi'_h} L'_h \to \tau^{\geqslant -1} L_h \cong L_h$$

is an obstruction theory. Since the fibers of $\mathbb{F}_h$ over $\lambda \in \mathbb{A}^1$ are

$$(\mathbb{F}_h)_\lambda = \begin{cases} \mathbb{F}_{X/Z} & \text{if } \lambda \neq 0 \\ \mathbb{F}_{X/Y} \oplus f^*\mathbb{F}_{Y/Z} & \text{if } \lambda = 0 \end{cases}$$

ϕ_h is also a $\mathbf{T}$-good obstruction theory. Hence we have a $\mathbf{T}$-equivariant virtual pullback

$$h^!_{\mathbf{T}} : A_*(M^\circ_{Y/Z}) \to A_*(X \times \mathbb{A}^1).$$

Since the $\mathbf{T}$-equivariant virtual pullbacks are bivariant by Proposition 7.1.14, we have

$$(g \circ f)^!_{\mathbf{T}} = (0_{\mathfrak{C}_{Y/Z}} \circ f)^!_{\mathbf{T}} \circ \mathrm{sp}^{\mathbf{T}}_{Y/Z}$$

where $(0_{\mathfrak{C}_{Y/Z}} \circ f)^!_{\mathbf{T}}$ is the $\mathbf{T}$-equivariant virtual pullback of the fiber of ϕ_h. As in the proof of Theorem 2.3.12, by a deformation argument, we have assume that the $\mathbf{T}$-good obstruction theory for $(0_{\mathfrak{C}_{Y/Z}} \circ f)$ is given by

$$f^*(\phi_{Y/Z}) \oplus \phi_{X/Y} : f^*(\mathbb{F}_{Y/Z}) \oplus \mathbb{F}_{X/Y} \to \tau^{\geqslant -1} f^*(L_{Y/Z}) \oplus L_{X/Y}.$$

Then Lemma 7.1.16 completes the proof. □

7.2 Localization of virtual cycles

Definition 7.2.1 (Induced obstruction theory)**.** Let X be a separated DM stack with $\mathbf{T}$-action. Let $\phi : \mathbb{F} \to L_X$ be a $\mathbf{T}$-equivariant perfect obstruction theory. Choose a reparemetrization $\mathbf{T}' \to \mathbf{T}$ such that $\mathbf{T}'$ acts trivially on the fixed locus $X^{\mathbf{T}}$. We define the *induced obstruction theory* as the composition

$$\phi_{X^{\mathbf{T}}} : \mathbb{F}|^{\mathrm{fix}}_{X^{\mathbf{T}}} \to \mathbb{F}|_{X^{\mathbf{T}}} \xrightarrow{\phi|_{X^{\mathbf{T}}}} L_X|_{X^{\mathbf{T}}} \to L_{X^{\mathbf{T}}}.$$

Lemma 7.2.2. *In the situation of Definition 7.2.1, the composition $\phi_{X^{\mathbf{T}}}$ is a perfect obstruction theory. Moreover, $\phi_{X^{\mathbf{T}}}$ is independent of the choice of a reparametrization $\mathbf{T}' \to \mathbf{T}$.*

Proof. The independence is trivial. We will show that $\phi_{X^{\mathbf{T}}}$ is an obstruction theory. By [AHR, Thm. 4.3], we may assume that X is an affine scheme. Then [GP, Prop. 1] proves the claim (cf. [CKL, Lem. 3.3]). Moreover, $\mathbb{F}|^{\mathrm{fix}}_{X^{\mathbf{T}}}$ is clearly of tor-amplitude $[-1,0]$. $\square$

If the perfect obstruction theory comes from a quasi-smooth derived enhancement, then the above lemma follows from the general description of the cotangent complexes of derived mapping stacks.

Remark 7.2.3 (Derived fixed locus)**.** Let $\mathbb{X}$ be a quasi-smooth derived Deligne-Mumford stack with $\mathbf{T}$-action. (More generally, let $\mathbb{X}$ be a homotopically finitely presented derived Artin stack.) We define the *homotopy fixed locus* as the equivariant derived mapping stack

$$\mathbb{X}^{h\mathbf{T}} := R\underline{\mathrm{Map}}^{\mathbf{T}}(\mathrm{Spec}(\mathbb{C}), \mathbb{X}).$$

Equivalently, we can define the homotopy fixed locus via a homotopy fiber diagram

$$\begin{tikzcd} \mathbb{X}^{h\mathbf{T}} \arrow[rr] \arrow[d] & & \mathrm{Spec}(\mathbb{C}) \arrow[d, "\mathrm{id}"] \\ M := R\underline{\mathrm{Map}}(BT, [X/\mathbf{T}]) \arrow[rr] & & R\underline{\mathrm{Map}}(B\mathbf{T}, B\mathbf{T}). \end{tikzcd}$$

Then we can easily compute the cotangent complex of $\mathbb{X}^{h\mathbf{T}}$ as follows: Let

$$\mathrm{ev} : M \times B\mathbf{T} \to [X/\mathbf{T}]$$

be the evaluation map and $\pi : M \times B\mathbf{T} \to M$ be the projection map. Then by [HLP], we have

$$\mathbb{L}_M = R\pi_* L\mathrm{ev}^* \mathbb{L}_{[X/T]} = \mathbb{L}_{[X/T]}|^{\mathrm{fix}}_{M\times B\mathbf{T}}.$$

Hence we have

$$\mathbb{L}_{\mathbb{X}^{h\mathbf{T}}} = \mathrm{cone}(\mathbb{L}_M|_{\mathbb{X}^{h\mathbf{T}}} \to \mathbb{L}_{R\underline{\mathrm{Map}}(B\mathbf{T},B\mathbf{T})}|_{\mathbb{X}^{h\mathbf{T}}}) = \mathbb{L}_{[\mathbb{X}/\mathbf{T}]/B\mathbf{T}}|^{\mathrm{fix}}_{\mathbb{X}^{h\mathbf{T}}\times B\mathbf{T}}.$$

In particular, this proves Lemma 7.2.2, when the obstruction theory is induced by a $\mathbf{T}$-equivariant derived enhancement.

Definition 7.2.4 (Virtual normal bundle)**.** Let X be a separated DM stack with $\mathbf{T}$-action. Let $\phi : \mathbb{F} \to L_X$ be a $\mathbf{T}$-equivariant perfect obstruction theory. Choose a reparemetrization $\mathbf{T}' \to \mathbf{T}$ such that $\mathbf{T}'$ acts trivially on the fixed locus $X^{\mathbf{T}}$. We define the *Euler class of the virtual normal bundle* as

$$e^{\mathbf{T}}(N^{\mathrm{vir}}) := e^{\mathbf{T}'}((\mathbb{F}|^{\mathrm{mov}}_{X^{\mathbf{T}}}[1])^{\vee}[1]) : A^{\mathbf{T}}(X^{\mathbf{T}})_{\mathbf{s}} \xrightarrow{\cong} A^{\mathbf{T}}(X^{\mathbf{T}})_{\mathbf{s}}$$

where $e^{\mathbf{T}'}((\mathbb{F}|^{\mathrm{mov}}_{X^{\mathbf{T}}}[1])^{\vee}[1])$ is the equivariant Euler class in Definition 7.1.6 and we identified $A^{\mathbf{T}'}_*(X^{\mathbf{T}}) = A^{\mathbf{T}}_*(X^{\mathbf{T}})$ via Lemma 7.1.4.

Theorem 7.2.5 (Localization of virtual cycles)**.** *Let X be a separated Deligne-Mumfor stack with $\mathbf{T}$-action. Let $\phi : \mathbb{F} \to L_X$ be a $\mathbf{T}$-equivariant perfect obstruction theory. Let $X^{\mathbf{T}}$ be the fixed locus and $\phi_{X^{\mathbf{T}}}$ be the induced perfect obstruction in Definition 7.2.1. Then we have*

$$[X]^{\mathrm{vir}} = i_*\left(\frac{[X^{\mathbf{T}}]^{\mathrm{vir}}}{e^{\mathbf{T}}(N^{\mathrm{vir}})}\right) \in A^{\mathbf{T}}_*(X)_{\mathbf{s}}$$

where $e^{\mathbf{T}}(N^{\mathrm{vir}})$ is the Euler class of the virtual normal bundle in Definition 7.2.4 and $i : X^{\mathbf{T}} \hookrightarrow X$ is the inclusion map.

Proof. Replacing $\mathbf{T}$ by a reparametrization, we may assume that $\mathbf{T}$ acts trivially on $X^{\mathbf{T}}$. Consider a morphism of distinguished triangles

$$\begin{tikzcd} \mathbb{F}|_{X^{\mathbf{T}}} \arrow[r] \arrow[d, "\phi|_{X^{\mathbf{T}}}"] & \mathbb{F}|^{\mathrm{fix}}_{X^{\mathbf{T}}} \arrow[r] \arrow[d, "\phi_{X^{\mathbf{T}}}"] & \mathbb{F}|^{\mathrm{mov}}_{X^{\mathbf{T}}}[1] \arrow[d, dotted] \\ L_X|_{X^{\mathbf{T}}} \arrow[r] & L_{X^{\mathbf{T}}} \arrow[r] & L'_{X^{\mathbf{T}}/X} \end{tikzcd}$$

for some dotted arrow where $L'_{X^{\mathbf{T}}/X} := \mathrm{cone}(L_X|_{X^{\mathbf{T}}} \to L_{X^{\mathbf{T}}})$. By Theorem 7.1.15, we have

$$[X^{\mathbf{T}}]^{\mathrm{vir}} = i^!_{\mathbf{T}}[X]^{\mathrm{vir}} \in A^{\mathbf{T}}_*(X^{\mathbf{T}})_{\mathbf{s}}.$$

Since $i_* : A^{\mathbf{T}}_*(X^{\mathbf{T}})_{\mathbf{s}} \to A^{\mathbf{T}}_*(X)_{\mathbf{s}}$ is an isomorphism by Proposition 7.1.3, we may write

$$[X]^{\mathrm{vir}} = i_*(\alpha)$$

for some $\alpha \in A^{\mathbf{T}}_*(X^{\mathbf{T}})_{\mathbf{s}}$. Then we have

$$[X^{\mathbf{T}}]^{\mathrm{vir}} = i^!_{\mathbf{T}} \circ i_*(\alpha) = (0_{\mathfrak{C}(\mathbb{F}|^{\mathrm{mov}}_{X^{\mathbf{T}}}[1])})^!_{\mathbf{T}} \circ (0_{\mathfrak{C}(\mathbb{F}|^{\mathrm{mov}}_{X^{\mathbf{T}}}[1])})_*(\alpha) = e^{\mathbf{T}}(N^{\mathrm{vir}})(\alpha).$$

Since the equivariant Euler class $e^{\mathbf{T}}(N^{\mathrm{vir}})$ is invertible, we have the desired equality. ◻

Remark 7.2.6 (Comparison to Graber-Pandharipande/Chang-Kiem-Li)**.** In Graber-Pandharipande [GP], Theorem 7.2.5 is shown when there exist a **T**-equivariant global embedding $X \hookrightarrow Y$ to a smooth Deligne-Mumford stack Y and a **T**-equivariant global resolution $\mathbb{F} \cong [F^{-1} \to F^0]$ by vector bundles F^{-1} and F^0. In Chang-Kiem-Li [CKL], Theorem 7.2.5 is shown when the virtual normal bundle $\mathbb{F}|_{X^{\mathbf{T}}}^{\mathrm{mov}}[1]$ has a global resolution $\mathbb{F}|_{X^{\mathbf{T}}}^{\mathrm{mov}}[1] \cong [N^{-2} \to N^{-1}]$ by vector bundle N^{-2} and N^{-1}. Theorem 7.2.5 fully removes these technical assumptions on global embeddings/resolutions.

Remark 7.2.7 (Positive characteristic)**.** Let X be an Artin stack with finite stabilizers over an algebraically closed field k of characteristic $p > 0$. We note that X is not necessarily a Deligne-Mumford stack since the stabilizers $\underline{\mathrm{Aut}}_X(x)$ may not be étale. Moreover, if there is a **T**-action on X, the canonical map

$$u : \underline{\mathrm{Map}}^{\mathbf{T}}(\mathrm{Spec}(\mathbb{C}), X) \to X$$

may not be a closed embedding (e.g. when $X = B\alpha_p$ with a natural non-trivial **T**-action, where $\alpha_p := \ker(\mathbb{G}_a \xrightarrow{(-)^p} \mathbb{G}_a)$). However, if "the ramifiedness of the stabilizers only lie on the fixed part", i.e. $\Omega^{\mathrm{mov}}_{\underline{\mathrm{Aut}}_X(x)} = 0$ for all $x \in X(k)$ in the image of the map u, then the map u is unramified. If we assume that u is quasi-compact, then u is finite unramified, and we still have the formula

$$[X]^{\mathrm{vir}} = i_* \left(\frac{[X^{\mathbf{T}}]^{\mathrm{vir}}}{e^{\mathbf{T}}(N^{\mathrm{vir}})} \right) \in A^{\mathbf{T}}_*(X)_{\mathbf{s}}$$

after a suitable reparametrization $\mathbf{T} \to \mathbf{T}$. We refer to [AKLPR] for details.

Chapter 8

Cosection localization via (-1)-shifted 1-forms

In this chapter, we revisit Kiem-Li's cosection localization [KL1] via *derived algebraic geometry* [ToVe]. This chapter is based on [BKP, Appendix A].

Summary In derived algebraic geometry, quasi-smooth derived schemes are natural analogs of schemes with perfect obstruction theories. Moreover, (-1)-shifted 1-forms are natural analogs of cosections.

Firstly, we prove *scheme-theoretical* cone reduction lemma for (-1)-shifted *closed* 1-forms. The key idea is to use the derived Poincare lemma, i.e. a (-1)-shifted closed 1-form is locally exact.

Secondly, we speculate that the cosection-localized virtual cycles for quasi-smooth derived schemes with (-1)-shifted closed 1-forms are the Oh-Thomas virtual cycles of the derived zero locus, which is a (-2)-shifted symplectic.

8.1 Three reductions

We consider the following hierarchy of structures

$$\begin{array}{c} \{\text{derived schemes}\} \\ \Big\downarrow{\scriptstyle (a)} \\ \{\text{schemes with obstruction theories}\} \\ \Big\downarrow{\scriptstyle (b)} \\ \left\{\begin{array}{c}\text{schemes with closed embedding of} \\ \text{their intrinsic normal cone into an abelian cone stack}\end{array}\right\}. \end{array}$$

More precisely, the above two arrows can be given as follows:

(a) For any homotopically finitely presented derived scheme $\mathbb{X}$, there is an induced obstruction theory

$$\phi : \mathbb{E} := \mathbb{L}_{\mathbb{X}}|_X \to \mathbb{L}_X \to L_X := \tau^{\geqslant -1}\mathbb{L}_X$$

on the classical truncation $X := \mathbb{X}_{\mathrm{cl}}$ by [STV, Prop. 1.2].

(b) For any scheme X with an obstruction theory $\phi : \mathbb{E} \to L_X$, there is an induced closed embedding

$$\iota : \mathfrak{C}_X \hookrightarrow \mathfrak{C}(\mathbb{E})$$

of the intrinsic normal cone $\mathfrak{C}_X$ into the abelian cone stack $\mathfrak{C}(\mathbb{E})$.

We note that the (-1)-*shifted* 1-*forms* are natural analogs of *cosections* in derived algebraic geometry. We have a similar hierarchy for them:

(a) Let $\alpha : \mathcal{O}_{\mathbb{X}} \to \mathbb{L}_{\mathbb{X}}[-1]$ be a (-1)-shifted 1-form on a homotopically finitely presented derived scheme $\mathbb{X}$. Then we have an induced *cosection*

$$\sigma := \alpha|_X^{\vee} : \mathbb{E}^{\vee}[1] \to \mathcal{O}_X$$

for the induced obstruction theory $\phi : \mathbb{E} := \mathbb{L}_{\mathbb{X}}|_X \to L_X$ on the classical truncation $X := \mathbb{X}_{\mathrm{cl}}$.

(b) Let $\sigma : \mathbb{E}^\vee[1] \to O_X$ be a cosection for an obstruction theory $\phi : \mathbb{E} \to L_X$ on a scheme X. Then we have an induced *linear function*

$$\mathfrak{l}_\sigma := \mathfrak{C}(\sigma^\vee[1]) : \mathfrak{C}(\mathbb{E}) \to \mathbb{A}^1_X$$

on the associated cone stack $\mathfrak{C}(\mathbb{E})$.

Now we state our main result in this subsection.

Theorem 8.1.1. *Let $\mathbb{X}$ be a homotopically finitely presented derived scheme. Let $\phi : \mathbb{E} \to \tau^{\geqslant -1}\mathbb{L}_X$ be the induced obstruction theory on the classical truncation $X := \mathbb{X}_{\mathrm{cl}}$ and let $\iota : \mathfrak{C}_X \hookrightarrow \mathfrak{C}(\mathbb{E})$ be the induced closed embedding.*

1. **(Cone reduction, [KL1])** *For any (-1)-shifted 1-form $\alpha : O_{\mathbb{X}} \to \mathbb{L}_{\mathbb{X}}[-1]$, we have a commutative diagram of cone stacks*

$$\begin{tikzcd} & & \mathfrak{C}(\mathbb{E}_\sigma) \arrow[r] \arrow[d, hook] & X \arrow[d, hook, "0"] \\ (\mathfrak{C}_X)_{\mathrm{red}} \arrow[r, hook] \arrow[urr, dotted, "\iota^{\mathrm{red}}"] & \mathfrak{C}_X \arrow[r, hook, "\iota"] & \mathfrak{C}(\mathbb{E}) \arrow[r, "\mathfrak{l}_\sigma"] & \mathbb{A}^1_X \end{tikzcd}$$

for a unique dotted arrow where $\sigma := \alpha|_X^\vee$ and $\mathbb{E}_\sigma := \mathrm{cone}(\sigma^\vee[1])$.

2. **(Obstruction theory reduction)** *For any (-1)-shifted* closed *1-form $\widetilde{\alpha}$, we have a commutative diagram of complexes*

$$\begin{tikzcd} O_X[1] \arrow[r, "\sigma^\vee[1]"] & \mathbb{E} \arrow[r] \arrow[d, "\phi"] & \mathbb{E}_\sigma \arrow[dl, dotted, "\phi^{\mathrm{red}}"] \\ & \tau^{\geqslant -1}\mathbb{L}_X & \end{tikzcd}$$

for a unique dotted arrow where $\alpha := \widetilde{\alpha}_0 : O_{\mathbb{X}} \to \mathbb{L}_{\mathbb{X}}[-1]$ is the underlying (-1)-shifted 1-form, $\sigma := \alpha|_X^\vee$, and $\mathbb{E}_\sigma := \mathrm{cone}(\sigma^\vee[1])$.

3. **(Derived reduction, [STV])** *For any (-1)-shifted* exact *1-form $\widetilde{\alpha} = d_{DR}(u)$, we have a homotopy commutative diagram of derived schemes*

$$\begin{tikzcd} & \mathbb{X}^{\mathrm{red}} \arrow[r] \arrow[d] & \mathbb{X} \arrow[d, "0"] \\ X \arrow[r] \arrow[ur] & \mathbb{X} \arrow[r, "u"] & \mathbb{A}^1_{\mathbb{X}}[-1] \end{tikzcd}$$

for some $\mathbb{X}^{\mathrm{red}}$ where the square is homotopy cartesian and the triangle induces isomorphisms $X \cong (\mathbb{X}^{\mathrm{red}})_{\mathrm{cl}} \cong \mathbb{X}_{\mathrm{cl}}$.

Before we prove Theorem 8.1.1, we explain how the three reductions in Theorem 8.1.1 are related.

Remark 8.1.2. In the situation of Theorem 8.1.1, we have the following:

1. The obstruction theory reduction in Theorem 8.1.1(2) is equivalent to the *scheme-theoretical cone reduction*, i.e., there exists a commutative diagram of cone stacks

$$\begin{tikzcd} & \mathfrak{C}(\mathbb{E}_\sigma) \arrow[r] \arrow[d, hook] & X \arrow[d, hook, "0"] \\ \mathfrak{C}_X \arrow[r, hook, "\iota"] \arrow[ur, dotted] & \mathfrak{C}(\mathbb{E}) \arrow[r, "\mathfrak{l}_\sigma"] & \mathbb{A}^1_X \end{tikzcd}$$

   for some dotted arrow. Hence the obstruction theory reduction in Theorem 8.1.1(2) clearly implies the cone reduction in Theorem 8.1.1(1).

2. The derived reduction in Theorem 8.1.1(3) implies the obstruction theory reduction in Theorem 8.1.1(2). Indeed, the commutative diagram of derived schemes induces a commutative diagram of cotangent complexes

$$\begin{tikzcd} \mathcal{O}_X[1] = \mathbb{L}_{\mathbb{X}^{\mathrm{red}}/\mathbb{X}}|_X[-1] \arrow[r, "d_{DR}(u)_0"] & \mathbb{L}_{\mathbb{X}}|_X \arrow[r] \arrow[d] & \mathbb{L}_{\mathbb{X}^{\mathrm{red}}}|_X \arrow[dl, dotted] \\ & \mathbb{L}_X & \end{tikzcd}$$

   where $\mathbb{L}_{\mathbb{X}^{\mathrm{red}}/\mathbb{X}}|_X = \mathbb{L}_{\mathbb{X}/\mathbb{A}^1_{\mathbb{X}}[-1]}|_X = \mathcal{O}_X[2]$. By composing with the canonical map $\mathbb{L}_X \to \tau^{\geqslant -1}\mathbb{L}_X$, we obtain the desired obstruction theory reduction.

Note that the cone reduction in Theorem 8.1.1(1) is shown by Kiem-Li [KL1] (see Proposition 3.1.9) and the derived reduction in Theorem 8.1.1(3) is trivial. (This approach was first introduced by Schürg-Toën-Vezzosi [STV].) Thus the essential part of Theorem 8.1.1 is the obstruction theory reduction in Theorem 8.1.1(2). We will deduce this from the *derived Poincaré lemma* (which was essentially shown by Brav-Bussi-Joyce [BBJ]):

Proposition 8.1.3. *Let $\mathbb{X}$ be a homotopically finitely presented derived scheme. Let $\widetilde{\alpha}$ be a (-1)-shifted closed 1-form. Then there exists a Zariski open cover $\mathbb{U}_i \to \mathbb{X}$ such that $\widetilde{\alpha}|_{\mathbb{U}_i}$ are (-1)-shifted exact 1-forms.*

Proof. This is essentially shown in [BBJ, Prop. 5.6]. Indeed, it follows from the arguments in [BBJ, Prop. 5.7(a)], if we replace the closed 2-form by a closed 1-form. □

Now Theorem 8.1.1 is a direct corollary.

Proof of Theorem 8.1.1(2). The uniqueness follows from

$$\mathrm{Hom}_X(\mathcal{O}_X[2], \tau^{\geq -1}\mathbb{L}_X) = 0.$$

For the existence, it suffice to show that

$$\phi \circ \sigma^\vee[1] \in \mathrm{Hom}_X(\mathcal{O}_X[1], \tau^{\geq -1}\mathbb{L}_X) = \Gamma(X, h^{-1}(\mathbb{L}_X))$$

vanishes. Hence the statement is local. By the derived Poincare lemma in Proposition 8.1.3, we may assume that $\widetilde{\alpha}$ is exact. Then Theorem 8.1.1(3) completes the proof by Remark 8.1.2(2). □

8.2 Localized virtual cycles

Recall from Chapter 4 that the Oh-Thomas virtual cycles are constructed from Kiem-Li's cosection localization. We speculate that the converse is also true. We provide a simple proof of the speculation in the local model case.

We first fix some notations.

Notation 8.2.1 (Twisted shifted cotangent bundle)**.** Let $\mathbb{X}$ be a quasi-smooth derived scheme and α be a (-1)-shifted *closed* 1-form. Consider a homotopy fiber diagram

$$\begin{tikzcd} \mathbb{X}(\alpha) \arrow[r] \arrow[d] & \mathbb{X} \arrow[d, "\alpha_0"] \\ \mathbb{X} \arrow[r, "0"] & \Omega_{\mathbb{X}}[-1] \end{tikzcd}$$

where $\Omega_{\mathbb{X}}[-1]$ is the (-1)-shifted cotangent bundle, $\alpha_0 : \mathcal{O}_{\mathbb{X}} \to \mathbb{L}_{\mathbb{X}}[-1]$ is the underlying (-1)-shifted 1-form of α, and $\mathbb{X}(\alpha)$ is the derived zero locus of α_0. The derived zero locus $\mathbb{X}(\alpha)$ is sometimes called the α_0-*twisted* (-2)-shifted contangent bundle, and denoted by

$$\mathbb{X}(\alpha) := \Omega_{\mathbb{X}}[-2]_{\alpha_0}.$$

The shifted cotangent bundle $\Omega_{\mathbb{X}}[-1]$ has a canonical (-1)-shifted symplectic structure by [PTVV, Prop. 1.21] and the two sections $0, \alpha_0 : \mathbb{X} \to \Omega_{\mathbb{X}}[-1]$ have Lagrangian structures associated to the closing structures by [Cal, Thm. 2.22].

Hence the twisted shifted cotangent bundle $\mathbb{X}(\alpha)$ also has a canonical (-2)-shifted symplectic structure by [PTVV, Thm. 2.9].

Let $X := \mathbb{X}_{\mathrm{cl}}$ be the classical truncation and let $\phi : \mathbb{E} := \mathbb{L}_{\mathbb{X}}|_X \to \mathbb{L}_X \to L_X$ be the induced perfect obstruction theory. Since the classical truncation commutes with fiber products, we have a fiber diagram of closed embeddings

$$\begin{tikzcd} X(\sigma) \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook, "\sigma"] \\ X \arrow[r, hook, "0"] & C(\mathrm{Ob}) \end{tikzcd}$$

where $\mathrm{Ob} := h^1(\mathbb{E}|_X^\vee)$ is the obstruction sheaf, $\sigma := h^0(\alpha_0|_X^\vee) : \mathrm{Ob} \to \mathcal{O}_X$ is the induced cosection, and $X(\sigma)$ is the zero locus of σ in X.

Speculation 8.2.2 (Localized virtual cycles are Oh-Thomas virtual cycles)**.** *Let $\mathbb{X}$ be a quasi-smooth derived scheme and α be a (-1)-shifted closed 1-form. We use the notations in Notation 8.2.1. Then we have*

$$[X]^{\mathrm{loc}}_{KL} = [X(\sigma)]^{\mathrm{vir}}_{OT} \in A_*(X(\sigma))$$

where $[X]^{\mathrm{loc}}_{KL}$ is the cosection-localized virtual cycle for the induced obstruction theory ϕ and the induced cosection σ, and $[X(\sigma)]^{\mathrm{vir}}_{OT}$ is the Oh-Thomas virtual cycle of the (-2)-shifted symplectic derived scheme $\mathbb{X}(\alpha)$.

Remark 8.2.3 (Evidence)**.** Consider the following local model: Let U be a smooth scheme, E be a vector bundle on U, $s \in \Gamma(U, E)$ be a section, and $\sigma : E \to \mathcal{O}_U$ be a cosection such that $\sigma \circ s = 0$. Let $X := U(s)$ be the zero locus of s in U,

$$\begin{tikzcd} & E \arrow[r, "\sigma"] \arrow[d] & \mathbb{A}^1_U \\ X \arrow[r, hook] & U. \arrow[u, bend right, "s"'] & \end{tikzcd}$$

Then the cosection-localized virtual cycle is

$$[X]^{\mathrm{loc}}_{KL} = 0^!_{E|_X, \sigma|_X}[C_{X/U}] \in A_*(X(\sigma)).$$

On the other hand, the Oh-Thomas virtual cycle of $X(\sigma)$ is

$$[X(\sigma)]^{\mathrm{vir}}_{OT} = \sqrt{e}((E \oplus E^\vee), (s, \sigma))[U] \in A_*(X(\sigma)).$$

Consider the compostion

$$X(\sigma) \times \mathbb{A}^1 \hookrightarrow X \times \mathbb{A}^1 \hookrightarrow M^\circ_{X/U}.$$

By a deformation argument, we can show that

$$\sqrt{e}((E \oplus E^\vee), (s,\sigma))[U] = \sqrt{e}((E \oplus E^\vee)|_{C_{X/U}}, (\tau,\sigma))[C_{X/U}]$$

where $\tau \in \Gamma(C_{X/U}, E|_{C_{X/U}})$ is the tautological section. In [KP2, Lem. 5.5], it is shown that

$$\sqrt{e}((E \oplus E^\vee)|_{E(\sigma)}, (\tau,\sigma)) = 0^!_{E,\sigma} : A_*(E(\sigma)) \to A_*(X(\sigma)).$$

Therefore, by the bivariance of localized square root Euler classes in Proposition 4.1.25, we have

$$[X]^{\mathrm{loc}}_{KL} = [X(\sigma)]^{\mathrm{vir}}_{OT} \in A_*(X(\sigma))$$

in this case.

The author expects that a similar argument will prove Speculation 8.2.2 in the general case. We plan to give the details in [KP3].

Chapter 9

Virtual cycles in algebraic cobordism

In this chapter, we generalize virtual pullbacks [Man] and cosection localization [KL1] to algebraic cobordism. As a corollary, we extend the torus localization formula [GP] to virtual cobordism classes of Shen [Shen]. This is based on [KP1].

Summary We observed in Chapter 2 that virtual intersection theory [BF, Man] is a generalization of Fulton's intersection theory [Ful] to algebraic stacks. Similarly, we need to extend algebraic cobordism for schemes to algebraic stacks. Here we use a shortcut, called *limit algebraic cobordism*, introduced in [KP1]. This limit algebraic cobordism is still incomplete for serving a general theory for algebraic stacks, but it is sufficient for defining virtual pullbacks and cosection localization, and proving the functoriality.

9.1 Limit algebraic cobordism

We recall from [KP1] the notion of *limit algebraic cobordism* for algebraic stacks.

9.1.1 Definition and basic properties

Definition 9.1.1. Let X be an algebraic stack. Let $\mathsf{Sm}_{/X}$ denote the category of all pairs (T,t) of quasi-projective schemes T and smooth morphisms $t: T \to X$. The morphisms in $\mathsf{Sm}_{/X}$ are given by the morphisms over X.

Definition 9.1.2 (Limit algebraic cobordism)**.** Let X be an algebraic stack. Consider the functor

$$\mathsf{Sm}_X^{\mathrm{op}} \to \mathsf{Ab} : T \mapsto \Omega_*(T)$$

where each morphism $s : (T_1, t_1) \to (T_2, t_2)$ in $\mathsf{Sm}_{/X}$ maps to the lci pullback $s^* : \Omega_*(T_2) \to \Omega_{*+\dim(t_1)-\dim(t_2)}(T_1)$.

1. Define the *limit algebraic coboridsm* of degree $d \in \mathbb{Z}$ as the abelian group

$$\widehat{\Omega}_d(X) := \varprojlim_{(T,t)\in\mathsf{Sm}_{/X}} \Omega_{d+d(t)}(T)$$

   where $d(t)$ is the relative dimension of $t : T \to X$.

2. Define the *limit algebraic coboridsm* as the graded abelian group

$$\widehat{\Omega}_*(X) := \bigoplus_{d\in Z} \widehat{\Omega}_d(X).$$

For any $\alpha \in \widehat{\Omega}_d(X)$ and $(T,t) \in \mathsf{Sm}_{/X}$, we denote by $\alpha(t) \in \Omega_{d+d(t)}(T)$ the corresponding class.

Definition 9.1.3 (Projective pushforward)**.** Let $f : X \to Y$ be a projective morphism of algebraic stacks. We define the *pushforward*

$$f_* : \widehat{\Omega}_*(X) \to \widehat{\Omega}_*(Y)$$

via the formula

$$(f_*(\alpha))(s) = (f_S)_*(\alpha(s_X))$$

for $\alpha \in \widehat{\Omega}_*(X)$ and $(S,s) \in \mathsf{Sm}_{/Y}$. Here the two maps f_S and s_X are given by the fiber diagram

$$\begin{tikzcd}
X \times_Y S \arrow[r, "f_S"] \arrow[d, "s_X"] & T \arrow[d, "s"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where $X \times_Y S$ is a quasi-projective scheme, s_X is a smooth morphism, and f_S is a projective morphism.

The projective pushforward for limit algebraic cobordism $\widehat{\Omega}$ in Definition 9.1.3 is well-defined since projective pushforwards commute with lci pullbacks in ordinary algebraic cobordism Ω.

Definition 9.1.4 (Smooth pullback)**.** Let $f : X \to Y$ be a smooth morphism of algebraic stacks. We define the *pullback*

$$f^* : \widehat{\Omega}_*(Y) \to \widehat{\Omega}_{*+\dim(f)}(X)$$

via the formula

$$f^*(\beta)(t) = \beta(f \circ t)$$

for $\beta \in \widehat{\Omega}_*(X)$ and $(T, t) \in \mathsf{Sm}_{/X}$.

The smooth pullback in Definition 9.1.4 is clearly well-defined.

Definition 9.1.5 (Chern class)**.** Let E be a vector bundle on an algegbraic stack X. We define the i-th *Chern class*

$$c_i(E) : \widehat{\Omega}_*(X) \to \widehat{\Omega}_{*-i}(X)$$

via the formula

$$c_i(E)(\alpha)(t) = c_i(E|_T)(\alpha(t))$$

for $\alpha \in \widehat{\Omega}_*(X)$ and $(T, t) \in \mathsf{Sm}_{/X}$.

The Chern classes for limit algebraic cobordism $\widehat{\Omega}$ in Definition 9.1.5 is well-defined since the Chern classes commute with lci pullbacks in ordinary algebraic cobordism Ω.

Proposition 9.1.6 (Basic properties)**.** *The limit algebraic cobordism $\widehat{\Omega}$ in Definition 9.1.2 satisfies the following properties:*

1. *If $f : X \to Y$ and $Y \to Z$ are projective morphisms of algebraic stacks, then we have*
$$(g \circ f)_* = g_* \circ f_* : \widehat{\Omega}_*(X) \to \widehat{\Omega}_*(Z).$$
2. *If $f : X \to Y$ and $Y \to Z$ are smooth morphisms of algebraic stacks, then we have*
$$(g \circ f)^* = f^* \circ g^* : \widehat{\Omega}_*(Z) \to \widehat{\Omega}_{*+\dim(g\circ f)}(X).$$
3. *Consider a fiber diagram of algebraic stacks*
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$
If f is projective and g is smooth, then we have
$$g^* \circ f_* = (f')_* \circ (g')^* : \widehat{\Omega}_*(X) \to \widehat{\Omega}_{*+\dim(g)}(Y').$$

4. *If $f : X \to Y$ is a projective morphism and E is a vector bundle on Y, then we have*
$$c_i(E) \circ f_* = f_* \circ c_i(f^*(E)) : \widehat{\Omega}_*(X) \to \widehat{\Omega}_{*-i}(Y).$$

5. *If $f : X \to Y$ is a smooth morphism and E is a vector bundle on Y, then we have*
$$f^* \circ c_i(E) = c_i(f^*(E)) \circ f^* : \widehat{\Omega}_*(Y) \to \widehat{\Omega}_{*+\dim(f)-i}(X).$$

We omit the proof of Proposition 9.1.6 since it follows directly from the basic properties of the orinary algebraic cobordism Ω for schemes.

Example 9.1.7 (Equivariant algebraic cobordism)**.** Let $X = [P/G]$ be the quotient stack of a quasi-projective scheme by a linear action of a linear algebraic group G. Let $EG_i/G \to BG$ be Totaro's approximation [Tot]. Then we have a canonical isomorphism
$$\widehat{\Omega}_d(X) \cong \varprojlim_{i\to\infty} \Omega_{d+\dim(EG_i)}(P \times_G EG_i) =: \Omega^G_{d+dim(G)}(P)$$
where the last term $\Omega^G_*(P)$ is the *equivariant algebraic cobordism* of Heller-Malagon-Lopez [HML] and Krishna [Kri1]. We refer to [KP1, Cor. 3.8] for the proof of the above comparison.

Remark 9.1.8. One small technical advantage of using the limit algebraic cobordism for a global quotient stack X is that the definition of $\widehat{\Omega}_*(X)$ is stated without any specific choice of a presentation $X \cong [P/G]$ or an approximation $EG_i/G \to BG$. Moreover, the limit algebraic cobordism also behaves well for cone stacks and vector bundle stacks over global quotient stacks. Since they are the main objects in the virtual intersection theory, the limit algebraic cobordism is sufficient in this thesis.

Remark 9.1.9. The main limitation of the limit algebraic cobordism $\widehat{\Omega}$ is that there is no *excision sequence*, even for global quotient stacks.[1] Thus we need some tricks to avoid using the excision sequence in the subsequent sections.

[1]In [HML, Thm. 20], it is claimed that there is an excision sequence, but the author thinks the proof is not correct. In general, the Mittag-Leffler condition for each term in a right exact sequence of inverse systems is *not* sufficient for its completion being right exact.

9.1.2 Gysin maps for vector bundle stacks

Proposition 9.1.10 ((Extended) homotopy property)**.** *Let $\pi : E \to X$ be a vector bundle torsor on an algebraic stack X. Then the smooth pullback*

$$\pi^* : \widehat{\Omega}_*(X) \to \widehat{\Omega}_{*+\dim(\pi)}(E)$$

is an isomorphism.

Proof. By Lemma 9.1.11 below, the smooth pullback π^* can be identified to

$$\varprojlim_{(T,t)\in \mathsf{Sm}_{/X}} (\pi_T)^* : \varprojlim_{(T,t)\in \mathsf{Sm}_{/X}} \Omega_*(T) \to \varprojlim_{(T,t)\in \mathsf{Sm}_{/X}} \Omega_*(E_T)$$

where $E_T := t^*(E)$ and $\pi_T : E_T \to T$ is the base change of π to T. Since each $(\pi_T)^*$ is an isomorphism, so is its inverse limit π^*. □

We need the following lemma to complete the proof of Proposition 9.1.10.

Lemma 9.1.11. *Let $f : X \to Y$ be a smooth quasi-projective morphism of algebraic stacks. Consider the functor*

$$\mathsf{Sm}_{/Y} \to \mathsf{Sm}_{/X} : (s : S \to Y) \mapsto (s_X : S \times_Y X \to X).$$

Then the the induced map on the limits

$$\widehat{\Omega}_d(\mathcal{X}) = \varprojlim_{(T,t)\in \mathsf{Sm}_{/X}} \Omega_{d+\dim(t)}(T) \to \varprojlim_{(S,s)\in \mathsf{Sm}_{/Y}} \Omega_{d+\dim(t)}(S \times_Y X)$$

is an isomorphism.

Proof. We first express the above canonical map

$$\Phi : \varprojlim_{(T,t)\in \mathsf{Sm}_{/X}} \Omega_{d+\dim(t)}(T) \to \varprojlim_{(S,s)\in \mathsf{Sm}_{/Y}} \Omega_{d+\dim(t)}(S \times_Y X)$$

as the following formula

$$\Phi(\alpha)(s : S \to Y) = \alpha(S \times_Y X \xrightarrow{p_2} X) \in \Omega_*(S \times_Y X)$$

where p_2 is the second projection map.

We claim that the inverse of Φ is the map

$$\Psi : \varprojlim_{(S,s)\in \mathsf{Sm}_{/Y}} \Omega_{d+\dim(t)}(S_X) \to \varprojlim_{(T,t)\in \mathsf{Sm}_{/X}} \Omega_{d+\dim(t)}(T)$$

given by the formula

$$\Psi(\beta)(t : T \to X) = \Gamma_t^*(\beta(f \circ t : T \to X)) \in \Omega_*(T)$$

where $\Gamma_t : T \to T \times_Y X$ is the graph of $t : T \to X$ over Y.

It is easy to show $\Psi \circ \Phi = \mathrm{id}$. Indeed, from the definitions, we have

$$\Psi \circ \Phi(\alpha)(t) = \Gamma_t^*(\Phi(\alpha)(f \circ t)) = \Gamma_t^*(\alpha(T \times_Y X \xrightarrow{p_2} X)) = \alpha(t)$$

for $\alpha \in \varprojlim_{(T,t)\in \mathsf{Sm}_{/X}} \Omega_{d+\dim(t)}(T)$ and $(T, t) \in \mathsf{Sm}_{/X}$.

The other direction is quite subtle. We first write

$$\Phi \circ \Psi(\beta)(s) = \Psi(\beta)(S \times_Y X \xrightarrow{p_2} X) = \Gamma_{p_2}^*(\beta(f \circ p_2 : S \times_Y X \to Y))$$

where $\Gamma_{p_2} = (p_1, p_2, p_2) : S \times_Y X \to S \times_Y X \times_Y X$. Since

$$f \circ p_2 = s \circ p_1 : S \times_Y X \to Y,$$

and $\beta(-)$ commutes with the transition maps in $\varprojlim_{(S,s)\in \mathsf{Sm}_{/Y}} \Omega_{d+\dim(t)}(S \times_Y X)$, we have

$$\beta(f \circ p_2) = \beta(s \circ p_1) = p_{13}^*(\beta(s)).$$

where $p_{13} : S \times_Y X \times_Y X \to S \times_Y X$ is the projection map to the first and the third factor. Then we have

$$\Gamma_t^* \circ p_{13}^*(\beta(s)) = \mathrm{id}_{S \times_Y X}^*(\beta(s)) = \beta(s).$$

Hence $\Psi \circ \Phi = \mathrm{id}$ as desired. □

Corollary 9.1.12. *Let $\pi : \mathfrak{E} \to X$ be a vector bundle stack on an algebraic stack X. Assume that $\mathfrak{E}$ is globally presented, i.e., $\mathfrak{E} \cong [E_1/E_0]$ for some vector bundle E_0 and E_1. Then the smooth pullback*

$$\pi^* : \widehat{\Omega}_*(X) \to \widehat{\Omega}_{*+\dim(\pi)}(\mathfrak{E})$$

is an isomorphism.

Proof. Let $p : E_1 \to \mathfrak{E} = [E_1/E_0]$ be the projection map. Then p is a E_0-torsor. By the functoriality of smooth pullbacks in Proposition 9.1.6, we have a commutative diagram

$$\begin{tikzcd} \widehat{\Omega}(X) \arrow[r, "\pi_{\mathfrak{E}}^*"] \arrow[rr, bend right, "\pi_{E_1}^*"'] & \widehat{\Omega}(\mathfrak{E}) \arrow[r, "p^*"] & \widehat{\Omega}(E_1). \end{tikzcd}$$

By the extended homotopy property in Proposition 9.1.10, p^* and $\pi_{E_1}^*$ are isomorphisms. Hence $\pi_{\mathfrak{E}}^*$ is also an isomorphism. □

Definition 9.1.13 (Gysin pullback)**.** Let $\pi_{\mathfrak{E}} : \mathfrak{E} \to X$ be a vector bundle stack on an algebraic stack X. We define the *Gysin pullback*

$$0^!_{\mathfrak{E}} : \widehat{\Omega}_*(\mathfrak{E}) \to \widehat{\Omega}_{*-\dim(\pi_{\mathfrak{E}})}(X)$$

of the zero section $0_{\mathfrak{E}} : X \to \mathfrak{E}$ as follows:

1. *Case 1.* Assume that X has the resolution property. Then the vector bundle stack $\mathfrak{E}$ is globally presented and the smooth pullback $\pi^*_{\mathfrak{E}} : \widehat{\Omega}(X) \to \widehat{\Omega}(\mathfrak{E})$ is an isomorphism by Corollary 9.1.12. We define the Gysin pullback as the inverse
$$0^!_{\mathfrak{E}} := (\pi^*_{\mathfrak{E}})^{-1} : \widehat{\Omega}_*(\mathfrak{E}) \to \widehat{\Omega}_{*-\dim(\pi_{\mathfrak{E}})}(X).$$

2. *Case 2.* Consider the general case. We define the *Gysin pullback*
$$0^!_{\mathfrak{E}} : \widehat{\Omega}_*(\mathfrak{E}) \to \widehat{\Omega}_{*-\dim(\pi_{\mathfrak{E}})}(X)$$
via the formula
$$0^!_{\mathfrak{E}}(\alpha)(t) = 0^!_{\mathfrak{E}|_T}(\alpha(t))$$
for $\alpha \in \widehat{\Omega}_*(\mathfrak{E})$ and $(T, t) \in \mathsf{Sm}_{/X}$, where $0^!_{\mathfrak{E}|_T} : \widehat{\Omega}_*(\mathfrak{E}|_T) \to \widehat{\Omega}_{*-\dim(\pi_{\mathfrak{E}|_T})}(T)$ is the Gysin pullback in Case 1.

9.2 Virtual pullbacks

In this section, we generalize virtual pullbacks in Chow groups (see Chapter 2) to algebraic cobordism. The main obstruction is

9.2.1 Specialization maps

Let $f : X \to Y$ be a morphism of quasi-projective schemes. Given a *global factorization* of f, i.e., a commutative triangle

$$\begin{tikzcd} & Z \arrow[d, "\bar{f}"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & Y \end{tikzcd}$$

where $\widetilde{f}$ is a closed closed embedding and $\overline{f}$ is a smooth morphism, the intrinsic normal cone can be written as a global quotient stack

$$\mathfrak{C}_{X/Y} = \left[\frac{C_{X/Z}}{T_{Z/Y}|_X}\right].$$

Heuristically, we can form the following diagram

$$\begin{tikzcd} Z \arrow[r, squiggly, "\mathrm{sp}_{X/Z}"] \arrow[d, "\overline{f}"] & C_{X/Z} \arrow[d, "k"] \\ Y \arrow[r, squiggly, "\mathrm{sp}_{X/Y}"] & \mathfrak{C}_{X/Y} \end{tikzcd}$$

where k is a $T_{Z/Y}$-torsor.

Definition 9.2.1 (Specialization map)**.** Let $f : X \to Y$ be a morphism of quasi-projective schemes. Given a global factorization

$$\begin{tikzcd} & Z \arrow[d, "\overline{f}"] \\ X \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & Y \end{tikzcd} \tag{9.2.1}$$

by a closed closed embedding $\widetilde{f}$ and a smooth morphism $\overline{f}$, we define the *specialization map* as the composition

$$\mathrm{sp}_{X/Y} : \Omega_*(Y) \xrightarrow{\overline{f}^*} \Omega_*(Z) \xrightarrow{\mathrm{sp}_{X/Z}} \Omega_*(C_{X/Z}) \xrightarrow{(k^*)^{-1}} \widehat{\Omega}_*(\mathfrak{C}_{X/Y})$$

where k^* is an isomorphism by the extended homotopy property.

Lemma 9.2.2 (Well-definedness)**.** *The specialization map* $\mathrm{sp}_{X/Y}$ *in Definition 9.2.1 is independent of the choice of the global factorization* (9.2.1).

Proof. Choose another factorization $X \xrightarrow{\widetilde{f}'} Z' \xrightarrow{\overline{f}'} Y$ of f by a closed immersion $\widetilde{f}'$ and a smooth morphism $\overline{f}'$ for some quasi-projective scheme Z'. After replacing Z' by $Z \times_Y Z'$, we may assume that there is a smooth morphism $a : Z' \to Z$ making the diagram

$$\begin{tikzcd} & Z' \arrow[d, "a"] \arrow[dr, "g'"] & \\ X \arrow[ur, hook] \arrow[r, hook] & Z \arrow[r, "g"] & Y \end{tikzcd}$$

commute. Then we have a commutative diagram

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d, "a"'] & M^\circ_{X/Z'} \arrow[d] & C_{X/Z'} \arrow[l, hook'] \arrow[d, "b"] \\ Z \arrow[r, hook] & M^\circ_{X/Z} & C_{X/Z} \arrow[l, hook'] \end{tikzcd}$$

where the square are cartesian and the vertical arrows are smooth. Since Gysin pullbacks commute with smooth pullbacks, we have the equality

$$b^* \circ \mathrm{sp}_{X/Z} = \mathrm{sp}_{X/Z'} \circ a^*.$$

Hence the functoriality of smooth pullbacks completes the proof. $\square$

Remark 9.2.3 (Generalization)**.** Let $f : X \to Y$ be a quasi-projective morphism of algebraic stacks. Assume that Y has a vector bundle torsor Y' which is a quasi-projective scheme. Then we can define the *specialization map* $\mathrm{sp}_{X/Y}$ via the commutative square

$$\begin{tikzcd} \widehat{\Omega}_*(Y) \arrow[r, dotted, "\mathrm{sp}_{X/Y}"] \arrow[d, "\cong"] & \widehat{\Omega}_*(\mathfrak{C}_{X/Y}) \arrow[d, "\cong"] \\ \Omega_*(Y') \arrow[r, "\mathrm{sp}_{X'/Y'}"] & \widehat{\Omega}_*(\mathfrak{C}_{X'/Y'}) \end{tikzcd}$$

where the vertical arrows are isomorphisms by the homotopy property in Proposition 9.1.10.

Proposition 9.2.4 (Bivariance)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a fiber diagram of quasi-projective schemes. Form a commutative diagram

$$\begin{tikzcd} \mathfrak{C}_{X'/Y'} \arrow[r, hook, "j"] \arrow[dr, "g'''"'] & \mathfrak{C}_{X/Y}|_{X'} \arrow[r] \arrow[d, "g''"] & X' \arrow[d, "g'"] \\ & \mathfrak{C}_{X/Y} \arrow[r] & X. \end{tikzcd}$$

1. *If g is a projective morphism, then we have*

$$\mathrm{sp}_{X/Y} \circ g_* = (g''')_* \circ \mathrm{sp}_{X'/Y'} : \Omega_*(Y') \to \widehat{\Omega}_*(\mathfrak{C}_{X/Y}).$$

2. *If g is a smooth morphism, then j is an isomorphism and we have*

$$\mathrm{sp}_{X'/Y'} \circ g^* = g^* \circ \mathrm{sp}_{X/Y} : \Omega_*(Y) \to \widehat{\Omega}_*(\mathfrak{C}_{X'/Y'}).$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
Y' \arrow[r, hook] \arrow[d, "g"'] & M^{\circ}_{X'/Y'} \arrow[d] & C_{X'/Y'} \arrow[l, hook'] \arrow[d, "k"] \\
Y \arrow[r, hook] & M^{\circ}_{X/Y} & C_{X/Y} \arrow[l, hook']
\end{tikzcd}$$

of cartesian squares. The statements follow from the fact that the refined Gysin pullbacks commute with projective pushforwards and smooth pullbacks. □

Proposition 9.2.5 (Vistoli's equivalence)**.** *Let*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

be a fiber diagram of quasi-projective schemes. Consider the fiber diagram

$$\begin{tikzcd}
\mathfrak{C}_{X/Y}|_{X'} \times_{X'} \mathfrak{C}_{Y'/Y}|_{X'} \arrow[r] \arrow[d] & \mathfrak{C}_{Y'/Y}|_{X'} \arrow[r, "f''"] \arrow[d] & \mathfrak{C}_{Y'/Y} \arrow[d] \\
\mathfrak{C}_{X/Y}|_{X'} \arrow[r] \arrow[d, "g''"] & X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
\mathfrak{C}_{X/Y} \arrow[r] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

and the canonical closed embeddings

$$\mathfrak{C}_{\mathfrak{C}_{X/Y}|_{X'}/\mathfrak{C}_{X/Y}} \xhookrightarrow{a} \mathfrak{C}_{X/Y}|_{X'} \times_{X'} \mathfrak{C}_{Y'/Y}|_{X'} \xhookleftarrow{b} \mathfrak{C}_{\mathfrak{C}_{Y'/Y}|_{X'}/\mathfrak{C}_{Y'/Y}}.$$

Then we have

$$a_* \circ \mathrm{sp}_{\mathfrak{C}_{X/Y}|_{X'}/\mathfrak{C}_{X/Y}} \circ \mathrm{sp}_{X/Y} = b_* \circ \mathrm{sp}_{\mathfrak{C}_{Y'/Y}|_{X'}/\mathfrak{C}_{Y'/Y}} \circ \mathrm{sp}_{Y'/Y}$$

(see Remark 9.2.3).

Proof. Choose factorizations

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[ur, hook] \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} & W \arrow[d] \\ Y' \arrow[ur, hook] \arrow[r, "g"] & Y \end{tikzcd}$$

by closed embeddings and smooth morphisms. Consider the induced cartesian square of closed embeddings

$$\begin{tikzcd} \widetilde{X}' := X' \arrow[r, hook] \arrow[d, hook] & \widetilde{Y} := Z \times_Y Y' \arrow[d, hook] \\ \widetilde{X} := X \times_Y W \arrow[r, hook] & \widetilde{Y} := Z \times_Y W. \end{tikzcd}$$

Then the argument in Proposition 2.1.22 also works for the above square and we have

$$\widetilde{a}_* \circ \mathrm{sp}_{\mathfrak{C}_{\widetilde{X}/\widetilde{Y}|_{\widetilde{X}'}}/\mathfrak{C}_{\widetilde{X}/\widetilde{Y}}} \circ \mathrm{sp}_{\widetilde{X}/\widetilde{Y}} = \widetilde{b}_* \circ \mathrm{sp}_{\mathfrak{C}_{\widetilde{Y}'/\widetilde{Y}}|_{\widetilde{X}'}/\mathfrak{C}_{\widetilde{Y}'/\widetilde{Y}}} \circ \mathrm{sp}_{\widetilde{Y}'/\widetilde{Y}}. \tag{9.2.2}$$

Form a commutative diagram

$$\begin{tikzcd} \mathfrak{C}_{\mathfrak{C}_{\widetilde{X}/\widetilde{Y}}|_{\widetilde{X}'}/\mathfrak{C}_{\widetilde{X}/\widetilde{Y}}} \arrow[r, hook, "\widetilde{a}"] \arrow[d] & \mathfrak{C}_{\widetilde{X}/\widetilde{Y}}|_{\widetilde{X}'} \times_{\widetilde{X}'} \mathfrak{C}_{\widetilde{Y}'/\widetilde{Y}}|_{X'} \arrow[d] & \mathfrak{C}_{\mathfrak{C}_{\widetilde{Y}'/\widetilde{Y}}|_{\widetilde{X}'}/\mathfrak{C}_{\widetilde{Y}'/\widetilde{Y}}} \arrow[l, hook', "\widetilde{b}"'] \arrow[d] \\ \mathfrak{C}_{\mathfrak{C}_{X/Y}|_{X'}/\mathfrak{C}_{X/Y}} \arrow[r, hook, "a"] & \mathfrak{C}_{X/Y}|_{X'} \times_{X'} \mathfrak{C}_{Y'/Y}|_{X'} & \mathfrak{C}_{\mathfrak{C}_{Y'/Y}|_{X'}/\mathfrak{C}_{Y'/Y}}. \arrow[l, hook', "b"'] \end{tikzcd}$$

where the horizontal maps are closed embeddings and the vertical maps are vector bundle torsors of same rank. Hence the squares are cartesian and the identity (9.2.2) proves the desired identity. □

Proposition 9.2.6 (Kim-Kresch-Pantev's equivalence)**.** *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

be a commutative diagram of quasi-projective schemes. Let

$$\begin{tikzcd} \mathfrak{C}_{X/\mathfrak{C}_{Y/Z}} \arrow[r, hook, "a"] & \mathfrak{C}_{X\times\mathbb{A}^1/M^\circ_{Y/Z}} & \mathfrak{C}_{X/Z} \arrow[l, hook', "b"'] \end{tikzcd}$$

be the canonical closed embeddings. Then we have

$$a_* \circ \mathrm{sp}_{X/\mathfrak{C}_{Y/Z}} \circ \mathrm{sp}_{Y/Z} = b_* \circ \mathrm{sp}_{X/Z} : \Omega_*(Z) \to \widehat{\Omega}(\mathfrak{C}_{X\times\mathbb{A}^1/M^\circ_{Y/Z}})$$

(see Remark 9.2.3).

Proof. Form a commutative diagram

$$\begin{tikzcd}
X \arrow[r, hook, "f'"] \arrow[dr, "f"'] & Y' \arrow[r, hook, "g''"] \arrow[d] & Z'' \arrow[d] \\
& Y \arrow[r, hook, "g'"] \arrow[dr, "g"'] & Z' \arrow[d] \\
& & Z
\end{tikzcd}$$

such that the horizontal arrows are closed immersions, the vertical arrows are smooth and the square is cartesian. Then we have an induced factorization

$$\begin{tikzcd}
X \times \mathbb{P}^1 \arrow[r] \arrow[dr] & M^\circ_{Y'/Z''} \arrow[d] \\
& M^\circ_{Y/Z}.
\end{tikzcd}$$

This gives us a commutative diagram

$$\begin{tikzcd}
C_{X/C_{Y'/Z''}} \arrow[r, hook] \arrow[d] & C_{X\times\mathbb{P}^1/M^\circ_{Y'/Z''}} \arrow[d] & C_{X/Z''} \arrow[l, hook'] \arrow[d] \\
\mathfrak{C}_{X/\mathfrak{C}_{Y/Z}} \arrow[r, hook, "a"] & \mathfrak{C}_{X\times\mathbb{P}^1/M^\circ_{Y/Z}} & \mathfrak{C}_{X/Z} \arrow[l, hook', "b"']
\end{tikzcd}$$

of cone stacks. Since the vertical arrows are torsors of vector bundles of the same rank, the two squares are cartesian. It suffices to prove the lemma for the closed immersions $f' : X \to Y'$ and $g'' : Y' \to Z''$ since specialization homomorphisms commute with smooth pullbacks. Then the usual arguments using the double deformation space $M^\circ_{X\times\mathbb{P}^1/M^\circ_{Y'/Z''}}$ (see Lemma 2.3.19 and the proof of Theorem 2.3.12) remain valid since all the deformations spaces and the cone stacks are quasi-projective schemes in this case. ◻

9.2.2 Virtual pullbacks and functoriality

Definition 9.2.7 (Virtual pullbacks)**.** Let $f : X \to Y$ be a morphism of quasi-projective schemes and let $\phi : \mathbb{F} \to L_{X/Y}$ be a perfect obstruction theory. We define the *virtual pullback* as the composition

$$f^! : \Omega_*(Y) \xrightarrow{\mathrm{sp}_{X/Y}} \widehat{\Omega}_*(\mathfrak{C}_{X/Y}) \xrightarrow{\iota_*} \widehat{\Omega}_*(\mathfrak{E}(\mathbb{F})) \xrightarrow{0^!_{\mathfrak{E}(\mathbb{F})}} A_*(X)$$

where $\iota : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}(\mathbb{F})$ denotes the closed embedding induced by the obstruction theory ϕ.

Definition 9.2.8 (Virtual cobordism classes)**.** Let X be a quasi-projective scheme equipped with a perfect obstruction theory $\psi : \mathbb{F} \to L_X$. We define the *virtual cobordism class* as

$$[X]^{\mathrm{vir}} := p^![\mathrm{Spec}(\mathbb{C})] \in \Omega_*(X)$$

where $p : X \to \mathrm{Spec}(\mathbb{C})$ denotes the projection map.

Remark 9.2.9 (Shen's construction)**.** The virtual pullback in Definition 9.2.7 can be defined without using the limit algebraic cobordism $\widehat{\Omega}$. Indeed, let $f : X \to Y$ be a morphism of quasi-projective schemes and let $\phi : \mathbb{F} \to L_{X/Y}$ be a perfect obstruction theory. Given a global factorization

$$\begin{tikzcd} & Z \arrow[d, "\bar{f}"] \\ X \arrow[ur, hook, "\tilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd}$$

and a global resolution $\mathbb{F} \cong [F_1/F_0]$, we can form a fiber diagram

$$\begin{tikzcd} D \arrow[r, two heads, "b"] \arrow[d, "a"] & C \arrow[r, hook, "i"] \arrow[d] & F_1 \arrow[d] \\ C_{X/Z} \arrow[r, two heads] & \mathfrak{C}_{X/Y} \arrow[r, hook] & \mathfrak{E}(\mathbb{F}_{X/Y}). \end{tikzcd}$$

Then we can define the virtual pullback as the composition

$$f^! : \Omega_*(Y) \xrightarrow{\mathrm{sp}_{X/Z}} \Omega_*(C_{X/Z}) \xrightarrow{a^*} \Omega_*(D) \xrightarrow{(b^*)^{-1}} \Omega_*(C) \xrightarrow{i_*} \Omega_*(F_1) \xrightarrow{0^!_{F_1}} \Omega_*(X).$$

(Here the obvious degree shifts are ignored.) This approach was introduced by Shen in [Shen] to define the virtual cobordism classes (when $Y = \mathrm{Spec}(\mathbb{C})$).

Proposition 9.2.10 (Bivariance)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a cartesian square of quasi-projective schemes. Let $\phi : \mathbb{F} \to L_{X/Y}$ be a perfect obstruction theory and let $r := \mathrm{rank}(\mathbb{F})$.

1. *If g is a prorjective morphism, then we have*

$$f^! \circ g_* = g'_* \circ (f')^! : \Omega_*(Y') \to \Omega_{*+r}(X).$$

2. *If g is a local complete intersection morphism, then we have*

$$(f')^! \circ g^! = (g')^! \circ f^! : \Omega_*(Y) \to \Omega_{*+r}(X').$$

Proof. It follows directly from Proposition 9.2.4. □

Proposition 9.2.11 (Commutativity)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a fiber diagram of quasi-projective schemes. Let $\phi_{X/Y} : \mathbb{F}_{X/Y} \to L_{X/Y}$ and $\phi_{Y'/Y} : \mathbb{F}_{Y'/Y} \to L_{Y'/Y}$ be perfect obstruction theories. Then we have

$$g^! \circ f^! = f^! \circ g^! : \Omega_*(Y) \to \Omega_{*+r(f)+r(g)}(X')$$

where $r(f) := \mathrm{rank}(\mathbb{F}_{X/Y})$ and $r(g) := \mathrm{rank}(\mathbb{F}_{Y'/Y})$.

Proof. It follows directly from Proposition 9.2.5. □

Theorem 9.2.12 (Functoriality)**.** *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

be a commutative diagram of quasi-projective schemes. Let

$$(\phi_{X/Y} : \mathbb{F}_{X/Y} \to L_{X/Y},\ \phi_{Y/Z} : \mathbb{F}_{Y/Z} \to L_{Y/Z},\ \phi_{X/Z} : \mathbb{F}_{X/Z} \to L_{X/Z})$$

be a compatible triple of perfect obstruction theories in the sense of Definition 2.3.11. Then we have

$$(g \circ f)^! = f^! \circ g^! : \Omega_*(Z) \to \Omega_{*+r(f)+r(g)}(X).$$

where $r(f) := \operatorname{rank}(\mathbb{F}_{X/Y})$ *and* $r(g) := \operatorname{rank}(\mathbb{F}_{Y/Z})$.

Proof. If follows directly from Propostion 9.2.6 □

Corollary 9.2.13 (Virtual pullback formula)**.** *Let* $f : X \to Y$ *be a morphism of quasi-projective schemes. Let* $(\phi_{X/Y}, \phi_Y, \phi_X)$ *be a compatible triple of perfect obstruction theoreis in the sense of Definition 2.3.11. Then we have*

$$[X]^{\mathrm{vir}} = f^![Y]^{\mathrm{vir}} \in \Omega_*(X).$$

Corollary 9.2.13 follows directly from Theorem 9.2.12 and Definition 9.2.8.

Remark 9.2.14. The compatibility condition in [KP1, Thm. 4.4] is not correct since the truncated cotangent complexes does not form a distinguished triangle in general. Thus we should use (1) the full cotangent complexes or (2) the compatibility in Definition 2.3.11 to make [KP1, Thm. 4.4] correct.

9.3 Cosection localization

In this section, we generalize Kiem-Li's *cosection localization* [KL1] to algebraic cobordism.

9.3.1 Cosection-localized Gysin map

The cosection-localized Gysin map in Chow groups are defined via blowup method and the *abstract blowup sequence* played a key role. We note that we also have an abstract blowup sequence in algebraic cobordism.

Proposition 9.3.1 (Abstract blowup sequence)**.** *Let*

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "q"] & \widetilde{X} \arrow[d, "p"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd}$$

be an abstract blowup square (see Definition A.2.6) of quasi-projective schemes. Then we have a right exact sequence

$$\Omega_*(E) \xrightarrow{(-j_*, q_*)} \Omega_*(\widetilde{X}) \oplus \Omega_*(Z) \xrightarrow{(p_*, i_*)} \Omega_*(X) \longrightarrow 0.$$

We refer to Vishik [Vish, Lem. 7.9] for the proof of Proposition 9.3.1.

Remark 9.3.2. An alternative proof of Proposition 9.3.1 is to use Voevodsky's algebraic cobordism MGL in [**?**]. Indeed, In [Lev2], Levine constructed its Borel-Moore version MGL' ([Lev2, Prop. 4.1]) and show that it has a long exact localization sequence ([Lev2, p. 559]). In [Lev3], Levine show that the canonical map

$$\Omega_* \to \mathrm{MGL}'_{2*,*}$$

is an isomorphism. Then the abstract blowup sequence follows immediately from the long exact localization sequence (as in [Ful, Prop. 18.3.2]).

Remark 9.3.3. We note that Proposition 9.3.1 also holds when X is a cone stack over a quasi-projective scheme. Indeed, the resolution property of quasi-projective schemes assure that X has a vector bundle torsor which is a quasi-projective scheme. Then the extended homotopy property in Proposition 2.3.1 reduce the situation to the case when X is a quasi-projective scheme.

Recall that $c_1(L^\vee) \neq -c_1(L)$ in algebraic cobordism due to the formal group law. Still we have a power series $g(u) \in \Omega_*(\mathrm{Spec}(\mathbb{C}))[[u]]$ such that

$$c_1(L^\vee) = c_1(L) \cdot g(c_1(L)) : \Omega_*(X) \to \Omega_{*-1}(X)$$

for any line bundle L on a quasi-projective scheme X.

Definition 9.3.4 (Intersection product with anti-effective divisor)**.** Let D be an effective Cartier divisor of a quasi-projective scheme X. We define the *intersection product* with the divisor $-D$ as

$$(-D)\cdot := g(c_1(L)) \circ i^! : \Omega_*(X) \to \Omega_{*-1}(D)$$

where $L := \mathcal{O}_X(D)$ and $i : D \hookrightarrow X$ is the inclusion map.

We can now define the *cosection-localized Gysin map* as in Definition 3.3.2.

Definition 9.3.5 (Cosection-localized Gysin map)**.** Let X be a quasi-projective scheme, $\mathbb{F}$ be a perfect complex of tor-amplitude $[-1,0]$, and $\sigma : \mathbb{F}^\vee[1] \to O_X$ be a cosection. We use the notations in the blowup diagram in subsection 3.3.1. We define the *cosection-localized Gysin map*

$$0^!_{\mathfrak{E}(\mathbb{F}),\sigma} :_* (\mathfrak{R}(\mathbb{F},\sigma)) \to A_*(X(\sigma))$$

as the unique map that fits into the commutative diagram

$$\begin{tikzcd}
\Omega_*(\mathfrak{E}(\mathbb{K}|_D)) \arrow[r, "(-a_*,b_*)"] \arrow[dr, "0"'] & \Omega_*(\mathfrak{E}(\mathbb{K})) \oplus \Omega_*(\mathfrak{E}(\mathbb{F}|_{X(\sigma)})) \arrow[r, "(c_*,d_*)"] \arrow[d, "(u,v)"] & \Omega_*(\mathfrak{R}(\mathbb{F},\sigma)) \arrow[r] \arrow[d, "0^!_{\mathfrak{E}(\mathbb{F})}\circ k_*"] \arrow[dl, dotted, "0^!_{\mathfrak{E}(\mathbb{F}),\sigma}"'] & 0 \\
 & \Omega_*(X(\sigma)) \arrow[r, "i_*"] & \Omega_*(X) &
\end{tikzcd}$$

where the top horizontal sequence is the abstract blowup sequence (see Remark 9.3.3) and the two maps u and v are given as follows:

$$u : \Omega_*(\mathfrak{E}(\mathbb{K})) \xrightarrow{0^!_{\mathfrak{E}(\mathbb{K})}} \Omega_*(\widetilde{X}) \xrightarrow{(-D)\cdot} \Omega_*(D) \xrightarrow{q_*} \Omega_*(X(\sigma))$$

$$v : \Omega_*(\mathfrak{E}(\mathbb{F}|_{X(\sigma)}) \xrightarrow{0^!_{\mathfrak{E}(\mathbb{F}|_{X(\sigma)})}} \Omega_*(X(\sigma)).$$

The cosection-localized Gysin map in Definition 9.3.5 is well-defined (cf. Lemma 3.3.3).

Proposition 9.3.6 (Bivariance)**.** *Let $f : Y \to X$ be a morphism of quasi-projective scheme. Let $\mathbb{F}$ be a perfect complex of tor-amplitude $[-1,0]$ on X and $\sigma : \mathbb{F}^\vee[1] \to O_X$ be a cosection. Form a fiber diagram*

$$\begin{tikzcd}
\mathfrak{R}(\mathbb{F}|_Y, \sigma|_Y) \arrow[r, "\widetilde{f}"] \arrow[d] & \mathfrak{R}(\mathbb{F},\sigma) \arrow[d] \\
Y \arrow[r, "f"] & X.
\end{tikzcd}$$

1. *If $f : Y \to X$ is a projective morphism, then we have*
$$f_* \circ 0^!_{\mathfrak{E}(\mathbb{F}|_Y),\sigma|_Y} = 0^!_{\mathfrak{E}(\mathbb{F}),\sigma} \circ \widetilde{f}_* : \Omega_*(\mathfrak{R}(\mathbb{F}|_Y,\sigma|_Y)) \to \Omega_*(X).$$
2. *If $f : Y \to X$ is a local complete intersection morphism, then we have*
$$f^! \circ 0^!_{\mathfrak{E}(\mathbb{F}|_Y,\sigma_Y)} = 0^!_{\mathfrak{E}(\mathbb{F}),\sigma} \circ f^! : \Omega_*(\mathfrak{R}(\mathbb{E},\sigma)) \to \Omega_*(Y).$$

We omit the proof of Proposition 9.3.6 since it is identical to that in Proposition 3.3.4.

9.3.2 Cosection-localized virtual cobordism classes

Definition 9.3.7 (Cosection-localized virtual cobordism class)**.** Let X be a quasi-projective scheme equipped with a perfect obstruction theory $\phi : \mathbb{F} \to L_X$ and a cosection $\sigma : \mathbb{F}^\vee[1] \to O_X$. We define the *cosection-localized virtual cobordism class* as

$$[X]^{\mathrm{loc}} := 0^!_{\mathfrak{E}(\mathbb{F}),\sigma} \circ \iota^{\sigma,\mathrm{red}} \circ \mathrm{sp}_{X/\mathrm{Spec}(\mathbb{C})}[\mathrm{Spec}(\mathbb{C})] \in \Omega_*(X(\sigma))$$

where $0^!_{\mathfrak{E}(\mathbb{F}),\sigma}$ is the cosection-localized Gysin map in Definition 9.3.5, $\iota^{\sigma,\mathrm{red}} : (\mathfrak{C}_X)_{\mathrm{red}} \hookrightarrow \mathfrak{K}(\mathbb{F},\sigma)$ is the inclusion map given by the cone reduction lemma in Proposition 3.1.9, and $\mathrm{sp}_{X/\mathrm{Spec}(\mathbb{C})}$ is the specialization map in Definition 9.2.1.

Proposition 9.3.8 (Deformation invariance)**.** *Let $f : \mathcal{X} \to \mathcal{B}$ be a morphism of quasi-projective schemes. Assume that $\mathcal{B}$ is smooth. Form a fiber diagram*

$$\begin{tikzcd} \mathcal{X}_b \arrow[r, hook] \arrow[d, "f_b"] & \mathcal{X} \arrow[d, "f"] \\ \{b\} \arrow[r, hook, "i_b"] & \mathcal{B} \end{tikzcd}$$

where $b \in \mathcal{B}$. Let $\phi : \mathbb{E} \to L_{\mathcal{X}/\mathcal{B}}$ be a perfect obstruction theory and $\sigma : \mathbb{E}^\vee[1] \to O_{\mathcal{X}}$ be a cosection. Assume that the composition

$$O_{\mathcal{X}} \xrightarrow{\sigma^\vee} \mathbb{E}[-1] \xrightarrow{\phi} L_{\mathcal{X}/B}[-1] \xrightarrow{\mathsf{KS}} \Omega_{\mathcal{B}}|_{\mathcal{X}}$$

vanishes. Then there exists $[\mathcal{X}]^{\mathrm{red}} \in \Omega_*(\mathcal{X}(\sigma))$ such that

$$[\mathcal{X}_b]^{\mathrm{vir}}_{\mathrm{loc}} = i_b^! [\mathcal{X}]^{\mathrm{vir}}_{\mathrm{loc}} \in \Omega_*(\mathcal{X}_b(\sigma_b))$$

for all $b \in \mathcal{B}$.

Remark 9.3.9 (Cosection-localized virtual pullback)**.** Let $f : X \to Y$ be a morphism of quasi-projective schemes equipped with a perfect obstruction theory $\phi : \mathbb{F} \to L_{X/Y}$ satisfying the cone reduction property (see Definition 3.1.7) and a cosection $\sigma : \mathbb{F}^\vee[1] \to O_X$. We define the *cosection-localized virtual pullback* as the composition

$$f^!_\sigma : \Omega_*(Y) \xrightarrow{\mathrm{sp}_{X/Y}} \widehat{\Omega}_*(\mathfrak{C}_{X/Y}) \cong \widehat{\Omega}_*((\mathfrak{C}_{X/Y})_{\mathrm{red}}) \xrightarrow{\iota^{\sigma,\mathrm{red}}_*} \widehat{\Omega}_*(\mathfrak{K}(\mathbb{F},\sigma)) \xrightarrow{0^!_{\mathfrak{E},\sigma}} \Omega_*(X(\sigma))$$

where $\iota^{\sigma,\mathrm{red}}_* : (\mathfrak{C}_{X/Y})_{\mathrm{red}} \hookrightarrow \mathfrak{K}(\mathbb{F},\sigma)$ is the inclusion map given by the cone reduction property.

9.3.3 Reduced virtual cobordism classes

Definition 9.3.10 (Reduced virtual cobordism class)**.** Let X be a quasi-projective scheme equipped with a perfect obstruction theory $\phi : \mathbb{F} \to L_X$ and a cosection $\sigma : \mathbb{F}^\vee[1] \to O_X$. Assume that $h^0(\sigma) : h^1(\mathbb{F}^\vee) \to O_X$ is surjective so that the kernel cone stack $\mathfrak{K}(\mathbb{E}, \sigma) = \mathfrak{E}(\mathbb{E}_\sigma)$ is a vector bundle stack. We define the *reduced virtual cycles* as

$$[X]^{\mathrm{red}} := 0^!_{\mathfrak{E}(\mathbb{F}_\sigma)} \circ \iota^{\sigma,\mathrm{red}} \circ \mathrm{sp}_{X/\mathrm{Spec}(\mathbb{C})}[\mathrm{Spec}(\mathbb{C})] \in \Omega_*(X)$$

where $0^!_{\mathfrak{E}(\mathbb{F}_\sigma)}$ is the Gysin map in Definition 9.1.13, $\iota^{\sigma,\mathrm{red}} : (\mathfrak{C}_X)_{\mathrm{red}} \hookrightarrow \mathfrak{E}(\mathbb{F}_\sigma)$ is the inclusion map given by the cone reduction lemma in Proposition 3.1.9, and $\mathrm{sp}_{X/\mathrm{Spec}(\mathbb{C})}$ is the specialization map in Definition 9.2.1.

Proposition 9.3.11 (Deformation invariance)**.** *Let $f : \mathcal{X} \to \mathcal{B}$ be a morphism of quasi-projective schemes. Assume that $\mathcal{B}$ is smooth. Form a fiber diagram*

$$\begin{tikzcd} \mathcal{X}_b \arrow[r, hook] \arrow[d, "f_b"] & \mathcal{X} \arrow[d, "f"] \\ \{b\} \arrow[r, hook, "i_b"] & \mathcal{B} \end{tikzcd}$$

where $b \in \mathcal{B}$. Let $\phi : \mathbb{E} \to L_{X/\mathcal{B}}$ be a perfect obstruction theory and $\sigma : \mathbb{E}^\vee[1] \to O_X$ be a cosection such that $h^0(\sigma) : h^1(\mathbb{E}^\vee) \to O_X$ is surjective. Assume that the composition

$$O_X \xrightarrow{\sigma^\vee} \mathbb{E}[-1] \xrightarrow{\phi} L_{X/B}[-1] \xrightarrow{\mathsf{KS}} \Omega_{\mathcal{B}}|_X$$

vanishes. Then there exists $[\mathcal{X}]^{\mathrm{red}} \in \Omega_*(\mathcal{X})$ such that

$$[\mathcal{X}_b]^{\mathrm{red}} = i_b^![\mathcal{X}]^{\mathrm{red}} \in \Omega_*(\mathcal{X}_b)$$

for all $b \in \mathcal{B}$.

Remark 9.3.12 (Reduced virtual pullback)**.** Let $f : X \to Y$ be a morphism of quasi-projective schemes equipped with a perfect obstruction theory $\phi : \mathbb{F} \to L_{X/Y}$ satisfying the cone reduction property (see Definition 3.1.7) and a cosection $\sigma : \mathbb{F}^\vee[1] \to O_X$. Assume that $\overline{\sigma} := h^0(\sigma) : h^1(\mathbb{F}^\vee) \to O_X$ is surjective. We define the *reduced virtual pullback* as the composition

$$f^!_\sigma : \Omega_*(Y) \xrightarrow{\mathrm{sp}_{X/Y}} \widehat{\Omega}_*(\mathfrak{C}_{X/Y}) \cong \widehat{\Omega}_*((\mathfrak{C}_{X/Y})_{\mathrm{red}}) \xrightarrow{\iota_*^{\sigma,\mathrm{red}}} \widehat{\Omega}_*(\mathfrak{E}(\mathbb{F}_\sigma)) \xrightarrow{0^!_{\mathfrak{E}(\mathbb{F}_\sigma)}} \Omega_*(X)$$

where $\iota_*^{\sigma,\mathrm{red}} : (\mathfrak{C}_{X/Y})_{\mathrm{red}} \hookrightarrow \mathfrak{K}(\mathbb{F}, \sigma) = \mathfrak{E}(\mathbb{F}_\sigma)$ is the inclusion map given by the cone reduction property and $\mathbb{F}_\sigma := \mathrm{cone}(\sigma^\vee[1] : O_X[1] \to \mathbb{F})$.

9.4 Torus localization

In this section, we generalize torus localization of Edidin-Graham [EG3] and virtual torus localization of Graber-Pandharipande [GP] in Chow groups to algebraic cobordism. We follow the ideas and notations in Chapter 7.

9.4.1 Localization of algebraic cobordism

Notations We fix some notations on torus equivariant algebraic cobordism.

1. Let $\mathbf{T} := \mathbb{G}_m^{\times r}$ be the r-dimensional torus.
2. Let $\widehat{\mathbf{T}} := \mathrm{Grp}(\mathbf{T}, \mathbb{G}_m) = \mathrm{Pic}^{\mathbf{T}}(\mathrm{Spec}(\mathbb{C})) = \mathbb{Z}^{\oplus r}$ be the character group.
3. Let $\Omega_*^{\mathbf{T}}(\mathrm{Spec}(\mathbb{C}))_{\mathrm{loc}} := \Omega_*^{\mathbf{T}}(\mathrm{Spec}(\mathbb{C}))[c_1(\zeta)^{-1}]_{\{\zeta \neq 0 \in \widehat{\mathbf{T}}\}}$.
4. For a quasi-projective scheme X with a linear action of $\mathbf{T}$, we let
$$\Omega_*^{\mathbf{T}}(X)_{\mathrm{loc}} := \Omega_*^{\mathbf{T}}(X) \otimes_{\Omega_*^{\mathbf{T}}(\mathrm{Spec}(\mathbb{C}))} \Omega_*^{\mathbf{T}}(\mathrm{Spec}(\mathbb{C}))_{\mathrm{loc}}.$$

Theorem 9.4.1 (Localization of algebraic cobordism)**.** *Let X be a quasi-projective scheme with a linear action of* $\mathbf{T}$. *Let $i : X^{\mathbf{T}} \hookrightarrow X$ be the inclusion map of the fixed locus and $j : X \backslash X^{\mathbf{T}} \hookrightarrow X$ be the inclusion map of the complement.*

1. *Then we have a short exact sequence*
$$0 \longrightarrow \Omega_*(X^{\mathbf{T}}) \xhookrightarrow{i_*} \Omega_*^{\mathbf{T}}(X) \xrightarrow{j^*} \Omega_*^{\mathbf{T}}(X \backslash X^{\mathbf{T}}) \longrightarrow 0.$$
2. *Moreover we have an isomorphism*
$$i_* : \Omega_*^{\mathbf{T}}(X^{\mathbf{T}})_{\mathrm{loc}} \to \Omega_*^{\mathbf{T}}(X)_{\mathrm{loc}}.$$

We refer to [KP1, Thm. 6.1] and [KP1, Cor. 6.4] for the proof of Theorem 9.4.1. The proof is quite complicated than the standard arguments for Chow groups and K-theory in Edidin-Graham [EG3] and Thomason [Tho]. The main difficulty is that there is no localization sequence for equivariant algebraic cobordism.

Remark 9.4.2. The results in Theorem 9.4.1 were originally claimed by Krishna in [Kri2, Thm. 4.1] and [Kri2, Thm. 7.1], using the localization sequence of Heller-Malogon-Lopez in [HML, Thm. 20]. However the author thinks the proof of [HML, Thm. 20] is not correct (see footnote 1 in Remark 9.1.9).

Corollary 9.4.3. *Let X be a quasi-projective scheme and E be a vector bundle on $X \times B\mathbf{T}$. Assume that $E^{\mathrm{fix}} = 0$. Then the equivariant Euler class*

$$e^{\mathbf{T}}(E) : \Omega_*^{\mathbf{T}}(X)_{\mathrm{loc}} \to \Omega_*^{\mathbf{T}}(X)_{\mathrm{loc}}$$

is an isomorphism.

Proof. By Theorem 9.4.1,

$$(0_E)_* : \Omega_*^{\mathbf{T}}(X)_{\mathrm{loc}} \to \Omega_*^{\mathbf{T}}(E)_{\mathrm{loc}}$$

is an isomorphism. The formula

$$e^{\mathbf{T}}(E) = 0_E^! \circ (0_E)_*$$

completes the proof since $0_E^!$ is an isomorphism by the homotopy property. □

9.4.2 Localization of virtual cobordism classes

In this subsection, we prove the virtual torus localization formula for virtual cobordism classes.

Definition 9.4.4 (Equivariant virtual pullback)**.** Let $f : X \to Y$ be a $\mathbf{T}$-equivariant morphism of quasi-projective schemes with linear $\mathbf{T}$-actions. Assume the $\mathbf{T}$-action on X is trivial. Let $\phi : \mathbb{F} \to L_{X/Y}$ be a $\mathbf{T}$-good obstruction theory (see Definition 7.1.12). Choose a resolution $\mathbb{F}^{\mathrm{mov}} \cong [F^{-2} \to F^{-1}]$. Then the composition

$$\psi : \mathbb{F}^{\mathrm{fix}} \oplus F^{-1}[1] \to \mathbb{F}^{\mathrm{fix}} \oplus \mathbb{F}^{\mathrm{mov}} = \mathbb{F} \to L_{X/Y}$$

is a perfect obstruction theory. We define the $\mathbf{T}$-*equivariant pullback* as the composition

$$f_{\mathbf{T}}^! : \Omega_*^{\mathbf{T}}(Y)_{\mathrm{loc}} \xrightarrow{f_\psi^!} \Omega^{\mathbf{T}}(X)_{\mathrm{loc}} \xrightarrow{(e^{\mathbf{T}}(F_2))^{-1}} \Omega^{\mathbf{T}}(X)_{\mathrm{loc}}$$

where $f_\psi^!$ is the virtual pullback associated to the perfect obstruction theory ψ and the equivariant Euler class $e^{\mathbf{T}}(F_2)$ of $F_2 := (F^{-2})^\vee$ is invertible by Proposition 9.4.3.

It is easy to show that the equivariant virtual pullback in Definition 9.4.4 is independent of the choice of resolution $\mathbb{F}^{\mathrm{mov}} \cong [F^{-2} \to F^{-1}]$.

Theorem 9.4.5 (Functoriality)**.** *Consider a commutative diagram of quasi-projective schemes with linear* $\mathbf{T}$*-actions*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

where f and g are $\mathbf{T}$*-equivariant morphisms. Assume that the* $\mathbf{T}$*-action on X is trivial. Let $\phi_{X/Y} : \mathbb{F}_{X/Y} \to L_{X/Y}$, $\phi_{X/Z} : \mathbb{F}_{X/Z} \to L_{X/Z}$ be* $\mathbf{T}$*-good obstruction theories and $\phi_{Y/Z} : \mathbb{F}_{Y/Z} \to L_{Y/Z}$ be a* $\mathbf{T}$*-equivariant perfect obstruction theory. Assume that there exists a morphism of distinguished triangles*

$$\begin{tikzcd} f^*(\mathbb{F}_{Y/Z}) \arrow[r] \arrow[d, "f^*(\phi_{Y/Z})"] & \mathbb{F}_{X/Z} \arrow[r] \arrow[d, "\phi_{X/Z}"] & \mathbb{F}_{X/Y} \arrow[r] \arrow[d, dotted, "\phi'_{X/Y}"] & {} \\ \tau^{\geqslant -1} f^*(L_{Y/Z}) \arrow[r] & L_{X/Z} \arrow[r] & L'_{X/Y} \arrow[r] & {} \end{tikzcd}$$

for some $\phi'_{X/Y}$ such that $\phi_{X/Y} = r \circ \phi'_{X/Y}$. Then we have

$$(g \circ f)^!_{\mathbf{T}} = f^!_{\mathbf{T}} \circ g^! : \Omega^{\mathbf{T}}_*(Z)_{\mathrm{loc}} \to \Omega^{\mathbf{T}}_*(X)_{\mathrm{loc}}.$$

Proof. Consider a resolution

$$\mathbb{F}^{\mathrm{mov}}_{X/Y} = [F^{-2} \to F^{-1} \to 0]$$

by vector bundles F^{-2} and F^{-1}. Then we can form a compatible triple of perfect obstruction theories as follows:

$$\begin{tikzcd} \mathbb{F}_{Y/Z}|_X \arrow[r] \arrow[d, equal] & \mathbb{F} \arrow[r] \arrow[d, dashed] \arrow[dd, dashed, bend left, "\psi_{X/Z}"] & \mathbb{F}^{\mathrm{fix}}_{X/Y} \oplus F^{-1}[1] \arrow[r] \arrow[d] \arrow[dd, bend left, "\psi_{X/Y}"] & \mathbb{F}_{Y/Z}|_X[1] \arrow[d, equal] \\ \mathbb{F}_{Y/Z}|_X \arrow[r] \arrow[d, "\phi_{Y/Z}"] & \mathbb{F}_{X/Z} \arrow[r] \arrow[d, "\phi_{X/Z}"] & \mathbb{F}_{X/Y} \arrow[r] \arrow[d, "\phi'_{X/Y}"] & \mathbb{F}_{Y/Z}|_X[1] \arrow[d, "\phi_{Y/Z}"] \\ \tau^{\geqslant -1} L_{Y/Z}|_X \arrow[r] & L_{X/Z} \arrow[r] & L'_{X/Y} \arrow[r] & \tau^{\geqslant -1} L_{Y/Z}|_X[1]. \end{tikzcd}$$

By the functoriality of ordinary virtual pullbacks in Theorem 2.3.12, we have

$$(g \circ f)^!_{\psi_{X/Z}} = f^!_{\psi_{X/Y}} \circ g^! : A^{\mathbf{T}}_*(Z) \to A^{\mathbf{T}}_*(X).$$

By the octahedral axiom, we obtain a distinguished triangle

$$\mathbb{F}^{\mathrm{mov}} \longrightarrow \mathbb{F}^{\mathrm{mov}}_{X/Z} \longrightarrow F^{-2}[2]\,.$$

Hence $\mathbb{F}^{\mathrm{mov}} = F[1]$ for some vector bundle F and $\mathbb{F}^{\mathrm{mov}}_{X/Z} = [F^{-2} \to F \to 0]$. Therefore by the definition of equivariant virtual pullbacks in Definition 9.4.4, we have

$$(g \circ f)^!_{\mathbf{T}} = e^{\mathbf{T}}(F_2)^{-1} \circ (g \circ f)^!_{\psi_{X/Z}} : \Omega^{\mathbf{T}}_*(Z)_{\mathrm{loc}} \to \Omega^{\mathbf{T}}_*(X)_{\mathrm{loc}}$$
$$f^!_{\mathbf{T}} = e^{\mathbf{T}}(F_2)^{-1} \circ f^!_{\psi_{X/Y}} : \Omega^{\mathbf{T}}_*(Y)_{\mathrm{loc}} \to \Omega^{\mathbf{T}}_*(X)_{\mathrm{loc}}$$

where $F_2 := (F^{-2})^\vee$. Then we have the desired equality since the equivariant Euler class $e^{\mathbf{T}}(F_2)^{-1}$ commutes with the virtual pullback $g^!$. □

Theorem 9.4.6 (Localization of virtual cobordism class)**.** *Let X be a quasi-projective scheme with a linear $\mathbf{T}$-action. Let $\phi : \mathbb{F} \to L_X$ be a $\mathbf{T}$-equivariant perfect obstruction theory. Let $X^{\mathbf{T}}$ be the fixed locus and $\phi_{X^{\mathbf{T}}}$ be the induced perfect obstruction in Definition 7.2.1. Then we have*

$$[X]^{\mathrm{vir}} = i_*\left(\frac{[X^{\mathbf{T}}]^{\mathrm{vir}}}{e^{\mathbf{T}}(N^{\mathrm{vir}})}\right) \in \Omega^{\mathbf{T}}_*(X)_{\mathrm{loc}}$$

where $e^{\mathbf{T}}(N^{\mathrm{vir}}) := e^{\mathbf{T}}(F_0)/e^{\mathbf{T}}(F_1)$ for a resolution $\mathbb{F}^{\mathrm{mov}} = [F^{-1} \to F^0]$, and and $i : X^{\mathbf{T}} \hookrightarrow X$ is the inclusion map.

Proof. It follows from Theorem 9.4.1 and Theorem 9.2.12, as in the proof of Theorem 7.2.5 □

Appendix A

Kimura sequence for Artin stacks

In this appendix, we extend the Kimura sequence [Kim] to Kresch's Chow groups [Kre2] of Artin stacks. This is based on [BP].

A.1 Kimura sequence for Artin stacks

The main result in this section is the *Kimura sequence* for Artin stacks.

Theorem A.1.1 (Kimura sequence)**.** *Let $p : Y \to X$ be a proper representable surjective morphism of algebraic stacks with affine stabilizers. Consider the induced diagram*

$$Y \times_X Y \underset{p_2}{\overset{p_1}{\rightrightarrows}} Y \xrightarrow{p} X$$

where p_1 and p_2 are the projection maps. Then we have a right exact sequence

$$A_*(Y \times_X Y) \xrightarrow{(p_1)_* - (p_2)_*} A_*(Y) \xrightarrow{\quad p_* \quad} A_*(X) \xrightarrow{\qquad} 0$$

where $(p_1)_*$, $(p_2)_*$, and p_* are the proper pushforwards of [BS, Appendix B].

Proof. Since the Chow groups are invariant under the nilpotent thickenings, we may assume that X and Y are reduced.

We first prove the surjectivity of the proper pushforward

$$p_* : A_*(Y) \to A_*(X).$$

We will use the Noetherian induction on X. Since X has affine stabilizers, there exists a non-empty open substack U which is the quotient stack of a quasi-projective scheme by a linear action of a linear algebraic group by [Kre2, Prop. 3.5.2]. Then we have a morphism of right exact sequences

$$\begin{tikzcd}
A_*(Z_Y) \arrow[r] \arrow[d] & A_*(Y) \arrow[r] \arrow[d, "p_*"] & A_*(U_Y) \arrow[r] \arrow[d] & 0 \\
A_*(Z) \arrow[r] & A_*(X) \arrow[r] & A_*(U) \arrow[r] & 0
\end{tikzcd}$$

where $Z = X \backslash U$ is the complement as a reduced closed substack of X. By the induction hypothsis, the left vertical arrow is surjective. By a diagram chasing argument, we may replace X by U, and assume that X is the quotient of a quasi-projective scheme by a linear action of a linear algebraic group. By replacing X by Totaro's approximation [Tot], we may assume that X is a quasi-projective scheme. Then Y is a separated algebraic space and hence there is a projective surjection $q : \widetilde{Y} \to Y$ from a quasi-projective scheme $\widetilde{Y}$ by [LMB, Cor. 16.6.1]. Since the composition

$$A_*(\widetilde{Y}) \xrightarrow{q_*} A_*(Y) \xrightarrow{p_*} A_*(X)$$

is surjective by [Kim, Prop. 1.3], $p_* : A_*(Y) \to A_*(X)$ is also surjective.

We then prove the exactness on the middle. As in the previous paragraph, we will use the Noetherian induction on X. Choose a non-empty open substack U of X which is the quotient stack of a quasi-projective scheme by a linear action of a linear algebraic group. By generic smoothness, we may further assume that U is smooth. Let $Z = X \backslash U$ be the complement as a reduced closed substack of X. Form a commutative diagram

$$\begin{tikzcd}
 & A_*(Z_Y \times_Z Z_Y) \arrow[r] \arrow[d] & A_*(Y \times_X Y) \arrow[r] \arrow[d] & A_*(U_Y \times_U U_Y) \arrow[r] \arrow[d] & 0 \\
A_*(U_Y; 1) \arrow[r] \arrow[d] & A_*(Z_Y) \arrow[r] \arrow[d] & A_*(Y) \arrow[r] \arrow[d] & A_*(U_Y) \arrow[r] \arrow[d] & 0 \\
A_*(U; 1) \arrow[r] \arrow[d] & A_*(Z) \arrow[r] \arrow[d] & A_*(X) \arrow[r] \arrow[d] & A_*(U) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 & 0 &
\end{tikzcd}$$

where the rows are exact by [Kre2, Prop. 4.2.1] since U and U_Y are global quotient stacks. The first column is well-defined by [BS, Appendix B.12] and is surjective by Lemma A.1.2 below, after replacing U by Totaro's approximation [Tot]. The second column is exact by the induction hypothesis. By a diagram chasing argument, it suffices to show that the fourth column is exact. Hence by replacing X by U, we may assume that X is a quotient stack of a quasi-projective scheme by a linear action of a linear algebraic group. Replacing U by Totaro's approximation [Tot], we may further assume that X is a quasi-projective scheme. Then Y is a separated algebraic space. By the Chow lemma in [LMB, Cor. 16.6.1], there exists a projective surjective map $q : \widetilde{Y} \to Y$ from a quasi-projective scheme $\widetilde{Y}$. Form a commutative diagram

$$\begin{tikzcd} A_*(\widetilde{Y} \times_X \widetilde{Y}) \arrow[r] \arrow[d] & A_*(\widetilde{Y}) \arrow[r] \arrow[d] & A_*(X) \arrow[r] \arrow[d, equal] & 0 \\ A_*(Y \times_X Y) \arrow[r] & A_*(Y) \arrow[r] & A_*(X) \arrow[r] & 0 \end{tikzcd}$$

where the upper row is exact by [Kim, Thm. 1.8] and the colmumns are surjective by the result in the first paragraph. A diagram chasing argument shows that the lower row is also exact. □

We need the following lemma to complete the proof of Theorem A.1.1.

Lemma A.1.2. *Let $p : Y \to X$ be a proper representable surjective morphism from a separated DM stack Y to a smooth quasi-projective scheme X. Then the proper pushforward (in [BS, Appendix B])*

$$p_* : A_*(Y; 1) \to A_*(X; 1)$$

is surjective.

Proof. We may assume that Y is a smooth quasi-projective scheme. Indeed, by the Chow lemma in [LMB, Cor. 16.6.1], there exists a projective surjective map $q : \widetilde{Y} \to Y$ from a quasi-projective scheme $\widetilde{Y}$. By resolution of singularities, we may further assume that $\widetilde{Y}$ is smooth. It suffices to show that the composition

$$A_*(\widetilde{Y}; 1) \xrightarrow{q_*} A_*(Y; 1) \xrightarrow{p_*} A_*(X; 1)$$

is surjective. Replace Y by $\widetilde{Y}$.

Choose an element $\beta \in A_*(X; 1)$. Since $p_* : A_*(Y) \to A_*(X)$ is surjective by [Kim, Prop. 1.3], there exists a cycle class $\alpha \in A_*(Y)$ such that

$$p_*(\alpha) = [X] \in A_*(X).$$

Consider a fiber diagram

$$\begin{tikzcd} Y \arrow[r, hook, "\Gamma"] \arrow[d, "p"] & Y \times X \arrow[d, "p\times \mathrm{id}_X"] \\ X \arrow[r, hook, "\Delta"] & X \times X \end{tikzcd}$$

where Δ is the diagonal map and Γ is the graph of p. Since both $Y \times X$ and $X \times X$ are smooth, we have

$$\beta = \Delta^!([X] \times \beta) = \Delta^! \circ (p \times \mathrm{id}_X)_*(\alpha \times \beta) = p_*(\Gamma^!(\alpha \times \beta)),$$

which completes the proof. □

There are technical obstructions for generalizing Theorem A.1.1 to proper DM morphisms.

Remark A.1.3 (Generalization to proper DM morphism)**.** We may want to generalize the Kimura sequence in Theorem A.1.1 to proper DM morphisms. There are two technical obstructions for doing this:

1. We need a localization sequence

$$A_*(U;1) \longrightarrow A_*(Z) \longrightarrow A_*(X) \longrightarrow A_*(U) \longrightarrow 0$$

for closed immersion $Z \hookrightarrow X$ of algebraic stacks with affine stablizers when $U := X \backslash Z$ is not a global quotient stack (for the quotient stack of a separated DM stack by an action of a linear algebraic group).

2. We need a pushforward

$$p_* : A_*(Y;1) \to A_*(X;1)$$

for a proper DM morphism $p : Y \to X$ of algebraic stacks with affine stabilizers.

Both of these obstructions can be resolved by comparing Kresch's Chow groups [Kre2] and Khan's motivic Borel-Moore spectra [Khan]. We plan to study this in [BP].

A.2 Chow lemma for Artin stacks

The Kimura sequence in Theorem A.1.1 is especially useful if an algebraic stack has a proper cover by an algebraic stack whose intersection theory is well-understood, e.g., global quotient stacks. For the Chow groups with rational coefficients, the DM stacks behave like schemes. Thus we will work with the quotient stacks of DM stacks by linear algebraic groups.

We introduce the following definition.

Definition A.2.1 (Proper cover by quotient stack)**.** We say that an algebraic stack X admits a *proper cover by a quotient stack* it there exists a proper representable surjective morphism

$$p : Y \to X$$

from the quotient stack $Y = [P/G]$ of a separated DM stack P by an action of a linear algebraic group G.

We observe that the class of algebraic stacks that admit proper covers by quotient stacks is stable under basic operations.

Proposition A.2.2. *Let X, Y, and Z be algeabraic stacks.*

1. *Let $f : X \to Y$ be a proper representable surjective morphism. If X admits a proper cover by a quotient stack, then so is Y.*

2. *Let $f : X \to Y$ be a separated DM morphism. If Y admits a proper cover by a quotient stack, then so is X.*

3. *LAssume that the diagonal of Z is separated. If X, Y, and Z admit proper covers by quotient stacks, then so is the fiber product $X \times_Z Y$.*

The Chow lemma for Artin stacks is that étale-locally quotient stacks admit proper covers by quotient stacks.

Proposition A.2.3 (Chow lemma)**.** *Let X be an algebraic stack. Assume that there exists a separated, representable, étale, surjective map $u : U \to X$ such that U is the quotient stack of a separated DM stack by an action of a linear algebraic group. Then X admits a proper cover by a quotient stack.*

Proposition A.2.3 follows by the arguments in [LMB, Cor. 16.6.1]. We refer to [BP] for the details.

Example A.2.4 (Stacks with reductive stabilzers)**.** Let X be an algebraic stack with reductive stabilizers and affine diagonal. Then X is étale-locally a quotient stack by [AHR]. Consequently, X admits a proper cover by a quotient stack.

Example A.2.5 (Cone stacks)**.** Let X be a separated DM stack. Let $\mathfrak{C}$ be a cone stack on X. Then $\mathfrak{C}$ is étale-locally a quotient stack. Hence $\mathfrak{C}$ admits a proper cover by a quotient stack.

We obtain the *abstract blowup sequence* as a direct corollary. We first fix the notion of an *abstract blowup square*.

Definition A.2.6 (Abstract blowup square)**.** We say that a cartesian square of algebraic stacks

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "q"] & \widetilde{X} \arrow[d, "p"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd}$$

is an *abstract blowup square* if

1. p is a projective morphism,
2. i is a closed embedding, and
3. $p|_{\widetilde{X}\backslash E} : \widetilde{X}\backslash E \to X\backslash Z$ is an isomorphism.

Corollary A.2.7 (Abstract blowup sequence)**.** *Let*

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "q"] & \widetilde{X} \arrow[d, "p"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd}$$

be an abstract blowup square of algebraic stacks with affine stabilizers. Assume that X admits a proper cover by a quotient stack. Then we have a right exact sequence

$$A_*(E) \xrightarrow{(-j_*, q_*)} A_*(\widetilde{X}) \oplus A_*(Z) \xrightarrow{(p_*, i_*)} A_*(X) \longrightarrow 0.$$

Proof. Choose a proper representable surjective map

$$f : Y \to X$$

from the quotient stack of a separated DM stack by an action of a linear algebraic group. Note that the abstract blowup square is stable under the base change. Form a commutative diagram

$$\begin{tikzcd}
A_*(E_Y \times_E E_Y) \arrow[r] \arrow[d] & A_*(\widetilde{X}_Y \times_{\widetilde{X}} \widetilde{X}_Y) \oplus A_*(Z_Y \times_Z Z_Y) \arrow[r] \arrow[d] & A_*(Y \times_X Y) \arrow[r] \arrow[d] & 0 \\
A_*(E_Y) \arrow[r] \arrow[d] & A_*(\widetilde{X}_Y) \oplus A_*(Z_Y) \arrow[r] \arrow[d] & A_*(Y) \arrow[r] \arrow[d] & 0 \\
A_*(E) \arrow[r] \arrow[d] & A_*(\widetilde{X}) \oplus A_*(Z) \arrow[r] \arrow[d] & A_*(X) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

where the columns are exact by Theorem A.1.1. A diagram chasing argument shows that the exactness of the top two rows will imply the exactness of the third row. Hence we may assume that X is the quotient stack of a separated DM stack by an action of a linear algeabraic group.

Replacing X by Totaro's approximation [Tot], we may assume that X is a separated DM stack. By [LMB, Cor. 16.6.1], there exists a projective surjective map $f : Y \to X$ from a quasi-projective scheme Y. By repeating the argument in the previous paragraph, we may assume that X is a quasi-projective scheme. Then the abstract blowup sequence follows by [Ful, Ex. 1.8.1]. □

Remark A.2.8 (Operational Chow groups)**.** In [BHPSS], operational Chow groups for Artin stacks are introduced. Using the Kimura sequence in Theorem A.1.1, we can show that the operation Chow groups equal to Kresch's Chow groups when the algebraic stack has a proper cover by a quotient stack. We refer to [BP] for the details.

Bibliography

[AHR] J. Alper, J. Hall and D. Rydh, *A Luna étale slice theorem for algebraic stacks*, Ann. of Math. (2) **191** (2020), no. 3, 675–738.

[AG] B. Antieau and D. Gepner, *Brauer groups and étale cohomology in derived algebraic geometry*, Geom. Topol. **18**, (2014), no. 2, 1149–1244.

[AKLPR] D. Aranha, A. Khan, A. Latyntsev, H. Park, and C. Ravi, *Localization theorems for algebraic stacks*, arXiv:2207.01652.

[AP] D. Aranha and P. Pstragowski, *The intrinsic normal cone for Artin stacks*, arXiv:1909.07478.

[BHPSS] Y. Bae, D. Holmes, R. Pandharipande, J. Schmitt, and R. Schwarz, *Pixton's formula and Abel-Jacobi theory on the Picard stack*, arXiv:2004.08676, to appear in Acta Math.

[BKP] Y. Bae, M. Kool and H. Park, *Counting surfaces on Calabi-Yau* 4*-folds I: foundations*, in preparation.

[BP] Y. Bae and H. Park, *A comparison theorem for cycle theories for Artin stacks*, in preparation.

[BS] Y. Bae and J. Schmitt, *Chow rings of stacks of prestable curves 1*, (with an appendix joint with J. Skowera), arXiv:2012.09887, to appear in Forum Math. Sigma.

[Beh1] K. Behrend, *Gromov-Witten invariants in algebraic geometry*, Invent. Math. **127** (1997), no. 3, 601–617.

[Beh2] K. Behrend, *Donaldson-Thomas type invariants via microlocal geometry*, Ann. of Math. (2) **170** (2009), no. 3, 1307–1338.

[BF] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. 128 (1997), 45–88.

[BM] K. Behrend and Y. Manin, *Stacks of stable maps and Gromov-Witten invariants*, Duke Math. J. **85** (1996), no. 1, 1–60.

[BBBJ] O. Ben-Bassat, C. Brav, V. Bussi and D. Joyce, *A 'Darboux theorem' for shifted symplectic structures on derived Artin stacks, with applications*, Geom. Topol. **19** (2015) no. 3, 1287–1359.

[BBDJS] C. Brav, V. Bussi, D. Dupont, D. Joyce, and B. Szendrői, *Symmetries and stabilization for sheaves of vanishing cycles*, J. Singul. **11**, (2015), 85–151.

[Blo] S. Bloch, *Semi-regularity and deRham cohomology*, Invent. Math. **17** (1972), 51–66.

[Boj] A. Bojko, *Wall-crossing for zero-dimensional sheaves and Hilbert schemes of points on Calabi–Yau 4-folds*, arXiv:2102.01056.

[BJ] D. Borisov and D. Joyce, *Virtual fundamental classes for moduli spaces of sheaves on Calabi-Yau four-folds*, Geom. Topol. 21 (2017), 3231–3311.

[BG] E. Bouaziz and I. Grojnowski, *A d-shifted Darboux theorem*, arXiv:1309.2197.

[BBJ] C. Brav, V. Bussi and D. Joyce, *A Darboux theorem for derived schemes with shifted symplectic structure*, J. Amer. Math. Soc. **32** (2019), no. 2, 399–443.

[Bri] T. Bridgeland, *Hall algebras and curve-counting invariants*, J. Amer. Math. Soc. **24** (2011), no. 4, 969–998.

[BuFl] R.-O. Buchweitz and H. Flenner, *A semiregularity map for modules and applications to deformations*, Compositio Math. 37 (2003), no. 2, 135–210.

[Cal] D. Calaque, *Shifted cotangent stacks are shifted symplectic*, Annales de la Faculté des sciences de Toulouse: Mathématiques, **28** (2019), no. 1, 67–90.

[CGJ] Y. Cao, J. Gross, and D. Joyce, *Orientability of moduli spaces of Spin(7)-instantons and coherent sheaves on Calabi-Yau 4-folds*, Adv. Math. 368 (2020), 107134.

[CK1] Y. Cao and M. Kool, *Zero-dimensional Donaldson-Thomas invariants of Calabi-Yau 4-folds*, Adv. Math. 338 (2018) 601–648.

[CK2] Y. Cao and M. Kool, *Curve counting and DT/PT correspondence for Calabi-Yau 4-folds*, Adv. Math. 375 (2020) 107371.

[CKM] Y. Cao, M. Kool and S. Monavari, *K-theoretic DT/PT correspondence for toric Calabi-Yau 4-folds*, arXiv:1906.07856.

[CL] Y. Cao and N. C. Leung, *Donaldson-Thomas theory for Calabi-Yau 4-folds*, arXiv:1407.7659.

[CMT1] Y. Cao, D. Maulik, and Y. Toda, *Genus zero Gopakumar-Vafa type invariants for Calabi-Yau 4-folds*, Adv. Math. 338 (2018) 41–92.

[CMT2] Y. Cao, D. Maulik, and Y. Toda, *Stable pairs and Gopakumar-Vafa type invariants for Calabi-Yau 4-folds*, J. Eur. Math. Soc. (2021).

[COT1] Y. Cao, G. Oberdieck, and Y. Toda, *Gopakumar-Vafa type invariants of holomorphic symplectic 4-folds*, arXiv:2201.10878.

[COT2] Y. Cao, G. Oberdieck, and Y. Toda, *Stable pairs and Gopakumar-Vafa type invariants on holomorphic symplectic 4-folds*, arXiv:2201.11540.

[CT19] Y. Cao, Y. Toda, *Curve counting via stable objects in derived categories of Calabi-Yau 4-folds*, arXiv:1909.04897.

[CT20] Y. Cao, Y. Toda, *Tautological stable pair invariants of Calabi-Yau 4-folds*, Adv. Math. **396** (2022), 108176.

[CT21] Y. Cao and Y. Toda, *Gopakumar-Vafa type invariants on Calabi-Yau 4-folds via descendent insertions*, Comm. Math. Phys. **383** (2021), no. 1, 281–310.

[CK] H. L. Chang and Y. H. Kiem, *Poincaré invariants are Seiberg-Witten invariants*, Geom. Topol. **17** (2013), no. 2, 1149–1163.

[CKL] H.-L. Chang, Y.-H. Kiem, and J. Li, *Torus localization and wall crossing for cosection localized virtual cycles*, Adv. Math. **308** (2017), 964-986.

[ChLi] H.-L. Chang and J. Li, *Gromov–Witten invariants of stable maps with fields*, Int. Math. Res. Not. IMRN (2012), no. 18, 4163–4217.

[Dai] S. Dai, *Algebraic cobordism and Grothendieck groups over singular schemes*, Homology Homotopy Appl. **12** (2010), no. 1, 93–110.

[DT] S. Donaldson, R. P. Thomas, *Gauge theory in higher dimensions*, The geometric universe (Oxford, 1996), 31–47, Oxford Univ. Press, Oxford, (1998).

[EG1] D. Edidin and W. Graham, *Characteristic classes and quadric bundles*, Duke Math. J. **78** (1995), no. 2, 277–299.

[EG2] D. Edidin and W. Graham, *Equivariant intersection theory*, Invent. Math. **131** (1998), no. 3, 595-634.

[EG3] D. Edidin and W. Graham, *Localization in equivariant intersection theory and the Bott residue formula*, Amer. J. Math. **120** (1998), no. 3, 619–636.

[EHKV] D. Edidin, B. Hassett, A. Kresch, and A. Vistoli, *Brauer groups and quotient stacks*, Amer. J. Math. **123** (2001), no. 4, 761–777.

[FT] S. Feyzbakhsh and R. P. Thomas, *Rank r DT theory from rank* 1, to appear in J. Amer. Math. Soc.

[Ful] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 2. Springer-Verlag, Berlin, 1998.

[GS] A. Gholampour and A. Sheshmani, *Donaldson-Thomas invariants of 2-dimensional sheaves inside threefolds and modular forms*, Adv. Math. **326** (2018), 79–107.

[GP] T. Graber and R. Pandharipande, *Localization of virtual cycles*, Invent. Math. **135** (1999), no. 2, 487–518.

[GJT] J. Gross, D. Joyce and Y. Tanaka, *Universal structures in $\mathbb{C}$-linear enumerative invariant theories. I*, arXiv:2005.05637.

[Gro1] A. Grothendieck, *On the de Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. **29**, (1966), 95–103.

[Gro2] A. Grothendieck, *Techniques de construction et théorémes d'existence en géométrie algébrique. IV. Les schémas de Hilbert*, Séminaire Bourbaki, Vol. 6, Exp. No. 221, 249–276, Soc. Math. France, Paris, 1995.

[GK1] J. L. Gonzalez and K. Karu, *Projectivity in algebraic cobordism*, Canad. J. Math. **67** (2015), no. 3, 639-–653.

[HLP] D. Halpern-Leistner and A. Preygel, *Mapping stacks and categorical notions of properness*, arXiv:1402.3204.

[HML] J. Heller and J. Malagón-López, *Equivariant algebraic cobordism*, J. Reine Angew. Math. **684** (2013), 87-112.

[HL] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of sheaves*, Cambridge University Press (2010).

[HT] D. Huybrechts and R. P. Thomas, *Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes*, Math. Ann. **346** (2010), no. 3, 545–569.

[Ill] L. Illusie, *Complexe cotangent et déformations I, II*, LNM. **239, 283**. Springer-Verlag, Berlin-New York, 1971, 1972.

[Ina] M.-a. Inaba, *Toward a definition of moduli of complexes of coherent sheaves on a projective scheme*, J. Math. Kyoto Univ. **42** (2002), no. 2, 317–329.

[Joy] D. Joyce, *Enumerative invariants and wall-crossing formulae in abelian categories*, arXiv:2111.04694.

[JS] D. Joyce and Y. Song, *A theory of generalized Donaldson-Thomas invariants*, Mem. Amer. Math. Soc. **217** (2012), no. 1020, p. iv+199.

[Khan] A. A. Khan, *Virtual fundamental classes for derived stacks I*, arXiv:1909.01332.

[KR] A. A. Khan and C. Ravi, *Generalized cohomology theories for algebraic stacks*, arXiv:2106.15001.

[KL1] Y.-H. Kiem and J. Li, *Localizing virtual cycles by cosections*, J. Amer. Math. Soc. **26** (2013), no. 4, 1025–1050.

[KL2] Y.-H. Kiem and J. Li, *Categorification of Donaldson-Thomas invariants via perverse sheaves*, arXiv:1212.6444.

[KP1] Y.-H. Kiem and H. Park, *Virtual intersection theories*, Adv. Math. **388** (2021), Paper No. 107858, 51 pp.

[KP2] Y.-H. Kiem and H. Park, *Localizing virtual cycles for Donaldson-Thomas invariants of Calabi-Yau* 4*-folds*, arXiv:2012.13167.

[KP3] Y.-H. Kiem and H. Park, *Cosection localized virtual cycles are Oh-Thomas virtual cycles*, in preparation.

[KKP] B. Kim, A. Kresch, and T. Pantev, *Functoriality in intersection theory and a conjecture of Cox, Katz and Lee*, J. Pure Appl. Algebra **179** (2003), 127–136.

[Kim] S.-i. Kimura, *Fractional intersection and bivariant theory*, Comm. Algebra **20** (1992) no. 1, 285–302.

[Kon] M. Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves, 335–368, Progr. Math., 129, Birkhäuser Boston, 1995.

[KT1] M. Kool and R. Thomas, *Reduced classes and curve counting on surfaces I*, Algebr. Geom. **1** (2014), no. 3, 334–383.

[KT2] M. Kool and R. Thomas, *Reduced classes and curve counting on surfaces II*, Algebr. Geom. **1** (2014), no. 3, 384–399.

[Kre1] A. Kresch, *Canonical rational equivalence of intersections of divisors*, Invent. Math. **136** (1999), no. 3, 483-496.

[Kre2] A. Kresch, *Cycle groups for Artin stacks*, Invent. Math. **138** (1999), no. 3, 495-536.

[Kri1] A. Krishna, *Equivariant cobordism of schemes*, Doc. Math. **17** (2012), 95-134.

[Kri2] A. Krishna, *Equivariant cobordism for torus actions*, Adv. Math. **231** (2012), no. 5, 2858–2891.

[LMB] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 39. Springer-Verlag, Berlin, 2000.

[Lev1] M. Levine, *Fundamental classes in algebraic cobordism*, K-theory **30** (2003), no.2, 129–135.

[Lev2] M. Levine, *Oriented Cohomology, Borel–Moore Homology, and Algebraic Cobordism*, Michigan Math. J. **57** (2008).

[Lev3] M. Levine, *Comparison of cobordism theories*, J. Algebra **322** (2009), 3291–3317.

[LM] M. Levine and F. Morel, *Algebraic cobordism*, Springer Monographs in Mathematics. Springer, Berlin, 2007.

[LP] M. Levine and R. Pandharipande, *Algebraic cobordism revisited*, Invent. Math. **176** (2009), no. 1, 63–130.

[Li1] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom. **57** (2001), no. 3, 509–578.

[Li2] J. Li, *A degeneration formula of GW-invariants*, J. Differential Geom. **60** (2002), no. 2, 199–293.

[Li3] J. Li, *Zero dimensional Donaldson-Thomas invariants of threefolds*, Geom. Topol. **10** (2006), 2117–2171.

[LT] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math .Soc. 11 (1998) 119–174.

[LW] J. Li and B. Wu, *Good degeneration of Quot-schemes and coherent systems*, Comm. Anal. Geom. **23** (2015), no. 4, 841–921.

[Lie] M. Lieblich, *Moduli of complexes on a proper morphism*, J. Algebraic Geom. **15** (2006), no. 1, 175–206.

[Man] C. Manolache, *Virtual pull-backs*, J. Algebraic Geom. 21 (2012), no. 2, 201–245.

[MNOP1] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory, I*, Compos. Math. **142** (2006), no. 5, 1263–1285.

[MNOP2] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory, II*, Compos. Math. **142** (2006), no. 5, 1286–1304.

[MPT] D. Maulik, R. Pandharipande and R. Thomas, *Curves on K3 surfaces and modular forms. With an appendix by A. Pixton*, J. Topol. **3** (2010), no. 4, 937–996.

[MT] D. Maulik and Y. Toda, *Gopakumar-Vafa invariants via vanishing cycles*, Invent. Math. **213**, (2018), no. 3, 1017–1097.

[OT] J. Oh and R. P. Thomas, *Counting sheaves on Calabi-Yau 4-folds, I*, arXiv:2009.05542.

[OT2] J. Oh and R. P. Thomas, *Counting sheaves on Calabi-Yau 4-folds, II*,

[PP] R. Pandharipande and A. Pixton, *Gromov-Witten/Pairs correspondence for the quintic 3-fold*, J. Amer. Math. Soc. **30**, (2017), no. 2, 389–449.

[PT1] R. Pandharipande and R. P. Thomas, *Curve counting via stable pairs in the derived category*, Invent. Math. **178** (2009), 407–447.

[PT2] R. Pandharipande and R. P. Thomas, *Stable pairs and BPS invariants*, J. Amer. Math. Soc. **23** (2010), no.1, 267–297.

[PTVV] T. Pantev, B. Töen, M. Vaquié, and G. Vezzosi, *Shifted symplectic structures*, Publ. Math. I.H.E.S. **117** (2013), 271–328.

[Park1] H. Park, *Virtual pullbacks in Donaldson-Thomas theory of Calabi-Yau 4-folds*, arXiv:2110.03631.

[Park2] H. Park, *Deformation invariance in Donaldson-Thomas theory of Calabi-Yau 4-folds*, in preparation.

[Pot1] J. Le Potier, *Systemes cohérents et structures de niveau*, Asterisque **214** (1993).

[Pot2] J. Le Potier, *Faisceaux semi-stables et systemes cohérents*, Vector bundles in algebraic geometry (Durham, 1993) **208**, (1995) 179–239.

[Qu] F. Qu, *Virtual pullbacks in K-theory*, Ann. Inst. Fourier, Grenoble. **68** (2018), no. 4, 1609-1641.

[Quil] D. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. **7** (1971), 29–56.

[Sav] M. Savvas, *Cosection Localization and Vanishing for Virtual Fundamental Classes of D-Manifolds*, Adv. Math. **398** (2022), 108232.

[STV] T. Schürg, B. Toën and G. Vezzosi, *Derived algebraic geometry, determinants of perfect complexes, and applications to obstruction theories for maps and complexes*, J. Reine Angew. Math. **702** (2015), 1–40.

[Shen] J. Shen, *Cobordism invariants of the moduli space of stable pairs*, J. Lond. Math. Soc. (2) **94** (2016), no. 2, 427–446.

[Sie] B. Siebert, *Virtual fundamental classes, global normal cones and Fulton's canonical classes*, Frobenius manifolds (2004), 341–358.

[TT] Y. Tanaka and R. Thomas, *Vafa-Witten invariants for projective surfaces I: stable case*, J. Algebraic Geom. **29** (2020), no. 4, 603–668.

[Tho] R. P. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K3 fibrations*, J. Differential Geom. 54 (2000), no. 2, 367–438.

[Tho1] R. W. Thomason, *Equivariant resolution, linearization, and Hilbert's fourteenth problem over arbitrary base schemes*, Adv. Math. **65** (1987), no. 1, 16–34.

[Toda] Y. Toda, *Curve counting theories via stable objects I. DT/PT*, J. Amer. Math. Soc. **23** (2010), no. 4, 1119–1157.

[ToVa] B. Toën and M. Vaquié. *Moduli of objects in dg-categories.* Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 3, 387–444.

[ToVe] B. Toën and G. Vezzosi, *Homotopical algebraic geometry. II. Geometric stacks and applications.* Mem. Amer. Math. Soc. **193** (2008), no. 902, x+224 pp.

[Tot] B. Totaro, *The Chow ring of a classifying space*, Proc. Sympos. Pure Math. 67, Amer. Math. Soc., Providence, RI, 1999.

[Vish] A. Vishik, *Stable and unstable operations in algebraic cobordism*, Ann. Sci. Éc. Norm. Supér. (4) **52** (2019), no. 3, 561–630.

[Vist] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), no. 3, 613–670.

국문초록

이 학위 논문의 주요 목적은 가상 당김과 여절단 국소화를 칼라비-야우 4차원 다양체의 도널드슨-토마스 이론으로 확장하는 것입니다. 이차적인 목적은 가상 당김과 여절단 국소화를 아틴 스택의 교차이론, 파생 대수 기하학 및 대수적 코보디즘을 이용하여 일반화하는 것입니다.

주요어휘: 가상 당김, 여절단 국소화, 4차원 칼라비-야우 다양체의 도널드슨-토마스 이론
학번: 2018-20625

감사의 글