이학박사 학위논문

Regularity results for Orlicz phase problems

(오리츠 위상 문제의 정칙성)

2022년 8월

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Regularity results for Orlicz phase problems

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Abstract

Regularity results for Orlicz phase problems

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In this thesis, we provide comprehensive regularity results and optimal conditions for a general class of functionals involving Orlicz multi-phase, which exhibits non-standard growth conditions and non-uniformly elliptic properties.

First, we give a unified treatment to show various regularity results for minima of Orlicz multi-phase type functionals with coefficient functions not necessarily Hölder continuous even for a lower level of the regularity. Moreover, assuming that minima of such functionals belong to better spaces such as $C^{0,\gamma}(\Omega)$ or $L^{\kappa}(\Omega)$ for some $\gamma \in (0,1)$ and $\kappa \in (1,\infty]$, we address optimal conditions on nonlinearity for each variant under which we build comprehensive regularity results.

Second, we prove local Calderón-Zygmund type estimates under the optimal conditions on the nonlinearity for distributional solutions to non-uniformly elliptic equations of Orlicz double phase and multi-phase type in divergence form with the coefficient functions not necessarily Hölder continuous.

Lastly, we establish an optimal $C^{1,\alpha}$-regularity for viscosity solutions of a class of degenerate/singular fully nonlinear elliptic equations by finding minimal regularity requirements on the associated operator.

Key words: Orlicz phase problem; regularity; non-standard growth; Calderón-Zygmund theory; fully nonlinear degenerate/singular equations; viscosity solutions
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Contents

Chapter 1

Introduction

The first part of this thesis is concerned with optimal and comprehensive regularity results for minima of functionals featuring a non-standard growth and a non-uniform ellipticity. The primary model keeping in mind under investigation is given by an Orlicz multi-phase functional

$$W^{1,1}(\Omega) \ni \upsilon \mapsto \mathcal{P}(\upsilon, \Omega) := \int_{\Omega} \Psi(x, |D\upsilon|)\, dx \tag{1.0.1}$$

for a bounded open domain $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$, where throughout the thesis we shall always denote by

$$\Psi(x,t) := G(t) + a(x)H_a(t) + b(x)H_b(t) \quad (x \in \Omega,\ t \geqslant 0) \tag{1.0.2}$$

for N-functions $G, H_a, H_b \in \mathcal{N}$ in the sense of Definition 2.1.1 and $0 \leqslant a(\cdot), b(\cdot) \in L^{\infty}(\Omega)$. The Orlicz multi-phase functional $\mathcal{P}$ in (1.0.1) is naturally defined for functions $\upsilon \in W^{1,1}(\Omega)$, which is natural one including the following examples of functionals for the regularity theory:

1. p-growth: $\Psi(x,t) \equiv t^p$ with $p > 1$, see for instance [82, 89, 109, 110, 112, 113, 133, 134].

2. Orlicz growth: $\Psi(x,t) \equiv G(t)$, see for instance [74, 75, 111].

3. (p,q)-double phase: $\Psi(x,t) \equiv t^p + a(x)t^q$ for $1 < p \leqslant q$, see for instance [20, 22, 57, 58].

4. Borderline case of double phase: $\Psi(x,t) \equiv t^p + a(x)t^p \log(1+t)$ for $1 < p$, see for instance [21, 39].

5. Multi-phase: $\Psi(x,t) \equiv t^p + a(x)t^q + b(x)t^s$ for $1 < p \leqslant q, s$, see for instance [71].

6. Orlicz double phase: $\Psi(x,t) \equiv G(t) + a(x)H_a(t)$, see for instance [12, 39].

7. Orlicz multi-phase: $\Psi(x,t) \equiv G(t) + a(x)H_a(t) + b(x)H_b(t)$, see for instance [12].

Over last several years a systematic analysis of the functionals aforementioned has been an object of intensive studies for the regularity theory. Among them (p,q)-double phase functional is a significant example given by

$$W^{1,1}(\Omega) \ni v \mapsto \mathcal{P}_{p,q}(v,\Omega) := \int_\Omega [|Dv|^p + a(x)|Dv|^q] \, dx, \quad 1 < p \leqslant q. \tag{1.0.3}$$

Another example is the so-called borderline case of double phase defined by

$$W^{1,1}(\Omega) \ni v \mapsto \mathcal{P}_{\log}(v,\Omega) := \int_\Omega [|Dv|^p + a(x)|Dv|^p \log(1+|Dv|)] \, dx, \quad 1 < p. \tag{1.0.4}$$

The last functional we would like to single out is the so-called multi-phase functional introduced in [71] is of type

$$W^{1,1}(\Omega) \ni v \mapsto \mathcal{P}_{p,q,s}(v,\Omega) := \int_\Omega [|Dv|^p + a(x)|Dv|^q + b(x)|Dv|^s] \, dx, \quad 1 < p \leqslant q, s. \tag{1.0.5}$$

The (p,q)-double phase functional was initially introduced by Zhikov [139, 140, 143] in order to study the feature of strongly anisotropic materials in the context of homogenization and nonlinear elasticity. A main common feature of the functionals $\mathcal{P}_{p,q}$, $\mathcal{P}_{p,q,s}$ and $\mathcal{P}_{\log}$ in (1.0.3)-(1.0.5) is that their integrand changes their growth and ellipticity ratio depending on the geometric behavior of the coefficient functions $a(\cdot)$ and $b(\cdot)$, which determine the geometry of the mixture of different materials. Each functional mentioned above belongs to a family of functionals with nonstandard growth conditions of (p,q)-type. These are functionals of type

$$W^{1,1}(\Omega) \ni v \mapsto \int_\Omega F(x, Dv)\, dx,$$

whose energy density $F(x,z)$ satisfies

$$|z|^p - 1 \lesssim F(x,z) \lesssim |z|^q + 1, \quad 1 < p < q,$$

according to Marcellini's terminology [114, 115, 116]. Over the several decades, functionals with nonstandard growth have been extensively investigated, see for instance [3, 33, 61, 78, 79, 80, 86, 87, 129, 130] and see also [117, 118] for an overview of the state of the art. Those functionals aforementioned give a relevant example of the energy overlying in the so-called Musielak-Orlicz space which will described in Chapter 2.

For the regularity theory, the optimal conditions for the gradient of a local minimizer v of the functional $\mathcal{P}_{p,q}$ in (1.0.3) to be Hölder continuous have been discovered in [22, 57, 58]. They are

$$\begin{cases} \dfrac{q}{p} \leqslant 1 + \dfrac{\alpha}{n} & \text{if } v \in W^{1,p}(\Omega), \quad (1.0.6a) \\ q \leqslant p + \alpha & \text{if } v \in W^{1,p}(\Omega) \cap L^\infty(\Omega), \quad (1.0.6b) \\ q < p + \dfrac{\alpha}{1-\gamma} & \text{if } v \in W^{1,p}(\Omega) \cap C^{0,\gamma}(\Omega) \text{ with } \gamma \in (0,1), \quad (1.0.6c) \end{cases}$$

where $0 \leqslant a(\cdot) \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1]$.

Remark 1.0.1. The conditions in (1.0.6a)-(1.0.6b) are essentially sharp in the sense of Lavrentiev gap (see (1.0.26) for the definition) for the functional $\mathcal{P}_{p,q}$. Indeed, as shown in [80, 86], for every $\varepsilon > 0$, it is possible to construct a suitable coefficient function $0 \leqslant a(\cdot) \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ to find exponents p, q with

$$n - \varepsilon < p < n < n + \alpha < q < n + \varepsilon \tag{1.0.7}$$

such that there exist bounded minima of the functional in (1.0.3) whose set of discontinuity points has Hausdorff dimension larger than $n - p - \varepsilon$, which means that minima of the functional in (1.0.3) are as bad as any other $W^{1,p}$

functions. The selection in (1.0.7) makes both conditions (1.0.6a) and (1.0.6b) to be failed. Furthermore, there are recent results concerning the absence of Lavrentiev phenomenon [18], which shows that the conditions (1.0.6b) and (1.0.6c) are sharp for the functional in (1.0.3) with a coefficient function $0 \leqslant a(\cdot) \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ by constructing appropriate counterexamples based on Zhikov's two-dimensional checkboard as introduced in [139].

On the other hand, letting $0 \leqslant a(\cdot) \in C^{\omega_a}(\Omega)$ with a continuous and concave function $\omega_a : [0,\infty) \to [0,\infty)$ vanishing at the origin in (1.0.4), the conditions for a local minimizer v of the functional $\mathcal{P}_{\log}$ in (1.0.4) to be regular have been discovered in [21], which are

$$
\begin{cases}
v \text{ is Hölder continuous with an exponent} \\
\text{if } \limsup\limits_{\rho\to 0^+} \omega_a(\rho)\log\left(\dfrac{1}{\rho}\right) < \infty, & \text{(1.0.8a)} \\
v \text{ is Hölder continuous with an arbitrary exponent} \\
\text{if } \limsup\limits_{\rho\to 0^+} \omega_a(\rho)\log\left(\dfrac{1}{\rho}\right) = 0, & \text{(1.0.8b)} \\
Dv \text{ is Hölder continuous} \\
\text{if } \omega_a(\rho) \lesssim \rho^\alpha \text{ with } \alpha \in (0,1]. & \text{(1.0.8c)}
\end{cases}
$$

Furthermore, the optimal condition for the gradient of minima of the multi-phase functional $\mathcal{P}_{p,q,s}$ in (1.0.5) to be Hölder continuous has been obtained in [71], that is

$$
\frac{q}{p} \leqslant 1 + \frac{\alpha}{n} \quad \text{and} \quad \frac{s}{p} \leqslant 1 + \frac{\beta}{n}, \tag{1.0.9}
$$

where $0 \leqslant a(\cdot) \in C^{0,\alpha}(\Omega)$ and $0 \leqslant b(\cdot) \in C^{0,\beta}(\Omega)$ for some $\alpha, \beta \in (0,1]$. In fact, the condition (1.0.9) is a natural outcome of (1.0.6a) and sharp via Remark 1.0.1. In the first part of the thesis, we intend to unify all conditions presented in (1.0.6a)-(1.0.6c), (1.0.8a)-(1.0.8c) and (1.0.9) by considering more general class of functionals modelled on Orlicz multi-phase energy functional (1.0.1) under more weakened assumptions that the coefficient functions $a(\cdot)$ and $b(\cdot)$ in (1.0.1) are not necessarily Hölder continuous even for a lower level of the regularity. Moreover, under newly found conditions on the nonlinearity depending upon a priori assumptions on minima for investigation, we prove various regularity results starting from local boundedness of minima up to Hölder continuity for the gradient of minima. More precisely, we consider a class of general functionals of type

$$W^{1,1}(\Omega) \ni v \mapsto \mathcal{F}(v,\Omega) := \int_{\Omega} F(x,v,Dv)\,dx, \tag{1.0.10}$$

where the integral density $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathe͂́odory map satisfying the double-sided bound with constants $0 < \nu \leqslant L < \infty$:

$$\nu\Psi(x,|z|) \leqslant F(x,y,z) \leqslant L\Psi(x,|z|) \quad (x \in \Omega,\ y \in \mathbb{R},\ z \in \mathbb{R}^n), \tag{1.0.11}$$

where Ψ is the same function as in (1.0.2). Under the growth conditions (1.0.11), local minima (Q-minima) of the functional $\mathcal{F}$ in (1.0.10) for some number $Q \geqslant 1$ is defined classically as follows:

Definition 1.0.1. A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer (Q-minimizer) of the functional $\mathcal{F}$ defined in (1.0.10) if $\Psi(x,|Du|) \in L^1(\Omega)$ and the minimality condition

$$\begin{aligned} &\mathcal{F}(u,\operatorname{supp}(u-v)) \leqslant \mathcal{F}(v,\operatorname{supp}(u-v)) \\ &(\mathcal{F}(u,\operatorname{supp}(u-v)) \leqslant Q\mathcal{F}(v,\operatorname{supp}(u-v))) \end{aligned}$$

is satisfied, whenever $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ with $\operatorname{supp}(u-v) \Subset \Omega$.

In what follows, we shall always assume $0 \leqslant a(\cdot) \in C^{\omega_a}(\Omega)$ and $0 \leqslant b(\cdot) \in C^{\omega_b}(\Omega)$, where $\omega_a, \omega_b : [0,\infty) \to [0,\infty)$ are continuous and concave functions such that $\omega_a(0) = 0$ and $\omega_b(0) = 0$, unless they are specified. Then we define the auxiliary function $\Lambda : (0,\infty) \times (0,\infty) \to (0,\infty)$ given by

$$\Lambda(\rho,t) := \frac{\omega_a(\rho)}{1+\omega_a(\rho)}\frac{H_a(t)}{G(t)} + \frac{\omega_b(\rho)}{1+\omega_b(\rho)}\frac{H_b(t)}{G(t)} \quad \text{for any} \quad \rho, t > 0. \tag{1.0.12}$$

We shall consider a local Q-minimizer u of the functional $\mathcal{P}$ in (1.0.1) or a local minimizer u of the functional $\mathcal{F}$ in (1.0.10) under each of the following basic assumptions:

$$\begin{cases} u \in W^{1,\Psi}(\Omega), \\ \lambda_1 := \sup\limits_{\rho>0} \Lambda\left(\rho, G^{-1}(\rho^{-n})\right) < \infty, \end{cases} \tag{1.0.13}$$

$$\begin{cases} u \in W^{1,\Psi}(\Omega) \cap L^{\infty}(\Omega), \\ \lambda_2 := \sup_{\rho>0} \Lambda\left(\rho, \dfrac{1}{\rho}\right) < \infty, \end{cases} \tag{1.0.14}$$

$$\begin{cases} u \in W^{1,\Psi}(\Omega) \cap C^{0,\gamma}(\Omega) \quad \text{for some} \quad \gamma \in (0,1), \\ \lambda_3 := \sup_{\rho>0} \Lambda\left(\rho^{\frac{1}{1-\gamma}}, \dfrac{1}{\rho}\right) < \infty. \end{cases} \tag{1.0.15}$$

Here G^{-1} is the inverse function of G. Let us neatly explain why those conditions $(1.0.13) - (1.0.15)$ make sense by considering some significant special cases. In the case $(G(t), H_a(t), H_b(t)) \equiv (t^p, t^q, t^s)$ with $1 < p \leqslant q, s$, $\omega_a(\rho) \equiv \rho^{\alpha}$ and $\omega_b(\rho) \equiv \rho^{\beta}$ for some $\alpha, \beta \in (0,1]$, direct calculations yield that

$$\text{the condition } (1.0.13)_2 \quad \Longleftrightarrow \quad \frac{q}{p} \leqslant 1 + \frac{\alpha}{n} \text{ and } \frac{s}{p} \leqslant 1 + \frac{\beta}{n}, \tag{1.0.16}$$

$$\text{the condition } (1.0.14)_2 \quad \Longleftrightarrow \quad q \leqslant p + \alpha \text{ and } s \leqslant p + \beta \tag{1.0.17}$$

and

$$\text{the condition } (1.0.14)_2 \quad \Longleftrightarrow \quad q \leqslant p + \frac{\alpha}{1-\gamma} \text{ and } s \leqslant p + \frac{\beta}{1-\gamma}. \tag{1.0.18}$$

In particular, in the case $b(\cdot) \equiv 0$, the conditions (1.0.13)-(1.0.15) are read as (1.0.6a)-(1.0.6c), respectively, except the borderline case of the last condition. Clearly, the condition in (1.0.16) is the same one as in (1.0.9). Moreover, in the case of $(G(t), H_a(t), H_b(t)) \equiv (t^p, t^p \log(1+t), t^p \log(1+t))$ with $p > 1$, we see that the conditions $(1.0.13)_2$ and $(1.0.14)_2$ are equivalent to

$$\limsup_{\rho \to 0^+} \left(\omega_a(\rho) \log\left(\frac{1}{\rho}\right) + \omega_b(\rho) \log\left(\frac{1}{\rho}\right) \right) < +\infty. \tag{1.0.19}$$

In particular, for $b(\cdot) \equiv 0$, the above condition is equivalent to (1.0.8a). Furthermore, we are also able to give more examples of functionals showing how a modulus of continuity of $a(\cdot)$ and $b(\cdot)$ is exactly adjusted to the size of the phase transition. The natural assumptions for showing further regularity properties of minima of the Zygmund multi-phase functional determined by $(G(t), H_a(t), H_b(t)) \equiv (t^p[\log(1+t)]^{p_0}, t^q[\log(1+t)]^{q_0}, t^s[\log(1+t)]^{s_0})$ with $p, q, s > 1$ and $s_0, q_0 \geqslant p_0 \geqslant 1$, are that

$$\begin{aligned}
&\text{the condition } (1.0.13)_2 \\
&\Longleftrightarrow \\
&\sup_{\rho>0} \frac{\omega_a\left(\rho^{-\frac{p}{n}}[\log(1+\rho)]^{-\frac{p_0}{n}}\right)}{1+\omega_a\left(\rho^{-\frac{p}{n}}[\log(1+\rho)]^{-\frac{p_0}{n}}\right)}\rho^{q-p}[\log(1+\rho)]^{q_0-p_0} \\
&\quad + \sup_{\rho>0} \frac{\omega_b\left(\rho^{-\frac{p}{n}}[\log(1+\rho)]^{-\frac{p_0}{n}}\right)}{1+\omega_b\left(\rho^{-\frac{p}{n}}[\log(1+\rho)]^{-\frac{p_0}{n}}\right)}\rho^{s-p}[\log(1+\rho)]^{s_0-p_0} < \infty,
\end{aligned} \tag{1.0.20}$$

$$\begin{aligned}
&\text{the condition } (1.0.14)_2 \\
&\Longleftrightarrow \\
&\sup_{\rho>0} \frac{\omega_a(\rho)}{1+\omega_a(\rho)}\rho^{p-q}\left[\log\left(1+\frac{1}{\rho}\right)\right]^{q_0-p_0} \\
&\quad + \sup_{\rho>0} \frac{\omega_b(\rho)}{1+\omega_b(\rho)}\rho^{p-s}\left[\log\left(1+\frac{1}{\rho}\right)\right]^{s_0-p_0} < \infty
\end{aligned} \tag{1.0.21}$$

and

$$\begin{aligned}
&\text{the condition } (1.0.15)_2 \\
&\Longleftrightarrow \\
&\sup_{\rho>0} \frac{\omega_a\left(\rho^{\frac{1}{1-\gamma}}\right)}{1+\omega_a\left(\rho^{\frac{1}{1-\gamma}}\right)}\rho^{p-q}\left[\log\left(1+\frac{1}{\rho}\right)\right]^{q_0-p_0} \\
&\quad + \sup_{\rho>0} \frac{\omega_b\left(\rho^{\frac{1}{1-\gamma}}\right)}{1+\omega_b\left(\rho^{\frac{1}{1-\gamma}}\right)}\rho^{p-s}\left[\log\left(1+\frac{1}{\rho}\right)\right]^{s_0-p_0} < \infty.
\end{aligned} \tag{1.0.22}$$

Another example of functionals can be determined by $(G(t), H_a(t), H_b(t)) \equiv (t^p, t^q \log\log(e+t), t^s \log\log(e+t))$ with $1 < p \leqslant q, s$. Straightforwardly, it can be seen that

$$\begin{aligned}
&\text{the condition } (1.0.13)_2 \\
&\Longleftrightarrow \\
&\limsup_{\rho\to 0^+} \omega_a(\rho)\rho^{n-\frac{nq}{p}} \log\log\left(e+\rho^{-\frac{n}{p}}\right) \\
&+\limsup_{\rho\to 0^+} \omega_b(\rho)\rho^{n-\frac{ns}{p}} \log\log\left(e+\rho^{-\frac{n}{p}}\right) < \infty,
\end{aligned} \tag{1.0.23}$$

$$\begin{aligned}
&\text{the condition } (1.0.14)_2 \\
&\Longleftrightarrow \\
&\limsup_{\rho\to 0^+} \omega_a(\rho)\rho^{p-q} \log\log\left(e+\frac{1}{\rho}\right) \\
&\quad+\limsup_{\rho\to 0^+} \omega_b(\rho)\rho^{p-s} \log\log\left(e+\frac{1}{\rho}\right) < \infty.
\end{aligned} \tag{1.0.24}$$

and

$$\begin{aligned}
&\text{the condition } (1.0.15)_2 \\
&\Longleftrightarrow \\
&\limsup_{\rho\to 0^+} \omega_a\left(\rho^{\frac{1}{1-\gamma}}\right)\rho^{p-q} \log\log\left(e+\frac{1}{\rho}\right) \\
&\quad+\limsup_{\rho\to 0^+} \omega_b\left(\rho^{\frac{1}{1-\gamma}}\right)\rho^{p-s} \log\log\left(e+\frac{1}{\rho}\right) < \infty.
\end{aligned} \tag{1.0.25}$$

The assumptions (1.0.13)-(1.0.15) lead to exhibiting new instances of Lavrantiev phenomenon [139, 140, 141, 142, 143]. According to the classical definition, the Lavrentiev gap for the functional $\mathcal{F}$ defined in (1.0.10) under the growth assumption (1.0.11) may appear if

$$\inf_{v\in v_0+W_0^{1,G}(B)} \mathcal{F}(v,B) < \inf_{v\in v_0+W_0^{1,G}(B)\cap W_{\mathrm{loc}}^{1,\Psi_\Omega^+}(B)} \mathcal{F}(v,B) \tag{1.0.26}$$

holds for a ball $B \Subset \Omega$ and a function $v_0 \in W^{1,\infty}(B)$, where Ψ_Ω^+ is defined in (2.1.3) below. That is, local minima of $\mathcal{F}$ may not belong to $W_{\mathrm{loc}}^{1,\Psi_\Omega^+}(B)$ in general. To see this more precisely, let us turn our attention to the classical case that $G(t) \equiv t^p$, $H_a(t) \equiv t^q$, $\omega_a(\rho) \equiv \rho^\alpha$ and $\omega_b(\cdot) \equiv 0$ for some $1 < p < q$ and $\alpha \in (0, 1]$ such that

$$1 < p < n < n + \alpha < q. \tag{1.0.27}$$

Under classical double phase setting together with (1.0.27), the results of [57, Theorem 4.1] and [80, Section 3] provide us the existence of a coefficient function $0 \leqslant a(\cdot) \in C^{0,\alpha}(\Omega)$ and a boundary datum $u_0 \in W^{1,p}(B) \cap L^\infty(B)$ such that the Lavrentiev phenomenon (1.0.26) is occurred. In this regard, we show that there is no Lavrentiev gap for the functional $\mathcal{F}$ in (1.0.10) satisfying the basic structure assumption (1.0.11) under each of assumptions $(1.0.13)_2$, $(1.0.14)_2$ and $(1.0.15)_2$, see Theorem 2.3.1.

We shall investigate various regularity results of a local minimizer u of the functional $\mathcal{F}$ in (1.0.10) comprehensively in Chapter 3 , the main contents of Chapter 3 are Theorem 3.1.1 and Theorem 3.1.2 for functionals modelled on Orlicz multi-phase energy (Theorem 3.6.1 and Theorem 3.6.2 for functionals modelled on Orlicz double phase energy), under each of assumptions (1.0.13)-(1.0.15) for minima. We note that Hölder regularity for the gradient of a local minimizer in Theorem 3.1.1 (Theorem 3.6.1) is already optimal in the classical p-Laplacian case that $G(t) \equiv t^p$ and $a(\cdot) \equiv b(\cdot) \equiv 0$ [133, 134]. The assumptions in (3.1.10a)-(3.1.10c) are optimal by Remark 1.0.1. The regularity results reported here complement in a unified way the main results of [21, 22, 57, 58, 71], where the functions in (1.0.3)-(1.0.5) are considered under the corresponding conditions we have discussed in (1.0.6a)-(1.0.6c),(1.0.8a)-(1.0.8c) and (1.0.9), respectively, and the arguments used in these papers are strongly dependent of the number of phases along with the Hölder continuity of the coefficient functions in the non-linearity. Our approaches for proving the above theorems are in fact independent of this weakness. The approaches we present in Chapter 3 lead to avoiding the use of difference quotient methods employed in [57, 58] for obtaining various regularity properties of minima of the functional in (1.0.3). In fact, the difference quotient techniques can deal with the case that the coefficient functions in the nonlinearity are Hölder continuous. On the other hand, we are treating the case of not necessarily having Hölder continuous coefficient functions in the nonlinearity by applying a Harmonic type approximation (see Lemma 2.5.1) for comparing a homogeneous equation with a limiting equation having the lipschitz regularity property (see Lemma 3.3.3 and Lemma 3.3.4).

The contents in Chapter 3 could provide a guideline to deal with a very general class of non-autonomous functionals whose energy density behaves like

$$F(x,y,z) \approx \Phi(x,|z|) \tag{1.0.28}$$

for Φ being a certain Young function as we shall introduce in Definition 2.1.1 below. The investigation of such problems has been a field of interest for research activities over the decades. In fact, a main difficulty lies in discovering the optimal conditions to be placed on $\Phi(x,t)$ with respect to (x,t)-variables. Here we mention a very recent and interesting paper [97] in which the authors give a reasonable answer to such a question by considering a class of functionals of Uhlenbeck type without any a priori assumption on minima involved. Essensially, the assumption [97, (VA1)] is not comparable with the assumption $(1.0.13)_2$. Moreover, the method used in [97] can not be applicable to treat the regularity of minima of the functional $\mathcal{F}$ in (1.0.10) having the solution dependence. Besides the papers mentioned before, there is a rich literature, see for instance [4, 5, 25, 62, 81, 108, 124, 125, 132] and reference therein. We also refer to a survey paper [117].

The second part of the thesis is devoted to analyzing the validity of local Calderón-Zygmund type estimates for distributional solutions to the equation of divergence form

$$\operatorname{div} A(x,Du) = \operatorname{div} B(x,F) \quad \text{in} \quad \Omega \tag{1.0.29}$$

for a bounded open subset $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$, where the vector field $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, differentiable with respect to the second variable $z \in \mathbb{R}^n \setminus \{0\}$, and satisfies the following structural conditions with fixed constants $0 < \nu \leqslant L < \infty$:

$$\begin{cases} |A(x,z)| + |D_z A(x,z)||z| \leqslant L\dfrac{\Psi(x,|z|)}{|z|}, \\ \nu\dfrac{\Psi(x,|z|)}{|z|^2}|\xi|^2 \leqslant \langle D_z A(x,z)\xi,\xi\rangle\,, \\ |A(x_1,z) - A(x_2,z)||z| \leqslant L\,|\Psi(x_1,|z|) - \Psi(x_2,|z|)|\,, \end{cases} \tag{1.0.30}$$

whenever $z \in \mathbb{R}^n \setminus \{0\}$, $\xi \in \mathbb{R}^n$, $x, x_1, x_2 \in \Omega$. On the right-hand side of the equation (1.0.29), we have that $B : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Caratheodory vector field satisfying

$$|z||B(x,z)| \leqslant L\Psi(x,|z|) \quad (x \in \Omega, z \in \mathbb{R}^n). \tag{1.0.31}$$

In the structure assumptions (1.0.30) and (1.0.31) above, Ψ is the same one as initially defined in (1.0.2). We shall consider a distributional solution u of (1.0.29) under the assumptions (1.0.13) or (1.0.14). A primary model keeping in mind of the equation (1.0.29) is of the form

$$\operatorname{div}\left(\partial_t \Psi(x,|Du|)\frac{Du}{|Du|}\right) = \operatorname{div} B(x,F) \quad \text{in} \quad \Omega, \tag{1.0.32}$$

where ∂_t stands for the partial derivative of $\Psi(x,t)$ with respect to t-variable, which is the Euler-Lagrange equation of the following functional

$$W^{1,1}(\Omega) \ni v \mapsto \mathcal{P}(v,\Omega) - \int_\Omega \langle B(x,F), Dv\rangle \, dx, \tag{1.0.33}$$

where the functional $\mathcal{P}$ is initially given as in (1.0.1).

The main purpose of the second part of the thesis is to discover and develop optimal conditions on both nonlinearity $A(x,z)$ and the coefficient function $a(\cdot)$ and $b(\cdot)$ (see (4.1.9) and (4.1.10)), that are not necessarily Hölder continuous, under which for any distributional solution $u \in W^{1,\Psi}(\Omega)$ to (1.0.29) the following local Calderón-Zygmund type implication

$$\Psi(x,|F|) \in L^{\Upsilon}_{\mathrm{loc}}(\Omega) \Longrightarrow \Psi(x,|Du|) \in L^{\Upsilon}_{\mathrm{loc}}(\Omega) \tag{1.0.34}$$

with

$$\fint_{B_{R/2}} \Upsilon(\Psi(x,|Du|))\, dx \lesssim \Upsilon\left(\fint_{B_R} \Psi(x,|Du|)\, dx\right) + \fint_{B_R} \Upsilon(\Psi(x,|F|))\, dx \tag{1.0.35}$$

holds for every $\Upsilon \in \mathcal{N}$ with an index $s(\Upsilon) \geqslant 1$ in the sense of Definition 2.1.1 and every ball $B_R \Subset \Omega$. Let us now discuss previous known results on Calderón-Zygmund type implications like (1.0.34) as special cases of the problem we consider:

1. For $\Psi(x,t) \equiv t^p$ with $p > 1$, there has been historical progress of

studying the regularity theory of non-linear p-Laplacian type equations of divergence form over the last several decades so that there is almost no possibility to mention all the works that have been done up to now. We only refer to some noteworthy results, see for instance [5, 10, 41, 44, 45, 47, 48, 49, 102, 103, 105].

2. For $\Psi(x,t) \equiv G(t)$ with $G \in \mathcal{N}$, the global Calderón-Zygmund estimates over the whole domain $\mathbb{R}^n$ have been achieved in [135] and the same result was proved for general equations involving a solution dependence over bounded non-smooth domains [16, 42, 50]. Moreover, the Lipschitz regularity has been proved in [54] for equations and [55] for systems.

3. For $\Psi(x,t) \equiv t^p + a(x)t^q$ with $1 < p \leqslant q$ and $0 \leqslant a(\cdot) \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1]$, the implication (1.0.34) has been obtained in [59, 70] under the main assumption (1.0.6a). The global implication of (1.0.34) is proved in [37] over a suitable smooth domain and [36] over nonsmooth domain under the same assumption (1.0.6a). We also note that the implication (1.0.34) is proved in [59] for bounded solutions of (1.0.29) under the assumption (1.0.6b), where the nonlinearity $A(\cdot)$ is of variational form as in (1.0.32) and an additional information on the vector field F in (1.0.29) is assumed (see [59, (1.27)]).

4. For $\Psi(x,t) \equiv t^p + a(x)t^p \log(1+t)$ with $1 < p$ and $0 \leqslant a(\cdot) \in C^{\omega_a}(\Omega)$ for $\omega_a : [0,\infty) \to [0,\infty)$ being a continuous and concave function vanishing at the origin, the global Calderón-Zygmund estimates like (1.0.34) is proved in [38] under the optimal assumption
$$\limsup_{\rho \to 0^+} \omega_a(\rho) \log\left(\frac{1}{\rho}\right) = 0.$$

5. For $\Psi(x,t) \equiv G(t) + a(x)H_a(t) + b(x)H_b(t)$ with $G, H_a, H_b \in \mathcal{N}$ and $0 \leqslant a(\cdot) \in C^{0,\alpha}(\Omega)$ and $0 \leqslant b(\cdot) \in C^{0,\beta}(\Omega)$ for some $\alpha, \beta \in (0,1]$, the above implication (1.0.34) has been proved in [14] under the main assumption
$$\sup_{\rho > 0} \frac{H_a(\rho)}{G(\rho) + [G(\rho)]^{1+\frac{\alpha}{n}}} < \infty \tag{1.0.36}$$

when $b(\cdot) \equiv 0$. If $b(\cdot) \not\equiv 0$, then the same result is proved in [15] under the main condition on the nonlinearity

$$\sup_{\rho>0}\left(\frac{H_a(\rho)}{G(\rho)+[G(\rho)]^{1+\frac{\alpha}{n}}}+\frac{H_b(t)}{G(\rho)+[G(\rho)]^{1+\frac{\beta}{n}}}\right)<\infty$$
$$\text{and}\quad \max\{\alpha,\beta\}\leqslant 2\min\{\alpha,\beta\}. \tag{1.0.37}$$

The second assumption in (1.0.37) is unavoidable according to the arguments and structure assumptions in (1.0.30). Recently, the author of [69] proved the implication (1.0.34) under the condition (1.0.9) when $G(t) \equiv t^p$, $H_a(t) \equiv t^q$ and $H_b(t) \equiv t^s$ for $1 < p \leqslant q, s$ and the nonlinearity $A(\cdot)$ is of variational form in (1.0.32). Notice that if the nonlinearity $A(\cdot)$ is of variational form like in (1.0.32), then there are advantages that solutions to corresponding homogeneous problems can be directly treated as minima of the functional under the consideration.

5. Lastly, we only mention our recent result of [17] on the validity of the implication of (1.0.34) with the estimate (1.0.35) for more general settings involving variable exponents like $\Psi(x,t) \equiv [G(t)]^{p(x)} + a(x)[H_a(t)]^{q(x)}$ with $G, H_a \in \mathcal{N}$, log-Hölder continuous functions $1 \leqslant p(\cdot), q(\cdot)$ and $0 \leqslant a(\cdot) \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1]$. We refer reader to [6, 122, 124, 125] for further regularity results on problems involving variable exponents

For the Orlicz double phase case, we assume that the vector field $A(\cdot)$ in (1.0.29) is general one satisfying (1.0.30) and, for the Orlicz multi-phase case, we let

$$A(x,z) := A_G(z) + a(x)A_{H_a}(z) + b(x)A_{H_b}(z) \quad (\forall x \in \Omega,\ z \in \mathbb{R}^n)$$

in (1.0.30) for some reasons to apply harmonic type approximation Lemma 2.5.1, where the vector fields $A_G, A_{H_a}, A_{H_b} : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the growth and ellipticity conditions (4.1.17) below, we prove the validity of implication (1.0.34) for any distributional solution $u \in W^{1,\Psi}(\Omega)$ of (1.0.29) under the main assumptions (1.0.13) or (1.0.14), see Theorem 4.1.1 and Theorem 4.1.2. Note that we are not allowed directly to apply the approaches employed in [14, 15, 59, 70, 69] as they strongly rely on a difference quotient argument which in turn strictly require the Hölder continuity of the modulating coefficient functions $a(\cdot)$ and $b(\cdot)$ that are not always assumed to be Hölder continuous in Chapter 4. The main tool for establishing (1.0.34)-(1.0.35) is a reverse Hölder type inequality

$$\left(\fint_{B_{R/2}} \left[\Psi^{-}_{B_R}(|Dw|)\right]^d \, dx \right)^{\frac{1}{d}} \lesssim \fint_{B_R} \Psi(x, |Dw|) \, dx \tag{1.0.38}$$

for every $d \in (1, \infty)$ and ball $B_R \Subset \Omega$, see Theorem 4.2.4, where $w \in W^{1,\Psi}(B_R)$ is the weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} A(x, Dw) = 0 \text{ in } B_R \\ w \in u + W^{1,\Psi}_0(B_R). \end{cases} \tag{1.0.39}$$

We bypass such a nontrivial obstruction by treating the solution of (1.0.39) as quasi-minima of the Orlicz multi-phase (double phase) energy functional in (1.0.1) and combining harmonic type approximation with some delicate decay estimates to conclude (1.0.38). We believe that, using approaches in Chapter 4 together with adapting methods presented in [36], the global Calderón-Zygmund type estimates like (1.0.34) can be (should be) proved on a non-smooth domain under each of assumptions (4.1.9) and (4.1.10). We also point out that problems with Orlicz growth and generalized Orlicz growth are central topics as natural generalizations of p-Laplacian problems which have been an object intensive studies over last decades. Besides the papers mentioned above, there is a richness of literature on regularity theory of elliptic/parabolic equations; see for instance, Lipschitz regularity for elliptic/parabolic equations [23, 54, 55, 77, 81], potential estimates [19, 32, 43], Hölder continuity [35, 94, 95, 96], obstacle problems [11, 31], Calderón-Zygmund estimates [7, 8, 40, 98, 138] and reference therein. We also refer to the recent textbook [56].

In the last part of the thesis, we provide a unified way for proving Hölder regularity for the gradient of viscosity solutions to fully nonlinear elliptic equations of the form

$$\Phi(x, |Du|) F(D^2 u) = f(x) \quad \text{in} \quad B_1, \tag{1.0.40}$$

where $B_1 \equiv B_1(0) \subset \mathbb{R}^n$ with $n \geqslant 2$ is the unit ball, $F : \mathcal{S}(n) \to \mathbb{R}$ is a uniformly (λ, Λ)-elliptic operator in the sense of (A1) below (see Chapter 5) and $\Phi : B_1 \times [0, \infty) \to [0, \infty)$ is a continuous map featuring a degeneracy and singularity for the gradient described as in (A2) below (see Chapter 5).

From a variational point of view, the fully nonlinear equation (1.0.40) is closely related to the energy functional

$$v \mapsto \int_{B_1} \varphi(x, |Dv|)\, dx \tag{1.0.41}$$

for a integral density $\varphi : B_1 \times [0, \infty) \to [0, \infty)$ in a way that the Euler-Lagrange equation corresposnding to the functional (1.0.41) forms an equation of type (1.0.40). The functional in (1.0.41) is a highly general non-autonomous functional with Uhlenbeck structure including significant models such as $p-$, Orlicz-, $p(x)-$, double phase- and Orlicz multi-phase growth and so on. For instance, Orlicz multi-phase functional in (1.0.1) is one of functionals of type in (1.0.41) which we consider in the first parts of the thesis. Hölder continuity for the gradient of local minima of the functional (1.0.41) under suitable optimal assumptions has been investigated in [97], where fundamental assumptions on the integral density function φ in (1.0.41) are that there exist constants $1 < p \leqslant q$ such that the map $t \mapsto \dfrac{\varphi(x,t)}{t^p}$ is almost non-decreasing and the map $t \mapsto \dfrac{\varphi(x,t)}{t^q}$ is almost non-increasing, see [97, Definition 3.1]. In this regard, our conditions on Φ in (1.0.40) introduced in (A2) is absolutely natural. Let us discuss known regularity results for viscosity solutions of equations in the form of (1.0.40) for significant special cases.

1. For $\Phi(x,t) \equiv t^p$ with $i(\Phi) = d(\Phi) = p > -1$ in condition (A2), fully nonlinear equations (1.0.40) with this type of $\Phi(x,t)$ have been studied in a series of papers. The authors of [26] proved the comparison principle and Liouville type theorems in the singular case ($-1 < p < 0$), and showed the regularity and uniqueness of the first eigenfunction in [27]. Alexandrov-Bakelman-Pucci estimates and the Harnack inequality have been also obtained in [63, 64, 99]. In particular, the authors of [100] proved local Hölder continuity for the gradient of viscosity solutions of (5.1.1) in the degenerate case ($p \geqslant 0$). Moreover, the authors of [9] proved the optimality of Hölder regularity for the gradient of viscosity solutions for the same problem in [100] by showing that viscosity solutions are $C^{1,\beta}_{\mathrm{loc}}$ with $\beta = \min\left\{\bar{\alpha}, \dfrac{1}{p+1}\right\}$ and $\beta \in (0, \bar{\alpha})$, where

$\bar{\alpha} \in (0,1)$ is an universal Hölder exponent coming from the Krylov-Safonov regularity for the homogeneous equation $F(D^2h) = 0$.

2. For $\Phi(x,t) \equiv t^p + a(x)t^q$ with $-1 < p, q$ and $0 \leqslant a(\cdot) \in C(B_1)$, the constants in (A2) can be determined as $i(\Phi) = \min\{p,q\}$ and $d(\Phi) = \max\{p,q\}$. The author of [67] proved the local $C^{1,\beta}-$regularity of viscosity solutions of (5.1.1) for $0 \leqslant p \leqslant q$. Moreover, in this degenerate case, the sharpness of the local $C^{1,\beta}$-regularity estimates for bounded viscosity solutions is shown in [65].

3. For $\Phi(x,t) = t^{p(x)}$ with $p(\cdot) \in C(B_1)$, $i(\Phi) = \inf\limits_{x \in B_1} p(x) > -1$ and $d(\Phi) = \sup\limits_{x \in B_1} p(x)$ in (A2), $C^{1,\beta}$-regularity of viscosity solutions has been studied in [34]. In this paper, we provide a novel way to prove Hölder continuity for the gradient of viscosity solutions of (5.1.1) for both degenerate/singular cases in the full generality.

4. For $\Phi(x,t) \equiv t^{p(x)} + a(x)t^{q(x)}$ with functions $0 \leqslant a(\cdot) \in C(B_1)$ and $-1 < p(\cdot), q(\cdot)$ in $C(B_1)$, the constants in (A2) are $i(\Phi) = \inf\limits_{x \in B_1}\{p(x), q(x)\}$ and $d(\Phi) = \sup\limits_{x \in B_1}\{p(x), q(x)\}$. In [85], local Hölder continuity for the gradient has been proved when $0 \leqslant p(\cdot) \leqslant q(\cdot)$.

For a variational point of these special cases we have discussed above, we refer to the recent survey paper [118] presenting important results in problems with nonstandard growth conditions. We also point out the very recent paper [104] dealing with viscosity solutions of an equation of the form

$$|Du|^{\beta(x,u,Du)} F(D^2u) = f(x) \quad \text{in} \quad B_1, \tag{1.0.42}$$

where $\beta : B_1 \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a map satisfying $0 < \beta_m \leqslant \beta(\cdot) \leqslant \beta_M$ for some positive constants β_m and β_M. In [104], local Hölder continuity for the gradient of viscosity solutions of (1.0.42) is obtained under general conditions on the exponent function $\beta(\cdot)$ for the degenerate case, while the singular case is not be treated due to the methods employed there and the equation (1.0.40) can not be represented as (1.0.42) in general. The main results of Chapter 5 are contained in Theorem 5.1.1, which are sharp in the view of an example given in [100]. As we have discussed above, the results of Theorem 5.1.1 cover the main results of the papers [34, 85, 67, 100] for both cases involving degenerate/singular terms in a unified way. Moreover, the results of Theorem 5.1.1 cover another important cases such as

1. $\Phi(x,t) \equiv t^p + a(x)t^p \log(e+t)$ with $-1 < p$ and $0 \leqslant a(\cdot) \in C(B_1)$, where the constants in (A2) are given by $i(\Phi) = p$ and $d(\Phi) = p + \varepsilon$ for any $\varepsilon > 0$,

2. $\Phi(x,t) \equiv \phi(t) + a(x)\psi(t)$ for suitable functions ϕ, ψ and $0 \leqslant a(\cdot) \in C(B_1)$.

The rest of the thesis organized as follows. In the next chapter, we introduce notations, functions spaces, analytic tools and basic results such as Absence of Lavrentiev phenomenon, Sobolev Poincaré type inequalities and Harmonic type approximation to be employed throughout the thesis. In chapter 3, we discuss various regularity results of minima of Orlicz multi-phase functionals. Chapter 4 is devoted to proving local Calderón-Zygmund estimates for Orlicz multi-phase problems. In last chapter, we investigate Hölder regularity for the gradient of viscosity solutions to a class of fully nonlinear equations.

Chapter 2

Preliminaries and auxiliary tools

2.1 Notations

Throughout the thesis, we shall always denote by c to mean a generic positive constant, possibly varying from line to line, while special constants will be denoted by $c_1, \bar{c}, c_*, c_\varepsilon$, and so on. All such constants will be always not smaller than one; moreover relevant dependencies on parameters will be emphasized using parentheses, that is, for example $c \equiv c(n, s(G), \nu, L)$ means that c depends only on $n, s(G), \nu, L$. We denote by $B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$ the open ball in $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ with a radius $R > 0$. If the center is clear in the context, we shall omit the center point by writing $B_R \equiv B_R(x_0)$. We shall also denote $B_1 \equiv B_1(0) \subset \mathbb{R}^n$ unless the center is specified. With $f : \mathcal{B} \to \mathbb{R}^N$ $(N \geqslant 1)$ being a measurable map for a measurable subset $\mathcal{B} \subset \mathbb{R}^n$ having finite and positive measure, we denote by

$$(f)_{\mathcal{B}} \equiv \fint_{\mathcal{B}} f(x)\,dx = \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} f(x)\,dx$$

its integral average over $\mathcal{B}$. For a measurable map $f : \Omega \to \mathbb{R}$ and an open subset $\mathcal{B} \subset \Omega$ with $\sigma : [0, \infty) \to [0, \infty)$ being a continuous and concave function such that $\sigma(0) = 0$, we shall use the notation as

$$[f]_{\sigma;\mathcal{B}} := \sup_{x,y \in \mathcal{B}, x \neq y} \frac{|f(x) - f(y)|}{\sigma(|x - y|)} \quad \text{and} \quad [f]_\sigma \equiv [f]_{\sigma;\Omega}.$$

We denote by $C^\sigma(\Omega)$ the space of uniformly continuous functions on Ω whose modulus of continuity does not exceed σ. The space $C^\sigma(\Omega)$ is endowed with the norm defined for a function f by

$$\|f\|_{C^\sigma(\Omega)} = \|f\|_{L^\infty(\Omega)} + [f]_{\sigma;\Omega}.$$

In particular, if $\sigma(\rho) = \rho^\alpha$ for some $\alpha \in (0,1]$, then we denote

$$[f]_{0,\alpha;\mathcal{B}} := \sup_{x,y\in\mathcal{B},x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \quad \text{and} \quad [f]_{0,\alpha} \equiv [f]_{0,\alpha;\Omega}.$$

For a given continuous and concave function $\sigma : [0,\infty) \to [0,\infty)$ vanishing at the origin, we shall use some elementary properties in the future as

$$\sigma(\lambda t) \leqslant \lambda\sigma(t) \quad \text{for every} \quad \lambda \geqslant 1 \quad \text{and} \quad t \geqslant 0 \tag{2.1.1}$$

and

$$\frac{1}{\sigma(\lambda t)} \leqslant \frac{1}{\sigma(t)} + \frac{1}{\lambda\sigma(t)} \quad \text{for every} \quad \lambda, t > 0 \quad \text{unless} \quad \sigma \text{ is constant.} \tag{2.1.2}$$

Throughout the thesis, for any given open subset $\mathcal{B} \subset \Omega$, we shall also use the notations by

$$\begin{aligned}
&a^-(\mathcal{B}) := \inf_{x\in\mathcal{B}} a(x), \quad a^+(\mathcal{B}) := \sup_{x\in\mathcal{B}} a(x), \\
&b^-(\mathcal{B}) := \inf_{x\in\mathcal{B}} b(x), \quad b^+(\mathcal{B}) := \sup_{x\in\mathcal{B}} b(x), \\
&\Psi^-_{\mathcal{B}}(t) := G(t) + \inf_{x\in\mathcal{B}} a(x)H_a(t) + \inf_{x\in\mathcal{B}} b(x)H_b(t), \\
&\Psi^+_{\mathcal{B}}(t) := G(t) + \sup_{x\in\mathcal{B}} a(x)H_a(t) + \sup_{x\in\mathcal{B}} b(x)H_b(t)
\end{aligned} \tag{2.1.3}$$

for every $t \geqslant 0$.

Definition 2.1.1. A measurable function $\Phi : \Omega \times [0,\infty) \to [0,\infty)$ is called an Young function if, for any fixed $x \in \Omega$, the function $\Phi(x,\cdot)$ increasing and convex such that

$$\Phi(x,0) = 0,\ \lim_{t\to\infty} \Phi(x,t) = +\infty,\ \lim_{t\to 0^+} \frac{\Phi(x,t)}{t} = 0 \text{ and } \lim_{t\to\infty} \frac{\Phi(x,t)}{t} = +\infty.$$

We denote by $\mathcal{N}(\Omega)$ the set of Young functions $\Phi : \Omega \times [0,\infty) \to [0,\infty)$ such that, for any fixed $x \in \Omega$, $\Phi(x,\cdot) \in C^1([0,\infty)) \cap C^2((0,\infty))$ and there exists a constant $s(\Phi) \geqslant 1$ with

$$\frac{1}{s(\Phi)} \leqslant \frac{\partial_{tt}^2 \Phi(x,t)t}{\partial_t \Phi(x,t)} \leqslant s(\Phi) \tag{2.1.4}$$

uniformly for all $x \in \Omega$ and $t > 0$, where in the future we shall call this number $s(\Phi)$ by an index of Φ. Furthermore, we denote also $\mathcal{N}$ to mean the set of Young functions $\Phi \in \mathcal{N}(\Omega)$ such that Φ does not depend on the first variable x.

As a direct consequence of the above definition, for any $\Phi \in \mathcal{N}(\Omega)$ with an index $s(\Phi) \geqslant 1$ and any fixed point $x \in \Omega$, we can observe

$$t^2 \partial_{tt}^2 \Phi(x,t) \approx t\partial_t \Phi(x,t) \approx \Phi(x,t) \tag{2.1.5}$$

for uniformly all $t > 0$, where note that all implied constants only depend only on $s(\Phi)$. Now we state some important properties of functions of $\mathcal{N}(\Omega)$, see [14, 15, 39] for their proofs.

Lemma 2.1.1. *Let $\Phi \in \mathcal{N}(\Omega)$ with an index $s(\Phi) \geqslant 1$. Then, for any fixed $x \in \Omega$, we have*

1. $\Lambda_0^{1+\frac{1}{s(\Phi)}} \Phi(x,t) \leqslant \Phi(x,\Lambda_0 t) \leqslant \Lambda_0^{s(\Phi)+1} \Phi(x,t)$ *for any* $\Lambda_0 \geqslant 1$ *and* $t \geqslant 0$.
2. $\lambda_0^{1+s(\Phi)} \Phi(x,t) \leqslant \Phi(x,\lambda_0 t) \leqslant \lambda_0^{\frac{1}{s(\Phi)}+1} \Phi(x,t)$ *for any* $0 < \lambda_0 \leqslant 1$ *and* $t \geqslant 0$.
3. $\Lambda_0^{\frac{1}{1+s(\Phi)}} \Phi_t^{-1}(x,t) \leqslant \Phi_t^{-1}(x,\Lambda_0 t) \leqslant \Lambda_0^{\frac{s(\Phi)}{1+s(\Phi)}} \Phi_t^{-1}(x,t)$ *for any* $\Lambda_0 \geqslant 1$ *and* $t \geqslant 0$.
4. $\lambda_0^{\frac{s(\Phi)}{1+s(\Phi)}} \Phi_t^{-1}(x,t) \leqslant \Phi_t^{-1}(x,\lambda_0 t) \leqslant \lambda_0^{\frac{1}{1+s(\Phi)}} \Phi_t^{-1}(x,t)$ *for any* $0 < \lambda_0 \leqslant 1$ *and* $t \geqslant 0$.

In the above lemma, for a fixed point $x \in \Omega$, $\Phi_t^{-1}(x,t)$ is understood by the inverse function of $\Phi(x,t)$ with respect to t-variable.

Remark 2.1.1. For a given $\Phi \in \mathcal{N}(\Omega)$ with an index $s(\Phi) \geqslant 1$, we notice useful but direct consequences of Lemma 2.1.1 as

$$\Phi(x,t+s) \leqslant \Phi(x,2t) + \Phi(x,2s) \leqslant 2^{1+s(\Phi)} \left(\Phi(x,t) + \Phi(x,s)\right) \tag{2.1.6}$$

for every $x \in \Omega$ and $t, s \geqslant 0$. Furthermore, for any fixed $x \in \Omega$, we have

$$\partial_t \Phi(x,t) \leqslant (1+s(\Phi))\frac{\Phi(x,t)}{t} \leqslant (1+s(\Phi))[\Phi(x,1)]^{\frac{s(\Phi)}{1+s(\Phi)}}[\Phi(x,t)]^{\frac{1}{1+s(\Phi)}}$$
for every $0 \leqslant t \leqslant 1$

and

$$\partial_t \Phi(x,t) \leqslant (1+s(\Phi))\frac{\Phi(x,t)}{t} \leqslant (1+s(\Phi))[\Phi(x,1)]^{\frac{1}{1+s(\Phi)}}[\Phi(x,t)]^{\frac{s(\Phi)}{1+s(\Phi)}}$$
for every $t \geqslant 1$.

Putting together the last two inequalities, we have the following very useful inequality which will be applied in the future

$$\begin{aligned} &\frac{\Phi(x,t)}{t} \approx \partial_t \Phi(x,t) \\ &\leqslant (1+s(\Phi))\left([\Phi(x,1)]^{\frac{s(\Phi)}{1+s(\Phi)}}[\Phi(x,t)]^{\frac{1}{1+s(\Phi)}} + [\Phi(x,1)]^{\frac{1}{1+s(\Phi)}}[\Phi(x,t)]^{\frac{s(\Phi)}{1+s(\Phi)}}\right) \end{aligned} \tag{2.1.7}$$

for every $x \in \Omega$ and $t \geqslant 0$.

Lemma 2.1.2. *Let $\Phi, \tilde{\Phi} \in \mathcal{N}(\Omega)$ with indices $s(\Phi), s(\tilde{\Phi}) \geqslant 1$. Then,*

1. *For any non-negative real numbers a, b satisfying $a+b>0$, $a\Phi + b\tilde{\Phi} \in \mathcal{N}(\Omega)$ with $s(a\Phi + b\tilde{\Phi}) = s(\Phi) + s(\tilde{\Phi})$ and $\Phi\tilde{\Phi} \in \mathcal{N}(\Omega)$ with $s(\Phi\tilde{\Phi}) = 4s(\Phi)s(\tilde{\Phi})(s(\Phi)+s(\tilde{\Phi}))$.*
2. *For any number $m \geqslant 1$, $\Phi^m \in \mathcal{N}(\Omega)$ with $s(\Phi^m) = s(\Phi) + (m-1)(s(\Phi)+1)$.*
3. *For any number $\mu \geqslant 0$, $\Phi_\mu(x,t) := t^\mu \Phi(x,t) \in \mathcal{N}(\Omega)$ with $s(\Phi_\mu) = \mu + 3[s(\Phi)]^2$.*
4. *There exists $\theta_0 \in (0,1)$ depending only on $s(\Phi)$ such that $\Phi^\theta \in \mathcal{N}(\Omega)$ for every $\theta \in (\theta_0, 1]$ with $s(\Phi^\theta)$ depending only on $s(\Phi)$ and θ.*

Lemma 2.1.3. *Let $\Phi \in \mathcal{N}$ with an index $s(\Phi) \geqslant 1$. Then $t \mapsto \Phi\left(t^{\frac{1}{s(\Phi)+1}}\right)$ is a concave function.*

Lemma 2.1.4. *Let $\Phi \in \mathcal{N}(\Omega)$ with an index $s(\Phi) \geqslant 1$. Then there exists a positive constant $c \equiv c(s(\Phi))$ such that*

$$s_1 \partial_t \Phi(x, s_2) + s_2 \partial_t \Phi(x, s_1) \leqslant \varepsilon \Phi(x, s_1) + \frac{c}{\varepsilon^{s(\Phi)}} \Phi(x, s_2)$$

holds for all $s_1, s_2 \geqslant 0$ and $0 < \varepsilon \leqslant 1$.

Lemma 2.1.5. *Let $\Phi_1, \Phi_2 \in \mathcal{N}$ with indices $s(\Phi_1), s(\Phi_2) \geqslant 1$. There exists a constant $d_0 \equiv d_0(s(\Phi_1), s(\Phi_2))$ such that the map*

$$t \mapsto \left(\Phi_1 \circ \Phi_2^{-1}\right) \left(t^{\frac{1}{d}}\right)$$

is concave in $(0, \infty)$ for every $d \geqslant d_0$.

Proof. Let us denote by $h_d(t) := \left(\Phi_1 \circ \Phi_2^{-1}\right) \left(t^{\frac{1}{d}}\right)$ and $g_d(t) := \Phi_2^{-1} \left(t^{\frac{1}{d}}\right)$. It suffices to check $h_d'' \leqslant 0$ in $(0, \infty)$ for every $d \geqslant d_0(s(\Phi_1), s(\Phi_2))$. Direct computations and (2.1.4) imply

$$\begin{aligned} h_d''(t) &= \Phi_1''(g_d(t))[g_d'(t)]^2 + \Phi_1'(g_d(t)) g_d''(t) \\ &\leqslant \Phi_1'(g(t)) \left((1 + s(\Phi_1)) \frac{[g_d'(t)]^2}{g_d(t)} + g_d''(t) \right). \end{aligned}$$

Since $\Phi_1' \geqslant 0$ in $(0, \infty)$, we calculate the term in the bracket. By again elementary calculations and recalling $\Phi_2'' \geqslant 0$ in $(0, \infty)$, we have

$$g_d'(t) = \frac{\frac{1}{d} t^{\frac{1}{d} - 1}}{\Phi_2'(g_d(t))} \quad \text{and} \quad g_d''(t) \leqslant \frac{\frac{1}{d} \left(\frac{1}{d} - 1\right) t^{\frac{1}{d} - 2}}{\Phi_2'(g_d(t))}.$$

Then inserting the content of last display into the previous one and using (2.1.5), we find

$$(1 + s(\Phi_1)) \frac{[g_d'(t)]^2}{g_d(t)} + g_d''(t) \leqslant \frac{t^{\frac{1}{d} - 2}}{d \Phi_2'(g_d(t))} \left(\frac{d_0}{d} - 1 \right) \leqslant 0$$

for every constant $d \geqslant d_0 \equiv d_0(s(\Phi_1), s(\Phi_2))$. □

Remark 2.1.2. We note that $\Psi \in \mathcal{N}(\Omega)$ with an index $s(\Psi) = s(G) + s(H_a) + s(H_b)$ by Lemma 2.1.2. In particular, for every open subset $\mathcal{B} \subset \Omega$, it holds that $\Psi^+_{\mathcal{B}}, \Psi^-_{\mathcal{B}} \in \mathcal{N}$ with indices $s\left(\Psi^+_{\mathcal{B}}\right) = s(G) + s(H_a) + s(H_b)$ and $s\left(\Psi^-_{\mathcal{B}}\right) = s(G) + s(H_a) + s(H_b)$.

For a given Young function $\Phi \in \mathcal{N}(\Omega)$ with an index $s(\Phi) \geqslant 1$, we define the vector field $V_\Phi : \Omega \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ as follows

$$V_\Phi(x, z) := \left[\frac{\partial_t \Phi(x, |z|)}{|z|}\right]^{\frac{1}{2}} z. \tag{2.1.8}$$

Furthermore, we shall often use the following inequalities that

$$\int_0^1 \frac{\Phi(x, |\theta z_1 + (1-\theta) z_2|)}{|\theta z_1 + (1-\theta) z_2|^2} \, d\theta \approx \frac{\Phi(x, |z_1| + |z_2|)}{(|z_1| + |z_2|)^2}, \tag{2.1.9}$$

$$\begin{aligned} |V_\Phi(x, z_1) - V_\Phi(x, z_2)|^2 &\approx \partial^2_{tt} \Phi(x, |z_1| + |z_2|) |z_1 - z_2|^2 \\ &\approx \frac{\partial_t \Phi(x, |z_1| + |z_2|)}{|z_1| + |z_2|} |z_1 - z_2|^2, \end{aligned} \tag{2.1.10}$$

$$\Phi(x, |z_1 - z_2|) \lesssim \Phi(x, |z_1| + |z_2|) \frac{|z_1 - z_2|}{|z_1| + |z_2|} \tag{2.1.11}$$

and

$$\left\langle \partial_t \Phi(x, |z_1|) \frac{z_1}{|z_1|} - \partial_t \Phi(x, |z_2|) \frac{z_2}{|z_2|}, z_1 - z_2 \right\rangle \approx |V_\Phi(x, z_1) - V_\Phi(x, z_2)|^2 \tag{2.1.12}$$

hold true, whenever $x \in \Omega$ and $z_1, z_2 \in \mathbb{R}^n \setminus \{0\}$, where all implied constants in (2.1.9)-(2.1.12) depend on n and $s(\Phi)$ (see [73] for further discussions). Moreover, we have the following useful inequality

$$|V_\Phi(x, z_2) - V_\Phi(x, z_1)|^2 \lesssim \int_0^1 |V_\Phi(x, \theta z_2 + (1-\theta) z_1) - V_\Phi(x, z_1)|^2 \, \frac{d\theta}{\theta} \quad (\forall x \in \Omega), \tag{2.1.13}$$

which follows from the following estimates that

$$
\begin{aligned}
&\int_0^1 |V_\Phi(x,\theta z_2+(1-\theta)z_1)-V_\Phi(x,z_1)|^2\,\frac{d\theta}{\theta}\\
&\overset{(2.1.10)}{\gtrsim} \int_0^1 \frac{\Phi(x,|\theta z_2+(1-\theta)z_1|+|z_1|)}{(|\theta z_2+(1-\theta)z_1|+|z_1|)^2}\theta|z_2-z_1|^2\,d\theta\\
&\gtrsim \frac{|z_2-z_1|^2}{(|z_2|+|z_1|)^2}\int_0^1 \Phi(x,|\theta z_2+(1-\theta)z_1|+|z_1|)\theta\,d\theta\\
&\gtrsim \frac{|z_2-z_1|^2}{(|z_2|+|z_1|)^2}\Phi\left(x,\int_0^1(|\theta z_2+(1-\theta)z_1|+|z_1|)\theta\,d\theta\right)\\
&\gtrsim \frac{|z_2-z_1|^2}{(|z_2|+|z_1|)^2}\Phi\left(x,|z_2|+|z_1|\right)\overset{(2.1.10)}{\approx}|V_\Phi(x,z_2)-V_\Phi(x,z_1)|^2
\end{aligned}
$$

hold with having all implied constants in the above display depending on n and $s(\Phi)$, whenever $x\in\Omega$ and $z_1,z_2\in\mathbb{R}^n\setminus\{0\}$, where in the third inequality of the last display we have applied Jensen's inequality to the convex function $\Phi(x,\cdot)$ with respect to measure $\theta\,d\theta$.

Lemma 2.1.1. *Let $\Phi\in\mathcal{N}(\Omega)$ with an index $s(\Phi)$. Then there exists a constant $c\equiv c(s(\Phi))$ such that*

$$\Phi(x,|z_1-z_2|)\le\varepsilon\Phi(x,|z_1|)+\frac{c}{\varepsilon}|V_\Phi(x,z_1)-V_\Phi(x,z_2)|^2$$

holds, whenever $\varepsilon\in(0,1)$, $x\in\Omega$ and $z_1,z_2\in\mathbb{R}^n\setminus\{0\}$.

2.2 Musielak-Orlicz and Musielak-Orlicz-Sobolev spaces

We now introduce the Musielak-Orlicz spaces (generalized Orlicz spaces), which generalize the Orlicz spaces. Let $\Phi:\Omega\times[0,\infty)\to[0,\infty)$ be an Young function. Here we present some definitions and properties associated to Young functions.

Definition 2.2.1. Let Φ be a Young function.

1. Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if there exists a positive number $\Delta_2(\Phi)$ such that $\Phi(x, 2t) \leqslant \Delta_2(\Phi)\,\Phi(x,t)$ for all $x \in \Omega$ and $t \geqslant 0$.

2. Φ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if there exists a positive number $\nabla_2(\Phi) > 1$ such that $\Phi(x, \nabla_2(\Phi)\,t) \geqslant 2\nabla_2(\Phi)\,\Phi(x,t)$ for all $x \in \Omega$ and $t \geq 0$.

3. We write $\Phi \in \Delta_2 \cap \nabla_2$ if $\Phi \in \Delta_2$ and $\Phi \in \nabla_2$.

For a given Young function Φ, we define the complementary function Φ^* of Φ by, for each $x \in \Omega$ and $t \geqslant 0$,

$$\Phi^*(x,t) = \sup\{st - \Phi(x,s) : s \geq 0\}.$$

Then Φ^* satisfies all the conditions to be a Young function. One can see that $(\Phi^*)^* = \Phi$ and that $\Phi \in \nabla_2$ if and only if $\Phi^* \in \Delta_2$ with $2\nabla_2(\Phi) = \Delta_2(\Phi^*)$.

For an Young function Φ, the Musielak-Orlicz class $K^\Phi(\Omega;\mathbb{R}^N)$, $N \geqslant 1$, consists of all measurable functions $v : \Omega \to \mathbb{R}^N$ satisfying

$$\int_\Omega \Phi(x, |v(x)|)\,dx < +\infty.$$

The Musielak-Orlicz space $L^\Phi(\Omega;\mathbb{R}^N)$ is the vector space generated by $K^\Phi(\Omega;\mathbb{R}^N)$. If $\Phi \in \Delta_2$, then $K^\Phi(\Omega;\mathbb{R}^N) = L^\Phi(\Omega;\mathbb{R}^N)$ and this space is a Banach space under the Luxemburg norm

$$\|v\|_{L^\Phi(\Omega;\mathbb{R}^N)} = \inf\left\{\sigma > 0 : \int_\Omega \Phi\left(x, \frac{|v(x)|}{\sigma}\right)\,dx \leq 1\right\}.$$

The Musielak-Orlicz-Sobolev space $W^{1,\Phi}(\Omega;\mathbb{R}^N)$ is the function space of all measurable functions $v \in L^\Phi(\Omega;\mathbb{R}^N)$ such that its distributional gradient vector Dv belongs to $L^\Phi(\Omega;\mathbb{R}^{Nn})$. For $v \in W^{1,\Phi}(\Omega;\mathbb{R}^N)$, we define its norm to be

$$\|v\|_{W^{1,\Phi}(\Omega;\mathbb{R}^N)} = \|v\|_{L^\Phi(\Omega;\mathbb{R}^N)} + \|Dv\|_{L^\Phi(\Omega;\mathbb{R}^{Nn})}.$$

The space $W_0^{1,\Phi}(\Omega;\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\Omega;\mathbb{R}^N)$ in $W^{1,\Phi}(\Omega;\mathbb{R}^N)$. For $N = 1$, we simply write $L^\Phi(\Omega) := L^\Phi(\Omega;\mathbb{R})$ and $W^{1,\Phi}(\Omega) := W^{1,\Phi}(\Omega;\mathbb{R})$.

For a detailed discussion of the Musielak-Orlicz spaces and the associated Sobolev spaces, we refer the reader to [2, 24, 56, 72, 92, 93, 119, 127, 131] and references therein.

We end up this preliminary section with presenting some standard technical lemmas which will be applied later, see for instance [90, 91, 109].

Lemma 2.2.1. *Let $h : [\rho_0, \rho_1] \to \mathbb{R}$ be a non-negative and bounded function, and $\theta \in (0,1)$, $A, B \geqslant 0$, $\gamma_1, \gamma_2 \geqslant 0$. Assume that*

$$h(t) \leqslant \theta h(s) + \frac{A}{(s-t)^{\gamma_1}} + \frac{B}{(s-t)^{\gamma_2}}$$

holds for $\rho_0 \leqslant t < s \leqslant \rho_1$. Then there exists a constant $c \equiv c(\theta, \gamma_1, \gamma_2)$ satisfying the following inequality

$$h(\rho_0) \leqslant \frac{cA}{(\rho_1 - \rho_0)^{\gamma_1}} + \frac{cB}{(\rho_1 - \rho_0)^{\gamma_2}}.$$

Lemma 2.2.2. *Let $\{Y_i\}_{i=0}^{\infty}$ be a sequence of nonnegative numbers satisfying the following recursive inequalities*

$$Y_{i+1} \leqslant C b^i Y_i^{1+\tau_0}$$

with some fixed positive constant C, $b > 1$ and $\tau_0 > 0$ for every $i = 0, 1, 2, \ldots$. If

$$Y_0 \leqslant C^{-\frac{1}{\tau_0}} b^{-\frac{1}{\tau_0^2}},$$

then $Y_i \to 0$ as $i \to \infty$.

Lemma 2.2.3. *Let $v \in W^{1,1}(B_\rho)$ for some ball $B_\rho \subset \mathbb{R}^n$. Then there exists $c \equiv c(n)$ such that*

$$(l-k)|B_\rho \cap \{v > l\}|^{1-\frac{1}{n}} \leqslant \frac{c|B_\rho|}{|B_\rho \setminus \{v > k\}|} \int_{B_\rho \cap \{k < v \leqslant l\}} |Dv| \, dx$$

holds, whenever l and k are real numbers with $l > k$.

2.3 Absence of Lavrentiev phenomenon

Here we deal with the absence of Lavrantiev phenomenon under the assumptions introduced in $(1.0.13)_2$, $(1.0.14)_2$ and $(1.0.15)_2$. The following theorem widely covers the results of [57, Theorem 4.1], [58, Proposition 3.6], [39, Theorem 3.1], [22, Theorem 4], [14, Theorem 4.1] and [15, Theorem 3.1].

Theorem 2.3.1. *Let $\mathcal{P}$ be the functional defined in* (1.0.1) *with $G, H_a, H_b \in \mathcal{N}$ and the coefficient functions $0 \leqslant a(\cdot) \in C^{\omega_a}(\Omega)$ and $0 \leqslant b(\cdot) \in C^{\omega_b}(\Omega)$ for the functions ω_a, ω_b being continuous, concave and vanishing at 0.*

1. *If the condition $(1.0.13)_2$ is satisfied, then for any function $v \in W^{1,\Psi}_{\mathrm{loc}}(\Omega)$ and ball $B_R \equiv B_R(x_0) \Subset \tilde{B} \Subset \Omega$ with $\mathcal{P}(v, \tilde{B}) < \infty$, there exists a sequence of functions $\{v_k\}_{k=1}^{\infty} \subset W^{1,\infty}(B_R)$ such that*
$$v_k \to v \quad in \quad W^{1,G}(B_R) \quad and \quad \mathcal{P}(v_k, B_R) \to \mathcal{P}(v, B_R). \tag{2.3.1}$$

2. *If the condition $(1.0.14)_2$ is satisfied, then for any function $v \in W^{1,\Psi}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ and ball $B_R \equiv B_R(x_0) \Subset \tilde{B} \Subset \Omega$ with $\mathcal{P}(v, \tilde{B}) < \infty$, there exists a sequence of functions $\{v_k\}_{k=1}^{\infty} \subset W^{1,\infty}(B_R)$ such that*
$$\begin{aligned} &v_k \to v \quad in \quad W^{1,G}(B_R), \quad \mathcal{P}(v_k, B_R) \to \mathcal{P}(v, B_R) \\ &and \quad \limsup_{k\to\infty} \|v_k\|_{L^{\infty}(B_R)} \leqslant \|v\|_{L^{\infty}(B_R)}. \end{aligned} \tag{2.3.2}$$

3. *Let $v \in W^{1,\Psi}(\Omega) \cap C^{0,\gamma}(\Omega)$ with some $\gamma \in (0,1)$ be a local Q-minimizer of the functional $\mathcal{P}$ under the assumption $(1.0.15)_2$. Then, for every ball $B_R \Subset \Omega$, there exists a sequence of functions $\{v_k\}_{k=1}^{\infty} \subset W^{1,\infty}(B_R)$ such that*
$$v_k \to v \quad in \quad W^{1,G}(B_R) \quad and \quad \mathcal{P}(v_k, B_R) \to \mathcal{P}(v, B_R). \tag{2.3.3}$$

Proof. Essentially, the proof for the first two parts is similar to the one of [39, Theorem 3.1]. Since our assumptions are weaker than the assumptions considered there, we provide the detailed proof in any case. First we fix $\varepsilon_0 \in (0,1)$ such that $B_R \Subset B_{R+\varepsilon_0} \Subset \tilde{B} \Subset \Omega$. Let $\rho \in C_0^{\infty}(B_1)$ be a non-negative standard mollifier with $\int\limits_{\mathbb{R}^n} \rho\,dx = 1$. Then we set $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$ for $x \in B_\varepsilon$

with $0 < \varepsilon < \varepsilon_0$. Clearly $\rho_\varepsilon \in C_0^\infty(B_\varepsilon)$, $\int\limits_{\mathbb{R}^n} \rho_\varepsilon \, dx = 1$, $0 \leqslant \rho_\varepsilon \leqslant c(n)\varepsilon^{-n}$ and $|D\rho_\varepsilon| \leqslant c(n)\varepsilon^{-(n+1)}$. For every $0 < \varepsilon < \varepsilon_0/2$, we consider the following functions:

$$v_\varepsilon(x) := (v * \rho_\varepsilon)(x), \quad a_\varepsilon(x) := \inf_{y \in B_{2\varepsilon}(x)} a(y), \quad b_\varepsilon(x) := \inf_{y \in B_{2\varepsilon}(x)} b(y) \tag{2.3.4}$$

and

$$\Psi_\varepsilon(x,t) := G(t) + a_\varepsilon(x)H_a(t) + b_\varepsilon(x)H_b(t) \tag{2.3.5}$$

for every $x \in B_R$ and $t \geqslant 0$.

1. By Jensen's inequality, for a fixed $x \in B_R$, we have

$$G(|Dv_\varepsilon(x)|) = G\left(|(Dv * \rho_\varepsilon)(x)|\right) \leqslant \int\limits_{\mathbb{R}^n} G(|Dv(x-y)|)\rho_\varepsilon(y)\, dy \leqslant c\varepsilon^{-n}.$$

   It follows from $(1.0.13)_2$ and the last display that

$$\begin{aligned} H_a(|Dv_\varepsilon(x)|) &= \frac{(H_a \circ G^{-1})\left(G(|Dv_\varepsilon(x)|)\right)}{G(|Dv_\varepsilon(x)|)} G(|Dv_\varepsilon(x)|) \\ &\leqslant \lambda_1 \left(1 + \left[\omega_a\left(\left[G(|Dv_\varepsilon(x)|)\right]^{-\frac{1}{n}}\right)\right]^{-1}\right) G(|Dv_\varepsilon(x)|) \\ &\leqslant c\left(1 + [\omega_a(\varepsilon)]^{-1}\right) G(|Dv_\varepsilon(x)|) \\ &\leqslant c\left(1 + [\omega_a(\varepsilon)]^{-1}\right) \Psi_\varepsilon(x, |Dv_\varepsilon(x)|). \end{aligned} \tag{2.3.6}$$

   Similarly as above, we have

$$H_b(|Dv_\varepsilon(x)|) \leqslant c\left(1 + [\omega_b(\varepsilon)]^{-1}\right) \Psi_\varepsilon(x, |Dv_\varepsilon(x)|). \tag{2.3.7}$$

2. Since v is locally bounded in Ω, we have

$$|Dv_\varepsilon(x)| = |(v * D\rho_\varepsilon)(x)| \leqslant \int\limits_{\mathbb{R}^n} |v(x-y)||D\rho_\varepsilon(y)|\, dy \leqslant c(n)\, \|v\|_{L^\infty(\tilde{B})}\, \varepsilon^{-1}.$$

Then, the assumption $(1.0.14)_2$ and the last display imply

$$
\begin{aligned}
H_a(|Dv_\varepsilon(x)|) &= \frac{H_a(|Dv_\varepsilon(x)|)}{G(|Dv_\varepsilon(x)|)} G(|Dv_\varepsilon(x)|) \\
&\leqslant \lambda_2 \left(1 + \left[\omega_a \left(|Dv_\varepsilon(x)|^{-1}\right)\right]^{-1}\right) G(|Dv_\varepsilon(x)|) \\
&\leqslant c \left(1 + [\omega_a(\varepsilon)]^{-1}\right) G(|Dv_\varepsilon(x)|) \\
&\leqslant c \left(1 + [\omega_a(\varepsilon)]^{-1}\right) \Psi_\varepsilon(x, |Dv_\varepsilon(x)|) \qquad (2.3.8)
\end{aligned}
$$

with some constant $c \equiv c\left(n, \lambda_2, \|v\|_{L^\infty(\tilde{B})}\right)$ for every $x \in B_R$. Arguing in the same way, for every $x \in B_R$, we have

$$
H_b(|Dv_\varepsilon(x)|) \leqslant c \left(1 + [\omega_b(\varepsilon)]^{-1}\right) \Psi_\varepsilon(x, |Dv_\varepsilon(x)|). \qquad (2.3.9)
$$

Using the continuity of the coefficient functions $a(\cdot)$ and $b(\cdot)$ and recalling the definition of Ψ_ε in (2.3.6), for every $x \in B_R$, we have

$$
\begin{aligned}
\Psi(x, |Dv_\varepsilon(x)|) &\leqslant \Psi_\varepsilon(x, |Dv_\varepsilon(x)|) + |a(x) - a_\varepsilon(x)| H_a(|Df_\varepsilon(x)|) \\
&\quad + |b(x) - b_\varepsilon(x)| H_b(|Df_\varepsilon(x)|) \\
&\leqslant \Psi_\varepsilon(x, |Dv_\varepsilon(x)|) + 4[a]_{\omega_a} \omega_a(\varepsilon) H_a(|Dv_\varepsilon(x)|) \\
&\quad + 4[b]_{\omega_b} \omega_b(\varepsilon) H_b(|Dv_\varepsilon(x)|). \qquad (2.3.10)
\end{aligned}
$$

Therefore, taking into account (2.3.6)-(2.3.7) when the first case comes into play, and (2.3.8)-(2.3.9) when the second case is considered, in any case, it follows from (2.3.10) that

$$
\begin{aligned}
\Psi(x, |Dv_\varepsilon(x)|) &\leqslant c\Psi_\varepsilon(x, |Dv_\varepsilon(x)|) + c\omega_a(\varepsilon)(1 + [\omega_a(\varepsilon)]^{-1})\Psi_\varepsilon(x, |Dv_\varepsilon(x)|) \\
&\quad c\omega_b(\varepsilon)(1 + [\omega_b(\varepsilon)]^{-1})\Psi_\varepsilon(x, |Dv_\varepsilon(x)|) \leqslant c\Psi_\varepsilon(x, |Dv_\varepsilon(x)|) \qquad (2.3.11)
\end{aligned}
$$

for some constant c being independent of ε. Therefore, by Jensen's inequality, we get

$$
\Psi_\varepsilon(x, |Dv_\varepsilon(x)|) \leqslant \int_{B_\varepsilon(x)} \Psi_\varepsilon(x, |Dv(y)|) \rho_\varepsilon(x - y)\, dy
$$

$$
\begin{aligned}
&\leqslant \int_{B_\varepsilon(x)} \Psi(y, |Dv(y)|)\rho_\varepsilon(x-y)\,dy \\
&= [\Psi(\cdot, |Dv(\cdot)|) * \rho_\varepsilon](x) =: [\Psi(\cdot, |Dv(\cdot)|]_\varepsilon(x). \qquad (2.3.12)
\end{aligned}
$$

Hence, in any case, using (2.3.11)-(2.3.12), we conclude that

$$
\Psi(x, |Dv_\varepsilon(x)|) \leqslant c[\Psi(\cdot, |Dv(\cdot)|)]_\varepsilon(x) \qquad (2.3.13)
$$

holds every $x \in B_R$ with a constant c independent of ε. Since $[\Psi(\cdot, |Dv(\cdot)|)]_\varepsilon \to \Psi(\cdot, |Dv(\cdot)|)$ strongly in $L^1(B_R)$, we are able to apply the general Lebesgue's dominated convergence theorem of [126, Theorem 19] to obtain a sequence of functions $\{v_k\} := \{v_{\varepsilon_k}\} \subset C_0^\infty(\tilde{B})$ satisfying (2.3.1) for the first case and $(2.3.2)_{1,2}$ for the second case with some suitable choice of $\varepsilon_k \to 0$. Clearly, the assertion $(2.3.2)_3$ comes from the very definition of mollification of v defined in (2.3.4).

3. Now we turn our attention to proving the last part of the theorem. Applying a Caccioppoli type inequality of Lemma 3.2.2 under the assumption $(1.0.15)_2$ below, we see that

$$
\fint_{B_\varepsilon(x)} \Psi_\varepsilon(x, |Dv(z)|)\,dz \leqslant c \fint_{B_{2\varepsilon}(x)} \Psi_\varepsilon\left(x, \left|\frac{v(z) - (v)_{B_{2\varepsilon}(x)}}{\varepsilon}\right|\right) dz \qquad (2.3.14)
$$

for a constant c independent of ε. Therefore, by the definition of the convolution, the fact that $\Psi_\varepsilon(x, \cdot)$ is convex for any fixed $x \in B_R$ and (2.3.14), we have

$$
\begin{aligned}
|Dv_\varepsilon(x)| &\leqslant c\,(\Psi_\varepsilon(x,\cdot))_t^{-1} \circ \Psi_\varepsilon\left(x, \fint_{B_\varepsilon(x)} |Dv(z)|\,dz\right) \\
&\leqslant c\,(\Psi_\varepsilon(x,\cdot))_t^{-1}\left(\fint_{B_\varepsilon(x)} \Psi_\varepsilon(x, |Dv(z)|)\,dz\right)
\end{aligned}
$$

$$\leqslant c\left(\Psi_\varepsilon(x,\cdot)\right)_t^{-1}\left(\fint_{B_{2\varepsilon}(x)} \Psi_\varepsilon\left(x,\left|\frac{v(z)-(v)_{B_{2\varepsilon}(x)}}{\varepsilon}\right|\right)dz\right) \leqslant c\varepsilon^{\gamma-1} \tag{2.3.15}$$

with some constant c independent of ε, whenever $x \in B_R$ and $\varepsilon \in (0, \varepsilon_0/4)$, where we have also used the assumption $v \in C^{0,\gamma}(\Omega)$ for some $\gamma \in (0,1)$ and Lemma 2.1.1 together with Remark 2.1.2. Recalling the definition of Ψ_ε in (2.3.5), using the modulus of continuity of functions $a(\cdot)$, $b(\cdot)$ and the assumption $(1.0.15)_2$, for every $x \in B_R$, we estimate

$$\begin{aligned}
\Psi(x,|Dv_\varepsilon(x)|) &\leqslant \Psi_\varepsilon(x,|Dv_\varepsilon(x)|) + |a_\varepsilon(x)-a(x)|H_a(|Dv_\varepsilon(x)|) \\
&\quad + |b_\varepsilon(x)-b(x)|H_b(|Dv_\varepsilon(x)|) \\
&\leqslant \Psi_\varepsilon(x,|Dv_\varepsilon(x)|) \\
&\quad + c\omega_a(\varepsilon)\left(1+\left[\omega_a\left(|Dv_\varepsilon(x)|^{-\frac{1}{1-\gamma}}\right)\right]^{-1}\right)G(|Dv_\varepsilon(x)|) \\
&\quad + c\omega_b(\varepsilon)\left(1+\left[\omega_b\left(|Dv_\varepsilon(x)|^{-\frac{1}{1-\gamma}}\right)\right]^{-1}\right)G(|Dv_\varepsilon(x)|) \\
&\leqslant c\Psi_\varepsilon(x,|Dv_\varepsilon(x)|)
\end{aligned}$$

for a constant c independent of ε, where we have also used (2.3.15). Then arguing in the same way as in (2.3.12)-(2.3.13), we find a sequence of functions $\{v_k\}_{k=1}^\infty \subset W^{1,\infty}(B_R)$ satisfying (2.3.3). The proof is now finished.

□

2.4 Sobolev-Poincaré type inequalities

In the present section we provide a Sobolev-Poincaré type inequality for functions $v \in W^{1,\Psi}(B_R)$ with some ball $B_R \subset \Omega$, which is one of key points for further investigations. For this, first we give a Sobolev-Poincaré type inequality for functions of $W^{1,\Phi}(B_R)$ with $\Phi \in \mathcal{N}$ and a ball $B_R \subset \mathbb{R}^n$.

Lemma 2.4.1. *Let $\Phi \in \mathcal{N}$ with an index $s(\Phi) \geqslant 1$. For any $d_0 \in \left[1, \dfrac{n}{n-1}\right)$,* *there exists $\theta \equiv \theta(n, s(\Phi), d_0) \in (0,1)$ such that*

$$\left(\fint_{B_R} \left[\Phi\left(\left| \frac{v-(v)_{B_R}}{R} \right| \right) \right]^{d_0} dx \right)^{\frac{1}{d_0}} \leqslant c \left(\fint_{B_R} [\Phi(|Dv|)]^{\theta}\, dx \right)^{\frac{1}{\theta}} \tag{2.4.1}$$

holds for some constant $c \equiv c(n, s(\Phi), d_0)$, whenever $v \in W^{1,\Phi}(B_R)$ and $B_R \subset \mathbb{R}^n$ is a ball. Moreover, the above estimate still holds with $v-(v)_{B_R}$ replaced by v if $v \in W_0^{1,\Phi}(B_R)$.

Proof. First note by Lemma $2.1.2_4$ that there exists $\theta \equiv \theta(n, s(\Phi), d_0) \in \left(\frac{(n-1)d_0}{n}, 1\right)$ such that $\Phi^\theta \in \mathcal{N}$ with an index $s(\Phi^\theta)$ depending on $n, s(\Phi), d_0$. Therefore, the following classical formula

$$|v(x)-(v)_{B_R}| \leqslant c(n) \int_{B_R} \frac{|Dv(y)|}{|x-y|^{n-1}}\, dy \tag{2.4.2}$$

holds for a.e $x \in B_R$, see for instance [91, Lemma 7.14]. Letting $E := \fint_{B_R} \Phi^\theta(|Dv|)\, dx$, one can assume that $E > 0$, otherwise v is constant on B_R and the inequality (2.4.1) is trivial. Using (2.4.2), the fact that Φ is increasing and Lemma 2.1.1, we have

$$I := \fint_{B_R} \left[\Phi\left(\left| \frac{v-(v)_{B_R}}{R} \right| \right) \right]^{d_0} dx \leqslant c \fint_{B_R} \left[\Phi\left(\int_{B_R} \frac{|Dv(y)|}{R|x-y|^{n-1}}\, dy \right) \right]^{d_0} dx$$

with $c \equiv c(n, s(\Phi), d_0)$. Since $\displaystyle\int_{B_R} \frac{1}{R|x-y|^{n-1}}\, dy < c(n)$, where this constant is independent of $x \in B_R$ and a ball B_R, we apply Jensen's inequality to the convex function Φ^θ with respect to the measure $R^{-1}|x-y|^{-(n-1)}\, dy$ to obtain

$$I \leqslant c \fint_{B_R} \left(\int_{B_R} \frac{[\Phi(|Dv(y)|)]^\theta}{R|x-y|^{n-1}}\, dy \right)^{\frac{d_0}{\theta}} dx$$

$$
\begin{aligned}
&= cR^{\frac{(n-1)d_0}{\theta}} E^{\frac{d_0}{\theta}} \fint_{B_R} \left(\fint_{B_R} \frac{[\Phi(|Dv(y)|)]^\theta}{|x-y|^{n-1}} E^{-1}\, dy \right)^{\frac{d_0}{\theta}} dx \\
&\leqslant cR^{\frac{(n-1)d_0}{\theta}} E^{\frac{d_0}{\theta}} \fint_{B_R} \fint_{B_R} \frac{[\Phi(|Dv(y)|)]^\theta}{|x-y|^{\frac{(n-1)d_0}{\theta}}} E^{-1}\, dy\, dx, \qquad (2.4.3)
\end{aligned}
$$

where in the last estimate we have applied again Jensen's inequality to the convex function $t \mapsto t^{\frac{d_0}{\theta}}$ with respect to the measure $E^{-1}\Phi^\theta(|Dv(y)|)\, dy$. We observe that

$$
\fint_{B_R} \frac{1}{|x-y|^{\frac{(n-1)d_0}{\theta}}}\, dx \leqslant \frac{1}{|B_R|} \int_{B_{2R}(y)} \frac{1}{|x-y|^{\frac{(n-1)d_0}{\theta}}}\, dx \leqslant c(n, s(\Phi), d) R^{-\frac{(n-1)d_0}{\theta}},
$$

which is possible since $\dfrac{(n-1)d_0}{\theta} < n$. Inserting the last estimate into (2.4.3), the inequality (2.4.1) follows. Finally, if we replace $v - (v)_{B_R}$ by v if $v \in W_0^{1,\Phi}(B_R)$, then the estimate (2.4.1) still holds true since the following classical formula

$$
|v(x)| \leqslant c(n) \int_{B_R} \frac{|Dv(y)|}{|x-y|^{n-1}}\, dy
$$

is valid for a.e $x \in B_R$, whenever $v \in W_0^{1,1}(B_R)$, see for instance [91, Lemma 7.14]. $\square$

Theorem 2.4.1. *Let $v \in W^{1,\Psi}(B_R)$ for a ball $B_R \subset \Omega$ with $R \leqslant 1$ under $G, H_a, H_b \in \mathcal{N}$ and $0 \leqslant a(\cdot) \in C^{\omega_a}(\Omega)$ and $0 \leqslant b(\cdot) \in C^{\omega_b}(\Omega)$ for the continuous and concave functions ω_a, ω_b vanishing at the origin. Then, for any $d \in \left[1, \dfrac{n^2}{n^2-1}\right)$, there exist constants $\theta \equiv \theta(n, s(G), s(H_a), s(H_b), d) \in (0,1)$ and $c \equiv c(n, s(G), s(H_a), s(H_b), \omega_a(1), \omega_b(1), d)$ such that the following Sobolev-Poincaré-type inequality holds:*

$$
\left[\fint_{B_R} \left[\Psi\left(x, \left| \frac{v-(v)_{B_R}}{R} \right| \right) \right]^d dx \right]^{\frac{1}{d}} \leqslant c\lambda_{sp} \left[\fint_{B_R} [\Psi(x, |Dv|)]^\theta\, dx \right]^{\frac{1}{\theta}}, \qquad (2.4.4)
$$

where

$$\lambda_{sp} = \begin{cases} 1+\lambda_1([a]_{\omega_a}+[b]_{\omega_b})\left(1+\left(\int\limits_{B_R} G(|Dv|)\,dx\right)^{\frac{1}{n}}\right) \\ \textit{if } v\in W^{1,\Psi}(B_R) \textit{ with } (1.0.13)_2. & (2.4.5a) \\ 1+\lambda_2([a]_{\omega_a}+[b]_{\omega_b})\Big(1+\|v\|_{L^\infty(B_R)}\Big) \\ \textit{if } v\in L^\infty(B_R) \textit{ with } (1.0.14)_2. & (2.4.5b) \\ 1+\lambda_3([a]_{\omega_a}+[b]_{\omega_b})\left(1+\left[R^{-\gamma}\operatorname*{osc}_{B_R} v\right]^{\frac{1}{1-\gamma}}\right) \\ \textit{if } v\in C^{0,\gamma}(B_R) \textit{ with } (1.0.15)_2. & (2.4.5c) \end{cases}$$

Moreover, the above estimate (2.4.4) *is still valid with* $v-(v)_{B_R}$ *replaced by* v *depending on which one of* (2.4.5a)-(2.4.5c) *comes into play if* $v\in W_0^{1,\Psi}(B_R)$.

Proof. The above theorem widely covers the results of [14, Theorem 4.2], [15, Theorem 32.] and also the results of [57, Theorem 1.6], which is a special case when $G(t)\equiv t^p$, $H_a(t)\equiv t^q$, $\omega_a(\rho)\equiv\rho^\alpha$ and $\omega_b(\cdot)\equiv 0$ for some constants $1<p\leqslant q$ and $\alpha\in(0,1]$. Then using the continuity of the coefficient functions $a(\cdot)$ and $b(\cdot)$, we find

$$\begin{aligned} I &:= \left(\fint_{B_R}\left[\Psi\left(x,\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}} \\ &\leqslant 18[a]_{\omega_a}\omega_a(R)\left(\fint_{B_R}\left[H_a\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}} \\ &\qquad + 18[b]_{\omega_b}\omega_b(R)\left(\fint_{B_R}\left[H_b\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}} \\ &\qquad + 9\left(\fint_{B_R}\left[\Psi^-_{B_R}\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}} \\ &=: 18[a]_{\omega_a}I_1+18[b]_{\omega_b}I_2+9I_3, \end{aligned} \tag{2.4.6}$$

where we have used the following elementary inequality

$$(t_1+t_2+t_3)^d \leqslant 3^d\left(t_1^d+t_2^d+t_3^d\right) \quad (\forall t_1,t_2,t_3 \geqslant 0).$$

We now estimate the terms I_i with $i \in \{1,2,3\}$ in (2.4.6) depending on which one of $(1.0.13)_2$, $(1.0.14)_2$ and $(1.0.15)_2$ is in force. In turn, using $(1.0.13)_2$ and (2.1.2), we see

$$\begin{aligned}
I_1 &= \omega_a(R)\left(\fint_{B_R}\left[\frac{(H_a\circ G^{-1})\left(G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right)}{G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)}G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}}\\
&\leqslant \lambda_1\omega_a(R)\left(\fint_{B_R}\left[\left(1+\left[\omega_a\left(\left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^{-\frac{1}{n}}\right)\right]^{-1}\right)\right.\right.\\
&\qquad \left.\left.\times\ G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}}\\
&\leqslant \lambda_1\omega_a(R)\left(\fint_{B_R}\left[\left(1+\left[\frac{1}{\omega_a(R)}+\frac{R}{\omega_a(R)}\left(G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right)^{\frac{1}{n}}\right]\right)\right.\right.\\
&\qquad \left.\left.\times\ G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}}\\
&\leqslant 9\lambda_1(1+\omega_a(1))\left(\fint_{B_R}\left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}}\\
&\qquad + 9\lambda_1 R\left(\fint_{B_R}\left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^{\left(1+\frac{1}{n}\right)d} dx\right)^{\frac{1}{d}},
\end{aligned} \tag{2.4.7}$$

where we have used also that $\omega_a(\cdot)$ is non-decreasing and $R \leqslant 1$. In the same way, we have

$$I_2 \leqslant 9\lambda_1(1+\omega_b(1))\left(\fint_{B_R}\left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}}$$

$$+ 9\lambda_1 R \left(\fint_{B_R} \left[G\left(\left| \frac{v-(v)_{B_R}}{R} \right| \right) \right]^{\left(1+\frac{1}{n}\right)d} dx \right)^{\frac{1}{d}}. \tag{2.4.8}$$

Adding the estimates coming from the last two displays and applying Lemma 2.4.1 with $\Phi \equiv G$ for $d_0 \equiv d$ and $d_0 \equiv \left(1+\dfrac{1}{n}\right)d < \dfrac{n}{n-1}$, there exists $\theta_1 \equiv \theta_1(n, s(G), d) \in (0,1)$ such that

$$\begin{aligned}
I_1 + I_2 &\leqslant c\lambda_1 \left(\fint_{B_R} \left[G\left(\left| \frac{v-(v)_{B_R}}{R} \right| \right) \right]^{d} dx \right)^{\frac{1}{d}} \\
&\quad + c\lambda_1 R \left(\fint_{B_R} \left[G\left(\left| \frac{v-(v)_{B_R}}{R} \right| \right) \right]^{\left(1+\frac{1}{n}\right)d} dx \right)^{\frac{1}{d}} \\
&\leqslant c\lambda_1 \left[\fint_{B_R} [G(|Dv|)]^{\theta_1} dx \right]^{\frac{1}{\theta_1}} + c\lambda_1 R \left[\fint_{B_R} [G(|Dv|)]^{\theta_1} dx \right]^{\left(1+\frac{1}{n}\right)\frac{1}{\theta_1}} \\
&\leqslant c\lambda_1 \left[1 + \left(\int_{B_R} G(|Dv|)\, dx \right)^{\frac{1}{n}} \right] \left[\fint_{B_R} G^{\theta_1}(|Dv|)\, dx \right]^{\frac{1}{\theta_1}}
\end{aligned} \tag{2.4.9}$$

for some constant $c \equiv c(n, s(G), \omega_a(1), \omega_b(1), d)$, where in the last inequality of the above display we have used Hölder's inequality. Since $\Psi^-_{B_R} \in \mathcal{N}$ with an index $s(\Psi) = s(G) + s(H_a) + s(H_b)$ by Remark 2.1.2, we are able to apply Lemma 2.4.1 with $\Phi \equiv \Psi^-_{B_R}$ for $d_0 \equiv d$. In turn, there exists $\theta_2 \equiv \theta_2(n, s(\Psi), d)$ such that

$$I_3 \leqslant c \left[\fint_{B_R} \left[\Psi^-_{B_R}(|Dv|) \right]^{\theta_2} dx \right]^{\frac{1}{\theta_2}} \tag{2.4.10}$$

with some constant $c \equiv c(n, s(\Psi), d)$. Inserting the estimates obtained in (2.4.9)-(2.4.10) into (2.4.6), recalling the very definition of $\Psi^-_{B_R}$ in (2.1.3) and setting $\theta := \max\{\theta_1, \theta_2\}$, we arrive at (2.4.5a). Now we turn our attention to proving (2.4.5b). For this, we estimate the terms I_i for $i \in \{1,2,3\}$ for $v \in L^\infty(B_R)$ under the assumption $(1.0.14)_2$. In turn, using (2.1.2) and the assumption $(1.0.14)_2$, we see

$$
\begin{aligned}
I_1 &= \omega_a(R) \left(\fint_{B_R} \left[\frac{H_a\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)}{G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)} G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx \right)^{\frac{1}{d}} \\
&\leqslant \lambda_2 \omega_a(R) \left(\fint_{B_R} \left[\left(1 + \left[\omega_a \left(\left(\left| \frac{v-(v)_{B_R}}{R} \right| \right)^{-1} \right) \right]^{-1} \right) \right.\right. \\
&\quad \left.\left. \times\ G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx \right)^{\frac{1}{d}} \\
&\leqslant \lambda_2 \omega_a(R) \left(\fint_{B_R} \left[\left(1 + \left[\frac{1}{\omega_a(R)} + \frac{|v-(v)_{B_R}|}{\omega_a(R)} \right] \right) G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx \right)^{\frac{1}{d}} \\
&\leqslant 2\lambda_2 \left(1 + \omega_a(1) + \|v\|_{L^\infty(B_R)} \right) \left(\fint_{B_R} \left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx \right)^{\frac{1}{d}}.
\end{aligned}
\tag{2.4.11}
$$

In a similar way, one can see

$$
I_2 \leqslant 2\lambda_2 \left(1 + \omega_b(1) + \|v\|_{L^\infty(B_R)} \right) \left(\fint_{B_R} \left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx \right)^{\frac{1}{d}}.
\tag{2.4.12}
$$

Adding the estimates in (2.4.11)-(2.4.12) and applying Lemma 2.4.1 with $\Phi \equiv G$ for $d_0 \equiv d$, there exists an exponent $\theta_1 \equiv \theta_1(n, s(G), d) \in (0,1)$ such that

$$
I_1 + I_2 \leqslant c\lambda_2 \left(1 + \|v\|_{L^\infty(B_R)} \right) \left[\fint_{B_R} [G(|Dv|)]^{\theta_1}\, dx \right]^{\frac{1}{\theta_1}}
\tag{2.4.13}
$$

for some constant $c \equiv c(n, s(G), \omega_a(1), \omega_b(1), d)$. This estimate together with (2.4.10) and the very definition of $\Psi^-_{B_R}$ in (2.1.3), we find (2.4.5b). It remains to prove (2.4.5c). Essentially, it can proved in a similar manner we have shown in (2.4.11)-(2.4.12). So using the assumption $(1.0.15)_2$ and again (2.1.2), we see

$$
\begin{aligned}
I_1 &= \omega_a(R) \left(\fint_{B_R} \left[\frac{H_a\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)}{G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)} G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right) \right]^d dx \right)^{\frac{1}{d}} \\
&\leqslant \lambda_3 \omega_a(R) \left(\fint_{B_R} \left[\left(1 + \left[\omega_a\left(\left(\left| \frac{v-(v)_{B_R}}{R} \right| \right)^{-\frac{1}{1-\gamma}} \right) \right]^{-1} \right) \right.\right. \\
&\qquad \left.\left. \times\ G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right) \right]^d dx \right)^{\frac{1}{d}} \\
&\leqslant \lambda_3 \omega_a(R) \left(\fint_{B_R} \left[\left(1 + \left[\frac{1}{\omega_a(R)} + \frac{R^{\frac{-\gamma}{1-\gamma}}}{\omega_a(R)} \left|v-(v)_{B_R}\right|^{\frac{1}{1-\gamma}} \right] \right) \right.\right. \\
&\qquad \left.\left. \times\ G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right) \right]^d dx \right)^{\frac{1}{d}} \\
&\leqslant 2\lambda_3 \left(1 + \omega_a(1) + \left[R^{-\gamma} \operatorname*{osc}_{B_R} v \right]^{\frac{1}{1-\gamma}} \right) \left(\fint_{B_R} \left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right) \right]^d dx \right)^{\frac{1}{d}} .
\end{aligned}
\tag{2.4.14}
$$

By arguing in the same way, we see

$$
I_1 + I_2 \leqslant c\lambda_3 \left(1 + \left[R^{-\gamma} \operatorname*{osc}_{B_R} v \right]^{\frac{1}{1-\gamma}} \right) \left(\fint_{B_R} \left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right) \right]^d dx \right)^{\frac{1}{d}} .
\tag{2.4.15}
$$

for some constant $c \equiv c(s(G), \omega_a(1), \omega_b(1))$. Finally, this estimate together with (2.4.10) leads to (2.4.5c). The proof is complete. $\square$

Remark 2.4.1. We here remark that choosing $d \equiv 1$ in a Sobolev-Poincaré type inequality of Theorem 2.4.1, there exist an exponent $\theta \equiv \theta(n, s(G), s(H_a), s(H_b))$ such that

$$\fint_{B_R} \Psi\left(x, \left|\frac{v-(v)_{B_R}}{R}\right|\right) dx \leqslant c\lambda_{sp}\left[\fint_{B_R} [\Psi(x, |Dv|)]^{\theta}\, dx\right]^{\frac{1}{\theta}}, \tag{2.4.16}$$

holds for some constant $c \equiv c(n, s(G), s(H_a), s(H_b), \omega_a(1), \omega_b(1))$, where

$$\lambda_{sp} = \begin{cases} 1+\lambda_1([a]_{\omega_a}+[b]_{\omega_b})\left(1+\left(\int_{B_R} G(|Dv|)\, dx\right)^{\frac{1}{n}}\right) & \\ \text{if } v \in W^{1,\Psi}(B_R) \text{ with } (1.0.13)_2. & (2.4.17a) \\ 1+\lambda_2([a]_{\omega_a}+[b]_{\omega_b})\left(1+\|v\|_{L^\infty(B_R)}\right) & \\ \text{if } v \in L^\infty(B_R) \text{ with } (1.0.14)_2. & (2.4.17b) \\ 1+\lambda_3([a]_{\omega_a}+[b]_{\omega_b})\left(1+\left[R^{-\gamma}\operatorname*{osc}_{B_R} v\right]^{\frac{1}{1-\gamma}}\right) & \\ \text{if } v \in C^{0,\gamma}(B_R) \text{ with } (1.0.15)_2. & (2.4.17c) \end{cases}$$

2.5 Harmonic type approximation

In this section, we discuss some important regularity results for the solution to the following Dirichlet boundary value problem:

$$\begin{cases} -\operatorname{div} A_0(Dh) = 0 \text{ in } B_R \\ h \in v + W_0^{1,\Psi_0}(B_R), \end{cases} \tag{2.5.1}$$

where $B_R \subset \mathbb{R}^n$ is a given ball with $n \geqslant 2$, $v \in W^{1,\Psi_0}(B_R)$ is a given function, and $A_0 : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field belonging to $C^0(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ and satisfies the following ellipticity and coercivity assumptions:

$$\begin{cases} |A_0(z)||z| + |D_z A_0(z)||z|^2 \leqslant L\Psi_0(|z|) \\ \nu\dfrac{\Psi_0(|z|)}{|z|^2}|\xi|^2 \leqslant \langle D_z A_0(z)\xi, \xi\rangle \end{cases} \tag{2.5.2}$$

for fixed constants $0 < \nu \leqslant L$, whenever $z \in \mathbb{R}^n \setminus \{0\}$ and $\xi \in \mathbb{R}^n$, in which the function Ψ_0 is given by

$$\Psi_0(t) := G(t) + a_0 H_a(t) + b_0 H_b(t) \quad (G, H_a, H_b \in \mathcal{N}) \tag{2.5.3}$$

with fixed constants $a_0, b_0 \geqslant 0$ for every $t \geqslant 0$. By Lemma $2.1.2_1$, we get the following

$$\frac{1}{s(G) + s(H_a) + s(H_b)} \leqslant \frac{\Psi_0''(t)t}{\Psi_0'(t)} \leqslant s(G) + s(H_a) + s(H_b) \tag{2.5.4}$$

for every $t > 0$, which means that $\Psi_0 \in \mathcal{N}$ with an index $s(\Psi_0) = s(G) + s(H_a) + s(H_b)$. Therefore, we note that the following monotonicity property that

$$\begin{aligned} &|V_G(z_1) - V_G(z_2)|^2 + a_0|V_{H_a}(z_1) - V_{H_a}(z_2)|^2 + b_0|V_{H_b}(z_1) - V_{H_b}(z_2)|^2 \\ &\approx |V_{\Psi_0}(z_1) - V_{\Psi_0}(z_2)|^2 \\ &\leqslant c \, \langle A_0(z_1) - A_0(z_2), z_1 - z_2 \rangle \end{aligned} \tag{2.5.5}$$

holds with some constant $c \equiv c(n, s(\Psi_0), \nu)$, whenever $z_1, z_2 \in \mathbb{R}^n \setminus \{0\}$, where the map V_Φ for a function $\Phi \in \mathcal{N}$ has been defined in (2.1.8).

Theorem 2.5.1. *Let $h \in W^{1,\Psi_0}(B_R)$ be the weak solution to* (2.5.1) *under the assumption* (2.5.2)*. Suppose that there exists a higher integrability exponent $\delta_1 > 0$ such that*

$$\Psi_0(|Dv|) \in L^{1+\delta_1}(B_R) \quad \text{and} \quad \|\Psi_0(|Dv|)\|_{L^1(B_R)} \leqslant L_0 \tag{2.5.6}$$

for some constant $L_0 \geqslant 0$. Then there exists a positive exponent $\delta_0 \leqslant \delta_1$ depending on $n, s(\Psi_0), \nu, L$ and δ_1 such that the following inequality

$$\left(\fint_{B_R} [\Psi_0(|Dh|)]^{1+\delta_0} \, dx \right)^{\frac{1}{1+\delta_0}} \leqslant c \left(\fint_{B_R} [\Psi_0(|Dv|)]^{1+\delta_0} \, dx \right)^{\frac{1}{1+\delta_0}} \tag{2.5.7}$$

holds for some constant $c \equiv c(n, s(\Psi_0), \nu, L, L_0, \delta_1)$.

Proof. First the standard energy estimate implies that

$$\int_{B_R} \Psi_0(|Dh|)\,dx \leqslant c\int_{B_R} \Psi_0(|Dv|)\,dx \leqslant cL_0 \tag{2.5.8}$$

holds with some constant $c \equiv c(n, s(\Psi_0), \nu, L)$. For a fixed ball $B_{2\rho} \subset B_R$, let $\eta \in C_0^1(B_{2\rho})$ be a standard cut-off function satisfying $\chi_{B_\rho} \leqslant \eta \leqslant \chi_{B_{2\rho}}$ and $|D\eta| \leqslant 4/\rho$. Let us take the function $\varphi = \eta^{s(\Psi_0)+1}\left(h - (h)_{B_{2\rho}}\right)$ as a test function in the equation (2.5.1). Then using the monotonicity property of $A_0(\cdot)$ and Lemma 2.1.4 with Ψ_0, we have

$$\begin{aligned}\int_{B_{2\rho}} \eta^{s(\Psi_0)+1}\Psi_0(|Dh|)\,dx &\leqslant c\int_{B_{2\rho}} \eta^{s(\Psi_0)}\Psi_0'(|Dh|)\left|\frac{h-(h)_{B_{2\rho}}}{\rho}\right| dx\\ &\leqslant c\int_{B_{2\rho}} \eta^{s(\Psi_0)}\left((\varepsilon\eta)\Psi_0(|Dh|) + \frac{1}{(\varepsilon\eta)^{s(\Psi_0)}}\Psi_0\left(\left|\frac{h-(h)_{B_{2\rho}}}{\rho}\right|\right)\right) dx.\end{aligned} \tag{2.5.9}$$

Choosing ε sufficiently small in the last display, we conclude that

$$\fint_{B_\rho} \Psi_0(|Dh|)\,dx \leqslant c\fint_{B_{2\rho}} \Psi_0\left(\left|\frac{h-(h)_{B_{2\rho}}}{\rho}\right|\right) dx \tag{2.5.10}$$

for a constant $c \equiv c(n, s(\Psi_0), \nu, L)$. By applying Lemma 2.4.1 to $\Phi \equiv \Psi_0$ with $d_0 \equiv 1$, there exists $\theta_0 \equiv \theta_0(n, s(\Psi_0)) \in (0,1)$ such that

$$\fint_{B_\rho} \Psi_0(|Dh|)\,dx \leqslant c\fint_{B_{2\rho}} \Psi_0\left(\left|\frac{h-(h)_{B_{2\rho}}}{\rho}\right|\right) dx \leqslant c\left(\fint_{B_{2\rho}} [\Psi_0(|Dh|)]^{\theta_0}\,dx\right)^{\frac{1}{\theta_0}} \tag{2.5.11}$$

holds for some constant $c \equiv c(n, s(\Psi_0), \nu, L)$, whenever $B_{2\rho} \subset B_R$ is a ball. Now we prove a version of the last inequality near the boundary of B_R. For this, let $B_{2\rho}(y) \subset \mathbb{R}^n$ be a ball such that $y \in B_R$ and $\frac{1}{10} < \frac{|B_{2\rho}(y) \setminus B_R|}{|B_{2\rho}(y)|}$. We take a test function by $\varphi \equiv \eta^{s(\Psi_0)+1}(h - v)$, where $\eta \in C_0^1(B_{2\rho})$ is a standard cut-off function as before so that $\chi_{B_\rho} \leqslant \eta \leqslant \chi_{B_{2\rho}}$ and $|D\eta| \leqslant 4/\rho$. This choice of φ is admissible since $\operatorname{supp}\varphi \Subset B_R \cap B_{2\rho}(y)$. Arguing similarly as we have done above, we have

$$
\begin{aligned}
\int_{B_R\cap B_{2\rho}(y)} \eta^{s(\Psi_0)+1}\Psi_0(|Dh|)\,dx &\leqslant c \int_{B_R\cap B_{2\rho}(y)} \eta^{s(\Psi_0)}\Psi_0'(|Dh|)\left|\frac{h-\upsilon}{\rho}\right| dx \\
&\quad + c \int_{B_R\cap B_{2\rho}(y)} \eta^{s(\Psi_0)}\Psi_0'(|Dh|)|D\upsilon|\,dx \\
&\leqslant c \int_{B_R\cap B_{2\rho}} \eta^{s(\Psi_0)}\left((\varepsilon\eta)\Psi_0(|Dh|) + \frac{1}{(\varepsilon\eta)^{s(\Psi_0)}}\Psi_0\left(\left|\frac{h-\upsilon}{\rho}\right|\right)\right) dx \\
&\qquad + c \int_{B_R\cap B_{2\rho}} \eta^{s(\Psi_0)}\left((\varepsilon\eta)\Psi_0(|Dh|) + \frac{1}{(\varepsilon\eta)^{s(\Psi_0)}}\Psi_0\left(|D\upsilon|\right)\right) dx.
\end{aligned}
\tag{2.5.12}
$$

Again choosing ε small enough and reabsorbing the terms, we find that

$$
\begin{aligned}
\fint_{B_R\cap B_{2\rho}(y)} \eta^{s(\Psi_0)+1}\Psi_0(|Dh|)\,dx &\leqslant c \fint_{B_R\cap B_{2\rho}(y)} \Psi_0\left(\left|\frac{h-\upsilon}{\rho}\right|\right) dx \\
&\quad + c \fint_{B_R\cap B_{2\rho}(y)} \Psi_0\left(|D\upsilon|\right) dx
\end{aligned}
$$

for some constant $c \equiv c(n, s(\Psi_0), \nu, L)$. Redefining $h - \upsilon \equiv 0$ on $B_{2\rho}(y) \setminus B_R$, we are able to apply Lemma 2.4.1 to $\Phi \equiv \Psi_0$ with $d_0 \equiv 1$. In turn, there exists $\theta_0 \equiv \theta_0(n, s(\Psi_0)) \in (0, 1)$ as appearing in (2.5.11) such that

$$
\begin{aligned}
\fint_{B_R\cap B_{2\rho}(y)} \Psi_0\left(\left|\frac{h-\upsilon}{\rho}\right|\right) dx &\leqslant \left(\fint_{B_R\cap B_{2\rho}(y)} [\Psi_0(|Dh - D\upsilon|)]^{\theta_0}\,dx\right)^{\frac{1}{\theta_0}} \\
&\leqslant \left(\fint_{B_R\cap B_{2\rho}(y)} [\Psi_0(|Dh|)]^{\theta_0}\,dx\right)^{\frac{1}{\theta_0}}
\end{aligned}
$$

$$+ c \fint_{B_R \cap B_{2\rho}(y)} [\Psi_0(|Dv|)]\, dx$$

for some constant $c \equiv c(n, s(\Psi_0))$, where for the last inequality we have used (2.1.6) and Hölder's inequality. Combining the last two displays and (2.5.11), we have

$$\fint_{B_\rho(y)} [V(x)]^{\frac{1}{\theta_0}}\, dx \leqslant c \left(\fint_{B_{2\rho}(y)} V(x)\, dx \right)^{\frac{1}{\theta_0}} + c \fint_{B_{2\rho}(y)} U(x)\, dx \tag{2.5.13}$$

for some $c \equiv c(n, s(\Psi_0), \nu, L)$, where

$$V(x) := [\Psi_0(|Dh|)]^{\theta_0} \chi_{B_{2\rho}(y)}(x) \quad \text{and} \quad U(x) := \Psi_0(|Dv|) \chi_{B_{2\rho}(y)}(x)$$

for every ball $B_{2\rho}(y) \subset \mathbb{R}^n$ satisfying either $B_{2\rho}(y) \subset B_R$ or $\dfrac{1}{10} < \dfrac{|B_{2\rho}(y) \setminus B_R|}{|B_{2\rho}(y)|}$ with $y \in B_R$. Applying a variant of Gehring's lemma and a standard covering argument, we arrive at the desired estimate (2.5.7). $\square$

Essentially, the inequality (2.5.7) can be shown with $1+\delta_0$ replaced by any number $\gamma > 1$ provided $\Psi_0(|Dv|) \in L^\gamma(B_r)$. This type of estimate follows from a combination of interior and boundary estimates of the same type via a standard flattening of the boundary and covering argument similarly as employed in [59, Theorem 5.1] along with arguments used in [14]. The flattening of the boundary is standard, we refer for instance to [106, 107] for more details. But the small higher integrability type estimate (2.5.7) is sufficient for proving Lemma 2.5.1 below.

Before going on further, we recall a classical truncation lemma due to [1]. The statement involves the Hardy-Littlewood maximal operator, defined as

$$M(f)(x) := \sup_{B_r(x) \subset \mathbb{R}^n} \fint_{B_r(x)} |f(y)|\, dy, \qquad x \in \mathbb{R}^n, \tag{2.5.14}$$

whenever $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Theorem 2.5.2 ([1])**.** *Let $B_R \subset \mathbb{R}^n$ be a ball and $f \in W^{1,1}_0(B_R)$. Then, for* *every $\lambda > 0$, there exists $f_\lambda \in W_0^{1,\infty}(B_R)$ such that*

$$\|Df_\lambda\|_{L^\infty(B_R)} \leqslant c\lambda \tag{2.5.15}$$

for some constant c depending only on n. Moreover, it holds that

$$\{x \in B_R : f_\lambda(x) \neq f(x)\} \subset \{x \in B_R : M(|Df(x)|) > \lambda\} \cup \text{ negligible set.} \tag{2.5.16}$$

We notice that in this theorem we may assume that f is defined on $\mathbb{R}^n$ by redefining $f \equiv 0$ on $\mathbb{R}^n \setminus B_R$. We are now ready to state the main result of this section.

Lemma 2.5.1 (Harmonic type approximation)**.** *Let $B_R \subset \mathbb{R}^n$ be a ball with $R \leqslant 1$, $\sigma \in (0,1)$ and $\upsilon \in W^{1,\Psi_0}(B_{2R})$ be a function satisfying*

$$\fint_{B_{2R}} \Psi_0(|D\upsilon|)\, dx \leqslant c_0 \tag{2.5.17}$$

and

$$\fint_{B_R} [\Psi_0(|D\upsilon|)]^{1+\delta_1}\, dx \leqslant c_1 \tag{2.5.18}$$

for some constants $c_0, c_1 \geqslant 1$ and $\delta_1 > 0$. Suppose that $\Psi_0(1) \geqslant 1$. We further assume that

$$\left| \fint_{B_R} \langle A_0(D\upsilon), D\varphi\rangle \, dx \right| \leqslant \sigma \|D\varphi\|_{L^\infty(B_R)} \text{ holds for } \varphi \in C_0^\infty(B_R). \tag{2.5.19}$$

Then there exists $h \in \upsilon + W_0^{1,\Psi_0}(B_R)$ such that

$$\fint_{B_R} \langle A_0(Dh), D\varphi\rangle \, dx = 0 \text{ for all } \varphi \in C_0^\infty(B_R), \tag{2.5.20}$$

$$\fint_{B_R} [\Psi_0(|Dh|)]^{1+\delta_0}\, dx \leqslant c(n, s(\Psi_0), \nu, L, \delta_1, c_0, c_1)$$

for some positive $\delta_0 \equiv \delta_0(n, s(\Psi_0), \nu, L, \delta_1)$, (2.5.21)

$$\fint_{B_R} |V_{\Psi_0}(D\upsilon) - V_{\Psi_0}(Dh)|^2 \, dx \leqslant \bar{c}\sigma^{s_1}, \tag{2.5.22}$$

and

$$\fint_{B_R} \Psi_0 \left(\left| \frac{\upsilon - h}{R} \right| \right) dx \leqslant \bar{c}\sigma^{s_0} \tag{2.5.23}$$

for some constants with dependence as $s_1 \equiv s_1(n, s(\Psi_0), \delta_1, c_0) > 0$, $s_0 \equiv s_0(n, s(\Psi_0), \delta_1, c_0) > 0$ *and* $\bar{c} \equiv \bar{c}(n, s(\Psi_0), \nu, L, \delta_1, c_0, c_1) \geqslant 1$.

Proof. By the standard approximation argument, if (2.5.19) holds for all functions $\varphi \in C_0^\infty(B_R)$, then it also holds for all functions $\varphi \in W_0^{1,\infty}(B_R)$. The proof falls in three steps.

Step 1: Truncation. The standard energy estimate and (2.5.17) give us

$$\fint_{B_R} \Psi_0(|Dh|) \, dx \leqslant \fint_{B_R} \Psi_0(|D\upsilon|) \, dx \leqslant c(n, s(\Psi_0), \nu, L)c_0. \tag{2.5.24}$$

By applying Theorem 2.5.1, there exists an exponent $\delta_0 \equiv \delta_0(n, s(\Psi_0), \nu, L, \delta_1)$ satisfying

$$\fint_{B_R} [\Psi_0(|Dh|)]^{1+\delta_0} \, dx \leqslant c \fint_{B_R} [\Psi_0(|D\upsilon|)]^{1+\delta_0} \, dx \leqslant c(n, s(\Psi_0), \nu, L, \delta_1, c_0, c_1), \tag{2.5.25}$$

which is (2.5.21). We now set $f := \upsilon - h \in W_0^{1,\Psi_0}(B_R)$ and let $\lambda \geqslant 1$ to be chosen later. We consider $f_\lambda \in W_0^{1,\infty}(B_R)$ provided by Theorem 2.5.2, which satisfies (2.5.15) and (2.5.16). By these properties, Chebyshev's inequality and then the maximal function theorem for Orlicz spaces (see for instance [87, Proposition 1.2]), we have

$$\frac{|\{f \neq f_\lambda\}|}{|B_R|} \leqslant \frac{|B_R \cap \{M(|Df|) > \lambda\}|}{|B_R|}$$

$$
\begin{aligned}
&\leqslant \frac{1}{[\Psi_0(\lambda)]^{1+\delta_0}} \fint_{B_R} [\Psi_0(M(|Df|))]^{1+\delta_0}\, dx \\
&\leqslant \frac{c}{[\Psi_0(\lambda)]^{1+\delta_0}} \fint_{B_R} [\Psi_0(|Df|)]^{1+\delta_0}\, dx \\
&\leqslant \frac{c}{[\Psi_0(\lambda)]^{1+\delta_0}} \left[\fint_{B_R} [\Psi_0(|Dv|)]^{1+\delta_0}\, dx + \fint_{B_R} [\Psi_0(|Dh|)]^{1+\delta_0}\, dx \right] \\
&\leqslant \frac{c}{[\Psi_0(\lambda)]^{1+\delta_0}} \qquad (2.5.26)
\end{aligned}
$$

with $c \equiv c(n, s(\Psi_0), \nu, L, \delta_1, c_0, c_1)$, where we have used (2.5.21) and (2.5.25). Now we test the equation (2.5.1) against f_λ to obtain

$$
\begin{aligned}
\Gamma_1 &:= \fint_{B_R} \langle A_0(Dv) - A_0(Dh), Df_\lambda \rangle \, \chi_{\{f=f_\lambda\}}\, dx \\
&= \fint_{B_R} \langle A_0(Dv), Df_\lambda \rangle \, dx - \fint_{B_R} \langle A_0(Dv) - A_0(Dh), Df_\lambda \rangle \, \chi_{\{f \neq f_\lambda\}}\, dx \\
&=: \Gamma_2 + \Gamma_3. \qquad (2.5.27)
\end{aligned}
$$

Next we estimate each term appearing in the last equality. By using (2.5.5), we have

$$
\Gamma_1 \geqslant c \fint_{B_R} |V_{\Psi_0}(Dv) - V_{\Psi_0}(Dh)|^2 \chi_{\{f=f_\lambda\}}\, dx
$$

with $c \equiv c(n, s(\Psi_0))$. Using (2.5.19), and then (2.5.15), we get

$$
|\Gamma_2| \leqslant \sigma \, \|Df_\lambda\|_{L^\infty(B_R)} \leqslant c(n)\sigma\lambda.
$$

For Γ_3, we fix $\varepsilon \in (0,1)$ to be chosen later and we estimate

$$
|\Gamma_3| \leqslant \fint_{B_R} (|A_0(Dh)| + |A_0(Dv)|)\, |Df_\lambda| \chi_{\{f \neq f_\lambda\}}\, dx
$$

$$
\begin{aligned}
&\overset{(2.5.2)}{\leqslant} L\,\|Df_\lambda\|_{L^\infty(B_R)} \fint_{B_R} \left[\frac{\Psi_0(|Dv|)}{|Dv|} + \frac{\Psi_0(|Dh|)}{|Dh|}\right] \chi_{\{f\neq f_\lambda\}}\,dx \\
&\leqslant \varepsilon \fint_{B_R} [\Psi_0(|Dv|) + \Psi_0(|Dh|)]\,dx + \frac{c}{\varepsilon^{s(\Psi_0)}} \Psi_0(\|Df_\lambda\|_{L^\infty(B_R)}) \frac{|\{f\neq f_\lambda\}|}{|B_R|} \\
&\leqslant c\left(\varepsilon + \frac{1}{[\Psi_0(\lambda)]^{\delta_0}\varepsilon^{s(\Psi_0)}}\right)
\end{aligned}
$$

with some $c \equiv c(n, s(\Psi_0), \nu, L, \delta_1, c_0, c_1)$, where in the last two inequalities we have used Lemma 2.1.1 together with (2.5.15) and (2.5.24). Merging the estimates for Γ_1, Γ_2 and Γ_3 with (2.5.27), we deduce that

$$
\begin{aligned}
&\fint_{B_R} |V_{\Psi_0}(Dv) - V_{\Psi_0}(Dh)|^2 \chi_{\{f=f_\lambda\}}\,dx \\
&\qquad \leqslant c_*\left(\sigma\lambda + \varepsilon + \frac{1}{[\Psi_0(\lambda)]^{\delta_0}\varepsilon^{s(\Psi_0)}}\right) =: S(\sigma,\lambda,\varepsilon) \qquad (2.5.28)
\end{aligned}
$$

for some constant $c_* \equiv c_*(n, s(\Psi_0), \nu, L, \delta_1, c_0, c_1)$, where $\varepsilon \in (0,1)$ is still to be chosen later. Now let us use a short notation for the simplicity

$$
Z^2 := |V_{\Psi_0}(Dv) - V_{\Psi_0}(Dh)|^2 \qquad (2.5.29)
$$

and fix $\theta \in (0,1)$, again to be chosen later. Hölder's inequality and (2.5.28) imply

$$
\left(\fint_{B_R} Z^{2\theta} \chi_{\{f=f_\lambda\}}\,dx\right)^{\frac{1}{\theta}} \leqslant S(\sigma,\lambda,\varepsilon). \qquad (2.5.30)
$$

Again using Hölder's inequality, we get

$$
\begin{aligned}
\left(\fint_{B_R} Z^{2\theta} \chi_{\{f\neq f_\lambda\}}\,dx\right)^{\frac{1}{\theta}} &\leqslant \left(\frac{|\{f\neq f_\lambda\}|}{|B_R|}\right)^{\frac{1-\theta}{\theta}} \fint_{B_R} Z^2\,dx \\
&\overset{(2.5.26)}{\leqslant} c[\Psi_0(\lambda)]^{-\frac{(1-\theta)(1+\delta_0)}{\theta}} \fint_{B_R} [\Psi_0(|Dv|) + \Psi_0(|Dh|)]\,dx
\end{aligned}
$$

$$\overset{(2.5.24)}{\leqslant} c[\Psi_0(\lambda)]^{-\frac{(1-\theta)(1+\delta_0)}{\theta}} \tag{2.5.31}$$

for some constant $c \equiv c(n, s(\Psi_0), \nu, L, \delta_1, c_0, c_1, \theta)$. Consequently, (2.5.30) and (2.5.31) yield that

$$\left(\fint_{B_R} Z^{2\theta}\, dx\right)^{\frac{1}{\theta}} \leqslant c\left(S(\sigma,\lambda,\varepsilon) + [\Psi_0(\lambda)]^{-\frac{(1-\theta)(1+\delta_0)}{\theta}}\right)$$

holds with again $c \equiv c(n, s(\Psi_0), \nu, L, \delta_1, c_0, c_1, \theta)$. Recalling $S(\sigma,\lambda,\varepsilon)$ in (2.5.28) and using Lemma 2.1.1, we find

$$\left(\fint_{B_R} Z^{2\theta}\, dx\right)^{\frac{1}{\theta}} dx \leqslant c\left(\sigma\lambda + \varepsilon + \lambda^{-\delta_0\left(\frac{1}{s(\Psi_0)}+1\right)}\varepsilon^{-s(\Psi_0)} + \lambda^{-\left(\frac{1}{s(\Psi_0)}+1\right)\frac{(1-\theta)(1+\delta_0)}{\theta}}\right),$$

where at this moment we have used the assumption that $\Psi_0(1) \geqslant 1$. Choosing $\lambda = \sigma^{-\frac{1}{2}}$ and $\varepsilon = \sigma^s$ with $s = \dfrac{\delta_0}{4s(\Psi_0)}\left(\dfrac{1}{s(\Psi_0)}+1\right)$, we obtain

$$\left(\fint_{B_R} \left(|V_{\Psi_0}(Dv) - V_{\Psi_0}(Dh)|^2\right)^{\theta}\, dx\right)^{\frac{1}{\theta}} \leqslant c\sigma^{m_0} \tag{2.5.32}$$

with constants

$$m_0 = \min\{\frac{1}{2}, \frac{\delta_0}{4s(\Psi_0)}\left(\frac{1}{s(\Psi_0)}+1\right), \left(\frac{1}{s(\Psi_0)}+1\right)\frac{(1-\theta)(1+\delta_0)}{2\theta}\}$$

and $c \equiv c(n, s(\Psi_0), \nu, L, \delta_1, c_0, c_1, \theta)$. Recall that θ is yet to be chosen.

Step 2: Proof of (2.5.22)**.** By taking θ properly, we can deduce (2.5.22) from (2.5.32). Hölder's inequality with exponents $\left(\dfrac{2(1+\delta_0)}{1+2\delta_0}, 2(1+\delta_0)\right)$ yields

$$\fint_{B_R} Z^2\, dx = \fint_{B_R} Z\cdot Z\, dx \leqslant \left(\fint_{B_R} Z^{\frac{2(1+\delta_0)}{1+2\delta_0}}\, dx\right)^{\frac{1+2\delta_0}{2(1+\delta_0)}} \left(\fint_{B_R} Z^{2(1+\delta_0)}\, dx\right)^{\frac{1}{2(1+\delta_0)}}. \tag{2.5.33}$$

We now choose $\theta := \dfrac{1+\delta_0}{1+2\delta_0} \in (0,1)$ in (2.5.32) in order to find that

$$\left(\fint_{B_R} Z^{\frac{2(1+\delta_0)}{1+2\delta_0}} \, dx \right)^{\frac{1+2\delta_0}{2(1+\delta_0)}} \leqslant c\sigma^{\frac{m_0}{2}}. \tag{2.5.34}$$

On the other hand, recalling (2.5.29) and (2.5.25), we have

$$\begin{aligned} &\fint_{B_R} Z^{2(1+\delta_0)} \, dx = \fint_{B_R} |V_{\Psi_0}(Dv) - V_{\Psi_0}(Dh)|^{2(1+\delta_0)} \, dx \\ &\leqslant c \fint_{B_R} [\Psi_0(|Dv|)]^{1+\delta_0} \, dx + c \fint_{B_R} [\Psi_0(|Dh|)]^{1+\delta_0} \, dx \leqslant c(n, s(\Psi_0), \nu, L, \delta_1, c_0, c_1). \end{aligned} \tag{2.5.35}$$

We combine the estimates (2.5.33)-(2.5.35) to discover

$$\fint_{B_R} \left(|V_{\Psi_0}(Dv) - V_{\Psi_0}(Dh)|^2 \right) \, dx \leqslant c\sigma^{s_1}, \tag{2.5.36}$$

where

$$s_1 = \frac{1}{2} \min \left\{ \frac{1}{2}, \frac{\delta_0}{4s(\Psi_0)} \left(\frac{1}{s(\Psi_0)} + 1 \right), \left(\frac{1}{s(\Psi_0)} + 1 \right) \frac{\delta_0}{2} \right\}$$

and $c \equiv c(n, s(\Psi_0), \nu, L, \delta_0, c_0, c_1)$.

Step 3: Proof of (2.5.23)**.** By applying Lemma 2.4.1 to $\Phi \equiv \Psi_0$ with $d_0 \equiv 1$, we see that there exists $\theta_0 \equiv \theta_0(n, s(\Psi_0)) \in (0,1)$ such that

$$\begin{aligned} &\fint_{B_R} \Psi_0 \left(\left| \frac{v-h}{R} \right| \right) \, dx \leqslant c \left(\fint_{B_R} [\Psi_0(|Dv - Dh|)]^{\theta_0} \, dx \right)^{\frac{1}{\theta_0}} \\ &\leqslant c \left(\fint_{B_R} \left([\Psi_0(|Dv| + |Dh|)]^{\frac{1}{2}} \frac{|Dv - Dh|}{(|Dv| + |Dh|)} \right)^{\theta_0} [\Psi_0(|Dv| + |Dh|)]^{\frac{\theta_0}{2}} \, dx \right)^{\frac{1}{\theta_0}} \end{aligned}$$

$$
\begin{aligned}
&\leqslant c\left(\fint_{B_R} \Psi_0(|Dv|+|Dh|)\frac{|Dv-Dh|^2}{(|Dv|+|Dh|)^2}\,dx\right)^{\frac{1}{2}}\\
&\quad\times\left(\fint_{B_R} [\Psi_0(|Dv|+|Dh|)]^{\frac{\theta_0}{2-\theta_0}}\,dx\right)^{\frac{2-\theta_0}{2\theta_0}}\\
&\leqslant c\left(\fint_{B_R} Z^2\,dx\right)^{\frac{1}{2}}\left(\fint_{B_R} \Psi_0(|Dv|+|Dh|)\,dx\right)^{\frac{1}{2}}\\
&\leqslant c\sigma^{\frac{s_1}{2}} = c\sigma^{s_0}
\end{aligned}
$$

for some $c \equiv c(n, s(\Psi_0), \nu, L, \delta_1, c_0, c_1)$, where in the last display we have applied Hölder's inequality with conjugate exponents $\left(\frac{2}{\theta_0}, \frac{2}{2-\theta_0}\right)$, and finally used (2.1.10) with (2.5.36). This proves (2.5.23). The proof is complete. $\square$

Chapter 3

Regularity of minima of Orlicz phase functionals

3.1 Hypotheses and Main results

It has been known that the assumption (1.0.11) is not enough already in the special case of $G(t) \equiv t^p$ for $p > 1$ together with $a(\cdot) \equiv 0$ and $b(\cdot) \equiv 0$ for obtaining higher regularity of minima of the functional $\mathcal{F}$ in (1.0.10). In this regard, we consider the energy density F in (1.0.10) of type

$$F(x,y,z) := F_G(x,y,z) + a(x)F_{H_a}(x,y,z) + b(x)F_{H_b}(x,y,z) \tag{3.1.1}$$

for every $x \in \Omega$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^n$, where $F_G(\cdot)$, $F_{H_a}(\cdot)$ and $F_{H_b}(\cdot)$ are continuous functions belonging to $C^2(\mathbb{R}^n \setminus \{0\})$ with respect to z-variable and satisfying the following structure assumptions with fixed constants $0 < \nu \leqslant L$:

$$\begin{cases} |D_z F_\Phi(x,y,z)||z| + |D^2_{zz} F_\Phi(x,y,z)||z|^2 \leqslant L\Phi(|z|), \\ \nu \dfrac{\Phi(|z|)}{|z|^2}|\xi|^2 \leqslant \left\langle D^2_{zz} F_\Phi(x,y,z)\xi, \xi \right\rangle, \\ |D_z F_\Phi(x_1,y,z) - D_z F_\Phi(x_2,y,z)||z| \leqslant L\omega(|x_1 - x_2|)\Phi(|z|), \\ |F_\Phi(x,y_1,z) - F_\Phi(x,y_2,z)| \leqslant L\omega(|y_1 - y_2|)\Phi(|z|) \end{cases} \tag{3.1.2}$$

for every $\Phi \in \{G, H_a, H_b\}$, whenever $x, x_1, x_2 \in \Omega$, $y, y_1, y_2 \in \mathbb{R}$, $z \in \mathbb{R}^n \setminus \{0\}$, $\xi \in \mathbb{R}^n$, here either

$$\omega(t) = \min\{t^\mu, 1\} \text{ with some } \mu \in (0,1) \text{ for all } t \geqslant 0 \tag{3.1.3}$$

or

$$\omega : [0, +\infty) \to [0, +\infty) \text{ is concave such that } \omega(0) = 0 \text{ and } \omega(\cdot) \leqslant 1. \quad (3.1.4)$$

The structure conditions in (3.1.2) are satisfied for instance by the model functional

$$W^{1,1}(\Omega) \ni v \mapsto \int_\Omega f(x, v)\Psi(x, |Dv|)\, dx,$$

where $0 < \nu_1 \leqslant f(x, y) \leqslant L_1$ for some constants ν_1, L_1 and for some suitable continuous function $f(\cdot)$ satisfying the following inequality

$$|f(x_1, y_1) - f(x_2, y_2)| \leqslant L\omega(|x_1 - x_2| + |y_1 - y_2|)$$

whenever $x_1, x_2 \in \mathbb{R}^n$ and $y_1, y_2 \in \mathbb{R}$, where ω is the same as defined in (3.1.3) or (3.1.4). We also remark that those general functionals mentioned above have not been considered in the present literature for the regularity theory as far as we are concerned, moreover the functionals in (1.0.10) with structure assumptions (1.0.11) and (3.1.2) is not differentiable with respect to the second variable and so it can not be treated by its Euler-Lagrange equation.

Let us now formulate the monotonicity properties of the vector field $D_zF(x, y, z)$ with respect to the gradient variable z and some growth properties of the integrand F defined in (1.0.10) in terms of the maps introduced in (2.1.8) .

Lemma 3.1.1. *Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function defined in* (3.1.1) *satisfying* (1.0.11) *and* (3.1.2)*.Then there exist positive constants $c_1, c_2 \equiv c_1, c_2(n, s(G), s(H_a), s(H_b), \nu)$ and $c_3 \equiv c_3(n, s(G), s(H_a), s(H_b), L)$ such that the following inequalities*

$$\begin{aligned} &|V_\Psi(x, z_1) - V_\Psi(x, z_2)|^2 \\ &\leqslant c_1 \langle D_zF(x, y, z_1) - D_zF(x, y, z_2), z_1 - z_2\rangle, \end{aligned} \quad (3.1.5)$$

$$\begin{aligned} &|V_\Psi(x, z_1) - V_\Psi(x, z_2)|^2 + c_2 \langle D_zF(x, y, z_1), z_2 - z_1\rangle \\ &\leqslant c_2[F(x, y, z_2) - F(x, y, z_1)] \end{aligned} \quad (3.1.6)$$

and

$$\begin{aligned}
&|F(x_1,y,z)-F(x_2,y,z)| \\
&\leqslant c_3\omega(|x_1-x_2|)\,[G(|z|)+\min\{a(x_1),a(x_2)\}H_a(|z|) \\
&\quad +\min\{b(x_1),b(x_2)\}H_b(|z|)] \\
&\quad + c_3|a(x_1)-a(x_2)|H_a(|z|)+c_3|b(x_1)-b(x_2)|H_b(|z|)
\end{aligned} \tag{3.1.7}$$

hold true, whenever $z,z_1,z_2\in\mathbb{R}^n\setminus\{0\}$, $x,x_1,x_2\in\Omega$ *and* $y\in\mathbb{R}$.

Proof. It follows from $(3.1.2)_2$ that

$$\begin{aligned}
&\langle D_zF(x,y,z_1)-D_zF(x,y,z_2),z_1-z_2\rangle \\
&=\int_0^1 \left\langle D^2_{zz}F(x,y,\theta z_1+(1-\theta)z_2)[z_1-z_2],z_1-z_2\right\rangle d\theta \\
&\geqslant \nu\int_0^1 \frac{\Psi\left(x,\theta z_1+(1-\theta)z_2\right)}{|\theta z_1+(1-\theta)z_2|^2}|z_1-z_2|^2\,d\theta \\
&\geqslant c|V_\Psi(x,z_1)-V_\Psi(x,z_2)|^2,
\end{aligned}$$

where in the last inequality of the last display we have used (2.1.9) and (2.1.10). Then (3.1.5) follows. The inequality (3.1.6) follows from the following observation that

$$\begin{aligned}
&[F(x,y,z_2)-F(x,y,z_1)]-\langle D_zF(x,y,z_1),z_2-z_1\rangle \\
&=\int_0^1 \langle D_zF(x,y,\theta z_2+(1-\theta)z_1)-D_zF(x,y,z_1),z_2-z_1\rangle\,d\theta \\
&\overset{(3.1.5)}{\geqslant} c\int_0^1 \frac{1}{\theta}|V_\Psi(x,\theta z_2+(1-\theta)z_1)-V_\Psi(x,z_1)|^2\,d\theta \\
&\overset{(2.1.13)}{\geqslant} c|V_\Psi(x,z_1)-V_\Psi(x,z_2)|^2.
\end{aligned}$$

Since $F(x,y,0)=0$ for every $x\in\Omega$ and $y\in\mathbb{R}$, using (3.1.1), we have

$$|F(x_1,y,z)-F(x_2,y,z)|$$

$$
\begin{aligned}
&= |(F(x_1,y,z) - F(x_1,y,0)) - (F(x_2,y,z) - F(x_2,y,0))| \\
&= \left| \int_0^1 \langle D_z F(x_1,y,\theta z), z\rangle \, d\theta - \int_0^1 \langle D_z F(x_2,y,\theta z), z\rangle \, d\theta \right| \\
&\leqslant \int_0^1 |D_z F(x_1,y,\theta z) - D_z F(x_2,y,\theta z)| \, |z| d\theta \\
&\leqslant \int_0^1 |D_z F_G(x_1,y,\theta z) - D_z F_G(x_2,y,\theta z)| \, |z| \, d\theta \\
&\quad + \int_0^1 |a(x_1) D_z F_{H_a}(x_1,y,\theta z) - a(x_2) D_z F_{H_a}(x_2,y,\theta z)| \, |z| \, d\theta \\
&\quad + \int_0^1 |b(x_1) D_z F_{H_b}(x_1,y,\theta z) - b(x_2) D_z F_{H_b}(x_2,y,\theta z)| \, |z| \, d\theta.
\end{aligned}
$$

Without loss of generality, we can assume $a(x_2) \leqslant a(x_1)$ and $b(x_2) \leqslant b(x_1)$. Then using the structure assumption (3.1.2), we find

$$
\begin{aligned}
&\int_0^1 |a(x_1) D_z F_{H_a}(x_1,y,\theta z) - a(x_2) D_z F_{H_a}(x_2,y,\theta z)| \, |z| \, d\theta \\
&\qquad \leqslant L|a(x_1) - a(x_2)| \int_0^1 \frac{H_a(\theta|z|)}{\theta} \, d\theta + a(x_2)\omega(|x_1 - x_2|) \int_0^1 \frac{H_a(\theta|z|)}{\theta} \, d\theta \\
&\qquad \leqslant c a(x_2) H_a(|z|) + c\omega(|x_1 - x_2|) H_a(|z|)
\end{aligned}
$$

for some constant $c \equiv c(s(H_a), L)$. Similarly, we get

$$
\begin{aligned}
&\int_0^1 |b(x_1) D_z F_{H_b}(x_1,y,\theta z) - b(x_2) D_z F_{H_b}(x_2,y,\theta z)| \, |z| \, d\theta \\
&\qquad \leqslant c b(x_2) H_b(|z|) + c\omega(|x_1 - x_2|) H_b(|z|),
\end{aligned}
$$

where the validity of the last display is ensured by $(3.1.2)_3$. Combining the last three displays, (3.1.7) follows. □

In order to shorten the notations in this chapter, for a given local minimizer u of the functional $\mathcal{F}$ in (1.0.10), we shall use a set of various basic parameters which is "data of the problem" depending on which assumption of (1.0.13)-(1.0.15) is considered as follows:

$$\mathbf{data} \equiv \begin{cases} \Big\{n, \lambda_1, s(G), s(H_a), s(H_b), \nu, L, \|a\|_{C^{\omega_a}(\Omega)}, \|b\|_{C^{\omega_b}(\Omega)}, \omega(\cdot), \\ \|\Psi(x,|Du|)\|_{L^1(\Omega)}, \|u\|_{L^1(\Omega)}, \omega_a(1), \omega_b(1)\Big\} \\ \text{if (1.0.13) is considered,} \\ \Big\{n, \lambda_2, s(G), s(H_a), s(H_b), \nu, L, \|a\|_{C^{\omega_a}(\Omega)}, \|b\|_{C^{\omega_b}(\Omega)}, \omega(\cdot), \\ \|u\|_{L^\infty(\Omega)}, \omega_a(1), \omega_b(1)\Big\} \\ \text{if (1.0.14) is considered,} \\ \Big\{n, \lambda_3, s(G), s(H_a), s(H_b), \nu, L, \|a\|_{C^{\omega_a}(\Omega)}, \|b\|_{C^{\omega_b}(\Omega)}, \omega(\cdot), \\ [u]_{0,\gamma}, \omega_a(1), \omega_b(1)\} \\ \text{if (1.0.15) is considered,} \end{cases} \tag{3.1.8}$$

where λ_1, λ_2, λ_3 are the same numbers as defined in (1.0.13)-(1.0.15) and $s(G), s(H_a), s(H_b)$ are indices of the functions G, H_a, H_b in the sense of Definition 2.1.1, respectively. For a given local Q-minimizer u of the functional $\mathcal{P}$, **data** is understood by the above set of parameters with the constants L, ν having been replaced by Q in any case of (1.0.13)-(1.0.15) into the consideration. With $\Omega_0 \Subset \Omega$ being a fixed open subset, we also denote by $\mathbf{data}(\Omega_0)$ the set of parameters in (3.1.8) together with $\operatorname{dist}(\Omega_0, \partial\Omega)$ under one of the assumptions (1.0.13)-(1.0.15):

$$\mathbf{data}(\Omega_0) \equiv \mathbf{data}, \operatorname{dist}(\Omega_0, \partial\Omega). \tag{3.1.9}$$

Now we are ready to state our main results in this chapter.

Theorem 3.1.1 (Maximal regularity). *Let $u \in W^{1,\Psi}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ defined in* (1.0.10)*, under the assumptions* (1.0.11)*,* (3.1.2) *and* (3.1.3)*. Suppose that $\omega_a(\rho) = \rho^\alpha$ and $\omega_b(\rho) = \rho^\beta$ for some*

$\alpha, \beta \in (0,1]$. *If one of the following assumptions*

$$\begin{cases} (1.0.13), & (3.1.10a) \\ (1.0.14), & (3.1.10b) \\ (1.0.15) \quad \textit{with} \quad \limsup_{\rho \to 0^+} \Lambda\left(\rho^{\frac{1}{1-\gamma}}, \frac{1}{\rho}\right) = 0 & (3.1.10c) \end{cases}$$

is satisfied, then there exists $\theta \in (0,1)$ *depending only on* $n, s(G), s(H_a), s(H_b), \nu, L, \alpha, \beta$ *and* μ *such that* $Du \in C^{0,\theta}_{\mathrm{loc}}(\Omega)$.

Theorem 3.1.2 (Morrey decay)**.** *Let* $u \in W^{1,\Psi}(\Omega)$ *be a local minimizer of the functional* $\mathcal{F}$ *defined in* (1.0.10)*, under the assumptions* (1.0.11), (3.1.2) *and* (3.1.4)*. If one of the following assumptions*

$$\begin{cases} (1.0.13) \quad \textit{with} \quad \limsup_{\rho \to 0^+} \Lambda\left(\rho, G^{-1}(\rho^{-n})\right) = 0, & (3.1.11a) \\ (1.0.14) \quad \textit{with} \quad \limsup_{\rho \to 0^+} \Lambda\left(\rho, \frac{1}{\rho}\right) = 0, & (3.1.11b) \\ (1.0.15) \quad \textit{with} \quad \limsup_{\rho \to 0^+} \Lambda\left(\rho^{\frac{1}{1-\gamma}}, \frac{1}{\rho}\right) = 0, & (3.1.11c) \\ (1.0.13) \quad \textit{with} \quad \omega_a(\rho) = \rho^{\alpha} \quad \textit{and} \quad \omega_b(\rho) = \rho^{\beta} & \\ \qquad\qquad\qquad \textit{for some} \quad \alpha, \beta \in (0,1], & (3.1.11d) \\ (1.0.14) \quad \textit{with} \quad \omega_a(\rho) = \rho^{\alpha} \quad \textit{and} \quad \omega_b(\rho) = \rho^{\beta} & \\ \qquad\qquad\qquad \textit{for some} \quad \alpha, \beta \in (0,1] & (3.1.11e) \end{cases}$$

is satisfied, then

$$u \in C^{0,\theta}_{\mathrm{loc}}(\Omega) \quad \textit{for every} \quad \theta \in (0,1). \tag{3.1.12}$$

Moreover, for every $\sigma \in (0,n)$*, there exists a positive constant* $c \equiv c(\boldsymbol{data}(\Omega_0), \sigma)$ *such that the decay estimate*

$$\int_{B_\rho} \Psi(x, |Du|)\, dx \leqslant c \left(\frac{\rho}{R}\right)^{n-\sigma} \int_{B_R} \Psi(x, |Du|)\, dx \tag{3.1.13}$$

holds for every concentric balls $B_\rho \subset B_R \subset \Omega_0 \Subset \Omega$ *with* $R \leqslant 1$*.*

Remark 3.1.1. We would like to point out, in the same spirit as this chapter, the results of Theorem 3.1.1 and Theorem 3.1.2 can be restated and proved for the functional having a finite number of phases with replacing the function in (1.0.2) by

$$\Psi(x,t) := G(t) + \sum_{i=1}^{N} a_i(x) H_i(t), \quad m \geqslant 1, \tag{3.1.14}$$

where $G, H_i \in \mathcal{N}$ in the sense of the Definition 2.1.1 and $a_i(\cdot) \in C^{\omega_i}(\Omega)$ with $\omega_i : [0,\infty) \to [0,\infty)$ being a continuous and concave function vanishing at the origin for every $i \in \{1,\ldots,N\}$. Under this setting we replace the function in (1.0.12) by

$$\Lambda(\rho,t) := \sum_{i=1}^{N} \frac{\omega_i(\rho)}{1+\omega_i(\rho)} \frac{H_i(t)}{G(t)} \quad \text{for every} \quad \rho, t > 0. \tag{3.1.15}$$

The coefficient functions in Theorem 3.1.1 along with (3.1.11d) and (3.1.11e) in Theorem 3.1.2 are understood by letting $\omega_i(\rho) = \rho^{\alpha_i}$ with some $\alpha_i \in (0,1]$ for every $i \in \{1,\ldots,N\}$.

3.2 Basic regularity results

We start this section by stating the following Caccioppoli inequality as a fundamental result for the further investigations. In what follows let $Q = L/\nu$ for the convenience in the future, but in general it could be any number larger than one.

Lemma 3.2.1 (Caccioppoli Inequality)**.** *Let $u \in W^{1,\Psi}(\Omega)$ be a local Q-minimizer of the functional $\mathcal{P}$ defined in* (1.0.1) *with $G, H_a, H_b \in \mathcal{N}$ and $0 \leqslant a(\cdot), b(\cdot) \in L^\infty(\Omega)$. Then there exists a constant $c \equiv c(n, s(G), s(H_a), s(H_b), Q)$ such that the following Caccioppoli inequality*

$$\int_{B_\rho} \Psi(x, |D(u-k)_\pm|)\,dx \leqslant c \int_{B_R} \Psi\left(x, \frac{(u-k)_\pm}{R-\rho}\right) dx \tag{3.2.1}$$

holds, whenever $B_\rho \Subset B_R \subset \Omega$ are concentric balls and $k \in \mathbb{R}$.

Proof. The proof is elementary as done for [39, Lemma 4.6]. The only difference lies in that we have an additional one phase. But the inequality (3.2.1) is still valid since $H_b \in \mathcal{N}$ with an index $s(H_b) \geqslant 1$. $\square$

Remark 3.2.1. As a direct consequence of Lemma 3.2.1, with $u \in W^{1,\Psi}(B_R)$ being a local Q-minimizer of the functional $\mathcal{P}$ defined in (1.0.1) under the assumptions of Lemma 3.2.1, there exists a constant $c \equiv c(n, s(G), s(H_a), s(H_b), Q)$ such that

$$\fint_{B_{R/2}} \Psi(x, |Du|)\, dx \leqslant c \fint_{B_R} \Psi\left(x, \left|\frac{u-(u)_{B_R}}{R}\right|\right) dx$$

holds, whenever $B_R \subset \Omega$ is a ball.

3.2.1 Local boundedness

Now we focus on local boundedness of a local Q-minimizer u of the functional $\mathcal{P}$ defined in (1.0.1) with obtaining precise estimates under the assumption $(1.0.13)_2$.

Theorem 3.2.1. *Let $u \in W^{1,\Psi}(\Omega)$ be a local Q-minimizer of the functional $\mathcal{P}$ defined in* (1.0.1) *under the assumption* (1.0.13)*. Then there exists a constant $c \equiv c(\boldsymbol{data})$ such that*

$$\left\|\Psi^-_{B_R}\left(\left|\frac{(u-(u)_{B_R})_\pm}{R}\right|\right)\right\|_{L^\infty(B_{R/2})} \leqslant c \fint_{B_R} \Psi\left(x, \left|\frac{(u-(u)_{B_R})_\pm}{R}\right|\right) dx \quad (3.2.2)$$

and

$$\Psi^-_{B_R}\left(\left|\frac{u(x_1)-u(x_2)}{R}\right|\right) \leqslant c \fint_{B_R} \Psi\left(x, |Du|\right) dx \quad \text{for a.e} \quad x_1, x_2 \in B_{R/2}, \quad (3.2.3)$$

whenever $B_R \equiv B_R(x_0) \subset \Omega$ is a ball with $R \leqslant 1$. In particular, $u \in L^\infty_{\mathrm{loc}}(\Omega)$.

Proof. Let us consider the following scaling:

$$\bar{u}(x) := \frac{u(x_0+Rx)-(u)_{B_R}}{R}, \quad \bar{a}(x) := a(x_0+Rx), \quad \bar{b}(x) := b(x_0+Rx),$$

$$
\begin{aligned}
&\bar{\Psi}(x,t) := G(t) + \bar{a}(x)H_a(t) + \bar{b}(x)H_b(t),\\
&\bar{A}(k,s) := B_s(0)\cap\{\bar{u}>k\} \quad \text{and} \quad \bar{B}(k,s) := B_s(0)\cap\{\bar{u}<k\}
\end{aligned}
\tag{3.2.4}
$$

for every $x \in B_1(0)$, $t \geqslant 0$, $s \in (0,1)$ and $k \in \mathbb{R}$. The rest of the proof falls in 3 steps.

Step 1: Sobolev-Poincaré inequality under the scaling. Before going on further, let us consider a Sobolev-Poincaré type inequality under the new scaling introduced in (3.2.4). So we prove that there exists a positive exponent $\theta \equiv \theta(n, s(G), s(H_a), s(H_b)) \in (0,1)$ such that

$$
\int_{B_1} \bar{\Psi}(x,|f|)\,dx \leqslant c\bar{k}_{sp}\left(\int_{B_1}[\bar{\Psi}(x,|Df|)]^{\theta}\,dx\right)^{\frac{1}{\theta}} \tag{3.2.5}
$$

for some constant $c \equiv c(n, s(G), s(H_a), s(H_b), \omega_a(1), \omega_b(1))$, whenever $f \in W_0^{1,\bar{\Psi}}(B_1)$, where

$$
\bar{\kappa}_{sp} = 1 + ([a]_{\omega_a} + [b]_{\omega_b})\left(\lambda_1 + \lambda_1 R\left(\int_{B_1} G(|Df|)\,dx\right)^{\frac{1}{n}}\right). \tag{3.2.6}
$$

Essentially, the proof of the inequality (3.2.5) comes from a careful revealing of the arguments used in (2.4.7)-(2.4.9). So using continuity properties of $\bar{a}(\cdot)$ and $\bar{b}(\cdot)$, we see

$$
\begin{aligned}
I :=& \int_{B_1} \bar{\Psi}(x,|f|)\,dx \leqslant 2[a]_{\omega_a}\omega_a(R)\int_{B_1} H_a(|f|)\,dx\\
&+ 2[b]_{\omega_b}\omega_b(R)\int_{B_1} H_b(|f|)\,dx + \int_{B_1} \bar{\Psi}^{-}_{B_1}(|f|)\,dx\\
&: 2[a]_{\omega_a}I_1 + 2[b]_{\omega_b}I_2 + I_3,
\end{aligned}
\tag{3.2.7}
$$

where

$$
\bar{\Psi}^{-}_{B_1}(t) := G(t) + \inf_{x\in B_1}\bar{a}(x)H_a(t) + \inf_{x\in B_1}\bar{b}(x)H_b(t) \quad \text{for every} \quad t \geqslant 0. \tag{3.2.8}
$$

Now we estimate the terms I_i for $i \in \{1,2,3\}$ similarly as in the proof of Theorem 2.4.1. In turn, using the assumption $(1.0.13)_2$ and (2.1.2), we have

$$\begin{aligned}
I_1 &= \omega_a(R) \int\limits_{B_1} \frac{H_a(|f|)}{G(|f|)} G(|f|)\,dx \\
&\leqslant \lambda_1 \omega_a(R) \int\limits_{B_1} \left(1 + \left[\omega_a\left([G(|f|)]^{-\frac{1}{n}}\right)\right]^{-1}\right) G(|f|)\,dx \\
&\leqslant \lambda_1 \omega_a(R) \int\limits_{B_1} \left(1 + \left[\frac{1}{\omega_a(R)} + \frac{R}{\omega_a(R)} [G(|f|)]^{\frac{1}{n}}\right]\right) G(|f|)\,dx \\
&\leqslant \lambda_1 (1 + \omega_a(1)) \int\limits_{B_1} G(|f|)\,dx + 2\lambda_1 R \int\limits_{B_1} [G(|f|)]^{1+\frac{1}{n}}\,dx. \qquad (3.2.9)
\end{aligned}$$

In a similar manner, we find

$$I_2 \leqslant \lambda_1 (1 + \omega_b(1)) \int\limits_{B_1} G(|f|)\,dx + \lambda_1 R \int\limits_{B_1} [G(|f|)]^{1+\frac{1}{n}}\,dx. \qquad (3.2.10)$$

Inserting the estimates (3.2.9)-(3.2.10) into (3.2.8), the inequality (3.2.5) follows from the similar arguments used in (2.4.9)-(2.4.10) and Lemma 2.4.1.

Step 2. Proof of (3.2.2)**.** Since $u - (u)_{B_R}$ is a local Q-minimizer of the functional $\mathcal{P}$, we use a Caccioppoli inequality of Lemma 3.2.1 to see that

$$\int\limits_{B_t} \bar{\Psi}(x, |D(\bar{u} - k)_\pm|)\,dx \leqslant c \int\limits_{B_s} \bar{\Psi}\left(x, \frac{(\bar{u} - k)_\pm}{s - t}\right) dx \qquad (3.2.11)$$

holds for some constant $c \equiv c(s(G), s(H_a), s(H_b), Q)$, whenever $0 < t < s \leqslant 1$ and $k \in \mathbb{R}$. Let us now consider the concentric balls $B_\rho \Subset B_t \Subset B_s$ with $1/2 \leqslant \rho < s \leqslant 1$ and $t := (\rho + s)/2$. Let $\eta \in C_0^\infty(B_t)$ be a standard cut-off function such that $\chi_{B_\rho} \leqslant \eta \leqslant \chi_{B_t}$ and $|D\eta| \leqslant \dfrac{2}{t - \rho} = \dfrac{4}{s - \rho}$. Now we apply inequality (3.2.5) from Step 1 above in order to have a positive exponent

$\theta \equiv \theta(n, s(G), s(H_a), s(H_b))$ such that

$$\begin{aligned}\int_{\bar{A}(k,\rho)} \bar{\Psi}(x, \bar{u}-k)\,dx &\leqslant \int_{B_1} \bar{\Psi}(x, \eta(\bar{u}-k)_+)\,dx \\ &\leqslant c\bar{k}_{sp}\left(\int_{B_1} \left[\bar{\Psi}(x, |D(\eta(\bar{u}-k)_+)|)\right]^{\theta}\,dx\right)^{\frac{1}{\theta}} \qquad (3.2.12)\end{aligned}$$

for some constant $c \equiv c(n, s(G), s(H_a), s(H_b), \omega_a(1), \omega_b(1))$, where

$$\bar{k}_{sp} = 1 + ([a]_{\omega_a} + [b]_{\omega_b})\left(\lambda_1 + \lambda_1 R\left(\int_{B_1} G(|D(\eta(\bar{u}-k)_+)|)\,dx\right)^{\frac{1}{n}}\right). \qquad (3.2.13)$$

By scaling back and using Lemma 2.1.1, for any $k \geqslant 0$, we have

$$\begin{aligned}\bar{\kappa}_{sp} &\leqslant c\left[1 + R\left(\fint_{B_R} G(|Du|)\,dx\right)^{\frac{1}{n}}\right. \\ &\qquad \left. + \frac{R}{(s-\rho)^{\frac{s(G)+1}{n}}}\left(\fint_{B_R} G\left(\left|\frac{u-(u)_{B_R}}{R}\right|\right)\,dx\right)^{\frac{1}{n}}\right] \\ &\leqslant \frac{c}{(s-\rho)^{s(G)+1}}\left[1 + \left(\int_{B_R} G(|Du|)\,dx\right)^{\frac{1}{n}}\right] \qquad (3.2.14)\end{aligned}$$

with a constant $c \equiv c(n, \lambda_1, [a]_{\omega_a} + [b]_{\omega_b})$, where we have also used Lemma 2.4.1 to $\Phi \equiv G$ for $d_0 \equiv 1$.

Then, inserting the last estimate into (3.2.12) and applying Hölder inequality together with (3.2.11) yield that

$$\int_{\bar{A}(k,\rho)} \bar{\Psi}(x, \bar{u}-k)\,dx \leqslant c\frac{1}{(s-\rho)^{1+s(G)}}|\bar{A}(k,t)|^{\frac{1-\theta}{\theta}}$$

$$
\begin{aligned}
&\times \int\limits_{\bar{A}(k,t)} \left(\bar{\Psi}(x, |D\bar{u}|) + \bar{\Psi}\left(x, \frac{\bar{u}-k}{s-\rho}\right)\right) dx \\
&\leqslant c \frac{1}{(s-\rho)^{1+s(G)}} |\bar{A}(k,s)|^{\frac{1-\theta}{\theta}} \int\limits_{\bar{A}(k,s)} \bar{\Psi}\left(x, \frac{\bar{u}-k}{s-\rho}\right) dx
\end{aligned}
\tag{3.2.15}
$$

holds with some constant $c \equiv c(\mathbf{data})$, where in the last display we have also used (2.1.6). By the definition of $\bar{A}$ in (3.2.4), we observe

$$
|\bar{A}(k,s)| \leqslant \int\limits_{\bar{A}(h,s)} \frac{\bar{\Psi}(x, \bar{u}-h)}{\bar{\Psi}(x, k-h)} dx \leqslant \frac{1}{\bar{\Psi}^-_{B_1}(k-h)} \int\limits_{\bar{A}(h,s)} \bar{\Psi}(x, \bar{u}-h)\, dx
$$

and

$$
\int\limits_{\bar{A}(k,s)} \bar{\Psi}(x, \bar{u}-k)\, dx \leqslant \int\limits_{\bar{A}(h,s)} \bar{\Psi}(x, \bar{u}-h)\, dx
$$

for any $h < k$. Putting the last two inequalities into (3.2.15) and applying Lemma 2.1.1, we have the following inequality:

$$
\begin{aligned}
\int\limits_{\tilde{A}(k,\rho)} \bar{\Psi}(x, \bar{u}-k)\, dx &\leqslant \frac{c}{[\bar{\Psi}^-_{B_1}(k-h)]^{\frac{1-\theta}{\theta}} (s-\rho)^{2(\max\{s(G), s(H_a), s(H_b)\}+1)}} \\
&\quad \times \left(\int\limits_{\bar{A}(h,s)} \bar{\Psi}(x, \bar{u}-h)\, dx \right)^{\frac{1}{\theta}}.
\end{aligned}
\tag{3.2.16}
$$

Now we set sequences of numbers as follows:

$$
\begin{aligned}
&\rho_i := \frac{1}{2}\left(1 + \frac{1}{2^i}\right), \qquad k_i := 2l_0 \left(1 - \frac{1}{2^{i+1}}\right) \\
&\text{and } M_i := \frac{1}{\bar{\Psi}^-_{B_1}(l_0)} \int\limits_{\bar{A}(k_i,\rho_i)} \bar{\Psi}(x, \bar{u}-k_i)\, dx
\end{aligned}
$$

for any integer $i \geqslant 0$ and some number $l_0 > 0$ to be chosen in a few lines.

Then applying (3.2.16) with the choices $k \equiv k_{i+1}$, $h \equiv k_i$, $\rho \equiv \rho_{i+1}$ and $s \equiv \rho_i$, we have, for every $i \geqslant 0$,

$$\begin{aligned} M_{i+1} &\leqslant \frac{c}{\left[\bar{\Psi}^-_{B_1}\left(\frac{l_0}{2^{i+1}}\right)\right]^{\frac{1-\theta}{\theta}}\left(\frac{1}{4^{i+2}}\right)^{\max\{s(G),s(H_a),s(H_b)\}+1}}\left[\bar{\Psi}^-_{B_1}(l_0)\right]^{\frac{1-\theta}{\theta}} M_i^{\frac{1}{\theta}} \\ &\leqslant c_0\left[4^{(\max\{s(G),s(H_a),s(H_b)\}+1)\frac{1}{\theta}}\right]^i M_i^{1+\frac{1-\theta}{\theta}} \end{aligned}$$

with $c_0 \equiv c_0(\mathbf{data})$, where in the last inequality of the last display we have used again Lemma 2.1.1. Now it's turn to apply a standard iteration of Lemma 2.2.2, which means that if

$$\frac{1}{\bar{\Psi}^-_{B_1}(l_0)}\int\limits_{\bar{A}(l_0,1)} \bar{\Psi}(x, \bar{u}-l_0)\,dx = M_0 \leqslant c_0^{-\frac{\theta}{1-\theta}} 4^{-(\max\{s(G),s(H_a),s(H_b)\}+1)\frac{\theta}{(1-\theta)^2}},$$

then we obtain

$$\|\bar{u}_+\|_{L^\infty(B_{1/2})} \leqslant 2l_0.$$

Consequently, choosing $l_0 > 0$ in such a way that

$$\bar{\Psi}^-_{B_1}(l_0) = c_0^{\frac{\theta}{1-\theta}} 4^{(\max\{s(G),s(H_a),s(H_b)\}+1)\frac{\theta}{(1-\theta)^2}} \int\limits_{B_1} \bar{\Psi}(x, \bar{u}_+)\,dx,$$

we have

$$\left\|\bar{\Psi}^-_{B_1}(\bar{u}_+)\right\|_{L^\infty(B_{1/2})} \leqslant c\int\limits_{B_1} \bar{\Psi}(x, \bar{u}_+)\,dx,$$

which implies that

$$\left\|\Psi^-_{B_R}\left(\frac{(u-(u)_{B_R})_+}{R}\right)\right\|_{L^\infty(B_{R/2})} \leqslant c \fint\limits_{B_R} \Psi\left(x, \frac{(u-(u)_{B_R})_+}{R}\right)\,dx$$

holds with $c \equiv c(\mathbf{data})$. Repeating the same argument for $-u$, which is also a local Q-minimizer of the functional $\mathcal{P}$ defined in (1.0.10), the last inequality holds with $(u-(u)_{B_R})_+$ replaced by $(u-(u)_{B_R})_-$.

Step 3. Proof of (3.2.3)**.** Using (3.2.2) and (2.1.6), for a.e $x_1, x_2 \in B_{R/2}$, we have

$$\begin{aligned}\Psi^-_{B_R}\left(\left|\frac{u(x_1)-u(x_2)}{R}\right|\right) &\leqslant c\Psi^-_{B_R}\left(\left|\frac{u(x_1)-(u)_{B_R}}{R}\right|\right) \\ &\quad + c\Psi^-_{B_R}\left(\left|\frac{u(x_2)-(u)_{B_R}}{R}\right|\right) \\ &\leqslant c \fint_{B_R} \Psi\left(x,\left|\frac{u-(u)_{B_R}}{R}\right|\right)dx \leqslant c\fint_{B_R}\Psi\left(x,|Du|\right)dx\end{aligned}$$

for some constant $c \equiv c(\mathbf{data})$, where in the last inequality of the above display we have used a Sobolev-Poincaré type inequality of Theorem 2.4.1. Clearly, the last display implies $u \in L^\infty_{\mathrm{loc}}(\Omega)$. The proof is complete. □

3.2.2 Almost standard Caccioppoli inequality

Now we present the primary results, the so-called almost standard Caccioppoli type inequality, for proving Hölder continuity of a local Q-minimizer of the functional $\mathcal{P}$.

Lemma 3.2.2 (Almost standard Caccioppoli inequality)**.** *Let $u \in W^{1,\Psi}(\Omega)$ be a local Q-minimizer of the functional $\mathcal{P}$ defined in* (1.0.1) *under one of the assumptions* (1.0.13), (1.0.14) *and* (1.0.15)*. Let $B_{2R} \equiv B_{2R}(x_0) \subset \Omega$ be a ball with $R \leqslant 1$. Then there exists a constant $c \equiv c(\boldsymbol{data})$ such that*

$$\begin{aligned}\int_{B_{R_1}} \Psi^-_{B_R}\left(|D(u-k)_\pm|\right)dx &\leqslant \int_{B_{R_1}} \Psi\left(x,|D(u-k)_\pm|\right)dx \\ &\leqslant c\left(\frac{R}{R_2-R_1}\right)^{s(\Psi)+1}\int_{B_{R_2}}\Psi^-_{B_R}\left(\frac{(u-k)_\pm}{R}\right)dx\end{aligned} \tag{3.2.17}$$

holds, whenever $B_{R_1} \Subset B_{R_2} \subset B_R(x_0)$ are concentric balls and $k \in \mathbb{R}$.

Proof. First we prove the inequality (3.2.17) for the values of $k \in \mathbb{R}$ with $\inf_{B_R} u \leqslant k \leqslant \sup_{B_R} u$, depending on which one of the assumptions (1.0.13)-(1.0.15) is in force. Firstly, by the definition of $\Psi^-_{B_R}$ in (2.1.3) and Lemma

3.2.1, we see

$$
\begin{aligned}
I &:= \int_{B_{R_1}} \Psi^-_{B_R}\left(|D(u-k)_\pm|\right) dx \\
&\leqslant \int_{B_{R_1}} \Psi\left(x, |D(u-k)_\pm|\right) dx \leqslant c_* \int_{B_{R_2}} \Psi\left(x, \frac{(u-k)_\pm}{R_2-R_1}\right) dx \\
&\leqslant c_*\omega_a(R) \int_{B_{R_2}} H_a\left(\frac{(u-k)_\pm}{R_2-R_1}\right) dx + c_*\omega_b(R) \int_{B_{R_2}} H_b\left(\frac{(u-k)_\pm}{R_2-R_1}\right) dx \\
&\quad + c_* \int_{B_{R_2}} \Psi^-_{B_R}\left(\frac{(u-k)_\pm}{R_2-R_1}\right) dx =: c_*\left(I_1+I_2+I_3\right)
\end{aligned}
\tag{3.2.18}
$$

for some constant $c_* \equiv c_*(n, s(G), s(H_a), s(H_b), Q, [a]_{\omega_a}, [b]_{\omega_b})$. Now we shall estimate each term I_i for $i \in \{1,2,3\}$ in the above display. Then using Lemma 2.1.1, the assumption $(1.0.13)_2$, (2.1.2) and (3.2.3) of Lemma 3.2.1, we see

$$
\begin{aligned}
I_1 &= \omega_a(R) \int_{B_{R_2}} \frac{H_a\left(\frac{(u-k)_\pm}{R_2-R_1}\right)}{G\left(\frac{(u-k)_\pm}{R_2-R_1}\right)} G\left(\frac{(u-k)_\pm}{R_2-R_1}\right) dx \\
&\leqslant \omega_a(R)\left(\frac{R}{R_2-R_1}\right)^{s(H_a)+1} \int_{B_{R_2}} \frac{\left(H_a\circ G^{-1}\right)\left(G\left(\frac{(u-k)_\pm}{R}\right)\right)}{G\left(\frac{(u-k)_\pm}{R}\right)} G\left(\frac{(u-k)_\pm}{R_2-R_1}\right) dx \\
&\leqslant \lambda_1\omega_a(R)\left(\frac{R}{R_2-R_1}\right)^{s(H_a)+1} \\
&\quad \times \int_{B_{R_2}} \left(1 + \left[\omega_a\left(\left[G\left(\frac{(u-k)_\pm}{R}\right)\right]^{-\frac{1}{n}}\right)\right]^{-1}\right) G\left(\frac{(u-k)_\pm}{R_2-R_1}\right) dx \\
&\leqslant c\,\omega_a(R)\left(\frac{R}{R_2-R_1}\right)^{s(H_a)+1} \\
&\quad \times \int_{B_{R_2}} \left[1 + \frac{1}{\omega_a(R)} + \frac{R}{\omega_a(R)}\left(G\left(\frac{\operatorname{osc}_{B_R} u}{R}\right)\right)^{\frac{1}{n}}\right] G\left(\frac{(u-k)_\pm}{R_2-R_1}\right) dx
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant c\left[1+\left(\int_{B_{2R}}\Psi(x,|Du|)\,dx\right)^{\frac{1}{n}}\right]\left(\frac{R}{R_2-R_1}\right)^{s(H_a)+1}\int_{B_{R_2}} G\left(\frac{(u-k)_\pm}{R_2-R_1}\right)dx\\
&\leqslant c\left(\frac{R}{R_2-R_1}\right)^{s(\Psi)+1}\int_{B_{R_2}} G\left(\frac{(u-k)_\pm}{R}\right)dx \qquad (3.2.19)
\end{aligned}
$$

for some constant $c \equiv c(\mathbf{data})$. In a totally similar way, it can be shown that

$$
I_2 \leqslant c\left(\frac{R}{R_2-R_1}\right)^{s(\Psi)+1}\int_{B_{R_2}} G\left(\frac{(u-k)_\pm}{R}\right)dx \qquad (3.2.20)
$$

with a constant $c \equiv c(\mathbf{data})$. Clearly, recalling Remark 2.1.2 and using Lemma 2.1.1, we have

$$
I_3 \leqslant \left(\frac{R}{R_2-R_1}\right)^{s(\Psi)+1}\int_{B_{R_2}} \Psi^-_{B_R}\left(\frac{(u-k)_\pm}{R}\right)dx \qquad (3.2.21)
$$

Inserting the estimates obtained in (3.2.19)-(3.2.21) into (3.2.18) and recalling the very definition of $\Psi^-_{B_R}$ in (2.1.3), we arrive at (3.2.17) under the assumption (1.0.13). The second part of the proof is to show (3.2.17) under the assumption (1.0.14). For this, we again estimate the terms I_i with $i \in \{1,2,3\}$ in (3.2.18). Applying Lemma 2.1.1, the assumption (1.0.14) and (2.1.2), we see

$$
\begin{aligned}
I_1 &= \omega_a(R)\int_{B_{R_2}}\left(\frac{H_a}{G}\right)\left(\frac{(u-k)_\pm}{R_2-R_1}\right)G\left(\frac{(u-k)_\pm}{R_2-R_1}\right)dx\\
&\leqslant \omega_a(R)\left(\frac{R}{R_2-R_1}\right)^{s(H_a)+1}\int_{B_{R_2}}\left(\frac{H_a}{G}\right)\left(\frac{(u-k)_\pm}{R}\right)G\left(\frac{(u-k)_\pm}{R_2-R_1}\right)dx\\
&\leqslant 2\lambda_2\omega_a(R)\left(\frac{R}{R_2-R_1}\right)^{s(H_a)+1}\\
&\quad\times\int_{B_{R_2}}\left(1+\left[\omega_a\left(\frac{R}{(u-k)_\pm}\right)\right]^{-1}\right)G\left(\frac{(u-k)_\pm}{R_2-R_1}\right)dx
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant c\lambda_2\omega_a(R)\left(\frac{R}{R_2-R_1}\right)^{s(H_a)+1} \\
&\quad\times\int\limits_{B_{R_2}}\left(1+\left[\frac{1}{\omega_a(R)}+\frac{\|u\|_{L^\infty(B_R)}}{\omega_a(R)}\right]\right)G\left(\frac{(u-k)_\pm}{R_2-R_1}\right)\,dx \\
&\leqslant c\left(\frac{R}{R_2-R_1}\right)^{s(\Psi)+1}\int\limits_{B_{R_2}}G\left(\frac{(u-k)_\pm}{R}\right)\,dx \qquad (3.2.22)
\end{aligned}
$$

for some constant $c \equiv c(\mathbf{data})$. Arguing similarly, we have

$$
I_2 \leqslant c\left(\frac{R}{R_2-R_1}\right)^{s(\Psi)+1}\int\limits_{B_{R_2}}G\left(\frac{(u-k)_\pm}{R}\right)\,dx \qquad (3.2.23)
$$

with a constant $c \equiv c(\mathbf{data})$. Plugging the estimates (3.2.21)-(3.2.23) into (3.2.18), we conclude with (3.2.17) under the assumption (1.0.14). Finally, the remaining part of the proof is to obtain the inequality (3.2.17) under the assumption (1.0.15). In fact, we continue to estimate the terms I_i with $i \in \{1,2,3\}$ in (3.2.18). Therefore, using the assumption (1.0.15) and (2.1.2), we find

$$
\begin{aligned}
I_1 &= \omega_a(R)\int\limits_{B_{R_2}}\left(\frac{H_a}{G}\right)\left(\frac{(u-k)_\pm}{R_2-R_1}\right)G\left(\frac{(u-k)_\pm}{R_2-R_1}\right)\,dx \\
&\leqslant \omega_a(R)\left(\frac{R}{R_2-R_1}\right)^{s(H_a)+1}\int\limits_{B_{R_2}}\left(\frac{H_a}{G}\right)\left(\frac{(u-k)_\pm}{R}\right)G\left(\frac{(u-k)_\pm}{R_2-R_1}\right)\,dx \\
&\leqslant 2\lambda_3\omega_a(R)\left(\frac{R}{R_2-R_1}\right)^{s(H_a)+1} \\
&\quad\times\int\limits_{B_{R_2}}\left(1+\left[\omega_a\left(\left[\frac{R}{(u-k)_\pm}\right]^{\frac{1}{1-\gamma}}\right)\right]^{-1}\right)G\left(\frac{(u-k)_\pm}{R_2-R_1}\right)\,dx \\
&\leqslant c\lambda_3\omega_a(R)\left(\frac{R}{R_2-R_1}\right)^{s(H_a)+1}\int\limits_{B_{R_2}}\left(1+\frac{1}{\omega_a(R)}\right)G\left(\frac{(u-k)_\pm}{R_2-R_1}\right)\,dx
\end{aligned}
$$

$$\leqslant c\left(\frac{R}{R_2-R_1}\right)^{s(\Psi)+1}\int_{B_{R_2}} G\left(\frac{(u-k)_\pm}{R}\right)\,dx \tag{3.2.24}$$

for some constant $c \equiv c(\mathbf{data})$, where we have used Lemma 2.1.1 several times. Using the same argument as above, we have

$$I_2 \leqslant c\left(\frac{R}{R_2-R_1}\right)^{s(\Psi)+1}\int_{B_{R_2}} G\left(\frac{(u-k)_\pm}{R}\right)\,dx \tag{3.2.25}$$

with a constant $c \equiv c(\mathbf{data})$. Inserting the estimates (3.2.21), (3.2.24)-(3.2.25) into (3.2.18), we arrive at (3.2.17) under the assumption (1.0.15). So we have proved the inequality (3.2.17) for the values of $k \in \mathbb{R}$ such that $\inf_{B_R} u \leqslant k \leqslant \sup_{B_R} u$. Now we consider the remaining cases. Suppose $k < \inf_{B_R} u$. In this case, using (3.2.17) with $k \equiv \inf_{B_R} u$, we have

$$\begin{aligned}
\int_{B_{R_1}} \Psi^-_{B_R}\left(|D(u-k)_+|\right)\,dx &= \int_{B_{R_1}} \Psi^-_{B_R}\left(\left|D(u-\inf_{B_R} u)_+\right|\right)\,dx \\
&\leqslant \int_{B_{R_1}} \Psi\left(x, \left|D(u-\inf_{B_R} u)_+\right|\right)\,dx \\
&\leqslant c\left(\frac{R}{R_2-R_1}\right)^{s(\Psi)+1}\int_{B_{R_2}} \Psi^-_{B_R}\left(\frac{\left(u-\inf_{B_R} u\right)_+}{R}\right)\,dx \\
&\leqslant c\left(\frac{R}{R_2-R_1}\right)^{s(\Psi)+1}\int_{B_{R_2}} \Psi^-_{B_R}\left(\frac{(u-k)_+}{R}\right)\,dx
\end{aligned} \tag{3.2.26}$$

for some constant $c \equiv c(\mathbf{data})$. Similarly, it can seen that (3.2.26) is valid for the values of $k > \sup_{B_R} u$. Since $-u$ is also the local Q-minimizer of the functional $\mathcal{P}$ in (1.0.1), the inequality (3.2.17) is valid for all $k \in \mathbb{R}$. The proof is complete. □

From now on also in the rest of the thesis, for a fixed ball $B_R \subset \Omega$, we say that

$$\begin{cases} G - \text{phase occurs in } B_R \\ \text{if } a^-(B_R) \leqslant 4[a]_{\omega_a}\omega_a(R) \text{ and } b^-(B_R) \leqslant 4[b]_{\omega_b}\omega_b(R). & (3.2.27a) \\ (G, H_a) - \text{phase occurs in } B_R \\ \text{if } a^-(B_R) > 4[a]_{\omega_a}\omega_a(R) \text{ and } b^-(B_R) \leqslant 4[b]_{\omega_b}\omega_b(R). & (3.2.27b) \\ (G, H_b) - \text{phase occurs in } B_R \\ \text{if } a^-(B_R) \leqslant 4[a]_{\omega_a}\omega_a(R) \text{ and } b^-(B_R) > 4[b]_{\omega_b}\omega_b(R). & (3.2.27c) \\ (G, H_a, H_b) - \text{phase occurs in } B_R \\ \text{if } a^-(B_R) > 4[a]_{\omega_a}\omega_a(R) \text{ and } b^-(B_R) > 4[b]_{\omega_b}\omega_b(R). & (3.2.27d) \end{cases}$$

Then we have the following lemma which will be applied later, see Section 3.4.

Lemma 3.2.3. *Let $u \in W^{1,\Psi}(\Omega)$ be a local Q-minimizer of the functional $\mathcal{P}$ defined in* (1.0.1) *under one of the assumptions* (1.0.13), (1.0.14) *and* (1.0.15). *Let $B_{2R} \equiv B_{2R}(x_0) \subset \Omega$ be a ball with $R \leqslant 1$. Then there exists a constant $c \equiv c(\boldsymbol{data})$ such that*

$$\begin{aligned} \int_{B_{R_1}} \Psi^-_{B_R}\left(|D(u-k)_\pm|\right) dx &\leqslant \int_{B_{R_1}} \Psi\left(x, |D(u-k)_\pm|\right) dx \\ &\leqslant c\left(\frac{R}{R_2 - R_1}\right)^{s(\Psi)+1} \int_{B_{R_2}} \Phi\left(\frac{(u-k)_\pm}{R}\right) dx \end{aligned} \tag{3.2.28}$$

holds, whenever $B_{R_1} \Subset B_{R_2} \subset B_R(x_0)$ are concentric balls and $k \in \mathbb{R}$, where

$$\Phi(t) = \begin{cases} G(t) & \\ \text{if (3.2.27a) is satisfied in } B_R, & (3.2.29a) \\ G(t) + a^-(B_R)H_a(t) & \\ \text{if (3.2.27b) is satisfied in } B_R, & (3.2.29b) \\ G(t) + b^-(B_R)H_b(t) & \\ \text{if (3.2.27c) is satisfied in } B_R, & (3.2.29c) \\ \Psi^-_{B_R}(t) & \\ \text{if (3.2.27d) is satisfied in } B_R, & (3.2.29d) \end{cases}$$

for every $t \geqslant 0$.

Proof. First we observe that if $a^-(B_R) > 4[a]_{\omega_a}\omega_a(R)$, then using the continuity of the function $a(\cdot)$, we have

$$\begin{aligned} a^-(B_R) \leqslant a(x) &= a(x) - a^-(B_R) + a^-(B_R) \\ &\leqslant 2[a]_{\omega_a}\omega_a(R) + a^-(B_R) \leqslant 2a^-(B_R) \end{aligned} \tag{3.2.30}$$

for every $x \in B_R$. On the other hand, if $a^-(B_R) \leqslant 4[a]_{\omega_a}\omega_a(R)$, then using again the continuity of $a(\cdot)$, we see

$$a(x) = a(x) - a^-(B_R) + a^-(B_R) \leqslant 6[a]_{\omega_a}\omega_a(R)$$

for every $x \in B_R$. Clearly, analogous estimates to the last two displays are valid for the function $b(\cdot)$ in B_R. After those observations, we argue similarly as in the proof of Lemma 3.2.2 depending on which case of (3.2.27b)-(3.2.27d) occurs in the ball B_R. □

3.2.3 Hölder continuity

In this subsection we prove some local boundedness and Hölder continuity assertions of a local Q-minimizer of the functional $\mathcal{P}$ in (1.0.1) with various constants having the precise dependencies.

Theorem 3.2.2. *Let $u \in W^{1,\Psi}(\Omega)$ be a local Q-minimizer of the functional $\mathcal{P}$ defined in* (1.0.1) *under the coefficient functions $a(\cdot) \in C^{\omega_a}(\Omega)$ and*

$b(\cdot) \in C^{\omega_b}(\Omega)$ for ω_a, ω_b being non-negative continuous and concave functions vanishing at the origin.

1. *If the assumption (1.0.13) is satisfied, then for every open subset $\Omega_0 \Subset \Omega$, there exists a Hölder continuity exponent $\gamma \equiv \gamma(\boldsymbol{data}(\Omega_0)) \in (0,1)$ such that*
$$\|u\|_{L^\infty(\Omega_0)} + [u]_{0,\gamma;\Omega_0} \leqslant c(\boldsymbol{data}(\Omega_0)) \tag{3.2.31}$$
and the oscillation estimate
$$\operatorname*{osc}_{B_\rho} u \leqslant c\left(\frac{\rho}{R}\right)^\gamma \operatorname*{osc}_{B_R} u \tag{3.2.32}$$
holds for some $c \equiv c(\boldsymbol{data}(\Omega_0))$ and all concentric balls $B_\rho \Subset B_R \Subset \Omega_0 \Subset \Omega$ with $R \leqslant 1$.

2. *If the assumption (1.0.14) is satisfied, then there exists a Hölder continuity exponent $\gamma \equiv \gamma(\boldsymbol{data}) \in (0,1)$ such that*
$$[u]_{0,\gamma;\Omega_0} \leqslant c(\boldsymbol{data}(\Omega_0)) \tag{3.2.33}$$
and the oscillation estimate
$$\operatorname*{osc}_{B_\rho} u \leqslant c\left(\frac{\rho}{R}\right)^\gamma \operatorname*{osc}_{B_R} u \tag{3.2.34}$$
holds for some $c \equiv c(\boldsymbol{data})$ and all concentric balls $B_\rho \Subset B_R \subset \Omega$ with $R \leqslant 1$.

Proof. Basically, we shall use De Giorgi's methods to prove the local Hölder continuity of u based on arguments employed in [39, 57]. For the convenience of the reader, we give a detailed proof. Note that, for any given ball $B_R \Subset \Omega$, either
$$\left|\left\{x \in B_{R/2} : u(x) > \sup_{B_R} u - \frac{1}{2}\operatorname*{osc}_{B_R} u\right\}\right| \leqslant \frac{1}{2}|B_{R/2}| \tag{3.2.35}$$
or
$$\left|\left\{x \in B_{R/2} : (-u(x)) > \sup_{B_R}(-u) - \frac{1}{2}\operatorname*{osc}_{B_R} u\right\}\right| \leqslant \frac{1}{2}|B_{R/2}| \tag{3.2.36}$$

holds true. It is enough to deal with only the case of (3.2.35) is valid since $-u$ is a local Q-minimizer of the functional $\mathcal{P}$. The proof falls in three steps. In what follows, let $B_{2R} \equiv B_{2R}(x_0) \subset \Omega_0 \Subset \Omega$ be a fixed ball such that $R \leqslant 1$. Let us also denote by

$$A(k,\rho) := \{x \in B_\rho : u(x) > k\} \quad \text{and} \quad B(k,\rho) := \{x \in B_\rho : u(x) < k\} \tag{3.2.37}$$

for every concentric ball $B_\rho \subset B_{2R}$ and $k \in \mathbb{R}$.

Step 1. We suppose that (3.2.35) is satisfied. Then in this step we prove that, for any $\varepsilon \in (0,1)$, there exists a natural number $m \equiv m(\mathbf{data}(\Omega_0), \varepsilon) \geqslant 3$ if (1.0.13) is assumed, and $m \equiv m(\mathbf{data}, \varepsilon) \geqslant 3$ if (1.0.14) is assumed, such that

$$\left| \left\{ x \in B_{R/2} : u(x) > \sup_{B_R} u - \frac{1}{2^m} \operatorname*{osc}_{B_R} u \right\} \right| \leqslant \varepsilon |B_{R/2}|. \tag{3.2.38}$$

Let $m \geqslant 3$ be a natural number to be determined in a few lines. For every $i \in \{1, 2, \dots, m\}$, we set

$$k_i := \sup_{B_R} u - \frac{1}{2^i} \operatorname*{osc}_{B_R} u, \quad \mathcal{D}_i := A(k_i, R/2) \setminus A(k_{i+1}, R/2)$$

and

$$w_i(x) := \begin{cases} k_{i+1} - k_i & \text{if} \quad u(x) > k_{i+1}, \\ u(x) - k_i & \text{if} \quad k_i < u(x) \leqslant k_{i+1}, \\ 0 & \text{if} \quad u(x) \leqslant k_i. \end{cases}$$

Clearly $w_i \in W^{1,\Psi}(B_{R/2})$ with $w_i \equiv 0$ in $B_{R/2} \setminus A(k_1, R/2)$ for all $i \in \{1, \dots, m\}$, and also $|B_{R/2} \setminus A(k_1, R/2)| \geqslant 1/2 |B_{R/2}|$. Then applying Hölder's inequality, Sobolev's inequality and Lemma 2.1.4, for every $\tau \in (0,1)$, we have

$$|A(k_{i+1}, R/2)| \, \Psi^-_{B_R}\left(\frac{k_{i+1} - k_i}{R} \right) \leqslant c \int_{A(k_i, R/2)} \Psi^-_{B_R}\left(\frac{w_i}{R} \right) dx$$

$$
\begin{aligned}
&\leqslant |A(k_i,R/2)|^{\frac{1}{n}} \left(\int_{A(k_i,R/2)} \left[\Psi^-_{B_R}\left(\frac{w_i}{R}\right)\right]^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \\
&\leqslant cR \left(\int_{A(k_i,R/2)} \left[\Psi^-_{B_R}\left(\frac{w_i}{R}\right)\right]^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \\
&\leqslant c \int_{\mathcal{D}_i} \partial_t \Psi^-_{B_R}\left(\frac{u-k_i}{R}\right) |Du|\, dx \\
&\leqslant \tau \int_{\mathcal{D}_i} \Psi^-_{B_R}(|Du|)\, dx + \frac{c}{\tau^{s(\Psi)}} \int_{\mathcal{D}_i} \Psi^-_{B_R}\left(\frac{u-k_i}{R}\right) dx. \qquad (3.2.39)
\end{aligned}
$$

Now we use a Caccioppoli type inequality of Lemma 3.2.2 in order to have

$$
\begin{aligned}
\int_{\mathcal{D}_i} \Psi^-_{B_R}(|Du|)\, dx \leqslant c \int_{A(k_i,R)} \Psi^-_{B_R}\left(\left|\frac{u-k_i}{R}\right|\right) dx \leqslant c \int_{A(k_i,R)} \Psi^-_{B_R}\left(\frac{\operatorname{osc}_{B_R} u}{2^i R}\right) dx \\
\leqslant c\Psi^-_{B_R}\left(\frac{k_{i+1}-k_i}{R}\right) |A(k_i,R)| \leqslant c\Psi^-_{B_R}\left(\frac{k_{i+1}-k_i}{R}\right) R^n.
\end{aligned}
$$

One can see that

$$
\int_{\mathcal{D}_i} \Psi^-_{B_R}\left(\frac{u-k_i}{R}\right) dx \leqslant \int_{\mathcal{D}_i} \Psi^-_{B_R}\left(\frac{k_{i+1}-k_i}{R}\right) dx = \Psi^-_{B_R}\left(\frac{k_{i+1}-k_i}{R}\right) |\mathcal{D}_i|.
$$

Using the estimates coming from the last two displays in (3.2.39), for every $i \in \{1,\dots,m-1\}$, we see

$$
A(k_{m-1},R/2) \leqslant A(k_{i+1},R/2) \leqslant c\tau R^n + \frac{c}{\tau^{s(\Psi)}}|\mathcal{D}_i|.
$$

Summing over $i \in \{1,\dots,m-1\}$, it yields that

$$
|A(k_{m-1},R/2)| \leqslant \left(c\tau + \frac{c}{(m-1)\tau^{s(\Psi)}} \right) R^n.
$$

Now taking small enough $\tau \equiv \tau(\mathbf{data}(\Omega_0),\varepsilon)$ and large enough number $m \equiv$

$m(\mathbf{data}(\Omega_0), \varepsilon)$, we arrive at (3.2.38) when (1.0.13) is assumed. But in the case that (1.0.14) is assumed, we choose small enough $\tau \equiv \tau(\mathbf{data}, \varepsilon)$ and large enough number $m \equiv m(\mathbf{data}, \varepsilon)$ to conclude (3.2.38).

Step 2. In this step, we prove that there exists a small positive $\varepsilon_0 \equiv \varepsilon_0(\mathbf{data}(\Omega_0)) \in (0, 1/2^{n+1})$ such that if

$$0 < \nu_0 < \frac{1}{2} \operatorname*{osc}_{B_R} u \quad \text{and} \quad \left| \{x \in B_{R/2} : u(x) > \sup_{B_R} u - \nu_0\} \right| \leqslant \varepsilon_0 |B_{R/2}|, \tag{3.2.40}$$

then we have

$$\sup_{B_{R/4}} u \leqslant \sup_{B_R} u - \nu_0/2. \tag{3.2.41}$$

Now we set the sequences by

$$\rho_i := \frac{R}{4}\left(1 + \frac{1}{2^i}\right) \quad \text{and} \quad k_i := \sup_{B_R} u - \left(\frac{1}{2} + \frac{1}{2^{i+1}}\right)\nu_0 \quad \text{for every } i = 0, 1, 2, \ldots,$$

and we define

$$\mathcal{D}_{i+1} := A(k_i, \rho_{i+1}) \setminus A(k_{i+1}, \rho_{i+1}) \quad \text{and} \quad Y_i := \frac{|A(k_i, \rho_i)|}{|B_{R/2}|}.$$

Applying Lemma 3.2.2 together with (3.2.40), we discover

$$\begin{aligned}\int_{A(k_i,\rho_{i+1})} \Psi^-_{B_R}(|Du|)\,dx &\leqslant c2^{(i+3)(s(\Psi)+1)} \int_{A(k_i,\rho_i)} \Psi^-_{B_R}\left(\frac{(u-k_i)_+}{R}\right) dx \\ &\leqslant c2^{i(s(\Psi)+1)} \Psi^-_{B_R}\left(\frac{\nu_0}{R}\right) |A(k_i, \rho_i)|,\end{aligned}$$

where we have also used the very definition of k_i and that $(u-k_i)_+ \leqslant \nu_0 \leqslant \|u\|_{L^\infty(B_R)}$. The last display and the convexity of $\Psi^-_{B_R}$ imply that

$$\Psi^-_{B_R}\left(\fint_{\mathcal{D}_{i+1}} |Du|\,dx\right) \leqslant \fint_{\mathcal{D}_{i+1}} \Psi^-_{B_R}(|Du|)\,dx \leqslant c2^{i(s(\Psi)+1)} \frac{|A(k_i,\rho_i)|}{|D_{i+1}|} \Psi^-_{B_R}\left(\frac{\nu_0}{R}\right)$$

$$\leqslant \Psi^-_{B_R}\left(c2^{i(s(\Psi)+1)}\frac{|A(k_i,\rho_i)|}{|D_{i+1}|}\frac{\nu_0}{R}\right).$$

Therefore, we have

$$\fint_{\mathcal{D}_{i+1}} |Du|\,dx \leqslant c2^{i(s(\Psi)+1)}\frac{|A(k_i,\rho_i)|}{|D_{i+1}|}\frac{\nu_0}{R}.$$

On the other hand, applying Lemma 2.2.3 together with $\varepsilon_0 \in (0, 1/2^{n+1})$, we discover

$$\begin{aligned}\int_{\mathcal{D}_{i+1}} |Du|\,dx &\geqslant c(k_{i+1}-k_i)|A(k_{i+1},\rho_{i+1})|^{1-\frac{1}{n}}|B_{\rho_{i+1}}\setminus A(k_i,\rho_{i+1})|\rho_{i+1}^{-n}\\ &\geqslant c2^{-i}\nu_0|A(k_{i+1},\rho_{i+1})|^{1-\frac{1}{n}}\left(|B_{R/4}|-\varepsilon_0|B_{R/2}|\right)R^{-n}\\ &\geqslant c2^{-i}\nu_0|A(k_{i+1},\rho_{i+1})|^{1-\frac{1}{n}}\\ &\geqslant c2^{-i}\nu_0R^{n-1}Y_{i+1}^{1-\frac{1}{n}}\end{aligned}$$

for some constant $c \equiv c(\mathbf{data}(\Omega_0))$. Combining last two displays, we conclude

$$Y_{i+1} \leqslant c_*\left(2^{\frac{n(s(\Psi)+2)}{n-1}}\right)^i Y_i^{1+\frac{1}{n-1}}$$

for some constant $c_* \equiv c_*(\mathbf{data}(\Omega_0))$. Now we apply Lemma 2.2.2 in order to have $Y_i \to 0$ as $i \to \infty$, provided

$$Y_0 = \frac{|A(k_0,R/2)|}{|B_{R/2}|} \leqslant \varepsilon_0 \leqslant c_*^{-(n-1)}2^{-n(n-1)(s(\Psi)+2)}.$$

Therefore, (3.2.41) is satisfied since

$$\left|A\left(\sup_{B_R} u - \frac{\nu_0}{2}, R/4\right)\right| = 0.$$

Step 3: Proof of Hölder continuity. Finally, we are now ready to prove a local Hölder continuity of u. For this, let $m \geqslant 3$ be the natural number satisfying (3.2.38) for the choice $\varepsilon \equiv \varepsilon_0 \in (0, 1/2^{n+1})$, where ε_0 is determined via (3.2.40). Then we have

$$\operatorname*{osc}_{B_{R/4}} u \leqslant \left(1 - \frac{1}{2^{m+1}}\right) \operatorname*{osc}_{B_R} u$$

with $m \equiv m(\mathbf{data}(\Omega_0))$, whenever $B_{2R} \subset \Omega_0$ is a ball with $R \leqslant 1$. Clearly, the above display implies that there exists a positive exponent $\gamma \equiv \gamma(\mathbf{data}(\Omega_0)) \in (0,1)$ such that, for any fixed ball $B_{8R_0} \subset \Omega_0$ with $8R_0 \leqslant 1$, the following oscillation

$$\operatorname*{osc}_{B_R} u \leqslant c \left(\frac{R}{R_0}\right)^{\gamma} \operatorname*{osc}_{B_{R_0}} u$$

holds with some constant $c \equiv c(\mathbf{data}(\Omega_0))$ for every $R \in (0, R_0]$. Here we note that in the case that the assumption (1.0.14) is in force, the constants appearing in the above lemma depend only on **data**, but otherwise are independent of the subset Ω_0. Finally, we have shown that

$$u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega_0)$$

if either the assumption (1.0.13) or (1.0.14) is satisfied. Therefore by a standard covering argument, the estimates (3.2.31) and (3.2.32) are satisfied. Clearly, if (1.0.14) is assumed instead of (1.0.13), γ in (3.2.33) depends only on **data** since $\|u\|_{L^\infty(\Omega_0)} \leqslant \|u\|_{L^\infty(\Omega)}$. The proof is complete. □

3.2.4 The Harnack inequality

In this subsection we prove the Harnack inequality for a local Q-minimizer u of the functional $\mathcal{P}$ in (1.0.1) under one of the assumptions (1.0.13), (1.0.14) and (1.0.15). The analysis similar to the one in Step 1 of the proof of Theorem 3.2.2 gives the following lemma.

Lemma 3.2.4. *Let $u \in W^{1,\Psi}(\Omega)$ be a non-negative local Q-minimizer of the functional $\mathcal{P}$ in* (1.0.1) *under the coefficient functions $a(\cdot) \in C^{\omega_a}(\Omega)$ and $b(\cdot) \in C^{\omega_b}(\Omega)$ for ω_a, ω_b being non-negative, continuous and concave functions vanishing at the origin. Suppose that one of the assumptions* (1.0.13), (1.0.14) *and* (1.0.15) *is satisfied. Let $B_{6R} \subset \Omega_0 \Subset \Omega$ be a ball with $6R \leqslant 1$. Then for any $\tau_1, \tau_2 \in (0,1)$, there exists a large number m depending on* ***data*** *and*

τ_1, τ_2 such that for any $0 < k \leqslant \|u\|_{L^\infty(B_{3R})}$, if

$$|\{x \in B_R : u(x) \geqslant k\}| \geqslant \tau_1 |B_R| \tag{3.2.42}$$

holds, then

$$\left|\{x \in B_{2R} : u(x) \leqslant 2^{-m}k\}\right| \leqslant \tau_2 \left|B_{2R}\right|. \tag{3.2.43}$$

Proof. Let $m \geqslant 3$ be a large number to be determined later. We set, for $i = 0, 1, \ldots, m$,

$$k_i := \frac{k}{2^i}, \quad \mathcal{D}_i := B(k_i, 2R) \setminus B(k_{i+1}, 2R)$$

and

$$w_i(x) := \begin{cases} k_i - k_{i+1} & \text{if} \quad u(x) < k_{i+1}, \\ u(x) - k_{i+1} & \text{if} \quad k_{i+1} \leqslant u(x) < k_i, \\ 0 & \text{if} \quad u(x) \geqslant k_i. \end{cases}$$

We observe that $\Psi^-_{B_{3R}}(w_i) \in W^{1,1}(B_{2R})$ and $\Psi^-_{B_{3R}}(w_i) \equiv 0$ on $B_{2R} \setminus B(k_0, 2R)$ for every $i \in \{0, 1, \ldots, m\}$ and $|B_{2R} \setminus B(k_0, 2R)| \geqslant \tau_1 |B_R|$. Then using Hölder's inequality, Sobolev's inequality and Lemma 2.1.4, we have that

$$\begin{aligned}
B(k_{i+1}, 2R)\Psi^-_{B_{3R}}\left(\frac{k_i - k_{i+1}}{3R}\right) &\leqslant \int\limits_{B(k_i,2R)} \Psi^-_{B_{3R}}\left(\frac{w_i}{3R}\right) dx \\
&\leqslant |B(k_i, 2R)|^{\frac{1}{n}} \left(\int\limits_{B(k_i,2R)} \left[\Psi^-_{B_{3R}}\left(\frac{w_i}{3R}\right)\right]^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \\
&\leqslant cR \left(\int\limits_{B(k_i,2R)} \left[\Psi^-_{B_{3R}}\left(\frac{w_i}{3R}\right)\right]^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \\
&\leqslant c \int\limits_{\mathcal{D}_i} \left(\Psi^-_{B_{3R}}\right)' \left(\frac{u - k_{i+1}}{3R}\right) |Du| \, dx
\end{aligned}$$

$$\leqslant \varepsilon \int_{\mathcal{D}_i} \Psi^-_{B_{3R}}(|Du|)\,dx + \frac{c}{\varepsilon^{s(\Psi)}} \int_{\mathcal{D}_i} \Psi^-_{B_{3R}}\left(\frac{u-k_{i+1}}{3R}\right) dx \quad (3.2.44)$$

for every $\varepsilon \in (0,1)$ and some constant $c \equiv c(\mathbf{data}, \tau_1)$, where we have used Remark 2.1.2 that $\Psi^-_{B_{3R}} \in \mathcal{N}$ with an index $s(\Psi) = s(G) + s(H_a) + s(H_b)$. It follows from the almost standard Caccioppoli inequality of Lemma 3.2.2 that

$$\begin{aligned}\int_{\mathcal{D}_i} \Psi^-_{B_{3R}}(|Du|)\,dx &\leqslant c \int_{B(k_i,2R)} \Psi^-_{B_{3R}}\left(\frac{k_i-u}{3R}\right) dx \\ &\leqslant c \int_{B(k_i,2R)} \Psi^-_{B_{3R}}\left(\left|\frac{2(k_i-k_{i+1})}{3R}\right|\right) dx \\ &\leqslant c|B(k_i,2R)|\Psi^-_{B_{3R}}\left(\left|\frac{2(k_i-k_{i+1})}{3R}\right|\right) \\ &\leqslant cR^n \Psi^-_{B_{3R}}\left(\left|\frac{k_i-k_{i+1}}{3R}\right|\right), \qquad (3.2.45)\end{aligned}$$

where we have also used the assumption that u is non-negative. Clearly, by the very definition of $\mathcal{D}_i$, one can see that

$$\begin{aligned}\int_{\mathcal{D}_i} \Psi^-_{B_{3R}}\left(\frac{u-k_{i+1}}{3R}\right) dx &\leqslant \int_{\mathcal{D}_i} \Psi^-_{B_{3R}}\left(\frac{k_i-k_{i+1}}{3R}\right) dx \\ &\leqslant c|\mathcal{D}_i| \int_{\mathcal{D}_i} \Psi^-_{B_{3R}}\left(\frac{k_i-k_{i+1}}{3R}\right) dx. \qquad (3.2.46)\end{aligned}$$

Combining the estimates obtained in (3.2.44)-(3.2.46), we find that

$$|B(k_m,2R)| \leqslant |B(k_{i+1},2R)| \leqslant c\varepsilon R^n + \frac{c}{\varepsilon^{s(\Psi)}}|\mathcal{D}_i|$$

holds for some constant $c \equiv c(\mathbf{data}, \tau_1)$, whenever $\varepsilon \in (0,1)$ and $i \in \{0,1,\dots,m-1\}$. Summing the last inequality above over the index i from 0 to $m-1$ implies

$$|B(k_m,2R)| \leqslant c\varepsilon R^n + \frac{c}{\varepsilon^{s(\Psi)}m}|B(k_0,2R)| \leqslant \left(c_*\varepsilon + \frac{c_*}{\varepsilon^{s(\Psi)}m}\right)|B_{2R}|$$

for some constant $c_* \equiv c_*(\mathbf{data}, \tau_1)$. Now choosing small enough $\varepsilon \equiv (\mathbf{data}, \tau_1, \tau_2)$ and sufficiently large $m \equiv m(\mathbf{data}, \tau_1, \tau_2)$ such that

$$c_* \varepsilon + \frac{c_*}{\varepsilon^{s(\Psi)} m} \leqslant \tau_2,$$

we arrive at the desired estimate (3.2.43).

$\square$

Lemma 3.2.5. *Under the assumptions of Lemma 3.2.4, let* $u \in W^{1,\Psi}(\Omega)$ *be a non-negative* Q*-minimizer of the functional* $\mathcal{P}$ *in* (1.0.1). *Suppose that one of the assumptions* (1.0.13), (1.0.14) *and* (1.0.15) *is satisfied. Then for any* $\tau \in (0,1)$, *there exists a small* $\delta_1 \equiv \delta_1(\boldsymbol{data}(\Omega_0))$ *such that for any* $0 < k \leqslant \|u\|_{L^\infty(B_{3R})}$, *if*

$$|\{x \in B_R : u(x) \geqslant k\}| \geqslant \tau |B_R| \tag{3.2.47}$$

holds, then

$$\inf_{B_R} u \geqslant \delta_1 k. \tag{3.2.48}$$

Proof. It's enough to prove the lemma for $\tau \in \left(0, 2^{-(n+1)}\right)$. Let us fix $m_0 \in \mathbb{N}$, and consider the sequences defined by

$$\rho_i := R\left(1 + \frac{1}{2^i}\right) \quad \text{and} \quad k_i := \left(\frac{1}{2} + \frac{1}{2^i}\right) 2^{-m_0} k \quad (i = 0, 1, 2, \ldots). \tag{3.2.49}$$

Next we also define

$$\mathcal{D}_{i+1}^- := B(k_i, \rho_{i+1}) \setminus B(k_{i+1}, \rho_{i+1}) \quad \text{and} \quad Y_i := \frac{|B(k_i, \rho_i)|}{|B_{\rho_i}|}, \tag{3.2.50}$$

where the definition of $B(k_i, \rho_i)$ has been introduced in (3.2.37). By using the assumption that u is non-negative, we observe $(u - k_i)_- \leqslant 2^{-m_0} k$. Then by applying Lemma 3.2.2, we see

$$\int\limits_{B(k_i, \rho_{i+1})} \Psi_{B_{2R}}^-(|Du|)\, dx \leqslant c 2^{(i+3)(s(\Psi)+1)} \int\limits_{B(k_i, \rho_i)} \Psi_{B_{2R}}^- \left(\frac{(u - k_i)_-}{2R}\right) dx$$

$$\leqslant c2^{(i+3)(s(\Psi)+1)}\Psi^-_{B_{2R}}\left(\frac{2^{-m_0}k}{R}\right)|B(k_i,\rho_i)|$$

for some constant $c \equiv c(\mathbf{data})$. This estimate together with the convexity of $\Psi^-_{B_{2R}}$ implies

$$\begin{aligned}
\Psi^-_{B_{2R}}\left(\fint_{\mathcal{D}^-_{i+1}} |Du|\,dx\right) &\leqslant \fint_{\mathcal{D}^-_{i+1}} \Psi^-_{B_{2R}}(|Du|)\,dx \\
&\leqslant c2^{i(s(\Psi)+1)}\frac{|B(k_i,\rho_i)|}{|\mathcal{D}^-_{i+1}|}\Psi^-_{B_{2R}}\left(\frac{2^{-m_0}k}{R}\right) \\
&\leqslant c\Psi^-_{B_{2R}}\left(2^{i(s(\Psi)+1)}\frac{|B(k_i,\rho_i)|}{|\mathcal{D}^-_{i+1}|}\frac{2^{-m_0}k}{R}\right)
\end{aligned}$$

for some constant $c \equiv c(\mathbf{data}(\Omega_0))$. Therefore, using the fact that the function $\Psi^-_{B_{2R}}$ is increasing and Lemma 2.1.1, we have

$$\fint_{\mathcal{D}^-_{i+1}} |Du|\,dx \leqslant c2^{i(s(\Psi)+1)}\frac{|B(k_i,\rho_i)|}{|\mathcal{D}^-_{i+1}|}\frac{2^{-m_0}k}{R}.$$

Now applying Lemma 2.2.3 together with the fact that $\tau \in \left(0, 2^{-(n+1)}\right)$, we see

$$\begin{aligned}
\int_{\mathcal{D}^-_{i+1}} |Du|\,dx &\geqslant c(k_i - k_{i+1})|B(k_{i+1},\rho_{i+1})|^{1-\frac{1}{n}}\left|B_{\rho_{i+1}} \setminus B(k_i,\rho_{i+1})\right|\rho^n_{i+1} \\
&\geqslant c2^{-i}2^{-m_0}k|B(k_{i+1},\rho_{i+1})|^{1-\frac{1}{n}}\left(|B_{2R}| - \tau|B_R|\right)R^{-n} \\
&\geqslant c2^{-i}2^{-m_0}k|B(k_{i+1},\rho_{i+1})|^{1-\frac{1}{n}} \\
&\leqslant c2^{-i}2^{-m_0}kR^{n-1}Y^{1-\frac{1}{n}}_{i+1}.
\end{aligned}$$

The combination of the last two displays yields

$$Y^{1-\frac{1}{n}}_{i+1} \leqslant c2^{i(s(\Psi)+1)}R^{-n}|B(k_i,\rho_i)| \leqslant c2^{i(s(\Psi)+1)}Y_i$$

and then we conclude

$$Y_{i+1} \leqslant c_* 2^{i\frac{(s(\Psi)+1)n}{n-1}} Y_i^{1+\frac{1}{n-1}}$$

for some constant $c \equiv c(\mathbf{data}(\Omega_0))$. Now applying Lemma 3.2.4, we find a large natural number $m_0 \equiv m_0(\mathbf{data}(\Omega_0))$ such that

$$\left|\{x \in B_{2R} : u(x) \leqslant 2^{-m_0}k\}\right| \leqslant c_*^{-(n-1)} 2^{-n(n-1)(s(\Psi)+1)}.$$

With keeping the above choice of m_0, we observe that

$$Y_0 = \frac{|B(k_0, 2R)|}{|B_{2R}|} = \frac{|x \in B_{2R} : u(x) \leqslant 2^{-m_0}k|}{|B_{2R}|} \leqslant c_*^{-(n-1)} 2^{-n(n-1)(s(\Psi)+1)}.$$

Now we are at stage in applying Lemma 2.2.2 to obtain that $Y_i \to 0$ as $i \to \infty$, which is equivalent to

$$|B(2^{-(m_0+1)}k, R)| = 0.$$

The last display implies the validity of (3.2.48) with the choice of $\delta_1 \equiv 2^{-(m_0+1)}$. $\square$

From Lemma 3.2.5 and the covering arguments in [108, Section 7], we obtain the following weak Harnack inequality for a local Q-minima of the functional $\mathcal{P}$ defined in (1.0.1). We also refer to [20, 39, 96] for the proof.

Theorem 3.2.3 (The weak Harnack inequality)**.** *Let $W^{1,\Psi}(\Omega)$ be a local non-negative Q-minimizer of the functional $\mathcal{P}$ defined in* (1.0.1) *with the coefficient functions $a(\cdot) \in C^{\omega_a}(\Omega)$ and $b(\cdot) \in C^{\omega_b}(\Omega)$ for functions ω_a, ω_b being continuous and concave which vanish at 0. Suppose one of the assumptions* (1.0.13), (1.0.14) *and* (1.0.15) *is satisfied. Let $B_{9R} \equiv B_{9R}(x_0) \subset \Omega_0 \Subset \Omega$ be a ball with $9R \leqslant 1$. Then there exist $q_- > 0$ and a constant c depending on* ***data***(Ω_0) *such that*

$$\inf_{x \in B_R} u(x) \geqslant \frac{1}{c} \left(\fint_{B_{2R}} u^{q_-}\, dx \right)^{\frac{1}{q_-}}. \tag{3.2.51}$$

To conclude the result of Theorem 3.2.4 below, we need to obtain a local sup-estimates for local quasiminizers of $\mathcal{P}$.

Lemma 3.2.6. *Under the assumptions of Lemma 3.2.4, let $u \in W^{1,\Psi}(\Omega)$ be a non-negative local Q-minimizer of the functional $\mathcal{P}$ in* (1.0.1) *with the coefficient functions $a(\cdot) \in C^{\omega_a}(\Omega)$ and $b(\cdot) \in C^{\omega_b}(\Omega)$ for functions ω_a, ω_b being continuous and concave vanishing at the origin. Suppose that one of the assumptions* (1.0.13), (1.0.14) *and* (1.0.15) *is satisfied. Let $B_{9R} \equiv B_{9R}(x_0) \subset \Omega_0 \Subset \Omega$ be a ball with $9R \leqslant 1$. Then for any $q_+ > 0$, the local estimate holds*

$$\sup_{B_R} u \leqslant c \left(\fint_{B_{2R}} |u|^{q_+}\, dx \right)^{\frac{1}{q_+}} \tag{3.2.52}$$

for some constant $c \equiv c(\boldsymbol{data}(\Omega_0))$.

Proof. The proof consists of two steps. For the convenience, let us consider the scaled functions

$$\bar{u}(x) := \frac{u(x_0 + Rx)}{R} \quad \text{for every} \quad x \in B_4. \tag{3.2.53}$$

Then the almost standard Caccioppoli inequality (3.2.17) of Lemma 3.2.2 can be written in the view of $\bar{u}$ as follows:

$$\int_{B_{r_1}} \Psi^-_{B_{2R}}(|D(\bar{u} - k)_\pm|)\, dx \leqslant \frac{c}{(r_2 - r_1)^{s(\Psi)+1}} \int_{B_{r_2}} \Psi^-_{B_{2R}}((\bar{u} - k)_\pm)\, dx \tag{3.2.54}$$

with some constant $c \equiv c(\mathbf{data})$, whenever $B_{r_1} \Subset B_{r_2} \subset B_2(0)$ are concentric balls and $k \in \mathbb{R}$. Next for $1 \leqslant t \leqslant s \leqslant 2$, we set sequences by

$$\rho_i := \left(t + \frac{s-t}{2^i} \right) \quad \text{and} \quad k_i := 2l_0 \left(1 - \frac{1}{2^{i+1}} \right) \tag{3.2.55}$$

for some constant $d_0 > 0$ to be determined later. We also define

$$\bar{\rho}_i := \frac{\rho_i + \rho_{i+1}}{2} \quad \text{and} \quad Y_i := \frac{1}{\Psi^-_{B_{2R}}(l_0)} \int_{\bar{A}(k_i, \rho_i)} \Psi^-_{B_{2R}}((u - k_i)_+)\, dx, \tag{3.2.56}$$

where

$$\bar{A}(k, \rho) := \{x \in B_\rho : \bar{u} > k\}. \tag{3.2.57}$$

Let $\eta_i \in C_0^\infty(B_{\bar\rho_i})$ be a cut-off function such that $0 \leqslant \eta_i \leqslant 1$, $\eta_i \equiv 1$ on $B_{\rho_{i+1}}$ and $|D\eta_i| \leqslant \dfrac{c(n)2^i}{(s-t)}$.

Then using Hölder's inequality, Sobolev's inequality and Lemma 2.1.4, we have

$$
\begin{aligned}
\Psi^-_{B_{2R}}(l_0)Y_{i+1} &\leqslant \int_{B_{\bar\rho_i}} \Psi^-_{B_{2R}}\left((\bar u - k_{i+1})\eta_i\right)\, dx \\
&\leqslant |\bar A(k_{i+1},\rho_i)|^{\frac{1}{n}} \left(\int_{B_{\bar\rho_i}} \left[\Psi^-_{B_{2R}}\left((\bar u - k_{i+1})_+\eta_i\right)\right]^{\frac{n}{n-1}}\, dx \right)^{\frac{n-1}{n}} \\
&\leqslant c|\bar A(k_{i+1},\rho_i)|^{\frac{1}{n}} \int_{B_{\bar\rho_i}} \left(\Psi^-_{B_{2R}}\right)'\left((\bar u - k_{i+1})_+\eta_i\right) \\
&\quad \times \left[|D(\bar u - k_{i+1})_+|\eta_i + (\bar u - k_{i+1})_+|D\eta_i|\right]\, dx \\
&\leqslant c|\bar A(k_{i+1},\rho_i)|^{\frac{1}{n}} \int_{B_{\bar\rho_i}} \left(\Psi^-_{B_{2R}}\right)'\left((\bar u - k_{i+1})_+\right)|D(\bar u - k_{i+1})_+|\, dx \\
&\qquad + c|\bar A(k_{i+1},\rho_i)|^{\frac{1}{n}} \frac{2^i}{s-t} \int_{B_{\bar\rho_i}} \left(\Psi^-_{B_{2R}}\right)'\left((\bar u - k_{i+1})_+\right)(\bar u - k_{i+1})_+\, dx \\
&\leqslant c|\bar A(k_{i+1},\rho_i)|^{\frac{1}{n}} \left(\int_{B_{\bar\rho_i}} \Psi^-_{B_{2R}}(|D(\bar u - k_{i+1})_+|)\, dx \right. \\
&\quad \left. + \frac{2^i}{s-t} \int_{B_{\bar\rho_i}} \Psi^-_{B_{2R}}((\bar u - k_{i+1})_+)\, dx \right) \\
&\leqslant c|\bar A(k_{i+1},\rho_i)|^{\frac{1}{n}} \left(\frac{2^i}{s-t}\right)^{s(\Psi)+1} \int_{B_{\rho_i}} \Psi^-_{B_{2R}}((\bar u - k_{i+1})_+)\, dx
\end{aligned}
$$

for some constant $c \equiv c(\mathbf{data})$, where in the last inequality of the above display we also have used (3.2.54) and (3.2.57). Now applying Lemma 2.1.1, we see that

$$
\begin{aligned}
|\bar{A}(k_{i+1},\rho_i)| &\leqslant \frac{1}{\Psi^-_{B_{2R}}(k_{i+1}-k_i)} \int_{\bar{A}(k_{i+1},\rho_i)} \Psi^-_{B_{2R}}(\bar{u}-k_i)\, dx \\
&\leqslant \frac{1}{\Psi^-_{B_{2R}}(l_0/2^{i+1})} \int_{\bar{A}(k_{i+1},\rho_i)} \Psi^-_{B_{2R}}(\bar{u}-k_i)\, dx \\
&\leqslant \frac{\Psi^-_{B_{2R}}(l_0)}{\Psi^-_{B_{2R}}(l_0/2^{i+1})} Y_i \leqslant 2^{(i+1)(s(\Psi)+1)} Y_i \leqslant c\left(\frac{2^i}{s-t}\right)^{s(\Psi)+1} Y_i.
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{B_{\rho_i}} \Psi^-_{B_{2R}}((\bar{u}-k_{i+1})_+)\, dx &= \int_{\bar{A}(k_{i+1},\rho_i)} \Psi^-_{B_{2R}}(\bar{u}-k_{i+1})\, dx \\
&\leqslant \int_{\bar{A}(k_i,\rho_i)} \Psi^-_{B_{2R}}(\bar{u}-k_i)\, dx = \Psi^-_{B_{2R}}(l_0)\, Y_i.
\end{aligned}
$$

Combining the last three displays, we conclude with the following recursive inequality:

$$
Y_{i+1} \leqslant c_0 \frac{2^{i\left(1+\frac{1}{n}\right)(s(\Psi)+1)}}{(s-t)^{\left(1+\frac{1}{n}\right)(s(\Psi)+1)}} Y_i^{1+\frac{1}{n}}
$$

for some constant $c_0 \equiv c_0(\mathbf{data})$. Now we are at the stage to apply Lemma 2.2.2. In turn, we have $Y_i \to 0$ as $i \to \infty$, provided

$$
Y_0 = \frac{1}{\Psi^-_{B_{2R}}(l_0)} \int_{\bar{A}(l_0,s)} \Psi^-_{B_{2R}}(\bar{u}-l_0)\, dx \leqslant \left[\frac{c_0}{(s-t)^{\left(1+\frac{1}{n}\right)(s(\Psi)+1)}}\right]^{-n} 2^{-n(1+n)(s(\Psi)+1)}.
$$

The inequality in the last display is satisfied if we choose $l_0 > 0$ in the following way

$$
\Psi^-_{B_{2R}}(l_0) = \frac{c_0^n 2^{n(1+n)(s(\Psi)+1)}}{(s-t)^{(1+n)(s(\Psi)+1)}} \int_{B_s} \Psi^-_{B_{2R}}((\bar{u})_+)\, dx.
$$

Therefore, we obtain $\bar{u} \leqslant 2l_0$ in B_t. This estimate together with the last display yields

$$\Psi^-_{B_{2R}}\left(\sup_{B_t}(\bar{u})_+\right) \leqslant \frac{c}{(s-t)^{(1+n)(s(\Psi)+1)}} \fint_{B_s} \Psi^-_{B_{2R}}\left((\bar{u})_+\right)\,dx. \tag{3.2.58}$$

Recalling $\Psi^-_{B_{2R}} \in \mathcal{N}$ with an index $s(\Psi)$ and applying Lemma 2.1.3 for $\Psi^-_{B_{2R}}$, one can see that $t \mapsto \Psi^-_{B_{2R}}\left(t^{\frac{1}{s(\Psi)+1}}\right)$ is a concave function. Using this one together with Jensen's inequality in (3.2.58), we see

$$\begin{aligned}
\Psi^-_{B_{2R}}\left(\sup_{B_t}(\bar{u})_+\right) &\leqslant \frac{c}{(s-t)^{(1+n)(s(\Psi)+1)}} \fint_{B_s} \Psi^-_{B_{2R}}\left((\bar{u})_+\right)\,dx \\
&= \frac{c}{(s-t)^{(1+n)(s(\Psi)+1)}} \fint_{B_s} \Psi^-_{B_{2R}}\left(\left[(\bar{u})_+^{s(\Psi)+1}\right]^{\frac{1}{s(\Psi)+1}}\right)\,dx \\
&\leqslant \frac{c}{(s-t)^{(1+n)(s(\Psi)+1)}} \Psi^-_{B_{2R}}\left(\left[\fint_{B_s}(\bar{u})_+^{s(\Psi)+1}\,dx\right]^{\frac{1}{s(\Psi)+1}}\right) \\
&\leqslant \Psi^-_{B_{2R}}\left(\frac{c}{(s-t)^{(1+n)(s(\Psi)+1)}}\left[\fint_{B_s}(\bar{u})_+^{s(\Psi)+1}\,dx\right]^{\frac{1}{s(\Psi)+1}}\right).
\end{aligned}$$

Since $\Psi^-_{B_{2R}}$ is the increasing function, the last display implies

$$\sup_{B_t}(\bar{u})_+ \leqslant \frac{c}{(s-t)^{(1+n)(s(\Psi)+1)}}\left[\fint_{B_s}(\bar{u})_+^{s(\Psi)+1}\,dx\right]^{\frac{1}{s(\Psi)+1}}.$$

Since $-u$ is a local Q-minimizer of the functional $\mathcal{P}$, we find

$$\sup_{B_t}|\bar{u}| \leqslant \frac{c}{(s-t)^{(1+n)(s(\Psi)+1)}}\left[\fint_{B_s}|\bar{u}|^{s(\Psi)+1}\,dx\right]^{\frac{1}{s(\Psi)+1}}.$$

Therefore, for $0 < q_+ < s(\Psi) + 1$, we discover from Young's inequality that

$$\begin{aligned}\sup_{B_t} |\bar{u}| &\leqslant \frac{c}{(s-t)^{(1+n)(s(\Psi)+1)}} \left(\sup_{B_s} |\bar{u}|\right)^{1-\frac{q_+}{s(\Psi)+1}} \left[\fint_{B_s} |\bar{u}|^{q_+}\,dx\right]^{\frac{1}{s(\Psi)+1}} \\ &\leqslant \frac{1}{2}\sup_{B_s}|\bar{u}| + \frac{c}{(s-t)^{\frac{(1+n)(s(\Psi)+1)^2}{q_+}}} \left[\fint_{B_2} |\bar{u}|^{q_+}\,dx\right]^{\frac{1}{q_+}}\end{aligned}$$

holds for every $1 \leqslant t < s \leqslant 2$. Then we apply Lemma 2.2.1 for $h(t) = \sup_{B_t} |\bar{u}|$ in order to have

$$\sup_{B_1} |\bar{u}| \leqslant c \left[\fint_{B_2} |\bar{u}|^{q_+}\,dx\right]^{\frac{1}{q_+}} \tag{3.2.59}$$

for $c \equiv c(\mathbf{data}, q_+)$. On the other hand, for $q_+ \geqslant s(\Psi) + 1$, the inequality (3.2.59) is still valid by using Hölder's inequality. Scaling back as we introduced in (3.2.53), we arrive at the desired estimate (3.2.52). □

Finally, the main result of the this section is the following:

Theorem 3.2.4. *Let $u \in W^{1,\Psi}(\Omega)$ be a non-negative local Q-minimizer u of the functional $\mathcal{P}$ defined in* (1.0.1) *under the coefficient functions $a(\cdot) \in C^{\omega_a}(\Omega)$ and $b(\cdot) \in C^{\omega_b}(\Omega)$ for ω_a, ω_b being non-negative, continuous and concave functions vanishing at the origin. Suppose that one of the assumptions* (1.0.13), (1.0.14) *and* (1.0.15) *is satisfied. For every ball B_R with $B_{9R} \subset \Omega_0$ with $\Omega_0 \Subset \Omega$ being an open subset, there exists a positive constant $c \equiv c(\boldsymbol{data}(\Omega_0))$ such that*

$$\sup_{B_R} u \leqslant c \inf_{B_R} u \tag{3.2.60}$$

holds.

Proof. The proof is essentially based on the results we have obtained so far. In fact, applying Theorem 3.2.3 and Lemma 3.2.6 with $q_- = q_+$, we obtain (3.2.60). □

Remark 3.2.2. The results of the above theorem refine the results of [95, Theorem 1.3] without any extra term in (3.2.60) under our multi-phase settings when the assumptions (1.0.13) and (1.0.14) come into play, and see also [94, 96].

3.2.5 Higher integrability results

Next, we provide a higher integrability result for a local minimizer of the functional $\mathcal{F}$ defined in (1.0.10).

Theorem 3.2.5 (Higher Integrability)**.** *Let $u \in W^{1,\Psi}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ defined in* (1.0.10) *under the assumption* (1.0.11)*. Assume that one of the assumptions* (1.0.13)*,* (1.0.14) *and* (1.0.15) *is satisfied. Then there exists a higher integrability exponent $\delta \equiv \delta(\boldsymbol{data}) \in (0,1)$ such that the following reverse type Hölder inequality*

$$\left(\fint_{B_{R/2}} [\Psi(x,|Du|)]^{1+\delta}\,dx\right)^{\frac{1}{1+\delta}} \leqslant c \fint_{B_R} \Psi(x,|Du|)\,dx \tag{3.2.61}$$

holds for a constant $c \equiv c(\boldsymbol{data})$, where $\boldsymbol{data}$ is clarified in (3.1.8)*, whenever $B_R \Subset \Omega$ is a ball with $R \leqslant 1$. In particular, for any open subset $\Omega_0 \Subset \Omega$, it holds that*

$$\|\Psi(x,|Du|)\|_{L^{1+\delta}(\Omega_0)} \leqslant c(\boldsymbol{data}(\Omega_0)). \tag{3.2.62}$$

Proof. Let $B_R \Subset \Omega$ be a ball with $R \leqslant 1$ as in the statement. Since u is a local $Q := L/\nu$-minimizer of the functional $\mathcal{P}$ in (1.0.1), we are able to apply Lemma 3.2.1 with the choices $\rho \equiv R/2$, $r \equiv R$ and $k \equiv (u)_{B_R}$ in order to get

$$\fint_{B_{R/2}} \Psi(x,|Du|)\,dx \leqslant c \fint_{B_R} \Psi\left(x, \left|\frac{u-(u)_{B_R}}{R}\right|\right)\,dx \tag{3.2.63}$$

with some constant $c \equiv c(n, s(G), s(H_a), s(H_b), \nu, L)$. Then, applying Remark 2.4.1 depending on which one of the assumptions (1.0.13), (1.0.14) and

(1.0.15) is assumed, we obtain the following reverse Hölder inequality:

$$\fint_{B_{R/2}} \Psi(x, |Du|)\, dx \leqslant c \left(\fint_{B_R} [\Psi(x, |Du|)]^{\theta}\, dx \right)^{\frac{1}{\theta}}, \tag{3.2.64}$$

where $c \equiv c(\mathbf{data})$, and $\theta \in (0,1)$ is the same appearing in Remark 2.4.1. At this point (3.2.61) follows using a variant of Gehring's lemma on reverse Hölder inequalities, see for instance [90, Theorem 6.6]. $\square$

3.3 Comparison estimates

Throughout this section we fix a ball $B_{2R} \equiv B_{2R}(x_0) \subset \Omega_0 \Subset \Omega$ with $R \leqslant 1$ and some open subset $\Omega_0 \Subset \Omega$. We consider the functional defined by

$$W^{1,1}(B_{2R}) \ni \upsilon \mapsto \mathcal{F}_{B_{2R}}(\upsilon) := \int_{B_{2R}} F(x, (u)_{B_{2R}}, D\upsilon)\, dx, \tag{3.3.1}$$

where u is a local minimizer of the functional $\mathcal{F}$ in (1.0.10). Now we consider a function $w \in u + W^{1,\Psi}_0(B_R)$ being the solution to the following variational Dirichlet problem:

$$\begin{cases} w \mapsto \min\limits_{\upsilon} \mathcal{F}_{B_{2R}}(\upsilon) \\ \upsilon \in u + W^{1,\Psi}_0(B_{2R}). \end{cases} \tag{3.3.2}$$

In the following we shall deal with first comparison estimates in order to remove u-dependence in the original functional $\mathcal{F}$ in (1.0.10).

Lemma 3.3.1. *Let $w \in W^{1,\Psi}(B_{2R})$ be the solution to the variational problem* (3.3.2) *under the assumptions* (1.0.11), (3.1.2) *and* (3.1.4). *Let the coefficient functions $a(\cdot) \in C^{\omega_a}(\Omega)$ and $b(\cdot) \in C^{\omega_b}(\Omega)$ for ω_a, ω_b being non-negative, continuous and concave functions vanishing at the origin. Assume that one of the assumptions* (1.0.13), (1.0.14) *and* (1.0.15) *is satisfied. Then there exists a constant $c \equiv c(\boldsymbol{data}(\Omega_0))$ such that*

$$\fint_{B_{2R}} |V_{\Psi}(x, Du) - V_{\Psi}(x, Dw)|^2\, dx \leqslant c\omega(R^{\gamma}) \fint_{B_{2R}} \Psi(x, |Du|)\, dx \tag{3.3.3}$$

holds, where $\gamma \equiv \gamma(\boldsymbol{data}(\Omega_0))$ *is the Hölder exponent determined via Theorem 3.2.2. Moreover, the following estimates hold true:*

$$\fint_{B_{2R}} \Psi(x, |Dw|)\, dx \leqslant \frac{L}{\nu} \fint_{B_{2R}} \Psi(x, |Du|)\, dx, \tag{3.3.4}$$

$$\|w\|_{L^\infty(B_{2R})} \leqslant \|u\|_{L^\infty(B_{2R})}, \tag{3.3.5}$$

$$\operatorname*{osc}_{B_{2R}} w \leqslant \operatorname*{osc}_{B_{2R}} u \tag{3.3.6}$$

and

$$\fint_{B_{2R}} \Psi\left(x, \left|\frac{u-w}{R}\right|\right) dx \leqslant c[\omega(R^\gamma)]^{\frac{1}{2}} \fint_{B_{2R}} \Psi(x, |Du|)\, dx \tag{3.3.7}$$

for some constant $c \equiv c(\boldsymbol{data}(\Omega_0))$*, where in the case that* (1.0.15) *is considered,* γ *appearing in* (3.3.3) *and* (3.3.7) *is the same as in the assumption* (1.0.15).

Proof. The proof is very standard and we shall follow the structure of the proof of [22, Lemma 4]. The Euler-Lagrange equation of the functional $\mathcal{F}_{B_{2R}}$, which is

$$\fint_{B_{2R}} \langle D_z F(x, (u)_{B_{2R}}, Dw), D\varphi \rangle \, dx = 0, \tag{3.3.8}$$

holds for any function $\varphi \in W_0^{1,\Psi}(B_{2R})$ (see for instance [14, Lemma 5.2]). The minimality and growth condition (1.0.11) imply that

$$\begin{aligned} \fint_{B_{2R}} \Psi(x, |Dw|)\, dx &\leqslant \frac{1}{\nu} \fint_{B_{2R}} F(x, (u)_{B_{2R}}, Dw)\, dx \\ &\leqslant \frac{1}{\nu} \fint_{B_{2R}} F(x, (u)_{B_{2R}}, Du)\, dx \leqslant \frac{L}{\nu} \fint_{B_{2R}} \Psi(x, |Du|)\, dx, \end{aligned} \tag{3.3.9}$$

which proves (3.3.4). Therefore, we conclude with

$$\fint_{B_{2R}} \langle D_z F(x,(u)_{B_{2R}}, Dw), Du - Dw\rangle \, dx = 0. \tag{3.3.10}$$

Letting $u^+_{B_{2R}} := \sup_{x\in B_{2R}} u(x)$ and $u^-_{B_{2R}} := \inf_{x\in B_{2R}} u(x)$, the minimality of w yields

$$\mathcal{F}_{B_{2R}}(w) \leqslant \mathcal{F}_{B_{2R}}\left(\min\{w, u^+_{B_{2R}}\}\right) \qquad \text{and} \qquad \mathcal{F}_{B_{2R}}(w) \leqslant \mathcal{F}_{B_{2R}}(\max\{w, u^-_{B_{2R}}\}).$$

Consequently, the last display together with (1.0.11) gives us

$$\int_{B_{2R}\cap\left\{w\geqslant u^+_{B_{2R}}\right\}} \Psi(x,|Dw|)\, dx = 0 \qquad \text{and} \qquad \int_{B_{2R}\cap\left\{w\leqslant u^-_{B_{2R}}\right\}} \Psi(x,|Dw|)\, dx = 0.$$

By coarea formula, we get that

$$\inf_{x\in B_{2R}} u \equiv u^-_{B_{2R}} \leqslant w(x) \leqslant u^+_{B_{2R}} \equiv \sup_{x\in B_{2R}} u(x) \qquad \text{a.e. } x \in B_{2R}. \tag{3.3.11}$$

This proves (3.3.5) and (3.3.6). Using (3.1.6) and (3.3.10) together with the minimality of u and w, we have that

$$\begin{aligned}
&\fint_{B_{2R}} |V_\Psi(x,Du) - V_\Psi(x,Dw)|^2\, dx \\
&\overset{(3.3.10)}{=} \fint_{B_{2R}} |V_\Psi(x,Du) - V_\Psi(x,Dw)|^2\, dx \\
&\qquad + c_* \fint_{B_{2R}} \langle D_z F(x,(u)_{B_{2R}}, Dw), Du - Dw\rangle \, dx \\
&\leqslant c_* \fint_{B_{2R}} \left[F(x,(u)_{B_{2R}}, Du) - F(x,(u)_{B_{2R}}, Dw)\right] dx \\
&= c_* \fint_{B_{2R}} \left[F(x,(u)_{B_{2R}}, Du) - F(x,u,Du)\right] dx
\end{aligned}$$

$$
\begin{aligned}
&+ c_* \fint_{B_{2R}} [F(x,u,Du) - F(x,w,Dw)]\,dx \\
&+ c_* \fint_{B_{2R}} [F(x,w,Dw) - F(x,(w)_{B_{2R}},Dw)]\,dx \\
&+ c_* \fint_{B_{2R}} [F(x,(w)_{B_{2R}},Dw) - F(x,(u)_{B_{2R}},Dw)]\,dx \\
=:&\ c_* \sum_{i=1}^{4} I_i
\end{aligned}
\tag{3.3.12}
$$

with $c_* \equiv c_*(n, s(G), s(H_a), s(H_b), \nu)$. Now we estimate each term I_i for $i \in \{1,2,3,4\}$ in the last display. We have

$$
\begin{aligned}
I_1 &\overset{(3.1.2)}{\leqslant} c \fint_{B_{2R}} \omega(|u-(u)_{B_{2R}}|)\Psi(x,|Du|)\,dx \\
&\overset{(3.2.31),(3.2.33)}{\leqslant} c\omega(2[u]_{0,\gamma;\Omega_0}R^\gamma) \fint_{B_{2R}} \Psi(x,|Du|)\,dx \\
&\overset{(2.1.1)}{\leqslant} c(\mathbf{data}(\Omega_0))\omega(R^\gamma) \fint_{B_{2R}} \Psi(x,|Du|)\,dx,
\end{aligned}
\tag{3.3.13}
$$

where in the last display we have also used the fact that $\omega(\cdot)$ is concave. The minimality of u implies

$$
I_2 \leqslant 0. \tag{3.3.14}
$$

We have therefore

$$
\begin{aligned}
I_3 &\overset{(3.1.2)}{\leqslant} c \fint_{B_{2R}} \omega(|w-(w)_{B_{2R}}|)\Psi(x,|Dw|)\,dx \\
&\leqslant c \fint_{B_R} \omega\left(\operatorname*{osc}_{B_{2R}} w\right)\Psi(x,|Dw|)\,dx
\end{aligned}
$$

$$\overset{(3.3.6)}{\leqslant} c\omega\left(\operatorname*{osc}_{B_{2R}} u\right) \fint_{B_{2R}} \Psi(x, |Dw|)\, dx$$

$$\overset{(3.2.31),(3.2.33)}{\leqslant} c\omega(2[u]_{0,\gamma;\Omega_0} R^\gamma) \fint_{B_{2R}} \Psi(x, |Du|)\, dx$$

$$\overset{(2.1.1)}{\leqslant} c(\mathbf{data}(\Omega_0))\omega(R^\gamma) \fint_{B_{2R}} \Psi(x, |Du|)\, dx. \tag{3.3.15}$$

Observing that

$$|(w)_{B_{2R}} - (u)_{B_{2R}}| \overset{(3.3.11)}{\leqslant} \operatorname*{osc}_{B_{2R}} u, \tag{3.3.16}$$

as in the estimate for I_1, we still have

$$I_4 \leqslant c\omega(R^\gamma) \fint_{B_{2R}} \Psi(x, |Du|)\, dx. \tag{3.3.17}$$

Inserting all the estimates obtained for I_i with $i \in \{1, 2, 3, 4\}$ into (3.3.12) completes the proof of (3.3.3).

Let us now prove (3.3.7). By applying Theorem 2.4.1 with $d \equiv 1$, there exists $\theta_1 \equiv \theta_1(n, s(G), s(H_a), s(H_b)) \in (0, 1)$ such that

$$\begin{aligned} J &:= \fint_{B_{2R}} \Psi\left(x, \left|\frac{u-w}{R}\right|\right) dx \leqslant c \left(\fint_{B_{2R}} [\Psi(x, |Du - Dw|)]^{\theta_1}\, dx\right)^{\frac{1}{\theta_1}} \\ &\leqslant c \left(\fint_{B_{2R}} \left([\Psi(x, |Du| + |Dw|)]^{\frac{1}{2}} \frac{|Du - Dw|}{|Du| + |Dw|}\right)^{\theta_1} [\Psi(x, |Du| + |Dw|)]^{\frac{\theta_1}{2}}\, dx\right)^{\frac{1}{\theta_1}}, \end{aligned} \tag{3.3.18}$$

where in the last inequality of the last display we have used (2.1.11) for Ψ. Applying Hölder's inequality with conjugate exponents $\left(\frac{2}{\theta_1}, \frac{2}{2-\theta_1}\right)$ to the right hand side of the last display and (2.1.10), we get

$$
\begin{aligned}
J &\leqslant c \left(\fint_{B_{2R}} \Psi(x, |Du| + |Dw|) \frac{|Du - Dw|^2}{(|Du| + |Dw|)^2} \, dx \right)^{\frac{1}{2}} \\
&\quad \left(\fint_{B_{2R}} [\Psi(x, |Du| + |Dw|)]^{\frac{\theta_1}{2-\theta_1}} \, dx \right)^{\frac{2-\theta_1}{2\theta_1}} \\
&\leqslant c \left(\fint_{B_{2R}} |V_\Psi(x, Du) - V_\Psi(x, Dw)|^2 \, dx \right)^{\frac{1}{2}} \left(\fint_{B_{2R}} \Psi(x, |Du| + |Dw|) \, dx \right)^{\frac{1}{2}} \\
&\leqslant c(\mathbf{data}(\Omega_0)) [\omega(R^\gamma)]^{\frac{1}{2}} \fint_{B_{2R}} \Psi(x, |Du|) \, dx, \qquad (3.3.19)
\end{aligned}
$$

where in the last inequality of the above display we have used (3.3.3), and then (3.3.4). Combining the last two displays we arrive at (3.3.7). □

Next we consider the functional defined by

$$
W^{1,1}(B_R) \ni v \mapsto \mathcal{F}_c(v) := \int_{B_R} F_c(x, Dv) \, dx, \qquad (3.3.20)
$$

where the density function is given by

$$
F_c(x, z) := F_G \left(x_c, (u)_{B_{2R}}, z\right) + a(x) F_{H_a} \left(x_c, (u)_{B_{2R}}, z\right) + b(x) F_{H_b} \left(x_c, (u)_{B_{2R}}, z\right) \qquad (3.3.21)
$$

for some fixed point $x_c \in B_R$ and for every $x \in \Omega$ and $z \in \mathbb{R}^n$. Now we consider a function $w_c \in w + W_0^{1,\Psi}(B_R)$ being the solution to the following variational Dirichlet problem:

$$
\begin{cases}
w_c \mapsto \min\limits_{v} \mathcal{F}_c(v) \\
v \in w + W_0^{1,\Psi}(B_R),
\end{cases} \qquad (3.3.22)
$$

where $w \in W^{1,\Psi}(B_{2R})$ is the solution to the variational problem (3.3.2).

Lemma 3.3.2. *Let $w_c \in W^{1,\Psi}(B_R)$ be the solution to the variational problem* (3.3.22) *under the assumptions* (1.0.11), (3.1.2) *and* (3.1.4)*. Let the coefficient functions $a(\cdot) \in C^{\omega_a}(\Omega)$ and $b(\cdot) \in C^{\omega_b}(\Omega)$ for ω_a, ω_b being non-negative, continuous and concave functions vanishing at the origin. Assume that one of the assumptions* (1.0.13), (1.0.14) *and* (1.0.15) *is satisfied. Then there exists a constant $c \equiv c(\boldsymbol{data}(\Omega_0))$ such that*

$$\fint_{B_R} |V_\Psi(x, Dw) - V_\Psi(x, Dw_c)|^2 \, dx \leqslant c\omega(R) \fint_{B_R} \Psi(x, |Du|) \, dx. \tag{3.3.23}$$

Moreover, the following estimates hold true:

$$\fint_{B_R} \Psi(x, |Dw_c|) \, dx \leqslant \frac{L}{\nu} \fint_{B_R} \Psi(x, |Dw|) \, dx, \tag{3.3.24}$$

$$\|w_c\|_{L^\infty(B_R)} \leqslant \|w\|_{L^\infty(B_R)}, \tag{3.3.25}$$

$$\operatorname*{osc}_{B_R} w_c \leqslant \operatorname*{osc}_{B_R} w \tag{3.3.26}$$

and

$$\fint_{B_R} \Psi\left(x, \left|\frac{w - w_c}{R}\right|\right) dx \leqslant c[\omega(R)]^{\frac{1}{2}} \fint_{B_R} \Psi(x, |Dw|) \, dx \tag{3.3.27}$$

for some constant $c \equiv c(\boldsymbol{data})$. Finally, there exists a higher integrability exponent $\delta_0 \equiv \delta_0(\boldsymbol{data})$ with $\delta_0 \leqslant \delta$ with δ having been determined via Theorem 3.2.5, and a constant $c \equiv c(\boldsymbol{data})$ such that

$$\left(\fint_{B_{R/2}} [\Psi(x, |Dw_c|)]^{1+\delta_0} \, dx\right)^{\frac{1}{1+\delta_0}} \leqslant c \fint_{B_R} \Psi(x, |Dw_c|) \, dx. \tag{3.3.28}$$

Proof. Essentially, the proof is similar to the proof of Lemma 3.3.1. The estimates (3.3.24)-(3.3.26) can be obtained as for (3.3.4)-(3.3.6). We now focus on proving (3.3.23). The Euler-Lagrange equation arising from the functional

$\mathcal{F}_c$ defined in (3.3.20)

$$\fint_{B_R} \langle D_z F_c(x, Dw_c), D\varphi \rangle \, dx = 0 \tag{3.3.29}$$

is valid, whenever $\varphi \in W_0^{1,\Psi}(B_R)$. Then using (3.1.2), we have

$$\begin{aligned}
&\fint_{B_R} |V_\Psi(x, Dw) - V_\Psi(x, Dw_c)|^2 \, dx \\
&\quad \leqslant c \fint_{B_R} \langle D_z F_c(x, Dw) - D_z F_c(x, Dw_c), Dw - Dw_c \rangle \, dx \\
&\quad \leqslant c \fint_{B_R} |D_z F_G(x_c, (u)_{B_{2R}}, Dw) - D_z F_G(x, (u)_{B_{2R}}, Dw)||Dw - Dw_c| \, dx \\
&\qquad + c \fint_{B_R} a(x)|D_z F_{H_a}(x_c, (u)_{B_{2R}}, Dw) - D_z F_{H_a}(x, (u)_{B_{2R}}, Dw)||Dw - Dw_c| \, dx \\
&\qquad + c \fint_{B_R} b(x)|D_z F_{H_b}(x_c, (u)_{B_{2R}}, Dw) - D_z F_{H_b}(x, (u)_{B_{2R}}, Dw)||Dw - Dw_c| \, dx \\
&\quad \leqslant c\omega(R) \fint_{B_R} \Psi(x, |Dw|) \, dx
\end{aligned} \tag{3.3.30}$$

for some constant $c \equiv c(n, s(G), s(H_a), s(H_b), \nu, L)$. This proves (3.3.23), and (3.3.27) follows from this estimate together with applying the arguments used in (3.3.18)-(3.3.19). Since w_c is a L/ν-minimizer of the functional $\mathcal{F}_c$ defined in (3.3.22), we are able to apply Lemma 3.2.1 with the choices of $\upsilon \equiv w_c$, $\rho \equiv R/2$, $r \equiv R$ and $k \equiv (w_c)_{B_R}$. In turn, it gives us that

$$\fint_{B_{R/2}} \Psi(x, |Dw_c|) \, dx \leqslant c \fint_{B_R} \Psi\left(x, \left|\frac{w_c - (w_c)_{B_R}}{R}\right|\right) \, dx \tag{3.3.31}$$

holds with $c \equiv c(n, s(G), s(H_a), s(H_b), L, \nu)$. Then applying Remark 2.4.1, there exists a positive exponent $\theta \equiv \theta(n, s(G), s(H_a), s(H_b)) \in (0, 1)$ such that

$$\fint_{B_{R/2}} \Psi\left(x, |Dw_c|\right)\, dx \leqslant c\bar{\kappa}_{sp} \left[\fint_{B_R} [\Psi(x, |Dw_c|)]^{\theta}\, dx \right]^{\frac{1}{\theta}} \tag{3.3.32}$$

holds with some constant $c \equiv c(n, s(G), s(H_a), s(H_b), L, \nu, \omega_a(1), \omega_b(1))$, where

$$\bar{\kappa}_{sp} = \begin{cases} 1 + \lambda_1([a]_{\omega_a} + [b]_{\omega_b}) \left(1 + \left(\int_{B_R} G(|Dw_c|)\, dx \right)^{\frac{1}{n}} \right) \\ \text{if (1.0.13) is considered,} & \text{(3.3.33a)} \\ 1 + \lambda_2([a]_{\omega_a} + [b]_{\omega_b}) \left(1 + \|w_c\|_{L^\infty(B_R)} \right) \\ \text{if (1.0.14) is considered,} & \text{(3.3.33b)} \\ 1 + \lambda_3([a]_{\omega_a} + [b]_{\omega_b}) \left(1 + \left[R^{-\gamma} \operatorname{osc}_{B_R} w_c \right]^{\frac{1}{1-\gamma}} \right) \\ \text{if (1.0.15) is considered.} & \text{(3.3.33c)} \end{cases}$$

Furthermore, taking into account (3.3.4)-(3.3.6) and (3.3.24)-(3.3.26) in the last display, we conclude that

$$\fint_{B_{R/2}} \Psi(x, |Dw_c|)\, dx \leqslant c \left[\fint_{B_R} [\Psi\left(x, |Dw_c|\right)]^{\theta}\, dx \right]^{\frac{1}{\theta}} \tag{3.3.34}$$

holds for some constant $\theta \equiv \theta(n, s(G), s(H_a), s(H_b)) \in (0, 1)$ and $c \equiv c(\textbf{data})$. The estimate (3.3.28) follows from applying a variant of Gehring's lemma. $\square$

To go further let us introduce the excess functional defined by

$$E(v, B_r) := \left(\Psi^-_{B_{2r}}\right)^{-1} \left(\fint_{B_r} \Psi^-_{B_{2r}} \left(\left| \frac{v - (v)_{B_r}}{2r} \right| \right)\, dx \right) \tag{3.3.35}$$

for any function $v \in L^1(B_{2r})$ and a ball $B_{2r} \subset \Omega$, where we note that $\left(\Psi^-_{B_{2r}}\right)^{-1}$ is the inverse function of $\Psi^-_{B_{2r}}$. By the convexity of $\Psi^-_{B_{2r}}$ together with Lemma

2.1.1 and Remark 2.1.2, one can see that

$$E(v,B_r) \leqslant c\left(\Psi^-_{B_{2r}}\right)^{-1}\left(\fint_{B_r} \Psi^-_{B_{2r}}\left(\left|\frac{v-v_0}{2r}\right|\right)\,dx\right) \tag{3.3.36}$$

for some constant $c \equiv c(s(\Psi))$, whenever $v_0 \in \mathbb{R}$ is an arbitrary number.

Lemma 3.3.3. *Let $u \in W^{1,\Psi}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ defined in* (1.0.10) *under the assumptions* (1.0.11), (3.1.2) *and* (3.1.4)*. Let $w_c \in W^{1,\Psi}(B_R)$ be the solution to the variational problem* (3.3.22)*. If one of the assumptions* (3.1.11a)-(3.1.11e) *is satisfied, then for every $\varepsilon^* \in (0,1)$, there exists a positive radius*

$$R^* \equiv R^*(\boldsymbol{data}(\Omega_0),\varepsilon^*) \tag{3.3.37}$$

such that

$$\begin{aligned}
&\fint_{B_{\tau R}} \Psi^-_{B_R}\left(\left|\frac{w_c-(w_c)_{B_{\tau R}}}{\tau R}\right|\right)dx\\
&\leqslant c\left(1+\tau^{-(n+s(\Psi)+1)}\varepsilon^*\right)\fint_{B_{R/2}} \Psi^-_{B_R}\left(\left|\frac{w_c-(w_c)_{B_{R/2}}}{R}\right|\right)dx
\end{aligned} \tag{3.3.38}$$

for some constant $c \equiv c\left(\boldsymbol{data}(\Omega_0)\right)$, whenever $\tau \in (0,1/16)$ and $R \leqslant R^*$.

Proof. We assume $E(w_c,B_{R/2}) > 0$, otherwise (3.3.38) is trivial. For the sake of simplicity during the proof, we write

$$E(R) := E(w_c,B_{R/2}) = \left(\Psi^-_{B_R}\right)^{-1}\left(\fint_{B_{R/2}} \Psi^-_{B_R}\left(\left|\frac{w_c-(w_c)_{B_{R/2}}}{R}\right|\right)dx\right). \tag{3.3.39}$$

The proof falls in several delicate steps.

Step 1: Initial settings on w_c. Applying Lemma 3.2.2 to $B_{R/2}$ with

$k \equiv (w_c)_{B_{R/2}}$, we have

$$\fint_{B_{R/4}} \Psi\left(x, |Dw_c|\right) dx \leqslant c \fint_{B_{R/2}} \Psi^{-}_{B_R}\left(\left|\frac{w_c - (w_c)_{B_{R/2}}}{R}\right|\right) dx \tag{3.3.40}$$

for some constant $c \equiv c(\mathbf{data})$. Moreover, it follows from Lemma 3.3.2 that there exists a higher integrability exponent $\delta_0 \equiv \delta_0(\mathbf{data})$ such that

$$\left(\fint_{B_{R/8}} \left[\Psi(x, |Dw_c|)\right]^{1+\delta_0} dx\right)^{\frac{1}{1+\delta_0}} \leqslant c \fint_{B_{R/4}} \Psi(x, |Dw_c|)\, dx \tag{3.3.41}$$

for a constant $c \equiv c(\mathbf{data})$.

Step 2: Scaling. We set the scaled functions of $w_c(\cdot)$, $a(\cdot)$ and $b(\cdot)$ in the ball B_1 by

$$\begin{cases} \bar{w}_c(x) := \dfrac{w_c(x_0 + Rx) - (w_c)_{B_{R/2}}}{E(R)R}, & (3.3.42\text{a}) \\ \bar{a}(x) := a(x_0 + Rx)\dfrac{H_a(E(R))}{\Psi^{-}_{B_R}(E(R))} & \\ \text{and} \quad \bar{b}(x) := b(x_0 + Rx)\dfrac{H_b(E(R))}{\Psi^{-}_{B_R}(E(R))}. & (3.3.42\text{b}) \end{cases}$$

for every $x \in B_1$. Now we define the control function and energy integrand associated to our scaling in (3.3.42a)-(3.3.42b) as

$$\begin{cases} \bar{\Psi}(x, |z|) := \bar{G}(|z|) + \bar{a}(x)\bar{H}_a(|z|) + \bar{b}(x)\bar{H}_b(|z|), & (3.3.43\text{a}) \\ \bar{F}(x, z) := \bar{F}_G(z) + \bar{a}(x)\bar{F}_{H_a}(z) + \bar{b}(x)\bar{F}_{H_b}(z), & (3.3.43\text{b}) \\ \bar{F}_G(z) := \dfrac{F_G(x_c, (u)_{B_{2R}}, E(R)z)}{\Psi^{-}_{B_R}(E(R))}, & \\ \bar{F}_{H_a}(z) := \dfrac{F_{H_a}(x_c, (u)_{B_{2R}}, E(R)z)}{H_a(E(R))}, & \\ \bar{F}_{H_b}(x, z) := \dfrac{F_{H_b}(x_c, (u)_{B_{2R}}, E(R)z)}{H_b(E(R))} & (3.3.43\text{c}) \\ \text{and} \quad \bar{A}(x, z) := D_z\bar{F}(x, z) & (3.3.43\text{d}) \end{cases}$$

for every $x \in B_1$ and $z \in \mathbb{R}^n$, where the point $x_c \in B_R$ has been fixed in (3.3.22) and

$$\bar{G}(t) := \frac{G(E(R)t)}{\Psi^-_{B_R}(E(R))}, \quad \bar{H}_a(t) := \frac{H_a(E(R)t)}{H_a(E(R))}, \quad \bar{H}_b(t) := \frac{H_b(E(R)t)}{H_b(E(R))} \tag{3.3.44}$$

for every $t \geqslant 0$. Clearly, one can see that $\bar{G}, \bar{H}_a, \bar{H}_b \in \mathcal{N}$ with indices $s(G), s(H_a), s(H_b)$, respectively, and also that

$$\bar{G}(1) \leqslant 1, \quad \bar{H}_a(1) = 1 \quad \text{and} \quad \bar{H}_b(1) = 1. \tag{3.3.45}$$

Then one can check that the function $\bar{w}_c$ minimizes the following functional

$$W^{1,\bar{\Psi}}(B_1) \ni v \mapsto \int_{B_1} \bar{F}(x, Dv)\, dx, \tag{3.3.46}$$

where the functions $\bar{\Psi}(\cdot)$ and $\bar{F}(\cdot)$ have been defined in (3.3.43a) and (3.3.43b), respectively. The Euler-Lagrange equation associated to the functional in (3.3.46) becomes

$$\fint_{B_1} \left\langle \bar{A}(x, D\bar{w}_c), D\varphi \right\rangle \, dx = \fint_{B_1} \left\langle D_z\bar{F}\left(x, D\bar{w}_c\right), D\varphi \right\rangle \, dx = 0 \tag{3.3.47}$$

for every $\varphi \in W^{1,\bar{\Psi}}_0(B_1)$. By the assumptions (1.0.11) and (3.1.2) via elementary computations, we have the following structure condition in the scaled settings:

$$\begin{cases} \nu\bar{\Psi}(x, |z|) \leqslant \bar{F}(x, z) \leqslant L\bar{\Psi}(x, |z|), & (3.3.48a) \\ |\bar{A}(x, z)||z| + |D_z\bar{A}(x, z)||z|^2 \leqslant L\bar{\Psi}(x, |z|), & (3.3.48b) \\ \nu\dfrac{\bar{\Psi}(x, |z|)}{|z|^2}|\xi|^2 \leqslant \left\langle D_z\bar{A}(x, z)\xi, \xi \right\rangle & (3.3.48c) \end{cases}$$

hold true for every $x \in B_1$ and $z \in \mathbb{R}^n \setminus \{0\}$.

Step 3: Freezing. Now we consider frozen functional and vector field associated to $\bar{F}(\cdot)$ and $\bar{A}(\cdot)$ defined in (3.3.43b)-(3.3.43d). Let $\bar{x}_a, \bar{x}_b \in \overline{B}_1$ be points such that $\bar{a}(\bar{x}_a) = \inf\limits_{x\in B_1} \bar{a}(x)$ and $\bar{b}(\bar{x}_b) = \inf\limits_{x\in B_1} \bar{b}(x)$. Then we denote by

$$\bar{F}_0(z) := \bar{F}_G(z) + \bar{a}(\bar{x}_a)\bar{F}_{H_a}(z) + \bar{b}(\bar{x}_b)\bar{F}_{H_b}(z), \tag{3.3.49}$$

$$\bar{A}_0(z) := D_z\bar{F}_0(z) \tag{3.3.50}$$

and

$$\bar{\Psi}_0(t) := \bar{G}(t) + \bar{a}(\bar{x}_a)\bar{H}_a(t) + \bar{b}(\bar{x}_b)\bar{H}_b(t) \tag{3.3.51}$$

for every $x \in B_1$, $z \in \mathbb{R}^n$ and $t \geqslant 0$. By the very definition in (3.3.43a)-(3.3.43d), straightforwardly one can see

$$\bar{\Psi}_0(1) = 1. \tag{3.3.52}$$

In our new scaled settings, we now consider the functional

$$W^{1,\bar{\Psi}_0}\left(B_{1/8}\right) \ni v \mapsto \int_{B_{1/8}} \bar{F}_0(Dv)\,dx. \tag{3.3.53}$$

We observe that the newly defined integrand $\bar{F}_0(\cdot)$ and vector field $\bar{A}_0(\cdot)$ satisfy the growth and ellipticity conditions as

$$\begin{cases} \nu\bar{\Psi}_0(|z|) \leqslant \bar{F}_0(z) \leqslant L\bar{\Psi}_0(|z|), & \text{(3.3.54a)}\\ |\bar{A}_0(z)||z| + |D_z\bar{A}_0(z)||z|^2 \leqslant L\bar{\Psi}_0(|z|), & \text{(3.3.54b)}\\ \nu\dfrac{\bar{\Psi}_0(|z|)}{|z|^2}|\xi|^2 \leqslant \left\langle D_z\bar{A}_0(z)\xi,\xi\right\rangle & \text{(3.3.54c)} \end{cases}$$

for every $z \in \mathbb{R}^n \setminus \{0\}$ and $\xi \in \mathbb{R}^n$. Therefore, the estimates (3.3.40) and (3.3.41) are written in the view of $\bar{w}_c$ as

$$\fint_{B_{1/4}} \bar{\Psi}(x,|D\bar{w}_c|)\,dx + \left(\fint_{B_{1/8}} [\bar{\Psi}(x,|D\bar{w}_c|)]^{1+\delta_0}\,dx\right)^{\frac{1}{1+\delta_0}} \leqslant c(\mathbf{data}). \tag{3.3.55}$$

Step 4: Harmonic type approximation. Let $\varphi \in W_0^{1,\infty}\left(B_{1/8}\right)$. Using

(3.3.47), we see

$$
\begin{aligned}
I_0 &:= \left| \fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}_c), D\varphi \right\rangle \, dx \right| = \left| \fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}_c) - \bar{A}(x, D\bar{w}_c), D\varphi \right\rangle \, dx \right| \\
&\leqslant \fint_{B_{1/8}} |\bar{A}_0(D\bar{w}_c) - \bar{A}(x, D\bar{w}_c)| \, dx \, \|D\varphi\|_{L^\infty(B_{1/8})} =: I_1 \, \|D\varphi\|_{L^\infty(B_{1/8})} \, .
\end{aligned} \tag{3.3.56}
$$

Now using (3.1.2), we see

$$
\begin{aligned}
I_1 &\leqslant L \fint_{B_{1/8}} |\bar{a}(x) - \bar{a}(\bar{x}_a)| \frac{\bar{H}_a(|D\bar{w}_c|)}{|D\bar{w}_c|} \, dx \\
&\quad + L \fint_{B_{1/8}} |\bar{b}(x) - \bar{b}(\bar{x}_b)| \frac{\bar{H}_b(|D\bar{w}_c|)}{|D\bar{w}_c|} \, dx =: L \left(I_{11} + I_{12} \right) .
\end{aligned} \tag{3.3.57}
$$

Now we estimate the terms appearing in the last display. In turn, using (2.1.7), (3.3.45) and (3.3.55), we have

$$
\begin{aligned}
I_{11} &\leqslant c \fint_{B_{1/8}} |\bar{a}(x) - \bar{a}(\bar{x}_a)| \left([\bar{H}_a(|D\bar{w}_c|)]^{\frac{1}{s(H_a)+1}} + [\bar{H}_a(|D\bar{w}_c|)]^{\frac{s(H_a)}{s(H_a)+1}} \right) \, dx \\
&\leqslant c \, \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{s(H_a)}{s(H_a)+1}} \left(\fint_{B_{1/8}} \bar{a}(x) \bar{H}_a(|D\bar{w}_c|) \, dx \right)^{\frac{1}{s(H_a)+1}} \\
&\quad + c \, \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_a)+1}} \left(\fint_{B_{1/8}} \bar{a}(x) \bar{H}_a(|D\bar{w}_c|) \, dx \right)^{\frac{s(H_a)}{s(H_a)+1}} \\
&\leqslant c(\mathbf{data}) \left(\|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_a)+1}} + \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{s(H_a)}{s(H_a)+1}} \right) ,
\end{aligned} \tag{3.3.58}
$$

where we have used also Hölder's inequality and the fact that $\bar{a}(\bar{x}_a) \leqslant \bar{a}(x)$

for every $x \in B_1$. In a similar way, we have

$$I_{12} \leqslant c(\mathbf{data}) \left(\left\|\bar{b} - \bar{b}(\bar{x}_b)\right\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_b)+1}} + \left\|\bar{b} - \bar{b}(\bar{x}_b)\right\|_{L^\infty(B_{1/8})}^{\frac{s(H_b)}{s(H_b)+1}} \right). \tag{3.3.59}$$

Inserting those estimates into (3.3.57) and then (3.3.56), we find

$$\begin{aligned} I_0 \leqslant{}& c(\mathbf{data}) \left(\|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_a)+1}} + \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{s(H_a)}{s(H_a)+1}} \right) \\ &+ c(\mathbf{data}) \left(\left\|\bar{b} - \bar{b}(\bar{x}_b)\right\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_b)+1}} + \left\|\bar{b} - \bar{b}(\bar{x}_b)\right\|_{L^\infty(B_{1/8})}^{\frac{s(H_b)}{s(H_b)+1}} \right). \end{aligned} \tag{3.3.60}$$

Now we estimate the terms $\|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}$ and $\left\|\bar{b} - \bar{b}(\bar{x}_b)\right\|_{L^\infty(B_{1/8})}$ depending on which assumption of (3.1.11a)-(3.1.11e) comes into play. Recalling the definition of $\bar{a}(\cdot)$, $\bar{b}(\cdot)$ in (3.3.42b) and the excess functional in (3.3.39), we have

$$I_a := \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})} \leqslant c\omega_a(R) \frac{H_a(E(R))}{\Psi^-_{B_R}(E(R))} \tag{3.3.61}$$

and

$$I_b := \left\|\bar{b} - \bar{b}(\bar{x}_b)\right\|_{L^\infty(B_{1/8})} \leqslant c\omega_b(R) \frac{H_b(E(R))}{\Psi^-_{B_R}(E(R))}. \tag{3.3.62}$$

Case 1: Assumption (3.1.11a) **is in force.** The assumption $(3.1.11a)_2$ implies that for any $\varepsilon \in (0,1)$ there exists $\mu_1 > 0$ depending on ε such that

$$\Lambda\left(t, G^{-1}\left(t^{-n}\right)\right) \leqslant \varepsilon \quad \text{for every} \quad t \in (0, \mu_1). \tag{3.3.63}$$

Then using this one and (1.0.13), we continue to estimate I_a in (3.3.61) as

$$\begin{aligned} I_a &\leqslant c\omega_a(R) \frac{(H_a \circ G^{-1})\left(\Psi^-_{B_R}(E(R))\right)}{\Psi^-_{B_R}(E(R))} \\ &\leqslant c\omega_a(R)\varepsilon \left(1 + \frac{1}{\omega_a\left(\left[\Psi^-_{B_R}(E(R))\right]^{-\frac{1}{n}}\right)} \right) + c\omega_a(R)\left(1 + \frac{1}{\omega_a(\mu_1)}\right) \end{aligned} \tag{3.3.64}$$

with $c \equiv c([a]_{\omega_a}, \lambda_1)$, where we have used the fact that $\left(\Psi^-_{B_R}\right)^{-1}(t) \leqslant G^{-1}(t)$ for every $t \geqslant 0$. Using (2.1.2) together with the energy estimates (3.3.24) and (3.3.4), we observe that

$$\frac{1}{\omega_a\left(\left[\Psi^-_{B_R}(E(R))\right]^{-\frac{1}{n}}\right)} \leqslant \frac{c}{\omega_a(R)} + \frac{c}{\omega_a(R)} \int_{B_{R/2}} \Psi^-_{B_R}\left(\left|\frac{w_c - (w_c)_{B_{R/2}}}{R}\right|\right) dx$$
$$\leqslant \frac{c}{\omega_a(R)} + \frac{c}{\omega_a(R)} \int_{B_{2R}} \Psi\left(x, |Du|\right) dx \leqslant \frac{c(\mathbf{data})}{\omega_a(R)}. \tag{3.3.65}$$

Combining the last two displays, we conclude

$$I_a \leqslant c\left(\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a(\mu_1)}\right)\right) \tag{3.3.66}$$

with some constant $c \equiv c(\mathbf{data})$. In the same manner, we see

$$I_b \leqslant c\left(\varepsilon + \omega_b(R)\left(1 + \frac{1}{\omega_b(\mu_1)}\right)\right) \tag{3.3.67}$$

for some constant $c \equiv c(\mathbf{data})$. Therefore, inserting the estimates in the last two displays into (3.3.60) and recalling (3.3.56), we have

$$\left| \fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}_c), D\varphi \right\rangle dx \right| \leqslant c(\mathbf{data}) p_1(\varepsilon, R) \left\| D\varphi \right\|_{L^\infty(B_{1/8})}, \tag{3.3.68}$$

where

$$p_1(\varepsilon, R) :=$$
$$\left[\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a(\mu_1)}\right)\right]^{\frac{1}{s(H_a)+1}} + \left[\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a(\mu_1)}\right)\right]^{\frac{s(H_a)}{s(H_a)+1}}$$
$$+ \left[\varepsilon + \omega_b(R)\left(1 + \frac{1}{\omega_b(\mu_1)}\right)\right]^{\frac{1}{s(H_b)+1}} + \left[\varepsilon + \omega_b(R)\left(1 + \frac{1}{\omega_b(\mu_1)}\right)\right]^{\frac{s(H_b)}{s(H_b)+1}}. \tag{3.3.69}$$

Case 2: Assumption (3.1.11b) **is in force.** From the assumption $(3.1.11\text{b})_2$ it holds that for every $\varepsilon \in (0,1)$ there exists $\mu_2 > 0$ depending on ε such that

$$\Lambda\left(t, \frac{1}{t}\right) \leqslant \varepsilon \quad \text{for every} \quad t \in (0, \mu_2). \tag{3.3.70}$$

Then by the very definition of $\Psi^-_{B_R}$ in (2.1.3) together with (3.3.70) and (1.0.14), we have

$$\begin{aligned} I_a &\leqslant c\omega_a(R)\frac{H_a(E(R))}{G(E(R))} \\ &\leqslant c\omega_a(R)\varepsilon\left(1 + \frac{1}{\omega_a\left([E(R)]^{-1}\right)}\right) + c\omega_a(R)\left(1 + \frac{1}{\omega_a\left(\mu_2\right)}\right). \end{aligned} \tag{3.3.71}$$

Again using (2.1.1) together with taking into account (3.3.25) and (3.3.5), we see

$$\frac{1}{\omega_a\left([E(R)]^{-1}\right)} \leqslant \frac{1}{\omega_a\left(\frac{R}{2\|w_c\|_{L^\infty(B_R)}}\right)} \leqslant \frac{c(\mathbf{data})}{\omega_a(R)}. \tag{3.3.72}$$

Combining the last two displays, we find

$$I_a \leqslant c\left(\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a(\mu_2)}\right)\right) \tag{3.3.73}$$

with some constant $c \equiv c(\mathbf{data})$. Similarly, it holds that

$$I_b \leqslant c\left(\varepsilon + \omega_b(R)\left(1 + \frac{1}{\omega_a(\mu_2)}\right)\right). \tag{3.3.74}$$

Then, plugging the estimates in the last two displays into (3.3.60) and recalling (3.3.56), we have

$$\left| \fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}_c), D\varphi \right\rangle \, dx \right| \leqslant c(\mathbf{data}) p_2(\varepsilon, R) \left\| D\varphi \right\|_{L^\infty(B_{1/8})}, \tag{3.3.75}$$

where

$$
\begin{aligned}
&p_2(\varepsilon, R) := \\
&\left[\varepsilon + \omega_a(R)\left(1+\frac{1}{\omega_a(\mu_2)}\right)\right]^{\frac{1}{s(H_a)+1}} + \left[\varepsilon + \omega_a(R)\left(1+\frac{1}{\omega_a(\mu_2)}\right)\right]^{\frac{s(H_a)}{s(H_a)+1}} \\
&\quad + \left[\varepsilon + \omega_b(R)\left(1+\frac{1}{\omega_b(\mu_2)}\right)\right]^{\frac{1}{s(H_b)+1}} + \left[\varepsilon + \omega_b(R)\left(1+\frac{1}{\omega_b(\mu_2)}\right)\right]^{\frac{s(H_b)}{s(H_b)+1}}.
\end{aligned}
\tag{3.3.76}
$$

Case 3: Assumption (3.1.11c) **is in force.** The assumption $(3.1.11\text{c})_2$ implies that for any $\varepsilon \in (0,1)$ there exists $\mu_3 > 0$ depending on ε such that

$$
\Lambda\left(t^{\frac{1}{1-\gamma}}, \frac{1}{t}\right) \leqslant \varepsilon \quad \text{for every} \quad t \in (0, \mu_3). \tag{3.3.77}
$$

This one together with recalling (3.3.61) and (1.0.15), we see

$$
\begin{aligned}
I_a &\leqslant c\omega(R)\frac{H_a(E(R))}{G(E(R))} \\
&\leqslant c\omega_a(R)\varepsilon\left(1+\frac{1}{\omega_a\left([E(R)]^{-\frac{1}{1-\gamma}}\right)}\right) + c\omega_a(R)\left(1+\frac{1}{\omega_a\left(\mu_3^{\frac{1}{1-\gamma}}\right)}\right).
\end{aligned}
\tag{3.3.78}
$$

Now using (3.3.26), (3.3.6) and (1.0.15), we have

$$
\frac{1}{\omega_a\left([E(R)]^{-\frac{1}{1-\gamma}}\right)} \leqslant \frac{1}{\omega_a\left(\left[\frac{\operatorname{osc}_{B_{2R}} u}{R}\right]^{-\frac{1}{1-\gamma}}\right)} \leqslant \frac{c(\mathbf{data})}{\omega_a(R)}. \tag{3.3.79}
$$

Combining the last two displays, we find

$$
I_a \leqslant c\left(\varepsilon + \omega_a(R)\left(1+\frac{1}{\omega_a\left(\mu_3^{\frac{1}{1-\gamma}}\right)}\right)\right) \tag{3.3.80}
$$

for some constant $c \equiv c(\mathbf{data})$. In the same way, we show

$$I_b \leqslant c\left(\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a\left(\mu_3^{\frac{1}{1-\gamma}}\right)}\right)\right) \tag{3.3.81}$$

for some constant $c \equiv c(\mathbf{data})$. Using the estimates (3.3.80)-(3.3.81) in (3.3.60), we conclude

$$\left| \fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}_c), D\varphi \right\rangle \, dx \right| \leqslant c(\mathbf{data}) p_3(\varepsilon, R) \left\| D\varphi \right\|_{L^\infty(B_{1/8})}, \tag{3.3.82}$$

where

$$\begin{aligned} p_3(\varepsilon, R) &:= \\ &\left[\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a\left(\mu_3^{\frac{1}{1-\gamma}}\right)}\right)\right]^{\frac{1}{s(H_a)+1}} + \left[\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a\left(\mu_3^{\frac{1}{1-\gamma}}\right)}\right)\right]^{\frac{s(H_a)}{s(H_a)+1}} \\ &+ \left[\varepsilon + \omega_b(R)\left(1 + \frac{1}{\omega_b\left(\mu_3^{\frac{1}{1-\gamma}}\right)}\right)\right]^{\frac{1}{s(H_b)+1}} + \left[\varepsilon + \omega_b(R)\left(1 + \frac{1}{\omega_b\left(\mu_3^{\frac{1}{1-\gamma}}\right)}\right)\right]^{\frac{s(H_b)}{s(H_b)+1}}. \end{aligned} \tag{3.3.83}$$

Case 4. Assumption (3.1.11d) **is in force.** We treat this case in a different way rather than the estimate used in (3.3.63)-(3.3.69). In fact, we take an advantage that $w_a(\cdot)$ is a power function. Then recalling I_a introduced in (3.3.61), we see that

$$I_a \leqslant cR^\alpha \frac{(H_a \circ G^{-1})\left(\Psi^-_{B_R}(E(R))\right)}{\Psi^-_{B_R}(E(R))}$$

$$
\begin{aligned}
&\leqslant cR^{\alpha}\left(1+\left[\fint_{B_{R/2}} \Psi^{-}_{B_R}\left(\left|\frac{w_c-(w_c)_{B_{R/2}}}{R}\right|\right)\,dx\right]^{\frac{\alpha}{n}}\right)\\
&\leqslant cR^{\alpha}+c\left(\int_{B_{R/2}} \Psi^{-}_{B_R}\left(|Dw_c|\right)\,dx\right)^{\frac{\alpha}{n}} \leqslant cR^{\alpha}+c\left(\int_{B_{2R}} \Psi\left(x,|Du|\right)\,dx\right)^{\frac{\alpha}{n}}\\
&\leqslant cR^{\alpha}+cR^{\frac{\alpha\delta}{1+\delta}}\left(\int_{B_{2R}} \left[\Psi\left(x,|Du|\right)\right]^{1+\delta}\,dx\right)^{\frac{\alpha}{n(1+\delta)}}\\
&\leqslant c(\mathbf{data}(\Omega_0))R^{\frac{\alpha\delta}{1+\delta}} \qquad (3.3.84)
\end{aligned}
$$

for a higher integrability exponent δ coming from Theorem 3.2.5, where we have used (3.3.24), (3.3.4) together with (3.2.62). By arguing similarly, we estimate I_b in (3.3.62) as

$$
I_b \leqslant c(\mathbf{data}(\Omega_0))R^{\frac{\beta\delta}{1+\delta}}. \qquad (3.3.85)
$$

Using estimates from the last two displays in (3.3.60) and recalling $R \leqslant 1$, we see

$$
\left|\fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}_c), D\varphi\right\rangle\,dx\right| \leqslant c(\mathbf{data}(\Omega_0))q_1(R)\left\|D\varphi\right\|_{L^{\infty}(B_{1/8})}, \qquad (3.3.86)
$$

where

$$
q_1(R) := R^{\frac{\alpha\delta}{(1+\delta)(1+s(H_a))}} + R^{\frac{\beta\delta}{(1+\delta)(1+s(H_b))}}. \qquad (3.3.87)
$$

Case 5: Assumption (3.1.11e) **is in force.** Again we estimate I_a and I_b introduced in (3.3.61)-(3.3.62). Using the assumption (1.0.14), (3.3.26) and (3.3.6), we have

$$
I_a \leqslant cR^{\alpha}\frac{H_a(E(R))}{G(E(R))}
$$

$$\leqslant cR^{\alpha}\left(1+\left[\left(\Psi^{-}_{B_R}\right)^{-1}\left(\fint_{B_{R/2}} \Psi^{-}_{B_R}\left(\left|\frac{w_c-(w_c)_{B_{R/2}}}{R}\right|\right)\,dx\right)\right]^{\alpha}\right)$$

$$\leqslant c\left(R^{\alpha}+\left[\operatorname*{osc}_{B_{2R}} u\right]^{\alpha}\right)\leqslant c(\mathbf{data}(\Omega_0))R^{\gamma\alpha}, \tag{3.3.88}$$

where we have also used (3.2.33) and the Hölder continuity exponent γ came from Theorem 3.2.2. Similarly, we see

$$I_b \leqslant c(\mathbf{data}(\Omega_0))R^{\gamma\beta}. \tag{3.3.89}$$

Inserting the estimates from the last two displays into (3.3.60) and recalling $R\leqslant 1$, we see

$$\left|\fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}_c), D\varphi\right\rangle\,dx\right| \leqslant c(\mathbf{data}(\Omega_0))q_2(R)\left\|D\varphi\right\|_{L^\infty(B_{1/8})}, \tag{3.3.90}$$

where

$$q_2(R) := R^{\frac{\alpha\gamma}{1+s(H_a)}} + R^{\frac{\beta\gamma}{1+s(H_b)}} \tag{3.3.91}$$

Collecting the estimates obtained in (3.3.68), (3.3.75),(3.3.82), (3.3.86) and (3.3.90), we conclude with

$$\left|\fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}_c), D\varphi\right\rangle\,dx\right| \leqslant c_h d(\varepsilon,R)\left\|D\varphi\right\|_{L^\infty(B_{1/8})} \tag{3.3.92}$$

for some constant $c_h\equiv c_h(\mathbf{data}(\Omega_0))$, whenever $\varphi\in W^{1,\infty}_0(B_{1/8})$, where

$$d(\varepsilon,R) := \begin{cases} p_1(\varepsilon,R) & \text{if (3.1.11a) is assumed,}\\ p_2(\varepsilon,R) & \text{if (3.1.11b) is assumed,}\\ p_3(\varepsilon,R) & \text{if (3.1.11c) is assumed,}\\ q_1(R) & \text{if (3.1.11d) is assumed,}\\ q_2(R) & \text{if (3.1.11e) is assumed,}\end{cases} \tag{3.3.93}$$

in which p_1, p_2, p_3, q_1 and q_2 have been defined in (3.3.69), (3.3.76), (3.3.83), (3.3.87) and (3.3.91), respectively. By (3.3.52), (3.3.54a)-(3.3.55) and (3.3.92), we are able to apply Lemma 2.5.1 with $A_0(z) \equiv \bar{A}_0(z)$, $\Psi_0(t) \equiv \bar{\Psi}_0(t)$ with $a_0 \equiv \bar{a}(\bar{x}_a)$ and $b_0 \equiv \bar{b}(\bar{x}_b)$, to discover that there exists $\bar{h} \in \bar{w}_c + W_0^{1,\bar{\Psi}_0}(B_{1/8})$ such that

$$\fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{h}), D\varphi \right\rangle dx = 0 \qquad \text{for all} \qquad \varphi \in W_0^{1,\infty}(B_{1/8}), \tag{3.3.94}$$

$$\fint_{B_{1/4}} \bar{\Psi}_0(|D\bar{h}|)\, dx + \fint_{B_{1/8}} [\bar{\Psi}_0(|D\bar{h}|)]^{1+\delta_1}\, dx \leqslant c \text{ for some } \delta_1 \leqslant \delta_0, \tag{3.3.95}$$

$$\begin{aligned} &\fint_{B_{1/8}} \left(|V_{\bar{G}}(D\bar{w}_c) - V_{\bar{G}}(D\bar{h})|^2 + \bar{a}(\bar{x}_a)|V_{\bar{H}_a}(D\bar{w}_c) - V_{\bar{H}_a}(D\bar{h})|^2 \right. \\ &\qquad \left. + \bar{b}(\bar{x}_b)|V_{\bar{H}_b}(D\bar{w}_c) - V_{\bar{H}_b}(D\bar{h})|^2 \right) dx \\ &\qquad \leqslant c[d(\varepsilon, R)]^{s_1} \end{aligned} \tag{3.3.96}$$

and finally

$$\fint_{B_{1/8}} \left(\bar{G}\left(|\bar{w}_c - \bar{h}|\right) + \bar{a}(\bar{x}_a)\bar{H}_a\left(|\bar{w}_c - \bar{h}|\right) + \bar{b}(\bar{x}_b)\bar{H}_b\left(|\bar{w}_c - \bar{h}|\right) \right) dx \leqslant c_d[d(\varepsilon, R)]^{s_0} \tag{3.3.97}$$

with some constants $c, c_d \equiv c, c_d(\mathbf{data}(\Omega_0)) \geqslant 1$ and $s_0, s_1 \equiv s_0, s_1(\mathbf{data}) \in (0,1)$, but they are all independent of R. Therefore, for a given $\varepsilon^* \in (0,1)$ as in the statement of our lemma, we choose small enough ε and R^* to satisfy

$$c_d \left[d(\varepsilon, R^*)\right]^{s_0} \leqslant \varepsilon^*. \tag{3.3.98}$$

Since the constants c_d and s_0 only depend on $\mathbf{data}(\Omega_0)$ and $\mathbf{data}$, respectively, the last display gives us the dependence of R^* as in (3.3.37). Furthermore, by (3.3.97), we conclude with

$$\fint_{B_{1/8}} \left[\bar{G}\left(|\bar{w}_c - \bar{h}|\right) + \bar{a}(\bar{x}_a)\bar{H}_a\left(|\bar{w}_c - \bar{h}|\right) + \bar{b}(\bar{x}_b)\bar{H}_b\left(|\bar{w}_c - \bar{h}|\right)\right] dx \leqslant \varepsilon^*. \tag{3.3.99}$$

Proof of (3.3.38)**.** We observe that by a standard density argument, the relation in (3.3.94) still holds for every $\varphi \in W_0^{1,1}(B_{1/8})$ with $\bar{\Psi}_0(|D\varphi|) \in L^1(B_{1/8})$. Recalling (3.3.49) and (3.3.50), we see that $\bar{h}$ is a local minimizer of the functional

$$W^{1,\bar{\Psi}_0}(B_{1/8}) \ni v \mapsto \int_{B_{1/8}} \bar{F}_0(Dv)\, dx. \tag{3.3.100}$$

Since the conditions (3.3.54a)-(3.3.54c) are satisfied for the integrand $\bar{F}_0(\cdot)$, we are in a position to apply the results from [111] to obtain the following a priori Lipschitz estimate:

$$\sup_{B_{1/16}} \bar{\Psi}_0(|D\bar{h}|) \leqslant c \fint_{B_{1/8}} \bar{\Psi}_0(|D\bar{h}|)\, dx \tag{3.3.101}$$

with some constant $c \equiv c(n, s(G), s(H_a), s(H_b), \nu, L)$. For any $\tau \in (0, 1/16)$, we have that

$$\begin{aligned}
\fint_{B_\tau} \bar{\Psi}_0\left(\left|\frac{\bar{w}_c - (\bar{w}_c)_{B_\tau}}{\tau}\right|\right) dx &\leqslant \fint_{B_\tau} \bar{\Psi}_0\left(\left|\frac{\bar{w}_c - (\bar{h})_{B_\tau}}{\tau}\right|\right) dx \\
&\leqslant \fint_{B_\tau} \bar{\Psi}_0\left(\left|\frac{\bar{h} - (\bar{h})_{B_\tau}}{\tau}\right|\right) dx + \fint_{B_\tau} \bar{\Psi}_0\left(\left|\frac{\bar{w}_c - \bar{h}}{\tau}\right|\right) dx \\
&\overset{(3.3.99)}{\leqslant} c \sup_{B_\tau} \bar{\Psi}_0(|D\bar{h}|) + c\tau^{-(n+s(\Psi)+1)}\varepsilon^* \\
&\overset{(3.3.101)}{\leqslant} c \fint_{B_{1/8}} \bar{\Psi}_0(|D\bar{h}|)\, dx + c\tau^{-(n+s(\Psi)+1)}\varepsilon^* \\
&\overset{(3.3.95)}{\leqslant} c + c\tau^{-(n+s(\Psi)+1)}\varepsilon^*.
\end{aligned} \tag{3.3.102}$$

By scaling back to w_c as introduced in (3.3.42a)-(3.3.42b), we obtain the desired estimate (3.3.38). The proof is complete. $\square$

Lemma 3.3.4. *Under the assumptions and notations of Lemma 3.3.3, let $w_c \in W^{1,\Psi}(B_R)$ be the solution to the problem defined in (3.3.22). If one of the assumptions (3.1.11a)-(3.1.11e) is satisfied, then there exists $h \in w_c + W_0^{1,\Psi^-_{B_R}}(B_{R/8})$ being a local minimizer of the functional defined by*

$$W^{1,1}(B_{R/8}) \ni v \mapsto \mathcal{F}_0(v) := \int_{B_{R/8}} F_0(Dv)\,dx, \tag{3.3.103}$$

where the integrand function is given by

$$F_0(z) := F_G\left(x_c, (u)_{B_{2R}}, z\right) + a(x_a)F_{H_a}\left(x_c, (u)_{B_{2R}}, z\right) + b(x_b)F_{H_b}\left(x_c, (u)_{B_{2R}}, z\right) \tag{3.3.104}$$

for some fixed point $x_c \in B_R$ having been fixed in (3.3.22) and $x_a, x_b \in \overline{B}_R$ being points such that $a(x_a) := \inf_{x \in B_R} a(x)$ and $b(x_b) := \inf_{x \in B_R} b(x)$, whenever $z \in \mathbb{R}^n$, such that

$$\begin{aligned}
&\fint_{B_{R/8}} \Big[|V_G(Du) - V_G(Dh)|^2 + a(x_a)|V_{H_a}(Du) - V_{H_a}(Dh)|^2 \\
&\qquad\qquad + b(x_b)|V_{H_b}(Du) - V_{H_b}(Dh)|^2\Big]\,dx \\
&\qquad\leqslant c\left(\omega\left(R^\gamma\right) + [d(\varepsilon, R)]^{s_1}\right) \fint_{B_{2R}} \Psi(x, |Du|)\,dx
\end{aligned} \tag{3.3.105}$$

for some constant $c \equiv c(\boldsymbol{data}(\Omega_0))$, where s_1 and $d(\varepsilon, R)$ have been defined in (3.3.96) and (3.3.93), respectively. Moreover, we have the energy estimate

$$\fint_{B_{R/8}} \Psi^-_{B_R}(|Dh|)\,dx \leqslant c \fint_{B_{2R}} \Psi(x, |Du|)\,dx \tag{3.3.106}$$

for some constant $c \equiv c(n, \nu, L)$.

Proof. We need to revisit the proof of Lemma 3.3.3, specially Step 3 and Step 4. Under the settings of the proof of Lemma 3.3.3, we consider a function

$\bar{h} \in \bar{w}_c + W_0^{1,\bar{\Psi}_0}(B_{1/8})$ satisfying (3.3.94)-(3.3.97). Let h be the scaled back function of $\bar{h}$ in $B_{R/8}$ as

$$h(x) := E(w_c, B_{R/2}) R \bar{h}\left(\frac{x - x_0}{R}\right) \quad \text{for every} \quad x \in B_{R/8}(x_0). \tag{3.3.107}$$

Clearly, $h \in w_c + W_0^{1,\Psi^-_{B_R}}(B_{R/8})$ is a local minimizer of the functional $\mathcal{F}_0$ defined in (3.3.103) which means that

$$\mathcal{F}_0(h) = \int_{B_{R/8}} F_0(Dh)\,dx \leqslant \int_{B_{R/8}} F_0(Dh + D\varphi)\,dx = \mathcal{F}_0(h + \varphi) \tag{3.3.108}$$

holds for every $\varphi \in W_0^{1,\Psi^-_{B_R}}(B_{R/8})$. As we have shown in (3.3.9), recalling (3.3.24) and (3.3.4), we see

$$\begin{aligned}\fint_{B_{R/8}} \Psi^-_{B_R}(|Dh|)\,dx &\leqslant \frac{L}{\nu} \fint_{B_{R/8}} \Psi^-_{B_R}(|Dw_c|)\,dx \leqslant \frac{8^n L}{\nu} \fint_{B_R} \Psi(x, |Dw_c|)\,dx \\ &\leqslant c(n,\nu,L) \fint_{B_R} \Psi(x, |Dw|)\,dx \leqslant c(n,\nu,L) \fint_{B_{2R}} \Psi(x, |Du|)\,dx,\end{aligned} \tag{3.3.109}$$

which proves (3.3.106). We write the inequality (3.3.96) in view of G, H_a, H_b, w_c and h in order to have

$$\fint_{B_{R/8}} \left[|V_G(Dw_c) - V_G(Dh)|^2 + a(x_a)|V_{H_a}(Dw_c) - V_{H_a}(Dh)|^2\right. \tag{3.3.110}$$

$$\begin{aligned}&\left.+ b(x_b)|V_{H_b}(Dw_c) - V_{H_b}(Dh)|^2\right]\,dx \\ &\leqslant c[d(\varepsilon,R)]^{s_1} \fint_{B_{R/2}} \Psi^-_{B_R}\left(\left|\frac{w_c - (w_c)_{B_{R/2}}}{R}\right|\right)\,dx \\ &\leqslant c[d(\varepsilon,R)]^{s_1} \fint_{B_{R/2}} \Psi^-_{B_R}\left(|Dw_c|\right)\,dx \leqslant c[d(\varepsilon,R)]^{s_1} \fint_{B_{R/2}} \Psi\left(x, |Du|\right)\,dx\end{aligned} \tag{3.3.111}$$

for some constant $c \equiv c(\mathbf{data}(\Omega_0))$, where we have applied Sobolev-Poincaré inequality and (3.3.109). Combining this estimate together with (3.3.3) and (3.3.23) via some elementary computations and recalling $R \leqslant 1$, we directly arrive at (3.3.105). $\square$

We finally finish this section with a crucial decay estimate on a local minimizer u of the functional $\mathcal{F}$.

Lemma 3.3.5. *Under the assumptions and notations of Lemma 3.3.3, if one of the conditions* (3.1.11a)-(3.1.11e) *is satisfied, then for every $\varepsilon_* \in (0,1)$, there exists a positive radius R_* with the dependence as*

$$R_* \equiv R_*(\boldsymbol{data}(\Omega_0), \varepsilon_*) \tag{3.3.112}$$

such that if $R \leqslant R_*$, then there exists a constant $c \equiv c(\boldsymbol{data}(\Omega_0))$ such that

$$\int_{B_{\tau R}} \Psi^-_{B_R}\left(\left|\frac{u-(u)_{B_{\tau R}}}{\tau R}\right|\right) dx \leqslant c\left(\tau^n + \tau^{-(s(\Psi)+1)}\varepsilon_*\right) \int_{B_{2R}} \Psi(x,|Du|)\, dx \tag{3.3.113}$$

holds for every $\tau \in (0,1/16)$.

Proof. First we apply Lemma 3.3.3 with $\varepsilon^* \in (0,1)$ to be determined in a few lines, and we can use (3.3.38) provided

$$R \leqslant R^* \equiv R^*(\mathbf{data}(\Omega_0), \varepsilon^*)$$

is found via (3.3.37). Therefore, using the convexity of $\Psi^-_{B_R}$, Lemma 3.3.3 and a Sobolev-Poincaré inequality of Lemma 2.4.1 via some elementary manipulations, for every $\tau \in (0,1/32)$, we have that

$$\begin{aligned}
\fint_{B_{\tau R}} \Psi^-_{B_R}\left(\left|\frac{u-(u)_{B_{\tau R}}}{\tau R}\right|\right) dx &\leqslant c \fint_{B_{\tau R}} \Psi^-_{B_R}\left(\left|\frac{u-(w_c)_{B_{\tau R}}}{\tau R}\right|\right) dx \\
&\leqslant c \fint_{B_{\tau R}} \Psi^-_{B_R}\left(\left|\frac{w_c-(w_c)_{B_{\tau R}}}{\tau R}\right|\right) dx \\
&\quad + c\tau^{-(n+s(\Psi)+1)} \fint_{B_R} \Psi^-_{B_R}\left(\left|\frac{u-w_c}{R}\right|\right) dx
\end{aligned}$$

$$
\begin{aligned}
&\leqslant c\left(1+\tau^{-(n+s(\Psi)+1)}\varepsilon^*\right) \fint_{B_{R/2}} \Psi^-_{B_R}\left(\left|\frac{w_c-(w_c)_{B_{R/2}}}{R}\right|\right) dx \\
&\qquad + c\tau^{-(n+s(\Psi)+1)} \fint_{B_R} \Psi^-_{B_R}\left(\left|\frac{u-w_c}{R}\right|\right) dx \\
&\leqslant c\left(1+\tau^{-(n+s(\Psi)+1)}\varepsilon^*\right) \fint_{B_R} \Psi^-_{B_R}\left(|Dw_c|\right) dx \\
&\qquad + c\tau^{-(n+s(\Psi)+1)} \fint_{B_R} \Psi^-_{B_R}\left(\left|\frac{u-w_c}{R}\right|\right) dx
\end{aligned} \tag{3.3.114}
$$

with some constant $c \equiv c(\mathbf{data}(\Omega_0))$, where throughout the last display we have repeatedly used (2.1.6) and (3.3.36). The last display, (3.3.7) and (3.3.27) with some elementary manipulations yield

$$
\begin{aligned}
&\int_{B_{\tau R}} \Psi^-_{B_R}\left(\left|\frac{u-(u)_{B_{\tau R}}}{\tau R}\right|\right) dx \\
&\leqslant c\left(\tau^n + \tau^{-(s(\Psi)+1)}\varepsilon^* + \tau^{-(s(\Psi)+1)}[\omega(R^\gamma)]^{\frac{1}{2}}\right) \int_{B_{2R}} \Psi(x,|Du|)\, dx
\end{aligned}
$$

for every $\tau \in (0,1/16)$ and some $c \equiv c(\mathbf{data}(\Omega_0))$. Then we choose $\varepsilon_* \equiv \varepsilon^*/2$ and $R_* \leqslant R^*$ in such a way that $[\omega(R_*^\gamma)]^{\frac{1}{2}} \leqslant \varepsilon_*/2$. This choice gives us the dependence as described in (3.3.112) and yields (3.3.113). $\square$

3.4 Proof of Theorem 3.1.2

Now we are ready to provide the proof of Theorem 3.1.2. In fact, it comes from the combination of Lemma 3.2.3 and Lemma 3.3.5.

Step 1: Different alternatives. Now we consider the different alternatives depending on which phase of (3.2.27a)-(3.2.27d) occurs in some fixed ball $B_R \equiv B_R(x_0) \subset \Omega_0 \Subset \Omega$ with $R \leqslant R_* \equiv R_*(\mathbf{data}(\Omega_0), \varepsilon_*)$, which will be determined via Lemma 3.3.5 depending on $\varepsilon_* \in (0,1)$.

Alternative 1. Let $\tau_{ab} \in (0,1/64)$ to be chosen in a few lines. Assume that G-phase occurs in the ball $B_{\tau_{ab}R}$, which means that (3.2.27a) happens in $B_{\tau_{ab}R}$. In this situation, we have

$$a^-(B_{2\tau_{ab}R}) \leqslant 8[a]_{\omega_a}\omega_a(\tau_{ab}R) \quad \text{and} \quad b^-(B_{2\tau_{ab}R}) \leqslant 8[b]_{\omega_b}\omega_b(\tau_{ab}R). \tag{3.4.1}$$

Then we are able to apply Lemma 3.2.3 in the ball $B_{2\tau_{ab}R}$. In turn, this one together with applying Lemma 3.3.5 implies that

$$\begin{aligned} \fint_{B_{\tau_{ab}R}} \Psi(x,|Du|)\,dx &\leqslant c \fint_{B_{2\tau_{ab}R}} G\left(\left|\frac{u-(u)_{B_{2\tau_{ab}R}}}{2\tau_{ab}R}\right|\right)\,dx \\ &\leqslant c \fint_{B_{2\tau_{ab}R}} \Psi^-_{B_R}\left(\left|\frac{u-(u)_{B_{2\tau_{ab}R}}}{2\tau_{ab}R}\right|\right)\,dx \\ &\leqslant c\left(\tau_{ab}^n + \tau_{ab}^{-(s(\Psi)+1)}\varepsilon_*\right)\fint_{B_R}\Psi(x,|Du|)\,dx \end{aligned} \tag{3.4.2}$$

for $c \equiv c(\mathbf{data}(\Omega_0))$, provided $R \leqslant R_*(\mathbf{data}(\Omega_0), \varepsilon_*)$. Then, for every $\sigma \in (0,n)$, we write down the last inequality in the following form

$$\fint_{B_{\tau_{ab}R}} \Psi(x,|Du|)\,dx \leqslant \tau_{ab}^{n-\sigma}\left(c_{ab}\tau_{ab}^{\sigma} + c_{ab}\tau_{ab}^{\sigma-(n+s(\Psi)+1)}\varepsilon_*\right)\fint_{B_R}\Psi(x,|Du|)\,dx$$

for some constant $c_{ab} \equiv c_{ab}(\mathbf{data}(\Omega_0))$. We select small enough τ_{ab}, ε_* depending on $\mathbf{data}(\Omega_0)$ and σ in such a way that $c_{ab}\tau_{ab}^{\sigma} \leqslant 1/2$ and $c_{ab}\tau_{ab}^{\sigma-(n+s(\Psi)+1)}\varepsilon_* \leqslant 1/2$. Then we have

$$\fint_{B_{\tau_{ab}R}} \Psi(x,|Du|)\,dx \leqslant \tau_{ab}^{n-\sigma}\fint_{B_R}\Psi(x,|Du|)\,dx \tag{3.4.3}$$

for every $R \leqslant R_{ab} \equiv R_{ab}(\mathbf{data}(\Omega_0), \sigma)$.

Alternative 2. Let $\tau_b \in (0, 1/64)$ also to be determined later. This time we assume that (G, H_a)-phase occurs in B_R ((3.2.27b) happens in B_R) and that $b^-(B_{\tau_b R}) \leqslant 4[b]_{\omega_b}\omega_b(\tau_b R)$. Then we have

$$b^-(B_{2\tau_b R}) \leqslant 8[b]_{\omega_b}\omega_b(\tau_b R). \tag{3.4.4}$$

Also we can observe that

$$a^-(B_{\tau_b R}) \geqslant a^-(B_R) > 4[a]_{\omega_a}\omega_a(R) \geqslant 4[a]_{\omega_a}\omega_a(\tau_b R) \tag{3.4.5}$$

and

$$a^-(B_R) \leqslant a(x) \leqslant 2[a]_{\omega_a}\omega_a(R) + a^-(B_R) \leqslant 2a^-(B_R) \quad (\forall x \in B_R). \tag{3.4.6}$$

Applying Lemma 3.2.3 and then Lemma 3.3.5 together with recalling (3.4.6), we have

$$\begin{aligned}
\int_{B_{\tau_b R}} \Psi(x,|Du|)\,dx &\leqslant c \int_{B_{2\tau_b R}} \left[G\left(\left|\frac{u-(u)_{B_{2\tau_b R}}}{2\tau_b R}\right|\right) + a^-(B_{2\tau_b R})H_a\left(\left|\frac{u-(u)_{B_{2\tau_b R}}}{2\tau_b R}\right|\right)\right] dx \\
&\leqslant c \int_{B_{2\tau_b R}} \Psi^-_{B_R}\left(\left|\frac{u-(u)_{B_{2\tau_b R}}}{2\tau_b R}\right|\right) dx \\
&\leqslant c\left(\tau_b^n + \tau_b^{-(s(\Psi)+1)}\varepsilon_*\right)\int_{B_R} \Psi(x,|Du|)\,dx \qquad (3.4.7)
\end{aligned}$$

for some constant $c \equiv c(\mathbf{data}(\Omega_0))$, provided $R \leqslant R_*(\mathbf{data}(\Omega_0), \varepsilon_*)$. Then, for every $\sigma \in (0,n)$, we write down the last display as

$$\int_{B_{\tau_b R}} \Psi(x,|Du|)\,dx \leqslant \tau_b^{n-\sigma}\left(c_b\tau_b^{\sigma} + c_b\tau_b^{\sigma-(n+s(\Psi)+1)}\varepsilon_*\right)\int_{B_R} \Psi(x,|Du|)\,dx$$

for some constant $c_b \equiv c_b(\mathbf{data}(\Omega_0))$. We select small enough τ_b, ε_* depending on $\mathbf{data}(\Omega_0)$ and σ in such a way that $c_b\tau_b^{\sigma} \leqslant 1/2$ and $c_b\tau_b^{\sigma-(n+s(\Psi)+1)}\varepsilon_* \leqslant 1/2$. Then we have

$$\int_{B_{\tau_b R}} \Psi(x,|Du|)\,dx \leqslant \tau_b^{n-\sigma}\int_{B_R} \Psi(x,|Du|)\,dx \tag{3.4.8}$$

for every $R \leqslant R_b \equiv R_b(\mathbf{data}(\Omega_0), \sigma)$.

Alternative 3. Let $\tau_a \in (0, 1/64)$ to be fixed later. Assume that (G, H_b)-phase occurs in B_R ((3.2.27c) happens in B_R) and $a^-(B_{\tau_a R}) \leqslant 4[a]_{\omega_a}\omega_a(\tau_a R)$.

Then we have

$$a^-(B_{2\tau_a R}) \leqslant 8[a]_{\omega_a}\omega_a(\tau_a R). \tag{3.4.9}$$

Applying Lemma 3.2.3 and then Lemma 3.3.5 together with recalling that $b^-(B_R) \leqslant b(x) \leqslant 2b^-(B_R)$ holds for every $x \in B_R$ if $b^-(B_R) > 4[b]_{\omega_b}\omega_b(R)$ likewise in (3.4.6), we have

$$\begin{aligned}
\int_{B_{\tau_a R}} \Psi(x,|Du|)\,dx &\leqslant c \int_{B_{2\tau_a R}} \left[G\left(\left| \frac{u-(u)_{B_{2\tau_a R}}}{2\tau_a R} \right| \right) + b^-(B_{2\tau_a R}) H_b \left(\left| \frac{u-(u)_{B_{2\tau_a R}}}{2\tau_a R} \right| \right) \right] dx \\
&\leqslant c \int_{B_{2\tau_a R}} \Psi^-_{B_R} \left(\left| \frac{u-(u)_{B_{2\tau_a R}}}{2\tau_a R} \right| \right) dx \\
&\leqslant c \left(\tau_a^n + \tau_a^{-(s(\Psi)+1)}\varepsilon_* \right) \int_{B_R} \Psi(x,|Du|)\,dx
\end{aligned} \tag{3.4.10}$$

for some constant $c \equiv c(\mathbf{data}(\Omega_0))$, provided $R \leqslant R_*(\mathbf{data}(\Omega_0), \varepsilon_*)$. Then, for every $\sigma \in (0,n)$, we write down the last display as

$$\int_{B_{\tau_a R}} \Psi(x,|Du|)\,dx \leqslant \tau_a^{n-\sigma} \left(c_a \tau_a^{\sigma} + c_a \tau_a^{\sigma-(n+s(\Psi)+1)}\varepsilon_* \right) \int_{B_R} \Psi(x,|Du|)\,dx$$

for some constant $c_a \equiv c_a(\mathbf{data}(\Omega_0))$. We select small enough τ_a, ε_* depending on $\mathbf{data}(\Omega_0)$ and σ in such a way that $c_a\tau_a^{\sigma} \leqslant 1/2$ and $c_a\tau_a^{\sigma-(n+s(\Psi)+1)}\varepsilon_* \leqslant 1/2$. Then we have

$$\int_{B_{\tau_a R}} \Psi(x,|Du|)\,dx \leqslant \tau_a^{n-\sigma} \int_{B_R} \Psi(x,|Du|)\,dx \tag{3.4.11}$$

for every $R \leqslant R_a \equiv R_a(\mathbf{data}(\Omega_0), \sigma)$.

Alternative 4. Let $\tau_0 \in (0, 1/64)$ to be chosen later. We assume that (G, H_a, H_b)-phase occurs in B_R, which means that (3.2.27d) happens in B_R. In this situation, from the observation in (3.4.6) we see that $a^-(B_R) \leqslant a(x) \leqslant 2a^-(B_R)$ and $b^-(B_R) \leqslant b(x) \leqslant 2b^-(B_R)$ for every $x \in B_R$. Then again applying Lemma 3.2.3 and Lemma 3.3.5, we find

$$
\begin{aligned}
\int_{B_{\tau_0 R}} \Psi(x,|Du|)\,dx &\leqslant c \int_{B_{2\tau_a R}} \Psi^-_{B_{2\tau_0 R}}\left(\left|\frac{u-(u)_{B_{2\tau_0 R}}}{2\tau_0 R}\right|\right)dx \\
&\leqslant c \int_{B_{2\tau_0 R}} \Psi^-_{B_R}\left(\left|\frac{u-(u)_{B_{2\tau_0 R}}}{2\tau_0 R}\right|\right)dx \\
&\leqslant c\left(\tau_0^n + \tau_0^{-(s(\Psi)+1)}\varepsilon_*\right)\int_{B_R}\Psi(x,|Du|)\,dx \qquad (3.4.12)
\end{aligned}
$$

for some constant $c \equiv c(\mathbf{data}(\Omega_0))$, provided $R \leqslant R_*(\mathbf{data}(\Omega_0), \varepsilon_*)$. Then, for every $\sigma \in (0,n)$, we write down the last display as

$$
\int_{B_{\tau_0 R}} \Psi(x,|Du|)\,dx \leqslant \tau_0^{n-\sigma}\left(c_0\tau_0^{\sigma} + c_0\tau_0^{\sigma-(n+s(\Psi)+1)}\varepsilon_*\right)\int_{B_R}\Psi(x,|Du|)\,dx
$$

for some constant $c_0 \equiv c_0(\mathbf{data}(\Omega_0))$. Then we choose τ_0, ε_* depending on $\mathbf{data}(\Omega_0)$ and σ in such a way that $c_0\tau_0^{\sigma} \leqslant 1/2$ and $c_0\tau_0^{\sigma-(n+s(\Psi)+1)}\varepsilon_* \leqslant 1/2$. Then we have

$$
\int_{B_{\tau_0 R}} \Psi(x,|Du|)\,dx \leqslant \tau_0^{n-\sigma}\int_{B_R}\Psi(x,|Du|)\,dx \qquad (3.4.13)
$$

for every $R \leqslant R_0 \equiv R_0(\mathbf{data}(\Omega_0), \sigma)$. Next we consider the double nested exit time argument based on the proof of [71, Theorem 2].

Step 2: Double nested exit time and iteration. Now we shall combine all the alternatives we have discussed with the estimates (3.4.3), (3.4.8), (3.4.11) and (3.4.13). Take a ball $B_R \subset \Omega_0 \Subset \Omega$ such that $R \leqslant R_m$, where $R_m = \min\{R_{ab}, R_a, R_b, R_0\}$ depends on $\mathbf{data}(\Omega_0)$ and σ. We consider G-phase in $B_{\tau_{ab}^{k+1}R}$ for every integer $k \geqslant 0$ and define the exit time index

$$
t_{ab} = \min\{k \in \mathbb{N} : G - \text{phase in the ball } B_{\tau_{ab}^{k+1}R} \text{ does not occur}\}. \qquad (3.4.14)
$$

If there does not exist such t_{ab}, then for any $0 < \rho < \tau_{ab}^2 R < R \leqslant R_m$, there exists an integer $m \geqslant 1$ such that $\tau_{ab}^{m+2}R \leqslant \rho < \tau_{ab}^{m+1}R$. Using iterative

(3.4.3), we have

$$
\begin{aligned}
\int_{B_\rho} \Psi(x,|Du|)\,dx &\leqslant \int_{B_{\tau_{ab}^{m+1}R}} \Psi(x,|Du|)\,dx \\
&\leqslant \tau_{ab}^{(m-1)(n-\sigma)} \int_{\tau_{ab}^{2}R} \Psi(x,|Du|)\,dx \\
&= \tau_{ab}^{(m+2)(n-\sigma)} \tau_{ab}^{-3(n-\sigma)} \int_{\tau_{ab}^{2}R} \Psi(x,|Du|)\,dx \\
&\leqslant c(\mathbf{data}(\Omega_0),\sigma) \left(\frac{\rho}{R}\right)^{n-\sigma} \int_{R} \Psi(x,|Du|)\,dx. \qquad (3.4.15)
\end{aligned}
$$

Clearly, the above inequality holds true when $\tau_{ab}^2 R \leqslant \rho \leqslant R \leqslant R_m$. So we consider the case of $t_{ab} < \infty$. For every $k \in \{1,\dots,t_{ab}\}$, we apply (3.4.3) repeatedly in order to obtain

$$
\int_{B_{\tau_{ab}^{k}R}} \Psi(x,|Du|)\,dx \leqslant \tau_{ab}^{k(n-\sigma)} \int_{B_R} \Psi(x,|Du|)\,dx. \qquad (3.4.16)
$$

By the very definition of τ_{ab} in (3.4.14), we have three different scenarios: either (G,H_a)-phase occurs in $B_{\tau_{ab}^{t_{ab}+1}R}$, (G,H_b)-phase occurs in $B_{\tau_{ab}^{t_{ab}+1}R}$ or (G,H_a,H_b)-phase occurs in $B_{\tau_{ab}^{t_{ab}+1}R}$. Clearly, the last condition is stable for shrinking balls. Since the first two conditions can be considered similarly, we shall focus on the occurrence of (G,H_a)-phase in the ball $B_{\tau_{ab}^{t_{ab}+1}R}$. Let us define a second exit time index

$$
t_b := \min\{k \in \mathbb{N} : (G,H_a) - \text{phase in the ball } B_{\tau_b^{k+1}\tau_{ab}^{t_{ab}+1}R} \text{ does not occur}\}. \qquad (3.4.17)
$$

Arguing similarly as in (3.4.15) by using (3.4.8) if there is no such a finite number $t_b \in \mathbb{N}$, we are able to arrive at the inequality (3.4.25) below. So we only focus on the case of $t_b < \infty$. Iterating (3.4.8) with B_R replaced by

$B_{\tau_{ab}^{t_{ab}+1}R}$, we have

$$\int\limits_{B_{\tau_b^k \tau_{ab}^{t_{ab}+1}R}} \Psi(x,|Du|)\,dx \leqslant \tau_b^{k(n-\sigma)} \int\limits_{B_{\tau_{ab}^{t_{ab}+1}R}} \Psi(x,|Du|)\,dx \tag{3.4.18}$$

for every $k \in \{1,\dots,t_b\}$. By again the very definition of t_b, there is only one chance that (G,H_a,H_b)-phase occurs in the ball $B_{\tau_b^{t_b+1}\tau_{ab}^{t_{ab}+1}R}$. But as this condition is stable, we can iterate (3.4.13) for every $k \in \mathbb{N}$ in order to have

$$\int\limits_{B_{\tau_0^k \tau_b^{t_b+1}\tau_{ab}^{t_{ab}+1}R}} \Psi(x,|Du|)\,dx \leqslant \tau_0^{k(n-\sigma)} \int\limits_{B_{\tau_b^{t_b+1}\tau_{ab}^{t_{ab}+1}R}} \Psi(x,|Du|)\,dx. \tag{3.4.19}$$

Now we have all the needed estimates (3.4.16), (3.4.18) and (3.4.19). For $0 < \rho < R \leqslant R_m$, we consider the following cases.

Case 1: $R > \rho \geqslant \tau_{ab}^{t_{ab}+1}R$**.** There exists $m \in \{0,1,\dots,t_{ab}\}$ such that $\tau_{ab}^{m+1}R \leqslant \rho < \tau_{ab}^{m}R$. Then from (3.4.16), we have

$$\begin{aligned}
\int\limits_{B_\rho} \Psi(x,|Du|)\,dx &\leqslant \int\limits_{B_{t_{ab}^m R}} \Psi(x,|Du|)\,dx \\
&\leqslant \tau_{ab}^{m(n-\sigma)} \int\limits_{B_R} \Psi(x,|Du|)\,dx \\
&\leqslant \tau_{ab}^{(m+1)(n-\sigma)} \tau_{ab}^{\sigma-n} \int\limits_{B_R} \Psi(x,|Du|)\,dx \\
&\leqslant c(\mathbf{data}(\Omega_0),\sigma) \left(\frac{\rho}{R}\right)^{n-\sigma} \int\limits_{B_R} \Psi(x,|Du|)\,dx,
\end{aligned} \tag{3.4.20}$$

where the last inequality is valid since τ_{ab} depends on $\mathbf{data}(\Omega_0)$ and σ.

Case 2: $\tau_{ab}^{t_{ab}+1}R > \rho \geqslant \tau_b \tau_{ab}^{t_{ab}+1}R$**.** In this case, using (3.4.20), we see

$$\int\limits_{B_\rho} \Psi(x,|Du|)\,dx \leqslant \int\limits_{B_{\tau_{ab}^{t_{ab}+1}R}} \Psi(x,|Du|)\,dx$$

$$
\begin{aligned}
&= \tau_{ab}^{(t_{ab}+1)(n-\sigma)} \int\limits_{B_R} \Psi(x,|Du|)\,dx \\
&\leqslant \left(\tau_b \tau_{ab}^{t_{ab}+1}\right)^{n-\sigma} \tau_b^{\sigma-n} \int\limits_{B_R} \Psi(x,|Du|)\,dx \\
&\leqslant c(\mathbf{data}(\Omega_0),\sigma) \left(\frac{\rho}{R}\right)^{n-\sigma} \int\limits_{B_R} \Psi(x,|Du|)\,dx, \qquad (3.4.21)
\end{aligned}
$$

where again the last inequality is possible by the dependencies of τ_b.

Case 3: $\tau_b \tau_{ab}^{t_{ab}+1} R > \rho \geqslant \tau_b^{t_b+1} \tau_{ab}^{t_{ab}+1} R$**.** Again there exists a natural number $m \in \{1, \ldots, t_b\}$ so that $\tau_b^m \tau_{ab}^{t_{ab}+1} R > \rho \geqslant \tau_b^{m+1} \tau_{ab}^{t_{ab}+1} R$. Therefore, using (3.4.18) and (3.4.20), we have

$$
\begin{aligned}
\int\limits_{B_\rho} \Psi(x,|Du|)\,dx &\leqslant \int\limits_{B_{\tau_b^m \tau_{ab}^{t_{ab}+1} R}} \Psi(x,|Du|)\,dx \\
&\leqslant \tau_b^{m(n-\sigma)} \int\limits_{B_{\tau_{ab}^{t_{ab}+1} R}} \Psi(x,|Du|)\,dx \\
&\leqslant \tau_b^{(m+1)(n-\sigma)} \tau_b^{\sigma-n} \tau_{ab}^{(t_{ab}+1)(n-\sigma)} \int\limits_{B_R} \Psi(x,|Du|)\,dx \\
&\leqslant c(\mathbf{data}(\Omega_0),\sigma) \left(\frac{\rho}{R}\right)^{n-\sigma} \int\limits_{B_R} \Psi(x,|Du|)\,dx. \qquad (3.4.22)
\end{aligned}
$$

Case 4: $\tau_b^{t_b+1} \tau_{ab}^{t_{ab}+1} R > \rho \geqslant \tau_b^{t_b+1} \tau_{ab}^{t_{ab}+1} \tau_0 R$**.** Now by (3.4.22), we find

$$
\begin{aligned}
\int\limits_{B_\rho} \Psi(x,|Du|)\,dx &\leqslant \int\limits_{B_{\tau_b^{t_b+1} \tau_{ab}^{t_{ab}+1} R}} \Psi(x,|Du|)\,dx \\
&\leqslant c \left(\tau_b^{t_b+1} \tau_{ab}^{t_{ab}+1}\right)^{n-\sigma} \int\limits_{B_R} \Psi(x,|Du|)\,dx \\
&\leqslant c \tau_0^{\sigma-n} \left(\tau_0 \tau_b^{t_b+1} \tau_{ab}^{t_{ab}+1}\right)^{n-\sigma} \int\limits_{B_R} \Psi(x,|Du|)\,dx
\end{aligned}
$$

$$\leqslant c(\mathbf{data}(\Omega_0), \sigma) \left(\frac{\rho}{R}\right)^{n-\sigma} \int_{B_R} \Psi(x, |Du|)\, dx. \qquad (3.4.23)$$

Case 5: $\tau_b^{t_b+1}\tau_{ab}^{t_{ab}+1}\tau_0 R > \rho > 0$**.** This condition implies that there exists a natural number $m \in \mathbb{N}$ such that $\tau_0^{m+1}\tau_b^{t_b+1}\tau_{ab}^{t_{ab}+1}R \leqslant \rho < \tau_0^{m}\tau_b^{t_b+1}\tau_{ab}^{t_{ab}+1}R$. This time we apply (3.4.19) and (3.4.23) in order to have

$$\begin{aligned}
\int_{B_\rho} \Psi(x, |Du|)\, dx &\leqslant \int_{B_{\tau_0^m \tau_b^{t_b+1}\tau_{ab}^{t_{ab}+1}R}} \Psi(x, |Du|)\, dx \\
&\leqslant \tau_0^{m(n-\sigma)} \int_{B_{\tau_b^{t_b+1}\tau_{ab}^{t_{ab}+1}R}} \Psi(x, |Du|)\, dx \\
&\leqslant \tau_0^{m(n-\sigma)} \left(\tau_b^{t_b+1}\tau_{ab}^{t_{ab}+1}\right)^{n-\sigma} \int_{B_R} \Psi(x, |Du|)\, dx \\
&\leqslant c\tau_0^{\sigma-n} \left(\frac{\rho}{R}\right)^{n-\sigma} \int_{B_R} \Psi(x, |Du|)\, dx \\
&= c(\mathbf{data}(\Omega_0), \sigma) \left(\frac{\rho}{R}\right)^{n-\sigma} \int_{B_R} \Psi(x, |Du|)\, dx. \qquad (3.4.24)
\end{aligned}$$

As we discussed earlier after (3.4.16), we can proceed the same for the occurrence of (G, H_a)-phase in the ball $B_{\tau_{ab}^{\tau_{ab}+1}R}$ instead of the occurrence of (G, H_b)-phase in the ball $B_{\tau_{ab}^{\tau_{ab}+1}R}$. Then we can directly jump to the case that (G, H_a, H_b)-phase occurs in the ball $B_{\tau_{ab}^{\tau_{ab}+1}R}$, which is trivial by (3.4.13). Moreover, if we start with the occurrence of (G, H_a, H_b)-phase in B_R, then the procedure will be much easier by (3.4.13). Taking into account all the possible cases that we considered above, we can conclude that, for every $\sigma \in (0, 1)$, there exists $c \equiv c(\mathbf{data}(\Omega_0), \sigma)$ such that

$$\int_{B_\rho} \Psi(x, |Du|)\, dx \leqslant c \left(\frac{\rho}{R}\right)^{n-\sigma} \int_{B_R} \Psi(x, |Du|)\, dx \qquad (3.4.25)$$

holds true, whenever $0 < \rho < R \leqslant R_m$, where R_m is some positive radius depending only on $\mathbf{data}(\Omega_0)$ and σ in the beginning of the proof. In order to complete the proof, we need to consider the remaining cases. If $0 < R_m \leqslant \rho < R \leqslant 1$, then we have

$$
\begin{aligned}
\int_{B_\rho} \Psi(x, |Du|)\, dx &\leqslant \left(\frac{\rho}{R}\right)^{n-\sigma} \left(\frac{R}{\rho}\right)^{n-\sigma} \int_{B_R} \Psi(x, |Du|)\, dx \\
&\leqslant \left(\frac{\rho}{R}\right)^{n-\sigma} \left(\frac{R}{R_m}\right)^{n-\sigma} \int_{B_R} \Psi(x, |Du|)\, dx \\
&\leqslant c(\mathbf{data}(\Omega_0), \sigma) \left(\frac{\rho}{R}\right)^{n-\sigma} \int_{B_R} \Psi(x, |Du|)\, dx, \qquad (3.4.26)
\end{aligned}
$$

where we have used the dependence of R_m. Finally, if $0 < \rho < R_m \leqslant R \leqslant 1$, then by (3.4.25) and (3.4.26), we see

$$
\begin{aligned}
\int_{B_\rho} \Psi(x, |Du|)\, dx &\leqslant c \left(\frac{\rho}{R_m}\right)^{n-\sigma} \int_{B_{R_m}} \Psi(x, |Du|)\, dx \\
&\leqslant c \left(\frac{\rho}{R_m}\right)^{n-\sigma} \left(\frac{R_m}{R}\right)^{n-\sigma} \int_{B_R} \Psi(x, |Du|)\, dx \\
&= c(\mathbf{data}(\Omega_0), \sigma) \left(\frac{\rho}{R}\right)^{n-\sigma} \int_{B_R} \Psi(x, |Du|)\, dx. \qquad (3.4.27)
\end{aligned}
$$

All in all, collecting the estimates obtained in (3.4.25)-(3.4.27), we arrive at the validity of the Morrey type inequality (3.1.13). The proof is complete.

Now we consider a crucial outcome of Theorem 3.1.2, which plays a crucial role for proving Theorem 3.1.1 afterwards.

Lemma 3.4.1. *Under the assumptions of Lemma 3.3.3, let $w_c \in W^{1,\Psi}(B_R)$ be the solution to the problem defined in* (3.3.22). *Suppose that* (3.1.11c) *is satisfied for $\omega_a(\rho) = \rho^\alpha$ and $\omega_b(\rho) = \rho^\beta$ with some $\alpha, \beta \in (0, 1]$. Then there exists $h \in w_c + W_0^{1,\Psi^-_{B_R}}(B_{R/8})$ being a local minimizer of the functional $\mathcal{F}_0$ defined in* (3.3.103) *such that*

$$
\fint_{B_{R/8}} \left[|V_G(Du) - V_G(Dh)|^2 + a(x_a)|V_{H_a}(Du) - V_{H_a}(Dh)|^2 \right.
$$

$$
\begin{aligned}
&+b(x_b)|V_{H_b}(Du) - V_{H_b}(Dh)|^2\big]\,dx \\
&\leqslant c\left(\omega(R^\gamma) + \left[R^{\frac{\alpha}{2(1+s(H_a))}} + R^{\frac{\beta}{2(1+s(H_b))}}\right]^{s_1}\right) \fint_{B_{2R}} \Psi(x,|Du|)\,dx \qquad (3.4.28)
\end{aligned}
$$

for some constant $c \equiv c(\boldsymbol{data}(\Omega_0))$ and $s_1 \equiv s_1(\boldsymbol{data})$, respectively. Moreover, the energy estimate

$$
\fint_{B_{R/8}} \Psi^-_{B_R}(|Dh|)\,dx \leqslant c \fint_{B_{2R}} \Psi(x,|Du|)\,dx \qquad (3.4.29)
$$

holds for some constant $c \equiv c(n,\nu,L)$.

Proof. Essentially, the above lemma is a special case of Lemma 3.3.4 since we consider a particular case that $\omega_a(\rho) = \rho^\alpha$ and $\omega_b(\rho) = \rho^\beta$ for some $\alpha, \beta \in (0,1)$. But our purpose here is to obtain an estimate such as (3.4.28) with a different multiplier containing some power of R, which will be used for proving Theorem 3.1.1. Therefore, we are able to apply Theorem 3.1.2. In turn, for every $\theta \in (0,1)$ and open sunset $\Omega_0 \Subset \Omega$, there exists a constant $c \equiv c(\mathbf{data}(\Omega_0), \theta)$ such that

$$
[u]_{0,\theta;\Omega_0} \leqslant c(\mathbf{data}(\Omega_0), \theta). \qquad (3.4.30)
$$

In particular, we choose $\theta := (\gamma+1)/2$. Now we need to revisit the proof of Lemma 3.3.3. Under the settings of the proof of Lemma 3.3.3, we turn our attention to estimating the terms I_a and I_b introduced in (3.3.59)-(3.3.60). Using (1.0.15), (3.3.26) and (3.3.6), we have

$$
\begin{aligned}
I_a &\leqslant cR^\alpha \left(1 + \left[\left(\Psi^-_{B_R}\right)^{-1}\left(\fint_{B_{R/2}} \Psi^-_{B_R}\left(\left|\frac{w_c - (w_c)_{B_{R/2}}}{R}\right|\right)dx\right)\right]^{\frac{\alpha}{1-\gamma}}\right) \\
&\leqslant c\left(R^\alpha + R^{-\frac{\alpha\gamma}{1-\gamma}}\left[\operatorname*{osc}_{B_{2R}} u\right]^{\frac{\alpha}{1-\gamma}}\right) \leqslant c(\mathbf{data}(\Omega_0))R^{\alpha/2}, \qquad (3.4.31)
\end{aligned}
$$

where we have used (3.4.30) with the choice of $\theta := (1+\gamma)/2$ and $B_{2R} \subset \Omega_0$ with $R \leqslant 1$. In the same way, we show

$$
I_b \leqslant c(\mathbf{data}(\Omega_0))R^{\beta/2}. \qquad (3.4.32)
$$

Inserting those estimates into (3.3.60), we see that

$$\left| \fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}_c), D\varphi \right\rangle \, dx \right| \leqslant c(\mathbf{data}(\Omega_0)) q_3(R) \left\| D\varphi \right\|_{L^\infty(B_{1/8})}, \tag{3.4.33}$$

whenever $\varphi \in W_0^{1,\infty}(B_{1/8})$, where

$$q_3(R) := R^{\frac{\alpha}{2(1+s(H_a))}} + R^{\frac{\beta}{2(1+s(H_b))}}. \tag{3.4.34}$$

Note that the vector field $\bar{A}_0$ has been defined in (3.3.50). We consider a function $\bar{h} \in \bar{w}_c + W_0^{1,\bar{\Psi}_0}(B_{1/8})$ satisfying (3.3.94)-(3.3.97) with the term $d(\varepsilon, R)$ replaced by $q_3(R)$. Let h be the scaled back function of $\bar{h}$ in $B_{R/8}$ as

$$h(x) := E(w_c, B_{R/2}) R \bar{h}\left(\frac{x - x_0}{R}\right) \quad \text{for every} \quad x \in B_{R/8}(x_0). \tag{3.4.35}$$

Clearly, $h \in w_c + W_0^{1,\Psi^-_{B_R}}(B_{R/8})$ is a local minimizer of the functional $\mathcal{F}_0$ defined in (3.3.103), which means that

$$\mathcal{F}_0(h) = \int_{B_{R/8}} F_0(Dh) \, dx \leqslant \int_{B_{R/8}} F_0(Dh + D\varphi) \, dx = \mathcal{F}_0(h + \varphi) \tag{3.4.36}$$

holds for every $\varphi \in W_0^{1,\Psi^-_{B_R}}(B_{R/8})$. As we have shown in (3.3.9), we recall (3.3.24) and (3.3.4) to see that

$$\begin{aligned} \fint_{B_{R/8}} \Psi^-_{B_R}(|Dh|) \, dx &\leqslant \frac{L}{\nu} \fint_{B_{R/8}} \Psi^-_{B_R}(|Dw_c|) \, dx \leqslant \frac{8^n L}{\nu} \fint_{B_R} \Psi(x, |Dw_c|) \, dx \\ &\leqslant c(n, \nu, L) \fint_{B_R} \Psi(x, |Dw|) \, dx \leqslant c(n, \nu, L) \fint_{B_{2R}} \Psi(x, |Du|) \, dx, \end{aligned} \tag{3.4.37}$$

which proves (3.3.106). We write the inequality (3.3.96) in view of G, H_a, H_b,

w_c and h in order to have

$$
\begin{aligned}
&\fint_{B_{R/8}} \big[|V_G(Dw_c) - V_G(Dh)|^2 + a(x_a)|V_{H_a}(Dw_c) - V_{H_a}(Dh)|^2 \\
&\qquad\quad + b(x_b)|V_{H_b}(Dw_c) - V_{H_b}(Dh)|^2\big]\, dx \\
&\qquad \leqslant c[q_3(R)]^{s_1} \fint_{B_{R/2}} \Psi^-_{B_R}\left(\left|\frac{w_c - (w_c)_{B_{R/2}}}{R}\right|\right) dx \\
&\qquad \leqslant c[q_3(R)]^{s_1} \fint_{B_{R/2}} \Psi^-_{B_R}\left(|Dw_c|\right) dx \\
&\qquad \leqslant c[q_3(R)]^{s_1} \fint_{B_{R/2}} \Psi\left(x, |Du|\right) dx
\end{aligned} \tag{3.4.38}
$$

for some constant $c \equiv c(\mathbf{data}(\Omega_0))$, where we have applied Sobolev-Poincaré inequality and (3.3.109). Combining this estimate together with (3.3.3) and (3.3.23) alongside some elementary computations, we arrive at the desired estimate (3.4.28). $\square$

3.5 Proof of Theorem 3.1.1.

Finally, we are ready to prove Theorem 3.1.1. First applying Theorem 3.1.2 and a standard covering argument, we find that for every open subset $\Omega_0 \Subset \Omega$ and any number $k > 0$, there exists a constant $c \equiv c(\mathbf{data}(\Omega_0), k)$ such that

$$
\fint_{B_{2R}} \Psi(x, |Du|)\, dx \leqslant cR^{-k}, \tag{3.5.1}
$$

whenever $B_{2R} \subset \Omega_0$ is a ball with $R \leqslant 1$. Now we fix an open subset $\Omega_0 \Subset \Omega$ and a ball $B_{2R} \equiv B_{2R}(x_0) \subset \Omega_0$ with $R \leqslant 1$. Then applying Lemma 3.3.4 and Lemma 3.4.1,

$$
\begin{aligned}
&\fint_{B_{R/8}} \big(|V_G(Du) - V_G(Dh)|^2 + a(x_a)|V_{H_a}(Du) - V_{H_a}(Dh)|^2 \\
&\qquad\quad + b(x_b)|V_{H_b}(Du) - V_{H_b}(Dh)|^2\big)\, dx
\end{aligned}
$$

$$\leqslant c\left(R^{\mu\gamma}+[q(R)]^{s_1}\right) \fint_{B_{2R}} \Psi(x,|Du|)\,dx \tag{3.5.2}$$

for some constant $c \equiv c(\mathbf{data}(\Omega_0))$ and $s_1 \equiv s_1(\mathbf{data})$, where

$$q(R) := \begin{cases} R^{\frac{\alpha\delta}{(1+\delta)(1+s(H_a))}} + R^{\frac{\beta\delta}{(1+\delta)(1+s(H_b))}} & \text{if (3.1.10a) is assumed,} \\ R^{\frac{\alpha\gamma}{1+s(H_a)}} + R^{\frac{\beta\gamma}{1+s(H_b)}} & \text{if (3.1.10b) is assumed,} \\ R^{\frac{\alpha}{2(1+s(H_a))}} + R^{\frac{\beta}{2(1+s(H_b))}} & \text{if (3.1.10c) is assumed,} \end{cases} \tag{3.5.3}$$

in which γ is the Hölder continuity exponent determined via Theorem 3.2.2 and δ is the higher integrability exponent coming from Theorem 3.2.5. We denote by

$$d \equiv d(\mathbf{data}(\Omega_0)) := \begin{cases} \min\left\{\mu\gamma, \dfrac{\alpha\delta s_1}{(1+\delta)(1+s(H_a))}, \dfrac{\beta\delta s_1}{(1+\delta)(1+s(H_b))}\right\} \\ \text{if (3.1.10a) is assumed,} \\ \min\left\{\mu\gamma, \dfrac{\alpha\gamma s_1}{1+s(H_a)}, \dfrac{\beta\gamma s_1}{1+s(H_b)}\right\} \\ \text{if (3.1.10b) is assumed,} \\ \min\left\{\mu\gamma, \dfrac{\alpha s_1}{2(1+s(H_a))}, \dfrac{\beta s_1}{2(1+s(H_b))}\right\} \\ \text{if (3.1.10c) is assumed,} \end{cases} \tag{3.5.4}$$

and $x_a, x_b \in \overline{B_R}$ are points such that $a(x_a) = \inf_{x\in B_R} a(x)$ and $b(x_b) = \inf_{x\in B_R} b(x)$. Now choosing $k \equiv d/4$ in (3.5.1), the inequality (3.5.2) can be written as

$$\begin{aligned} \fint_{B_{R/8}} & \big(|V_G(Du)-V_G(Dh)|^2 + a(x_a)|V_{H_a}(Du)-V_{H_a}(Dh)|^2 \\ & +b(x_b)|V_{H_b}(Du)-V_{H_b}(Dh)|^2\big)\,dx \\ & \leqslant cR^{3d/4} \end{aligned} \tag{3.5.5}$$

for some constant $c \equiv c(\mathbf{data}(\Omega_0))$, where we again recall that the function h has been defined via Lemma 3.3.4 and Lemma 3.4.1. Recalling, (3.3.106) and (3.4.29), we have the energy estimate

$$\fint_{B_{R/8}} \Psi^-_{B_R}(|Dh|)\,dx \leqslant c \fint_{B_{2R}} \Psi(x,|Du|)\,dx \tag{3.5.6}$$

with a constant $c \equiv c(n,\nu,L)$. Now using repeatedly (2.1.10), we have

$$\begin{aligned}
&\fint_{B_{R/8}} \Psi^-_{B_R}(|Du-Dh|)\,dx \\
&\quad \leqslant c \fint_{B_{R/8}} \left(\left[\Psi^-_{B_R}(|Du|+|Dh|)\right]^{\frac{1}{2}} \frac{|Du-Dh|}{|Du|+|Dh|}\right) \left[\Psi^-_{B_R}(|Du|+|Dh|)\right]^{\frac{1}{2}}\,dx \\
&\quad \leqslant c \left(\fint_{B_{R/8}} \Psi^-_{B_R}(|Du|+|Dh|) \frac{|Du-Dh|^2}{(|Du|+|Dh|)^2}\,dx \right)^{\frac{1}{2}} \\
&\qquad \times \left(\fint_{B_{R/8}} \Psi^-_{B_R}(|Du|+|Dh|)\,dx \right)^{\frac{1}{2}} \\
&\quad \overset{(2.1.10),(3.5.5)}{\leqslant} cR^{3d/8} \left(\fint_{B_{R/8}} \Psi^-_{B_R}(|Du|+|Dh|)\,dx \right)^{\frac{1}{2}} \overset{(3.5.6),(3.5.1)}{\leqslant} cR^{d/4}
\end{aligned} \tag{3.5.7}$$

with $c \equiv c(\mathbf{data}(\Omega_0))$, where d has been introduced in (3.5.3). Since h is a minimizer of functional $\mathcal{F}_0$ defined in (3.3.103), and this functional satisfies the growth and ellipticity conditions $(3.1.2)_{1,2}$ with $a(x) \equiv a(x_a)$ and $b(x) \equiv b(x_b)$, we are able to apply the theory in [111], which provides the gradient Hölder regularity with the estimates

$$\begin{aligned}
\fint_{B_\rho} \Psi^-_{B_R}(|Dh-(Dh)_{B_\rho}|)\,dx &\leqslant c\left(\frac{\rho}{R}\right)^{\beta_1} \fint_{B_{R/8}} \Psi^-_{B_R}(|Dh|)\,dx \\
&\overset{(3.5.6)}{\leqslant} c\left(\frac{\rho}{R}\right)^{\beta_1} \fint_{B_{2R}} \Psi(x,|Du|)\,dx,
\end{aligned} \tag{3.5.8}$$

whenever $0 < \rho \leqslant R/8$, where the constants c, β_1 depend only on n, $s(G)$, $s(H_a)$, $s(H_b)$, ν, L, but are independent of the values $a(x_a)$ and $b(x_b)$. Therefore, for every $0 < \rho \leqslant R/8$, we have

$$
\begin{aligned}
&\fint_{B_\rho} G(|Du - (Du)_{B_\rho}|)\,dx \\
&\quad \leqslant c \fint_{B_\rho} G(|Dh - (Dh)_{B_\rho}|)\,dx + c \fint_{B_\rho} G(|Du - Dh|)\,dx \\
&\quad \overset{(3.5.8)}{\leqslant} c\left(\frac{\rho}{R}\right)^{\beta_1} \fint_{B_{2R}} \Psi(x, |Du|)\,dx + c\left(\frac{R}{\rho}\right)^n \fint_{B_{R/8}} G(|Du - Dh|)\,dx \\
&\quad \overset{(3.5.1),(3.5.7)}{\leqslant} c\left(\frac{\rho}{R}\right)^{\beta_1} R^{-k} + c\left(\frac{R}{\rho}\right)^n R^{d/4}
\end{aligned}
\tag{3.5.9}
$$

with $c \equiv c(\mathbf{data}(\Omega_0), k)$. Notice that $k \in (0,1)$ is still arbitrary and d has been defined in (3.5.3) depending only on $\mathbf{data}(\Omega_0)$. Taking $k \equiv d\beta_1/(32n)$ and $\rho \equiv (R/8)^{1+d/(16n)}$ in the last display, after some elementary manipulations, we get

$$
\fint_{B_\rho} G(|Du - (Du)_{B_\rho}|)\,dx \leqslant c\rho^{\frac{d\beta_1}{64n}} \tag{3.5.10}
$$

for every $\rho \in (0, 1/8)$, provided $B_{8\rho} \Subset \Omega_0$. In particular, using Jensen's inequality and Lemma $2.1.1_1$, we have

$$
\fint_{B_\rho} |Du - (Du)_{B_\rho}|\,dx \leqslant c\rho^{\frac{d\beta_1}{64n}\left(1+\frac{1}{s(G)}\right)} \tag{3.5.11}
$$

for every $\rho \in (0, 1/8)$ with $B_{8\rho} \Subset \Omega_0$. By the integral characterization of Hölder continuity due to Campanato and Meyers and a standard covering argument alongside (3.5.11), $Du \in C^{0,\theta}_{\mathrm{loc}}(\Omega)$ for $\theta \equiv \dfrac{d\beta_1}{64n}\left(1 + \dfrac{1}{s(G)}\right)$. This proves the local Hölder continuity of Du. But the proof is not finished yet, since θ should be independent of Ω_0 as in the statement of Theorem 3.1.1. In order to obtain the full completeness, we apply some standard perturbation methods. Indeed, once we have that Du is locally bounded, we shall revisit the proof of Lemma 3.3.4 and Lemma 3.4.1. We also observe that the functional defined in (3.3.103) satisfies the bounded slope condition (see for instance [32]). Then there exists a constant $c \equiv c(n, s(G), s(H_a), s(H_b), \nu, L, \|Du\|_{L^\infty(B_R)})$ such that

$$\|Dh\|_{L^\infty(B_R)} \leqslant c.$$

Since Du is locally bounded, following the proof of Lemma 3.3.1, Lemma 3.3.4 and Lemma 3.4.1, specially the estimate in (3.3.3) can be modified with $\gamma \equiv 1$. Moreover, the estimates in (3.4.28) and (3.3.105) can be upgraded by

$$\begin{aligned} \fint_{B_{R/8}} & \big(|V_G(Du) - V_G(Dh)|^2 + a(x_a)|V_{H_a}(Du) - V_{H_a}(Dh)|^2 \\ & \quad + b(x_b)|V_{H_b}(Du) - V_{H_b}(Dh)|^2\big)\, dx \\ & \leqslant cR^{\min\{\mu,\alpha,\beta\}} \end{aligned} \tag{3.5.12}$$

with some constant c depending only on n, $s(G)$, $s(H_a)$, $s(H_b)$, ν, L, $\|a\|_{L^\infty(\Omega_0)}$, $\|b\|_{L^\infty(\Omega_0)}$ and $\|Du\|_{L^\infty(B_{2R})}$. In particular, the last estimate via (3.5.7) implies that

$$\fint_{B_R} G(|Du - Dh|)\, dx \leqslant cR^{\min\{\mu,\alpha,\beta\}/4}. \tag{3.5.13}$$

Therefore, (3.5.8) implies that

$$\fint_{B_\rho} G(|Dh - (Dh)_{B_\rho}|)\, dx \leqslant c\left(\frac{\rho}{R}\right)^{\beta_1}, \tag{3.5.14}$$

where β_1 depends on $n, s(G), s(H_a), s(H_b), \nu, L$ while the constant c depends only on $n, s(G),\ s(H_a), s(H_b), \nu,\ L,\ \|Du\|_{L^\infty(\Omega_0)},\ \|a\|_{L^\infty(\Omega_0)}$ and $\|b\|_{L^\infty(\Omega_0)}$. Combining the last two estimates similarly as shown in (3.5.9), we deduce the gradient Hölder continuity with the exponent depending only on n, $s(G)$, $s(H_a)$, $s(H_b)$, ν, L, α, β and μ, which is the desired dependence as described in the statement. The proof is finally complete.

3.6 Orlicz double phase problems

Let us consider a general class of functionals with double phase growth, which is essentially the case when $b(\cdot) \equiv 0$ in (1.0.2). The functionals we shall deal with is of type

$$W^{1,1}(\Omega) \ni v \mapsto \mathcal{F}_d(v, \Omega) := \int_\Omega F_d(x, v, Dv)\, dx, \tag{3.6.1}$$

where $F_d : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carátheódory function fulfilling the following double-sided growth

$$\nu \Psi_d(x, |z|) \leqslant F_d(x, y, z) \leqslant L\Psi_d(x, |z|), \tag{3.6.2}$$

whenever $x \in \Omega$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^n$, in which here and in the rest of the chapter we denote by

$$\Psi_d(x, t) := G(t) + a(x)H_a(t) \quad (\forall x \in \Omega,\ t \geqslant 0). \tag{3.6.3}$$

As we introduced before we assume $G, H_a \in \mathcal{N}$ with indices $s(G), s(H_a) \geqslant 1$ and $a(\cdot) \in C^{\omega_a}(\Omega)$ with $\omega_a : [0, \infty) \to [0, \infty)$ being a continuous and concave function such that $\omega_a(0) = 0$. We shall consider a local minimizer u of the functional $\mathcal{F}_d$ in (3.6.1) under one of the assumptions (1.0.13), (1.0.14) and (1.0.15) with $\omega_b(\cdot) \equiv 0$. Since the double sided growth assumption (3.6.3) is not enough for higher regularity properties of a local minimizer u of the functional $\mathcal{F}_d$, we shall assume that F_d is a continuous integrand belonging to the space $C^2(\mathbb{R}^n \setminus \{0\})$ with respect to z-variable and having the the following structure assumptions:

$$\begin{cases} |D_z F_d(x, y, z)||z| + |D^2_{zz} F_d(x, y, z)||z|^2 \leqslant L\Psi_d(x, |z|), \\ \nu \dfrac{\Psi_a(x, |z|)}{|z|^2} |\xi|^2 \leqslant \left\langle D^2_{zz} F_d(x, y, z)\xi, \xi \right\rangle, \\ |D_z F_d(x_1, y, z) - D_z F_d(x_2, y, z)||z| \leqslant L\omega(|x_1 - x_2|)[\Psi_d(x_1, |z|) + \Psi_d(x_2, |z|)] \\ \qquad\qquad\qquad\qquad\qquad\qquad + L|\Psi_d(x_1, |z|) - \Psi_d(x_2, |z|)|, \\ |F_d(x, y_1, z) - F_d(x, y_2, z)| \leqslant L\omega(|y_1 - y_2|)\Psi_d(x, |z|), \end{cases} \tag{3.6.4}$$

whenever $x, x_1, x_2 \in \Omega$, $y, y_1, y_2 \in \mathbb{R}$, $z \in \mathbb{R}^n \setminus \{0\}$, $\xi \in \mathbb{R}^n$, where $0 < \nu \leqslant L$ are fixed constants, and the function ω is the same as defined in (3.1.3) or

(3.1.4). The structure conditions in (3.6.4) are satisfied for instance by the model functional

$$W^{1,1}(\Omega) \ni v \mapsto \int_{\Omega} f_d(x,v)\Psi_d(x,|Dv|)\,dx, \tag{3.6.5}$$

where the continuous function $f_d(\cdot)$ satisfies $0 < \nu_0 \leqslant f(\cdot,\cdot) \leqslant L_0$ for some constants ν_0, L_0 and fulfills the following inequality

$$|f_d(x_1,y_1) - f_d(x_2,y_2)| \leqslant L_0\omega(|x_1-x_2| + |y_1-y_2|),$$

whenever $x_1, x_2 \in \mathbb{R}^n$ and $y_1, y_2 \in \mathbb{R}$, in which ω is the same as defined in (3.1.3) or (3.1.4). Another model case is given by

$$W^{1,1}(\Omega) \ni v \mapsto \int_{\Omega} [F_G(x,v,Dv) + a(x)F_{H_a}(x,v,Dv)]\,dx, \tag{3.6.6}$$

where $F_G(\cdot)$ and $F_{H_a}(\cdot)$ have $G-$growth and H_a-growth respectively, and satisfy the following suitable structure assumptions that

$$\begin{cases} |D_zF_\Phi(x,y,z)||z| + |D^2_{zz}F_\Phi(x,y,z)||z|^2 \leqslant L_0\Phi(|z|), \\ \nu_0\dfrac{\Phi(|z|)}{|z|^2}|\xi|^2 \leqslant \left\langle D^2_{zz}F_\Phi(x,y,z)\xi,\xi\right\rangle, \\ |D_zF_\Phi(x_1,y,z) - D_zF_\Phi(x_2,y,z)||z| \leqslant L_0\omega(|x_1-x_2|)\Phi(|z|), \\ |F_\Phi(x,y_1,z) - F_\Phi(x,y_2,z)| \leqslant L_0\omega(|y_1-y_2|)\Phi(|z|) \end{cases}$$

hold with $\Phi \in \{G, H_a\}$ for some positive constants ν_0, L_0, where ω is as in (3.1.3) or (3.1.4). The reason we consider the double phase case independently is that we have discussed the various regularity properties of the functional $\mathcal{F}$ in (1.0.10) in the sense of multi-phase of the type defined in (3.1.1) together with the structure assumptions (3.1.2), but this one is a special case of (3.6.1) together with the structure assumptions (3.6.4) in the sense of the double phase structures. Now we restate and prove Lemma 3.1.1 in the double phase settings which will be applied later.

Lemma 3.6.1. *Let $F_d : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function defined in* (3.6.1) *which satisfies* (3.6.2) *and* (3.6.4). *There exist positive constants $c_1, c_2 \equiv$*

$c_1, c_2(n, s(G), s(H_a), \nu)$ *such that the following inequalities*

$$\begin{aligned} &|V_G(z_1) - V_G(z_2)|^2 + a(x)|V_{H_a}(z_1) - V_{H_a}(z_2)|^2 \\ &\leqslant c_1 \langle D_z F_d(x,y,z_1) - D_z F_d(x,y,z_2), z_1 - z_2 \rangle, \end{aligned} \tag{3.6.7}$$

$$\begin{aligned} |V_G(z_1) - V_G(z_2)|^2 + a(x)|V_{H_a}(z_1) - V_{H_a}(z_2)|^2 + c_2 \langle D_z F_d(x,y,z_1), z_2 - z_1 \rangle \\ \leqslant c_2 [F_d(x,y,z_2) - F_d(x,y,z_1)] \end{aligned} \tag{3.6.8}$$

and

$$\begin{aligned} |F_d(x_1,y,z) - F_d(x_2,y,z)| \leqslant L\omega(|x_1 - x_2|)\,[\Psi_d(x_1,|z|) + \Psi_d(x_2,|z|)] \\ + L|a(x_1) - a(x_2)|H_a(|z|) \end{aligned} \tag{3.6.9}$$

hold true, whenever $z, z_1, z_2 \in \mathbb{R}^n \setminus \{0\}$, $x, x_1, x_2 \in \Omega$ *and* $y \in \mathbb{R}$.

Proof. The arguments of the proof for (3.6.7) and (3.6.8) are essentially the same as done for Lemma 3.1.1. Only difference lies in the one for (3.6.9). Since $F_d(x, y, 0) = 0$ for every $x \in \Omega$ and $y \in \mathbb{R}$, we have

$$\begin{aligned} &|F_d(x_1,y,z) - F_d(x_2,y,z)| \\ &= |(F_d(x_1,y,z) - F_d(x_1,y,0)) - (F_d(x_2,y,z) - F_d(x_2,y,0))| \\ &= \left| \int_0^1 \langle D_z F_d(x_1,y,\theta z), z\rangle \, d\theta - \int_0^1 \langle D_z F_d(x_2,y,\theta z), z\rangle \, d\theta \right| \\ &\leqslant \int_0^1 |D_z F_d(x_1,y,\theta z) - D_z F_d(x_2,y,\theta z)|\,|z| d\theta \\ &\leqslant L\omega(|x_1 - x_2|)\,[\Psi_d(x_1,|z|) + \Psi_d(x_2,|z|)] + L|a(x_1) - a(x_2)|H_a(|z|), \end{aligned}$$

where the last inequality of the last display is implied by $(3.6.4)_3$. This proves (3.6.9). □

In order to simplify the notations in the present section, we use the set of parameters for a minimizer u of the functional $\mathcal{F}_d$ depending on which one of the assumptions (1.0.13)-(1.0.15) under $\omega_b(\cdot) \equiv 0$ comes into play as the **data** in this section.

$$
\mathbf{data}_d \equiv \begin{cases} \Big\{n, \lambda_1, s(G), s(H_a), \nu, L, \|a\|_{C^{\omega_a}(\Omega)}, \omega(\cdot), \|\Psi(x,|Du|)\|_{L^1(\Omega)}, \\ \|u\|_{L^1(\Omega)}, \omega_a(1)\Big\} \\ \text{if (1.0.13) is considered under } \omega_b(\cdot) \equiv 0, \\ \Big\{n, \lambda_2, s(G), s(H_a), \nu, L, \|a\|_{C^{\omega_a}(\Omega)}, \omega(\cdot), \|u\|_{L^\infty(\Omega)}, \omega_a(1)\Big\} \\ \text{if (1.0.14) is considered under } \omega_b(\cdot) \equiv 0, \\ \Big\{n, \lambda_3, s(G), s(H_a), \nu, L, \|a\|_{C^{\omega_a}(\Omega)}, \omega(\cdot), [u]_{0,\gamma}, \omega_a(1)\Big\} \\ \text{if (1.0.15) is considered under } \omega_b(\cdot) \equiv 0, \end{cases} \tag{3.6.10}
$$

where λ_1, λ_2, λ_3 are the same as defined in (1.0.13)-(1.0.15) and $s(G), s(H_a)$ are indices of the functions G, H_a in the sense of Definition 2.1.1, respectively. With $\Omega_0 \Subset \Omega$ being a fixed open subset, we also denote by $\mathbf{data}_d(\Omega_0)$ the above set of parameters together with $\operatorname{dist}(\Omega_0, \partial\Omega)$:

$$
\mathbf{data}_d(\Omega_0) \equiv \mathbf{data}_d, \operatorname{dist}(\Omega_0, \partial\Omega). \tag{3.6.11}
$$

Now we provide the main results in this section, which correspond to Theorem 3.1.1 and Theorem 3.1.2.

Theorem 3.6.1 (Maximal regularity)**.** *Let $u \in W^{1,\Psi_d}(\Omega)$ be a local minimizer of the functional $\mathcal{F}_d$ defined in* (3.6.1) *under the assumptions* (3.6.2)*,* (3.6.4) *and* (3.1.3) *with $\omega_b(\cdot) \equiv 0$. Suppose that $\omega_a(\rho) = \rho^\alpha$ for some $\alpha \in (0,1]$. If one of the following assumptions*

$$
\begin{cases} (1.0.13), & (3.6.12\text{a}) \\ (1.0.14), & (3.6.12\text{b}) \\ (1.0.15) \quad \textit{with} \quad \limsup\limits_{\rho \to 0^+} \Lambda\left(\rho^{\frac{1}{1-\gamma}}, \dfrac{1}{\rho}\right) = 0 & (3.6.12\text{c}) \end{cases}
$$

is satisfied, then there exists $\theta \in (0,1)$ depending only on $n, s(G), s(H_a), \nu, L, \alpha$ and μ such that $Du \in C^{0,\theta}_{\mathrm{loc}}(\Omega)$.

Theorem 3.6.2 (Morrey decay)**.** *Let $u \in W^{1,\Psi_d}(\Omega)$ be a local minimizer of the functional $\mathcal{F}_d$ defined in* (3.6.1)*, under the assumptions* (3.6.2)*,* (3.6.4) *and* (3.1.4)*. Assume that $\omega_b(\cdot) \equiv 0$ in what follows. If one of the following* *assumptions*

$$\begin{cases} (1.0.13) \quad \textit{with} \quad \limsup_{\rho\to 0^+} \Lambda\left(\rho, G^{-1}(\rho^{-n})\right) = 0, & (3.6.13a) \\ (1.0.14) \quad \textit{with} \quad \limsup_{\rho\to 0^+} \Lambda\left(\rho, \dfrac{1}{\rho}\right) = 0, & (3.6.13b) \\ (1.0.15) \quad \textit{with} \quad \limsup_{\rho\to 0^+} \Lambda\left(\rho^{\frac{1}{1-\gamma}}, \dfrac{1}{\rho}\right) = 0, & (3.6.13c) \\ (1.0.13) \quad \textit{with} \quad \omega_a(\rho) = \rho^\alpha \quad \textit{for some} \quad \alpha \in (0,1], & (3.6.13d) \\ (1.0.14) \quad \textit{with} \quad \omega_a(\rho) = \rho^\alpha \quad \textit{for some} \quad \alpha \in (0,1] & (3.6.13e) \end{cases}$$

is satisfied, then

$$u \in C^{0,\theta}_{\mathrm{loc}}(\Omega) \quad \textit{for every} \quad \theta \in (0,1). \tag{3.6.14}$$

Moreover, for every $\sigma \in (0,n)$, there exists a positive constant $c \equiv c(\boldsymbol{data}_d(\Omega_0), \sigma)$ such that the decay estimate

$$\int_{B_\rho} \Psi_d(x, |Du|)\,dx \leqslant c\left(\frac{\rho}{R}\right)^{n-\sigma} \int_{B_R} \Psi_d(x, |Du|)\,dx \tag{3.6.15}$$

holds for every concentric balls $B_\rho \subset B_R \subset \Omega_0 \Subset \Omega$ with $R \leqslant 1$.

The above theorems completely cover the main results of [22], where the special case that $G(t) = t^p$, $H_a(t) = t^q$ and $\omega_a(\rho) = \rho^\alpha$ with some constants $q \geqslant p > 1$ and $\alpha \in (0,1]$ is considered. Also the results of [21] can be considered for a general class of functionals not only for the model functional in (1.0.4). Let us now briefly overview our arguments employed in proving the above theorems comparing with the ones used in [21, 22]. We do not distinguish between the G-phase, where an inequality of the type $a(\cdot) \leqslant M\omega_a(R)$ is satisfied, and (G, H_a)-phase, where a complementary inequality $a(\cdot) \geqslant M\omega_a(R)$ holds in a certain ball B_R under consideration for some suitable large constant M, which has a drawback to deal with the multi-phase type problems and even double phase type problems that we consider. Instead we consider the function $[\Psi_d]^-_{B_R}(\cdot)$ defined in (3.6.29) for a ball $B_R \subset \Omega$ under the investigation to obtain various estimates, and the advantage of considering this function is that $[\Psi_d]^-_{B_R} \in \mathcal{N}$ with an index $s(\Psi_d) = s(G) + s(H_a)$ by Remark 2.1.2, which is independent of the considered ball B_R. Also the approach introduced in this chapter may open a gate to study parabolic double phase equations of type

$$u_t - \operatorname{div}\left(G'(|Du|)\frac{Du}{|Du|} + a(x,t)H_a'(|Du|)\frac{Du}{|Du|} \right) = 0,$$

which would be one of attracting topics for the regularity theory in the future, we refer some recent results on this topic [35, 68]. Essentially, the idea of the proofs of Theorem 3.6.2 and Theorem 3.6.1 is based on the arguments previously used for proving Theorem 3.1.2 and Theorem 3.1.1, but the functional $\mathcal{F}_d$ in (3.6.1) is much more general than the functional $\mathcal{F}$ in (1.0.10) for the consideration under the double phase settings. In this regard, we need to take care of some points in more detail depending on the structure assumptions (3.6.4), specially Lemma 3.6.3 below. Since $u \in W^{1,\Psi_d}(\Omega)$ is a local L/ν-minimizer of the functional $\mathcal{P}$ defined in (1.0.1) with $b(\cdot) \equiv 0$ if u is a minimizer of the functional $\mathcal{F}_d$ in (3.6.1), we are able to rewrite the results together with their proofs under the double phase settings up to the end of Section 3.2. Starting by Section 3.3, we shall investigate in a different way.

In what follows let $B_R \equiv B_R(x_0)$ be a ball such that $B_{2R} \subset \Omega_0 \Subset \Omega$, where Ω_0 is some fixed open subset of Ω. We define a functional given by

$$W^{1,1}(B_{2R}) \ni \upsilon \mapsto \mathcal{F}_{d,B_{2R}}(\upsilon) := \int_{B_{2R}} F_d(x,(u)_{B_{2R}}, D\upsilon)\,dx \tag{3.6.16}$$

with u being a local minimizer of the functional $\mathcal{F}_d$ defined in (3.6.1). Now we consider a function $w \in u + W_0^{1,\Psi_d}(B_{2R})$ being the solution to the following variational Dirichlet problem:

$$\begin{cases} w \mapsto \min\limits_{\upsilon} \mathcal{F}_{d,B_{2R}}(\upsilon) \\ \upsilon \in u + W_0^{1,\Psi_d}(B_{2R}). \end{cases} \tag{3.6.17}$$

As in Lemma 3.3.1 we shal consider the first comparison estimates in order to remove u-dependence in the original functional $\mathcal{F}_d$ defined in (3.6.1).

Lemma 3.6.2. *Let $w \in W^{1,\Psi}(B_{2R})$ be the solution to the variational problem* (3.6.17) *under the assumptions* (3.6.2), (3.6.4) *and* (3.1.4). *Let the coefficient function $a(\cdot) \in C^{\omega_a}(\Omega)$ for ω_a being non-negative, continuous and concave function vanishing at the origin. Assume that one of the assumptions*

(1.0.13), (1.0.14) *and* (1.0.15) *under* $\omega_b(\cdot) \equiv 0$ *is satisfied. Then there exists a constant* $c \equiv c(\boldsymbol{data}_d(\Omega_0))$ *such that*

$$\begin{aligned}\fint_{B_{2R}} \left(|V_G(Du) - V_G(Dw)|^2 + a(x)|V_{H_a}(Du) - V_{H_a}(Dw)|^2\right) dx \\ \leqslant c\omega(R^\gamma) \fint_{B_{2R}} \Psi_d(x, |Du|)\, dx \qquad (3.6.18)\end{aligned}$$

holds, where $\gamma \equiv \gamma(\boldsymbol{data}_d(\Omega_0))$ *is the Hölder exponent determined via Theorem 3.2.2 in the double phase settings. Moreover, the following estimates holds true:*

$$\fint_{B_{2R}} \Psi_d(x, |Dw|)\, dx \leqslant \frac{L}{\nu} \fint_{B_{2R}} \Psi_d(x, |Du|)\, dx, \qquad (3.6.19)$$

$$\|w\|_{L^\infty(B_{2R})} \leqslant \|u\|_{L^\infty(B_{2R})}, \qquad (3.6.20)$$

$$\operatorname*{osc}_{B_{2R}} w \leqslant \operatorname*{osc}_{B_{2R}} u \qquad (3.6.21)$$

and

$$\fint_{B_{2R}} \Psi_d\left(x, \left|\frac{u-w}{R}\right|\right) dx \leqslant c[\omega(R^\gamma)]^{\frac{1}{2}} \fint_{B_{2R}} \Psi_d(x, |Du|)\, dx \qquad (3.6.22)$$

for some constant $c \equiv c(\boldsymbol{data}_d(\Omega_0))$. *Moreover, there exist a positive higher integrability exponent* $\delta_0 \equiv \delta_0(\boldsymbol{data}_d)$ *with* $\delta_0 \leqslant \delta$, *where* δ *has been determined via Theorem 3.2.5 under the double phase settings, and a constant* $c \equiv c(\boldsymbol{data}_d)$ *satisfying the following reverse Hölder inequalities:*

$$\left[\fint_{B_{R/2}} [\Psi_d(x, |Dw|)]^{1+\delta_0}\, dx\right]^{\frac{1}{1+\delta_0}} \leqslant c \fint_{B_R} \Psi_d(x, |Dw|)\, dx. \qquad (3.6.23)$$

Here, in the case that (1.0.15) *is considered,* γ *appearing in* (3.6.18) *and*

(3.6.22) *is the same as in the assumption* (1.0.15).

Proof. First of all the meaning of $\mathbf{data}_d$ and $\mathbf{data}_d(\Omega_0)$ has been defined in (3.6.10) and (3.6.11), respectively. The proofs for (3.6.19)-(3.6.22) can be done by arguing similarly as in the proof Lemma 3.3.1 together with Lemma 3.6.1. Since w is a L/ν-minimizer of the functional $\mathcal{F}_{d,B_{2R}}$ defined in (3.6.16), we are able to apply Lemma 3.2.1 under the double phase settings. In turn, it gives us that

$$\fint_{B_{R/2}} \Psi_d(x, |Dw|)\, dx \leqslant c \fint_{B_R} \Psi_d\left(x, \left|\frac{w-(w)_{B_R}}{R}\right|\right) dx \tag{3.6.24}$$

holds with $c \equiv c(n, s(G), s(H_a), L, \nu)$. Then applying Theorem 2.4.1, there exists $\theta \equiv \theta(n, s(G), s(H_a)) \in (0,1)$ such that

$$\fint_{B_{R/2}} \Psi_d\left(x, |Dw|\right)\, dx \leqslant c\bar{\kappa}_{sp} \left[\fint_{B_R} [\Psi_d(x,|Dw|)]^{\theta}\, dx\right]^{\frac{1}{\theta}} \tag{3.6.25}$$

holds with some constant $c \equiv c(n, s(G), s(H_a), L, \nu, \omega_a(1))$, where

$$\bar{\kappa}_{sp} = \begin{cases} 1 + \lambda_1[a]_{\omega_a} + \lambda_1[a]_{\omega_a} \left(\int_{B_R} G(|Dw|)\, dx\right)^{\frac{1}{n}} & \\ \text{if } (1.0.13) \text{ with } \omega_b(\cdot) \equiv 0, & (3.6.26a) \\ 1 + \lambda_2[a]_{\omega_a} + \lambda_2[a]_{\omega_a} \|w\|_{L^\infty(B_R)} & \\ \text{if } (1.0.14) \text{ with } \omega_b(\cdot) \equiv 0, & (3.6.26b) \\ 1 + \lambda_3[a]_{\omega_a} + \lambda_3[a]_{\omega_a} \left[R^{-\gamma} \operatorname*{osc}_{B_R} w\right]^{\frac{1}{1-\gamma}} & \\ \text{if } (1.0.15) \text{ with } \omega_b(\cdot) \equiv 0. & (3.6.26c) \end{cases}$$

Furthermore, taking into account (3.6.19)-(3.6.22) in the last display, we conclude that

$$\fint_{B_{R/2}} \Psi_d(x, |Dw|)\, dx \leqslant c \left[\fint_{B_R} [\Psi_d\,(x, |Dw|)]^{\theta}\, dx \right]^{\frac{1}{\theta}} \tag{3.6.27}$$

holds for some constants $\theta \equiv \theta(n, s(G), s(H_a)) \in (0,1)$ and $c \equiv c(\mathbf{data}_d)$. The last display follows (3.6.23) by applying a variant of Gehring's lemma. $\square$

At this stage, we do not need to consider Lemma 3.3.2 because we shall freeze x-variable in the non-linearity at once. For this, let us consider the excess functional given by

$$E_d(v, B_r) := \left([\Psi_d]^-_{B_{2r}}\right)^{-1} \left(\fint_{B_r} [\Psi_d]^-_{B_{2r}} \left(\left| \frac{v - (v)_{B_r}}{2r} \right| \right) dx \right) \tag{3.6.28}$$

for any function $v \in L^1(B_{2r})$ and ball $B_{2r} \subset \Omega$, where now and in the rest of this section for every open subset $\mathcal{B} \subset \Omega$, we shall denote by

$$[\Psi_d]^-_{\mathcal{B}}\,(t) := G(t) + \inf_{x \in \mathcal{B}} a(x) H_a(t) \quad (\forall t \geqslant 0), \tag{3.6.29}$$

and $\left([\Psi_d]^-_{\mathcal{B}}\right)^{-1}$ is the inverse function of $[\Psi_d]^-_{\mathcal{B}}$. By convexity of the function $[\Psi_d]^-_{B_{2r}}$ and Lemma 2.1.1, there is a constant $c \equiv c(s(G) + s(H_a))$ such that

$$E_d(v, B_r) \leqslant c \left([\Psi_d]^-_{B_{2r}}\right)^{-1} \left(\fint_{B_r} [\Psi_d]^-_{B_{2r}} \left(\left| \frac{v - v_0}{2r} \right| \right) dx \right) \tag{3.6.30}$$

holds for every $v_0 \in \mathbb{R}$. Now we consider the estimates corresponding to the outcome of Lemma 3.3.3 under our double phase settings.

Lemma 3.6.3. *Let $u \in W^{1,\Psi_d}(\Omega)$ be a local minimizer of the functional $\mathcal{F}_d$ defined in* (3.6.1) *under the assumptions* (3.6.2), (3.6.4) *and* (3.1.4). *Let $w \in W^{1,\Psi}(B_{2R})$ be the solution to the variational problem* (3.6.17). *Suppose $\omega_b(\cdot) \equiv 0$ in what follows. If one of the assumptions* (3.6.13a)-(3.6.13e) *is satisfied, then for every $\varepsilon^* \in (0,1)$, there exists a positive radius*

$$R^* \equiv R^*(\boldsymbol{data}_d(\Omega_0), \varepsilon^*) \tag{3.6.31}$$

such that

$$\begin{aligned}&\fint_{B_{\tau R}} [\Psi_d]^-_{B_R}\left(\left|\frac{w-(w)_{B_{\tau R}}}{\tau R}\right|\right)dx\\&\qquad\leqslant c\left(1+\tau^{-(n+s(\Psi_d)+1)}\varepsilon^*\right)\fint_{B_{R/2}} [\Psi_d]^-_{B_R}\left(\left|\frac{w-(w)_{B_{R/2}}}{R}\right|\right)dx \qquad (3.6.32)\end{aligned}$$

for some constant $c \equiv c\left(\boldsymbol{data}_d(\Omega_0)\right)$*, whenever* $\tau \in (0, 1/16)$ *and* $R \leqslant R^*$*.*

Proof. Again note that the meaning of $\mathbf{data}_d$ and $\mathbf{data}_d(\Omega_0)$ already has been introduced in (3.6.10)-(3.6.11). We can always assume $E_d(w, B_{R/2}) > 0$, otherwise there is nothing to prove in (3.6.32). For the simplicity, we shall write

$$E_d(R) := E_d(w, B_{R/2}), \qquad (3.6.33)$$

where the notion E_d has been defined in (3.6.28). The proof falls in several steps, similarly as we have done in the proof of Lemma 3.3.3. For the sake of completeness, we provide the proof in a full detail.

Step 1: Initial information on w**.** Applying Lemma 3.2.2 under the double phase settings to $B_{R/2}$ with $k \equiv (w)_{B_{R/2}}$, we have

$$\fint_{B_{R/4}} \Psi_d\left(x, |Dw|\right) dx \leqslant c \fint_{B_{R/2}} [\Psi_d]^-_{B_R}\left(\left|\frac{w-(w)_{B_{R/2}}}{R}\right|\right)dx \qquad (3.6.34)$$

for some constant $c \equiv c(\mathbf{data}_d)$. Moreover, it follows from Lemma 3.6.2 that there exists a higher integrability exponent $\delta_0 \equiv \delta_0(\mathbf{data}_d)$ such that

$$\left(\fint_{B_{R/8}} [\Psi_d(x, |Dw|)]^{1+\delta_0}\, dx\right)^{\frac{1}{1+\delta_0}} \leqslant c \fint_{B_{R/4}} \Psi_d(x, |Dw|)\, dx \qquad (3.6.35)$$

for a constant $c \equiv c(\mathbf{data}_d)$.

Step 2: Scaled functions. We consider scaled functions of $w(\cdot)$ and $a(\cdot)$

in the ball B_1 by setting

$$\begin{cases} \bar{w}(x) := \dfrac{w(x_0 + Rx) - (w)_{B_{R/2}}}{E_d(R)R}, & (3.6.36a) \\ \bar{a}(x) := a(x_0 + Rx)\dfrac{H_a(E_d(R))}{[\Psi_d]^-_{B_R}(E_d(R))} & (3.6.36b) \end{cases}$$

for every $x \in B_1$. Now we introduce the control function and energy density associated to our scaling introduced above in (3.6.36a)-(3.6.36b) as

$$\begin{cases} \bar{\Psi}_d(x, |z|) := \bar{G}(|z|) + \bar{a}(x)\bar{H}_a(|z|), & (3.6.37a) \\ \bar{F}_d(x, z) := \dfrac{F_d(x_0 + Rx, (u)_{B_{2R}}, E_d(R)z)}{[\Psi_d]^-_{B_R}(E_d(R))} & \\ \text{and} \quad \bar{A}_d(x, z) := D_z\bar{F}_d(x, z) & (3.6.37b) \end{cases}$$

for every $x \in B_1$ and $z \in \mathbb{R}^n$, where to the end of the proof of this lemma, we always shall understand by

$$\bar{G}(t) := \frac{G(E(R)t)}{[\Psi_d]^-_{B_R}(E_d(R))} \quad \text{and} \quad \bar{H}_a(t) := \frac{H_a(E(R)t)}{H_a(E_d(R))} \tag{3.6.38}$$

for every $t \geqslant 0$. By elementary computations, we can observe that $\bar{G}, \bar{H}_a \in \mathcal{N}$ with indices $s(G), s(H_a)$, respectively, and also that

$$\bar{G}(1) \leqslant 1 \quad \text{and} \quad \bar{H}_a(1) = 1. \tag{3.6.39}$$

Clearly, the function $\bar{w}$ minimizes the following functional

$$W^{1,\bar{\Psi}_d}(B_1) \ni v \mapsto \int_{B_1} \bar{F}_d(x, Dv)\, dx, \tag{3.6.40}$$

where the functions $\bar{\Psi}_d(\cdot)$ and $\bar{F}_d(\cdot)$ have been defined in (3.6.37a) and (3.6.37b), respectively. The Euler-Lagrange equation arising from the functional in (3.6.40) can be written as

$$\fint_{B_1} \left\langle \bar{A}_d(x, D\bar{w}), D\varphi \right\rangle dx = \fint_{B_1} \left\langle D_z\bar{F}_d\left(x, D\bar{w}\right), D\varphi \right\rangle dx = 0 \tag{3.6.41}$$

for every $\varphi \in W_0^{1,\bar{\Psi}_d}(B_1)$. By the assumptions (3.6.2) and (3.6.4) via elementary computations, we have the following structure conditions in the scaled settings:

$$\begin{cases} \nu\bar{\Psi}_d(x,|z|) \leqslant \bar{F}_d(x,z) \leqslant L\bar{\Psi}_d(x,|z|), & (3.6.42a) \\ |\bar{A}_d(x,z)||z| + |D_z\bar{A}_d(x,z)||z|^2 \leqslant L\bar{\Psi}_d(x,|z|), & (3.6.42b) \\ \nu\dfrac{\bar{\Psi}_d(x,|z|)}{|z|^2}|\xi|^2 \leqslant \left\langle D_z\bar{A}_d(x,z)\xi,\xi\right\rangle, & (3.6.42c) \\ \left|\bar{A}_d(x_1,z) - \bar{A}_d(x_2,z)\right||z| & \\ \leqslant L\omega(R|x_1-x_2|)\left[\bar{\Psi}_d(x_1,|z|) + \bar{\Psi}_d(x_2,|z|)\right] & \\ \qquad\qquad + L|\bar{a}(x_1) - \bar{a}(x_2)|\bar{H}_a(|z|) & (3.6.42d) \end{cases}$$

for every $x, x_1, x_2 \in B_1$ and $z \in \mathbb{R}^n \setminus \{0\}$.

Step 3: Freezing. Now we shall consider frozen functional and vector field associated to $\bar{F}_d(\cdot)$ and $\bar{A}_d(\cdot)$ defined in (3.6.37b). Let $\bar{x}_a \in \overline{B}_1$ such that $\bar{a}(\bar{x}_a) = \inf_{x\in B_1} \bar{a}(x)$. Then we denote by

$$\bar{F}_0(z) := \bar{F}_d(\bar{x}_a,z), \quad \bar{A}_0(z) := D_z\bar{F}_d(\bar{x}_a,z), \tag{3.6.43}$$

and

$$\bar{\Psi}_0(t) := \bar{G}(t) + \bar{a}(\bar{x}_a)\bar{H}_a(t) \tag{3.6.44}$$

for every $x \in B_1$, $z \in \mathbb{R}^n$ and $t \geqslant 0$. Here we single out that here is a difference between Step 3 of the proof for Lemma 3.3.3 and our present situation. By the very definition in (3.6.37a) and (3.6.38), one can check

$$\bar{\Psi}_0(1) = 1. \tag{3.6.45}$$

In our newly scaled environment, let us now consider the functional

$$W^{1,\bar{\Psi}_0}\left(B_{1/8}\right) \ni v \mapsto \int_{B_{1/8}} \bar{F}_0(Dv)\,dx. \tag{3.6.46}$$

We observe that the newly defined integrand $\bar{F}_0(\cdot)$ and vector field $\bar{A}_0(\cdot)$

satisfy the growth and ellipticity conditions as

$$\begin{cases} \nu\bar{\Psi}_0(|z|) \leqslant \bar{F}_0(z) \leqslant L\bar{\Psi}_0(|z|), & (3.6.47a) \\ |\bar{A}_0(z)||z| + |D_z\bar{A}_0(z)||z|^2 \leqslant L\bar{\Psi}_0(|z|), & (3.6.47b) \\ \nu\dfrac{\bar{\Psi}_0(|z|)}{|z|^2}|\xi|^2 \leqslant \left\langle D_z\bar{A}_0(z)\xi, \xi\right\rangle & (3.6.47c) \end{cases}$$

for every $z \in \mathbb{R}^n \setminus \{0\}$ and $\xi \in \mathbb{R}^n$. Therefore, the energy and higher integralibility estimates in (3.6.34) and (3.6.35) can be seen in the view of $\bar{w}$ as

$$\fint_{B_{1/4}} \bar{\Psi}_d(x, |D\bar{w}|)\, dx + \left(\fint_{B_{1/8}} [\bar{\Psi}_d(x, |D\bar{w}|)]^{1+\delta_0}\, dx\right)^{\frac{1}{1+\delta_0}} \leqslant c(\mathbf{data}_d). \quad (3.6.48)$$

Step 4: Harmonic type approximation. Let $\varphi \in W_0^{1,\infty}\left(B_{1/8}\right)$ be any fixed function. Using (3.6.41), we see

$$\begin{aligned} I_0 := \left|\fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}), D\varphi\right\rangle\, dx\right| &= \left|\fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}) - \bar{A}_d(x, D\bar{w}), D\varphi\right\rangle\, dx\right| \\ &\leqslant \fint_{B_{1/8}} |\bar{A}_0(D\bar{w}) - \bar{A}_d(x, D\bar{w})|\, dx\, \|D\varphi\|_{L^\infty(B_{1/8})} =: I_1\, \|D\varphi\|_{L^\infty(B_{1/8})}\,. \end{aligned} \quad (3.6.49)$$

Now we estimate I_1 in the last display using (3.6.42d). In turn, we have

$$\begin{aligned} I_1 &\leqslant L\omega(R) \fint_{B_{1/8}} \left(\frac{\bar{\Psi}_d(\bar{x}_a, |D\bar{w}|)}{|D\bar{w}|} + \frac{\bar{\Psi}_d(x, |D\bar{w}|)}{|D\bar{w}|}\right)\, dx \\ &\quad + L \fint_{B_{1/8}} |\bar{a}(x) - \bar{a}(\bar{x}_a)| \frac{\bar{H}_a(|D\bar{w}|)}{|D\bar{w}|}\, dx \\ &\leqslant 2L\omega(R) \fint_{B_{1/8}} \frac{\bar{\Psi}_d(\bar{x}_a, |D\bar{w}|)}{|D\bar{w}|}\, dx \end{aligned}$$

$$
\begin{aligned}
&+ 2L(1+\omega(R)) \fint_{B_{1/8}} |\bar{a}(x) - \bar{a}(\bar{x}_a)| \frac{\bar{H}_a(|D\bar{w}|)}{|D\bar{w}|}\, dx \\
&=: 2L\omega(R) I_{11} + 2L(1+\omega(R)) I_{12}. \qquad (3.6.50)
\end{aligned}
$$

Now we estimate the terms appearing in the last display. Recalling (3.6.44) and (3.6.45) together with (2.1.7), we find

$$
\begin{aligned}
I_{11} &\leqslant c \fint_{B_{1/8}} \bar{\Psi}_0'(|D\bar{w}|)\, dx \leqslant c \left[\bar{\Psi}_0(1)\right]^{\frac{s(\Psi_d)}{1+s(\Psi_d)}} \fint_{B_{1/8}} \left[\bar{\Psi}_0(|D\bar{w}|)\right]^{\frac{1}{1+s(\Psi_d)}}\, dx \\
&\quad + c \left[\bar{\Psi}_0(1)\right]^{\frac{1}{1+s(\Psi_d)}} \fint_{B_{1/8}} \left[\bar{\Psi}_0(|D\bar{w}|)\right]^{\frac{s(\Psi_d)}{1+s(\Psi_d)}}\, dx \\
&\leqslant c \left(\fint_{B_{1/8}} \bar{\Psi}_0(|D\bar{w}|)\, dx \right)^{\frac{1}{1+s(\Psi_d)}} + c \left(\fint_{B_{1/8}} \bar{\Psi}_0(|D\bar{w}|)\, dx \right)^{\frac{s(\Psi_d)}{1+s(\Psi_d)}} \\
&\leqslant c(\mathbf{data}_d), \qquad (3.6.51)
\end{aligned}
$$

where we have applied the Hölder's inequality together with (3.6.48) and the fact that $\bar{\Psi}_0 \in \mathcal{N}$ with an index $s(\Psi_d) = s(G) + s(H_a)$ by Remark 2.1.2. Next we shall deal with estimating the second term I_{12} in (3.6.50). In turn, using (2.1.7) and (3.6.39), we have

$$
\begin{aligned}
I_{12} &\leqslant c \fint_{B_{1/8}} |\bar{a}(x) - \bar{a}(\bar{x}_a)| \left([\bar{H}_a(|D\bar{w}|)]^{\frac{1}{s(H_a)+1}} + [\bar{H}_a(|D\bar{w}|)]^{\frac{s(H_a)}{s(H_a)+1}} \right) dx \\
&\leqslant c \, \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{s(H_a)}{s(H_a)+1}} \left(\fint_{B_{1/8}} \bar{a}(x) \bar{H}_a(|D\bar{w}|)\, dx \right)^{\frac{1}{s(H_a)+1}} \\
&\quad + c \, \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_a)+1}} \left(\fint_{B_{1/8}} \bar{a}(x) \bar{H}_a(|D\bar{w}|)\, dx \right)^{\frac{s(H_a)}{s(H_a)+1}} \\
&\leqslant c(\mathbf{data}_d) \left(\|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_a)+1}} + \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{s(H_a)}{s(H_a)+1}} \right), \qquad (3.6.52)
\end{aligned}
$$

where we have used also Hölder's inequality and the fact that $\bar{a}(\bar{x}_a) \leqslant \bar{a}(x)$ for every $x \in B_1$. Inserting those estimates coming from the last two displays into (3.6.50) and then (3.6.49), we find

$$
\begin{aligned}
I_0 &\leqslant c(\mathbf{data}_d(\Omega_0)) \\
&\times \left[\omega(R) + (1+\omega(R)) \left(\|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_a)+1}} + \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{s(H_a)}{s(H_a)+1}} \right)\right].
\end{aligned}
\tag{3.6.53}
$$

Now we shall estimate the term $\|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}$ depending on which one of the assumptions (3.6.13a)-(3.6.13e) comes into play. Recalling the definition of $\bar{a}(\cdot)$ in (3.6.36b) and the excess functional in (3.6.33), we have

$$
I_a := \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})} \leqslant c\omega_a(R) \frac{H_a(E_d(R))}{[\Psi_d]^-_{B_R}(E_d(R))}. \tag{3.6.54}
$$

Case 1: Assumption (3.6.13a) **is in force.** It follows from the assumption $(3.6.13\text{a})_2$ that for any $\varepsilon \in (0,1)$ there exists $\mu_1 > 0$ depending on ε such that

$$
\Lambda\left(\rho, G^{-1}\left(\rho^{-n}\right)\right) \leqslant \varepsilon \quad \text{for every} \quad \rho \in (0, \mu_1). \tag{3.6.55}
$$

Then using the last display, (1.0.13) and the fact that $\left([\Psi_d]^-_{B_R}\right)^{-1}(t) \leqslant G^{-1}(t)$ for every $t \geqslant 0$, I_a in (3.6.54) can be estimated as

$$
\begin{aligned}
I_a &\leqslant c\omega_a(R) \frac{(H_a \circ G^{-1})\left([\Psi_d]^-_{B_R}(E_d(R))\right)}{[\Psi_d]^-_{B_R}(E_d(R))} \\
&\leqslant c\omega_a(R)\varepsilon \left(1 + \frac{1}{\omega_a\left(\left[[\Psi_d]^-_{B_R}(E_d(R))\right]^{-\frac{1}{n}}\right)}\right) + c\omega_a(R)\left(1 + \frac{1}{\omega_a(\mu_1)}\right)
\end{aligned}
\tag{3.6.56}
$$

with $c \equiv c([a]_{\omega_a}, \lambda_1)$. Using (2.1.2) and the energy estimate (3.6.19), we see

$$
\frac{1}{\omega_a\left(\left[[\Psi_d]^-_{B_R}(E_d(R))\right]^{-\frac{1}{n}}\right)} \leqslant \frac{c}{\omega_a(R)} + \frac{c}{\omega_a(R)} \int_{B_{R/2}} [\Psi_d]^-_{B_R}\left(\left|\frac{w - (w)_{B_{R/2}}}{R}\right|\right) dx
$$

$$\leqslant \frac{c}{\omega_a(R)} + \frac{c}{\omega_a(R)} \int_{B_{2R}} \Psi_d\left(x, |Du|\right)\, dx \leqslant \frac{c(\mathbf{data}_d)}{\omega_a(R)}. \tag{3.6.57}$$

Combining the last two displays, we conclude

$$I_a \leqslant c\left(\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a(\mu_1)}\right)\right) \tag{3.6.58}$$

with some constant $c \equiv c(\mathbf{data}_d)$. Therefore, inserting the estimates in the last two displays into (3.6.53) and recalling (3.6.49), we have

$$\left| \fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}), D\varphi \right\rangle \, dx \right| \leqslant c(\mathbf{data}_d) P_1(\varepsilon, R) \left\| D\varphi \right\|_{L^\infty(B_{1/8})}, \tag{3.6.59}$$

where

$$\begin{aligned} P_1(\varepsilon, R) := \omega(R) + (1+\omega(R))\left[\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a(\mu_1)}\right)\right]^{\frac{1}{s(H_a)+1}} \\ + (1+\omega(R))\left[\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a(\mu_1)}\right)\right]^{\frac{s(H_a)}{s(H_a)+1}} \end{aligned} \tag{3.6.60}$$

Case 2: Assumption (3.6.13b) **is in force.** From the assumption $(3.6.13\mathrm{b})_2$ it holds that for every $\varepsilon \in (0,1)$ there exists $\mu_2 > 0$ depending on ε such that

$$\Lambda\left(\rho, \frac{1}{\rho}\right) \leqslant \varepsilon \quad \text{for every} \quad \rho \in (0, \mu_2). \tag{3.6.61}$$

Then by the very definition of $[\Psi_d]^-_{B_R}$ in (3.6.29) together with (3.6.61) and (1.0.14) under $\omega_b \equiv 0$, we have

$$\begin{aligned} I_a &\leqslant c\omega_a(R)\frac{H_a(E_d(R))}{G(E_d(R))} \\ &\leqslant c\omega_a(R)\varepsilon\left(1 + \frac{1}{\omega_a\left([E_d(R)]^{-1}\right)}\right) + c\omega_a(R)\left(1 + \frac{1}{\omega_a\left(\mu_2\right)}\right). \end{aligned} \tag{3.6.62}$$

Again using (2.1.1) together with taking into account (3.6.20), we see

$$\frac{1}{\omega_a\left([E_d(R)]^{-1}\right)} \leqslant \frac{1}{\omega_a\left(\frac{R}{2\|w\|_{L^\infty(B_R)}}\right)} \leqslant \frac{c(\mathbf{data}_d)}{\omega_a(R)}. \tag{3.6.63}$$

Combining the last two displays, we find

$$I_a \leqslant c\left(\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a(\mu_2)}\right)\right) \tag{3.6.64}$$

with some constant $c \equiv c(\mathbf{data}_d)$. Then, plugging the estimates in the last two displays into (3.6.53) and recalling (3.6.49), we have

$$\left| \fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}), D\varphi \right\rangle \, dx \right| \leqslant c(\mathbf{data}_d) P_2(\varepsilon, R) \left\| D\varphi \right\|_{L^\infty(B_{1/8})}, \tag{3.6.65}$$

where

$$\begin{aligned} P_2(\varepsilon, R) := \omega(R) + (1+\omega(R)) \left[\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a(\mu_2)}\right)\right]^{\frac{1}{s(H_a)+1}} \\ + (1+\omega(R)) \left[\varepsilon + \omega_a(R)\left(1 + \frac{1}{\omega_a(\mu_2)}\right)\right]^{\frac{s(H_a)}{s(H_a)+1}}. \end{aligned} \tag{3.6.66}$$

Case 3: Assumption (3.6.13c) **is in force.** The assumption $(3.6.13\text{c})_2$ implies that for any $\varepsilon \in (0,1)$, there exists $\mu_3 > 0$ depending on ε such that

$$\Lambda\left(\rho^{\frac{1}{1-\gamma}}, \frac{1}{\rho}\right) \leqslant \varepsilon \quad \text{for every} \quad \rho \in (0, \mu_3). \tag{3.6.67}$$

This one together with using (3.6.54) and (1.0.15) under $\omega_b(\cdot) \equiv 0$ implies

$$I_a \leqslant c\omega_a(R)\frac{H_a(E_d(R))}{G(E_d(R))}$$

$$\leqslant c\omega_a(R)\varepsilon\left(1+\frac{1}{\omega_a\left([E_d(R)]^{-\frac{1}{1-\gamma}}\right)}\right)+c\omega_a(R)\left(1+\frac{1}{\omega_a\left(\mu_3^{\frac{1}{1-\gamma}}\right)}\right). \tag{3.6.68}$$

Now using (3.6.21) and (1.0.15), we have

$$\frac{1}{\omega_a\left([E_d(R)]^{-\frac{1}{1-\gamma}}\right)}\leqslant\frac{1}{\omega_a\left(\left[\frac{\operatorname{osc}_{B_{2R}} u}{R}\right]^{-\frac{1}{1-\gamma}}\right)}\leqslant\frac{c(\mathbf{data}_d)}{\omega_a(R)}. \tag{3.6.69}$$

Combining the last two displays, we find

$$I_a\leqslant c\left(\varepsilon+\omega_a(R)\left(1+\frac{1}{\omega_a\left(\mu_3^{\frac{1}{1-\gamma}}\right)}\right)\right) \tag{3.6.70}$$

for some constant $c\equiv c(\mathbf{data}_d)$. Using the estimate (3.6.70) in (3.6.53), we conclude

$$\left|\fint_{B_{1/8}}\left\langle\bar{A}_0(D\bar{w}),D\varphi\right\rangle\,dx\right|\leqslant c(\mathbf{data}_d)P_3(\varepsilon,R)\left\|D\varphi\right\|_{L^\infty(B_{1/8})}, \tag{3.6.71}$$

where

$$\begin{aligned}P_3(\varepsilon,R):=\omega(R)+(1+\omega(R))&\left[\varepsilon+\omega_a(R)\left(1+\frac{1}{\omega_a\left(\mu_3^{\frac{1}{1-\gamma}}\right)}\right)\right]^{\frac{1}{s(H_a)+1}}\\&+(1+\omega(R))\left[\varepsilon+\omega_a(R)\left(1+\frac{1}{\omega_a\left(\mu_3^{\frac{1}{1-\gamma}}\right)}\right)\right]^{\frac{s(H_a)}{s(H_a)+1}}.\end{aligned} \tag{3.6.72}$$

Case 4. Assumption (3.6.13d) **is in force.** Now we take the advantage that $w_a(\cdot)$ is the power function. Recalling I_a denoted in (3.6.54), we see that

$$\begin{aligned}
I_a &\leqslant cR^{\alpha}\frac{(H_a\circ G^{-1})\left([\Psi_d]^-_{B_R}(E_d(R))\right)}{[\Psi_d]^-_{B_R}(E_d(R))} \\
&\leqslant cR^{\alpha}\left(1+\left[\fint_{B_{R/2}}[\Psi_d]^-_{B_R}\left(\left|\frac{w-(w)_{B_{R/2}}}{R}\right|\right)\,dx\right]^{\frac{\alpha}{n}}\right) \\
&\leqslant cR^{\alpha}+c\left(\int_{B_{R/2}}[\Psi_d]^-_{B_R}(|Dw|)\,dx\right)^{\frac{\alpha}{n}} \\
&\leqslant cR^{\alpha}+cR^{\frac{\alpha\delta_0}{1+\delta_0}}\left(\int_{B_{R/2}}\left[[\Psi_d]^-_{B_R}(|Dw|)\right]^{1+\delta_0}\,dx\right)^{\frac{\alpha}{n(1+\delta_0)}} \\
&\leqslant c(\mathbf{data}_d(\Omega_0))R^{\frac{\alpha\delta_0}{1+\delta_0}},
\end{aligned}\tag{3.6.73}$$

where we have used the higher integrability estimates (3.2.62) of Theorem 3.2.5 under the double phase settings. Using estimates from the last display in (3.6.53) and recalling $R\leqslant 1$, we see

$$\left|\fint_{B_{1/8}}\left\langle \bar{A}_0(D\bar{w}),D\varphi\right\rangle\,dx\right|\leqslant c(\mathbf{data}_d(\Omega_0))Q_1(R)\left\|D\varphi\right\|_{L^{\infty}(B_{1/8})},\tag{3.6.74}$$

where

$$Q_1(R):=\omega(R)+(1+\omega(R))R^{\frac{\alpha\delta_0}{(1+\delta_0)(1+s(H_a))}}.\tag{3.6.75}$$

Case 5: Assumption (3.6.13e) **is in force.** Using the assumption (1.0.14) and (3.6.21), I_a in (3.6.54) can be estimated as

$$I_a\leqslant cR^{\alpha}\frac{H_a(E_d(R))}{G(E_d(R))}$$

$$\leqslant cR^{\alpha}\left(1+\left[\left([\Psi_d]^-_{B_R}\right)^{-1}\left(\fint_{B_{R/2}}[\Psi_d]^-_{B_R}\left(\left|\frac{w-(w)_{B_{R/2}}}{R}\right|\right)\,dx\right)\right]^{\alpha}\right)$$
$$\leqslant c\left(R^{\alpha}+\left[\operatorname*{osc}_{B_{2R}} u\right]^{\alpha}\right)\leqslant c(\mathbf{data}_d(\Omega_0))R^{\gamma\alpha}, \tag{3.6.76}$$

where we have also used (3.2.33) and γ is the Hölder continuity exponent coming from Theorem 3.2.2 under the double phase settings. Inserting the estimate from the last display into (3.6.53) and recalling $R\leqslant 1$, we see

$$\left|\fint_{B_{1/8}}\left\langle \bar{A}_0(D\bar{w}),D\varphi\right\rangle\,dx\right|\leqslant c(\mathbf{data}_d(\Omega_0))Q_2(R)\left\|D\varphi\right\|_{L^\infty(B_{1/8})}, \tag{3.6.77}$$

where

$$Q_2(R):=\omega(R)+(1+\omega(R))R^{\frac{\alpha\gamma}{1+s(H_a)}}. \tag{3.6.78}$$

Collecting the estimates obtained in (3.6.59), (3.6.65),(3.6.71), (3.6.74) and (3.6.77), we conclude with

$$\left|\fint_{B_{1/8}}\left\langle \bar{A}_0(D\bar{w}),D\varphi\right\rangle\,dx\right|\leqslant c_h D(\varepsilon,R)\left\|D\varphi\right\|_{L^\infty(B_{1/8})} \tag{3.6.79}$$

for some constant $c_h\equiv c_h(\mathbf{data}_d(\Omega_0))$ for every $\varphi\in W^{1,\infty}_0(B_{1/8})$, where

$$D(\varepsilon,R):=\begin{cases} P_1(\varepsilon,R) & \text{if (3.6.13a) is assumed,}\\ P_2(\varepsilon,R) & \text{if (3.6.13b) is assumed,}\\ P_3(\varepsilon,R) & \text{if (3.6.13c) is assumed,}\\ Q_1(R) & \text{if (3.6.13d) is assumed,}\\ Q_2(R) & \text{if (3.6.13e) is assumed,}\end{cases} \tag{3.6.80}$$

in which P_1,P_2, P_3, Q_1 and Q_2 have been defined in (3.6.60), (3.6.66), (3.6.72), (3.6.75) and (3.6.78), respectively. By (3.6.45), (3.6.47a)-(3.6.47c) and (3.6.79), we are able to apply Lemma 2.5.1 with $A_0(z)\equiv\bar{A}_0(z)$, $\Psi_0(t)\equiv\bar{\Psi}_0(t)$ with $a_0\equiv\bar{a}(\bar{x}_a)$ and $b_0\equiv 0$. By Lemma 2.5.1, there exists $\bar{h}\in$

$\bar{w} + W_0^{1,\bar{\Psi}_0}(B_{1/8})$ such that

$$\fint_{B_{1/8}} \langle \bar{A}_0(D\bar{h}), D\varphi \rangle \, dx = 0 \qquad \text{for all} \qquad \varphi \in W_0^{1,\infty}(B_{1/8}), \tag{3.6.81}$$

$$\fint_{B_{1/4}} \bar{\Psi}_0(|D\bar{h}|) \, dx + \fint_{B_{1/8}} [\bar{\Psi}_0(|D\bar{h}|)]^{1+\delta_1} \, dx \leqslant c \text{ for some } \delta_1 \leqslant \delta_0, \tag{3.6.82}$$

$$\fint_{B_{1/8}} \left(|V_{\bar{G}}(D\bar{w}) - V_{\bar{G}}(D\bar{h})|^2 + \bar{a}(\bar{x}_a)|V_{\bar{H}_a}(D\bar{w}) - V_{\bar{H}_a}(D\bar{h})|^2\right) \, dx \leqslant c[D(\varepsilon, R)]^{s_1} \tag{3.6.83}$$

and finally

$$\fint_{B_{1/8}} \left(\bar{G}\left(|\bar{w} - \bar{h}|\right) + \bar{a}(\bar{x}_a)\bar{H}_a\left(|\bar{w} - \bar{h}|\right)\right) \, dx \leqslant c_d[D(\varepsilon, R)]^{s_0} \tag{3.6.84}$$

with some constants $c, c_d \equiv c, c_d(\mathbf{data}_d(\Omega_0)) \geqslant 1$ and $s_0, s_1 \equiv s_0, s_1(\mathbf{data}_d) \in (0,1)$, but they are all independent of R. The rest of the proof is similar as the argument after (3.3.98) of Lemma 3.3.3. $\square$

Lemma 3.6.4. *Under the assumptions of Lemma 3.6.3, let $w \in W^{1,\Psi}(B_{2R})$ be the solution to the problem defined in* (3.6.17). *If one of the assumptions* (3.6.13a)-(3.6.13e) *is satisfied, then there exists $h \in w + W_0^{1,[\Psi_d]^-_{B_R}}(B_{R/8})$ being a local minimizer of the functional defined by*

$$W^{1,1}(B_{R/8}) \ni v \mapsto \mathcal{F}_0(v) := \int_{B_{R/8}} F_0(Dv) \, dx, \tag{3.6.85}$$

where the integrand function is given by

$$F_0(z) := F\left(x_a, (u)_{B_{2R}}, z\right) \tag{3.6.86}$$

for $x_a \in \overline{B}_R$ being a point such that $a(x_a) := a^-(B_R)$, whenever $z \in \mathbb{R}^n$, such *that*

$$\fint_{B_{R/8}} \left[|V_G(Du) - V_G(Dh)|^2 + a(x_a)|V_{H_a}(Du) - V_{H_a}(Dh)|^2\right] \, dx$$
$$\leqslant c\left(\omega\left(R^{\gamma}\right) + [D(\varepsilon, R)]^{s_1}\right) \fint_{B_{2R}} \Psi_d(x, |Du|)\, dx \tag{3.6.87}$$

for some constant $c \equiv c(\boldsymbol{data}_d(\Omega_0))$, where s_1 and $D(\varepsilon, R)$ have been defined in (3.6.83) *and* (3.6.80)*, respectively. Moreover, we have the energy estimate*

$$\fint_{B_{R/8}} [\Psi_d]^-_{B_R}(|Dh|)\, dx \leqslant c \fint_{B_{2R}} \Psi_d(x, |Du|)\, dx \tag{3.6.88}$$

for some constant $c \equiv c(n, \nu, L)$.

Proof. We need to revisit the proof of Lemma 3.6.3, specially Step 3 and Step 4. We consider a function $\bar{h} \in \bar{w} + W_0^{1,\bar{\Psi}_0}(B_{1/8})$ satisfying (3.6.81)-(3.6.84). Let h be the scaled back function of $\bar{h}$ in $B_{R/8}$ as

$$h(x) := E_d(w, B_{R/2}) R \bar{h}\left(\frac{x - x_0}{R}\right) \quad \text{for every} \quad x \in B_{R/8}(x_0). \tag{3.6.89}$$

Clearly, $h \in w + W_0^{1,[\Psi_d]^-_{B_R}}(B_{R/8})$ is a local minimizer of the functional $\mathcal{F}_0$ defined in (3.6.85) which means that

$$\mathcal{F}_0(h) = \int_{B_{R/8}} F_0(Dh)\, dx \leqslant \int_{B_{R/8}} F_0(Dh + D\varphi)\, dx \leqslant \mathcal{F}_0(h + \varphi) \tag{3.6.90}$$

holds for every $\varphi \in W_0^{1,[\Psi_d]^-_{B_R}}(B_{R/8})$. As shown in (3.3.9), we recall (3.6.19) to discover that

$$\fint_{B_{R/8}} [\Psi_d]^-_{B_R}(|Dh|)\, dx \leqslant \frac{L}{\nu} \fint_{B_{R/8}} [\Psi_d]^-_{B_R}(|Dw|)\, dx$$
$$\leqslant \frac{8^n L}{\nu} \fint_{B_R} \Psi_d(x, |Dw|)\, dx \leqslant c(n, \nu, L) \fint_{B_{2R}} \Psi_d(x, |Du|)\, dx, \tag{3.6.91}$$

which proves (3.6.88). We write the inequality (3.6.83) in view of G, H_a, w and h in order to have

$$
\begin{aligned}
&\fint_{B_{R/8}} \left[|V_G(Du) - V_G(Dh)|^2 + a(x_a)|V_{H_a}(Du) - V_{H_a}(Dh)|^2\right]\, dx \\
&\qquad \leqslant c[D(\varepsilon, R)]^{s_1} \fint_{B_{R/2}} [\Psi_d]^-_{B_R}\left(\left|\frac{w-(w)_{B_{R/2}}}{R}\right|\right)\, dx \\
&\qquad \leqslant c[D(\varepsilon, R)]^{s_1} \fint_{B_{R/2}} [\Psi_d]^-_{B_R}\left(|Dw|\right)\, dx \\
&\qquad \leqslant c[D(\varepsilon, R)]^{s_1} \fint_{B_{R/2}} \Psi_d\left(x, |Du|\right)\, dx
\end{aligned} \tag{3.6.92}
$$

for some constant $c \equiv c(\mathbf{data}_d(\Omega_0))$, where we have applied the Sobolev-Poincaré inequality and (3.6.91). Combining this estimate together with (3.6.18) via some elementary computations, we directly reach (3.6.87). $\square$

We finally finish the present subsection with a crucial decay estimate on u.

Lemma 3.6.5. *Under the assumptions of Lemma 3.6.3, if one of the conditions* (3.6.13a)-(3.6.13e) *is satisfied, then for every $\varepsilon_* \in (0,1)$, there exists a positive radius R_* with the dependence as*

$$R_* \equiv R_*(\boldsymbol{data}_d(\Omega_0), \varepsilon_*) \tag{3.6.93}$$

such that if $R \leqslant R_*$, then there exists a constant $c_G \equiv c_G(\boldsymbol{data}_d(\Omega_0))$ such that

$$\int_{B_{\tau R}} [\Psi_d]^-_{B_R}\left(\left|\frac{u-(u)_{B_{\tau R}}}{\tau R}\right|\right)\, dx \leqslant c_G\left(\tau^n + \tau^{-(s(\Psi_d)+1)}\varepsilon_*\right)\int_{B_{2R}} \Psi_d(x, |Du|)\, dx \tag{3.6.94}$$

holds for every $\tau \in (0, 1/32)$.

Proof. For the proof, we apply Lemma 3.6.3 with $\varepsilon^* \in (0,1)$ to be determined in a few lines, and we can use (3.6.32) provided

$$R \leqslant R^* \equiv R^*(\mathbf{data}_d(\Omega_0), \varepsilon^*)$$

is found via (3.6.31). For every $\tau \in (0, 1/32)$ with some elementary manipulations, we see that

$$\begin{aligned}
\fint_{B_{\tau R}} [\Psi_d]^-_{B_R}\left(\left|\frac{u-(u)_{B_{\tau R}}}{\tau R}\right|\right) dx &\leqslant c \fint_{B_{\tau R}} [\Psi_d]^-_{B_R}\left(\left|\frac{u-(w)_{B_{\tau R}}}{\tau R}\right|\right) dx \\
&\leqslant c \fint_{B_{\tau R}} [\Psi_d]^-_{B_R}\left(\left|\frac{w-(w)_{B_{\tau R}}}{\tau R}\right|\right) dx + c\tau^{-(n+s(\Psi_d)+1)} \fint_{B_R} [\Psi_d]^-_{B_R}\left(\left|\frac{u-w}{R}\right|\right) dx \\
&\leqslant c\left(1+\tau^{-(n+s(\Psi_d)+1)}\varepsilon^*\right) \fint_{B_{R/2}} [\Psi_d]^-_{B_R}\left(\left|\frac{w-(w)_{B_{R/2}}}{R}\right|\right) dx \\
&\quad + c\tau^{-(n+s(\Psi_d)+1)} \fint_{B_R} [\Psi_d]^-_{B_R}\left(\left|\frac{u-w}{R}\right|\right) dx \\
&\leqslant c\left(1+\tau^{-(n+s(\Psi_d)+1)}\varepsilon^*\right) \fint_{B_R} [\Psi_d]^-_{B_R}(|Dw|)\, dx \\
&\quad + c\tau^{-(n+s(\Psi_d)+1)} \fint_{B_R} [\Psi_d]^-_{B_R}\left(\left|\frac{u-w}{R}\right|\right) dx
\end{aligned} \tag{3.6.95}$$

with some constant $c \equiv c(\mathbf{data}_d(\Omega_0))$, where throughout the last display we repeatedly used (2.1.6) and (3.3.36). The last display and (3.6.22) along with some elementary manipulations yield

$$\begin{aligned}
&\int_{B_{\tau R}} [\Psi_d]^-_{B_R}\left(\left|\frac{u-(u)_{B_{\tau R}}}{\tau R}\right|\right) dx \\
&\qquad \leqslant c\left(\tau^n + \tau^{-(s(\Psi_d)+1)}\varepsilon^* + \tau^{-(s(\Psi_d)+1)}[\omega(R^\gamma)]^{\frac{1}{2}}\right) \int_{B_{2R}} \Psi_d(x, |Du|)\, dx
\end{aligned}$$

for every $\tau \in (0, 1/16)$ and some $c \equiv c(\mathbf{data}_d(\Omega_0))$. Then we choose $\varepsilon^* \equiv \varepsilon_*/2$ and $R_* \leqslant R^*$ in such a way that $[\omega(R_*^\gamma)]^{\frac{1}{2}} \leqslant \varepsilon_*/2$. This choice gives us the dependence as described in (3.6.93) and yields (3.6.94). $\square$

We have now discovered all the necessary tools. They are Lemma 3.6.2, Lemma 3.6.3 and Lemma 3.6.5 in the double phase settings for proving Theorem 3.6.1 and Theorem 3.6.2. Applying those lemmas with arguing in a similar manner as in the proofs of Theorem 3.1.2 and Theorem 3.1.1, we are able to prove Theorem 3.6.1 and Theorem 3.6.2. For the sake of the completeness, we provide a sketch of the proofs.

Proof of Theorem 3.6.2. The proof of Theorem 3.6.2 can be done similarly as for the proof of Theorem 3.1.2. We just combine Lemma 3.2.3 under the double phase settings and Lemma 3.6.5, as we already have done in (3.4.1)-(3.4.27).

Lemma 3.6.6. *Under the assumptions and notations of Lemma 3.6.3 and Lemma 3.6.4, let* $w \in W^{1,\Psi_d}(B_R)$ *be the function defined in* (3.6.17)*. Suppose that* (3.6.13c) *is satisfied for* $\omega_a(t) = t^\alpha$ *with some* $\alpha \in (0,1]$*. Then there exists a function* $h \in w + W_0^{1,[\Psi_d]^-_{B_R}}(B_{R/8})$ *being a local minimizer of the functional* $\mathcal{F}_0$ *defined in* (3.6.85) *such that*

$$\fint_{B_{R/8}} \left[|V_G(Du) - V_G(Dh)|^2 + a(x_a)|V_{H_a}(Du) - V_{H_a}(Dh)|^2\right] dx \leqslant c\left(\omega\left(R^\gamma\right) + \left[\omega(R) + (1+\omega(R))R^{\frac{\alpha}{2(1+s(H_a))}}\right]^{s_1}\right)\fint_{B_{2R}} \Psi_d(x,|Du|)\,dx \tag{3.6.96}$$

for some constant $c \equiv c(\boldsymbol{data}_d(\Omega_0))$ *and* $s_1 \equiv s_1(\boldsymbol{data}_d)$*, respectively. Moreover, the energy estimate*

$$\fint_{B_{R/8}} [\Psi_d]^-_{B_R}(|Dh|)\,dx \leqslant c \fint_{B_{2R}} \Psi_d(x,|Du|)\,dx \tag{3.6.97}$$

holds for some constant $c \equiv c(n,\nu,L)$*.*

Proof. First we apply Theorem 3.6.2 in order to obtain that, for every $\theta \in (0,1)$ and every open subset $\Omega_0 \Subset \Omega$, there exists a constant $c \equiv c(\mathbf{data}_d(\Omega_0),\theta)$ such that

$$[u]_{0,\theta;\Omega_0} \leqslant c(\mathbf{data}_d(\Omega_0),\theta). \tag{3.6.98}$$

In particular, we choose $\theta \equiv (\gamma+1)/2$. By revisiting the proof of Lemma 3.6.3, we shall estimate the term I_a introduced in (3.6.54). Using (1.0.15) and (3.6.21), we have

$$I_a \leqslant cR^{\alpha}\left(1+\left[\left([\Psi_d]^-_{B_R}\right)^{-1}\left(\fint_{B_{R/2}} [\Psi_d]^-_{B_R}\left(\left|\frac{w-(w)_{B_{R/2}}}{R}\right|\right)\,dx\right)\right]^{\frac{\alpha}{1-\gamma}}\right)$$

$$\leqslant c\left(R^{\alpha}+R^{-\frac{\alpha\gamma}{1-\gamma}}\left[\operatorname*{osc}_{B_{2R}} u\right]^{\frac{\alpha}{1-\gamma}}\right) \leqslant c(\mathbf{data}_d(\Omega_0))R^{\alpha/2}, \tag{3.6.99}$$

where we have used (3.6.98) with the choice of $\theta \equiv (1+\gamma)/2$ and $B_{2R}\subset\Omega_0$ with $R\leqslant 1$. Plugging this estimate in (3.6.53), we find

$$\left|\fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}), D\varphi\right\rangle\,dx\right| \leqslant c(\mathbf{data}_d(\Omega_0))Q_3(R)\,\|D\varphi\|_{L^\infty(B_{1/8})}\,, \tag{3.6.100}$$

where

$$Q_3(R) := \omega(R)+(1+\omega(R))R^{\frac{\alpha}{2(1+s(H_a))}}, \tag{3.6.101}$$

where the vector field $\bar{A}_0$ has been defined in (3.6.43). We consider a function $\bar{h}\in\bar{w}+W^{1,\bar{\Psi}_0}_0(B_{1/8})$ satisfying (3.6.81)-(3.6.84) with the term $D(\varepsilon,R)$ replaced by $Q_3(R)$ defined above. Let h be the scaled back function of $\bar{h}$ in $B_{R/8}$ as

$$h(x) := E_d(w,B_{R/2})R\bar{h}\left(\frac{x-x_0}{R}\right) \quad \text{for every} \quad x\in B_{R/8}(x_0). \tag{3.6.102}$$

Clearly, $h\in w+W^{1,[\Psi_d]^-_{B_R}}_0(B_{R/8})$ is a local minimizer of the functional $\mathcal{F}_0$ defined in (3.6.85) which means that

$$\mathcal{F}_0(h)=\int_{B_{R/8}} F_0(Dh)\,dx \leqslant \int_{B_{R/8}} F_0(Dh+D\varphi)\,dx \leqslant \mathcal{F}_0(h+\varphi) \tag{3.6.103}$$

holds for every $\varphi\in W^{1,[\Psi_d]^-_{B_R}}_0(B_{R/8})$. Arguing similarly as in the proof of

Lemma 3.6.2 together with recalling (3.6.19), we see

$$\begin{aligned}\fint_{B_{R/8}} [\Psi_d]^-_{B_R}(|Dh|)\,dx &\leqslant \frac{L}{\nu}\fint_{B_{R/8}} [\Psi_d]^-_{B_R}(|Dw|)\,dx\\ &\leqslant \frac{8^n L}{\nu}\fint_{B_R} \Psi_d(x,|Dw|)\,dx\\ &\leqslant c(n,\nu,L)\fint_{B_{2R}} \Psi_d(x,|Du|)\,dx, \qquad (3.6.104)\end{aligned}$$

which proves (3.6.97). We write the inequality (3.6.83) in view of G, H_a, w and h in order to have

$$\begin{aligned}&\fint_{B_{R/8}} \left[|V_G(Dw)-V_G(Dh)|^2 + a(x_a)|V_{H_a}(Dw)-V_{H_a}(Dh)|^2\right]\,dx\\ &\leqslant c[Q_3(R)]^{s_1}\fint_{B_{R/2}} [\Psi_d]^-_{B_R}\left(\left|\frac{w-(w)_{B_{R/2}}}{R}\right|\right)\,dx\\ &\leqslant c[Q_3(R)]^{s_1}\fint_{B_{R/2}} [\Psi_d]^-_{B_R}(|Dw|)\,dx\\ &\leqslant c[Q_3(R)]^{s_1}\fint_{B_{2R}} \Psi_d(x,|Du|)\,dx \qquad (3.6.105)\end{aligned}$$

for some constant $c \equiv c(\mathbf{data}_d(\Omega_0))$, where we have applied the Sobolev-Poincaré inequality and (3.6.91). Combining this estimate together with (3.6.18) via some elementary computations implies (3.6.96). □

Proof of Theorem 3.6.1. It follows from Theorem 3.6.2 and a standard covering argument that, for every open subset $\Omega_0 \Subset \Omega$ and any number $k > 0$, there exists a constant $c \equiv c(\mathbf{data}_d(\Omega_0), k)$ such that

$$\fint_{B_{2R}} \Psi_d(x,|Du|)\,dx \leqslant cR^{-k} \qquad (3.6.106)$$

for every $B_{2R} \subset \Omega_0$ with $R \leqslant 1$. Now we fix an open subset $\Omega_0 \Subset \Omega$ and a ball $B_{2R} \equiv B_{2R}(x_0) \subset \Omega_0$ with $R \leqslant 1$. Then applying Lemma 3.6.4 and Lemma 3.6.6,

$$\fint_{B_{R/8}} \left(|V_G(Du) - V_G(Dh)|^2 + a(x_a)|V_{H_a}(Du) - V_{H_a}(Dh)|^2\right) dx$$
$$\leqslant c\left(R^{\mu\gamma} + [Q(R)]^{s_1}\right) \fint_{B_{2R}} \Psi_d(x, |Du|)\, dx \tag{3.6.107}$$

for some constant $c \equiv c(\mathbf{data}_d(\Omega_0))$ and $s_1 \equiv s_1(\mathbf{data}_d)$, where

$$Q(R) := \begin{cases} R^\mu + (1+R^\mu)R^{\frac{\alpha\delta_0}{(1+\delta_0)(1+s(H_a))}} & \text{if (3.6.12a) is assumed,} \\ R^\mu + (1+R^\mu)R^{\frac{\alpha\gamma}{1+s(H_a)}} & \text{if (3.6.12b) is assumed,} \\ R^\mu + (1+R^\mu)R^{\frac{\alpha}{2(1+s(H_a))}} & \text{if (3.6.12c) is assumed,} \end{cases} \tag{3.6.108}$$

in which γ is the Hölder continuity exponent determined via Theorem 3.2.2 under the double phase settings and δ_0 is the higher integrability exponent coming from Lemma 3.6.2. Denoting by

$$d \equiv d(\mathbf{data}_d(\Omega_0)) := \begin{cases} \min\left\{\mu\gamma, s_1\mu, \dfrac{\alpha\delta_0 s_1}{(1+\delta_0)(1+s(H_a))}\right\} \\ \text{if (3.6.12a) is assumed,} \\ \min\left\{\mu\gamma, s_1\mu, \dfrac{\alpha\gamma s_1}{1+s(H_a)}\right\} \\ \text{if (3.6.12b) is assumed,} \\ \min\left\{\mu\gamma, s_1\mu, \dfrac{\alpha s_1}{2(1+s(H_a))}\right\} \\ \text{if (3.6.12c) is assumed,} \end{cases} \tag{3.6.109}$$

and choosing $k \equiv d/4$ in (3.6.106), the inequality (3.6.107) can be written as

$$\fint_{B_{R/8}} \left(|V_G(Du) - V_G(Dh)|^2 + a(x_a)|V_{H_a}(Du) - V_{H_a}(Dh)|^2\right) dx \leqslant cR^{3d/4} \tag{3.6.110}$$

for some constant $c \equiv c(\mathbf{data}_d(\Omega_0))$, where we again recall that the function h has been defined via Lemma 3.6.4 and Lemma 3.6.6. Recalling (3.6.88) and

(3.6.97), we have the energy estimate

$$\fint_{B_{R/8}} [\Psi_d]^-_{B_R}(|Dh|)\,dx \leqslant c \fint_{B_{2R}} \Psi_d(x,|Du|)\,dx \tag{3.6.111}$$

with a constant $c \equiv c(n,\nu,L)$. Once we arrive at this stage, the rest of the proof can be done in the same way as argued in the proof of Theorem 3.1.1. The proof is complete.

3.7 Regularity results under additional integrability

We turn our attention to studying properties of a local Q-minimizer of the functional $\mathcal{P}$ defined in (1.0.1) under some additional Lebesgue integrability assumption. We shall consider a local Q-minimizer u of the functional $\mathcal{P}$ under the following assumptions:

$$\begin{cases} u \in W^{1,\Psi}(\Omega) \cap L^{\kappa}(\Omega) \quad (\kappa \geqslant 1) \\ \lambda_4(\kappa) := \sup\limits_{\rho>0} \Lambda\left(\rho^{\frac{\kappa}{n+\kappa}}, \dfrac{1}{\rho}\right) < \infty, \end{cases} \tag{3.7.1}$$

where the function $\Lambda : (0,\infty)\times(0,\infty) \to (0,\infty)$ has been defined in (1.0.12) together with $\omega_a, \omega_b : [0,\infty) \to [0,\infty)$ being continuous and concave functions vanishing at the origin such that $a(\cdot) \in C^{0,\omega_a}(\Omega)$ and $b(\cdot) \in C^{0,\omega_b}(\Omega)$. To see the meaning of the assumption $(3.7.1)_2$, let us consider the standard double phase that $G(t) = t^p$, $H_a(t) = t^q$ and $\omega_a(\rho) = \rho^{\alpha}$, $b(\cdot) \equiv 0$ for $1 < p \leqslant q$ and $\alpha \in (0,1]$. Under these standard double phase settings, the assumption $(3.7.1)_2$ is equivalent to the following one:

$$q \leqslant p + \frac{\alpha\kappa}{n+\kappa}. \tag{3.7.2}$$

A local Q-minimizer $u \in W^{1,\Psi}(\Omega)$ implies that $u \in W^{1,p}(\Omega)$. It is clearly interesting point that $p < n$, otherwise we can prove $u \in L^{\infty}_{\mathrm{loc}}(\Omega)$ by using Morrey-Embedding properties for $p > n$ and using a higher integrability for $p = n$. Then, for $1 < p < n$, applying Sobolev embedding properties, one can see that $u \in L^{\frac{np}{n-p}}_{\mathrm{loc}}(\Omega)$. Choosing $\kappa \equiv \dfrac{np}{n-p}$, the condition (3.7.2) is equivalent to the following one

$$q \leqslant p + \frac{\alpha p}{n},$$

which generates the same condition as (1.0.6a), as we have discussed in the introduction part. Now if $\kappa > \frac{np}{n-p}$, then we would have

$$q \leqslant p + \frac{\alpha p}{n} < p + \frac{\alpha \kappa}{n+\kappa},$$

which tells us the possible range of q is larger than the one in (1.0.6a). Considering a local Q-minima of the functional $\mathcal{P}$ under the assumption (3.7.1) , we shall show that $u \in L^{\infty}_{\mathrm{loc}}(\Omega)$. To do this, we start by proving a Sobolev-Poincaré inequality under the assumption $(3.7.1)_2$.

Theorem 3.7.1. *Let $v \in W^{1,\Psi}(B_R) \cap L^{\kappa}(B_R)$ for a ball $B_R \subset \Omega$ with $R \leqslant 1$ under the assumption $(3.7.1)_2$. Then, for any $d \in \left[1, \frac{n(n+\kappa)}{n(n+\kappa)-\kappa}\right)$, there exist constants $\theta \equiv \theta(n, s(G), s(H_a), s(H_b), \kappa, d) \in (0,1)$ and $c \equiv c(n, s(G), s(H_a), s(H_b), \omega_a(1), \omega_b(1), \kappa, d)$ such that the following Sobolev-Poincaré-type inequality holds:*

$$\left[\fint_{B_R} \left[\Psi\left(x, \left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx \right]^{\frac{1}{d}} \leqslant c\lambda_{sp} \left[\fint_{B_R} [\Psi(x,|Dv|)]^{\theta}\, dx \right]^{\frac{1}{\theta}}, \quad (3.7.3)$$

where

$$\lambda_{sp} = 1 + ([a]_{\omega_a} + [b]_{\omega_b}) \left(\lambda_4(\kappa) + \lambda_4(\kappa) \left(\int_{B_R} |v|^{\kappa}\, dx \right)^{\frac{1}{n+\kappa}} \right) \quad (3.7.4)$$

Moreover, the above estimate (3.7.3) is still valid with $v-(v)_{B_R}$ replaced by v if $v \in W^{1,\Psi}_0(B_R) \cap L^{\kappa}(B_R)$.

Proof. Note that the above theorem covers [123, Theorem 3.1], which is a special case when $G(t) = t^p$, $H(t) = t^q$, $\omega_a(\rho) = \rho^{\alpha}$ and $\omega_b(\cdot) \equiv 0$ for some constants $1 < p \leqslant q$ and $\alpha \in (0,1]$. Also our proof is much more elementary comparing with the approach used there. Using the continuity of the coefficient functions $a(\cdot)$ and $b(\cdot)$ and arguing in the same way as in (2.4.6), we find

$$\begin{aligned}
I &:= \left(\fint_{B_R} \left[\Psi \left(x, \left| \frac{v-(v)_{B_R}}{R} \right| \right) \right]^d dx \right)^{\frac{1}{d}} \\
&\leqslant 6[a]_{\omega_a} \omega_a(R) \left(\fint_{B_R} \left[H_a \left(\left| \frac{v-(v)_{B_R}}{R} \right| \right) \right]^d dx \right)^{\frac{1}{d}} \\
&\quad + 6[b]_{\omega_b} \omega_b(R) \left(\fint_{B_R} \left[H_b \left(\left| \frac{v-(v)_{B_R}}{R} \right| \right) \right]^d dx \right)^{\frac{1}{d}} \\
&\quad + 3 \left(\fint_{B_R} \left[\Psi^-_{B_R} \left(\left| \frac{v-(v)_{B_R}}{R} \right| \right) \right]^d dx \right)^{\frac{1}{d}} \\
&=: 6[a]_{\omega_a} I_1 + 6[b]_{\omega_b} I_2 + 3 I_3.
\end{aligned} \tag{3.7.5}$$

We now shall deal with estimating the terms I_i with $i \in 1,2,3$ in (3.7.5) using the additional a priori assumption $u \in L^\kappa(B_R)$ under $(3.7.1)_2$. In turn, using (2.1.2) and the assumption $(3.7.1)_2$, we see

$$\begin{aligned}
I_1 &= \omega_a(R) \left(\fint_{B_R} \left[\frac{H_a \left(\left| \frac{v-(v)_{B_R}}{R} \right| \right)}{G \left(\left| \frac{v-(v)_{B_R}}{R} \right| \right)} G \left(\left| \frac{v-(v)_{B_R}}{R} \right| \right) \right]^d dx \right)^{\frac{1}{d}} \\
&\leqslant \lambda_4(\kappa) \omega_a(R) \left(\fint_{B_R} \left[\left(1 + \left[\omega_a \left(\left(\left| \frac{v-(v)_{B_R}}{R} \right| \right)^{-\frac{\kappa}{n+\kappa}} \right) \right]^{-1} \right) \right. \right. \\
&\quad \left. \left. G \left(\left| \frac{v-(v)_{B_R}}{R} \right| \right) \right]^d dx \right)^{\frac{1}{d}} \\
&\leqslant \lambda_4(\kappa) \omega_a(R) \left(\fint_{B_R} \left[\left(1 + \left[\frac{1}{\omega_a(R)} + \frac{R}{\omega_a(R)} \left| \frac{v-(v)_{B_R}}{R} \right|^{\frac{\kappa}{n+\kappa}} \right] \right) \right. \right.
\end{aligned}$$

$$G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}}$$

$$\leqslant c_*\lambda_4(\kappa)\left(\fint_{B_R}\left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}}$$

$$+ c_*\lambda_4(\kappa)R^{\frac{n}{n+\kappa}}\left(\fint_{B_R}|v-(v)_{B_R}|^{\frac{d\kappa}{n+\kappa}}\left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}}$$

for the constant $c_* = 2(1+\omega_a(1))$. Using Hölder's inequality with conjugate exponents $\left(\frac{n+\kappa}{d}, \frac{n+\kappa}{n+\kappa-d}\right)$, we have

$$R^{\frac{n}{n+\kappa}}\left(\fint_{B_R}|v-(v)_{B_R}|^{\frac{d\kappa}{n+\kappa}}\left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}}$$

$$\leqslant R^{\frac{n}{n+\kappa}}\left(\fint_{B_R}|v-(v)_{B_R}|^{\kappa} dx\right)^{\frac{1}{n+\kappa}}\left(\fint_{B_R}\left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^{\frac{(n+\kappa)d}{n+\kappa-d}} dx\right)^{\frac{n+\kappa-d}{(n+\kappa)d}}$$

$$\leqslant c\left(\int_{B_R}|v|^{\kappa} dx\right)^{\frac{1}{n+\kappa}}\left(\fint_{B_R}\left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^{\frac{(n+\kappa)d}{n+\kappa-d}} dx\right)^{\frac{n+\kappa-d}{(n+\kappa)d}}$$

for some constant $c \equiv c(n)$. Combining the last two displays and arguing similarly for I_2, we discover

$$I_1 + I_2 \leqslant$$

$$c\lambda_4(\kappa)\left(\fint_{B_R}\left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^d dx\right)^{\frac{1}{d}}$$

$$+ c\lambda_4(\kappa)\left(\int_{B_R}|v|^{\kappa} dx\right)^{\frac{1}{n+\kappa}}\left(\fint_{B_R}\left[G\left(\left|\frac{v-(v)_{B_R}}{R}\right|\right)\right]^{\frac{(n+\kappa)d}{n+\kappa-d}} dx\right)^{\frac{n+\kappa-d}{(n+\kappa)d}}$$

for some constant $c \equiv c(n, \omega_a(1), \omega_b(1))$. Now we apply Lemma 2.4.1 to $\Phi \equiv G$ with $d_0 \equiv d$ and $d_0 \equiv \dfrac{n+\kappa-d}{(n+\kappa)d}$ in order to have an exponent $\theta_1 \equiv \theta_1(n, s(G), \kappa, d) \in (0,1)$ such that

$$I_1 + I_2 \leqslant c \left(\lambda_4(\kappa) + \lambda_4(\kappa) \left(\int_{B_R} |v|^{\kappa} \, dx \right)^{\frac{1}{n+\kappa}} \right) \left(\fint_{B_R} \left[G\left(|Dv|\right)\right]^{\theta_1} \, dx \right)^{\frac{1}{\theta_1}} \tag{3.7.6}$$

holds for some constant $c \equiv c(n, s(G), \omega_a(1), \omega_b(1), \kappa, d)$. On the other hand, since $\Psi^-_{B_R} \in \mathcal{N}$ with an index $s(\Psi) = s(G) + s(H_a) + s(H_b)$ by Remark 2.1.2, we are able to apply Lemma 2.4.1 with $\Phi \equiv \Psi^-_{B_R}$ for $d_0 \equiv d$. In turn, there exists $\theta_2 \equiv \theta_2(n, s(\Psi), d)$ such that

$$I_3 \leqslant c \left[\fint_{B_R} \left[\Psi^-_{B_R}(|Dv|) \right]^{\theta_2} \, dx \right]^{\frac{1}{\theta_2}} \tag{3.7.7}$$

with some constant $c \equiv c(n, s(\Psi), d)$. Taking into account the estimates obtained in (3.7.6)-(3.7.7) into (3.7.5), recalling the very definition of $\Psi^-_{B_R}$ in (2.1.3) and setting $\theta := \max\{\theta_1, \theta_2\}$, we arrive at (3.7.3). The proof is finished. □

Remark 3.7.1. We here remark that choosing $d \equiv 1$ in a Sobolev-Poincaré type inequality of Theorem 3.7.1, we see that there exists an exponent $\theta \equiv \theta(n, s(G), s(H_a), s(H_b), \kappa)$ such that

$$\fint_{B_R} \Psi\left(x, \left|\frac{v - (v)_{B_R}}{R}\right|\right) \, dx \leqslant c\lambda_{sp} \left[\fint_{B_R} [\Psi(x, |Dv|)]^{\theta} \, dx \right]^{\frac{1}{\theta}} \tag{3.7.8}$$

holds for some constant $c \equiv c(n, s(G), s(H_a), s(H_b), \omega_a(1), \omega_b(1), \kappa)$, where λ_{sp} is the one same as in (3.7.4).

Remark 3.7.2. With $u \in W^{1,\Psi}(\Omega)$ being a local Q-minimizer of the functional $\mathcal{P}$, we here point out that it is also possible to suppose a priori $u \in W^{\Phi}(\Omega)$ for some Young function Φ. In this case, discovering a relevant assumption like $(3.7.1)_2$ would be an interesting point to find how it is connected to

Embedding properties in Orlicz-Sobolev spaces [51, 52, 53] likewise we have discussed above in Lebesgue settings. Moreover, proving various regularity results under a new relevant condition may generate a different phenomenon even for a Lavrentiev gap. We can also a priori assume that local Q-minima belong to certain Campanato, BMO, VMO, or some other spaces. Under all those a priori assumptions, it should be necessary to discover out the relevant optimal conditions under which various regularity results are obtainable.

For a local Q-minimizer u of the functional $\mathcal{P}$ under the assumption (3.7.1), the data of the problem is understood by the following set of parameters:

$$\mathbf{data}_i \equiv \{n, \lambda_4(\kappa), \kappa, s(G), s(H_a), s(H_b), \omega_a(1), \omega_b(1), \|u\|_{L^\kappa(\Omega)}, Q\}. \quad (3.7.9)$$

As usual, for any open subset $\Omega_0 \Subset \Omega$, we denote by $\mathbf{data}_i(\Omega_0)$ the set of parameters defined above together with $\operatorname{dist}(\Omega_0, \partial\Omega)$. Now we focus on showing local boundedness estimates of a local Q-minimizer u of the functional $\mathcal{P}$ in (1.0.1) under the assumption (3.7.1).

Theorem 3.7.2. *Let $u \in W^{1,\Psi}(\Omega)$ be a local Q-minimizer of the functional $\mathcal{P}$ in* (1.0.1) *under the assumption* (3.7.1)*. Then there exists a constant $c \equiv c(\boldsymbol{data}_i)$ such that*

$$\left\| \Psi^-_{B_R}\left(\left| \frac{(u-(u)_{B_R})_\pm}{R} \right| \right) \right\|_{L^\infty(B_{R/2})} \leqslant c \fint_{B_R} \Psi\left(x, \left| \frac{(u-(u)_{B_R})_\pm}{R} \right| \right) dx \quad (3.7.10)$$

and

$$\Psi^-_{B_R}\left(\left| \frac{u(x_1)-u(x_2)}{R} \right| \right) \leqslant c \fint_{B_R} \Psi(x, |Du|)\, dx \quad \text{for a.e} \quad x_1, x_2 \in B_{R/2}, \quad (3.7.11)$$

whenever $B_R \equiv B_R(x_0) \subset \Omega$ is a ball with $R \leqslant 1$. In particular, $u \in L^\infty_{\rm loc}(\Omega)$.

Proof. The meaning of $\mathbf{data}_i$ under the assumption (3.7.1), already has been introduced in (3.7.9). As in the proof of Theorem 3.2.1, we consider the following scaled functions as:

$$
\begin{aligned}
&\bar{u}(x) := \frac{u(x_0+Rx)-(u)_{B_R}}{R}, \quad \bar{a}(x) := a(x_0+Rx), \quad \bar{b}(x) := b(x_0+Rx),\\
&\bar{\Psi}(x,t) := G(t)+\bar{a}(x)H(t)+\bar{b}(x)H(t),\\
&\bar{A}(k,s) := B_s(0)\cap\{\bar{u}>k\} \quad \text{and} \quad \bar{B}(k,s) := B_s(0)\cap\{\bar{u}<k\} \qquad (3.7.12)
\end{aligned}
$$

for every $x \in B_1(0)$, $t \geqslant 0$, $s \in (0,1)$ and $k \in \mathbb{R}$. The remaining part of the proof consists of 3 steps as in the proof of Theorem 3.2.1.

Step 1: Sobolev-Poincaré under the scaling in (3.7.12)**.** In this step, we prove that there exists a positive exponent $\theta \equiv \theta(n, s(G), s(H_a), s(H_b), \kappa) \in (0,1)$ such that

$$
\fint_{B_1} \bar{\Psi}(x,|f|)\,dx \leqslant c\bar{\kappa}_{sp}\left(\fint_{B_1}[\bar{\Psi}(x,|Df|)]^{\theta}\,dx\right)^{\frac{1}{\theta}} \qquad (3.7.13)
$$

for some constant $c \equiv c(n, s(G), s(H_a), s(H_b), \omega_a(1), \omega_b(1), \kappa)$, whenever $f \in W_0^{1,\bar{\Psi}}(B_1) \cap L^{\kappa}(B_1)$, where

$$
\bar{\kappa}_{sp} = 1 + ([a]_{\omega_a}+[b]_{\omega_b})\left(\lambda_1+\lambda_1 R\left(\fint_{B_1}|f|^{\kappa}\,dx\right)^{\frac{1}{n+\kappa}}\right).
$$

Using the continuity properties of $\bar{a}(\cdot)$ and $\bar{b}(\cdot)$, we see

$$
\begin{aligned}
I :&= \fint_{B_1} \bar{\Psi}(x,|f|)\,dx\\
&\leqslant 2[a]_{\omega_a}\omega_a(R)\fint_{B_1} H_a(|f|)\,dx + 2[b]_{\omega_b}\omega_b(R)\fint_{B_1} H_b(|f|)\,dx + \fint_{B_1}\bar{\Psi}^{-}_{B_1}(|f|)\,dx\\
&=: 2[a]_{\omega_a}I_1 + 2[b]_{\omega_b}I_2 + I_3,
\end{aligned}
$$

where

$$
\bar{\Psi}^{-}_{B_1}(t) := G(t) + \inf_{x\in B_1}\bar{a}(x)H_a(t) + \inf_{x\in B_1}\bar{b}(x)H_b(t) \quad \text{for every} \quad t \geqslant 0.
$$

Now we estimate the terms I_i for $i \in \{1,2,3\}$ similarly as in the proof of Theorem 3.7.1. In turn, using the assumption $(3.7.1)_2$ and (2.1.2), we have

$$
\begin{aligned}
I_1 &= \omega_a(R) \int_{B_1} \frac{H_a(|f|)}{G(|f|)} G(|f|)\, dx \\
&\leqslant \lambda_4(\kappa)\omega_a(R) \int_{B_1} \left(1 + \left[\omega_a\left(|f|^{-\frac{\kappa}{n+\kappa}}\right)\right]^{-1}\right) G\left(|f|\right)\, dx \\
&\leqslant \lambda_4(\kappa)\omega_a(R) \int_{B_1} \left(1 + \left[\frac{1}{\omega_a(R)} + \frac{R}{\omega_a(R)} \left|f\right|^{\frac{\kappa}{n+\kappa}}\right]\right) G\left(|f|\right)\, dx \\
&\leqslant \lambda_4(\kappa)(1+\omega_a(1)) \int_{B_1} G\left(|f|\right)\, dx + 2\lambda_4(\kappa) R \int_{B_1} |f|^{\frac{\kappa}{n+\kappa}} G\left(|f|\right)\, dx.
\end{aligned}
$$

Arguing in the same way, we have

$$
I_2 \leqslant \lambda_4(\kappa)(1+\omega_b(1)) \int_{B_1} G\left(|f|\right)\, dx + 2\lambda_4(\kappa) R \int_{B_1} |f|^{\frac{\kappa}{n+\kappa}} G\left(|f|\right)\, dx.
$$

Then the inequality (3.7.13) follows from the arguments used in the proof of Theorem 3.7.1 and Lemma 2.4.1.

Step 2. Proof of (3.7.10)**.** Since $u-(u)_{B_R}$ is a local Q-minimizer of the functional $\mathcal{P}$ in (1.0.1), using a Caccioppoli inequality of Lemma 3.2.1, one can see that

$$
\int_{B_t} \bar{\Psi}(x, |D(\bar{u}-k)_{\pm}|)\, dx \leqslant c \int_{B_s} \bar{\Psi}\left(x, \frac{(\bar{u}-k)_{\pm}}{s-t}\right)\, dx \tag{3.7.14}
$$

holds for some constant $c \equiv c(s(G), s(H_a), s(H_b), Q)$, whenever $0 < t < s \leqslant 1$ and $k \in \mathbb{R}$. Let us now consider the concentric balls $B_\rho \Subset B_t \Subset B_s$ with $1/2 \leqslant \rho < s \leqslant 1$ and $t := (\rho+s)/2$. Let $\eta \in C_0^\infty(B_t)$ be a standard cut-off function such that $\chi_{B_\rho} \leqslant \eta \leqslant \chi_{B_t}$ and $|D\eta| \leqslant \dfrac{2}{t-\rho} = \dfrac{4}{s-\rho}$. Now we apply inequality (3.7.13) from Step 1 above in order to have a positive exponent

$\theta \equiv \theta(n, s(G), s(H_a), s(H_b), \kappa)$ such that

$$\int_{\bar{A}(k,\rho)} \bar{\Psi}(x, \bar{u}-k)\,dx \leqslant \int_{B_1} \bar{\Psi}(x, \eta(\bar{u}-k)_+)\,dx$$

$$\leqslant c\bar{\kappa}_{sp}\left(\int_{B_1} [\bar{\Psi}(x, |D(\eta(\bar{u}-k)_+)|)]^{\theta}\,dx\right)^{\frac{1}{\theta}}$$

for some constant $c \equiv c(n, s(G), s(H_a), s(H_b), \omega_a(1), \omega_b(1), \kappa)$, where

$$\bar{\kappa}_{sp} = 1 + ([a]_{\omega_a} + [b]_{\omega_b})\left(\lambda_4(\kappa) + \lambda_4(\kappa) R\left(\int_{B_1} [\eta(\bar{u}-k)_+)]^{\kappa}\ dx\right)^{\frac{1}{n+\kappa}}\right).$$

By recalling the definition of $\bar{u}$ in (3.7.12), we have

$$\bar{\kappa}_{sp} \leqslant c\left[1 + R\left(\fint_{B_R} \left|\frac{u-(u)_{B_R}}{R}\right|^{\kappa}\ dx\right)^{\frac{1}{n+\kappa}}\right] \leqslant c\left[1 + \left(\int_{B_R} |u|^{\kappa}\,dx\right)^{\frac{1}{n+\kappa}}\right]$$

with a constant $c \equiv c(n, \lambda_4(\kappa), [a]_{\omega_a} + [b]_{\omega_b})$. Once we arrive at this stage the rest of the proof can be proceed in the same way as in the proof of Theorem 3.2.1. $\square$

Theorem 3.7.3. *Let $u \in W^{1,\Psi}(\Omega)$ be a local Q-minimizer of the functional $\mathcal{P}$ defined in* (1.0.1) *under the coefficient functions $a(\cdot) \in C^{\omega_a}(\Omega)$ and $b(\cdot) \in C^{\omega_b}(\Omega)$ for ω_a, ω_b being non-negative concave functions vanishing at the origin. If the assumption* (3.7.1) *is satisfied, then for for every open subset $\Omega_0 \Subset \Omega$, there exists a Hölder continuity exponent $\gamma \equiv \gamma(\boldsymbol{data}_i(\Omega_0)) \in (0,1)$ such that*

$$\|u\|_{L^\infty(\Omega_0)} + [u]_{0,\gamma;\Omega_0} \leqslant c(\boldsymbol{data}_i(\Omega_0)) \tag{3.7.15}$$

and the oscillation estimate

$$\operatorname*{osc}_{B_\rho} u \leqslant c\left(\frac{\rho}{R}\right)^{\gamma} \operatorname*{osc}_{B_R} u \tag{3.7.16}$$

holds for some $c \equiv c(\boldsymbol{data}_i(\Omega_0))$ *and all concentric balls* $B_\rho \Subset B_R \Subset \Omega_0 \Subset \Omega$ *with* $R \leqslant 1$.

Proof. First let us observe that, for every $t \geqslant 1$, we have

$$\begin{aligned}\frac{\omega_a(t)}{1+\omega_a(t)}\frac{1+\omega_a\left(t^{\frac{\kappa}{n+\kappa}}\right)}{\omega_a\left(t^{\frac{\kappa}{n+\kappa}}\right)} &\leqslant 1+\frac{\omega_a(t)}{\omega_a\left(t^{\frac{\kappa}{n+\kappa}}\right)+\omega_a(t)\omega_a\left(t^{\frac{\kappa}{n+\kappa}}\right)} \\ &\leqslant 1+\frac{1}{\omega_a\left(t^{\frac{\kappa}{n+\kappa}}\right)} \leqslant 1+\frac{1}{\omega_a(1)}.\end{aligned}$$

This same inequality holds true also for ω_b. Therefore, for every $t \geqslant 1$, we see that

$$\begin{aligned}\Lambda\left(t,\frac{1}{t}\right) &\leqslant \lambda_4(\kappa)\left(\frac{\omega_a(t)}{1+\omega_a(t)}\frac{1+\omega_a\left(t^{\frac{\kappa}{n+\kappa}}\right)}{\omega_a\left(t^{\frac{\kappa}{n+\kappa}}\right)}+\frac{\omega_b(t)}{1+\omega_b(t)}\frac{1+\omega_b\left(t^{\frac{\kappa}{n+\kappa}}\right)}{\omega_b\left(t^{\frac{\kappa}{n+\kappa}}\right)}\right) \\ &\leqslant \lambda_4(\kappa)\left(1+\frac{1}{\omega_a(1)}+\frac{1}{\omega_b(1)}\right) =: \lambda_2,\end{aligned}$$

where we have used the assumption $(3.7.1)_2$. On the other hand, recalling that the functions ω_a and ω_b are increasing, we have

$$\Lambda\left(t,\frac{1}{t}\right) \leqslant \Lambda\left(t^{\frac{\kappa}{n+\kappa}},\frac{1}{t}\right) \leqslant \lambda_4(\kappa) \leqslant \lambda_2$$

for every $t \in (0,1]$. Recalling that $u \in L^\infty_{\mathrm{loc}}(\Omega)$ by Theorem 3.7.2 and taking into account the last two displays, we are able to apply Theorem 3.2.2 in order to have (3.7.15) and (3.7.16). □

Remark 3.7.3. As a consequence of the last two theorems like we have that if $u \in W^{1,\Psi}(\Omega)$ is a local Q-minimizer of the functional $\mathcal{P}$ under the assumption (3.7.1), then $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and (1.0.14) is satisfied. Therefore, the results of Theorem 3.1.1, Theorem 3.1.2, Theorem 3.6.1 and Theorem 3.6.2 are still available under the assumption (3.7.1). Furthermore, the results of the present section can be considered under multi-phase settings, as we have pointed out in Remark 3.1.1.

Chapter 4

Calderón-Zygmund theory for Orlicz phase problems

4.1 Hypotheses and Main results

In this chapter we investigate the local Calderón-Zygmund type estimates for distributional solutions to the equation of the divergence form

$$\operatorname{div} A(x, Du) = \operatorname{div} B(x, F) \quad \text{in} \quad \Omega \tag{4.1.1}$$

for a bounded open subset $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$, where the vector field $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, differentiable with respect to the second variable $z \in \mathbb{R}^n \setminus \{0\}$, and satisfies the following structural conditions with fixed constants $0 < \nu \leqslant L < \infty$:

$$\begin{cases} |A(x,z)| + |D_z A(x,z)||z| \leqslant L \dfrac{\Psi(x,|z|)}{|z|}, \\ \nu \dfrac{\Psi(x,|z|)}{|z|^2} |\xi|^2 \leqslant \langle D_z A(x,z)\xi, \xi \rangle \,, \\ |A(x_1,z) - A(x_2,z)||z| \leqslant L \left| \Psi(x_1,|z|) - \Psi(x_2,|z|) \right| , \end{cases} \tag{4.1.2}$$

whenever $z \in \mathbb{R}^n \setminus \{0\}$, $\xi \in \mathbb{R}^n$, $x, x_1, x_2 \in \Omega$. On the right-hand side of the equation (4.1.1), we have that $B : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Caratheodory vector field satisfying

$$|z||B(x,z)| \leqslant L\Psi(x,|z|) \quad (x \in \Omega, z \in \mathbb{R}^n). \tag{4.1.3}$$

In the structure assumptions (4.1.2), (4.1.3) and the rest of the chapter we shall always use the notation Ψ is the same one as in (1.0.2) for $G, H_a, H_b \in \mathcal{N}$ in the sense of Definition 2.1.1 and the coefficient function initially $0 \leqslant a(\cdot), b(\cdot) \in L^\infty(\Omega)$. As a consequence of $(4.1.2)_2$, there exists a constant $c \equiv c(n, s(G), s(H_a), s(H_b), \nu, L)$ such that

$$|V_\Psi(x, z_1) - V_\Psi(x, z_2)|^2 \leqslant c \langle A(x, z_1) - A(x, z_2), z_1 - z_2 \rangle . \tag{4.1.4}$$

for all $x \in \Omega$ and $z_1, z_2 \in \mathbb{R}^n \setminus \{0\}$, where the vector field V_Ψ has been defined in (2.1.8).

A primary model in mind of the equation (4.1.1) is of the form

$$\operatorname{div}\left(\partial_t \Psi(x, |Du|) \frac{Du}{|Du|}\right) = \operatorname{div}\left(\partial_t \Psi(x, |F|) \frac{F}{|F|}\right) \quad \text{in} \quad \Omega, \tag{4.1.5}$$

which is the Euler-Lagrange equation of the following functional

$$W^{1,1}(\Omega) \ni v \mapsto \mathcal{P}(v, \Omega) - \int_\Omega \left\langle \partial_t \Psi(x, |F|) \frac{F}{|F|}, Dv \right\rangle dx, \tag{4.1.6}$$

where the Orlicz double phase functional $\mathcal{P}$ is given as in (1.0.1). The main purpose of the present chapter is to discover and develop optimal conditions on both nonlinearity $A(x, z)$ and the coefficient functions $a(\cdot)$ and $b(\cdot)$, not necessarily Hölder continuous, under which for any distributional solution $u \in W^{1,\Psi}(\Omega)$ to (1.0.29) the following local Calderón-Zygmund type implication

$$\Psi(x, |F|) \in L^\Upsilon_{\mathrm{loc}}(\Omega) \Longrightarrow \Psi(x, |Du|) \in L^\Upsilon_{\mathrm{loc}}(\Omega) \tag{4.1.7}$$

holds for every $\Upsilon \in \mathcal{N}$. Throughout the chapter, we shall always assume that

$$0 \leqslant a(\cdot) \in C^{\omega_a}(\Omega) \quad \text{and} \quad 0 \leqslant b(\cdot) \in C^{\omega_b}(\Omega) \tag{4.1.8}$$

for some continuous and concave functions $\omega_a, \omega_b : [0, \infty) \to [0, \infty)$ with $\omega_a(0) = \omega_b(0) = 0$. Then we shall consider a distributional solution $u \in W^{1,\Psi}(\Omega)$ to the equation (4.1.1) under one of the following main assumptions:

$$\begin{cases} u \in W^{1,\Psi}(\Omega), \\ \lambda_1 := \sup_{\rho > 0} \Lambda\left(\rho, G^{-1}\left(\rho^{-n}\right)\right) < \infty \end{cases} \tag{4.1.9}$$

and

$$\begin{cases} u \in W^{1,\Psi}(\Omega) \cap L^{\infty}(\Omega), \\ \lambda_2 := \sup_{\rho>0} \Lambda\left(\rho, \dfrac{1}{\rho}\right) < \infty, \end{cases} \tag{4.1.10}$$

where $\Lambda : (0,\infty) \times (0,\infty) \to (0,\infty)$ is the same map introduced in (1.0.12). For the sake of convenience, we use a set of parameters for a distributional solution $u \in W^{1,\Psi}(\Omega)$ to (4.1.1), which is "basic data of the problem" as follows:

$$\mathbf{data}_d \equiv \begin{cases} \Big\{n, \lambda_1, s(G), s(H_a), \nu, L, \|a\|_{C^{\omega_a}(\Omega)}, \omega_a(\cdot), \\ \|\Psi(x,|Du|)\|_{L^1(\Omega)}, \|u\|_{L^1(\Omega)}\Big\} \\ \text{if (4.1.9) is assumed and } b(\cdot) \equiv 0. & \text{(4.1.11a)} \\ \Big\{n, \lambda_2, s(G), s(H_a), \nu, L, \|a\|_{C^{\omega_a}(\Omega)}, \omega_a(\cdot), \|u\|_{L^{\infty}(\Omega)}\Big\} \\ \text{if (4.1.10) is assumed and } b(\cdot) \equiv 0. & \text{(4.1.11b)} \end{cases}$$

Here $s(G)$, $s(H_a)$ and $s(H_b)$ are indices of G,H_a and H_b, respectively, in the sense of Definition 2.1.1, respectively, while λ_1 and λ_2 are as in (4.1.9)-(4.1.10).

The first main results of this chapter is the local Calderón-Zygmund type implication (4.1.7) for Orlicz double phase problems.

Theorem 4.1.1 ([13]). *Suppose that Ψ is given as in (1.0.2) with $b(\cdot) \equiv 0$, $G, H_a \in \mathcal{N}$ in the sense of Definition 2.1.1 and $0 \leqslant a(\cdot) \in C^{\omega_a}(\Omega)$ for some continuous and concave function $\omega_a : [0,\infty) \to [0,\infty)$ with $\omega_a(0) = 0$. Let $u \in W^{1,\Psi}(\Omega)$ be a distributional solution to (4.1.1) with the assumptions (4.1.2) and (4.1.3). Suppose that any of the following assumptions is satisfied:*

$$\begin{cases} \text{(4.1.9)} \quad \textit{with} \quad \limsup_{\rho\to 0^+} \Lambda\left(\rho, G^{-1}\left(\rho^{-n}\right)\right) = 0, & \text{(4.1.12a)} \\ \text{(4.1.10)} \quad \textit{with} \quad \limsup_{\rho\to 0^+} \Lambda\left(\rho, \dfrac{1}{\rho}\right) = 0, & \text{(4.1.12b)} \\ \text{(4.1.9)} \quad \textit{with} \quad \omega_a(\rho) \equiv \rho^{\alpha} \textit{ for some } \alpha \in (0,1]. & \text{(4.1.12c)} \end{cases}$$

Then there holds that

$$\Psi(x,|F|) \in L^{\Upsilon}_{\mathrm{loc}}(\Omega) \Longrightarrow \Psi(x,|Du|) \in L^{\Upsilon}_{\mathrm{loc}}(\Omega) \quad \text{for every } \Upsilon \in \mathcal{N} \text{ with an index } s(\Upsilon) \geqslant 1. \tag{4.1.13}$$

Moreover, for every $\Upsilon \in \mathcal{N}$ with an index $s(\Upsilon) \geqslant 1$ and for every open subset $\Omega_0 \Subset \Omega$, there exist a radius $R_0 > 0$ and a constant $c > 0$ which depend on $\boldsymbol{data}_{db}(\Omega_0)$ and $s(\Upsilon)$ such that the following inequality

$$\fint_{B_{R/2}} \Upsilon\left[\Psi(x,|Du|)\right] dx \leqslant c\Upsilon\left[\fint_{B_R} \Psi(x,|Du|)\, dx\right] + c\fint_{B_R} \Upsilon\left[\Psi(x,|F|)\right] dx \tag{4.1.14}$$

holds for every ball $B_R \subset \Omega_0$ with $R \leqslant R_0$, where

$$\boldsymbol{data}_{db}(\Omega_0) \equiv \begin{cases} \boldsymbol{data}_d & \text{for } (4.1.12a) \\ \boldsymbol{data}_d & \text{for } (4.1.12b) \\ \boldsymbol{data}_d, \operatorname{dist}(\Omega_0,\partial\Omega), \|\Upsilon[\Psi(x,|F|)]\|_{L^1(\Omega_1)} & \text{for } (4.1.12c) \end{cases} \tag{4.1.15}$$

in which $\Omega_1 := \{x \in \Omega : \operatorname{dist}(x,\Omega_0) < 1/2 \operatorname{dist}(\Omega_0,\partial\Omega)\}$.

Now consider the case $b(\cdot) \not\equiv 0$ in (4.1.9). For the reason to apply Harmonic type approximation, we consider the map $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$A(x,z) := A_G(z) + a(x)A_{H_a}(z) + b(x)A_{H_b}(z), \tag{4.1.16}$$

where the continuous vector fields $A_G, A_{H_a}, A_{H_b} : \mathbb{R}^n \to \mathbb{R}^n$ are of a class $C^1(\mathbb{R}^n \setminus \{0\})$ and satisfy the following structure assumptions with fixed constants $0 < \nu \leqslant L$:

$$\begin{cases} |A_\Phi(z)| + |D_z A_\Phi(z)||z| \leqslant L\dfrac{\Phi(|z|)}{|z|}, \\ \nu\dfrac{\Phi(|z|)}{|z|^2}|\xi|^2 \leqslant \langle D_z A_\Phi(z)\xi,\xi\rangle \end{cases} \tag{4.1.17}$$

for every $\Phi \in \{G, H_a, H_b\}$, whenever $z \in \mathbb{R}^n \setminus \{0\}$, $\xi \in \mathbb{R}^n$. Clearly, the vector field given by (4.1.16) satisfies the structure assumptions (4.1.2) with constants ν, L in (4.1.17). Then we consider a distributional solution $u \in W^{1,\Psi}(\Omega)$ of the equation

$$\operatorname{div}\left(A_G(Du) + a(x)A_{H_a}(Du) + b(x)A_{H_b}(Du)\right) = \operatorname{div} B(x,F) \tag{4.1.18}$$

under any of assumptions (4.1.9) and (4.1.10), where $B : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Caratheodory vector field satisfying (4.1.3). For the simplicity of writing, we use a set of parameters for a distributional solution $u \in W^{1,\Psi}(\Omega)$ to (4.1.18), which is "basic data of the problem" in this chapter as follows:

$$\textbf{data} \equiv \begin{cases} \Big\{ n, \lambda_1, s(G), s(H_a), s(H_b), \nu, L, \|a\|_{C^{\omega_a}(\Omega)}, \|b\|_{C^{\omega_b}(\Omega)}, \\ \omega_a(\cdot), \omega_b(\cdot), \|\Psi(x,|Du|)\|_{L^1(\Omega)}, \|u\|_{L^1(\Omega)} \Big\} \\ \text{if (4.1.9) is assumed.} & \text{(4.1.19a)} \\ \Big\{ n, \lambda_2, s(G), s(H_a), s(H_b), \nu, L, \|a\|_{C^{\omega_a}(\Omega)}, \|b\|_{C^{\omega_b}(\Omega)}, \\ \omega_a(\cdot), \omega_b(\cdot), \|u\|_{L^\infty(\Omega)} \Big\} \\ \text{if (4.1.10) is assumed.} & \text{(4.1.19b)} \end{cases}$$

The second main result of the chapter reads as follows:

Theorem 4.1.2. *Suppose that Ψ is given as in* (1.0.2) *with $G, H_a, H_b \in \mathcal{N}$ in the sense of Definition 2.1.1, $0 \leqslant a(\cdot) \in C^{\omega_a}(\Omega)$ and $0 \leqslant b(\cdot) \in C^{\omega_b}(\Omega)$ for some continuous and concave function $\omega_a, \omega_b : [0,\infty) \to [0,\infty)$ with $\omega_a(0) = \omega_b(0) = 0$. Let $u \in W^{1,\Psi}(\Omega)$ be a distributional solution to* (4.1.18) *with the assumptions* (4.1.3) *and* (4.1.17)*. Suppose that any of the following assumptions is satisfied:*

$$\begin{cases} \text{(4.1.9)} \quad \textit{with} \quad \limsup\limits_{\rho \to 0^+} \Lambda\left(\rho, G^{-1}\left(\rho^{-n}\right)\right) = 0, & \text{(4.1.20a)} \\ \text{(4.1.10)} \quad \textit{with} \quad \limsup\limits_{\rho \to 0^+} \Lambda\left(\rho, \dfrac{1}{\rho}\right) = 0, & \text{(4.1.20b)} \\ \text{(4.1.9)} \quad \textit{with} \quad \omega_a(\rho) \equiv \rho^\alpha \ \textit{and}\ \omega_b(\rho) \equiv \rho^\beta \\ \qquad\qquad \textit{for some}\ \alpha, \beta \in (0,1]. & \text{(4.1.20c)} \end{cases}$$

Then there holds that

$$\Psi(x,|F|) \in L^{\Upsilon}_{\mathrm{loc}}(\Omega) \Longrightarrow \Psi(x,|Du|) \in L^{\Upsilon}_{\mathrm{loc}}(\Omega)$$

$$\textit{for every } \Upsilon \in \mathcal{N} \textit{ with an index } s(\Upsilon) \geqslant 1. \tag{4.1.21}$$

Moreover, for every $\Upsilon \in \mathcal{N}$ with an index $s(\Upsilon) \geqslant 1$ and for every open subset $\Omega_0 \Subset \Omega$, there exist a radius $R_0 > 0$ and a constant $c > 0$ which depend on $\boldsymbol{data}_b(\Omega_0)$ and $s(\Upsilon)$ such that the following inequality

$$\fint_{B_{R/2}} \Upsilon\left[\Psi(x, |Du|)\right] dx \leqslant c\Upsilon\left[\fint_{B_R} \Psi(x, |Du|)\, dx\right] + c\fint_{B_R} \Upsilon\left[\Psi(x, |F|)\right] dx \tag{4.1.22}$$

holds for every ball $B_R \subset \Omega_0$ with $R \leqslant R_0$, where

$$\boldsymbol{data}_b(\Omega_0) \equiv \begin{cases} \boldsymbol{data} & \textit{for } (4.1.20\text{a}) \\ \boldsymbol{data} & \textit{for } (4.1.20\text{b}) \\ \boldsymbol{data}, \operatorname{dist}(\Omega_0, \partial\Omega), \|\Upsilon[\Psi(x,|F|)]\|_{L^1(\Omega_1)} & \textit{for } (4.1.20\text{c}) \end{cases} \tag{4.1.23}$$

in which $\Omega_1 := \{x \in \Omega : \operatorname{dist}(x, \Omega_0) < 1/2 \operatorname{dist}(\Omega_0, \partial\Omega)\}$.

Remark 4.1.1. We remark that the results of Theorem 4.1.2 can be restated and proved for the equation exhibiting a finite number of phases with replacing the function in (1.0.2) by

$$\Psi_N(x,t) = G(t) + \sum_{k=1}^{N} a_k(x) H_k(t), \quad N \geqslant 1, \tag{4.1.24}$$

where $G, H_k \in \mathcal{N}$ in the sense of Definition 2.1.1 and $0 \leqslant a_k(\cdot) \in C^{\omega_k}(\Omega)$ with $\omega_k : [0,\infty) \to [0,\infty)$ being a continuous and concave function vanishing at the origin for every $k \in \{1, \ldots, N\}$. We also replace the function in (1.0.12) by

$$\Lambda_N(\rho, t) := \sum_{k=1}^{N} \frac{\omega_k(\rho)}{1+\omega_k(\rho)} \frac{H_k(t)}{G(t)} \text{ for every } \rho, t > 0. \tag{4.1.25}$$

Under this setting, with the same spirit as in the chapter, we are able to prove the results of Theorem 4.1.2 for a distributional solution $u \in W^{1,\Psi_N}(\Omega)$ to the equation of type

$$\operatorname{div}\left(A_G(Du)+\sum_{k=1}^{N} a_k(x)A_{H_k}(Du)\right)=\operatorname{div} B(x,F) \quad \text{in} \quad \Omega, \tag{4.1.26}$$

where the continuous vector fields $A_G, A_{H_k} : \mathbb{R}^n \to \mathbb{R}^n$ are $C^1(\mathbb{R}^n \setminus \{0\})$ and satisfy the structure assumptions with fixed constants $0 < \nu \leqslant L$:

$$\begin{cases} |A_\Phi(z)| + |D_z A_\Phi(z)||z| \leqslant L\dfrac{\Phi(|z|)}{|z|}, \\ \nu\dfrac{\Phi(|z|)}{|z|^2}|\xi|^2 \leqslant \langle D_z A_\Phi(z)\xi, \xi\rangle \end{cases} \tag{4.1.27}$$

for every $\Phi \in \{G, H_1, \dots, H_N\}$, whenever $z \in \mathbb{R}^n \setminus \{0\}$, $\xi \in \mathbb{R}^n$. Note that the coefficient functions in Theorem 4.1.2 along with (4.1.20c) are understood by letting $\omega_k(\rho) = \rho^{\alpha_k}$ with some $\alpha_k \in (0,1]$ for every $k \in \{1, \dots, N\}$.

4.2 Homogeneous equations

Proposition 4.2.1 (Existence of weak solutions). *Suppose that Ψ is given as in* (1.0.2) *with $G, H_a, H_b \in \mathcal{N}$ in the sense of Definition 2.1.1, $0 \leqslant a(\cdot) \in C^{\omega_a}(\Omega)$ and $0 \leqslant b(\cdot) \in C^{\omega_b}(\Omega)$ for some continuous and concave function $\omega_a, \omega_b : [0,\infty) \to [0,\infty)$ with $\omega_a(0) = \omega_b(0) = 0$. Suppose that either* $(4.1.9)_2$ *or* $(4.1.10)_2$ *is satisfied. Let*

$$w_0 \in \begin{cases} W^{1,\Psi}(B_R) & \text{if} \quad (4.1.9)_2 \text{ is assumed,} \\ W^{1,\Psi}(B_R) \cap L^\infty(B_R) & \text{if} \quad (4.1.10)_2 \text{ is assumed} \end{cases} \tag{4.2.1}$$

a given ball $B_R \subset \Omega$. Then there exists a unique weak solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div} A(x, Dw) = 0 \text{ in } B_R \\ w \in w_0 + W_0^{1,\Psi}(B_R), \end{cases} \tag{4.2.2}$$

where the vector field $A : \Omega \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ *is same one satisfying* $(4.1.2)_{1,2}$, *with energy estimates*

$$\fint_{B_R} \Psi(x, |Dw|)\, dx \leqslant c \fint_{B_R} \Psi(x, |Dw_0|)\, dx \tag{4.2.3}$$

and

$$\|w\|_{L^\infty(B_R)} \leqslant \|w_0\|_{L^\infty(B_R)} \tag{4.2.4}$$

for some constant $c \equiv c(n, s(\Psi), \nu, L)$.

Proof. First let us consider the case of the condition $(4.1.9)_2$ is in force. Letting $v := w - w_0$, we rewrite (4.2.2) as

$$\begin{cases} -\operatorname{div} A(x, Dv + Dw_0) = 0 \text{ in } B_R \\ v \in W_0^{1,\Psi}(B_R). \end{cases} \tag{4.2.5}$$

By the structure assumptions $(4.1.2)_{1,2}$, we observe that the operator $T : W_0^{1,\Psi}(B_R) \to \left(W_0^{1,\Psi}(B_R)\right)^*$ given by

$$(T(v))(\varphi) = \int_{B_R} \langle A(x, Dv + Dw_0), D\varphi \rangle \, dx$$

is continuous monotone operator. Since $W_0^{1,\Psi}(B_R)$ is a separable reflexive Banach space with endowed norm $\|D\varphi\|_{L^\Psi(B_R)}$, where $\varphi \in W_0^{1,\Psi}(B_R)$ is any, via Poincaré type inequality of Theorem 2.4.1, we are able to apply classical monotonicity method in order to find $v \in W_0^{1,\Psi}(B_R)$ such that $T(v)(\varphi) = 0$ holds true for every $\varphi \in W_0^{1,\Psi}(B_R)$. As a consequence, $w = v + w_0$ is a weak solution of (4.2.2). If there are weak solutions $w_1, w_2 \in w_0 + W_0^{1,\Psi}(B_R)$ of (4.2.2), then via (4.1.4), we have

$$\begin{aligned} 0 &= \fint_{B_R} \langle A(x, Dw_1) - A(x, Dw_2), Dw_1 - Dw_2 \rangle \, dx \\ &\geqslant c|V_\Psi(x, z_1) - V_\Psi(x, z_2)|^2 \end{aligned} \tag{4.2.6}$$

for some constant $c \equiv c(n, s(\Psi), \nu, L)$. Thus, $w_1 \equiv w_2$.

To see (4.2.3), we take $\varphi := w - w_0$ as a test function to the equation (4.2.2) together with using the structure assumptions (4.1.17) and applying Young's inequality of Lemma 2.1.4. In turn, we have

$$\begin{aligned}\nu \fint_{B_R} \Psi(x,|Dw|)\,dx &\leqslant \fint_{B_R} \langle A(x,Dw), Dw\rangle\,dx = \fint_{B_R} \langle A(x,Dw), Dw_0\rangle\,dx \\ &\leqslant L \fint_{B_R} \frac{\Psi(x,|Dw|)}{|Dw|}|Dw_0|\,dx \\ &\leqslant \varepsilon \fint_{B_R} \Psi(x,|Dw|)\,dx + \frac{c}{\varepsilon^{s(\Psi)}} \fint_{B_R} \Psi(x,|Dw_0|)\,dx \qquad (4.2.7)\end{aligned}$$

for some constant $c \equiv c(s(\Psi), L)$ and every $\varepsilon \in (0,1)$. Then choosing ε small enough, we see (4.2.3). If $w_0 \notin L^\infty(B_R)$, then (4.2.4) is valid trivially. Suppose $w_0 \in L^\infty(B_R)$. Taking $\varphi := (w - \sup_{B_R} w_0)_+$ and $\varphi := (w - \inf_{B_R} w_0)_-$ as a test function in (4.2.2) and following the arguments in (4.2.16) below, we find (4.2.4).

Now we consider the case of the condition (4.1.10) is in force. In fact, we are not allowed to employ the monotonicity arguments as above since a constant appearing in Sobolev-Poincaré type inequality of Theorem 2.4.1 for a function $\varphi \in W_0^{1,\Psi}(B_R) \cap L^\infty(B_R)$ depends on $\|\varphi\|_{L^\infty(B_R)}$. The absence of Lavrentiev phenomenon discussed in Theorem 2.3.1 allows us to find a sequence of functions $\{w_m\}_{m=1}^\infty \in W^{1,\infty}(B_R)$ such that

$$\begin{aligned}&w_k \to w_0 \quad \text{in} \quad W^{1,G}(B_R), \qquad \int_{B_R} \Psi(x,|Dw_k|)\,dx \to \int_{B_R} \Psi(x,|Dw_0|)\,dx \\ &\text{and} \quad \limsup_{k\to\infty} \|w_k\|_{L^\infty(B_R)} \leqslant \|w_0\|_{L^\infty(B_R)}. \qquad (4.2.8)\end{aligned}$$

Then we define the new vector fields

$$A_m(x,z) := A(x,z) + \varepsilon_m \partial_t \Psi^+_{B_R}(|z|)\frac{z}{|z|} \quad (x \in B_R,\ z \in \mathbb{R}^n \setminus \{0\}), \quad (4.2.9)$$

where the function $\Psi^+_{B_R}(\cdot)$ has been defined in (2.1.3) and $\{\varepsilon_m\}_{m=1}^\infty$ is the sequence of real numbers defined as

$$\varepsilon_m := \left(m + \left[\int_{B_R} \Psi^+_{B_R}(|Dw_m|)\, dx \right]^2 \right)^{-1}, \tag{4.2.10}$$

the functions $v_m \in w_m \in W_0^{1,\Psi^+_{B_R}}(B_R)$ as the unique solutions of the Dirichlet problem

$$\begin{cases} -\operatorname{div} A_m(x, Dv_m) = 0 \text{ in } B_R \\ v_m \in w_m + W_0^{1,\Psi^+_{B_R}}(B_R). \end{cases} \tag{4.2.11}$$

The existence of such sequence of functions $\{v_m\}_{m=1}^{\infty}$ follows by standard monotonicity methods as we have discussed above since the newly defined vector fields $A_m(\cdot)$ in (4.2.9) are coercive and monotone in $W^{1,\Psi^+_{B_R}}$ by $\varepsilon_m > 0$. The weak form of $(4.2.11)_1$ is

$$\int_{B_R} \langle A_m(x, Dv_m), D\varphi \rangle \; dx = 0 \quad \text{for all} \quad \varphi \in W_0^{1,\Psi^+_{B_R}}(B_R) \tag{4.2.12}$$

By taking $\varphi := v_m - w_m$ as a test function in (4.2.12) and arguing similarly as in (4.2.7) we see

$$\int_{B_R} [\Psi(x, |Dv_m|) + \varepsilon_m \Psi^+_{B_R}(|Dv_m|)]\, dx \leqslant c \int_{B_R} [\Psi(x, |Dw_m|) + \varepsilon_m \Psi^+_{B_R}(|Dw_m|)]\, dx \tag{4.2.13}$$

for a constant $c \equiv c(n, s(\Psi), \nu, L)$. For m large enough, (4.2.8) implies

$$\int_{B_R} \Psi(x, |Dw_m|)\, dx \leqslant 2 \int_{B_R} \Psi(x, |Dw_0|)\, dx =: L_0 \tag{4.2.14}$$

and recalling also (4.2.10) we have

$$\int_{B_R} \Psi(x, |Dv_m|)\, dx \leqslant c \int_{B_R} \Psi(x, |Dw_0|)\, dx + c \leqslant c(L_0 + 1) \tag{4.2.15}$$

again for $c \equiv c(n, s(\Psi), \nu, L)$. Therefore, we can conclude that up to passing to not relabelled subsequences, $v_m \rightharpoonup w$ in $W^{1,\Psi}(B_R)$ for some $w \in w_0 + W_0^{1,\Psi}(B_R)$. By lower semi-continuity in (4.2.13) and (4.2.15), and again recalling (4.2.10), we find (4.2.3). On the other hand, testing the equation (4.2.12) against $\varphi := (v_m - \sup_{B_R} w_m)_+$, we have

$$
\begin{aligned}
&\nu \fint_{B_R} \Psi(x, |D(v_m - \sup_{B_R} w_m)_+|)\,dx \\
&\leqslant \fint_{B_R} \left\langle A_m(x, D(v_m - \sup_{B_R} w_m)_+), D(v_m - \sup_{B_R} w_m)_+ \right\rangle dx \\
&= \fint_{B_R} \left\langle A_m(x, Dv_m), D(v_m - \sup_{B_R} w_m)_+ \right\rangle dx = 0. \qquad (4.2.16)
\end{aligned}
$$

Using the co-area formula, we see that $v_m \leqslant \sup_{B_R} w_m$ in B_R. Similarly, by taking a test function $\varphi := (v_m - \inf_{B_R} w_m)_-$ to the equation (4.2.11), we see that $v_m \geqslant \inf_{B_R} w_m$ in B_R. Combining those estimates and recalling (4.2.8), we find (4.2.4). Finally, the uniqueness of weak solutions to (4.2.2) can shown similarly as in (4.2.6). □

Lemma 4.2.1 (Density lemma)**.** *Suppose that Ψ is given as in* (1.0.2) *with $G, H_a, H_b \in \mathcal{N}$ in the sense of Definition 2.1.1, $0 \leqslant a(\cdot) \in C^{\omega_a}(\Omega)$ and $0 \leqslant b(\cdot) \in C^{\omega_b}(\Omega)$ for some continuous and concave function $\omega_a, \omega_b : [0,\infty) \to [0,\infty)$ with $\omega_a(0) = \omega_b = 0$. Let a measurable vector field $S : B \to \mathbb{R}^n$ for some ball $B \equiv B_r \Subset \Omega$ be a distributional solution to the equation*

$$
-\operatorname{div} T(x, S) = 0 \text{ in } B \qquad (4.2.17)
$$

with $\Psi(x, |S|) \in L^1(B)$, where the vector field $T : B \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the growth condition

$$
|T(x,z)| \leqslant L \frac{\Psi(x, |z|)}{|z|} \qquad (4.2.18)
$$

for every $x \in B$ and $z \in \mathbb{R}^n$. Then if the condition $(4.1.9)_2$ *is satisfied, then* *every* $\varphi \in W_0^{1,1}(B)$ *with* $\Psi(x,|D\varphi|) \in L^1(B)$ *satisfies*

$$\int_B \langle T(x,S), D\varphi\rangle \, dx = 0. \tag{4.2.19}$$

Also, if the condition $(4.1.10)_2$ *is satisfied, then* (4.2.19) *holds for every* $\varphi \in W_0^{1,1}(B) \cap L^\infty(B)$ *with* $\Psi(x,|D\varphi|) \in L^1(B)$ *.*

Proof. An idea of the proof is similar to the proof of [14, Lemma 5.2]. Clearly the proof can be reduced to the case $B_r \equiv B_1(0)$ by dilation and translation, and we can assume $\varphi \in W_0^{1,\Psi}(\mathbb{R}^n)$ by zero extention outside of B. There exists $\varepsilon_0 > 0$ such that $B_{1+\varepsilon_0}(0) \Subset \Omega$. Let $\rho \in C_0^\infty(B_1(0))$ be a non-negative standard mollifier with $\int_{\mathbb{R}^n} \rho \, dx = 1$. Then we set $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$ for every $x \in B_\varepsilon(0)$. Directly, we observe that $\rho_\varepsilon \in C_0^\infty(B_\varepsilon(0))$, $\int_{\mathbb{R}^n} \rho_\varepsilon \, dx = 1$, $0 \leqslant \rho_\varepsilon \leqslant c(n)\varepsilon^{-n}$ and $|D\rho_\varepsilon| \leqslant c(n)\varepsilon^{-(n+1)}$. For every $0 < \varepsilon < \frac{\varepsilon_0}{2(1+\varepsilon_0)}$, we define

$$\tilde{\varphi}_\varepsilon(x) := \varphi\left(\frac{x}{1-2\varepsilon}\right), \quad \tilde{a}_\varepsilon(x) := a\left(\frac{x}{1-2\varepsilon}\right), \quad \tilde{b}_\varepsilon(x) := b\left(\frac{x}{1-2\varepsilon}\right),$$
$$\varphi_\varepsilon(x) := (\tilde{\varphi} * \rho_\varepsilon)(x), \quad a_\varepsilon(x) := \inf_{y \in B_\varepsilon(x)} \tilde{a}_\varepsilon(y), \quad b_\varepsilon(x) := \inf_{y \in B_\varepsilon(x)} \tilde{b}_\varepsilon(y) \quad \text{and}$$
$$\Psi_\varepsilon(x,t) := G(t) + a_\varepsilon(x)H_a(t) + b_\varepsilon(x)H_b(t) \tag{4.2.20}$$

for every $x \in B_1$ and $t \geqslant 0$. It follows from the Jensen's inequality and properties of the convolution that

$$G(|D\varphi_\varepsilon(x)|) \leqslant G\left(|(D\tilde{\varphi}_\varepsilon * \rho_\varepsilon)(x)|\right) \leqslant \int_{\mathbb{R}^n} G(|D\tilde{\varphi}_\varepsilon(x-y)|)\rho_\varepsilon(y)\, dy \leqslant c\varepsilon^{-n} \tag{4.2.21}$$

and

$$|D\varphi_\varepsilon(x)| = |(\tilde{\varphi}_\varepsilon * D\rho_\varepsilon)(x)| \leqslant \int_{\mathbb{R}^n} |\tilde{\varphi}_\varepsilon(x-y)||D\rho_\varepsilon(y)|\, dy \leqslant c(n)\, \|\varphi\|_{L^\infty(B_1)}\, \varepsilon^{-1}. \tag{4.2.22}$$

Arguing similarly as in [14, (5.4)], we have

$$\Psi_\varepsilon\left(x, |D\varphi_\varepsilon(x)|\right) \leqslant c\left[\Psi\left(\frac{\cdot}{1-2\varepsilon}, \left|D\varphi\left(\frac{\cdot}{1-2\varepsilon}\right)\right|\right) * \rho_\varepsilon\right](x). \tag{4.2.23}$$

for some constant $c \equiv c(s(\Psi))$ and for every $x \in B_1$. Suppose now that the condition $(4.1.9)_2$ is satisfied. Then using $(4.1.9)_2$ and (4.2.21),we see

$$\begin{aligned}
\Psi(x, |D\varphi_\varepsilon(x)|) &\leqslant |a(x) - a_\varepsilon(x)| H_a(|D\varphi_\varepsilon(x)|) + |b(x) - b_\varepsilon(x)| H_b(|D\varphi_\varepsilon(x)|) \\
&\quad + \Psi_\varepsilon(x, |D\varphi_\varepsilon(x)|) \\
&\leqslant 2\lambda_1 [a]_{\omega_a} \omega_a(\varepsilon) \left(1 + \frac{1}{\omega_a\left([G(|D\varphi_\varepsilon(x)|)]^{-\frac{1}{n}}\right)}\right) G(|D\varphi_\varepsilon(x)|) \\
&\quad + 2\lambda_1 [b]_{\omega_b} \omega_b(\varepsilon) \left(1 + \frac{1}{\omega_b\left([G(|D\varphi_\varepsilon(x)|)]^{-\frac{1}{n}}\right)}\right) G(|D\varphi_\varepsilon(x)|) \\
&\quad + \Psi_\varepsilon(x, |D\varphi_\varepsilon(x)|) \\
&\leqslant c\Psi_\varepsilon(x, |D\varphi_\varepsilon(x)|)
\end{aligned} \tag{4.2.24}$$

for every $x \in B_1$ with some constant c independent of ε. If the condition $(4.1.10)_2$ is satisfied, then using this one and (4.2.22) we have

$$\begin{aligned}
\Psi(x, |D\varphi_\varepsilon(x)|) &\leqslant |a(x) - a_\varepsilon(x)| H_a(|D\varphi_\varepsilon(x)|) + |b(x) - b_\varepsilon(x)| H_b(|D\varphi_\varepsilon(x)|) \\
&\quad + \Psi_\varepsilon(x, |D\varphi_\varepsilon(x)|) \\
&\leqslant 2\lambda_2 [a]_{\omega_a} \omega_a(\varepsilon) \left(1 + \frac{1}{\omega_a\left(|D\varphi_\varepsilon(x)|^{-1}\right)}\right) G(|D\varphi_\varepsilon(x)|) \\
&\quad 2\lambda_2 [b]_{\omega_b} \omega_b(\varepsilon) \left(1 + \frac{1}{\omega_b\left(|D\varphi_\varepsilon(x)|^{-1}\right)}\right) G(|D\varphi_\varepsilon(x)|) \\
&\quad + \Psi_\varepsilon(x, |D\varphi_\varepsilon(x)|) \\
&\leqslant c\Psi_\varepsilon(x, |D\varphi_\varepsilon(x)|)
\end{aligned} \tag{4.2.25}$$

for every $x \in B_1$ with some constant c independent of ε but depending on $\|\varphi\|_{L^\infty(B_1)}$. Once we arrive at this stage, the rest of the proof can be argued in the same way as in the proof of [14, Lemma 5.2]. □

Theorem 4.2.1 (Higher integrability)**.** *Let $u \in W^{1,\Psi}(\Omega)$ be a distributional solution to* (4.1.18) *under the assumptions* (4.1.3) *and* (4.1.17)*. Suppose ei-*

ther (4.1.9) *or* (4.1.10) *is satisfied. Suppose also that* $\Psi(x,|F|) \in L^{\Upsilon}_{\mathrm{loc}}(\Omega)$ *for some* $\Upsilon \in \mathcal{N}$ *with an index* $s(\Upsilon) \geqslant 1$. *Then there exists a positive higher integrability exponent* $\delta \equiv \delta(\boldsymbol{data}, s(\Upsilon))$ *such that* $\Psi(x,|Du|) \in L^{1+\delta}_{\mathrm{loc}}(\Omega)$. *Moreover, the following inequality*

$$\left(\fint_{B_r} [\Psi(x,|Du|)]^{1+\delta}\,dx\right)^{\frac{1}{1+\delta}} \leqslant c\fint_{B_{2r}} \Psi(x,|Du|)\,dx + c\left(\fint_{B_{2r}} [\Psi(x,|F|)]^{1+\delta}\,dx\right)^{\frac{1}{1+\delta}} \tag{4.2.26}$$

holds for a constant $c \equiv c(\boldsymbol{data})$, *whenever* $B_{2r} \subset \Omega$ *is a ball with* $2r \leqslant 1$. *In particular, for every open subset* $\Omega_0 \Subset \Omega_1 \Subset \Omega$ *with* $\operatorname{dist}(\Omega_0,\partial\Omega) \approx \operatorname{dist}(\Omega_1,\partial\Omega) \approx \operatorname{dist}(\Omega_0,\partial\Omega_1)$, *we have*

$$\|\Psi(x,|Du|)\|_{L^{1+\delta}(\Omega_0)} \leqslant c\left(\boldsymbol{data}, \operatorname{dist}(\Omega_0,\partial\Omega), s(\Upsilon), \|\Upsilon[\Psi(x,|F|)]\|_{L^1(\Omega_1)}\right). \tag{4.2.27}$$

Proof. Let $B_{2\rho} \subset \Omega$ be a fixed ball with $2\rho \leqslant 1$ and $\eta \in C_0^{\infty}(B_{2\rho})$ be a cut-off function such that $\chi_{B_\rho} \leqslant \eta \leqslant \chi_{B_{2\rho}}$ with $|D\eta| \leqslant \dfrac{2}{\rho}$. Applying the methods employed in [14, Theorem 6.1] or [15, Theorem 5.1], we have

$$\int_{B_{2\rho}} \Psi(x,|Du|)\eta^{s(\Psi)+1}\,dx \leqslant c\int_{B_{2\rho}} \Psi\left(x, \left|\frac{u-(u)_{B_{2\rho}}}{\rho}\right|\right)\,dx + c\int_{B_{2\rho}} \Psi(x,|F|)\,dx \tag{4.2.28}$$

for some constant $c \equiv c(n, s(\Psi), \nu, L)$. Now we apply Theorem 2.4.1 depending on which assumption of (4.1.9) and (4.1.10) comes into play. In turn, we have

$$\fint_{B_\rho} \Psi(x,|Du|)\,dx \leqslant c\left(\fint_{B_{2\rho}} [\Psi(x,|Du|)]^{\theta}\,dx\right)^{\frac{1}{\theta}} + c\fint_{B_{2\rho}} \Psi(x,|F|)\,dx \tag{4.2.29}$$

with some $c \equiv c(\mathbf{data})$ and $\theta \equiv \theta(n, s(\Psi)) \in (0,1)$. Applying Lemma 2.1.1 for Υ, for every open subset $\Omega_0 \Subset \Omega$, one can show that

$$\int_{\Omega_0} [\Psi(x,|F|)]^{1+\frac{1}{s(\Upsilon)}}\,dx \leqslant |\Omega_0| + c\int_{\Omega_0} \Upsilon[\Psi(x,|F|)]\,dx < +\infty. \tag{4.2.30}$$

Therefore, there exists a higher integrability exponent $\delta \equiv \delta(\mathbf{data}, s(\Upsilon))$ fulfilling the inequality (4.2.26) by a variant of Gehring's lemma. Finally, the estimate (4.2.27) implies from (4.2.26) together with a standard covering argument. □

In the rest of the section, we shall always suppose that Ψ is given as in (1.0.2) with $G, H_a, H_b \in \mathcal{N}$ in the sense of Definition 2.1.1, $0 \leqslant a(\cdot) \in C^{\omega_a}(\Omega)$ and $0 \leqslant b(\cdot) \in C^{\omega_b}(\Omega)$ for some continuous and concave functions $\omega_a, \omega_b : [0,\infty) \to [0,\infty)$ with $\omega_a(0) = \omega_b(0) = 0$, unless we specify. We also consider the following Dirichlet boundary value problem:

$$\begin{cases} -\operatorname{div} A(x,Dw) = 0 \quad \text{in} \quad B_R \equiv B_R(x_0), \\ w \in u + W_0^{1,\Psi}(B_R) \end{cases} \tag{4.2.31}$$

for some fixed ball $B_R \subset \Omega_0 \Subset \Omega$ with $R \leqslant 1$, where

$$A(x,z) = A_G(z) + a(x)A_{H_a}(z) + b(x)A_{H_b}(z) \quad (x \in \Omega,\ z \in \mathbb{R}^n) \tag{4.2.32}$$

as we introduced in (4.1.16) and $u \in W^{1,\Psi}(B_R)$ is a distributional solution to (4.1.18). Furthermore, we shall always assume that $\Psi(x,|F|) \in L^{\Upsilon}_{\mathrm{loc}}(\Omega)$ for some $\Upsilon \in \mathcal{N}$ with an index $s(\Upsilon)$.

4.2.1 Local boundedness estimates

Next we start with the following direct outcome of the equation (4.2.31).

Proposition 4.2.2. *Let $w \in W^{1,\Psi}(B_R)$ be the weak solution to* (4.2.31) *under the assumptions* (4.1.17). *Suppose either* (4.1.9) *or* (4.1.10) *is satisfied. There exists a constant $Q \equiv Q(s(\Psi), \nu, L) \geqslant 1$ such that w is a Q-minimizer of the functional*

$$W^{1,1}(B_R) \ni v \mapsto \mathcal{P}(v, B_R) = \int_{B_R} \Psi(x,|Dv|)\,dx. \tag{4.2.33}$$

In particular, there exists a constant $c \equiv c(s(\Psi), \nu, L)$ such that

$$\fint_{B_R} \Psi(x, |Dw|)\, dx \leqslant c \fint_{B_R} \Psi(x, |Du|)\, dx. \tag{4.2.34}$$

Moreover, if $u \in L^\infty(B_R)$, then it holds that

$$\operatorname*{osc}_{B_R} w \leqslant \operatorname*{osc}_{B_R} u \quad \textit{and} \quad \|w\|_{L^\infty(B_R)} \leqslant \|u\|_{L^\infty(B_R)}. \tag{4.2.35}$$

Proof. Let $\varphi \in W_0^{1,1}(B_R)$ with $\mathcal{P}(\varphi, B_R) < \infty$ if (4.1.9) is assumed or $\varphi \in W_0^{1,1}(B_R) \cap L^\infty(B_R)$ with $\mathcal{P}(\varphi, B_R) < \infty$ if (4.1.10) is assumed , which can be a test function in (4.2.31) by Lemma 4.2.1 below. Then by testing the equation (4.2.31) by φ and using the structure assumption (4.1.17) together with Young's type inequality of Lemma 2.1.4, we have

$$\begin{aligned}
\nu \int_{B_R} \Psi(x, |Dw|)\, dx &\leqslant \int_{B_R} \langle A(x, Dw), Dw \rangle\, dx = \int_{B_R} \langle A(x, Dw), Dw + D\varphi \rangle\, dx \\
&\leqslant L \int_{B_R} \frac{\Psi(x, |Dw|)}{|Dw|} |Dw + D\varphi|\, dx \\
&\leqslant \frac{\nu}{2} \int_{B_R} \Psi(x, |Dw|)\, dx + c \int_{B_R} \Psi(x, |Dw + D\varphi|)\, dx
\end{aligned} \tag{4.2.36}$$

for some constant $c \equiv c(s(\Psi), L, \nu)$. For showing (4.2.35), if $u \notin L^\infty(B_R)$, the estimates in (4.2.35) are trivial. Suppose $u \in L^\infty(B_R)$. Then we take a test function $\varphi := (w - \sup_{B_R} u)_+$ which is admissible by $w \in u + W_0^{1,\Psi}(B_R)$ and $u \in L^\infty(B_R)$ via Lemma 4.2.1. Then using (4.1.17), we have

$$\begin{aligned}
&\nu \fint_{B_R} \Psi(x, |D(w - \sup_{B_R} u)_+|)\, dx \\
&\leqslant \fint_{B_R} \left\langle A(x, D(w - \sup_{B_R} u)_+), D(w - \sup_{B_R} u)_+ \right\rangle dx
\end{aligned}$$

$$= \fint_{B_R} \left\langle A(x,Dw), D(w-\sup_{B_R} u)_+ \right\rangle dx = 0.$$

Using the co-area formula, we see that $w \leqslant \sup_{B_R} u$ in B_R. Similarly, by taking a test function $\varphi := (w - \inf_{B_R} u)_-$ to the equation (4.2.31), we see that $w \geqslant \inf_{B_R} u$ in B_R. Combining those estimates, we find (4.2.35). □

Proposition 4.2.3. *Let $w \in W^{1,\Psi}(B_R)$ be the weak solution to* (4.2.31) *under the assumption* (4.1.17). *Suppose that either* (4.1.9) *or* (4.1.10) *is satisfied. Then there exists a higher integrability exponent $\delta_0 \equiv \delta_0(\boldsymbol{data}, s(\Upsilon)) \leqslant \delta$ such that*

$$\fint_{B_R} [\Psi(x,|Dw|)]^{1+\delta_0}\, dx \leqslant c \fint_{B_R} [\Psi(x,|Du|)]^{1+\delta_0}\, dx \tag{4.2.37}$$

for some constant $c \equiv c(\mathbf{data})$, where δ is the higher integrability exponent determined by Theorem 4.2.1.

Proof. Since we have already obtained a Sobolev-Poincaré type inequality of Theorem 2.4.1 under either (4.1.9) or (4.1.10), we follow the arguments employed in the proof of [14, Lemma 5.3 and Lemma 5.4]. □

Since $w \in W^{1,\Psi}(B_R)$ is a Q-minimizer of the functional in (4.2.33), we are able to derive a Caccioppoli inequality for w, see Lemma 3.2.1.

Proposition 4.2.4. *Let $w \in W^{1,\Psi}(B_R)$ be the weak solution to* (4.2.31) *under the assumption* (4.1.17). *Suppose either* (4.1.9) *or* (4.1.10) *is satisfied. Then there exists a constant $c \equiv c(s(\Psi),\nu,L) \geqslant 1$ such that the following Caccioppoli inequality*

$$\int_{B_\rho} \Psi(x,|D(w-k)_\pm|)\, dx \leqslant c \int_{B_r} \Psi\left(x, \frac{(w-k)_\pm}{r-\rho}\right) dx, \tag{4.2.38}$$

holds, whenever $B_\rho \equiv B_\rho(y) \Subset B_r(y) \equiv B_r \subset B_R$ are balls and $k \in \mathbb{R}$.

Theorem 4.2.2. *Let $w \in W^{1,\Psi}(B_R)$ be the weak solution to* (4.2.31) *under the assumption* (4.1.17). *Suppose that either* (4.1.9) *or* (4.1.10) *is satisfied.*

Then there exists a constant $c \equiv c(\boldsymbol{data})$ such that

$$\left\| \Psi^{-}_{B_r}\left(\frac{(w-(w)_{B_r})_{\pm}}{r}\right)\right\|_{L^{\infty}(B_{r/2})} \leqslant c \fint_{B_r} \Psi\left(x, \frac{(w-(w)_{B_r})_{\pm}}{r}\right) dx \quad (4.2.39)$$

and

$$\Psi^{-}_{B_r}\left(\left|\frac{w(x_1)-w(x_2)}{r}\right|\right) \leqslant c \fint_{B_r} \Psi\left(x, \left|\frac{w-(w)_{B_r}}{r}\right|\right) dx \quad \text{for a.e} \quad x_1, x_2 \in B_{r/2}, \quad (4.2.40)$$

whenever $B_r \equiv B_r(y) \subset B_R$ is a ball.

Proof. We omit the proof since it is similar to the proof of Theorem 3.2.1 by using the estimates (4.2.34) and (4.2.35) of Proposition 4.2.2. □

Let us also restate the results of Lemma 3.2.2 and Lemma 3.2.3 for w.

Lemma 4.2.2. *Let $w \in W^{1,\Psi}(B_R)$ be the weak solution to* (4.2.31) *under assumptions* (4.1.17)*. Suppose that either* (4.1.9) *or* (4.1.10) *is satisfied. Let $B_{2r} \equiv B_{2r}(y) \subset B_R$ be any fixed ball. Then there exists a constant $c \equiv c(\boldsymbol{data})$ such that*

$$\begin{aligned}\int_{B_{r_1}} \Psi^{-}_{B_r}(|D(w-k)_{\pm}|)\, dx &\leqslant \int_{B_{r_1}} \Psi(x, |D(w-k)_{\pm}|)\, dx \\ &\leqslant c\left(\frac{r}{r_2-r_1}\right)^{s(\Psi)+1} \int_{B_{r_2}} \Psi^{-}_{B_r}\left(\frac{(w-k)_{\pm}}{r}\right) dx,\end{aligned} \quad (4.2.41)$$

whenever $B_{r_1} \Subset B_{r_2} \subset B_r$ are concentric balls and $k \in \mathbb{R}$.

Lemma 4.2.3. *Let $w \in W^{1,\Psi}(B_R)$ be the weak solution to* (4.2.31) *under the assumption* (4.1.17)*. Suppose either the assumption* (4.1.9) *or* (4.1.10) *is satisfied. Let $B_{2r} \equiv B_{2r}(y) \subset B_R$ be some fixed ball. Then there exists a constant $c \equiv c(\boldsymbol{data})$ such that*

$$\int_{B_{r_1}} \Psi^{-}_{B_r}\left(|D(w-k)_{\pm}|\right) dx \leqslant \int_{B_{r_1}} \Psi\left(x, |D(w-k)_{\pm}|\right) dx$$

$$\leqslant c \left(\frac{r}{r_2 - r_1} \right)^{s(\Psi)+1} \int_{B_{r_2}} \Phi \left(\frac{(w-k)_\pm}{r} \right) dx, \tag{4.2.42}$$

whenever $B_{r_1} \Subset B_{r_2} \subset B_r(y)$ *are concentric balls and* $k \in \mathbb{R}$*, where*

$$\Phi(t) = \begin{cases} G(t) & \\ \text{if (3.2.27a) is satisfied in } B_r, & (4.2.43a) \\ G(t) + a^-(B_r) H_a(t) & \\ \text{if (3.2.27b) is satisfied in } B_r, & (4.2.43b) \\ G(t) + b^-(B_r) H_b(t) & \\ \text{if (3.2.27c) is satisfied in } B_r, & (4.2.43c) \\ \Psi^-_{B_r}(t) & \\ \text{if (3.2.27d) is satisfied in } B_r, & (4.2.43d) \end{cases}$$

for every $t \geqslant 0$*.*

4.2.2 Decay estimates

We continue to consider the function $w \in W^{1,\Psi}(B_R)$ defined in (4.2.31) for the fixed ball $B_R \equiv B_R(x_0) \subset \Omega_0 \Subset \Omega$ with $R \leqslant 1$. Throughout the present subsection let us consider the excess functional given by

$$E(w, B_r) := \left(\Psi^-_{B_{2r}}\right)^{-1} \left(\fint_{B_r} \Psi^-_{B_{2r}} \left(\left| \frac{w - (w)_{B_r}}{r} \right| \right) dx \right) \tag{4.2.44}$$

for any ball $B_{2r} \equiv B_{2r}(y) \subset B_R$. Using the convexity of $\Psi^-_{B_{2r}}$ together with Lemma 2.1.1, one can see that

$$E(w, B_r) \leqslant c \left(\Psi^-_{B_{2r}}\right)^{-1} \left(\fint_{B_r} \Psi^-_{B_{2r}} \left(\left| \frac{w - w_0}{r} \right| \right) dx \right) \tag{4.2.45}$$

for some constant $c \equiv c(s(\Psi))$ and for every $w_0 \in \mathbb{R}$.

Lemma 4.2.4. *Let $w \in W^{1,\Psi}(B_R)$ be the weak solution to* (4.2.31) *under the assumption* (4.1.17). *Let $B_r \equiv B_r(y) \subset B_R$ be any fixed ball. If one of the assumptions* (4.1.20a)-(4.1.20c) *is satisfied, then for every $\varepsilon^* \in (0,1)$, there exists a positive radius*

$$r^* \equiv r^*(\boldsymbol{data}_b(\Omega_0), \varepsilon^*) \tag{4.2.46}$$

such that

$$\begin{aligned} &\fint_{B_{\lambda r}} \Psi^-_{B_r}\left(\left|\frac{w-(w)_{B_{\lambda r}}}{\lambda r}\right|\right) dx \\ &\leqslant c\left(1+\lambda^{-(n+s(\Psi)+1)}\varepsilon^*\right) \fint_{B_{r/2}} \Psi^-_{B_r}\left(\left|\frac{w-(w)_{B_{r/2}}}{r}\right|\right) dx \end{aligned} \tag{4.2.47}$$

holds for some constant $c \equiv c(\boldsymbol{data}_b(\Omega_0))$, whenever $\lambda \in (0,1/16)$ and $r \leqslant r^*$.

Proof. First note that the meaning of $\mathbf{data}_b(\Omega_0)$ has been defined in (4.1.23). We can always assume that $E(w, B_{r/2}) > 0$ otherwise the inequality (4.2.47) is trivial. For the abbreviation, we shall denote

$$E(r) := E(w, B_{r/2}). \tag{4.2.48}$$

The proof consists of several steps.

Step 1: Initial settings on w. Applying Lemma 4.2.2 in the ball $B_{r/4}$ with $k \equiv (w)_{B_{r/2}}$, we find

$$\fint_{B_{r/4}} \Psi(x,|Dw|)\,dx \leqslant c \fint_{B_{r/2}} \Psi^-_{B_r}\left(\left|\frac{w-(w)_{B_{r/2}}}{r}\right|\right) dx \tag{4.2.49}$$

for some constant $c \equiv c(\mathbf{data})$. Moreover, by Theorem 3.2.5, there exists a higher integrability exponent $\delta_0 \equiv \delta_0(\mathbf{data})$ such that

$$\left(\fint_{B_{r/8}} [\Psi(x,|Dw|)]^{1+\delta_0}\,dx\right)^{\frac{1}{1+\delta_0}} \leqslant c \fint_{B_{r/4}} \Psi(x,|Dw|)\,dx \tag{4.2.50}$$

for a constant $c \equiv c(\mathbf{data})$.

Step 2: Scaling and freezing. Now we consider the scaled functions of $w(\cdot)$, $a(\cdot)$ and $b(\cdot)$ in the unit ball $B_1(0)$ by setting

$$\begin{aligned} &\bar{w}(x) := \frac{w(y+rx) - (w)_{B_{r/2}}}{E(r)r}, \quad \bar{a}(x) := a(y+rx)\frac{H_a(E(r))}{\Psi^-_{B_r}(E(r))} \\ &\text{and} \quad \bar{b}(x) := b(y+rx)\frac{H_b(E(r))}{\Psi^-_{B_r}(E(r))} \end{aligned} \tag{4.2.51}$$

for every $x \in B_1$. Now we define the vector field and energy density associated to the scaling in (4.2.51) by

$$\begin{aligned} \bar{A}(x,z) &:= \frac{A(y+rx, E(r)z)}{\Psi^-_{B_r}(E(r))} \\ &= \frac{A_G(E(r)z)}{\Psi^-_{B_r}(E(r))} + \bar{a}(x)\frac{A_{H_a}(E(r)z)}{H_a(E(r))} + \bar{b}(x)\frac{A_{H_b}(E(r)z)}{H_b(E(r))} \\ &\text{and } \bar{\Psi}(x,|z|) := \bar{G}(|z|) + \bar{a}(x)\bar{H}_a(|z|) + \bar{b}(x)\bar{H}_b(|z|) \end{aligned} \tag{4.2.52}$$

for every $x \in B_1$ and $z \in \mathbb{R}^n$, where

$$\bar{G}(t) := \frac{G(E(r)t)}{\Psi^-_{B_r}(E(r))}, \quad \bar{H}_a(t) := \frac{H_a(E(r)t)}{H_a(E(r))} \quad \text{and} \quad \bar{H}_b(t) := \frac{H_b(E(r)t)}{H_b(E(r))}. \tag{4.2.53}$$

One can check via elementary calculations that $\bar{G}, \bar{H}_a, \bar{H}_b \in \mathcal{N}$ with indices $s(G)$, $s(H_a)$, $s(H_b)$ respectively, and that

$$\bar{G}(1) \leqslant 1, \quad \bar{H}_a(1) = 1 \quad \text{and} \quad \bar{H}_b(1) = 1. \tag{4.2.54}$$

Then we see that $\bar{w} \in W^{1,\bar{\Psi}}(B_1)$ in (4.2.51) is a weak solution to the following equation that

$$\fint_{B_1} \left\langle \bar{A}(x, D\bar{w}), D\varphi \right\rangle dx = 0 \quad \text{for all} \quad \varphi \in W^{1,\bar{\Psi}}_0(B_1), \tag{4.2.55}$$

and the vector field $\bar{A}$ in (4.2.52) satisfies the following structure assumptions via (4.1.17):

$$\begin{cases} |\bar{A}(x,z)||z| + |D_z\bar{A}(x,z)||z|^2 \leqslant L\bar{\Psi}(x,|z|), & (4.2.56a) \\ \nu\dfrac{\bar{\Psi}(x,|z|)}{|z|^2}|\xi|^2 \leqslant \left\langle D_z\bar{A}(x,z)\xi,\xi\right\rangle, & (4.2.56b) \end{cases}$$

for every $x, x_1, x_2 \in B_1$, $\xi \in \mathbb{R}^n$ and $z \in \mathbb{R}^n \setminus \{0\}$. Furthermore, the inequalities (4.2.49)-(4.2.50) can we written in the view of the scaling in (4.2.51) as

$$\fint_{B_{1/4}} \bar{\Psi}(x,|D\bar{w}|)\,dx + \left(\fint_{B_{1/8}} [\bar{\Psi}(x,|D\bar{w}|)]^{1+\delta_0}\,dx\right)^{\frac{1}{1+\delta_0}} \leqslant c(\mathbf{data}). \tag{4.2.57}$$

Let $\bar{x}_a, \bar{x}_b \in \overline{B_1}$ be points such that $\bar{a}(\bar{x}_a) = \inf\limits_{x\in B_1} \bar{a}(x)$ and $\bar{b}(\bar{x}_b) = \inf\limits_{x\in B_1} \bar{b}(x)$. Then we consider the associated vector field and frozen functional denoted by

$$\begin{aligned} &\bar{A}_0(z) := \frac{A_G(E(r)z)}{\Psi^-_{B_r}(E(r))} + \bar{a}(\bar{x}_a)\frac{A_{H_a}(E(r)z)}{H_a(E(r))} + \bar{b}(\bar{x}_b)\frac{A_{H_b}(E(r)z)}{H_b(E(r))} \\ &\text{and} \quad \bar{\Psi}_0(t) := \bar{G}(t) + \bar{a}(\bar{x}_a)\bar{H}_a(t) + +\bar{b}(\bar{x}_b)\bar{H}_b(t) \quad (z\in\mathbb{R}^n,\ t\geqslant 0). \end{aligned} \tag{4.2.58}$$

From the definition in (4.2.51)-(4.2.53), one can see that

$$\begin{cases} |\bar{A}_0(z)||z| + |D_z\bar{A}_0(z)||z|^2 \leqslant L\bar{\Psi}_0(|z|), & (4.2.59a) \\ \nu\dfrac{\bar{\Psi}_0(|z|)}{|z|^2}|\xi|^2 \leqslant \left\langle D_z\bar{A}_0(z)\xi,\xi\right\rangle, & (4.2.59b) \\ \bar{\Psi}_0(1) = 1 & (4.2.59c) \end{cases}$$

for every $z \in \mathbb{R}^n \setminus \{0\}$ and $\xi \in \mathbb{R}^n$.

Step 4: Harmonic approximation. In the following let $\varphi \in W^{1,\infty}_0(B_{1/8})$ be a fixed function. Then using (4.2.55) and (4.1.17), we see

$$\left|\fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}), D\varphi\right\rangle\,dx\right| = \left|\fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}) - \bar{A}(x,D\bar{w}), D\varphi\right\rangle\,dx\right|$$

$$
\begin{aligned}
&\leqslant \fint_{B_{1/8}} |\bar{A}_0(D\bar{w}) - \bar{A}(x, D\bar{w})| \, dx \, \|D\varphi\|_{L^\infty(B_{1/8})} \\
&\leqslant L \fint_{B_{1/8}} |\bar{a}(x) - \bar{a}(\bar{x}_a)| \frac{\bar{H}_a(|D\bar{w}|)}{|D\bar{w}|} \, dx \, \|D\varphi\|_{L^\infty(B_{1/8})} \\
&\quad + L \fint_{B_{1/8}} |\bar{b}(x) - \bar{b}(\bar{x}_b)| \frac{\bar{H}_b(|D\bar{w}|)}{|D\bar{w}|} \, dx \, \|D\varphi\|_{L^\infty(B_{1/8})} \\
&=: (I_1 + I_2) \, \|D\varphi\|_{L^\infty(B_{1/8})} \,.
\end{aligned}
\tag{4.2.60}
$$

Now we estimate the terms I_i with $i \in \{1, 2\}$ via (2.1.7), (4.2.54) and (4.2.57) in order to have that

$$
\begin{aligned}
I_1 &\leqslant c \fint_{B_{1/8}} |\bar{a}(x) - \bar{a}(\bar{x}_a)| \left([\bar{H}_a(|D\bar{w}|)]^{\frac{1}{s(H_a)+1}} + [\bar{H}_a(|D\bar{w}|)]^{\frac{s(H_a)}{s(H)_a+1}} \right) dx \\
&\leqslant c \, \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{s(H_a)}{s(H_a)+1}} \left(\fint_{B_{1/8}} \bar{a}(x) \bar{H}_a(|D\bar{w}|) \, dx \right)^{\frac{1}{s(H_a)+1}} \\
&\quad + c \, \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_a)+1}} \left(\fint_{B_{1/8}} \bar{a}(x) \bar{H}_a(|D\bar{w}|) \, dx \right)^{\frac{s(H_a)}{s(H_a)+1}} \\
&\leqslant c(\mathbf{data}) \left(\|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_a)+1}} + \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{s(H_a)}{s(H_a)+1}} \right),
\end{aligned}
\tag{4.2.61}
$$

where we have used Hölder's inequality and the fact that $\bar{a}(\bar{x}_a) \leqslant \bar{a}(x)$ for every $x \in B_1$. Similarly as above, we have

$$
I_2 \leqslant c(\mathbf{data}) \left(\left\|\bar{b} - \bar{b}(\bar{x}_b)\right\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_b)+1}} + \left\|\bar{b} - \bar{b}(\bar{x}_b)\right\|_{L^\infty(B_{1/8})}^{\frac{s(H_b)}{s(H_b)+1}} \right). \tag{4.2.62}
$$

Inserting the inequalities in the last two displays into (4.2.60), we find

$$\begin{aligned}
&\left| \fint_{B_{1/8}} \langle \bar{A}_0(D\bar{w}), D\varphi \rangle \, dx \right| \\
&\leqslant c(\mathbf{data}) \left(\|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_a)+1}} + \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})}^{\frac{s(H_a)}{s(H_a)+1}} \right) \|D\varphi\|_{L^\infty(B_{1/8})} \\
&\quad + c(\mathbf{data}) \left(\left\|\bar{b} - \bar{b}(\bar{x}_b)\right\|_{L^\infty(B_{1/8})}^{\frac{1}{s(H_b)+1}} + \left\|\bar{b} - \bar{b}(\bar{x}_b)\right\|_{L^\infty(B_{1/8})}^{\frac{s(H_b)}{s(H_b)+1}} \right) \|D\varphi\|_{L^\infty(B_{1/8})} .
\end{aligned} \tag{4.2.63}$$

By the definition of $\bar{a}(\cdot)$ and $\bar{b}(\cdot)$ in (4.2.51) and the excess functional in (4.2.48), we find

$$I_a := \|\bar{a} - \bar{a}(\bar{x}_a)\|_{L^\infty(B_{1/8})} \leqslant c\omega_a(r) \frac{H_a(E(r))}{\Psi_{B_r}^-(E(r))}. \tag{4.2.64}$$

and

$$I_b := \left\|\bar{b} - \bar{b}(\bar{x}_b)\right\|_{L^\infty(B_{1/8})} \leqslant c\omega_b(r) \frac{H_b(E(r))}{\Psi_{B_r}^-(E(r))}. \tag{4.2.65}$$

Next we shall estimate the resulting terms of the last two display depending on which one of (4.1.20a)-(4.1.20c) comes into play.

Case 1: Assumption (4.1.20a) **is in force.** The assumption $(4.1.20\text{a})_2$ implies that for every $\varepsilon \in (0,1)$, there exists a constant $\mu_1 \equiv \mu_1(\varepsilon) > 0$ such that

$$\Lambda\left(\rho, G^{-1}(\rho^{-n})\right) \leqslant \varepsilon \quad \text{for every} \quad \rho \in (0, \mu_1). \tag{4.2.66}$$

Then using this one and (4.1.9), we continue to estimate I_a in (4.2.64) as

$$\begin{aligned}
I_a &\leqslant c\omega_a(r) \frac{(H_a \circ G^{-1})\left(\Psi_{B_r}^-(E(r))\right)}{\Psi_{B_r}^-(E(r))} \\
&\leqslant c\omega_a(r)\varepsilon \left(1 + \frac{1}{\omega_a\left([\Psi_{B_r}^-(E(r))]^{-\frac{1}{n}} \right)} \right) + c\omega_a(r) \left(1 + \frac{1}{\omega\left(\mu_1\right)} \right)
\end{aligned} \tag{4.2.67}$$

with $c \equiv c([a]_{\omega_a}, \lambda_1)$, where we have used the fact that $\left(\Psi_{B_r}^-\right)^{-1}(t) \leqslant G^{-1}(t)$ for every $t \geqslant 0$. Using (2.1.2) and recalling (4.2.48) together with (4.2.44) and (4.2.34), we have

$$\begin{aligned}\frac{1}{\omega_a\left(\left[\Psi_{B_r}^-(E(r))\right]^{-\frac{1}{n}}\right)} &\leqslant \frac{c}{\omega_a(r)} + \frac{c}{\omega_a(r)}\left[\int_{B_{r/2}} \Psi_{B_r}^-\left(\left|\frac{w-(w)_{B_{r/2}}}{r}\right|\right) dx\right]^{\frac{1}{n}} \\ &\leqslant \frac{c}{\omega_a(r)} + \frac{c}{\omega_a(r)}\left[\int_{B_R} \Psi\left(x, |Du|\right) dx\right]^{\frac{1}{n}} \leqslant \frac{c(\mathbf{data})}{\omega_a(r)}.\end{aligned} \tag{4.2.68}$$

Combining the last two displays, we conclude

$$I_a \leqslant c\left(\varepsilon + \omega_a(r)\left(1 + \frac{1}{\omega_a(\mu_1)}\right)\right) \tag{4.2.69}$$

with some constant $c \equiv c(\mathbf{data})$. In a similar way as we have shown (4.2.67)-(4.2.68), we also have

$$I_b \leqslant c\left(\varepsilon + \omega_b(r)\left(1 + \frac{1}{\omega_b(\mu_1)}\right)\right) \tag{4.2.70}$$

with some constant $c \equiv c(\mathbf{data})$. Therefore, inserting the estimates in the last two displays into (4.2.64) and (4.2.65) and recalling (4.2.63), we have

$$\left|\fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}), D\varphi\right\rangle dx\right| \leqslant c(\mathbf{data}_b(\Omega_0))P_1(\varepsilon, r)\left\|D\varphi\right\|_{L^\infty(B_{1/8})}, \tag{4.2.71}$$

where

$$\begin{aligned}&P_1(\varepsilon, r) := \\ &\left[\varepsilon + \omega_a(r)\left(1 + \frac{1}{\omega_a(\mu_1)}\right)\right]^{\frac{1}{s(H_a)+1}} + \left[\varepsilon + \omega_a(r)\left(1 + \frac{1}{\omega_a(\mu_1)}\right)\right]^{\frac{s(H_a)}{s(H_a)+1}}\end{aligned}$$

$$+\left[\varepsilon+\omega_b(r)\left(1+\frac{1}{\omega_b(\mu_1)}\right)\right]^{\frac{1}{s(H_b)+1}}+\left[\varepsilon+\omega_b(r)\left(1+\frac{1}{\omega_b(\mu_1)}\right)\right]^{\frac{s(H_b)}{s(H_b)+1}}. \tag{4.2.72}$$

Case 2: Assumption (4.1.20b) **is in force.** From the assumption $(4.1.20\text{b})_2$, for every $\varepsilon \in (0,1)$, we see that there exists a constant $\mu_2 \equiv \mu_2(\varepsilon) > 0$ such that

$$\Lambda\left(\rho, \frac{1}{\rho}\right) \leqslant \varepsilon \quad \text{for every} \quad \rho \in (0, \mu_2). \tag{4.2.73}$$

This one together with (4.1.10) yields

$$I_a \leqslant c\omega_a(r)\frac{H_a(E(r))}{G(E(r))} \leqslant c\omega_a(r)\varepsilon\left(1+\frac{1}{\omega_a\left([E(r)]^{-1}\right)}\right)+c\omega_a(r)\left(1+\frac{1}{\omega_a(\mu_2)}\right). \tag{4.2.74}$$

Now recalling (4.2.48) together with (4.2.44) and (4.2.35), we have

$$\frac{1}{\omega_a\left([E(r)]^{-1}\right)} \leqslant \frac{1}{\omega_a\left(\frac{r}{2\|w\|_{L^\infty(B_r)}}\right)} \leqslant \frac{c(\mathbf{data}_b(\Omega_0))}{\omega_a(r)}. \tag{4.2.75}$$

Inserting the estimate from the last display into (4.2.74), we find

$$I_a \leqslant c(\mathbf{data}_b(\Omega_0))\left(\varepsilon+\omega_a(r)\left(1+\frac{1}{\omega_a(\mu_2)}\right)\right). \tag{4.2.76}$$

Similarly, we also find

$$I_b \leqslant c(\mathbf{data}_b(\Omega_0))\left(\varepsilon+\omega_b(r)\left(1+\frac{1}{\omega_b(\mu_2)}\right)\right). \tag{4.2.77}$$

Plugging this one in (4.2.63) together with recalling (4.2.64) and (4.2.65), we see

$$\left|\fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}), D\varphi\right\rangle\, dx\right| \leqslant c(\mathbf{data}_b(\Omega_0))P_2(\varepsilon, r)\left\|D\varphi\right\|_{L^\infty(B_{1/8})}, \tag{4.2.78}$$

where

$$
\begin{aligned}
&P_2(\varepsilon, r) := \\
&\left[\varepsilon + \omega_a(r)\left(1 + \frac{1}{\omega_a(\mu_2)}\right)\right]^{\frac{1}{s(H_a)+1}} + \left[\varepsilon + \omega_a(r)\left(1 + \frac{1}{\omega_a(\mu_2)}\right)\right]^{\frac{s(H_a)}{s(H_a)+1}} \\
&+ \left[\varepsilon + \omega_b(r)\left(1 + \frac{1}{\omega_b(\mu_2)}\right)\right]^{\frac{1}{s(H_b)+1}} + \left[\varepsilon + \omega_b(r)\left(1 + \frac{1}{\omega_b(\mu_2)}\right)\right]^{\frac{s(H_b)}{s(H_b)+1}}.
\end{aligned}
\tag{4.2.79}
$$

Case 3: Assumption (4.1.20c) **is in force.** At this point we shall take an advantage that $\omega_a(\cdot)$ and $\omega_b(\cdot)$ are power functions. Recalling I_a denoted in (4.2.64), we have

$$
\begin{aligned}
I_a &\leqslant cr^{\alpha} \frac{(H \circ G^{-1})(\Psi^-_{B_r}(E(r)))}{\Psi^-_{B_r}(E(r))} \\
&\leqslant cr^{\alpha}\left(1 + \left[\fint_{B_{r/2}} \Psi^-_{B_r}\left(\left|\frac{w - (w)_{B_{r/2}}}{r}\right|\right) dx\right]^{\frac{\alpha}{n}}\right) \\
&\leqslant cr^{\alpha} + c\left(\int_{B_{r/2}} \Psi^-_{B_r}(|Dw|)\, dx\right)^{\frac{\alpha}{n}} \\
&\leqslant cr^{\alpha} + cr^{\frac{\alpha\delta_0}{1+\delta_0}}\left(\int_{B_{r/2}} [\Psi(x, |Dw|)]^{1+\delta_0}\, dx\right)^{\frac{\alpha}{n(1+\delta_0)}} \\
&\leqslant cr^{\alpha} + cr^{\frac{\alpha\delta_0}{1+\delta_0}}\left(\int_{B_R} [\Psi(x, |Du|)]^{1+\delta_0}\, dx\right)^{\frac{\alpha}{n(1+\delta_0)}} \leqslant c(\mathbf{data}_b(\Omega_0))r^{\frac{\alpha\delta_0}{1+\delta_0}},
\end{aligned}
\tag{4.2.80}
$$

where we have applied a Poincaré type inequality of Lemma 2.4.1 and Proposition 4.2.3. Again similarly, we see

$$
I_b \leqslant c(\mathbf{data}_b(\Omega_0))r^{\frac{\beta\delta_0}{1+\delta_0}}, \tag{4.2.81}
$$

Inserting the resulting estimates from the last two displays into (4.2.63) and then (4.2.60), we find

$$\left| \fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}), D\varphi \right\rangle \, dx \right| \leqslant c(\mathbf{data}_b(\Omega_0)) P_3(r) \, \|D\varphi\|_{L^\infty(B_{1/8})} , \tag{4.2.82}$$

where

$$P_3(r) := r^{\frac{\alpha\delta_0}{(1+\delta_0)(1+s(H_a))}} + r^{\frac{\beta\delta_0}{(1+\delta_0)(1+s(H_b))}}. \tag{4.2.83}$$

Summarizing all the cases we considered so far, we conclude with

$$\left| \fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}), D\varphi \right\rangle \, dx \right| \leqslant c_h P(\varepsilon, r) \, \|D\varphi\|_{L^\infty(B_{1/8})} \tag{4.2.84}$$

for some constant $c_h \equiv c_h(\mathbf{data}_b(\Omega_0))$, whenever $\varphi \in W_0^{1,\infty}(B_{1/8})$, where

$$P(\varepsilon, R) := \begin{cases} P_1(\varepsilon, r) & \text{if (4.1.20a) is assumed,} \\ P_2(\varepsilon, r) & \text{if (4.1.20b) is assumed,} \\ P_3(r) & \text{if (4.1.20c) is assumed,} \end{cases} \tag{4.2.85}$$

in which the functions P_1, P_2 and P_3 have been defined in (4.2.72), (4.2.79) and (4.2.83), respectively. Taking into account (4.2.57), (4.2.59a)-(4.2.59c) and (4.2.84), it is possible to apply Lemma 2.5.1 by setting $A_0(z) := \bar{A}_0(z)$, $\Psi_0(t) := \bar{\Psi}_0(t)$,$a_0 := \bar{a}(\bar{x}_a)$ and $a_0 := \bar{b}(\bar{x}_b)$. In turn, there exists $\bar{h} \in \bar{w} + W_0^{1,\bar{\Psi}_0}(B_{1/8})$ such that

$$\fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{h}), D\varphi \right\rangle \, dx = 0 \qquad \text{for all} \qquad \varphi \in W_0^{1,\infty}(B_{1/8}), \tag{4.2.86}$$

$$\fint_{B_{1/4}} \bar{\Psi}_0(|D\bar{h}|) \, dx + \fint_{B_{1/8}} [\bar{\Psi}_0(|D\bar{h}|)]^{1+\delta_1} \, dx \leqslant c \text{ for some } \delta_1 \leqslant \delta_0, \tag{4.2.87}$$

$$\fint_{B_{1/8}} |V_{\bar{\Psi}_0}(D\bar{w}) - V_{\bar{\Psi}_0}(D\bar{h})|^2 \, dx \leqslant c[P(\varepsilon, r)]^{s_1} \tag{4.2.88}$$

and finally

$$\fint_{B_{1/8}} \bar{\Psi}_0 \left(|\bar{w} - \bar{h}|\right) \, dx \leqslant c_d [P(\varepsilon, r)]^{s_0} \tag{4.2.89}$$

with some constants c, c_d which depend on $\mathbf{data}_b(\Omega_0)$ and $s_0, s_1 \equiv s_0, s_1(\mathbf{data}) \in (0,1)$, but they are all independent of r, ε. Therefore, for a given $\varepsilon^* \in (0,1)$ as in the statement of our lemma, we choose ε and r^* small enough to satisfy

$$c_d \left[P(\varepsilon, r^*)\right]^{s_0} \leqslant \varepsilon^*. \tag{4.2.90}$$

Taking into account the dependence of the constants c_d and s_0 as mentioned above, the last display gives us the dependence of r^* as in the statement of the present lemma. Furthermore, by (4.2.89), we conclude with

$$\fint_{B_{1/8}} \bar{\Psi}_0 \left(|\bar{w} - \bar{h}|\right) \, dx \leqslant \varepsilon^*. \tag{4.2.91}$$

Proof of (4.2.47)**.** We observe that by a standard density argument, the relation in (4.2.86) still holds for every $\varphi \in W_0^{1,1}(B_{1/8})$ with $\bar{\Psi}_0(|D\varphi|) \in L^1(B_{1/8})$. Since the structure conditions (4.2.59a)-(4.2.59b) are satisfied for the vector field $\bar{A}_0(\cdot)$ with respect to N-function $\bar{\Psi}_0$ which also belongs to $\mathcal{N}$ with an index independent of $\bar{a}(\bar{x}_a)$ and $\bar{b}(\bar{x}_b)$, we are in a position to apply the results from [111] to obtain the following a priori Lipschitz estimate:

$$\sup_{B_{1/16}} \bar{\Psi}_0(|D\bar{h}|) \leqslant c \fint_{B_{1/8}} \bar{\Psi}_0(|D\bar{h}|) \, dx \tag{4.2.92}$$

with some constant $c \equiv c(n, s(\Psi), \nu, L)$. For any $\tau \in (0, 1/16)$, we have that

$$\fint_{B_\tau} \bar{\Psi}_0 \left(\left| \frac{\bar{w} - (\bar{w})_{B_\tau}}{\tau} \right| \right) \, dx \leqslant \fint_{B_\tau} \bar{\Psi}_0 \left(\left| \frac{\bar{w} - (\bar{h})_{B_\tau}}{\tau} \right| \right) \, dx$$

$$
\begin{aligned}
&\leqslant c \fint_{B_\tau} \bar{\Psi}_0\left(\left|\frac{\bar{h}-(\bar{h})_{B_\tau}}{\tau}\right|\right) dx + c\fint_{B_\tau} \bar{\Psi}_0\left(\left|\frac{\bar{w}-\bar{h}}{\tau}\right|\right) dx \\
&\overset{(4.2.91)}{\leqslant} c\sup_{B_\tau} \bar{\Psi}_0(|D\bar{h}|) + c\tau^{-(n+s(\Psi)+1)}\varepsilon^* \\
&\overset{(4.2.92)}{\leqslant} c\fint_{B_{1/8}} \bar{\Psi}_0(|D\bar{h}|)\, dx + c\tau^{-(n+s(\Psi)+1)}\varepsilon^* \\
&\overset{(4.2.87)}{\leqslant} c + c\tau^{-(n+s(\Psi)+1)}\varepsilon^*. \qquad (4.2.93)
\end{aligned}
$$

By returning back to w as introduced in (4.2.51), we obtain the desired estimate (4.2.47). The proof is complete. □

4.2.3 Morrey decay estimate

Here we discuss an important outcome of Lemma 4.2.4, the so-called Morrey decay estimate, which will play a crucial role later.

Theorem 4.2.3. *Let $w \in W^{1,\Psi}(B_R)$ be the weak solution to* (4.2.31) *under the assumption* (4.1.17). *If one of the assumptions* (4.1.20a)-(4.1.12c) *is satisfied, then for every $\sigma \in (0,n)$, there exists a constant $c \equiv c(\boldsymbol{data}_b(\Omega_0),\sigma)$ such that the following decay estimate*

$$
\int_{B_\rho} \Psi(x,|Dw|)\, dx \leqslant c\left(\frac{\rho}{r}\right)^{n-\sigma} \int_{B_r} \Psi(x,|Dw|)\, dx \qquad (4.2.94)
$$

holds, whenever $B_\rho(y) \Subset B_r(y) \subset B_R(x_0)$.

Proof. In fact, the proof can be proceeded similarly as for the proof of Theorem 3.1.2. We only show alternatives discussed in the proof of Theorem 3.1.2 for w. Let $B_R \equiv B_R(x_0) \subset \Omega_0 \Subset \Omega$ be a ball with $R \leqslant 1$ as fixed in (4.2.31). Let $B_r(y) \subset B_R$ be any fixed ball with $r \leqslant r^*(\mathbf{data}_b(\Omega_0),\sigma)$ which will be determined in a few lines.

Alternative 1: G-phase. Let $\tau_{ab} \in (0,1/64)$. Assume that G-phase occurs in the ball $B_{\tau_{ab}r}$ (see (3.2.27a) for the definition). In this case we have

$$
a^-(B_{2\tau_{ab}r}) \leqslant 8[a]_{\omega_a}\omega_a(\tau_{ab}r) \quad \text{and} \quad b^-(B_{2\tau_{ab}r}) \leqslant 8[b]_{\omega_b}\omega_b(\tau_{ab}r). \qquad (4.2.95)
$$

Applying Lemma 4.2.3 in the ball $B_{2\tau_{ab}r}$ and Lemma 4.2.4, we find

$$\begin{aligned}\int_{B_{\tau_{ab}r}} \Psi(x,|Dw|)\,dx \leqslant & c\int_{B_{2\tau_{ab}r}} G\left(\left|\frac{w-(w)_{B_{2\tau_{ab}r}}}{2\tau_{ab}r}\right|\right)dx \\ & \leqslant c\int_{B_{2\tau_{ab}r}} \Psi^-_{B_r}\left(\left|\frac{w-(w)_{B_{2\tau_{ab}r}}}{2\tau_{ab}r}\right|\right)dx \\ & \leqslant c\left(\tau_{ab}^n+\tau_{ab}^{-(s(\Psi)+1)}\varepsilon^*\right)\int_{B_r}\Psi(x,|Dw|)\,dx \qquad (4.2.96)\end{aligned}$$

for some constant $c \equiv c(\mathbf{data}_b(\Omega_0))$, provided $r \leqslant r^*(\mathbf{data}_b(\Omega_0),\varepsilon^*)$. Now for every $\sigma \in (0,n)$, we rewrite the last display in the following form

$$\int_{B_{\tau_{ab}r}} \Psi(x,|Dw|)\,dx \leqslant \tau_{ab}^{n-\sigma}\left(c_{ab}\tau_{ab}^{\sigma}+c_{ab}\tau_{ab}^{\sigma-(n+s(\Psi)+1)}\varepsilon^*\right)\int_{B_r}\Psi(x,|Dw|)\,dx. \tag{4.2.97}$$

Here we choose parameters τ_{ab},ε^* having the dependence on $\mathbf{data}_b(\Omega_0)$ and σ in such a way that $c_{ab}\tau_{ab}^{\sigma} \leqslant 1/2$ and $c_{ab}\tau_{ab}^{\sigma-(n+s(\Psi)+1)}\varepsilon^* \leqslant 1/2$. With those choices, we conclude with

$$\int_{B_{\tau_{ab}r}} \Psi(x,|Dw|)\,dx \leqslant \tau_{ab}^{n-\sigma}\int_{B_r}\Psi(x,|Dw|)\,dx, \tag{4.2.98}$$

provided $r \leqslant r_{ab}(\mathbf{data}_b(\Omega_0),\sigma)$.

Alternative 2: (G,H_a)**-phase.** Let $\tau_b \in (0,1/64)$. Suppose that (G,H_a)-phase occurs in B_r and that $b^-(B_{\tau_b r}) \leqslant 4[b]_{\omega_b}\omega_b(\tau_b r)$. Clearly, we have

$$b^-(B_{2\tau_b r}) \leqslant 8[b]_{\omega_b}\omega_b(\tau_b r). \tag{4.2.99}$$

On the other hand, we also see

$$a^-(B_{\tau_b r}) \geqslant a^-(B_r) > 4[a]_{\omega_a}\omega_a(r) \geqslant 4[a]_{\omega_a}\omega_a(\tau_b r) \tag{4.2.100}$$

and

$$a^-(B_r) \leqslant a(x) \leqslant 2[a]_{\omega_a}\omega_a(r) + a^-(B_r) \leqslant 2a^-(B_r) \quad (\forall x \in B_r). \tag{4.2.101}$$

Applying Lemma 4.2.3 in the ball $B_{2\tau_b r}$ and then Lemma 4.2.4 together with recalling (4.2.101), we have

$$\begin{aligned}
&\int_{B_{\tau_b r}} \Psi(x, |Dw|)\, dx \\
&\leqslant c \int_{B_{2\tau_b r}} \left[G\left(\left| \frac{w - (w)_{B_{2\tau_b r}}}{2\tau_b r} \right| \right) + a^-(B_{2\tau_b r}) H_a \left(\left| \frac{w - (w)_{B_{2\tau_b r}}}{2\tau_b r} \right| \right) \right] dx \\
&\leqslant c \int_{B_{2\tau_b r}} \Psi^-_{B_r} \left(\left| \frac{w - (w)_{B_{2\tau_b r}}}{2\tau_b r} \right| \right) dx \\
&\leqslant c \left(\tau_b^n + \tau_b^{-(s(\Psi)+1)} \varepsilon^* \right) \int_{B_r} \Psi(x, |Dw|)\, dx
\end{aligned} \tag{4.2.102}$$

for some constant $c \equiv c(\mathbf{data}_b(\Omega_0))$, provided $r \leqslant r^*(\mathbf{data}_b(\Omega_0), \varepsilon^*)$. Then, for every $\sigma \in (0, n)$, we write down the last display as

$$\int_{B_{\tau_b r}} \Psi(x, |Dw|)\, dx \leqslant \tau_b^{n-\sigma} \left(c_b \tau_b^{\sigma} + c_b \tau_b^{\sigma-(n+s(\Psi)+1)} \varepsilon^* \right) \int_{B_r} \Psi(x, |Dw|)\, dx$$

for some constant $c_b \equiv c_b(\mathbf{data}_b(\Omega_0))$. We select small enough τ_b, ε^* depending on $\mathbf{data}_b(\Omega_0)$ and σ in such a way that $c_b \tau_b^{\sigma} \leqslant 1/2$ and $c_b \tau_b^{\sigma-(n+s(\Psi)+1)} \varepsilon^* \leqslant 1/2$. In turn, we find

$$\int_{B_{\tau_b r}} \Psi(x, |Du|)\, dx \leqslant \tau_b^{n-\sigma} \int_{B_r} \Psi(x, |Du|)\, dx \tag{4.2.103}$$

for every $r \leqslant r_b \equiv r_b(\mathbf{data}_b(\Omega_0), \sigma)$.

Alternative 3: (G, H_b)**-phase.** Let $\tau_a \in (0, 1/64)$ to be fixed later. Assume that (G, H_b)-phase occurs in B_r ((3.2.27c) happens in B_r) and $a^-(B_{\tau_a r}) \leqslant$

$4[a]_{\omega_a}\omega_a(\tau_a r)$. Then we have

$$a^-(B_{2\tau_a R}) \leqslant 8[a]_{\omega_a}\omega_a(\tau_a R). \tag{4.2.104}$$

Applying Lemma 4.2.3 and then Lemma 4.2.4 together with recalling that $b^-(B_r) \leqslant b(x) \leqslant 2b^-(B_r)$ holds for every $x \in B_r$ if $b^-(B_r) > 4[b]_{\omega_b}\omega_b(r)$ likewise in (4.2.101), we have

$$\begin{aligned}
&\int_{B_{\tau_a R}} \Psi(x, |Dw|)\, dx \\
&\qquad \leqslant c \int_{B_{2\tau_a r}} \left[G\left(\left| \frac{w-(w)_{B_{2\tau_a r}}}{2\tau_a r} \right| \right) + b^-(B_{2\tau_a r}) H_b \left(\left| \frac{w-(w)_{B_{2\tau_a r}}}{2\tau_a r} \right| \right) \right] dx \\
&\qquad \leqslant c \int_{B_{2\tau_a r}} \Psi^-_{B_r} \left(\left| \frac{w-(w)_{B_{2\tau_a r}}}{2\tau_a r} \right| \right) dx \\
&\qquad \leqslant c \left(\tau_a^n + \tau_a^{-(s(\Psi)+1)}\varepsilon^* \right) \int_{B_r} \Psi(x, |Dw|)\, dx
\end{aligned} \tag{4.2.105}$$

for some constant $c \equiv c(\mathbf{data}_b(\Omega_0))$, provided $r \leqslant r^*(\mathbf{data}_b(\Omega_0), \varepsilon^*)$. Then, for every $\sigma \in (0, n)$, we write down the last display as

$$\int_{B_{\tau_a r}} \Psi(x, |Dw|)\, dx \leqslant \tau_a^{n-\sigma} \left(c_a \tau_a^{\sigma} + c_a \tau_a^{\sigma-(n+s(\Psi)+1)}\varepsilon^* \right) \int_{B_r} \Psi(x, |Dw|)\, dx$$

for some constant $c_a \equiv c_a(\mathbf{data}_b(\Omega_0))$. We select small enough τ_a, ε^* depending on $\mathbf{data}_b(\Omega_0)$ and σ in such a way that $c_a\tau_a^{\sigma} \leqslant 1/2$ and $c_a \tau_a^{\sigma-(n+s(\Psi)+1)}\varepsilon^* \leqslant 1/2$. Then we have

$$\int_{B_{\tau_a r}} \Psi(x, |Dw|)\, dx \leqslant \tau_a^{n-\sigma} \int_{B_R} \Psi(x, |Dw|)\, dx \tag{4.2.106}$$

for every $r \leqslant r_a \equiv r_a(\mathbf{data}_b(\Omega_0), \sigma)$.

Alternative 4 : (G, H_a, H_b)-phase. Let $\tau_0 \in (0, 1/64)$ to be chosen later. We assume that (G, H_a, H_b)-phase occurs in B_r, which means that (3.2.27d) happens in B_r. In this situation, from the observation in (4.2.101) we see that $a^-(B_r) \leqslant a(x) \leqslant 2a^-(B_r)$ and $b^-(B_r) \leqslant b(x) \leqslant 2b^-(B_r)$ for every $x \in B_r$.

Then again applying Lemma 4.2.3 and Lemma 4.2.4, we find

$$\begin{aligned}
\int_{B_{\tau_0 r}} \Psi(x,|Dw|)\,dx &\leqslant c \int_{B_{2\tau_0 r}} \Psi^-_{B_{2\tau_0 R}}\left(\left|\frac{w-(w)_{B_{2\tau_0 r}}}{2\tau_0 r}\right|\right) dx \\
&\leqslant c \int_{B_{2\tau_0 r}} \Psi^-_{B_r}\left(\left|\frac{w-(w)_{B_{2\tau_0 r}}}{2\tau_0 r}\right|\right) dx \\
&\leqslant c\left(\tau_0^n + \tau_0^{-(s(\Psi)+1)}\varepsilon^*\right)\int_{B_r} \Psi(x,|Dw|)\,dx \qquad (4.2.107)
\end{aligned}$$

for some constant $c \equiv c(\mathbf{data}_b(\Omega_0))$, provided $r \leqslant r^*(\mathbf{data}_b(\Omega_0), \varepsilon^*)$. Then, for every $\sigma \in (0,n)$, we write down the last display as

$$\int_{B_{\tau_0 r}} \Psi(x,|Dw|)\,dx \leqslant \tau_0^{n-\sigma}\left(c_0\tau_0^{\sigma} + c_0\tau_0^{\sigma-(n+s(\Psi)+1)}\varepsilon_*\right)\int_{B_r} \Psi(x,|Dw|)\,dx$$

for some constant $c_0 \equiv c_0(\mathbf{data}_b(\Omega_0))$. Then we choose τ_0, ε^* depending on $\mathbf{data}_b(\Omega_0)$ and σ in such a way that $c_0\tau_0^{\sigma} \leqslant 1/2$ and $c_0\tau_0^{\sigma-(n+s(\Psi)+1)}\varepsilon^* \leqslant 1/2$. Then we have

$$\int_{B_{\tau_0 r}} \Psi(x,|Dw|)\,dx \leqslant \tau_0^{n-\sigma}\int_{B_r} \Psi(x,|Dw|)\,dx \qquad (4.2.108)$$

for every $r \leqslant r_0 \equiv r_0(\mathbf{data}_b(\Omega_0), \sigma)$.

Conclusion. Since we have alternatives discussed above, the remaining part of the proof can be argued in a similar way as starting from Step 2 until the end of the proof of Theorem 3.1.2. The proof is complete. □

4.2.4 Gradient estimates.

Now we shall focus on the gradient estimates of $w \in W^{1,\Psi}(B_R)$, the solution to (4.2.31) under each assumption of (4.1.20a)-(4.1.20c).

Theorem 4.2.4. *Let* $w \in W^{1,\Psi}(B_R)$ *be the weak solution to* (4.2.31) *under the assumption* (4.1.17). *Suppose that one of the assumptions* (4.1.20a)-(4.1.20c) *is satisfied.*

1. *Then for every $d \geqslant 1$, there exists a constant $c \equiv c(\boldsymbol{data}_b(\Omega_0), d)$ such that*

$$\left(\fint_{B_r} \left[\Psi^-_{B_{4r}}(|Dw|) \right]^d dx \right)^{\frac{1}{d}} \leqslant c \fint_{B_{4r}} \Psi(x, |Dw|)\, dx \tag{4.2.109}$$

holds true, whenever $B_{4r} \equiv B_{4r}(y) \subset B_R$ is a ball.

2. *Then there exists a constant $c_{ab} \equiv c_{ab}(\boldsymbol{data}_b(\Omega_0))$ such that*

$$\fint_{B_r} |\Psi^+_{B_r}(|Dw|) - \Psi(x, |Dw|)|\, dx \leqslant c_{ab} Q(\varepsilon, r) \fint_{B_{8r}} \Psi(x, |Dw|)\, dx \tag{4.2.110}$$

holds, whenever $B_{8r} \equiv B_{8r}(y) \subset B_R$ is a ball and $\varepsilon \in (0,1)$ is arbitrary, where

$$Q(\varepsilon, r) := \begin{cases} \varepsilon + \omega_a(r) \left(1 + \dfrac{1}{\omega_a(\mu_1(\varepsilon))} \right) + \omega_b(r) \left(1 + \dfrac{1}{\omega_b(\mu_1(\varepsilon))} \right) \\ \textit{if } (4.1.20\text{a}) \textit{ is assumed,} \\ \varepsilon + \omega_a(r) \left(1 + \dfrac{1}{\omega_a(\mu_2(\varepsilon))} \right) + \omega_b(r) \left(1 + \dfrac{1}{\omega_b(\mu_2(\varepsilon))} \right) \\ \textit{if } (4.1.20\text{b}) \textit{ is assumed,} \\ r^{\frac{\alpha\delta_0}{1+\delta_0}} + r^{\frac{\beta\delta_0}{1+\delta_0}} \\ \textit{if } (4.1.20\text{c}) \textit{ is assumed} \end{cases} \tag{4.2.111}$$

for some constants μ_1 and μ_2 depending only on ε.

Proof. First we prove (4.2.109). Let $x_1, x_2 \in B_r$ be any points. Then applying Theorem 4.2.2 and then Theorem 4.2.3, we have

$$\begin{aligned} \Psi^-_{B_{2|x_1-x_2|}(x_1)} \left(\frac{|w(x_1) - w(x_2)|}{|x_1 - x_2|} \right) &\leqslant c \fint_{B_{2|x_1-x_2|}(x_1)} \Psi(x, |Dw|)\, dx \\ &\leqslant c \left(\frac{r}{|x_1 - x_2|} \right)^{\sigma} \fint_{B_{2r}(x_1)} \Psi(x, |Dw|)\, dx \end{aligned} \tag{4.2.112}$$

for some $c \equiv c(\mathbf{data}_b(\Omega_0), \sigma)$, whenever $\sigma \in (0, n)$. Then last display implies that

$$\left[\fint_{B_r} \left[\Psi^-_{B_{4r}}\left(\frac{|w(x_1) - w(x_2)|}{|x_1 - x_2|}\right)\right]^d dx_1\right]^{\frac{1}{d}}$$

$$\leqslant c \left(\fint_{B_r} \left(\frac{r}{|x_1 - x_2|}\right)^{\sigma d} dx_1\right)^{\frac{1}{d}} \fint_{B_{4r}} \Psi(x, |Dw|)\, dx \tag{4.2.113}$$

holds for some constant $c \equiv c(\mathbf{data}_b(\Omega_0), \sigma)$. By a standard calculation, we observe that

$$\left(\fint_{B_r} \left(\frac{r}{|x_1 - x_2|}\right)^{\sigma d} dx_1\right)^{\frac{1}{d}} \leqslant \left(\frac{c(n)}{n - \sigma d}\right)^{\frac{1}{d}} \tag{4.2.114}$$

holds, whenever $\sigma d < n$. Now using the last display in (4.2.113) and choosing $\sigma = n/2d$, we see that

$$\left[\fint_{B_r} \left[\Psi^-_{B_{4r}}\left(\frac{|w(x_1) - w(x_2)|}{|x_1 - x_2|}\right)\right]^d dx_1\right]^{\frac{1}{d}} \leqslant c(\mathbf{data}_b(\Omega_0), d) \fint_{B_{4r}} \Psi(x, |Dw|)\, dx \tag{4.2.115}$$

holds for a.e $x_2 \in B_r$. Finally, applying Fatou's lemma, we arrive at the desired estimate (4.2.109). Now we turn our attention to proving (4.2.110). Using the definition of $\omega_a(\cdot)$ and $\omega_b(\cdot)$, we see

$$I := \fint_{B_r} |\Psi^+_{B_r}(|Dw|) - \Psi(x, |Dw|)|\, dx \tag{4.2.116}$$

$$\leqslant 2[a]_{\omega_a}\omega_a(r) \fint_{B_r} H_a(|Dw|)\, dx + 2[b]_{\omega_b}\omega_b(r) \fint_{B_r} H_b(|Dw|)\, dx$$

$$=: I_a + I_b \tag{4.2.117}$$

Now we estimate the terms I_a and I_b in the above display. For this, we shall consider three cases depending on which one of the assumptions (4.1.20a)-(4.1.20c) comes into play.

Case 1: (4.1.20a) **is in force.** Recalling (4.2.66), for every $\varepsilon \in (0,1)$ there exists $\mu_1 \equiv \mu_1(\varepsilon)$ such that

$$\begin{aligned} I_a &= 2[a]_{\omega_a}\omega_a(r) \fint_{B_r} \frac{(H_a \circ G^{-1})(G(|Dw|))}{G(|Dw|)} G(|Dw|)\,dx \\ &\leqslant c\omega_a(r)\varepsilon \fint_{B_r} \left(1 + \frac{1}{\omega_a\left([G(|Dw|)]^{-\frac{1}{n}}\right)}\right) G(|Dw|)\,dx \\ &\quad + c\omega_a(r)\left(1 + \frac{1}{\omega_a(\mu_1)}\right) \fint_{B_r} G(|Dw|)\,dx \end{aligned} \tag{4.2.118}$$

for some constant $c \equiv c([a]_\omega, \lambda_1)$. Arguing similarly as we have done in (4.2.68) and using (4.2.109) together with (4.2.34), we have

$$\begin{aligned} &\fint_{B_r} \left(1 + \frac{1}{\omega_a\left([G(|Dw|)]^{-\frac{1}{n}}\right)}\right) G(|Dw|)\,dx \\ &\leqslant \fint_{B_r} \left(1 + \frac{1}{\omega_a(r)} + \frac{r[G(|Dw|)]^{\frac{1}{n}}}{\omega_a(r)}\right) G(|Dw|)\,dx \\ &\leqslant \left(1 + \frac{1}{\omega_a(r)}\right) \fint_{B_r} G(|Dw|)\,dx \\ &\quad + \frac{c(\mathbf{data}_b(\Omega_0))}{\omega_a(r)} \left(\int_{B_{4r}} \Psi(x,|Dw|)\,dx\right)^{\frac{1}{n}} \fint_{B_{4r}} \Psi(x,|Dw|)\,dx \\ &\leqslant \left(1 + \frac{c(\mathbf{data}_b(\Omega_0))}{\omega_a(r)}\right) \fint_{B_{4r}} \Psi(x,|Dw|)\,dx. \end{aligned} \tag{4.2.119}$$

Inserting the last display into (4.2.118) and recalling $R \leqslant 1$, we have

$$I_a \leqslant c(\mathbf{data}_b(\Omega_0)) \left(\varepsilon + \omega_a(r)\left(1 + \frac{1}{\omega_a(\mu_1(\varepsilon))}\right)\right) \fint_{B_{8r}} \Psi(x, |Dw|)\, dx. \tag{4.2.120}$$

In the exactly same way, we have

$$I_b \leqslant c(\mathbf{data}_b(\Omega_0)) \left(\varepsilon + \omega_b(r)\left(1 + \frac{1}{\omega_b(\mu_1(\varepsilon))}\right)\right) \fint_{B_{8r}} \Psi(x, |Dw|)\, dx. \tag{4.2.121}$$

Plugging the estimates of the last two displays into (4.2.116), we arrive at the validity of (4.2.110) when (4.1.20a) is in force.

Case 2: (4.1.20b) **is in force.** First applying Lemma 2.1.5, there exists a constant $d \equiv d(s(G), s(H_a), s(H_b))$ such that the maps

$$t \mapsto \left(H_a \circ \left(\Psi^-_{B_{4r}}\right)^{-1}\right)\left(t^{\frac{1}{d}}\right) \quad \text{and} \quad t \mapsto \left(H_b \circ \left(\Psi^-_{B_{4r}}\right)^{-1}\right)\left(t^{\frac{1}{d}}\right)$$

are concave in $(0, \infty)$. Now applying Jensen's inequality and (4.2.109), we see

$$\begin{aligned} I_a &:= 2[a]_{\omega_a}\omega_a(r) \fint_{B_r} \left(H_a \circ \left(\Psi^-_{B_{4r}}\right)^{-1}\right)\left(\left([\Psi^-_{B_{4r}}(|Dw|)]^d\right)^{\frac{1}{d}}\right) dx \\ &\leqslant 2[a]_{\omega_a}\omega_a(r) \left(H_a \circ \left(\Psi^-_{B_{4r}}\right)^{-1}\right)\left(\left(\fint_{B_r} [\Psi^-_{B_{4r}}(|Dw|)]^d\, dx\right)^{\frac{1}{d}}\right) \\ &\leqslant c\omega_a(r) \left(H_a \circ \left(\Psi^-_{B_{4r}}\right)^{-1}\right)\left(\fint_{B_{4r}} \Psi(x, |Dw|)\, dx\right) \end{aligned} \tag{4.2.122}$$

for some constant $c \equiv c(\mathbf{data}_b(\Omega_0))$. Recalling (4.2.73) and letting $M := \fint_{B_{4r}} \Psi(x, |Dw|)\, dx$ for the simplicity of writing, we continue to estimate the last display as follows

$$
\begin{aligned}
I &\leqslant c\omega_a(r)\frac{\left(H\circ\left(\Psi^-_{B_{4r}}\right)^{-1}\right)(M)}{\left(G\circ\left(\Psi^-_{B_{4r}}\right)^{-1}\right)(M)}M \\
&\leqslant c\left[\omega_a(r)\varepsilon\left(1+\frac{1}{\omega_a\left(\left[\left(\Psi^-_{B_{4r}}\right)^{-1}(M)\right]^{-1}\right)}\right)+\omega_a(r)\left(1+\frac{1}{\omega_a(\mu_2)}\right)\right]M
\end{aligned}
\tag{4.2.123}
$$

for some constant $c \equiv c(\mathbf{data}_b(\Omega_0))$ and $\mu_2 \equiv \mu_2(\varepsilon)$, whenever $\varepsilon \in (0,1)$. At this moment, we use a Caccioppoli type inequality of Proposition 4.2.4 and then (4.1.10) to have

$$
\begin{aligned}
M &\leqslant c \fint_{B_{8r}} \Psi\left(x, \left|\frac{w-(w)_{B_{8r}}}{r}\right|\right) dx \\
&\leqslant c\omega_a(r)\fint_{B_{8r}} H_a\left(\left|\frac{w-(w)_{B_{8r}}}{r}\right|\right) dx + c\omega_b(r)\fint_{B_{8r}} H_b\left(\left|\frac{w-(w)_{B_{8r}}}{r}\right|\right) dx \\
&\quad + c\fint_{B_{8r}} \Psi^-_{B_{4r}}\left(\left|\frac{w-(w)_{B_{8r}}}{r}\right|\right) dx \\
&\leqslant c\omega_a(r)\fint_{B_{8r}}\left(1+\frac{1}{\omega_a\left(\left|\frac{w-(w)_{B_{8r}}}{r}\right|^{-1}\right)}\right) G\left(\left|\frac{w-(w)_{B_{8r}}}{r}\right|\right) dx \\
&\quad + c\omega_b(r)\fint_{B_{8r}}\left(1+\frac{1}{\omega_b\left(\left|\frac{w-(w)_{B_{8r}}}{r}\right|^{-1}\right)}\right) G\left(\left|\frac{w-(w)_{B_{8r}}}{r}\right|\right) dx \\
&\quad + c\fint_{B_{8r}} \Psi^-_{B_{4r}}\left(\left|\frac{w-(w)_{B_{8r}}}{r}\right|\right) dx \\
&\leqslant c\fint_{B_{8r}} \Psi^-_{B_{4r}}\left(\left|\frac{w-(w)_{B_{8r}}}{r}\right|\right) dx \leqslant c\Psi^-_{B_{4r}}\left(\frac{1}{r}\right)
\end{aligned}
\tag{4.2.124}
$$

for some constant $c \equiv c(\mathbf{data}_b(\Omega_0))$, where we have used (4.2.35). Inserting the resulting estimate of the previous display into (4.2.123) and recalling $R \leqslant 1$, we find

$$I_a \leqslant c(\mathbf{data}_b(\Omega_0)) \left(\varepsilon + \omega_a(r)\left(1 + \frac{1}{\omega_a(\mu_2)}\right)\right) \fint_{B_{8r}} \Psi(x, |Dw|)\, dx \quad (4.2.125)$$

for some constant $\mu_2 \equiv \mu_2(\varepsilon)$, whenever $\varepsilon \in (0,1)$. In a similar way, we also have

$$I_b \leqslant c(\mathbf{data}_b(\Omega_0)) \left(\varepsilon + \omega_b(r)\left(1 + \frac{1}{\omega_b(\mu_2)}\right)\right) \fint_{B_{8r}} \Psi(x, |Dw|)\, dx. \quad (4.2.126)$$

Inserting the estimates in the last two displays into (4.2.116), we see (4.2.110) when the condition (4.1.20b) is assumed.

Case 3: (4.1.20c) **is in force.** As before, in this case we shall take an advantage of $\omega_a(\rho) = \rho^\alpha$ and $\omega_b(\rho) = \rho^\beta$ for some $\alpha, \beta \in (0,1]$. Then using (4.1.9) and applying (4.2.109), we have

$$\begin{aligned}
I_a &= 2[a]_\alpha r^\alpha \fint_{B_r} H_a(|Dw|)\, dx \\
&\leqslant c r^\alpha \fint_{B_r} \left[G(|Dw|) + [G(|Dw|)]^{1+\frac{\alpha}{n}}\right] dx \\
&\leqslant c r^\alpha \fint_{B_r} G(|Dw|)\, dx + c r^\alpha \left(\fint_{B_{4r}} \Psi(x, |Dw|)\, dx\right)^{\frac{1+\alpha}{n}} \\
&\leqslant c \left(r^\alpha + r^{\frac{\alpha\delta_0}{1+\delta_0}} \left(\int_{B_{4r}} [\Psi(x, |Dw|)]^{1+\delta_0}\, dx\right)^{\frac{1}{1+\delta_0}}\right) \fint_{B_{4r}} \Psi(x, |Dw|)\, dx \\
&\leqslant c(\mathbf{data}_b(\Omega_0)) r^{\frac{\alpha\delta_0}{1+\delta_0}} \fint_{B_{4r}} \Psi(x, |Dw|)\, dx, \qquad (4.2.127)
\end{aligned}$$

where in the last two inequalities of the last display we have used Proposition

4.2.3 together with Theorem 4.2.1. In the same way, we find

$$I_b \leqslant c(\mathbf{data}_b(\Omega_0)) r^{\frac{\beta\delta_0}{1+\delta_0}} \fint_{B_{4r}} \Psi(x,|Dw|)\,dx. \tag{4.2.128}$$

Again plugging the content of the last two displays into (4.2.116), we arrive at the estimate (4.2.110) when the condition (4.1.20c) is assumed. □

4.3 Proof of Theorem 4.1.2

Basically, the structure of the proof is similar as the proof of [14, Theorem 2.1] or [15, Theorem 1.1], which is initially introduced in [3, 59]. The proof of Theorem 4.1.2 consists of several steps.

Step 1: Exit time and covering of the level sets. This step is essentially classical and we provide it for the completeness. Let $B_R \equiv B_R(x_0) \subset \Omega_0 \Subset \Omega$ be a fixed ball with $R \leqslant R_0$. The size of R_0 will be determined by the end of the proof. Now consider radii $R/2 \leqslant R_1 < R_2 \leqslant R$ and consider the level sets

$$E^s_\lambda := \{x \in B_s(x_0) : \Psi(x,|Du|) > \lambda\} \quad \text{for every} \quad R/2 \leqslant s \leqslant R \quad \text{and} \quad \lambda > 0. \tag{4.3.1}$$

Let us consider the map defined by

$$T(B_r(y)) := \fint_{B_r(y)} [\Psi(x,|Du|) + M\Psi(x,|F|)]\,dx \tag{4.3.2}$$

for every ball $B_r(y) \subset B_R$ and some $M \geqslant 1$ to be determined later. Then it's clear that

$$\lim_{r\to 0^+} T(B_r(y)) > \lambda \quad \text{for a.e.} \quad y \in E^s_\lambda, \quad R/2 \leqslant s \leqslant R. \tag{4.3.3}$$

If $y \in B_{R_1}$ and $r \in \left[\frac{R_2 - R_1}{80}, R_2 - R_1\right]$, then we see

$$T(B_r(y)) \leqslant \frac{80^n R_2^n}{(R_2-R_1)^n} \fint_{B_{R_2}} [\Psi(x,|Du|) + M\Psi(x,|F|)]\,dx := \lambda_0. \tag{4.3.4}$$

Taking into account (4.3.3) and (4.3.4), for the values of $\lambda > \lambda_0$ and for almost every $y \in E_\lambda^{R_1}$, there exists a radius $r_y < \dfrac{R_2 - R_1}{80}$ such that

$$T(B_{r_y}(y)) = \lambda \quad \text{and} \quad T(B_r(y)) < \lambda \text{ for every } r \in (r_y, R_2 - R_1]. \tag{4.3.5}$$

The last display implies that the family $\left\{B_{r_y(y)}\right\}$ covers $E_\lambda^{R_1}$ up to a negligible set, and then applying Vitali's covering theorem, there exists a countable family of mutually disjoint balls $\left\{B_{r_{y_k}(y_k)}\right\}_{k=1}^{\infty} \equiv \{\tilde{B}_k\}_{k=1}^{\infty}$ such that

$$E_\lambda^{R_1} \subset \bigcup_{k=1}^{\infty} 5\bar{B}_k \tag{4.3.6}$$

and

$$T(B_{r_{y_k}}(y_k)) = \lambda \quad \text{and} \quad T(B_r(y_k)) < \lambda \text{ for every } r \in (r_{y_k}, R_2 - R_1] \quad (\forall k \in \mathbb{N}). \tag{4.3.7}$$

In the rest of the proof, we shall denote

$$B_k \equiv 5B_{r_{y_k}}(y_k) \quad \text{and} \quad r_k = 5r_{y_k}. \tag{4.3.8}$$

By this construction, we here notice that

$$80\tilde{B}_k = 16B_k \subset B_{R_2}, \quad r_k = 5r_{y_k} \leqslant \frac{R_2 - R_1}{16} \tag{4.3.9}$$

and that

$$\begin{cases} T(\tilde{B}_k) = \displaystyle\fint_{\tilde{B}_k} [\Psi(x,|Du|) + M\Psi(x,|F|)]\,dx = \lambda \\ T(16B_k) = \displaystyle\fint_{16B_k} [\Psi(x,|Du|) + M\Psi(x,|F|)]\,dx \leqslant \lambda \end{cases} \tag{4.3.10}$$

Step 2: Comparison estimates. Let us start with the following Dirichlet boundary value problem

$$\begin{cases} -\operatorname{div} A(x, Dw_k) = 0 \text{ in } 16B_k, \\ w_k \in u + W_0^{1,\Psi}(16B_k). \end{cases} \tag{4.3.11}$$

Using the arguments employed in the proof of [15, Theorem 1.1], we discover that, for every $\theta \in (0,1)$, there exists a constant $c \equiv c(n, s(\Psi), \nu, L, \theta)$ such that

$$\fint_{16B_k} |V_\Psi(x, Dw_k) - V_\Psi(x, Du)|^2 \, dx \leqslant \theta \fint_{16B_k} \Psi(x, |Du|) \, dx + c_\theta \fint_{16B_k} \Psi(x, |F|) \, dx, \tag{4.3.12}$$

where the vector field V_Ψ has been defined in (2.1.8). At this moment applying Lemma 2.1.1 together with the last display, we can show that

$$\fint_{16B_k} \Psi(x, |Du - Dw_k|) \, dx \leqslant \theta \fint_{16B_k} \Psi(x, |Du|) \, dx + c_\theta \fint_{16B_k} \Psi(x, |F|) \, dx \tag{4.3.13}$$

holds for some constant $c_\theta \equiv c_\theta(n, s(\Psi), \nu, L, \theta)$, whenever $\varepsilon \in (0,1)$. Now let $x_{a_k}, x_{b_k} \in \overline{2B_k}$ be points such that $a(x_{a_k}) = \sup_{x \in 2B_k} a(x)$ and $b(x_{b_k}) = \sup_{x \in 2B_k} b(x)$. Then we consider the following Dirichlet problem

$$\begin{cases} -\operatorname{div} A_k(Dv_k) = 0 \text{ in } 2B_k, \\ v_k \in w_k + W_0^{1,\Psi^+_{2B_k}}(2B_k), \end{cases} \tag{4.3.14}$$

where

$$A_k(z) := A_G(z) + a(x_{a_k}) A_{H_a}(z) + b(x_{b_k}) A_{H_b}(z) \quad (\forall z \in \mathbb{R}^n \setminus \{0\}) \tag{4.3.15}$$

and

$$\Psi^+_{2B_k}(t) = G(t) + a(x_{a_k}) H_a(t) + b(x_{b_k}) H_b(t) \quad (\forall t \geqslant 0). \tag{4.3.16}$$

The existence of the weak solution $v_k \in W^{1,\Psi^+_{2B_k}}(2B_k)$ to (3.5.14) is ensured by Theorem 4.2.4 and Proposition 4.2.1. The weak formulation of the equations (4.3.11) and (4.3.14) can be written as

$$\fint_{2B_k} \langle A_k(Dv_k) - A_k(Dw_k), D\varphi\rangle \, dx = \fint_{2B_k} \langle A(x, Dw) - A_k(Dw_k), D\varphi\rangle \, dx \tag{4.3.17}$$

for every $\varphi \in C_0^\infty(2B_k)$. Taking into account Proposition 4.2.1 and Lemma 4.2.1, we find that the function $\varphi = v_k - w_k$ is admissible in (4.3.17). Therefore, using the structure assumption (4.1.17) and Young's type inequality of Lemma 2.1.4, we see

$$\begin{aligned}
&\fint_{2B_k} |V_{\Psi^+_{2B_k}}(Dv_k) - V_{\Psi^+_{2B_k}}(Dw_k)|^2 \, dx \\
&\leqslant c \fint_{2B_k} (a(x_{a_k}) - a(x)) \frac{H_a(|Dw_k|)}{|Dw_k|} |Dw_k - Dv_k| \, dx \\
&\quad + c \fint_{2B_k} (b(x_{b_k}) - b(x)) \frac{H_b(|Dw_k|)}{|Dw_k|} |Dw_k - Dv_k| \, dx \\
&\leqslant \tau_0 \fint_{2B_k} a(x_{a_k}) H_a(|Dw_k - Dv_k|) \, dx + \frac{c}{\tau_0^{s(H_a)}} \fint_{2B_k} (a(x_{a_k}) - a(x)) H_a(|Dw_k|) \, dx \\
&\quad + \tau_0 \fint_{2B_k} b(x_{b_k}) H_b(|Dw_k - Dv_k|) \, dx + \frac{c}{\tau_0^{s(H_b)}} \fint_{2B_k} (b(x_{b_k}) - b(x)) H_b(|Dw_k|) \, dx
\end{aligned} \tag{4.3.18}$$

for some constant $c \equiv c(s(G), s(H_a), s(H_b), \nu, L)$, whenever $\tau_0 \in (0, 1)$, where we have also used the fact that $a(x) \leqslant a(x_{a_k})$ and $b(x) \leqslant b(x_{b_k})$ for every $x \in 2B_k$. Applying Lemma 2.1.1 for Ψ_k defined in (4.3.16) together with the last display, we have

$$\begin{aligned}
&\fint_{2B_k} \Psi^+_{2B_k}(|Dw_k - Dv_k|) \, dx \\
&\leqslant \tau \fint_{2B_k} \Psi^+_{2B_k}(|Dw_k|) \, dx + \frac{c}{\tau} \fint_{2B_k} |V_{\Psi^+_{2B_k}}(Dw_k) - V_{\Psi^+_{2B_k}}(Dv_k)|^2 \, dx
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \tau \fint_{2B_k} \Psi^{+}_{2B_k}(|Dw_k|)\,dx + \frac{c_*}{\tau}\tau_0 \fint_{2B_k} \Psi^{+}_{2B_k}(|Dw_k - Dv_k|)\,dx \\
&\quad + \frac{c_*}{\tau\tau_0^{s(H_a)}} \fint_{2B_k} (a(x_{a_k}) - a(x))H_a(|Dw_k|)\,dx \\
&\quad + \frac{c_*}{\tau\tau_0^{s(H_b)}} \fint_{2B_k} (b(x_{b_k}) - a(x))H_b(|Dw_k|)\,dx
\end{aligned} \tag{4.3.19}
$$

for some constant $c_* \equiv c(s(G), s(H_a), s(H_b), \nu, L) \geqslant 1$, whenever $\tau, \tau_0 \in (0,1)$. Choosing $\tau_0 = \dfrac{\tau}{2c_*}$ and reabsorbing the terms in the last display, for every $\tau \in (0,1)$, we have

$$
\begin{aligned}
&\fint_{2B_k} \Psi^{+}_{2B_k}(|Dw_k - Dv_k|)\,dx \\
&\leqslant \tau \fint_{2B_k} \Psi^{+}_{2B_k}(|Dw_k|)\,dx + \frac{c}{\tau^{s(\Psi)+1}} \fint_{2B_k} |\Psi^{+}_{2B_k}(|Dw_k|) - \Psi(x,|Dw_k|)|\,dx \\
&\leqslant \tau \fint_{2B_k} \Psi(x,|Dw_k|)\,dx + \frac{c}{\tau^{s(\Psi)+1}} \fint_{2B_k} |\Psi^{+}_{2B_k}(|Dw_k|) - \Psi(x,|Dw_k|)|\,dx
\end{aligned} \tag{4.3.20}
$$

for some constant $c \equiv c(s(G), s(H_a), s(H_b), \nu, L)$, where $s(\Psi) = s(G) + s(H_a) + s(H_b)$ (see Remark 2.1.2). At this moment we apply (4.2.110) of Theorem 4.2.4 depending on which one of the assumptions (4.1.20a)-(4.1.20c) comes into play. In turn, we have

$$
\begin{aligned}
\fint_{2B_k} \Psi^{+}_{2B_k}(|Dw_k - Dv_k|)\,dx &\leqslant \tau \fint_{2B_k} \Psi(x,|Dw_k|)\,dx + c\frac{Q(\varepsilon,R)}{\tau^{s(\Psi)+1}} \fint_{16B_k} \Psi(x,|Dw_k|)\,dx \\
&\leqslant c\left(\tau + \frac{Q(\varepsilon,R)}{\tau^{s(\Psi)+1}}\right) \fint_{16B_k} \Psi(x,|Du|)\,dx
\end{aligned} \tag{4.3.21}
$$

for some constant $c \equiv c(\mathbf{data}_b(\Omega_0))$, where

$$Q(\varepsilon,R) := \begin{cases} \varepsilon + \omega_a(R)\left(1+\dfrac{1}{\omega_a(\mu_1(\varepsilon))}\right) + \omega_b(R)\left(1+\dfrac{1}{\omega_b(\mu_1(\varepsilon))}\right) \\ \text{if (4.1.20a) is assumed,} \\ \varepsilon + \omega_a(R)\left(1+\dfrac{1}{\omega_a(\mu_2(\varepsilon))}\right) + +\omega_b(R)\left(1+\dfrac{1}{\omega_b(\mu_2(\varepsilon))}\right) \\ \text{if (4.1.20b) is assumed,} \\ R^{\frac{\alpha\delta_0}{1+\delta_0}} + R^{\frac{\beta\delta_0}{1+\delta_0}} \\ \text{if (4.1.20c) is assumed} \end{cases} \tag{4.3.22}$$

for any $\varepsilon \in (0,1)$. The constants μ_1, μ_2 are determined by Theorem 4.2.4, and δ_0 is a higher integrability exponent coming from Proposition 4.2.3 and Theorem 4.2.1. Combining the estimates (4.3.13) and (4.3.21) and using (2.1.6), we have

$$\begin{aligned} &\fint_{2B_k} \Psi(x, |Du - Dv_k|)\,dx \\ &\leqslant 2^{s(\Psi)+1} \fint_{2B_k} \Psi(x, |Du - Dw_k|)\,dx + 2^{s(\Psi)+1} \fint_{2B_k} \Psi(x, |Dw_k - Dv_k|)\,dx \\ &\leqslant c_0\left(\theta + \tau + \frac{Q(\varepsilon,R)}{\tau^{s(\Psi)+1}}\right) \fint_{16B_k} \Psi(x,|Du|)\,dx + c_\theta \fint_{16B_k} \Psi(x,|F|)\,dx \end{aligned} \tag{4.3.23}$$

for some constants $c_0 \equiv c_0(\mathbf{data}_b(\Omega_0))$ and $c_\theta \equiv c_\theta(n, s(G), s(H_a), s(H_b), \nu, L, \theta)$, whenever $\theta, \tau, \varepsilon \in (0,1)$, where the function $Q(\varepsilon,R)$ has been defined in (4.3.22) depending on which one of the assumptions (4.1.20a)-(4.1.20c) is under consideration. We use the auxiliary notation

$$S(\theta,\tau,\varepsilon,R,M) := c_0\left(\theta + \tau + \frac{Q(\varepsilon,R)}{\tau^{s(\Psi)+1}}\right) + \frac{c_\theta}{M} \tag{4.3.24}$$

and then using (4.3.10) directly in (4.3.23) to discover the desired estimate

$$\fint_{2B_k} \Psi(x,|Du - Dv_k|)\,dx \leqslant S(\theta,\tau,\varepsilon,R,M)\lambda, \tag{4.3.25}$$

which is valid for all the balls B_k from the covering constructed in (4.3.8).

Step 3: A priori estimate for Dv_k. The energy estimates for v_k and w_k together with Theorem 4.2.4 imply that

$$\begin{aligned}\fint_{2B_k} \Psi^+_{2B_k}(|Dv_k|)\,dx &\leqslant c\fint_{2B_k} \Psi^+_{2B_k}(|Dw_k|)\,dx \leqslant c\fint_{16B_k} \Psi(x,|Dw_k|)\,dx \\ &\leqslant c\fint_{16B_k} \Psi(x,|Du|)\,dx \leqslant c\lambda \end{aligned} \tag{4.3.26}$$

for some constant $c \equiv c(\mathbf{data}_b(\Omega_0))$. Then we apply the classical result of [111, Theorem 1.2] together with the last display to have the Lipschitz estimate

$$\sup_{x\in B_k} \Psi(x,|Dv_k|) \leqslant \sup_{x\in B_k} \Psi^+_{2B_k}(|Dv_k(x)|) \leqslant c\fint_{2B_k} \Psi^+_{2B_k}(|Dv_k|)\,dx \leqslant c_l\lambda \tag{4.3.27}$$

with $c_l \equiv c_l(\mathbf{data}_b(\Omega_0))$.

Step 4: Estimates involving level sets. By using (2.1.6) and elementary calculations, we discover

$$\begin{aligned} &2^{1+s(\Psi)}c_l\lambda|B_k\cap\{\Psi(x,|Du|)>2^{2+s(\Psi)}c_l\lambda\}| + \frac{1}{2}\int_{B_k\cap\{\Psi(x,|Du|)>2^{2+s(\Psi)}c_l\lambda\}} \Psi(x,|Du|)\,dx \\ &\quad\leqslant \int_{B_k\cap\{\Psi(x,|Du|)>2^{2+s(\Psi)}c_l\lambda\}} \Psi(x,|Du|)\,dx \\ &\quad\leqslant 2^{1+s(\Psi)}\int_{B_k} \Psi(x,|Du-Dv_k|)\,dx \\ &\qquad + 2^{1+s(\Psi)}\int_{B_k\cap\{\Psi(x,|Du|)>2^{2+s(\Psi)}c_l\lambda\}} \Psi(x,|Dv_k|)\,dx \\ &\quad\leqslant 2^{1+s(\Psi)}\int_{B_k} \Psi(x,|Du-Dv_k|)\,dx \\ &\quad 2^{1+s(\Psi)}c_l\lambda|B_k\cap\{\Psi(x,|Du|)>2^{2+s(\Psi)}c_l\lambda\}|, \end{aligned} \tag{4.3.28}$$

where we have also used (4.3.27) to get the last estimate. Therefore, we have

$$\int\limits_{B_k\cap\{\Psi(x,|Du|)>2^{2+s(\Psi)}c_l\lambda\}} \Psi(x,|Du|)\,dx \leqslant 2^{2+s(\Psi)}|2B_k| \fint\limits_{2B_k} \Psi(x,|Du-Dv_k|)\,dx \tag{4.3.29}$$

Recalling (4.3.24) and (4.3.8) that $|2B_k| = 10^n|\tilde{B}_k|$, we get

$$\int\limits_{B_k\cap\{\Psi(x,|Du|)>2^{2+s(\Psi)}c_l\lambda\}} \Psi(x,|Du|)\,dx \leqslant 2^{2+s(\Psi)}10^n S(\theta,\tau,\varepsilon,R,M)\lambda|\tilde{B}_k|. \tag{4.3.30}$$

Recalling (4.3.10), we find that

$$|\tilde{B}_k| = \frac{1}{\lambda}\int\limits_{\tilde{B}_k} (\Psi(x,|Du|) + M\Psi(x,|F|))\,dx. \tag{4.3.31}$$

Next, we estimate

$$|\tilde{B}_k| \leqslant \frac{1}{\lambda} \int\limits_{\tilde{B}_k\cap\{\Psi(x,|Du|)>\frac{\lambda}{4}\}} \Psi(x,|Du|)\,dx + \frac{1}{\lambda} \int\limits_{\tilde{B}_k\cap\{\Psi(x,|F|)>\frac{\lambda}{4M}\}} M\Psi(x,|F|)\,dx + \frac{|\tilde{B}_k|}{2}, \tag{4.3.32}$$

and hence

$$|\tilde{B}_k| \leqslant \frac{2}{\lambda} \int\limits_{\tilde{B}_k\cap\{\Psi(x,|Du|)>\frac{\lambda}{4}\}} \Psi(x,|Du|)\,dx + \frac{2}{\lambda} \int\limits_{\tilde{B}_k\cap\{\Psi(x,|F|)>\frac{\lambda}{4M}\}} M\Psi(x,|F|)\,dx. \tag{4.3.33}$$

The last display in (4.3.30) yields

$$\begin{aligned}
&\int\limits_{B_k\cap\{\Psi(x,|Du|)>2^{2+s(\Psi)}c_l\lambda\}} \Psi(x,|Du|)\,dx \\
&\leqslant 2^{3+s(\Psi)}10^n S(\theta,\tau,\varepsilon,R,M) \int\limits_{\tilde{B}_k\cap\{\Psi(x,|Du|)>\frac{\lambda}{4}\}} \Psi(x,|Du|)\,dx
\end{aligned}$$

$$+ 2^{3+s(\Psi)} 10^n S(\theta,\tau,\varepsilon,R,M) \int\limits_{\tilde{B}_k \cap \{\Psi(x,|F|) > \frac{\lambda}{4M}\}} M\Psi(x,|F|)\,dx. \tag{4.3.34}$$

Since $\{B_k\}_{k=1}^{\infty}$ is a covering of $E_{\lambda}^{R_1}$ and $E_{2^{2+s(\Psi)}c_l\lambda}^{R_1} \subset E_{\lambda}^{R_1}$, summing up over the covering $\{B_k\}_{k=1}^{\infty}$, we find

$$\int\limits_{E_{2^{2+s(\Psi)}c_l\lambda}^{R_1}} \Psi(x,|Du|)\,dx \leqslant \sum_{k=1}^{\infty} \int\limits_{B_k \cap \{\Psi(x,|Du|) > 2^{2+s(\Psi)}c_l\lambda\}} \Psi(x,|Du|)\,dx. \tag{4.3.35}$$

Before going on, let us introduce the short notation

$$D_{\lambda}^{s} := \{x \in B_s(x_0) : \Psi(x,|F(x)|) > \lambda\}, \qquad R/2 \leqslant s \leqslant R, \qquad \lambda > 0. \tag{4.3.36}$$

Then recalling that the balls $\{\tilde{B}_k\}$ are disjoint and using (4.3.9), we sum up (4.3.34) over indices k to have

$$\begin{aligned}
&\int\limits_{E_{\lambda}^{R_1}} \Psi(x,|Du|)\,dx \\
&\leqslant 2^{3+s(\Psi)} 10^n S(\theta,\tau,\varepsilon,R,M) \int\limits_{E^{R_2}_{\frac{\lambda}{2^{4+s(\Psi)}c_l}}} \Psi(x,|Du|)\,dx \\
&\quad + 2^{3+s(\Psi)} 10^n S(\theta,\tau,\varepsilon,R,M) \int\limits_{D^{R_2}_{\frac{\lambda}{2^{3+s(\Psi)}c_l M}}} M\Psi(x,|F|)\,dx
\end{aligned} \tag{4.3.37}$$

for all $\lambda > 0$ such that

$$\lambda \geqslant \lambda_1 := 2^{2+s(\Psi)} c_l \lambda_0 = \frac{2^{3+s(\Psi)} c_l 80^n R_2^n}{(R_2 - R_1)^n} \fint_{B_{R_2}} [\Psi(x,|Du|) + M\Psi(x,|F|)]\,dx.$$

Step 5: Conclusion. Let us define the truncated functions

$$[\Psi(x,|Du|)]_t := \min\{\Psi(x,|Du|), t\} \quad \text{for} \quad t \geqslant 0. \tag{4.3.38}$$

Then for $t \geqslant 2^{4+s(\Psi)}c_l\lambda_0$, we multiply the inequality (4.3.37) by $\Upsilon''(\lambda)$ which is positive since $\Upsilon \in \mathcal{N}$, and then integrate over λ to obtain

$$\begin{aligned}
&\int_{2^{2+s(\Psi)}c_l\lambda_0}^{t} \Upsilon''(\lambda) \int_{E_\lambda^{R_1}} \Psi(x,|Du|)\,dx\,d\lambda \\
&\leqslant 2^{3+s(\Psi)}10^n S(\theta,\tau,\varepsilon,R,M) \int_{2^{2+s(\Psi)}c_l\lambda_0}^{t} \Upsilon''(\lambda) \int_{E^{R_2}_{\frac{\lambda}{2^{4+s(\Psi)}c_l}}} \Psi(x,|Du|)\,dx\,d\lambda \\
&\quad + 2^{3+s(\Psi)}10^n S(\theta,\tau,\varepsilon,R,M) \int_{2^{2+s(\Psi)}c_l\lambda_0}^{t} \Upsilon''(\lambda) \int_{D^{R_2}_{\frac{\lambda}{2^{4+s(\Psi)}c_l M}}} M\Psi(x,|F|)\,dx\,d\lambda.
\end{aligned} \tag{4.3.39}$$

Fubini's theorem to the term on the left hand side of the last display yields

$$\begin{aligned}
\int_{2^{2+s(\Psi)}c_l\lambda_0}^{t} \Upsilon''(\lambda) \int_{E_\lambda^{R_1}} \Psi(x,|Du|)\,dx\,d\lambda &= \int_{B_{R_1}} \Upsilon'\left([\Psi(x,|Du|)]_t\right)\Psi(x,|Du|)\,dx \\
&\quad - \int_0^{2^{2+s(\Psi)}c_l\lambda_0} \Upsilon''(\lambda) \int_{E_\lambda^{R_1}} \Psi(x,|Du|)\,dx\,d\lambda.
\end{aligned} \tag{4.3.40}$$

Using the fact $\Upsilon'(0)=0$ and Fubini's theorem, we have

$$\begin{aligned}
&\int_{B_{R_1}} \Upsilon'\left([\Psi(x,|Du|)]_t\right)\Psi(x,|Du|)\,dx = \int_{B_{R_1}} \int_0^{\Psi(x,|Du|)} \left[\Upsilon'\left(\min\{\lambda,t\}\right)\right]' \Psi(x,|Du|)\,d\lambda\,dx \\
&= \int_{B_{r_1}} \int_0^{\infty} |\{(x,\lambda)\in B_{R_1}\times(0,\infty) : \Psi(x,|Du|)>\lambda\}| \left[\Upsilon'\left(\min\{\lambda,t\}\right)\right]' \Psi(x,|Du|)\,d\lambda\,dx
\end{aligned}$$

$$= \int_0^t \Upsilon''(\lambda) \int_{E_\lambda^{R_1}} \Psi(x, |Du|)\, dx\, d\lambda.$$

Recalling the definition of λ_0 in (4.3.4), we estimate the last integral in (4.3.40) as follows:

$$\begin{aligned}
&\int_0^{2^{2+s(\Psi)}c_l\lambda_0} \Upsilon''(\lambda) \int_{E_\lambda^{R_1}} \Psi(x, |Du|)\, dx\, d\lambda \\
&\leqslant \int_0^{2^{2+s(\Psi)}c_l\lambda_0} \Upsilon''(\lambda)\, d\lambda \int_{B_{R_1}} \Psi(x, |Du|)\, dx \\
&\leqslant \Upsilon'\left(2^{2+s(\Psi)}c_l\lambda_0\right) \int_{B_{r_2}} \Psi(x, |Du|)\, dx \\
&\leqslant (s(\Upsilon)+1)\frac{\Upsilon\left(2^{2+s(\Psi)}c_l\lambda_0\right)}{2^{2+s(\Psi)}c_l\lambda_0}\lambda_0|B_{r_2}| \\
&\leqslant (s(\Upsilon)+1)(2^{2+s(\Psi)}c_l)^{s(\Upsilon)}\Upsilon(\lambda_0)|B_{r_2}|, \qquad (4.3.41)
\end{aligned}$$

where we have used $2^{2+s(\Psi)}c_l \geqslant 1$.

Now we treat the remaining terms in (4.3.39) similarly. By changing variables we have

$$\begin{aligned}
&\int_{2^{2+s(\Psi)}c_l\lambda_0}^{t} \Upsilon''(\lambda) \int_{E^{R_2}_{\frac{\lambda}{2^{4+s(\Psi)}c_l}}} \Psi(x, |Du|)\, dx\, d\lambda \\
&\qquad \leqslant 2^{4+s(\Psi)}c_l \int_0^{\frac{t}{2^{4+s(\Psi)}c_l}} \Upsilon''\left(2^{4+s(\Psi)}c_l\lambda\right) \int_{E_\lambda^{R_2}} \Psi(x, |Du|)\, dx\, d\lambda \\
&\qquad \leqslant [s(\Upsilon)]^3(2^{4+s(\Psi)}c_l)^{s(\Upsilon)} \int_0^{\frac{t}{2^{4+s(\Psi)}c_l}} \Upsilon''(\lambda) \int_{E_\lambda^{R_2}} \Psi(x, Du)\, dx\, d\lambda
\end{aligned}$$

$$\leqslant [s(\Upsilon)]^3 (2^{4+s(\Psi)} c_l)^{s(\Upsilon)} \int_{B_{R_2}} \Upsilon' \left([\Psi(x,|Du|)]_{\frac{t}{2^{4+s(\Psi)} c_l}} \right) \Psi(x,|Du|)\, dx$$
$$\leqslant [s(\Upsilon)]^3 (2^{4+s(\Psi)} c_l)^{s(\Upsilon)} \int_{B_{R_2}} \Upsilon' \left([\Psi(x,|Du|)]_t \right) \Psi(x,|Du|)\, dx, \tag{4.3.42}$$

where in the last inequality we have used the trivial fact that

$$\Upsilon' \left([\Psi(x,|Du|)]_{\frac{t}{2^{4+s(\Psi)} c_l}} \right) \leqslant \Upsilon' \left([\Psi(x,|Du|)]_t \right)$$

holds, whenever $2^{4+s(\Psi)} c_l \geqslant 1$. Arguing as for (4.3.42), we use Fubini's theorem again for the last term in (4.3.39) to get

$$\int_{2^{2+s(\Psi)} c_l \lambda_0}^{t} \Upsilon''(\lambda) \int_{D^{R_2}_{\frac{\lambda}{2^{4+s(\Psi)} c_l M}}} \Psi(x,|F|)\, dx\, d\lambda$$
$$\leqslant \int_0^\infty \Upsilon''(\lambda) \int_{D^{r_2}_{\frac{\lambda}{2^{4+s(\Psi)} c_l M}}} \Psi(x,|F|)\, dx\, d\lambda$$
$$\leqslant [s(\Upsilon)]^3 (2^{4+s(\Psi)} c_l M)^{s(\Upsilon)} \int_{B_{R_2}} \Upsilon\left(\Psi(x,|F|) \right)\, dx. \tag{4.3.43}$$

Putting the estimates in (4.3.41)-(4.3.43) into (4.3.39) and manipulating the terms in a standard way, we deduce that

$$\fint_{B_{R_1}} \Upsilon' \left([\Psi(x,|Du|)]_t \right) \Psi(x,|Du|)\, dx$$
$$\leqslant c_f^{s(\Upsilon)+1} c_l^{s(\Upsilon)} S(\theta,\tau,\varepsilon,R,M) \fint_{B_{R_2}} \Upsilon' \left([\Psi(x,|Du|)]_t \right) \Psi(x,|Du|)\, dx$$
$$+ c_f^{s(\Upsilon)+1} c_l^{s(\Upsilon)} M^{s(\Upsilon)} S(\theta,\tau,\varepsilon,R,M) \fint_{B_{R_2}} \Upsilon\left([\Psi(x,|F|)] \right)\, dx$$

$$+ c_f^{s(\Upsilon)+1} c_l^{s(\Upsilon)} \Upsilon(\lambda_0), \tag{4.3.44}$$

for every $t \geqslant 2^{4+s(\Psi)} c_l \lambda_0$, where $c_l \equiv c_l(\mathbf{data}_b(\Omega_0))$ was given in (4.3.27) and $c_f \equiv c_f(n, s(\Psi))$. This last inequality holds for any $M \geqslant 1$, $R \leqslant 1$ and $\theta, \tau, \varepsilon \in (0,1)$. Now we choose those constants in a such way to satisfy

$$c_f^{s(\Upsilon)+1} c_l^{s(\Upsilon)} S(\theta, \tau, \varepsilon, R, M) \leqslant \frac{1}{2}, \tag{4.3.45}$$

where the quantity $S(\theta, \tau, \varepsilon, R, M)$ has been defined in (4.3.24). Indeed, we take $\theta, \tau \equiv \theta, \tau(\mathbf{data}_b(\Omega_0), s(\Upsilon)) \in (0,1)$ such that

$$\theta = \tau := \frac{1}{8 c_0 c_f^{s(\Upsilon)+1} c_l^{s(\Upsilon)+1}}. \tag{4.3.46}$$

Since θ is a fixed constant depending on $\mathbf{data}_b(\Omega_0)$ and $s(\Upsilon)$, we select $M \geqslant 1$ satisfying

$$c_0 c_f^{s(\Upsilon)+1} c_l^{s(\Upsilon)} \frac{c_\theta}{M} \leqslant \frac{1}{8}. \tag{4.3.47}$$

Finally, we choose ε, R small enough depending on $\mathbf{data}_b(\Omega_0)$ and $s(\Upsilon)$ such that

$$c_0 c_f^{s(\Upsilon)+1} c_l^{s(\Upsilon)} \frac{Q(\varepsilon, R)}{\tau^{s(\Psi)+1}} \leqslant \frac{1}{8}. \tag{4.3.48}$$

All the above choices of constants as in (4.3.46)-(4.3.48) ensure that (4.3.45) holds. Inserting these choices of constants in (4.3.46)-(4.3.48) into (4.3.44) and using the definition of λ_0 in (4.3.4), we conclude that

$$\begin{aligned} &\fint_{B_{R_1}} \Upsilon'\left([\Psi(x,|Du|)]_t\right) \Psi(x,|Du|)\, dx \\ &\qquad \leqslant \frac{1}{2} \fint_{B_{R_2}} \Upsilon'\left([\Psi(x,|Du|)]_t\right) \Psi(x,|Du|)\, dx + c \fint_{B_R} \Upsilon\left([\Psi(x,|F|)]\right)\, dx \end{aligned}$$

$$+\frac{cR^{n(s(\Upsilon)+1)}}{(R_2-R_1)^{n(s(\Upsilon)+1)}}\Upsilon\left(\fint_{B_R}[\Psi(x,|Du|)+M\Psi(x,|F|)]\,dx\right) \tag{4.3.49}$$

for some constants $c \equiv c(\mathbf{data}_b(\Omega_0), s(\Upsilon))$ and $M \equiv M(\mathbf{data}_b(\Omega_0), s(\Upsilon))$. Notice that $t \geqslant 2^{4+s(\Psi)}c_l\lambda_0$ and all constants appearing in the above estimates are independent of t. We now apply Lemma 2.2.1 for a function

$$h(s) := \fint_{B_s}\Upsilon'\left([\Psi(x,|Du|)]_t\right)\Psi(x,|Du|)\,dx$$

with $\gamma_1 \equiv n(s(\Upsilon)+1)$ and $\gamma_2 \equiv 0$, which is a non-negative and bounded on $[R/2, R]$, to discover the following estimate:

$$\fint_{B_{R/2}}\Upsilon'\left([\Psi(x,|Du|)]_t\right)\Psi(x,|Du|)\,dx \leqslant c\Upsilon\left(\fint_{B_R}[\Psi(x,|Du|)+M\Psi(x,|F|)]\,dx\right) + c\fint_{B_R}\Upsilon\left(\Psi(x,|F|)\right)\,dx$$

with $c \equiv c(\mathbf{data}_b(\Omega_0), s(\Upsilon))$. After some manipulations including Jensen's inequality, we conclude that

$$\fint_{B_{R/2}}\Upsilon'\left([\Psi(x,|Du|)]_t\right)\Psi(x,|Du|)\,dx \leqslant c\Upsilon\left(\fint_{B_R}\Psi(x,|Du|)\,dx\right) + c\fint_{B_R}\Upsilon\left(\Psi(x,|F|)\right)\,dx \tag{4.3.50}$$

with $c \equiv c(\mathbf{data}_b(\Omega_0), s(\Upsilon))$. Letting $t \to \infty$ in the last display, we conclude

$$\fint_{B_{R/2}}\Upsilon\left(\Psi(x,|Du|)\right)\,dx \leqslant c\Upsilon\left(\fint_{B_R}\Psi(x,|Du|)\,dx\right) + c\fint_{B_R}\Upsilon\left(\Psi(x,|F|)\right)\,dx \tag{4.3.51}$$

for all $R \leqslant R_0$ with $R_0 \equiv R_0(\mathbf{data}_b(\Omega_0), s(\Upsilon))$, where $c \equiv c(\mathbf{data}_b(\Omega_0), s(\Upsilon))$. This proves (4.1.22), and then (4.1.21) follows by a standard covering argument. This completes the proof.

4.4 Proof of Theorem 4.1.1

The proof of Theorem 4.1.1 can be done by following the proof of Theorem 4.1.2. A main difficulty lies in obtaining results of Lemma (4.2.4) for the weak solution $w \in W^{1,\Psi}(B_R)$ of the equation (4.2.31), where the vector field $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is not only in the form of (4.2.32). Essentially, the vector field $A(\cdot)$ is of general type satisfying (4.1.2) under the Orlicz double phase settings ($b(\cdot) \equiv 0$). All the results of Section 4.2 can be restated and proved for the weak solution w except the result of Lemma 4.2.4. For the completeness, we provide the proof of Lemma 4.2.4 under the Orlicz double phase settings.

Lemma 4.4.1. *Let Ψ is given as in* (1.0.2) *with $b(\cdot) \equiv 0$, $G, H_a \in \mathcal{N}$ in the sense of Definition 2.1.1, $0 \leqslant a(\cdot) \in C^{\omega_a}(\Omega)$ for some continuous and concave function $\omega_a : [0,\infty) \to [0,\infty)$ with $\omega_a(0) = 0$. Suppose $w \in W^{1,\Psi}(B_R)$ be the weak solution to* (4.2.31) *under the assumption* (4.1.2). *Let $B_r \equiv B_r(y) \subset B_R$ be any fixed ball. If one of the assumptions* (4.1.12a)-(4.1.12c) *is satisfied, then for every $\varepsilon^* \in (0,1)$, there exists a positive radius*

$$r^* \equiv r^*(\boldsymbol{data}_{db}(\Omega_0), \varepsilon^*) \tag{4.4.1}$$

such that

$$\begin{aligned} &\fint_{B_{\lambda r}} \Psi^-_{B_r}\left(\left|\frac{w-(w)_{B_{\lambda r}}}{\lambda r}\right|\right) dx \\ &\leqslant c\left(1+\lambda^{-(n+s(\Psi)+1)}\varepsilon^*\right) \fint_{B_{r/2}} \Psi^-_{B_r}\left(\left|\frac{w-(w)_{B_{r/2}}}{r}\right|\right) dx \end{aligned} \tag{4.4.2}$$

holds for some constant $c \equiv c(\boldsymbol{data}_{db}(\Omega_0))$, whenever $\lambda \in (0, 1/16)$ and $r \leqslant r^*$.

Proof. Now we shall revisit the proof of Lemma 4.2.4 and keep the same notations and steps employed there. In (4.2.58), we consider the vector field and frozen functional by

$$\bar{A}_0(z) := \frac{\bar{A}(\bar{x}_a, E(r)z)}{\Psi^{-}_{B_r}(E(r))} \text{and} \quad \bar{\Psi}_0(t) := \bar{G}(t) + \bar{a}(\bar{x}_a)\bar{H}_a(t) \quad (z \in \mathbb{R}^n,\, t \geqslant 0), \tag{4.4.3}$$

where $\bar{A}(\cdot)$, $\bar{G}$, $\bar{H}_a$, $\bar{a}(\cdot)$ are defined as in (4.2.51)-(4.2.53) under the Orlicz double phase settings. Using $(4.1.2)_3$, for every $\varphi \in W_0^{1,\infty}(B_{1/8})$, we have

$$\begin{aligned}
\left| \fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}), D\varphi \right\rangle \, dx \right| &= \left| \fint_{B_{1/8}} \left\langle \bar{A}_0(D\bar{w}) - \bar{A}(x, D\bar{w}), D\varphi \right\rangle \, dx \right| \\
&\leqslant \fint_{B_{1/8}} |\bar{A}_0(D\bar{w}) - \bar{A}(x, D\bar{w})| \, dx \, \|D\varphi\|_{L^\infty(B_{1/8})} \\
&\leqslant L \fint_{B_{1/8}} |\bar{a}(x) - \bar{a}(\bar{x}_a)| \frac{\bar{H}_a(|D\bar{w}|)}{|D\bar{w}|} \, dx \, \|D\varphi\|_{L^\infty(B_{1/8})} \\
&=: I_1 \, \|D\varphi\|_{L^\infty(B_{1/8})} \, ,
\end{aligned} \tag{4.4.4}$$

which is a key difference comparing with the estimate (4.2.60) in the proof of Lemma 4.2.4, where we only use $(4.1.2)_3$ without setting the form as in (4.2.32). The remaining part can be argued by following the remainder of the proof of Lemma 4.2.4 with $b(\cdot) \equiv 0$. □

Chapter 5

Regularity for degenerate/singular fully nonlinear elliptic equations

5.1 Hypotheses and Main results

We consider viscosity solutions to fully nonlinear elliptic equations of the form

$$\Phi(x, |Du|)F(D^2u) = f(x) \quad \text{in} \quad B_1, \tag{5.1.1}$$

where $B_1 \equiv B_1(0) \subset \mathbb{R}^n$ with $n \geqslant 2$ is the unit ball, $F : \mathcal{S}(n) \to \mathbb{R}$ is a uniformly (λ, Λ)-elliptic operator in the sense of (A1) and $\Phi : B_1 \times [0, \infty) \to [0, \infty)$ is a continuous map featuring a degeneracy and singularity for the gradient described as in (A2). Let us recall main assumptions for the problem (5.1.1) in this chapter for the simplicity of writing as we have introduced in the introduction part:

(A1) The operator $F : \mathcal{S}(n) \to \mathbb{R}$ in (5.1.1) is continuous and uniformly (λ, Λ)-elliptic in the sense that

$$\lambda \mathrm{tr}(N) \leqslant F(M) - F(M + N) \leqslant \Lambda \mathrm{tr}(N)$$

holds with some constants $0 < \lambda \leqslant \Lambda$ and $F(0) = 0$, whenever $M, N \in \mathcal{S}(n)$ with $N \geqslant 0$, where we denote by $\mathcal{S}(n)$ to mean the set of $n \times n$ real symmetric matrices.

(A2) $\Phi : B_1 \times [0, \infty) \to [0, \infty)$ is a continuous map satisfying the following properties:

1. There exist constants $d(\Phi) \geqslant i(\Phi) > -1$ such that the map $t \mapsto \dfrac{\Phi(x,t)}{t^{i(\Phi)}}$ is almost non-decreasing with constant $L \geqslant 1$ in $(0, \infty)$ and the map $t \mapsto \dfrac{\Phi(x,t)}{t^{d(\Phi)}}$ is almost non-increasing with constant $L \geqslant 1$ in $(0, \infty)$ for all $x \in B_1$.
2. There exist constants $0 < \nu_0 \leqslant \nu_1$ such that $\nu_0 \leqslant \Phi(x, 1) \leqslant \nu_1$ for all $x \in B_1$.

(A3) The term f on the right hand side of (5.1.1) belongs to $C(B_1) \cap L^\infty(B_1)$.

The Pucci extremal operators $P^{\pm}_{\lambda,\Lambda} : \mathcal{S}(n) \to \mathbb{R}$ are defined as

$$P^{+}_{\lambda,\Lambda}(M) := -\lambda \sum_{\lambda_k > 0} \lambda_k - \Lambda \sum_{\lambda_k < 0} \lambda_k$$

and

$$P^{-}_{\lambda,\Lambda}(M) := -\Lambda \sum_{\lambda_k > 0} \lambda_k - \lambda \sum_{\lambda_k < 0} \lambda_k,$$

where $\{\lambda_k\}_{k=1}^{n}$ are the eigenvalues of the matrix M. The (λ, Λ)-ellipticity of the operator F via the Pucci extremal operators can be formulated as

$$P^{-}_{\lambda,\Lambda}(N) \leqslant F(M + N) - F(M) \leqslant P^{+}_{\lambda,\Lambda}(N)$$

for all $M, N \in \mathcal{S}(n)$.

In what follows, for any vector $\xi \in \mathbb{R}^n$, we define a map $G_\xi : B_1 \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ by

$$G_\xi(x, p, M) := \Phi(x, |\xi + p|) F(M) - f(x) \tag{5.1.2}$$

under the assumptions prescribed in (A1)-(A3). Then we shall focus on viscosity solutions of the equation

$$G_\xi(x, Du, D^2 u) = 0 \text{ in } B_1. \tag{5.1.3}$$

Now we give the definition of a viscosity solution u of the equation (5.1.3) as follows.

Definition 5.1.1. A lower semicontinuous function v is called a viscosity supersolution of (5.1.3) if for all $x_0 \in B_1$ and $\varphi \in C^2(B_1)$ such that $v - \varphi$ has a local minimum at x_0, then

$$G_\xi(x_0, D\varphi(x_0), D^2\varphi(x_0)) \geqslant 0.$$

An upper semicontinuous function w is called is a viscosity subsolution of (5.1.3) if for all $x_0 \in B_1$ and $\varphi \in C^2(B_1)$ such that $w - \varphi$ has a local maximum at x_0, there holds

$$G_\xi(x_0, D\varphi(x_0), D^2\varphi(x_0)) \leqslant 0.$$

We say that $u \in C(B_1)$ is a viscosity solution of (5.1.3) if u is a viscosity supersolution and a subsolution simultaneously.

Also we recall a concept of superjet and subjet introduced in [60].

Definition 5.1.2. Let $v : B_1 \to \mathbb{R}$ be an upper semicontinuous function and $w : B_1 \to \mathbb{R}$ be a lower semicontinuous function.

1. A couple $(p, M) \in \mathbb{R}^n \times \mathcal{S}(n)$ is a superjet of v at $x \in B_1$ if

$$v(x + y) \leqslant v(x) + \langle p, y\rangle + \frac{1}{2}\langle My, y\rangle + O(|y|^2).$$

2. A couple $(p, M) \in \mathbb{R}^n \times \mathcal{S}(n)$ is a subjet of w at $x \in B_1$ if

$$w(x + y) \geqslant w(x) + \langle p, y\rangle + \frac{1}{2}\langle My, y\rangle + O(|y|^2).$$

3. A couple $(p, M) \in \mathbb{R}^n \times \mathcal{S}(n)$ is a limiting superjet of v at $x \in B_1$ if there exists a sequence $\{x_k, p_k, M_k\} \to \{x, p, M\}$ as $k \to \infty$ in a such way that $\{p_k, M_k\}$ is a superjet of v at x_k and $\lim\limits_{k\to\infty} v(x_k) = v(x)$.

4. A couple $(p, M) \in \mathbb{R}^n \times \mathcal{S}(n)$ is a limiting subjet of w at $x \in B_1$ if there exists a sequence $\{x_k, p_k, M_k\} \to \{x, p, M\}$ as $k \to \infty$ in such a way that $\{p_k, M_k\}$ is a subjet of v at x_k and $\lim\limits_{k\to\infty} w(x_k) = w(x)$.

Finally, let us recall a consequence of the classical Krylov-Safonov Harnack inequality, see [46], that viscosity solutions to the homogeneous equation

$$F(D^2h) = 0 \quad \text{in} \quad B_1, \tag{5.1.4}$$

under the assumption that $F : \mathcal{S}(n) \to \mathbb{R}$ satisfies (A1), are locally of class $C^{1,\bar{\alpha}}(B_1)$ for a universal constant $\bar{\alpha} \equiv \bar{\alpha}(n, \lambda, \Lambda) \in (0, 1)$ with the estimate

$$\|h\|_{C^{1,\bar{\alpha}}(B_{1/2})} \leqslant c\,\|h\|_{L^\infty(B_1)} \tag{5.1.5}$$

for some constant $c \equiv c(n, \lambda, \Lambda)$. The main results of this chapter read as follows.

Theorem 5.1.1 (Hölder continuity of the gradient)**.** *Let $u \in C(B_1)$ be a viscosity solution of* (5.1.1) *under the assumptions (A1)-(A3). Then $u \in C^{1,\beta}_{\mathrm{loc}}(B_1)$ for all $\beta > 0$ satisfying*

$$\beta < \begin{cases} \min\left\{\bar{\alpha}, \dfrac{1}{1+d(\Phi)}\right\} & \text{if } i(\Phi) \geqslant 0, \\ \min\left\{\bar{\alpha}, \dfrac{1}{1+d(\Phi)-i(\Phi)}\right\} & \text{if } -1 < i(\Phi) < 0, \end{cases} \tag{5.1.6}$$

where $\bar{\alpha}$ is given in (5.1.5)*. Moreover, for every β in* (5.1.6)*, there exists a constant $c \equiv c(n, \lambda, \Lambda, i(\Phi), L, \beta)$ such that*

$$\|u\|_{L^\infty(B_{1/2})} + \sup_{x\neq y \in B_{1/2}} \frac{|Du(x) - Du(y)|}{|x-y|^\beta} \leqslant c\left(1 + \|u\|_{L^\infty(B_1)} + \left\|\frac{f}{\nu_0}\right\|_{L^\infty(B_1)}^{\frac{1}{1+i(\Phi)}}\right). \tag{5.1.7}$$

5.2 Basic regularity results

5.2.1 Small regime

Here we verify that, for a viscosity solution u of (5.1.3), we are able to assume

$$\operatorname*{osc}_{B_1} u \leqslant 1 \quad \text{and} \quad \|f\|_{L^\infty(B_1)} \leqslant \varepsilon_0 \tag{5.2.1}$$

for some constant $0 < \varepsilon_0 < 1$ small enough, and also $\nu_0 = \nu_1 = 1$ without loss of generality. In order to consider the problem in a small regime as in

(5.2.1), for a fixed ball $B_R(x_0) \subset B_1$, we define $\bar{u} : B_1 \to \mathbb{R}$ by

$$\bar{u}(x) := \frac{u(x_0 + Rx)}{K} \tag{5.2.2}$$

for positive constants $K \geqslant 1 \geqslant R$ to be determined later. It can be seen that $\bar{u}$ is a viscosity solution of

$$\bar{G}_{\bar{\xi}}(x, D\bar{u}, D^2\bar{u}) := \bar{\Phi}(x, |\bar{\xi} + D\bar{u}|)\bar{F}(D^2\bar{u}) - \bar{f}(x) = 0, \tag{5.2.3}$$

where

$$\begin{aligned} \bar{\Phi}(x,t) &:= \frac{\Phi\left(x_0 + Rx, \frac{K}{R}t\right)}{\Phi\left(x_0 + Rx, \frac{K}{R}\right)}, \\ \bar{F}(M) &:= \frac{R^2}{K} F\left(\frac{K}{R^2}M\right), \\ \bar{f}(x) &:= \frac{R^2}{\Phi\left(x_0 + Rx, \frac{K}{R}\right) K} f(x_0 + Rx) \text{ and } \bar{\xi} := \frac{R}{K}\xi. \end{aligned}$$

Note that $\bar{F}$ is still a uniformly (λ, Λ)-elliptic operator, the map $t \mapsto \dfrac{\bar{\Phi}(x,t)}{t^{i(\Phi)}}$ is almost non-decreasing and the map $t \mapsto \dfrac{\bar{\Phi}(x,t)}{t^{d(\Phi)}}$ is almost non-increasing with the same constants $L \geqslant 1$ and $d(\Phi) \geqslant i(\Phi) > -1$ as in (A2), and $\bar{\Phi}(x, 1) = 1$ for all $x \in B_1$. Moreover, the assumption (A2) implies

$$\left\|\bar{f}\right\|_{L^\infty(B_1)} \leqslant \frac{LR^{2+i(\Phi)}}{\nu_0 K^{1+i(\Phi)}} \left\|f\right\|_{L^\infty(B_1)} \leqslant \frac{L}{\nu_0} \left\|f\right\|_{L^\infty(B_1)}.$$

By recalling $i(\Phi) > -1$ and setting

$$K := 2\left(1 + \|u\|_{L^\infty(B_1)} + \left[\frac{L}{\nu_0}\|f\|_{L^\infty(B_1)}\right]^{\frac{1}{1+i(\Phi)}}\right)$$

and

$$R := \varepsilon_0^{\frac{1}{2+i(\Phi)}},$$

we see that $\bar{u}$ solves the equation (5.2.3) in the same class as (5.1.3) under the small regime in (5.2.1).

5.2.2 Auxiliary tools

In this subsection, we state some basic regularity results for (5.1.3). The first key tool to be employed later is the classical Ishii-Lions lemma, see [60].

Lemma 5.2.1 (Ishii-Lions Lemma)**.** *Let u be a viscosity solution of* (5.1.3) *with $\operatorname{osc}_{B_1} u \leqslant 1$ and $\|f\|_{L^\infty(B_1)} \leqslant \varepsilon_0 \ll 1$ under the assumptions (A1)-(A3), where $\xi \in \mathbb{R}^n$ is any vector. Suppose that $\mathcal{B} \subset B_1$ is an open subset and $\psi \in C^2(\mathcal{B} \times \mathcal{B})$. Define a map $v : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ as*

$$v(x, y) := u(x) - u(y).$$

Suppose further $(\bar{x}, \bar{y}) \in \mathcal{B} \times \mathcal{B}$ is a local maximum point of $v - \psi$ in $\mathcal{B} \times \mathcal{B}$. Then, for each $\delta > 0$, there exist matrices $X_\delta, Y_\delta \in \mathcal{S}(n)$ such that

$$G_\xi(\bar{x}, D_x\psi(\bar{x}, \bar{y}), X_\delta) \leqslant 0 \leqslant G_\xi(\bar{y}, -D_y\psi(\bar{x}, \bar{y}), Y_\delta)$$

and

$$-\left(\frac{1}{\delta} + \|A\|\right) I \leqslant \begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix} \leqslant A + \delta A^2$$

with $A := D^2\psi(\bar{x}, \bar{y})$.

Another important result to be applied afterwards is the results of [101] in our settings.

Theorem 5.2.1 (Imbert-Silvestre)**.** *Let $u \in C(B_1)$ be a viscosity solution to* (5.1.3) *for some $\xi \in \mathbb{R}^n$. Suppose there exists $\gamma > 0$ such that*

1. *for all $(x, p) \in B_1 \times \mathbb{R}^n$ with $|p| > \gamma$, it holds that*

$$G_\xi(x, p, 0) \leqslant c_0|p|$$

   for some constant $c_0 > 0$ and

2. *for any fixed $(x, p) \in B_1 \times \mathbb{R}^n$ with $|p| > \gamma$, $G_\xi(x, p, M)$ is uniformly elliptic with respect to M.*

Then $u \in C^{0,\alpha}_{\mathrm{loc}}(B_1)$ for some $\alpha \in (0, 1)$. In particular, the following estimate

$$\|u\|_{C^{0,\alpha}(B_{1/2})} \leqslant c\,\|u\|_{L^\infty(B_1)}$$

holds true for some constant $c > 0$. The constants $\alpha \in (0,1)$ and $c > 0$ depending on n, the ellipticity constants and the parameter $\gamma > 0$.

5.3 Hölder continuity

In this section we provide Hölder regularity for solutions of (5.1.3), where ξ is any vector, under the small regime.

Lemma 5.3.1 (Hölder continuity)**.** *Let u be a viscosity solution of* (5.1.3) *under the assumptions (A1)-(A3) with $\operatorname{osc}_{B_1} u \leqslant 1$, $\|f\|_{L^\infty(B_1)} \leqslant \varepsilon_0 < 1$ and $\nu_0 = \nu_1 = 1$. Let $B_R \equiv B_R(x_0) \subset B_1$ be any ball. Then, we have the following regularity results:*

(R1) *If $-1 < i(\Phi) < 0$ and $|\xi| = 0$, then u is Lipschitz continuous in $B_{R/2}$ with the estimate*

$$[u]_{C^{0,1}(B_{R/2})} \leqslant C_{sl} \tag{5.3.1}$$

for some constant $C_{sl} \equiv C_{sl}(n, \lambda, \Lambda, i(\Phi), L, R)$.

(R2) *If $i(\Phi) \geqslant 0$ and $|\xi| > A_0$ with $A_0 \equiv A_0(n, \lambda, \Lambda, i(\Phi), L, R)$, then u is Lipschitz continuous in $B_{R/2}$ with the estimate*

$$[u]_{C^{0,1}(B_{R/2})} \leqslant C_{dl} \tag{5.3.2}$$

for some constant $C_{dl} \equiv C_{dl}(n, \lambda, \Lambda, i(\Phi), L, R)$.

(R3) *If $i(\Phi) \geqslant 0$ and $|\xi| \leqslant A_0$, then $u \in C^{0,\beta}(B_{R/2})$ with the estimate*

$$[u]_{C^{0,\beta}(B_{R/2})} \leqslant C_{ds}, \tag{5.3.3}$$

where $\beta \equiv \beta(n, \lambda, \Lambda, R, A_0) \in (0,1)$ and $C_{ds} \equiv C_{ds}(n, \lambda, \Lambda, R, A_0)$.

Proof. For the proof of (R1) and (R2), it suffices to show that there exist positive constants L_1 and L_2 such that

$$\mathcal{L} := \sup_{x,y \in B_R} \left(u(x) - u(y) - L_1\omega(|x-y|) - L_2\left(|x-z_0|^2 + |y-z_0|^2\right)\right) \leqslant 0 \tag{5.3.4}$$

for every $z_0 \in B_{R/2}$, where

$$\omega(t) = \begin{cases} t - \omega_0 t^{\frac{3}{2}} & \text{if } t \leqslant t_0 := \left(\dfrac{2}{3\omega_0}\right)^2, \\ \omega(t_0) & \text{if } t \geqslant t_0. \end{cases} \tag{5.3.5}$$

We choose $\omega_0 \in (0, 2/3)$ in such a way that $t_0 \geqslant 1$. For instance, we take any constant $\omega_0 \leqslant 1/3$. By the contradiction, suppose that there are no such positive constants L_1 and L_2 satisfying (5.3.4) for every $z_0 \in B_{R/2}$. Then there exists a point $z_0 \in B_{R/2}$ so that $\mathcal{L} > 0$ for all numbers $L_1 > 0$ and $L_2 > 0$. Now we define two auxiliary functions $\phi, \psi : \overline{B_R} \times \overline{B_R} \to \mathbb{R}$ given by

$$\psi(x, y) := L_1 \omega(|x - y|) + L_2 \left(|x - z_0|^2 + |y - z_0|^2\right) \tag{5.3.6}$$

and

$$\phi(x, y) := u(x) - u(y) - \psi(x, y). \tag{5.3.7}$$

Let $(\bar{x}, \bar{y}) \in \overline{B_R} \times \overline{B_R}$ be a maximum point for ϕ. Then we have

$$\phi(\bar{x}, \bar{y}) = \mathcal{L} > 0$$

and

$$L_1 \omega(|\bar{x} - \bar{y}|) + L_2 \left(|\bar{x} - z_0|^2 + |\bar{y} - z_0|^2\right) \leqslant \operatorname*{osc}_{B_1} u \leqslant 1.$$

Now we select

$$L_2 := \frac{64}{R^2}.$$

This choice of L_2 ensures

$$|\bar{x} - z_0| + |\bar{y} - z_0| \leqslant \frac{R}{4} \quad \text{and} \quad |\bar{x} - \bar{y}| \leqslant \frac{R}{4}. \tag{5.3.8}$$

This means that the points $\bar{x}$ and $\bar{y}$ belong to the open ball B_R and also we are able to assume that $\bar{x} \neq \bar{y}$; otherwise $\mathcal{L} \leqslant 0$ clearly. The rest of the proof is divided into several steps.

Step 1. We are in a position to apply Lemma 5.2.1 in order to ensure the existence of a limiting subjet $(\xi_{\bar{x}}, X_\delta)$ of u at $\bar{x}$ and a limiting superjet $(\xi_{\bar{y}}, Y_\delta)$ of u at $\bar{y}$, where

$$\xi_{\bar{x}} := D_x\psi(\bar{x},\bar{y}) = L_1\omega'(|\bar{x}-\bar{y}|)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|} + 2L_2(\bar{x}-z_0)$$

and

$$\xi_{\bar{y}} := -D_y\psi(\bar{x},\bar{y}) = L_1\omega'(|\bar{x}-\bar{y}|)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|} - 2L_2(\bar{y}-z_0),$$

such that matrices X_δ and Y_δ satisfy the matrix inequality

$$\begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix} \leqslant \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + (2L_2+\delta)I, \tag{5.3.9}$$

where

$$\begin{aligned} Z &:= L_1 D^2(\omega(|\cdot|))(\bar{x}-\bar{y}) \\ &= L_1\left[\frac{\omega'(|\bar{x}-\bar{y}|)}{|\bar{x}-\bar{y}|}I + \left(\omega''(|\bar{x}-\bar{y}|) - \frac{\omega'(|\bar{x}-\bar{y}|)}{|\bar{x}-\bar{y}|}\right)\frac{(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^2}\right] \end{aligned}$$

and the constant $\delta > 0$ only depends on the norm of Z, which can be selected sufficiently small. Applying the inequality (5.3.9) for vectors of the form $(z,z) \in \mathbb{R}^{2n}$, we find

$$\langle (X_\delta - Y_\delta)z, z\rangle \leqslant (4L_2 + 2\delta)|z|^2.$$

The last inequality yields that all the eigenvalues of the matrix $(X_\delta - Y_\delta)$ are not larger than $4L_2+2\delta$. On the other hand, applying again (5.3.9) for the vector $\bar{z} := \left(\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}, \frac{\bar{y}-\bar{x}}{|\bar{x}-\bar{y}|}\right)$, we have

$$\begin{aligned} \left\langle (X_\delta - Y_\delta)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}, \frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}\right\rangle &\leqslant (4L_2 + 2\delta + 4L_1\omega''(|\bar{x}-\bar{y}|))\left|\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}\right|^2 \\ &= \left(4L_2 + 2\delta - \frac{6\omega_0 L_1}{|\bar{x}-\bar{y}|^{1/2}}\right)\left|\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}\right|^2 \\ &\leqslant (4L_2 + 2\delta - 6\omega_0 L_1)\left|\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}\right|^2, \end{aligned}$$

where we have used the definition of ω in (5.3.5) together with $|\bar{x}-\bar{y}| \leqslant 1/4$ in (5.3.8). So at least one eigenvalue of $(X_\delta - Y_\delta)$ is not larger than $4L_2+2\delta-6\omega_0 L_1$, where this quantity can be negative for large values of L_1. By the definition of the extremal Pucci operator, we see

$$\begin{aligned} P^-_{\lambda,\Lambda}(X_\delta - Y_\delta) &\geqslant -\lambda(4L_2+2\delta-6\omega_0 L_1) - \Lambda(n-1)(4L_2+2\delta) \\ &\geqslant -(\lambda+(n-1)\Lambda)(4L_2+2\delta) + 6\omega_0\lambda L_1. \end{aligned}$$

From two viscosity inequalities and the uniform ellipticity, we have

$$\Phi(\bar{x}, |\xi+\xi_{\bar{x}}|)F(X_\delta) \leqslant f(\bar{x}), \quad \Phi(\bar{y}, |\xi+\xi_{\bar{y}}|)F(Y_\delta) \geqslant f(\bar{y})$$

and

$$F(X_\delta) \geqslant F(Y_\delta) + P^-_{\lambda,\Lambda}(X_\delta - Y_\delta).$$

Combining last three displays, we have

$$\begin{aligned} 6\omega_0\lambda L_1 &\leqslant (\lambda+(n-1)\Lambda)(4L_2+2\delta) \\ &\quad + \frac{f(\bar{x})}{\Phi(\bar{x}, |\xi+\xi_{\bar{x}}|)} - \frac{f(\bar{y})}{\Phi(\bar{y}, |\xi+\xi_{\bar{y}}|)}. \end{aligned} \tag{5.3.10}$$

At this stage, we shall separate it into several cases depending on the quantity of $|\xi|$ and the positiveness of $i(\Phi)$.

Step 2: Proof of (R1). Suppose $-1 < i(\Phi) < 0$ and $\xi = 0$. By triangle inequality and (5.3.8), we observe that

$$|\xi_{\bar{x}}| \leqslant L_1\left(1+\frac{3}{2}\omega_0\right) + 2L_2 \leqslant \frac{7}{4}L_1 \tag{5.3.11}$$

and

$$|\xi_{\bar{x}}| \geqslant L_1\left(1-\frac{3\omega_0}{2}|\bar{x}-\bar{y}|^{\frac{1}{2}}\right) - 3L_2 \geqslant \frac{3L_1}{4} - 3L_2 \geqslant 3L_2 \tag{5.3.12}$$

for all $L_1 \geqslant 8L_2$. In the exactly same way, we see

$$|\xi_{\bar{y}}| \leqslant \frac{7}{4}L_1 \quad \text{and} \quad |\xi_{\bar{y}}| \geqslant 2L_2 \tag{5.3.13}$$

for all $L_1 \geqslant 8L_2$. Then we have

$$\frac{f(\bar{x})}{\Phi(\bar{x}, |\xi_{\bar{x}}|)} \leqslant c\frac{\|f\|_{L^\infty(B_1)}}{|\xi_{\bar{x}}|^{i(\Phi)}} \leqslant \frac{c}{L_1^{i(\Phi)}} \tag{5.3.14}$$

and

$$\frac{-f(\bar{y})}{\Phi(\bar{y}, |\xi_{\bar{y}}|)} \leqslant c\frac{\|f\|_{L^\infty(B_1)}}{|\xi_{\bar{y}}|^{i(\Phi)}} \leqslant \frac{c}{L_1^{i(\Phi)}} \tag{5.3.15}$$

for a constant $c \equiv c(i(\Phi), L)$. Using the last two displays in (5.3.10), we obtain

$$6\omega_0\lambda L_1 \leqslant (\lambda + (n-1)\Lambda)(4L_2 + 2\delta) + \frac{c}{L_1^{i(\Phi)}}$$

for a constant $c \equiv c(n, \lambda, \Lambda, i(\Phi), L, R)$. Recalling $-1 < i(\Phi) < 0$ and taking L_1 large enough, depending only on $n, \lambda, \Lambda, i(\Phi), L$ and R, we get a contradiction. Then the first part of the lemma is proved.

Step 3: Proof of (R2). We suppose that $i(\Phi) \geqslant 0$ and $|\xi| > A_0$ for a constant A_0 to be determined in a moment. We set

$$A_0 := \frac{35L_1}{2} \tag{5.3.16}$$

for $L_1 > 1$ to be selected soon. This choice of A_0 together with (5.3.11) and (5.3.14) leads to

$$|\xi + \xi_{\bar{x}}| \geqslant A_0 - \frac{A_0}{10} = \frac{9A_0}{10} \quad \text{and} \quad |\xi + \xi_{\bar{y}}| \geqslant \frac{9A_0}{10}.$$

Therefore, we have

$$\frac{f(\bar{x})}{\Phi(\bar{x}, |\xi + \xi_{\bar{x}}|)} \leqslant c\frac{\|f\|_{L^\infty(B_1)}}{|\xi + \xi_{\bar{x}}|^{i(\Phi)}} \leqslant \frac{c}{A_0^{i(\Phi)}}$$

and

$$\frac{-f(\bar{y})}{\Phi(\bar{y}, |\xi + \xi_{\bar{y}}|)} \leqslant c\frac{\|f\|_{L^\infty(B_1)}}{|\xi + \xi_{\bar{y}}|^{i(\Phi)}} \leqslant \frac{c}{A_0^{i(\Phi)}}$$

for a constant $c \equiv c(i(\Phi), L)$. Again using the last two displays in (5.3.10), we obtain

$$6\omega_0 \lambda L_1 \leqslant (\lambda + (n-1)\Lambda)(4L_2 + 2\delta) + \frac{c}{L_1^{i(\Phi)}}$$

for a constant $c \equiv c(n, \lambda, \Lambda, i(\Phi), L, R)$. By choosing L_1 large enough, depending only on $n, \lambda, \Lambda, i(\Phi), L$ and R, we have again a contradiction. Indeed, we have proved the second part of the lemma.

Step 4: Proof of (R3). Finally, we shall focus on proving (R3). Suppose now $|\xi| \leqslant A_0$, where A_0 has been determined in (5.3.16). We consider the operator

$$G_\xi(x, p, M) := \Phi(x, |\xi + p|)F(M) - f(x).$$

In fact, $G_\xi(x, p, M)$ is uniformly elliptic, whenever $|p| > 2A_0$. At this stage, we apply Theorem 5.2.1 to conclude the last part of the Lemma. The proof is complete.

□

5.4 Approximation

Now we prove a key approximation lemma, which plays a crucial role in later arguments.

Lemma 5.4.1. *Let $u \in C(B_1)$ be a viscosity solution of (5.1.3) with $\operatorname{osc}_{B_1} \leqslant 1$, where $\xi \in \mathbb{R}^n$ is arbitrarily given. Suppose (A1)-(A3) hold true for $i(\Phi) \geqslant 0$ and $\nu_0 = \nu_1 = 1$. Then, for any $\mu > 0$, there exists a constant $\delta \equiv \delta(n, \lambda, \Lambda, i(\Phi), L, \mu)$ such that if*

$$\|f\|_{L^\infty(B_1)} \leqslant \delta, \tag{5.4.1}$$

then one can find $h \in C^{1,\bar\alpha}(B_{3/4})$ with the estimate $\|h\|_{C^{1,\bar\alpha}(B_{3/4})} \leqslant c \equiv c(n, \lambda, \Lambda)$, for some $0 < \bar\alpha < 1$, satisfying

$$\|u - h\|_{L^\infty(B_{1/2})} \leqslant \mu. \tag{5.4.2}$$

Proof. By contradiction, we suppose the conclusion of the lemma fails. Then there exist $\mu_0 > 0$ and sequences of $\{F_k\}_{k=1}^{\infty}$, $\{\Phi_k\}_{k=1}^{\infty}$, $\{f_k\}_{k=1}^{\infty}$, and $\{u_k\}_{k=1}^{\infty}$ and a sequence of vectors $\{\xi_k\}_{k=1}^{\infty}$ such that

(C1) $F_k \in C(\mathcal{S}(n), \mathbb{R})$ is uniformly (λ, Λ)-elliptic,

(C2) $\Phi_k \in C(B_1 \times [0,\infty), [0,\infty))$ such that the map $t \mapsto \dfrac{\Phi_k(x,t)}{t^{i(\Phi)}}$ is almost non-decreasing and the map $t \mapsto \dfrac{\Phi(x,t)}{t^{d(\Phi)}}$ is almost non-increasing with constant $L \geqslant 1$, and $\Phi_k(x,1) = 1$ for all $x \in B_1$,

(C3) $f_k \in C(B_1)$ with $\|f_k\|_{L^\infty(B_1)} \leqslant \dfrac{1}{k}$ and

(C4) $u_k \in C(B_1)$ with $\operatorname*{osc}_{B_1} u_k \leqslant 1$ solves the equation

$$\Phi_k(x, |\xi_k + Du_k|) F_k(D^2 u_k) = f_k(x), \tag{5.4.3}$$

but

$$\sup_{x \in B_{1/2}} |u_k(x) - h(x)| > \mu_0 \tag{5.4.4}$$

for all $h \in C^{1,\bar{\alpha}}(B_{3/4})$ and every $0 < \bar{\alpha} < 1$.

The condition (C1) implies that F_k converges to some uniformly (λ, Λ)-elliptic operator $F_\infty \in C(\mathcal{S}(n), \mathbb{R})$. Applying Lemma 5.3.1, $u_k \in C^{0,\beta}_{\text{loc}}(B_1) \cap C(B_1)$ for some $\beta \in (0,1)$. Using (5.3.2), (5.3.3) and Arzela-Ascoli theorem, we have that the sequence $\{u_k\}_{k=1}^{\infty}$ converges to a function u_∞ locally uniformly in B_1. In particular, there holds that

$$u_\infty \in C(B_1) \quad \text{and} \quad \operatorname*{osc}_{B_1} u_\infty \leqslant 1. \tag{5.4.5}$$

Now we prove that the limiting function u_∞ is a viscosity solution of the homogeneous equation

$$F_\infty(D^2 u_\infty) = 0 \quad \text{in} \quad B_{3/4}. \tag{5.4.6}$$

For this, first we verify that u_∞ is a viscosity supersolution. Let

$$\varphi(x) := \frac{1}{2}\langle M(x-y), x-y\rangle + \langle b, x-y\rangle + u_\infty(y)$$

be a quadratic polynomial touching u_∞ from below at a point $y \in B_{3/4}$. Without loss of generality, let us assume $|y| = u_\infty(y) = 0$. Then there exists a sequence $x_k \to 0$ as $k \to \infty$ such that $u_k - \varphi$ has a local minimum at x_k. Observe that $D\varphi(x_k) \to b$ and $D^2\varphi(x_k) \to M$. Since u_k is a viscosity solution of (5.4.3), we have

$$\Phi_k(x_k, |\xi_k + D\varphi(x_k)|)F_k(D^2\varphi(x_k)) \geqslant f_k(x_k). \tag{5.4.7}$$

For the ease of presentation, from now on we shall consider several cases depending on the boundedness of sequence $\{\xi_k\}_{k=1}^{\infty}$.

Case 1: Sequence $\{\xi_k\}_{k=1}^{\infty}$ is unbounded. In this case, we can assume $|\xi_k| \to \infty$ (up to a subsequence). As a consequence, we can show (up to a subsequence) that

$$|\xi_k + D\varphi(x_k)| \geqslant |\xi_k| - |D\varphi(x_k)| \geqslant |\xi_k| - (|b| + 1) \geqslant 1, \tag{5.4.8}$$

which implies that

$$\begin{aligned} F_\infty(M) = \lim_{k\to\infty} F_k(D^2\varphi(x_k)) &\geqslant \lim_{k\to\infty} \frac{f_k(x_k)}{\Phi_k(x_k, |\xi_k + D\varphi(x_k)|)} \\ &\geqslant -\lim_{k\to\infty} \frac{L}{k|\xi_k + D\varphi(x_k)|^{i(\Phi)}} = 0, \end{aligned}$$

where we have used (C2) and (5.4.7).

Case 2: Sequence $\{\xi_k\}_{k=1}^{\infty}$ bounded In the case we may assume $\xi_k \to \xi_\infty$ (up to a subsequence). Therefore, for the case $|\xi_\infty + b| \neq 0$, in the exactly same way as in (5.4.8), we infer that $F_\infty(M) \geqslant 0$. Then we focus on the case $|\xi_\infty + b| = 0$. There are two possibilities as $|b| = |\xi_\infty| = 0$ or $b = -\xi_\infty$ with $|b|, |\xi_\infty| > 0$. In those scenarios, we prove that $F_\infty(M) \geqslant 0$. By contradiction suppose

$$F_\infty(M) < 0. \tag{5.4.9}$$

From the uniformly ellipticity condition of F_∞, the matrix M has at least one positive eigenvalue. Let $\mathbb{R}^n = E \oplus Q$, where $E = \mathrm{span}\{e_1, \ldots, e_m\}$ is the space consisting of those eigenvectors corresponding to positive eigenvalues of M.

Case 2-1: $b = -\xi_\infty$ with $|b|, |\xi_\infty| > 0$. Let $\gamma > 0$ and set

$$p_\gamma(x) := \varphi(x) + \gamma|P_E(x)| = \frac{1}{2}\langle Mx, x\rangle + \langle b, x\rangle + \gamma|P_E(x)|,$$

where P_E stands for the orthogonal projection on E. Since $u_k \to u_\infty$ locally uniformly in B_1 and $\varphi(x)$ touches $u_\infty(x)$ from below at the origin, for γ small enough, $p_\gamma(x)$ touches $u_k(x)$ from below at a point $x_k^\gamma \in B_r$ (B_r is a small neighborhood of the origin). Moreover, there holds that $x_k^\gamma \to x_\infty^\gamma$ for some x_∞^γ as $k \to \infty$. At this point we consider two scenarios: $P_E(x_k^\gamma) = 0$ for all $k \in \mathbb{N}$ (up to a subsequence) or $P_E(x_k^\gamma) \neq 0$ for all $k \in \mathbb{N}$ (up to a subsequence).

Scenario 1: $P_E(x_k^\gamma) = 0$ for all $k \in \mathbb{N}$ (up to a subsequence). In this scenario, first we note that

$$\bar{p}_\gamma(x) := \frac{1}{2}\langle Mx, x\rangle + \langle b, x\rangle + \gamma\langle e, P_E(x)\rangle$$

touches u_k from below at x_k^γ for every $e \in \mathbb{S}^{n-1}$. A straightforward computation gives us

$$D\bar{p}_\gamma(x_k^\gamma) = Mx_k^\gamma + b + \gamma P_E(e) \quad \text{and} \quad D^2\bar{p}_\gamma(x_k^\gamma) = M.$$

Now we select $e \in E \cap \mathbb{S}^{n-1}$ such that $P_E(e) = e$. Therefore, by u_k being a viscosity solution of (5.4.3), we see

$$\Phi_k(x_k^\gamma, |\xi_k + Mx_k^\gamma + b + \gamma e|)F_k(M) \geqslant f_k(x_k^\gamma).$$

We also notice that if $Mx_\infty^\gamma = 0$, then for k enough large, we have

$$|\xi_k + Mx_k^\gamma + b| \leqslant \gamma/2 \quad \text{and} \quad 3\gamma/2 \geqslant |\xi_k + Mx_k^\gamma + b + \gamma e| \geqslant \gamma/2.$$

Therefore, combining the last two displays and using (C2) together with $\gamma \ll 1$, we have

$$\begin{aligned} F_k(M) &\geqslant \frac{f_k(x_k^\gamma)}{\Phi_k(x_k^\gamma, |\xi_k + Mx_k^\gamma + b + \gamma e|)} \\ &\geqslant \frac{-L|f_k(x_k^\gamma)|}{|\xi_k + Mx_k^\gamma + b + \gamma e|^{s(\Phi)}} \geqslant -\frac{L}{k}\left(\frac{2}{3\gamma}\right)^{s(\Phi)}. \end{aligned}$$

Letting $k \to \infty$ in the last display, we obtain $F_\infty(M) \geqslant 0$. Let us now consider the situation $|Mx^\gamma_\infty| > 0$. First we consider the case of $E \equiv \mathbb{R}^n$ and select $e \in \mathbb{S}^{n-1}$ such that

$$|Mx^\gamma_\infty + \gamma P_E(e)| = |Mx^\gamma_\infty + \gamma e| > 0.$$

There hold that, for k large enough,

$$|Mx^\gamma_k + \gamma e| > \frac{1}{2}|Mx^\gamma_\infty + \gamma e| \quad \text{and} \quad |\xi_k + b| < \frac{1}{4}|Mx^\gamma_\infty + \gamma e|.$$

As a consequence, we see

$$|\xi_k + Mx^\gamma_k + b + \gamma P_E(e)| > \frac{1}{4}|Mx^\gamma_\infty + \gamma e| > 0.$$

Again applying (C2) and taking into account the last display, we have

$$\begin{aligned} F_k(M) &\geqslant \frac{f_k(x^\gamma_k)}{\Phi_k(x^\gamma_k, |\xi_k + Mx^\gamma_k + b + \gamma e|)} \\ &\geqslant -\left(\frac{L}{|\xi_k + Mx^\gamma_k + b + \gamma e|^{i(\Phi)}} + \frac{L}{|\xi_k + Mx^\gamma_k + b + \gamma e|^{s(\Phi)}} \right) |f_k(x^\gamma_k)| \\ &\geqslant \frac{-L4^{s(\Phi)}}{k} \left(\frac{1}{|Mx^\gamma_\infty + \gamma e|^{i(\Phi)}} + \frac{1}{|Mx^\gamma_\infty + \gamma e|^{s(\Phi)}} \right). \end{aligned} \tag{5.4.10}$$

Again letting $k \to \infty$ in the last display, we again arrive at $F_\infty(M) \geqslant 0$. On the other hand, if $E \not\equiv \mathbb{R}^n$, then there exists $e \in \mathbb{S}^{n-1} \cap E^\perp$ so that

$$|Mx^\gamma_\infty + \gamma P_E(e)| = |Mx^\gamma_\infty| > 0.$$

Therefore, for large enough k, there hold that

$$|Mx^\gamma_k| > \frac{1}{2}|Mx^\gamma_\infty| \quad \text{and} \quad |\xi_k + b| < \frac{1}{4}|Mx^\gamma_\infty|.$$

Using the last display, we get

$$|\xi_k + Mx^\gamma_k + b + \gamma P_E(e)| > \frac{1}{4}|Mx^\gamma_\infty| > 0.$$

Repeating the same arguments as in (5.4.10), we arrive at $F_\infty(M) \geqslant 0$.

Scenario 2: $P_E(x_k^\gamma) \neq 0$ *for all* $k \in \mathbb{N}$ *(up to a subsequence).* In this scenario, we note that the map $x \mapsto |P_E(x)|$ is smooth and convex in a small neighborhood of x_k^γ. Let us denote

$$\zeta_k^\gamma := \frac{P_E(x_k^\gamma)}{|P_E(x_k^\gamma)|}.$$

A direct computation yields

$$D(|P_E(\cdot)|)(x_k^\gamma) = \zeta_k^\gamma \quad \text{and} \quad D^2(P_E(|\cdot|))(x_k^\gamma) = \frac{1}{|P_E(x_k^\gamma)|}\left(I - \zeta_k^\gamma \otimes \zeta_k^\gamma\right).$$

Hence, with u_k being a viscosity solution of (5.4.3), we have the following viscosity inequality

$$\Phi_k(x_k^\gamma, |\xi_k + Mx_k^\gamma + b + \gamma\zeta_k^\gamma|)F_k\left(M + \frac{1}{|P_E(x_k^\gamma)|}\left(I - \zeta_k^\gamma \otimes \zeta_k^\gamma\right)\right) \geqslant f_k(x_k^\gamma).$$

Observing that $|\zeta_k^\gamma| = 1$ and letting $e := \zeta_k^\gamma$, we can perform the same procedure as in the first scenario of $P_E(x_k^\gamma) = 0$ by considering the cases of $Mx_\infty^\gamma = 0$ and $Mx_\infty^\gamma \neq 0$. Finally, we conclude that $F_\infty(M) \geqslant 0$ when $b = -\xi_\infty \neq 0$, which contradicts to (5.4.9).

Case 2-2: $b = \xi_\infty = 0$**.** In fact, this case is much easier to handle. Since $\frac{1}{2}\langle Mx, x\rangle$ touches $u_\infty(x)$ from below at the origin and $u_k \to u_\infty$ locally uniformly, the function

$$\hat{p}_\gamma(x) := \frac{1}{2}\langle Mx, x\rangle + \gamma|P_E(x)|$$

touches u_k from below at a point $\hat{x}_k^\gamma \in B_r$ (B_r is a small neighborhood of the origin) for $\gamma > 0$ sufficiently small. Again the sequence $\{\hat{x}_k^\gamma\}$ is uniformly bounded. As in **Case 3**, we analyze those two scenarios $P_E(\hat{x}_k^\gamma) = 0$ for all $k \in \mathbb{N}$ (up to a subsequence) and $P_E(\hat{x}_k^\gamma) \neq 0$ for all $k \in \mathbb{N}$ (up to a subsequence). All in all, we conclude $F_\infty(M) \geqslant 0$ in this case.

Finally, taking into account all cases we have analyzed above, we have shown that u_∞ is a viscosity supersolution of (5.4.6). In order to prove that u_∞ is a viscosity subsolution of (5.4.6), we show that $-u_\infty$ is a viscosity supersolution of $\hat{F}_\infty(D^2h) = 0$, where $\hat{F}_\infty(M) = -F_\infty(-M)$ is uniformly (λ, Λ)-elliptic operator as well. Therefore, u_∞ is a viscosity solution of (5.4.6).

From the regularity results of [46, Chap. 5], we see $u_\infty \in C^{1,\bar{\alpha}}_{\mathrm{loc}}(B_{3/4})$ for some $\bar{\alpha} \in (0,1)$. Moreover, $\|u_\infty\|_{C^{1,\bar{\alpha}}(B_{1/2})} \leqslant c \equiv c(n,\lambda,\Lambda)$ via (5.4.5). So choosing $h := u_\infty$ in (5.4.4), we have a contradiction. The proof is complete.

□

5.5 Proof of Theorem 5.1.1

Now we provide a proof of Theorem 5.1.1. Let $u \in C(B_1)$ be a viscosity solution with $\operatorname*{osc}_{B_1} u \leqslant 1$, $\|f\|_{L^\infty(B_1)} \leqslant \delta \ll 1$ for a constant $\delta \equiv \delta(n,\lambda,\Lambda,i(\Phi),L)$ to be determined in a moment and $\nu_0 = \nu_1 = 1$. The proof is divided into two main parts, where in the first part we shall deal with the case $i(\Phi) \geqslant 0$ and the remaining case $-1 < i(\Phi) < 0$ will be investigated in the second part.

Part 1: $i(\Phi) \geqslant 0$. Let us first fix a point $y \in B_{1/2}$ and an exponent with

$$0 < \beta < \min\left\{\bar{\alpha}, \frac{1}{1+d(\Phi)}\right\}. \tag{5.5.1}$$

We prove that there exist universal constants $0 < r \ll 1$, $C_0 > 1$ and a sequence of affine functions

$$l_k(x) := a_k + \langle b_k, x\rangle, \tag{5.5.2}$$

where $\{a_k\}_{k=1}^\infty \subset \mathbb{R}$ and $\{b_k\}_{k=1}^\infty \subset \mathbb{R}^n$, such that for every $k \in \mathbb{N}$:

(E1) $\sup_{x \in B_{r^k}(y)} |u(x) - l_k(x)| \leqslant r^{k(1+\beta)}$,

(E2) $|a_k - a_{k-1}| \leqslant C_0 r^{(k-1)(1+\beta)}$ and

(E3) $|b_k - b_{k-1}| \leqslant C_0 r^{(k-1)\beta}$.

We show these estimates by mathematical induction. For the simplicity, we divide the proof into several steps.

Step 1. Basis of induction. Without loss of generality we can assume $y = 0$ by translating $x \mapsto y + \dfrac{1}{2}x$. Let us set

$$l_1(x) := h(0) + \langle Dh(0), x\rangle,$$

where h is the approximation function coming from Lemma 5.4.1 for a certain constant $\mu > 0$ to be determined in a few lines. Then there exists a constant $C_0 \equiv C_0(n, \lambda, \Lambda) > 1$ such that

$$\|h\|_{C^{1,\bar{\alpha}}(B_{3/8})} \leqslant C_0 \quad \text{and} \quad \sup_{x \in B_r} |h(x) - l_1(x)| \leqslant C_0 r^{1+\bar{\alpha}}$$

for every $r \leqslant 3/8$. The triangle inequality yields

$$\sup_{x \in B_r} |u(x) - l_1(x)| \leqslant \mu + C_0 r^{1+\bar{\alpha}}.$$

We first select a universal constant $0 < r \ll 1$ satisfying

$$r^{\beta} \leqslant \frac{1}{2}, \quad C_0 r^{1+\bar{\alpha}} \leqslant \frac{1}{2} r^{1+\beta} \quad \text{and} \quad r^{1-\beta(1+d(\Phi))} \leqslant 1, \tag{5.5.3}$$

which is possible by (5.5.1). In a sequel, we select a constant $\mu > 0$ as

$$\mu := \frac{1}{2} r^{1+\beta}, \tag{5.5.4}$$

which fixes an arbitrary constant $\mu > 0$ in Lemma 5.4.1. In turn, there exists a constant $\delta \equiv \delta(n, \lambda, \Lambda, i(\Phi), L, \beta)$ verifying the smallness assumption $\|f\|_{L^\infty(B_1)} \leqslant \delta$, but such a smallness assumption can be assumed without loss of generality. Therefore, to conclude this step we set

$$a_0 := 0, \quad a_1 := h(0), \quad b_0 = 0 \quad \text{and} \quad b_1 := Dh(0).$$

These choices with (5.5.3) and (5.5.4) verify that the estimates (E1)-(E3) are satisfied for $k = 1$.

Step 2: Induction process. Now we suppose that the hypotheses of the induction have been established for $k = 1, 2, \ldots, m$ for $m \geqslant 1$. We show that the estimates (E1)-(E3) hold true for $k = m + 1$. For this, we introduce an auxiliary function as

$$w_m(x) := \frac{u(r^m x) - l_m(r^m x)}{r^{m(1+\beta)}}.$$

We note that w_m solves the following equation in the viscosity sense

$$\Phi_m(x, |r^{-m\beta} b_m + Dw_m|) F_m(D^2 w_m) = f_m(x),$$

where

$$F_m(M) := r^{m(1-\beta)}F(r^{(\beta-1)m}M),$$

which is uniformly (λ,Λ)-operator, the function

$$\Phi_m(x,t) := \frac{\Phi(r^m x, r^{m\beta}t)}{\Phi(r^m x, r^{m\beta})} \quad (x \in B_1, t > 0)$$

still satisfies the properties that the map $t \mapsto \dfrac{\Phi_m(x,t)}{t^{i(\Phi)}}$ is almost non-decreasing, the map $t \mapsto \dfrac{\Phi_m(x,t)}{t^{d(\Phi)}}$ is almost non-increasing with the same constant $L \geqslant 1$ and $\Phi_m(x,1) = 1$ for all $x \in B_1$, and

$$f_m(x) := \frac{r^{m(1-\beta)}f(r^m x)}{\Phi(r^m x, r^{m\beta})}.$$

Using (A2) and (5.5.1), we notice that

$$\|f_m\|_{L^\infty(B_1)} \leqslant \frac{Lr^{m(1-\beta)}\,\|f\|_{L^\infty(B_1)}}{r^{m\beta d(\Phi)}} \leqslant L\delta r^{m(1-(1+d(\Phi))\beta)} \leqslant L\delta.$$

Therefore, we are in a position to apply Lemma 5.4.1 to w_m. In turn, there exists a function $\bar{h} \in C^{1,\bar{\alpha}}(B_{3/4})$ such that

$$\sup_{x\in B_r} |w_m(x) - \bar{h}(x)| \leqslant \mu.$$

Arguing as in **Step 1**, we show that

$$\sup_{x\in B_r} |w_m(x) - \bar{l}(x)| \leqslant r^{1+\beta},$$

where

$$\bar{l}(x) := \bar{a} + \left\langle \bar{b}, x \right\rangle \quad \text{for some} \quad \bar{a} \in \mathbb{R} \text{ and } \bar{b} \in \mathbb{R}^n.$$

Denoting

$$l_{m+1} := l_m(x) + r^{m(1+\beta)}\bar{l}(r^{-m}x),$$

we see

$$\sup_{x\in B_{r^{m+1}}} |u(x) - l_{m+1}(x)| \leqslant r^{(m+1)(1+\beta)}$$

and

$$|a_{m+1} - a_m| + r^m |b_{m+1} - b_m| \leqslant C_0 r^{m(1+\beta)}.$$

Therefore, the $(m+1)$-th step of the induction is complete.

Step 3: Conclusion. Once we have the existence of universal constants $0 < r \ll 1$, $C_0 > 1$ and a sequence of affine functions in (5.5.2) verifying the estimates (E1)-(E3), the remaining part of the proof is very standard, see for instance [100, 67]. Therefore, the proof of (5.1.7) is complete when $i(\Phi) \geqslant 0$.

Part 2: $-1 < i(\Phi) < 0$. Now we shall with the case of $-1 < i(\Phi) < 0$. Again we fix a point $y \in B_{1/2}$. Without loss of generality, we may assume $y = 0$ by using the translation $x \mapsto y + \dfrac{1}{2}x$. Now we apply (R1) of Lemma 5.3.1 in order to ensure that

$$[u]_{C^{0,1}(B_{3/4})} \leqslant C_{sl} \tag{5.5.5}$$

for a constant $C_{sl} \equiv C_{sl}(n, \lambda, \Lambda, i(\Phi), L)$. Therefore, it can be seen that u is a viscosity solution of the equation

$$\tilde{\Phi}(x, |Dv|)F(D^2 v) = \tilde{f}(x) \quad \text{in} \quad B_{3/4},$$

where

$$\tilde{\Phi}(x,t) := t^{-i(\Phi)}\Phi(x,t) \quad (x \in B_1, t > 0),$$

which satisfies the properties that the map $t \mapsto \tilde{\Phi}(x,t)$ is almost non-increasing, the map $t \mapsto \dfrac{\tilde{\Phi}(x,t)}{t^{d(\Phi)-i(\Phi)}}$ is almost non-increasing with constant $L \geqslant 1$, $\tilde{\Phi}(x,1) = 1$ for all $x \in B_1$, and

$$\tilde{f}(x) = |Du(x)|^{-i(\Phi)} f(x).$$

Using the estimate (5.5.5) together with $\|f\|_{L^\infty B_1} \leqslant \delta \ll 1$, we see

$$\left\|\tilde{f}\right\|_{L^\infty(B_{3/4})} \leqslant C_{sl}^{-i(\Phi)}\delta.$$

So we are able to apply **Part 1** of the proof in order to have (E1)-(E3). This means that we have the estimate (5.1.7) for $-1 < i(\Phi) < 0$. The proof is complete.

Bibliography

[1] E. Acerbi and N. Fusco, *An harmonic approximation lemma for $W^{1,p}$ functions*, Material Instabilities in Continuum Mechanics(Edinburg, 1985-1986), pp.1-5. Oxford Science Publications, Oxford University Press, New York (1988).

[2] R.A. Adams and J.J.F. Fournier, *Sobolev spaces*, 2nd ed., Pure and Applied Mathematics (Amsterdam) **140**, Elsevier/Academic Press, Amsterdam, 2003.

[3] E. Acerbi and G. Mingione, *Regularity results for a class of functionals with non-standard growth*, Arch. Ration. Mech. Anal., **156**, (2001) 121-140.

[4] E. Acerbi and G. Mingione, *Regularity results for electrorheological fluids: the stationary case*, C. R. Math. Acad. Sci. Paris, **334**, (2002) 817-822.

[5] E. Acerbi and G. Mingione, *Gradient estimates for a class of parabolic systems*, Duke Math. J., **136** (2) (2007), 285-320.

[6] E. Acerbi and G. Mingione, *Gradient estimates for the $p(x)$-Laplacean system*, J. Reine Angew. Math **584** (2005), 117-148.

[7] K. Adimurthi and S. Byun, *Gradient weighted estimates at the natural exponent for quasilinear parabolic equations*, Adv. Math. **348** (2019), 456-511.

[8] K. Adimurthi, S. Byun and J. Park, *Sharp gradient estimates for quasilinear elliptic equations with $p(x)$ growth on nonsmooth domains*, J. Funct. Anal. **274** (2018), 3411-3439.

[9] D. J. Araújo, G. Ricarte and E. V. Teixeira, *Geometric gradient estimates for solutions to degenerate elliptic equations*, Calc. Var. Partial Differ. Equ **53** (2015), 605-625.

[10] B. Avelin, T. Kuusi and G. Mingione, *Nonlinear Calderón-Zygmund theory in the limiting case*, Arch. Ration. Mech. Anal. **227** (2018), no. 2, 663-714.

[11] S. Baasandorj and S. Byun, *Irregular obstacle problems for Orlicz double phase*, J. Math. Anal. Appl. **507** (1) (2022), 125791.

[12] S. Baasandorj and S. Byun, *Regularity for Orlicz phase problems*, submitted, `https://arxiv.org/abs/2106.15131`.

[13] S. Baasandorj and S. Byun, *Gradient estimates for Orlicz double phase problems*, Ann. Sc. Norm. Super. Pisa Cl. Sci., to appear.

[14] S. Baasandorj, S. Byun and J. Oh, *Calderón-Zygmund estimates for generalized double phase problems*, J. Funct. Anal. **279** (7) (2020), 108670.

[15] S. Baasandorj, S. Byun and J. Oh, *Gradient estimates for multi-phase problems*, Calc. Var. Partial Differential Equations, **60** (104) (2021).

[16] S. Baasandorj, S. Byun and H. Lee, *Global gradient estimates for a general class of quasilinear elliptic equations with Orlicz growth*, Proc. Amer. Math. Soc., **149** (10) (2021), 4189-4206.

[17] S. Baasandorj, S. Byun and H. Lee, *Gradient estimates for Orlicz double phase problems with variable exponents*, Nonlinear Analysis **221** (2022), 112891.

[18] Anna Kh. Balci, L. Diening and M. Surnachev, *New Examples on Lavrentiev Gap Using Fractals*, Calc. Var. Partial Differential Equations **59:180** (2020).

[19] P. Baroni, *Riesz potential estimates for a general class of quasilinear equations*, Calc. Var. Partial Differential Equations **53** (2015), 803-846.

[20] P. Baroni, M. Colombo and G. Mingione, *Harnack inequalities for double phase functionals*, Nonlinear Anal. **121** (2015), 206-222.

[21] P. Baroni, M. Colombo and G. Mingione, *Non-autonomous functionals, borderline cases and related function classes*, St. Petersburg Math. J. **27** (2016), 347-379.

[22] P. Baroni, M. Colombo and G. Mingione, *Regularity for general functionals with double phase*, Calc. Var. Partial Differential Equations **57** (2) (2018), Art. 62, 48 pp.

[23] L. Beck and G. Mingione, *Optimal Lipschitz criteria and local estimates for non-uniformly elliptic problems*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **30** (2) (2019), 223-236.

[24] A. Benkirane and M. Sidi El Vally, *Variational inequalities in Musielak-Orlicz-Sobolev spaces*, Bull. Belg. Math. Soc. Simon Stevin **21** (5) (2014), 787-811.

[25] P. Bildhauer and M. Fuchs, *$C^{1,\alpha}$-solutions to non-autonomous anisotropic variational problems*, Calc. Var. Partial Differential Equations **24**, (2005), 309-340.

[26] I. Birindelli and F. Demengel, *Comparison principle and Liouville type results for singular fully nonlinear operators*, Ann. Fac. Sci. Toulouse Math. **13** (2004), 261-287.

[27] I. Birindelli and F. Demengel, *Regularity and uniqueness of the first eigenfunction for singular fully nonlinear operators*, J. Differential Equations **249** (2010), 1089-1110.

[28] I. Birindelli and F. Demengel, *Regularity for radial solutions of degenerate fully nonlinear equations*, Nonlinear Anal. **75** (2012), 6237-6249.

[29] I. Birindelli and F. Demengel, *$C^{1,\beta}$ regularity for Dirichlet problems associated to fully nonlinear degenerate elliptic equations*, ESAIM Control OPtim. Calc. Var. **20** (2014), 1009-1024.

[30] I. Birindelli, F. Demengel and F. Leoni, *$C^{1.\gamma}$ regularity for singular or degenerate fully nonlinear equations and applications*, NoDEA Nonlinear Differential Equations Appl. **26** (2019), Paper 40.

[31] V. Bögelein, F. Duzaar and G. Mingione, *Degenerate problems with irregular obstacles*, J. Reine Angew. Math., **650** (2011), 107-160.

[32] P. Bousquet and L. Brasco, *Global Lipschitz continuity for minima of degenerate problems*, Math. Ann. **366**, (2016), 1403-1450.

[33] D. Breit, *New regularity theorems for non-autonomous variational integrals with (p, q)-growth*, Calc. Var. Partial Differential Equations **44** (1-2) (2012), 101-129.

[34] A. C. Bronzi, E. A. Pimentel, G. C. Rampasso and E. V. Teixeira, *Regularity of solutions to a class of variable-exponent fully nonlinear elliptic equations*, J. Func. Anal. **279** (2020), 108781.

[35] Kateryna O. Buryachenko and Igor I. Skrypnik, *Local continuity and Harnack's inequality for double-phase parabolic equations*, Potential Analysis **23** (2020).

[36] S. Byun and M. Lim, *Calderón-Zygmund estimates for non-uniformly elliptic equations with discontinuous nonlinearities on nonsmooth domains*, J. Differential Equations **312** (2022), 374-406.

[37] S. Byun and J. Oh, *Global gradient estimates for non-uniformly elliptic equations*, Calc. Var. Partial Differential Equations **56** (2) (2017), Art. 46, 36 pp.

[38] S. Byun and J. Oh, *Global gradient estimates for the borderline case of double phase problems with BMO coefficients in nonsmooth domains*, J. Differential Equations **263** (2) (2017), 1643-1693.

[39] S. Byun and J. Oh, *Regularity results for generalized double phase functionals*, Anal. PDE **13** (5) (2020), 1269-1300.

[40] S. Byun and W. Kim, *Global Calderón-Zygmund estimate for p-Laplacian parabolic system*, Math. Anal. **369** (2020).

[41] S. Byun and S. Ryu, *Global weighted estimates for the gradient of solutions to nonlinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **30** (2) (2013), 291-313.

[42] S. Byun and Y. Cho, *Nonlinear gradient estimates for generalized elliptic equations with nonstandard growth in nonsmooth domains*, Nonlinear Analysis, **140** (2016), 145-165.

[43] S. Byun and Y. Youn, *Riesz potential estimates for a class of double phase problems*, J. Differential Equations **264** (2) (2018), 1263-1316.

[44] S. Byun and L. Wang, *Elliptic equations with BMO coefficients in Reifenberg domains*, Comm. Pure Appl. Math. **57** (10)(2004), 1283-1310.

[45] S. Byun and L. Wang, *Nonlinear gradient estimates for elliptic equations of general type*, Calc. Var. Partial Differential Equations **45** (3-4)(2012), 403-419.

[46] L. Caffarelli and C. Cabré, *Fully nonlinear equations*, American Mathematical Society Colloquium Publications 43 (American Mathematical Society, Providence, RI, 1995).

[47] L. Caffarelli and I. Peral, *On $W^{1,p}$ estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math. **51** (1)(1998), 1-21.

[48] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85-139.

[49] A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289-309.

[50] Y. Cho, *Global gradient estimates for divergence-type elliptic problems involving general nonlinear operators*, J. Differential Equations **264** (2018), 6152-6190.

[51] A. Cianchi and L. Pick, *Sobolev embeddings into spaces of Campanato, Morrey, and Hölder type*, J. Math. Anal. Appl. **282** (1) (2003), 128-150.

[52] A. Cianchi, *Optimal Orlicz-Sobolev embeddings*, Rev. Mat. Iberoamericana **20** (2004), 427-474.

[53] A. Cianchi, *Orlicz-Sobolev boundary trace embeddings*, Mathematische Zeitschrift **266** (2010), 431-449.

[54] A.Cianchi and V. G. Mazya, *Global Lipschitz regularity for a class of quasilinear elliptic equations*, Comm. Partial Differential Equations **36** (2011), no. 1, 100-133.

[55] A.Cianchi and V. G. Mazya, *Global Lipschitz regularity of the gradient for a class of nonlinear elliptic systems*, Arch. Ration. Mech. Anal. **212** (2014), no. 1, 129-177.

[56] I. Chlebicka, P. Gwiazda, A. Świerczewska-Gwiazda and A. Wróblewska-Kamińska, *Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces*, Springer Monographs in Mathematics, 2021.

[57] M. Colombo and G. Mingione, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal. **215** (2) (2015), 443-496.

[58] M. Colombo and G. Mingione, *Bounded minimisers of double phase variational integrals*, Arch. Ration. Mech. Anal. **218** (1) (2015), 219-273.

[59] M. Colombo and G. Mingione, *Calderón-Zygmund estimates and non-uniformly elliptic operators*, J. Funct. Anal. **270** (4) (2016), 1416-1478.

[60] M. G. Crandall, H. Ishii and P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), 1-67.

[61] G. Cupini, *Local boundedness of minimizers with limit growth conditions*, J. Optim. Theory Appl. **166** (1) (2015), 1-22.

[62] G. Cupini, P. Marcellini, E. Mascolo, *Regularity of minimizers under limit growth conditions*, Nonlinear Anal. **153** (2017), 294-310.

[63] J. Dávila, P. Felmer and A. Quaas, *Alexandroff-Bakelman-Pucci estimate for singular or degenerate fully nonlinear elliptic equations*, C. R. Math. Acad. Sci. Paris **347** (2009), 1165-1168.

[64] J. Dávila, P. Felmer and A. Quaas, *Harnack inequality for singular fully nonlinear operators and some existence results*, Calc. Var. Partial Differ. Equ. **39** (2010), 557-578.

[65] J. V. Da Silva and G. Ricarte, *Geometric gradient estimates for fully nonlinear models with nonhomogeneous degeneracy and applications*, Calc. Var. Partial Differ. Equ. **59** (2020), Paper 161.

[66] J. V. Da Silva and H. Vivas, *Sharp regularity for degenerate obstacle type problems: a geometric approach*, Discrete Contin. Dyn. Syst. **41** (2021), 1359-1385.

[67] C. De Filippis, *Regularity for solutions of fully nonlinear elliptic equations with nonhomogeneous degeneracy*, Proc. Roy. Soc. Edinburg Sect. A **151** (2021), 110-132.

[68] C. De Filippis, *Gradient bounds for solutions to irregular parabolic equations with (p, q)-growth*, **59** (5) (2020), 1-32.

[69] C. De Filippis, *Optimal gradient estimates for multi-phase integrals*, Mathematics in Engineering **4** (5) (2022), 1-36.

[70] C. De Filippis and G. Mingione, *A borderline case of Calderón-Zygmund estimates for nonuniformly elliptic problems*, St. Peterburg Math. J. **31** (3) (2019), 82-115.

[71] C. De Filippis and J. Oh, *Regularity for multi-phase variational problems*, J. Differential Equations **267** (3) (2019), 1631-1670.

[72] L. Diening, *Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces*, Bull. Sci. Math. **129** (8) (2005), 657-700.

[73] L. Diening and F. Ettwein, *Fractional estimates for non-differentiable elliptic systems with general growth*, Forum Math. **20** (3) (2008), 523-556.

[74] L. Diening, B. Stroffolini and A. Verde, *Everywhere regularity of functionals with φ-growth*, Manuscripta Math. **129** (4) (2009), 449-481.

[75] L. Diening, B. Stroffolini and A. Verde, *The φ-harmonic approximation and the regularity of φ-harmonic maps*, J. Differ. Equ. **253** (2012), 1943-1958.

[76] F. Duzaar and G. Mingione, *Harmonic type approximation lemmas*, J. Math. Anal. Appl. **352** (2009), 301-335.

[77] F. Duzaar, G. Mingione and K. Steffen, *Parabolic systems with polynomial growth and regularity*, Mem. Amer. Math. Soc., **214** (1005) (2011).

[78] L. Esposito, F. Leonetti and G. Mingione, *Regularity for minimizers of functionals with $p - q$ growth*, Nonlinear Differential Equations Appl. **6** (2) (1999), 133-148.

[79] L. Esposito, F. Leonetti and G. Mingione, *Regularity results for minimizers of irregular integrals with (p, q) growth*, Forum Math. **14** (2) (2002), 245-272.

[80] L. Esposito, F. Leonetti and G. Mingione, *Sharp regularity for functionals with (p, q) growth*, J. Differential Equations **204** (1) (2004), 5-55.

[81] L. Esposito, G. Mingione and C. Trombetti, *On the Lipschitz regularity for certain elliptic problems*, Forum Math. **18** (2) (2006), 263-292.

[82] L. C. Evans, *A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic equations*, J. Differential Equations **45** (3) (1982), 356-373.

[83] X. Fan, *An imbedding theorem for Musielak-Sobolev spaces*, Nonlinear Anal. **75** (4) (2012), 1959-1971.

[84] X. Fan and C.-X. Guan, *Uniform convexity of Musielak-Orlicz-Sobolev spaces and applications*, Nonlinear Anal. **73** (1) (2010), 163-175.

[85] Y. Fang, V. D. Radulescu and C. Zhang, *Regularity of solutions to degenerate fully nonlinear elliptic equations with variable exponent*, Bull. London Math. Soc. **0** (2021), 1-16.

[86] I. Fonseca, J. Malý and G. Mingione, *Scalar minimizers with fractal singular sets*, Arch. Ration. Mech. Anal. **172** (2) (2004), 295-307.

[87] N. Fusco and C. Sbordone, *Higher integrability of the gradient of minimizers of functionals with nonstandard growth conditions*, Comm. Pure Appl. Math. **43** (5) (1990), 673-683.

[88] F. W. Gehring, *The L^p-integrability of the partial derivatives of a quasiformal mapping*, Acta Math. **130** (1973), 265-277.

[89] M. Giaquinta and E. Guisti, *On the regularity of the minima of variational integrals*, Acta Math. **148** (1982), 31-46.

[90] E. Giusti, *Direct methods in the calculus of variations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[91] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations od second order*, Grundlehren der Mathematischen Wissenschaften, vol.**224**, Springer, Berlin, 1977

[92] P. Harjulehto and P. Hästö, *Orlicz Spaces and Generalized Orlicz Spaces*, Lecture notes in mathematics, **2236**, Springer, 2019.

[93] P. Harjulehto, P. Hästö and R. Klén, *Generalized Orlicz spaces and related PDE*, Nonlinear Anal. **143** (2016), 155-173.

[94] P. Harjulehto, P. Hästö and M. Lee *Hölder continuity of quasiminimizers and ω-minimizers of functionals with generalized Orlicz growth*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **XXII** (2021), no. 2, 549-582.

[95] P. Harjulehto, P. Hästö and O. Toivanen, *Hölder regularity of quasiminimizers under generalized growth conditions*, Calc. Var. Partial Differential Equations **56** (2) (2017), Art. 22, 26 pp.

[96] P. Harjulehto, T. Kuusi, T. Lukkari, N. Marola and M. Parviainen, *Harnack's inequality for quasiminimizers with nonstandard growth conditions*, J. Math. Anal. Appl. **344**:1 (2008), 504-520.

[97] P. Hästö and J. Ok, *Maximal regularity for local minimizers of non-autonomous functionals*, J. Eur. Math. Soc., to appear.

[98] P. Hästö and J. Ok, *Calderón-Zygmund estimates in generalized Orlicz spaces*, J. Differential Equations **267** (2019), 2792-2823.

[99] C. Imbert, *Alexandroff-Bakelman-Pucci estimate and Harnack inequality for degenerate/singular fully non-linear equations*, J. Differential Equations **250** (2011), 1553-1574.

[100] C. Imbert and L. Silvestre, *$C^{1,\alpha}$ regularity of solutions of some degenerate fully non-linear elliptic equations*, Adv. Math. **233** (2013), 196-206.

[101] C. Imbert and L. Silvestre, *Estimates on elliptic equations that hold only where the gradient is large*, J. Eur. Math. Soc. **18** (2016), 1321-1338.

[102] T. Iwaniec, *On L^p-integrability in PDEs and quasiregular mappings for large exponents*, Ann. Acad. Sci. Fenn. Ser. A I Math. **7** (2)(1982), 301-322.

[103] T. Iwaniec, *Projections onto gradient fields and L^p-estimates for degenerated elliptic operators*, Studia Math. **75** (3)(1983), 293-312.

[104] D. Jesus, *A degenerate fully nonlinear free transmission problem with variable exponents*, Calc. Var. Partial Differ. Equ. **61** (2022), Paper 29.

[105] J. Kinnunen and S. Zhou, *A local estimate for nonlinear equations with discontinuous coefficients*, Comm. Partial Differential Equations **24** (11-12)(1999), 2043-2068.

[106] J. Kristensen and G. Mingione, *The singular set of minima of integral functionals*, Arch. Ration. Mech. Anal. **180** (2006), 331-398.

[107] J. Kristensen and G. Mingione, *Boundary regularity in variational problems*, Arch. Ration. Mech. Anal. **198** (2010), 369-455.

[108] J. Kinnunen and N. Shanmugalingam, *Regularity of quasi-minimizers of metric spaces*, Manuscripta Math. **105**:3 (2001), 401-423.

[109] O. A. Ladyzhenskaya and N. N. Uraltseva, *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968.

[110] J. L. Lewis, *Regularity of the derivatives of solutions to certain degenerate elliptic equations*, Indiana Univ. Math. J. **32** (6) (1983),849-858.

[111] G. M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations. **16** (1991), 311-361.

[112] P. Lindqvist, *Notes on the p-Laplace equation*, Report Univ. Jyväskylä, vol. 102, Univ. Jyväskylä,Jyväskylä, 2006.

[113] J. J. Manfredi, *Regularity for minima of functionals with p-growth*, J. Differential Equations **76** (2) (1988), 203-212.

[114] P. Marcellini, *On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (5) (1986), 391-409.

[115] P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Ration. Mech. Anal. **105** (3) (1989), 267-284.

[116] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p, q-growth conditions*, J. Differential Equations **90** (1) (1991), 1-30.

[117] G. Mingione, *Regularity of minima: an invitation to the dark side of the calculus of variations*, Appl. Math. **51** (2006), 355-425.

[118] G. Mingione and V. Radulescu, *Recent developments in problems with nonstandard growth and nonuniformly ellipticity*, J. Math. Anal. Appl. **501** (2021), 125197.

[119] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, 1034, Springer-Verlag, Berlin, 1983.

[120] T. Nguyen and T. Phan, *Interior gradient estimates for quasilinear elliptic equations*, Calc. Var. Partial Differential Equations **55**, (3)(2016) Art. 59, 33.

[121] J. Ok, *Regularity of ω-minimizers for a class of functionals with non-standard growth*, Calc. Var. Partial Differential Equations **56** (2) (2017), Art. 48, 31 pp.

[122] J. Ok, *Partial Hölder regularity for elliptic systems with non-standard growth*, J. Funct. Anal. **274** (3) (2018), 723-768.

[123] J. Ok, *Regularity for double phase problems under additional integrability assumptions*, Nonlinear Analysis **194** (2020), 111408.

[124] M.A. Ragusa and A. Tachikawa, *Partial regularity of $p(x)$-harmonic maps*, Trans. Am. Math. Soc. **365** (2013), 3329-3353.

[125] M.A. Ragusa and A. Tachikawa, *Regularity regularity of minimizers of $p(x)$-energy functionals*, Ann. IHP Anal. non Linéare **33** (2017), 451-476.

[126] H.L. Royden and P.M. Fitzpatrick, *Real Analysis*, Upper Saddle River, NJ : Pearson Prentice Hall, 3rd ed.,(2010).

[127] M. M. Rao and Z. D. Ren *Theory of Orlicz spaces*, Monogr. Textbooks Pure Appl. Math. **146**, Dekker, Newyork, 1991.

[128] R.E. Showalter, *Monotone operators in Banach spaces and Nonlinear Partial Differential equations*, Mathematical Surveys and Monographs, Vol. 49, American Mathematical Society, Providence, RI, 1997.

[129] T. Schmidt, *Regularity of minimizers of $W^{1,p}$-quasiconvex variational integrals with (p,q)-growth*, Calc. Var. Partial Differential Equations **32** (1) (2008), 1-24.

[130] T. Schmidt, *Regularity of relaxed minimizers of quasiconvex variational integrals with (p,q)-growth*, Arch. Ration. Mech. Anal. **193** (2) (2009), 311-337.

[131] M. Sidi El Vally, *Strongly nonlinear elliptic problems in Musielak-Orlicz-Sobolev spaces*, Adv. Dyn. Syst. Appl. **8** (1) (2013), 115-124.

[132] A. Tachikawa and K.Usuba, *Regularity results up to the boundary for minimizers of $p(x)$-energy with $p(x) > 1$*, Manus.Math. **152** (2017), 127-151.

[133] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. **138**, (1977), 219-240.

[134] N.N. Ural'tseva, *Degenerate quasilinear elliptic systems*, Zap. Na. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **7**, (1968), 184-222.

[135] A. Verde, *Calderón-Zygmund estimates for systems of ϕ-growth*, J. Convex Anal. **18** (2011), no. 1, 67-84.

[136] F. Yao, *Regularity theory in Orlicz spaces for the parabolic polyharmonic equations*, Arch. Math. (Basel) **90** (5) (2008), 429-439.

[137] F. Yao, H. Jia, L. Wang and S. Zhou, *Regularity theory in Orlicz spaces for the Poisson and heat equations*, Commun. Pure Appl. Anal. **7** (2) (2008), 407-416.

[138] F. Yao and S. Zhou, *Calderón-Zygmund estimates for a class of quasilinear elliptic equations*, J. Funct. Anal. 272 (2017), 1524-1552.

[139] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (4) (1986), 675-710.

[140] V.V. Zhikov, *Lavrentiev phenomenon and homogenization for some variational problems*, C. R. Acad. Sci. Paris Sér. I Math. **316** (5) (1993), 435-439.

[141] V.V. Zhikov, *On Lavrentiev's phenomenon*, Russian J. Math. Phys. **3** (2) (1995), 249-269.

[142] V.V. Zhikov, *On some variational problems*, Russian J. Math. Phys. **5** (1) (1997), 105-116.

[143] V.V. Zhikov, S.M. Kozlov and O.A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994.

국문초록

이 학위논문에서는 오리츠 다상 문제를 포함하고 비표준 성장 조건 및 불균일한 타원형 특성을 나타내는 일반적인 종류의 범함수에 대한 종합적인 정칙성 결과와, 이를 위한 최적의 조건에 대해 조사한다. 우선, 조절 계수가 횔더 연속보다 약화된 경우의 오리츠 다상 범함수의 최소자에 대한 다양한 정칙성 결과를 보이기 위한 통일된 논의를 새롭게 이용한다. 더 나아가, 이러한 범함수의 최소자에 대해 특정 르벡 공간에 포함되거나 횔더 연속이라는 추가 조건이 있을 경우, 정칙성 결과들을 얻기 위해 비선형성에 주어져야 할 최적의 조건들을 찾는다.

두 번째로, 오리츠 이중 위상 및 다중 위상 형태의 발산형 타원 방정식을 고려한다. 비선형성에 최소의 조건을 부여하면서, 이러한 타원 방정식의 분포해에 대한 국소적 칼데론-지그문드 추정을 얻는다. 마지막으로 축퇴/특이 완전 비선형 타원 방정식의 점성 해에 대해 관련 연산자의 최소 정칙성 조건을 찾아, 이 해의 그래디언트 횔더 정칙성을 보인다.

주요어휘: 오리츠 위상 문제, 정칙성, 비표준 성장, 칼데론-지그문드 이론, 축퇴/특이 완전 비선형 방정식, 점성해
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