NEW CONCENTRATION PHENOMENA FOR RADIAL SIGN-CHANGING SOLUTIONS OF FULLY NONLINEAR ELLIPTIC EQUATIONS
NUOVI FENOMENI DI CONCENTRAZIONE PER SOLUZIONI RADIALI DI SEGNO VARIABILE DI EQUAZIONI ELLITTICHE COMPLETAMENTE NON LINEARI

FABIANA LEONI

ABSTRACT. We present recent results about radial sign-changing solutions of a class of fully nonlinear elliptic Dirichlet problems posed in a ball, driven by the extremal Pucci's operators and provided with power zero order terms. We show that new critical exponents appear, related to the existence or nonexistence of sign-changing solutions and due to the fully nonlinear character of the considered problem. Furthermore, we analyze the new concentration phenomena occurring as the exponents approach the critical values.

SUNTO. Vengono presentati alcuni risultati recenti riguardanti soluzioni radiali di segno variabile per una classe di problemi di Dirichlet completamente non lineari, posti in domini sferici, aventi gli operatori estremali di Pucci come parte principale e termini di ordine zero di tipo potenza. Mostreremo come l'esistenza o non esistenza di soluzioni sia regolata da nuovi esponenti critici tipici del carattere completamente non lineare del problema considerato. Analizzeremo inoltre i nuovi fenomeni di concentrazione che si verificano quando gli esponenti convergono ai valori critici.

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Dipartimento di Matematica, Università di Bologna

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1. Introduction

We focus on the existence of radially symmetric, sign-changing solutions of fully nonlinear uniformly elliptic Dirichlet problems of the form

$$
\begin{cases} -\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) = |u|^{p-1}u & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \tag{1}
$$

where

- $B = B_1(0)$ is the unit ball in $\mathbb{R}^n$;
- $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ are Pucci's extremal operators with ellipticity constants $0 < \lambda \leq \Lambda$;
- $p > 1$.

Our aim is to present the recent results obtained in [15] about the optimal threshold to be imposed on the exponent p for the existence of radial sign-changing solutions of problems (1).

Pucci's extremal operators $\mathcal{M}^{-}_{\lambda,\Lambda}$ and $\mathcal{M}^{+}_{\lambda,\Lambda}$ are respectively defined as

$$
\mathcal{M}^{-}_{\lambda,\Lambda}(X) = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AX) = \lambda \sum_{\mu_i>0} \mu_i + \Lambda \sum_{\mu_i<0} \mu_i
$$

$$
\mathcal{M}^{+}_{\lambda,\Lambda}(X) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AX) = \Lambda \sum_{\mu_i>0} \mu_i + \lambda \sum_{\mu_i<0} \mu_i
$$

where $\mathcal{A}_{\lambda,\Lambda} = \{A \in \mathcal{S}_n \,:\, \lambda\, I_n \leq A \leq \Lambda\, I_n\}$, ($I_n$ identity matrix), and $\mu_1, \dots, \mu_n$ are the eigenvalues of the matrix $X \in \mathcal{S}_n$. They have been introduced in [23], and extensively studied in [6]. They are the prototypes of second order, fully nonlinear uniformly elliptic operators. Being extremal not only with respect to linear operators but in the whole class of uniformly elliptic operators with the same ellipticity constants, they play a crucial role in the elliptic regularity theory for fully nonlinear equations, see [6]. Moreover, they frequently arise in the context of optimal stochastic control problems, see e.g. [13, 19], with special application to mathematical finance.

Pucci's extremal operators are mutually related by the relationship

$$
\mathcal{M}^{-}_{\lambda,\Lambda}(-X) = -\mathcal{M}^{+}_{\lambda,\Lambda}(X)
$$

Moreover, they can be seen as a generalization of Laplace operator, since

$$\mathcal{M}^-_{\lambda,\lambda}(X) = \mathcal{M}^+_{\lambda,\lambda}(X) = \lambda \operatorname{tr}(X)$$

However, as soon as $\lambda < \Lambda$, the operators $\mathcal{M}^\pm_{\lambda,\lambda}$ are neither linear nor in divergence form. As an example, observe that for the homogeneous planar equation, one has

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = 0 \quad \text{in } \mathbb{R}^2$$

$$\Longleftrightarrow$$

$$\Delta u = \left(\sqrt{\frac{\Lambda}{\lambda}} - \sqrt{\frac{\lambda}{\Lambda}}\right)\sqrt{-\det(D^2u)}$$

For a general uniformly elliptic operator F and for a general domain $\Omega \subset \mathbb{R}^n$, without any symmetry assumption, the best known existence result states that positive solutions of Dirichlet problems of the form

$$\begin{cases} -F(D^2u) = u^p \text{ in } \Omega \\ u > 0 \text{ in } \Omega\,,\ u = 0 \text{ on } \partial\Omega \end{cases}$$

do exist if $p \le \frac{\tilde{n}_-}{\tilde{n}_- - 2}$, see [24, 25]. In case F is the Pucci operator $\mathcal{M}^+_{\lambda,\Lambda}$, then the condition on p improves up to $p \le \frac{\tilde{n}_+}{\tilde{n}_+ - 2}$, where $\tilde{n}_\pm$ are the dimension like parameters associated with the operators $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$ respectively, defined as

$$\begin{aligned} \tilde{n}_+ &= \tfrac{\lambda}{\Lambda}(n-1) + 1 \le n \quad \text{for } \mathcal{M}^+_{\lambda,\Lambda} \\ \tilde{n}_- &= \tfrac{\Lambda}{\lambda}(n-1) + 1 \ge n \quad \text{for } \mathcal{M}^-_{\lambda,\Lambda} \end{aligned} \tag{2}$$

We will call $\tilde{n}_\pm$ the "effective dimensions", using the terminology introduced in [22], and we will always assume that $\tilde{n}_\pm > 2$. The effective dimensions play a key role in existence results for fully nonlinear uniformly elliptic equations. Note that if $\Lambda = \lambda$ then $\tilde{n}_+ = \tilde{n}_- = n$.

The above mentioned existence result relies on the Liouville type theorems proved in [7], stating that

$$\exists\, u>0\,,\ -\mathcal{M}^-_{\lambda,\Lambda}(D^2u)\geq u^p \ \text{ in } \mathbb{R}^n \Longleftrightarrow p>\tfrac{\tilde n_-}{\tilde n_- -2}$$

$$\exists\, u>0\,,\ -\mathcal{M}^+_{\lambda,\Lambda}(D^2u)\geq u^p \ \text{ in } \mathbb{R}^n \Longleftrightarrow p>\tfrac{\tilde n_+}{\tilde n_+ -2}$$

The thresholds $\frac{\tilde n_\pm}{\tilde n_\pm -2}$ are optimal for the existence of supersolutions in $\mathbb{R}^n$ and of solutions in $\mathbb{R}^n\setminus\{0\}$, see [1], and they both reduce, in the semilinear case $\Lambda=\lambda$, to the Serrin exponent $\frac{n}{n-2}$. However, for the semilinear equation $-\Delta u=u^p$, as it is well known, the optimal condition for the existence of entire positive solutions is $p\geq p^*=\frac{n+2}{n-2}$, where $p^*=2^*-1$ is the so called Sobolev exponent, and this in turn implies that positive solutions in bounded domains do exist if $p<p^*$. These existence results are intimately related to the (lack of) compactness properties of the Sobolev embeddings. Moreover, the entire solutions existing in the critical case $p=p^*$ realize the best constant in the Sobolev inequality, see [27]. In other words, the critical nature of the Sobolev exponent p^* may be largely interpreted in view of the structural properties of the operator and the functional setting behind the equation.

In the fully nonlinear framework, more precise information and optimal threshold exponents for the existence of solutions can be obtained for Pucci's operators in the radial setting. In particular, in [12] some critical exponents $p^*_\pm$ acting as thresholds for the existence of entire positive radial solutions or positive solutions in balls have been proved to exist. At least for positive radial solutions, the exponents $p^*_\pm$ play the same role for Pucci's operators as the critical Sobolev exponent $p^*=\frac{n+2}{n-2}$ for the Laplacian. Though their appearance is motivated exclusively as threshold for the existence of entire radial solutions or solutions in balls, the recent results of [3], where some weighted energies associated with radial solutions of (1) are introduced and proved to be asymptotically preserved by almost critical solutions, suggest that the critical exponents reflect some intrinsic properties of the operators, maybe beyond the radial setting. This motivated the further investigation pursued in [15] about radial sign-changing solutions.

We proved in [15] that for problem (1) with operator $\mathcal{M}^+_{\lambda,\Lambda}$ a new critical exponent p^{**}_+ appears as optimal threshold for the existence of radial sign-changing solutions, and it satisfies, as long as $\lambda < \Lambda$,

$$p_-^* < p_+^{**} < p_+^* \,.$$

This new feature is essentially due to the nonlinear character of the involved operators: when considering sign-changing solutions for a single problem with the operator given by either $\mathcal{M}^-_{\lambda,\Lambda}$ or $\mathcal{M}^+_{\lambda,\Lambda}$, the negative part of the solutions are positive solutions for the other operator, either $\mathcal{M}^+_{\lambda,\Lambda}$ or $\mathcal{M}^-_{\lambda,\Lambda}$ respectively. Hence, for each single problem both critical exponents related to both operators come into play in order to determine the optimal threshold for the existence of solutions.

Furthermore, when studying the asymptotic behavior of almost critical sign-changing solutions, new concentration phenomena and new limit solutions occur. In particular, the asymptotic analysis of radial nodal solutions with any number m of nodal domains shows that the behavior can be different in each nodal region and may also depend on m being even or odd. Indeed, while in some nodal domain there is blow up and concentration, in others the solutions are bounded and converge to a finite limit, radically differently from what happens for the classical semilinear case.

In Section 2 we recall the relevant results for positive solutions, with special reference to solutions in balls, see [3], annular domains, see [16], exterior domains, see [14] (see also [18]). In Section 3 we describe the results of [15] about sign-changing solutions.

2. Positive solutions

For radial functions $u(x) = u(|x|)$, the fully nonlinear equations

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) = u^p$$

reduce to ordinary differential equations, since the eigenvalues of the hessian matrix $D^2u(x)$ are nothing but

- $u''(r)$, which is simple
- $\frac{u'(r)}{r}$, which has with multiplicity $n-1$.

Let us focus initially on the operator $\mathcal{M}^+_{\lambda,\Lambda}$. If $u(x) = u(|x|)$ is a C^2 positive radial solution of $-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = u^p$, then u satisfies

- if $u'' \leq 0$ and $u' \geq 0$, then

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \lambda\, u''(r) + \Lambda(n-1)\frac{u'(r)}{r} = -u(r)^p$$

- if $u'' \leq 0$ and $u' \leq 0$, then

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \lambda\Delta u = \lambda\, u''(r) + \lambda(n-1)\frac{u'(r)}{r} = -u(r)^p$$

- if $u'' \geq 0$ and $u' \leq 0$, then

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \Lambda\, u''(r) + \lambda(n-1)\frac{u'(r)}{r} = -u(r)^p$$

In other words, the coefficients of u'' and u' jumps according to the changes of concavity and monotonicity of u respectively, and we can write

- where $u(r)$ is concave and increasing

$$u''(r) + (\tilde{n}_- - 1)\frac{u'(r)}{r} = -\frac{u(r)^p}{\lambda} \tag{3}$$

- where $u(r)$ is concave and decreasing

$$u''(r) + (n-1)\frac{u'(r)}{r} = -\frac{u(r)^p}{\lambda} \tag{4}$$

- where $u(r)$ is convex and decreasing

$$u''(r) + (\tilde{n}_+ - 1)\frac{u'(r)}{r} = -\frac{u(r)^p}{\Lambda} \tag{5}$$

where $\tilde{n}_\pm$ are the dimension like parameters defined in (2). For the operator $\mathcal{M}^-_{\lambda,\Lambda}$, we obtain analogous equations with Λ and λ (and $\tilde{n}_+$, $\tilde{n}_-$) interchanged. Let us stress the fact that the different equations arising in the different regimes of monotonicity/convexity properties of the solution u are structurally identical: they differ just for the constant coefficients in front of the lower order terms.

We further notice that when looking for radial solutions either in balls or in the whole space, any positive radial solution u is always deceasing and therefore only two ODEs come into play : the ones with n and $\tilde{n}_+$ for $\mathcal{M}^+_{\lambda,\Lambda}$, and the ones with n and $\tilde{n}_-$ for $\mathcal{M}^-_{\lambda,\Lambda}$.

In the semilinear case $\Lambda = \lambda$, the problem is ruled by the single equation

$$u'' + (n-1)\frac{u'}{r} = -u^p\,, \tag{6}$$

and it is well known that positive solutions defined in the whole interval $[0, +\infty)$ exist if and only if

$$p \geq p^* = \frac{n+2}{n-2}\,.$$

For $p < p^*$, any solution u of (6) becomes zero at a first finite radius, independently on the initial value $u(0)$. These functions correspond to positive solutions of the semilinear equation $-\Delta u = u^p$ in balls, vanishing on the boundary.
For $p = p^* = \frac{n+2}{n-2}$, the solutions of (6) are explicitly known. They are the radial solutions of

$$-\Delta u = u^{\frac{n+2}{n-2}} \qquad \text{in } \mathbb{R}^N\,,$$

and they are functions of the form

$$U_\alpha(x) = \left(\frac{\alpha}{1 + \frac{\alpha |x|^2}{n(n-2)}} \right)^{\frac{n-2}{2}} \tag{7}$$

for any $\alpha \geq 0$. The functions U_α are also known as the "talentian" functions and, as it is well known, they realize the best constant S in the Sobolev inequality, that is

$$S = \frac{\|DU_\alpha\|_{L^2}}{\|U_\alpha\|_{L^{2^*}}} = \inf_{u \in H^1(\mathbb{R}^n)\setminus\{0\}} \frac{\|Du\|_{L^2}}{\|u\|_{L^{2^*}}}\,. \tag{8}$$

For $p > p^*$, the positive solutions u of (6) existing for all $r \geq 0$ correspond to radial entire solutions of the semilinear problem $-\Delta u = u^p$ in $\mathbb{R}^n$, and they are known to satisfy

$$\exists \lim_{r \to +\infty} r^{\frac{2}{p-1}} u(r) > 0\,.$$

Let us further recall that the condition $p < p^* = \frac{n+2}{n-2}$ suffices for the existence of a positive solution of the semilinear Dirichlet problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for any bounded domain $\Omega \subset \mathbb{R}^n$. Moreover, the assumption $p < p^* = \frac{n+2}{n-2}$ is also necessary in case Ω is a starshaped domain, as it follows by the Pohozaev identity. The nonexistence of solutions for $p \geq p^* = \frac{n+2}{n-2}$ is strictly related to the lack of compactness for the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*=p^*+1}(\Omega)$. Thus, by the concentration–compactness principle, see [20, 21], a concentration phenomenon for almost critical solutions occurs. The problem has been largely studied and there are many many contributions in the literature, see e.g. [2, 5, 26]. In the particular case when the $\Omega = B$ is a ball, then the unique positive (radial) solutions u_ϵ of the homogeneous Dirichlet problem with exponent $p_\epsilon = \frac{n+2}{n-2} - \epsilon$ blow up and concentrate at the center of the ball as $\epsilon \to 0$, while their energy satisfies

$$J(u_\epsilon) = \left(\frac{1}{2} - \frac{1}{p_\epsilon + 1}\right) \int_B u_\epsilon^{p_\epsilon+1}\, dx \ \stackrel{\epsilon\to 0}{\longrightarrow}\ \frac{1}{n} S^n,$$

where S is the constant defined in (8). The local profile of u_ϵ, suitably rescaled, is that of the talentian function $U(x) = U_1(x)$ defined in (7).

In the fully nonlinear case $\Lambda > \lambda$, the Cauchy problem associated with the ODEs (4), (5) has been thoroughly analyzed in [12], where it is proved that there exist unique radial critical exponents $p^*_\pm$ associated with $\mathcal{M}^\pm_{\lambda,\Lambda}$ respectively, such that

$$\exists\, u > 0 \text{ radial solution of } \ -\mathcal{M}^\pm_{\lambda,\Lambda}(D^2u) = u^p \ \text{ in } \mathbb{R}^n$$

$$\Longleftrightarrow$$

$$p \geq p^*_\pm$$

The radial critical exponents $p^*_\pm$ are not explicitly known as functions of the effective dimensions, but they are proved to satisfy

$$\max\left\{\tfrac{\tilde{n}_+}{\tilde{n}_+-2}, \tfrac{n+2}{n-2}\right\} < p^*_+ < \tfrac{\tilde{n}_++2}{\tilde{n}_+-2} \qquad \text{for } \mathcal{M}^+_{\lambda,\Lambda}$$

$$\tfrac{\tilde{n}_-+2}{\tilde{n}_--2} < p^*_- < \tfrac{n+2}{n-2} \qquad \text{for } \mathcal{M}^-_{\lambda,\Lambda}$$

Moreover, one has

- for $p < p^*_\pm$ there exist positive solutions in balls vanishing on the boundary. In this case, the radial setting is not a restriction since the moving plane technique of Gidas, Ni, Nirenberg applies to positive solutions, see [8]. For other partial symmetry results in the fully nonlinear framework, also for sign-changing solutions, see [4].
- for $p = p^*_\pm$ the critical solutions U are showed to be fast decaying as $r \to +\infty$, meaning that

$$\exists \lim_{r\to+\infty} r^{\tilde{n}_\pm - 2} U(r) > 0$$

Even if the critical solutions U are not explicitly known as in the semilinear case, in [3] we proved that for each U there exists a unique radius $r_0 > 0$ such that $U''(r_0) = 0$, and there exist positive constants $c, C > 0$ depending on $\lim_{r\to+\infty} r^{\tilde{n}_\pm-2}U(r)$ such that

$$\frac{U(r_0)}{\left(1 + C(r^2 - (r_0)^2)\right)^{\frac{\tilde{n}_\pm-2}{2}}} \le U(r) \le \frac{U(r_0)}{\left(1 + c(r^2 - (r_0)^2)\right)^{\frac{\tilde{n}_\pm-2}{2}}}$$

for all $r \ge r_0$.

- for $p > p^*_\pm$ the existing entire solutions u may be, depending on p, either slow decaying, that is satisfying

$$\exists \lim_{r\to+\infty} r^{2/p-1}u(r) > 0$$

or pseudo–slow decaying, that is satisfying

$$0 < \liminf_{r\to+\infty} r^{2/p-1}u(r) < \limsup_{r\to+\infty} r^{2/p-1}u(r) < +\infty$$

In the latter case, the functions $u''(r)$ have infinitely many zeros and solutions u change convexity infinitely many times. Let us emphasize that the pseudo slow behavior at infinity never occurs in the semilinear case.

More than that, in [3] we analyzed the asymptotic behavior as $\epsilon \to 0$ of almost critical solutions u_ϵ, defined as the positive solutions in the unit ball, vanishing on the boundary, with almost critical exponents $p_\epsilon = p^*_\pm - \epsilon$. We obtained the following result, for the proof of which we refer to [3].

Theorem 2.1. *Let $\{u_\epsilon\}$ be a sequence of positive almost critical solutions, and let $M_\epsilon = u_\epsilon(0) = \|u_\epsilon\|_\infty$. Then:*

i) $\lim_{\varepsilon\to 0} M_\varepsilon = +\infty$;

ii) $u_\varepsilon \to 0$ *in* $C^2_{\mathrm{loc}}(\overline{B}\setminus\{0\})$ *as* $\varepsilon \to 0$;

iii) *the rescaled functions*

$$\tilde{u}_\varepsilon(r) = \frac{1}{M_\varepsilon} u_\varepsilon\left(\frac{r}{M_\varepsilon^{\frac{p_\varepsilon-1}{2}}}\right), \qquad r = |x| < M_\varepsilon^{\frac{p_\varepsilon-1}{2}}$$

converge in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ *to* U_1*, the fast decaying entire critical solution satisfying* $U_1(0) = 1$;

iv) $(M_\varepsilon)^{\frac{p_\varepsilon(\tilde n_\pm-2)-\tilde n_\pm}{2}} u_\varepsilon(r) \to c\left(\frac{1}{r^{\tilde n_\pm-2}}-1\right) \quad in\ C^2_{\rm loc}(\overline{B}\setminus\{0\})$.

Remark 2.1. *By a continuity argument, the exponents $p_*^\pm$ are proved to be the optimal thresholds for the existence of positive solutions also in some bounded domains obtained as small perturbations of balls, see* [11]. *But, in general, the conditions on p for the existence of positive solutions depend on the topology of the domain. For annular domains, (radial) positive solutions exist for all $p>1$ and for any rotationally invariant uniformly elliptic operator of the form $F(x,D^2u)$, see* [16].

The critical character of the exponents $p^*_\pm$ is confirmed by the existence and uniqueness results obtained in [14] for the exterior Dirichlet problem associated with Pucci's operators. Namely, for the existence of positive solutions of the problem

$$-\mathcal{M}^\pm_{\lambda,\Lambda}(D^2u)=u^p \text{ in } \mathbb{R}^n\setminus\overline{B}\,,\quad u=0 \text{ on } \partial B \tag{9}$$

one has the following result.

Theorem 2.2. *There exist positive radial solutions of problem* (9) *if and only if $p>p^*_\pm$. Moreover, for any $p>p^*_\pm$, problem* (9) *has a unique positive radial solution u^* satisfying*

$$\lim_{r\to+\infty} r^{\tilde n_\pm-2}u^*(r)=C>0\,, \tag{10}$$

and infinitely many positive radial solutions u satisfying

$$\lim_{r\to+\infty} r^{\tilde n_\pm-2}u(r)=+\infty\,. \tag{11}$$

Consistently with the case of entire solutions, the solution u^* satisfying (10) will be referred to as the fast decaying solution. It is characterized by an initial critical slop

$$(u^*)'(1)=\alpha^*_\pm(p)>0\,. \tag{12}$$

As far as solutions u satisfying (11) are concerned, they are proved to satisfy either

$$\lim_{r\to+\infty} r^{\frac{2}{p-1}}u(r)=c>0\,,$$

in which case they will be called slow decaying solutions, or

$$0<\liminf_{r\to+\infty} r^{\frac{2}{p-1}}u(r)<\limsup_{r\to+\infty} r^{\frac{2}{p-1}}u(r)<+\infty\,,$$

in which case they will be named pseudo-slow decaying solutions.

Let us observe that when studying the existence of radial solutions in exterior domains (as well as in annular domains), all the three equations (3), (4) and (5) corresponding to the different regimes of monotonicity/convexity properties of the solution come into play. Moreover, differently from the case of solutions in balls or in the whole space, in this case the initial slope imposed to any solution of the Cauchy problem associated with the ODEs (3), (4) and (5) plays a crucial role, at least when the exponent p satisfies $p > p^*_\pm$: a sufficiently large initial velocity will produce a solution in an annular domain, whereas small enough initial slopes (precisely $u'(1) \leq \alpha^*_\pm(p)$) yield solutions of the exterior Dirichlet problem.

In the semilinear case, a proof of Theorem 2.2 can be found in [17]. For the fully nonlinear case, the proof is given in [14] and it relies on different arguments for $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$: for $\mathcal{M}^-_{\lambda,\Lambda}$, we exploited the fact that $p^*_- > \frac{\tilde{n}_-+2}{\tilde{n}_--2}$ and we used some properties of solutions of supercritical semilinear problems; for $\mathcal{M}^+_{\lambda,\Lambda}$, for which $p^*_+ < \frac{\tilde{n}_++2}{\tilde{n}_+-2}$, a different proof is obtained as an application of Gauss-Green Theorem. However, let us stress the fact in the fully nonlinear setting a reflection argument analogous to the Kelvin transform which reduces a supercritical exterior Dirichlet problem to a subcritical Dirichlet problem in the punctured ball as in the semilinear case is not available.

Remark 2.2. *In the semilinear case, for the exterior Dirichlet problem, the Sobolev exponent $p^* = \frac{n+2}{n-2}$ acts as critical threshold for the existence of positive solutions even for non radial exterior domains, as proved in* [9]*, where solutions are constructed by using a perturbative method.*

3. Sign-changing solutions

The results of the previous section can be applied in order to obtain existence results for radial sign-changing solutions in a ball. Indeed, sign-changing solutions of problem (1) are nothing but constant sign radial solutions in balls and annuli glued together.

As before, let us first recall the known results for semilinear equations. For the Dirichlet problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } B \\ u(0) > 0\,, \quad u = 0 & \text{on } \partial B \end{cases} \tag{13}$$

radial sign-changing solutions do exist if and only if $p < p^* = \frac{n+2}{n-2}$. Moreover, for any fixed integer $m \geq 1$, if u_p^m denotes the solution having m nodal regions, then, for $p \to p^*$, one has

$$u_p^m \to 0 \quad \text{in } C^2_{loc}(B \setminus \{0\})\,,$$

$$\int_B |\nabla u_p^m(x)|^2\, dx = \int_B |u_p^m(x)|^{p+1}\, dx \to mS^n\,.$$

Furthermore, suitable rescalings of the restriction of u_p^m to each nodal region converge, up to the sign, to the entire positive solution U_1 of the critical equation defined by (7). We refer to [10] for the proof of the above results and further properties of the radial sign-changing solutions of problem (13).

In the fully nonlinear framework, as showed in [15], new critical exponents appear as threshold for the existence of sign-changing radial solutions and new concentration phenomena occur. The main difference with respect to the semilinear problem relies on the fact when considering sign-changing solutions of a single problem, with a fixed operator given by either $\mathcal{M}^+_{\lambda,\Lambda}$ or $\mathcal{M}^-_{\lambda,\Lambda}$, the negative part of a solution is a positive solution for the other operator, either $\mathcal{M}^-_{\lambda,\Lambda}$ or $\mathcal{M}^+_{\lambda,\Lambda}$ respectively. Thus, both the two different critical exponents $p^*_\pm$ come into play when analyzing each single problem. Keeping in mind that $p^*_- < p^*_+$ as soon as $\Lambda > \lambda$, one has the following existence result, whose proof can be found in [15].

Theorem 3.1. *We have:*

i) *for the operator $\mathcal{M}^-_{\lambda,\Lambda}$, radial sign-changing solutions of* (1) *with any number of nodal domains exist if and only if*

$$p < p^*_-\,;$$

ii) *for the operator* $\mathcal{M}^+_{\lambda,\Lambda}$*, there exists a new critical exponent* p^{**} *satisfying*

$$p^*_- < p^{**} < p^*_+ \,,$$

such that no radial sign-changing solutions to (1) *exist for* $p \geq p^{**}$*, while radial sign-changing solutions to* (1) *with any number of nodal domains exist at least for a sequence of exponents* $p_k \nearrow p^{**}$*.*

The difference between the two previous existence results is due to the facts that we are considering sign-changing solutions satisfying $u(0) > 0$ and that $p^*_- < p^*_+$. Indeed, for the operator $\mathcal{M}^-_{\lambda,\Lambda}$, the first nodal component of a solution is a positive solution relative to $\mathcal{M}^-_{\lambda,\Lambda}$ in some ball, and therefore it exists if and only if $p < p^*_-$. On the other hand, for $p < p^*_- < p^*_+$, negative solutions for the operator $\mathcal{M}^-_{\lambda,\Lambda}$ in annular domains, which, up to the sign, correspond to positive solutions for $\mathcal{M}^+_{\lambda,\Lambda}$ in annular domains, exist independently of their initial slope, according to Theorem 2.2. Hence, gluing together a positive solution for $\mathcal{M}^-_{\lambda,\Lambda}$ in some ball with negative and positive solutions for $\mathcal{M}^-_{\lambda,\Lambda}$ in annular domains, we obtain sign-changing solutions with any number of nodal domains if and only if $p < p^*_-$.

The above argument does not apply to the operator $\mathcal{M}^+_{\lambda,\Lambda}$, and in this latter case the proof of the existence of sign-changing solutions is much more delicate. Indeed, the existence of the first positive nodal component for a solution relative to $\mathcal{M}^+_{\lambda,\Lambda}$ requires the condition $p < p^*_+$. On the other hand, the existence of a second negative nodal component, which has initial slope given by the final slope of the first nodal component, and , up to the sign, corresponds to a positive solution for $\mathcal{M}^-_{\lambda,\Lambda}$ in an annular domain, is not guaranteed for $p > p^*_-$, according to Theorem 2.2. In particular, for $p < p^*_+$ but close to p^*_+, the positive solutions of the Dirichlet problem for $\mathcal{M}^+_{\lambda,\Lambda}$ in the unit ball B are almost critical solutions, and their slopes at the boundary ∂B converge to 0, according to Theorem 2.1. Thus, for p close to p^*_+, the slope of the first nodal component at the first nodal radius (which can be fixed at 1 up to a scaling) is an initial slope for the second nodal component not large enough in order to yield a positive solution for $\mathcal{M}^-_{\lambda,\Lambda}$ in an annular domain, meaning that no radial sign-changing solutions exist for p close to p^*_+. The new critical exponent p^{**} given by Theorem 3.1 is then defined as the supremum of the exponents p for which there exist sign-changing radial solutions, and it is easily proved to satisfy $p^{**} > p_-^*$. The exponent p^{**} is, roughly speaking, the exponent satisfying $p_-^+ < p^{**} < p_+^*$ for which the positive solution u of the Dirichlet problem

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = u^{p^{**}} & \text{in } B_1 \\ \qquad u = 0 \quad \text{on } \partial B_1 \end{cases}$$

satisfies

$$u'(1) = -\alpha_-^*(p^{**})\,,$$

where $\alpha_-^*(p)$ is the critical initial slope of the positive fast decaying solution of the exterior Dirichlet problem relative to $\mathcal{M}^-_{\lambda,\Lambda}$ in $\mathbb{R}^n \setminus B_1$, defined by (12).

The difference between the semilinear and the fully nonlinear cases, and between the two cases relative to $\mathcal{M}^-_{\lambda,\Lambda}$ and $\mathcal{M}^+_{\lambda,\Lambda}$, is even more evident in the study of the asymptotic behavior of almost critical radial sign-changing solutions.

Let us consider almost critical solutions u_ϵ with a fixed number $m \geq 2$ of nodal regions, that are solutions of problem (1) with $p = p_\epsilon$ and

$$p_\epsilon \to p_-^* \quad \text{as } \epsilon \to 0\,, \qquad \text{for } \mathcal{M}^-_{\lambda,\Lambda}$$

$$p_\epsilon \to p^{**} \quad \text{as } \epsilon \to 0\,, \qquad \text{for } \mathcal{M}^+_{\lambda,\Lambda}$$

The asymptotic analysis performed in [15] leads to the following concentration results.

Theorem 3.2. *For the operator $\mathcal{M}^-_{\lambda,\Lambda}$, up to a subsequence, as $\epsilon \to 0^+$, only the restriction of u_ϵ to its first nodal region blows up at the origin, and $u_\epsilon \to \bar{u}$ in $C^2_{loc}(\overline{B} \setminus \{0\})$, where $\bar{u}$ is a radial sign-changing solution of*

$$\begin{cases} -\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = |u|^{p_-^*-1}u & \textit{in } B \\ \qquad u = 0 & \textit{on } \partial B \\ \qquad u(0) < 0 \end{cases}$$

with $(m-1)$ nodal regions.

The result of the above theorem can be easily explained in light of the critical role played in this case by p_-^*: as $p_\epsilon \to p_-^*$, the only nodal component which can no longer exist is the first one, since it is an almost critical positive solution in a ball for the operator

$\mathcal{M}^-_{\lambda,\Lambda}$. Hence, the blow up behavior occurs only for the first nodal component, whereas the other nodal components rearrange in order to yield a sign-changing radial solution with one less nodal component.

As far as almost critical sign-changing radial solutions relative to $\mathcal{M}^+_{\lambda,\Lambda}$ are concerned, the concentration result reads as follows.

Theorem 3.3. *For the operator $\mathcal{M}^+_{\lambda,\Lambda}$, up to a subsequence, as $\epsilon \to 0^+$, one has:*

i) *if m is even, then the restrictions of u_ϵ to each nodal region blow up at the origin, and $u_\epsilon \to 0$ in $C^2_{loc}(\overline{B} \setminus \{0\})$;*

ii) *if m is odd, then the restrictions of u_ϵ to each of the first $m-1$ nodal regions blow up at the origin, and $u_\epsilon \to \bar{v}$ in $C^2_{loc}(\overline{B} \setminus \{0\})$, where $\bar{v}$ is the unique positive solution of*

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = u^{p^{**}} & \text{in } B \\ \qquad\qquad\qquad u = 0 & \text{on } \partial B. \end{cases}$$

Also for the operator $\mathcal{M}^+_{\lambda,\Lambda}$, an explanation of the above concentration result can be obtained by considering the critical character of the exponent p^{**}. As $p_\epsilon \to p^{**}$, the solutions which can no longer exist are the negative nodal components, which are, up to the sign, positive solutions relative to $\mathcal{M}^-_{\lambda,\Lambda}$ in annular domains whose initial slope converges to the critical one $\alpha^*_-(p^{**})$. On the other hand, by continuous dependence on initial conditions, there exists a sort of hierarchy in the blow up behavior of the nodal components: bounded nodal components keep bounded the subsequent ones. Hence, since the negative nodal components must blow up and concentrate at the origin, all the nodal components must blow up and concentrate at the centre of the ball as well when m is even. When m is odd, the last positive nodal components are the only ones which "survive" and they converge to a positive solution in the unit ball for the operator $\mathcal{M}^+_{\lambda,\Lambda}$ at the level $p = p^{**}$.

Another explanation of the concentration results stated in Theorems 3.2 and 3.3 can be obtained by considering a suitably defined energy associated with radial solutions of the considered problems, see [3, 15].

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Dipartimento di Matematica, Sapienza Università di Roma, P.le Aldo Moro 2, I–00185 Roma, Italy.

E-mail address: leoni@mat.uniroma1.it