The numeric-vectorial component for analysing the infinite limit of a sequence in preservice teachers

M. Arnal-Palacián [a], J. Claros-Mellado[b] , M.T. Sánchez-Compaña[c]

[a] University of Zaragoza, Mathematics Department. marnalp@unizar.es (Corresponding author)
[b] IES Calderón de la Barca de Pinto. fclaros@iescalderon.es
[c] University of Malaga, Didactics of Mathematics Department, teresasanchez@uma.es

Abstract: This article studies how preservice teachers consider the concept of the infinite limit of a sequence in relation to their future teaching practice through the numeric-vectorial component. During the research, 27 prospective secondary education and Baccalaureate (secondary school)-level teachers were organised into groups and were asked to debate different fragments setting out the above-mentioned limit that had been extracted from textbooks from different publishers. In the analysis, phenomenology was considered in the sense defined by Freudenthal, from two possible approaches—intuitive and formal, and by applying four systems of representation: verbal, tabular, graphic, and symbolic; in addition, Elementary and Advanced Mathematical Thinking levels were considered in order to classify the phenomena chosen. Based on this analysis, we were able to determine five individual and three group level phenomenological profiles. We have used these data to offer some insights into teaching and learning of the infinite limit of a sequence from the perspective of prospective teachers.

Keywords: Infinite limit of a sequence. Phenomenology. Teaching profile. Mathematics teacher training. Numeric-vectorial component.

1. Introduction

The infinite limit of a sequence is a concept studied by mathematical analysis. The difficulties generated in the process of its teaching and learning has formed the basis of one of the lines of research in the area of mathematical didactics in recent decades. However, in most cases, the limit is studied in a general way (**Tall & Schwarzenberger, 1978**; **Tall & Vinner, 1981**; **Cornu, 2002**; **Fernández et al., 2017**; **Douglas, 2018**; **Marufi et al., 2018**). Here, we started from the premise that each limit must be carefully studied, avoiding generalities in every case. Thus, in this research we deal with the infinite limit of a sequence (**Arnal-Palacián, 2019**). Through this study we have found evidence for how preservice mathematics teachers identify which fragments of the infinite limit of a sequence they wish to use in their future teaching practice. We took the stance that preservice teachers should complete mathematical activities at the same level as their future students, by debating and reflecting upon these problems in small groups (**Goffree & Oonk, 1999**). In addition, these teachers were asked to focus on different levels of justification for fragments involving both intuitive and formal approaches which promote elementary or advanced mathematical thinking (**Dreyfus, 1991**; **Tall, 1991**; **Garbin, 2015**) and which use of different representation systems (**Janvier, 1987**; **Duval, 1998**; **Molina, 2014**; **Fernandez-Plaza et al., 2015**). In the context of this scientific panorama related to the concept of the infinite limit of a sequence, and focusing on prospective secondary education and Baccalaureate (secondary school)-level teachers, our objective was to identify and describe the phenomenological profiles of these preservice teachers in terms of their vision of this limit, based on information obtained through discussion groups which had enabled them to reflect upon this topic.

2. Theoretical Framework

We started our bibliographic review by studying the difficulties related to the concept of the limit. Next, we present some of the ways in which it can be approached through phenomenology. Finally, we analyse this notion from the perspective of preservice teachers—a focus of interest in recent years in the didactics of mathematics.

Difficulties related to learning about the limit

Knowledge of the difficulties, obstacles, and errors related to the concept of a limit was essential to our approach in this present work. For this, we must start with the studies published by **Tall & Vinner (1981)**, **Cornu (1983)**, **Sierpinska (1985)**, **Hitt (2003)**, **Vrancken et al. (2006)**, **Morales et al. (2013)**, and **Irazoqui & Medina (2013)**, among others. **Irazoqui & Medina (2013)** classified difficulties related to learning the limit as epistemological (focusing on the notion itself), didactic (related to teaching), and cognitive (related to the cognitive structures required in students in order to learn). Among the epistemological obstacles, **Cornu (1983)** identified the common sense of the word 'limit' itself, generalisation of the properties of finite processes to infinite ones, the metaphysical aspect of the notion of a limit, and the idea of infinitely large and infinitely small quantities. **Vrancken et al. (2006)** stated that one way to approach limit problems is to use different representations, although the current trend tends to take an algorithmic and algebraic approach. These authors emphasised that the appearance of errors when using exclusively an algebraic representation prevents students from identifying where the error is; thus, graphical representations are only sparely used, and they do not consider them helpful in algebraic processes. In their study, these authors addressed the finite limit of a function at one point, the infinite limit of a function at one point, the finite limit of a function at infinity, and the infinite limit of a function at infinity. **Hitt (2003)** showed that the learning obstacles of the limit are linked precisely to the word 'limit' itself as well as the term 'tend towards', because they are not used in the same context: while the former is accurate, the latter can be called more intuitive. Along the same lines, **Tall & Vinner (1981)** showed the difficulties that everyday expressions such as 'tend', 'approach', 'get near to', etc. cause when considering the meaning of the limit. Furthermore, **Morales et al. (2013)** affirmed that the concept of a limit plays an especially important role in understanding most of the content of the areas of calculus and mathematical analysis.

Phenomenology of the infinite limit

When we refer to phenomenology, we do so in the sense it was used by **Freudenthal (1983)** as a component of his didactic analysis. Freudenthal gave this name to the method of analysis of mathematical content based on the idea of opposition between noumena, objects of thought, and phenomena: i.e., the situations that these mathematical objects organise. Regarding the limit, **Claros (2010)** characterised the phenomena organised from the finite limit of a sequence, **Sánchez (2012)** referred to those organised by the finite limit of a function at a point, and **Arnal-Palacián (2019)** looked at those determined by the infinite limit of a sequence, in the latter case, unlimited intuitive growth (u.i.-g.), unlimited intuitive decrease (u.i.-d.), and successive one-way and return (o.w.r.i.s.) infinite limit sequences. U.i.-g. is observed when the values of the sequence become greater and greater as the sequence advances, and as a consequence, it can be intuited that the sequence is increasing and not bounded at the top; in other words, it grows without limits. U.i.-d. is observed when the values of the sequence become lower and lower as the sequence advances, and as a consequence it can be intuited that the sequence is decreasing and it does not have a lower bound; that is, it decreases without limits. The o.w.r.i.s. phenomenon in infinite limit sequences is determined by two processes: in the first, when we take H from K for each element, there is a natural number v; in the second process, we consider that we have $a_n > H$, for all $n \geq v$ (**Arnal-Palacián et al., 2020**). The u.i.-g. and u.i.-d. phenomena are associated with Elementary Mathematical Thinking (EMT), while the o.w.r.i.s. phenomenon belongs to Advanced Mathematical Thinking (AMT). EMT is characterised by routine tasks and the definition of already known concepts, while AMT involves processes such as representation, translation, abstraction, generalisation, and synthesis (**Garbin, 2015**). Furthermore, these phenomena occur in different systems of representation. **Janvier (1987)** and **Rico (2009)**, consider that every mathematical concept requires a variety of representations for its capture, understanding, and structuring, therefore requiring the establishment of relationships between different verbal, tabular, graphic, and symbolic representation systems. In this work, although a multitude of classifications exist, we considered these aforementioned classification of representation systems which emphasise some properties of the concept but make others difficult. The greater an individual's knowledge of the representations and properties of a mathematical concept, the better their understanding of that concept can be (**Molina, 2014**).

The limit from the teacher training viewpoint

According to **Hill et al. (2008)**, the knowledge that a teacher must have to be able to carry out their teaching, Mathematical Knowledge for Teaching (MKT), includes both the mathematical knowledge common to anyone working in various professions related to the subject, as well as that specialised to the teaching profession. The same authors concluded that there is a relationship between what a teacher knows, how they know it, and what they can do in an educational context. As indicated by **Jacob et al. (2017)**, teacher training must be specific and it should be designed to improve the mathematical knowledge of these individuals and allow them to generate increased thinking and reasoning abilities in their prospective students during mathematics classes. In addition, this training focuses on helping teachers to learn more mathematics, as well as understanding how their students learn mathematics, how to use formative assessment to develop an understanding of what students know and do not know, and how to develop an effective instruction style in mathematics classrooms to solve student problems. In the creation of the basic knowledge for teaching and learning mathematics, preservice teachers must combine experiences preserved during this stage of learning as a student with their own reflections on these processes (**García-Blanco, 2005**). However, **Alsina (2010)** determined that interacting with others is one of the most important aspects in the promotion of reflective learning because interactions during dialogue favoured changes in attitude towards mathematics among preservice teachers. Specifically in relation to the concept of the limit, **Fernández et al. (2018)** analysed the anticipation by teaching staff in training of the responses of secondary school students to problems about the limit of a function with different characteristics in terms of their understanding. They went on to propose problems that would allow them to measure the conceptual progress of their students in this sense. Furthermore, we would like to highlight the study by **Sánchez (2012)**, who conducted interviews with active teachers to determine their phenomenological profiles based on their responses regarding the notion of the finite limit of a function at a point; the categorisation established by this author was the starting point for establishing the categories and dimensions of this present study.

3. Methods

In this section we present a description of the sample we analysed, the material we provided to the preservice teachers to begin the debate, action protocol followed by both the preservice teachers and the research team, and dimensions and categories we used for further analysis.

Sample

The sample consisted of 27 prospective mathematics teachers organised into 8 groups. All of them had completed specific training that enabled them to be secondary education and Baccalaureate-level mathematics teachers. Despite being trained and qualified to be a mathematics teacher, only 52% of them had completed a bachelor's or extended degree in this speciality, compared to 48% who had carried out other scientific studies that also allowed them to teach this subject. Given that the preservice teachers had accessed this specialty from different training itineraries, we considered their knowledge to be remarkably diverse which may have been decisive in their use of different approaches to teaching the limit of a sequence.

Material used for the discussions

The individuals in the sample were provided with an initial document in which their personal data was collected, and which also provided details about the ongoing investigation and guidelines and suggestions. This document also facilitated the common thread of their discourse: establish the courses or levels at which the concept would be worked on, difficulties that they thought they might encounter, and preferences to approach the infinite limit of a sequence in the classroom. After reading the first document and filling out the data, an initial questionnaire comprising different fragments was given to the preservice teachers. In these fragments the three phenomena characterised for the infinite limit of a sequence were identified: unlimited intuitive growth (u.i.-g.), unlimited intuitive decrease (u.i.-d.), and successive one-way and return (o.w.r.i.s.) infinite limit sequences. We also identified the intuitive and formal approaches, four systems of representation (verbal, graphic, tabular, and symbolic), and two formats—definition and example. The combination of all of them resulted in 24 fragments with different characteristics. To avoid the questionnaire being excessively long, only 11 of these fragments were presented, as shown in the Annex. Some of them came from textbooks from different publishers used in classrooms, while our research team prepared others, including the fragments in the tabular and graphic representation system (in a formal approach). This was because, although the graphical representation system had been presented in the textbooks we consulted, it was not represented independently from other systems and so it was not in a suitable format for use in a formal approach. Nonetheless, we felt that including this system in the study would provide additional points for reflection both by secondary school students and the preservice teachers. Furthermore, no fragment was considered in the symbolic representation system for the intuitive approach, because none appeared in the textbooks we considered or in previous studies regarding other limits (**Claros, 2010**; **Sánchez, 2012**; **Claros, Sánchez & Coriat, 2016**). In some of the examples we presented a definition format and in others we used an example format (that is, the fragments we used presented one of the described phenomena—either with a formal or intuitive approach—and used one of the previously described representation systems, presented either through an example or a definition). We only considered the latter formats (definition and example) for the formal approach and the verbal representation system because both of these types are very commonly described in mathematics textbooks. Furthermore, this also allowed us to balance the number of formal fragments (5) with the number of intuitive fragments (6), as shown in Table 1.

Table 1. Fragments, phenomena, and representation systems used in the questionnaire.

	Formal approach	Intuitive approach	
	$+\infty$	$+\infty$	$-\infty$
Graphic	Fragment E*	Fragment D	Fragment J
Tabular	Fragment H*	Fragment K	Fragment C
Verbal	Fragment F Fragment G	Fragment B	Fragment I
Symbolic	Fragment A	–	–

* Fragments prepared by the research team.

The letters attributed to each of the fragments and used to identify them were assigned in a completely random way, avoiding following an established pattern that could have guided the preservice teachers in some way.

Application protocol

We developed three pilot tests to validate this protocol. The first test was carried out individually and in writing, after first completing a mathematical knowledge activity based on the topic of study; the second was conducted as a group, maintaining the previous mathematical knowledge activity, and also collecting the data by recording the audio; and in the third test we did not use the mathematical knowledge activity but maintained the debate which was entirely focused on the didactic approach to the fragments, although the distance between the preservice teachers made it difficult to record the audio information. The final experimental study comprised four phases: presentation, completion of the first questionnaire, explanation of the infinite limit of a sequence study in progress, and completion of the second questionnaire. For this, we established discussion groups of 3–4 people chosen at random. We collected the data by recording audio files. To achieve this, each group was provided with a recorder and a mobile device. An additional space, supplementary to the usual classroom, was set up to create an environment conducive to debate for the preservice teachers. We aimed to make this a space in which they would not be interrupted, and which would also allow the research team to collect the audio data without any difficulties. In the first (presentation) phase, each participant had to identify themselves for the audio recording so that their comments could later be analysed together with the personal data they had provided. In the second (spontaneous) phase, the preservice teachers first had to individually reflect upon their acceptance or rejection of their future use in teaching work of each of the fragments they had been presented, and second, to discuss these fragments. In the third (explanation) phase, we entered the classroom to present the research, including the theoretical framework we had used, our choice of a definition of the infinite limit of a sequence created by mathematical experts, advancing towards a mathematical correction, and the characterisation of the resulting phenomena organised by that definition. In the fourth (spontaneous) phase, the prospective teachers were asked to debate these same fragments again, but this time also identifying the phenomena at play for each one.

Categorisation and response types

Following the instrument developed by Sánchez (2012), we considered different categories for the different types of acceptance or rejection comments: C1, for those considered generic; C2, when educational levels or specific students were referred to; C3, for comments that indicated a specific difficulty; and C3*, when any of the characterised phenomena types were identified. The representation system mentioned by the preservice teachers (or failure to mention any system) was also encoded when the above-mentioned categories were established for each comment type. Thus, the dimensions of each of the comments were categorised as follows for fragment use: ‘used’ (U) or ‘not used’ (NU) and for the phrase in which it occurred: ‘spontaneous’ (SP) or ‘induced’ (IND). Consequently, we used the following codes: SP/U (for mention of a system in a spontaneous phase); IND/U (for mentioning a system an induced phase); SP/NU (for when no system was mentioned in a spontaneous phase); and IND/NU (to refer to induced phrases in which no system was mentioned).

Analysis instrument

We used the analysis instrument previously constructed and validated by **Macías et al. (2017)**. To do this, we built what these authors called a ‘numeric-vectorial component’. This component comprises two vectors: one for the intuitive approach, where the comments associated with both the u.i.-g. and the u.i.-d. phenomenon must be combined, and another for the formal approach in which the comments associated with the o.w.r.i.s. phenomenon are found. In addition, each of them consisted of two components, the first for acceptance comments and the second for rejection comments, as shown in Table 2.

Table 2. Vectors of the numeric-vectorial component (**Macías et al., 2017**).

Vectors	First component	Second component
Intuitive vector	2 "Number of 'use' comments in the spontaneous phase" + "Number of 'use' comments in the induced phase".	2 "Number of 'non-use' comments in the spontaneous phase" + "Number of 'non-use' comments in the induced phase".
Formal vector	2 "Number of 'use' comments in the spontaneous phase" + "Number of 'use' comments in the induced phase".	2 "Number of 'non-use' comments in the spontaneous phase" + "Number of 'non-use' comments in the induced phase".

As shown in the table above, in each of the components of both vectors, twice the weight was given to the comments made in the spontaneous phase compared to those made in the induced phase. Furthermore, since the number of comments was always a positive value, and this will occur in the two components of each of the vectors, it will be located in the first quadrant of the Cartesian plane.

An example is the data from the intuitive approach of a discussion group in which 13 comments on use were provided in the spontaneous phase and 5 were given in the induced phase, in addition to 2 non-use comments in the spontaneous phase and none in the induced phase.

$$v_{intuitive} = \left(v_{use}, v_{non\_use}\right) = (2 \cdot 13 + 5, 2 \cdot 2 + 0) = (31, 4)$$

Which is completely analogous for comments with a formal approach.

$$v_{formal} = \left(v_{use}, v_{non\_use}\right) = (2 \cdot 8 + 1, 2 \cdot 9 + 4) = (17, 22)$$

Subsequently, the phenomenological profile—constructed from this instrument—was determined by a vector with three components:

- The first component takes the values N+, for $Arg\ v_{intuitive} \leq 45°$; N−, for $Arg\ v_{intuitive} > 45°$; and 0, when there is no comment in the intuitive approach.
- The second component can take the following values: D+, for $Arg\ v_{formal} \leq 45°$; D−, for $Arg\ v_{formal} > 45°$; and 0, when no comment is obtained for the formal approach.
- The third component may take the following values: M, for $|\,Int.Use - Int.Non\_use| > |Form.use - Form.non\_use|$, with the intuitive approach prevailing over the formal one; and m, for $|Int.Use - Int.non\_use| \leq |Form.use - Form.non\_use|$, where the formal approach prevails over the intuitive one.

The argument for each vector is the angle it makes with the axis 'OX'; because they are vectors in which both components are positive, the angle will be between 0° and 90°.

For the group that served as an example, the first component takes the value N+ because it has $Arg\ v_{intuitive} = 7.35° \leq 45$, the second component takes the value D−, because $Arg\ v_{formal} = 52.31° > 45°$, and M is the third component, because $|\,31 - 4| > |17 - 22|$, thereby giving rise to the phenomenological profile (N+, D−, M).
This numeric-vectorial component allows us to determine different phenomenological profiles (Table 3).

Table 3. Possible teaching profiles.

	M			m		
	N+	**N−**	**0**	**N+**	**N−**	**0**
D+	(N+,D+,M)	(N−,D+,M)	(0,D+,M)	(N+,D+,m)	(N−,D+,m)	(0,D+,m)
D−	(N+,D−,M)	(N−,D−,M)	(0,D−,M)	(N+,D−,m)	(N−,D−,m)	(0,D−,m)
0	(N+,0,M)	(N−,0,M)	Not considered	(N+,0,m)	(N−,0,m)	Not considered

Therefore, there were 18 different possible phenomenological profiles. Two of them—(0,0,M) and (0,0,m)—were not considered because they were generated by the preservice teacher not making any comments about the intuitive or formal approach fragments. Therefore, in this study we actually considered 16 phenomenological profiles.

4. Results

This section presents our analyses of the results we obtained after implementing the debate protocol described above. To do this, on the one hand we considered the different comments categories (C1, C2, C3, and C3*) and on the other, the different representation systems (verbal, tabular, graphic, and symbolic). In both cases, we analysed the phase dimension in which each comment occurred as well as whether the comment was a use acceptance or rejection. We also present the profiles of the preservice teachers for teaching the infinite limit of a sequence.

Results of the comment categorisation

Table 4 shows the distributions for each of the phases and their use dimensions, considering both the intuitive and formal approaches.

Table 4. Percentages of the different types of comments noted during the debates.

Focus	SP/U	IND/U	SP/NU	IND/NU	Subtotal
Intuitive	27.51	3.70	4.76	0.00	35.98
Formal	18.52	4.76	29.63	11.11	64.02
Subtotal	46.03	8.47	34.39	11.11	100

There were more comments on the fragments with a formal focus during the debates than for ones that had an intuitive focus. This was because the preservice teachers spent more time debating these fragments, leading them to describe difficulties associated with teaching them and to specify the types of students to whom these lessons could or should be directed. In fact, most of the comments regarding the intuitive focus fragments were to accept their use, while more of the comments made about the formal approach fragments were against their use. There were many more spontaneous phase comments compared to induced phase comments (80.42% versus 19.58%). We also present the data for the intuitive approach (Table 5) and formal approach (Table 6), according to the type of comment made.

Table 5. Percentages of comments associated with the intuitive approach.

Categories	SP/U	IND/U	SP/NU	IND/NU	Subtotal
C1	51.47	0	2.94	0	54.41
C2	17.65	1.47	0	0	19.11
C3	5.88	0	8.83	0	14.71
C3*	1.47	8.82	1.47	0	11.76
Subtotal	76.47	10.29	13.24	0	100

More than half of the comments about fragments that had an intuitive approach were category C1 (54.41%), while only 11.76% of them were C3*, having justified their decision according to the description of the phenomena described by Arnal-Palacián et al. (2020). In addition, 76.47% of the comments produced in the spontaneous phase were to accept the use of the fragment in question in their future teaching practice; none of the preservice teachers contemplated rejection of the fragment in the induced phase.

Table 6. Percentages of comments associated with the formal approach.

Categories	SP/U	IND/U	SP/NU	IND/NU	Subtotal
C1	9.92	0.00	19.83	4.96	34.71
C2	10.74	3.31	12.40	4.13	30.58
C3	5.79	0.83	12.40	1.65	20.66
C3*	2.48	3.31	1.65	6.61	14.88
Subtotal	28.93	7.44	46.28	17.36	100

The predominant type of comments on the formal approach fragments were C1 (34.71%) or C2 (30.58%), which combined, accounted for more than 65% of the comments. In this case, more preservice teachers indicated the difficulties associated with the formal approach fragments compared to intuitive approach fragments. Thus, category C3 and C3* comments were more common than for the latter case. This was probably because prospective teachers more frequently made category C3 comments about the use of this type of fragment for a higher educational levels and because category C3* comments emerged for the induced phase in which the students had identified the o.w.r.i.s. phenomenon. Considering these categories, phases, and uses, the highest number of comments again occurred in the spontaneous phase (46.28%), but in this case, to reject its use in their future teaching practice. In fact, this rejection of the formal approach occurred in more than half (36.64%) of the comments that emerged in the debates.

Results of the representation systems

Given the theoretical framework considered for this study, which considers the importance of representation systems, here we present our analysis of these results. As previously mentioned, the representation systems presented were verbal, tabular, or graphic for the intuitive approach. However, on some occasions some preservice teachers did not consider the representation system, and we also considered these 'NR' comments. See Table 7.

Table 7. Percentages of comments associated with the intuitive approach for each representation system, phase, and use.

Rep. Syst.	SP/U	IND/U	SP/NU	IND/NU	Subtotal
Verbal	30.00	4.44	2.22	0.00	36.67
Tabular	11.11	2.22	7.78	0.00	20.00
Graphic	22.22	1.11	1.11	0.00	24.44
NR	15.56	2.22	0.00	0.00	17.78
Subtotal	78.89	8.89	11.11	0.00	100.00

As might be expected, the 'subtotal' percentages do not coincide with those in Table 5 because some of the comments had to be simultaneously considered in several representation systems, thus producing duplications. The verbal representation system produced the most comments by preservice teachers (36.67%). Furthermore, this representation system also had the highest acceptance rate. In contrast, the tabular representation system generated the highest rejection rate, usually because two variables in the sequence were presented, which is common in the textbooks used by secondary school students. In the same way we present the results for verbal, tabular, graphic, or symbolic representation systems for the formal approach (Table 8).

Table 8. Percentages of comments associated with the formal approach for each representation system, phase, and use.

Rep. Syst.	SP/U	IND/U	SP/NU	IND/NU	Subtotal
Verbal	10.74	3.31	15.70	6.61	36.36
Tabular	1.65	0.83	7.44	2.48	12.40
Graphic	1.65	0.00	4.96	4.13	10.74
Symbolic	11.57	1.65	7.44	4.13	24.79
NR	3.31	1.65	10.74	0.00	15.70
Subtotal	28.93	7.44	46.28	17.36	100

Once again, the verbal representation system was the most frequently used in the prospective teachers' comments, with a percentage similar to that of the intuitive approach (36.36%); this representation system also generated the highest rejection rates. The symbolic representation system was the only one that presented more acceptance than rejection. In fact, the acceptance rates only accounted for 2.48% and 1.65% of the comments for the tabular and graphic representation systems, respectively.

Profiles of the prospective teachers

After applying the numeric-vectorial component described by **Macías et al. (2017)** to all of the groups with categorised comments, we identified three preservice teacher profiles at the group level, (N+, D+, M), (N+, D−, M), and (N +, D−, m), as shown in Table 9.

Table 9. Profiles of the preservice teachers at the group level.

	M			m		
	N+	**N−**	**0**	**N+**	**N−**	**0**
D+	22.22	0.00	0.00	0.00	0.00	0.00
D−	44.45	0.00	0.00	33.33	0.00	0.00
0	0.00	0.00	Not considered	0.00	0.00	Not considered

A total of 44.45% accepted the intuitive approach fragments, rejected the formal approach fragments, and preferred to use the intuitive approach, giving rise to the (N+, D−, M) phenomenological profile; 33.33% accepted the intuitive fragments, rejected the formal ones, and preferred to reject the formal fragments rather than accept the intuitive ones, giving rise to a (N+, D−, m) profile; and 22.22% accepted both the intuitive and formal approaches, but preferred to use the intuitive approach, resulting in the (N+, D+, M) phenomenological profile. These preservice teacher profiles are also presented individually in Table 10.

Table 10. Profiles of preservice teachers at the individual level.

	M			m		
	N+	**N−**	**0**	**N+**	**N−**	**0**
D+	14.81	0.00	0.00	0.00	0.00	7.41
D−	40.74	0.00	0.00	25.93	11.11	0.00
0	0.00	0.00	Not considered	0.00	0.00	Not considered

At the individual level we found five phenomenological profiles; the three we had already identified for the group phenomenological profiles as well as the (N−, D−, m) profile for preservice teachers who rejected both the intuitive and formal fragments, but rejected the formal ones more than the intuitive ones, and the (0, D+, m) profile for the preservice teachers who did not offer any comments for the intuitive fragments and accepted the formal fragments, thereby prioritising acceptance above the intuitive approach. Again, the predominant profile (40.74% of cases) accepted the intuitive approach, rejected the formal one, and opted for acceptance over rejection, thus giving rise to the (N+, D−, M) phenomenological profile; the preservice teachers with the least common profile (7.41%) did not comment on the intuitive fragments and accepted the formal ones, giving them a (0, D+, m) phenomenological profile.

5. Discussion and Conclusions

Based on our analysis of the group and individual reports for 8 groups and 27 participating students, we can state that we achieved the objective we set out for this research. We identified five individual phenomenological profiles and three group phenomenological profiles for the preservice teachers when they considered teaching the infinite limits of a sequence. Furthermore, of note, the three group phenomenological profiles (N+, D+, M), (N+, D−, M), and (N+, D−, m) formed part of the five individual phenomenological profiles (N+, D+, M), (N+, D−, M), (N+, D−, m), (N−, D−, m), and (0, D+, m).

Among the five individual phenomenological profiles we identified, the predominant one was (N+, D−, M), representing preservice teachers that accepted the intuitive phenomena, rejected the formal phenomenon, and preferred the use of intuitive phenomena rather than rejection of the formal phenomena. In addition, individual profiles (N+, D+, M) were identified in which the preservice teachers accepted the use of the intuitive and formal phenomena, but prioritised acceptance of the intuitive ones; in the (N+, D−, m) profile the intuitive phenomena were accepted and the formal one was rejected, but the rejection was prioritised; in (N−, D−, m), the intuitive and formal phenomena were rejected, and rejection of the latter was preferred; and in (0, D+, m), the preservice teachers did not comment on the intuitive phenomena and accepted the formal ones, and consequently, for the third component this acceptance was preferred.

Furthermore, among the group profiles, (N+, D−, M) the phenomenological profile was again predominant, representing preservice teacher groups that accepted use of the intuitive fragments and rejected the formal ones, prioritising rejection of the latter; the (N+, D+, M) profile defined groups that accepted the use of intuitive and formal phenomena, but prioritised acceptance of intuitive phenomena; and in the (N+, D−, m) phenomenological profile, use of the intuitive fragments was accepted while the formal ones were rejected, with acceptance of the former being preferred.

In relation to mathematical thinking, the (N+, D−, M) and (N+, D−, m) phenomenological profiles were linked to elementary mathematical thought, because they are associated with the intuitive approach and they reject the formal approach. In contrast, the (N+, D+, M) and (0, D+, m) phenomenological profiles were associated with advanced mathematical thinking (**Dreyfus, 1991**; **Tall, 1991**; **Garbin, 2015**) because they accept both the intuitive and formal approaches. The (N−, D−, m) profile did not reach the level of elementary mathematical thought because it does not even consider an intuitive approach to the fragment.

The phenomena of unlimited intuitive growth, unlimited intuitive decrease, and successive one-way and return infinite limit sequences described by **Arnal-Palacián (2019)** and **Arnal-Palacián et al. (2020)** influenced the profiles produced by the comments that emerged in the preservice teacher debates. Furthermore, these comments were determined by two approaches: the intuitive approach and the formal one. Most of the comments that had an intuitive approach were unjustified and were to accept the use of the fragment; for the formal approach, the comments were somewhat more elaborate and allowed the preservice teachers to justify their rejection based on the difficulty of the concepts involved in the fragment.

The preservice teachers accepted the use of fragments that took an intuitive approach. In fact, all the participating groups accepted the use of these fragments and none rejected their use in the classroom: 25 accepted it and 2 did not comment on the matter and so we considered their opinion to be the same as that of the group. However, the preservice teachers rejected the formal approach. Most of the groups and individual student teachers (77.78% in both cases) rejected the use of fragments that contained successive one-way and return infinite limit sequences.

The comments of acceptance and rejection did not appear in the same way for all representation systems, even though **Janvier (1987)** affirmed that every mathematical concept requires a variety of representations to be absorbed, understood, and structured, thereby requiring the establishment of relationships between different representation systems.

The dominant representation system was verbal—either to accept it in the case of the intuitive approach or to reject it for the formal approach. We attribute the predominance of comments within this representation system to the fact that it is the most common form used in textbooks to introduce the concept of the infinite limit of a sequence. In addition, rejection of the presentation of the concept as a tabular representation system stood out, which may be because this is the least used format in the different secondary school textbooks (**Arnal-Palacián, 2019**).

Given all the above, this research broadens the view of teaching and learning of the infinite limit of a sequence from the perspective of preservice teachers, and shows their preferences for the use or non-use of different types of representations of the phenomena characterised by this concept.

6. Acknowledgements

This work was carried out by the HUM-324 group in the PAIDI of the Autonomous Community of Andalusia and by the S60_20R-Research in Mathematics Education group within the scope of the Autonomous Community of Aragon for the 2020–2022 period.

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Annexes

Fragment A

$\lim_{n\to\infty} a_n = +\infty \Leftrightarrow \forall M > 0$ you can find a $n_0 \epsilon \mathbb{N}$ such that if $n > n_0 \Rightarrow a_n > M$.

Vizmanos, J.R., Hernández, J., & Alcaide, F. (2008). *Mathematics 2. Science and technology*. Editorial: SM.

Fragment B

$1, 4, 9, 16, 25, \ldots$ This sequence, whose general term is $a_n = n^2$, tends to infinity, since its terms can be made as large as you would like in order to sufficiently advance the sequence.

Martín, M.A., Morán, M., Rey, J.M., Reyes, M. (2001). *Mathematics. Baccalaureate 1. Science of nature and health*. Technology. Editorial: Bruño.

Fragment C

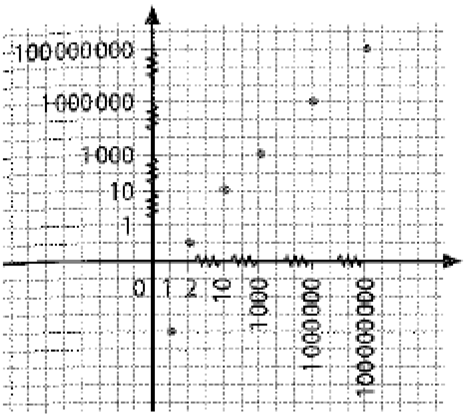
If n grows larger and larger, to what value do the terms of the sequence approach $a_n = -n^2 + 1$?
The following table is obtained by assigning increasing values to n:

n	1	10	100	1 000	10 000	…	tends to $+\infty$
a_n	0	-99	-9 999	-999 999	-99 999 999	…	tends to $-\infty$

Vizmanos, J.R., & Anzola, M. (2002). *Algorithm. Mathematics applied to the social sciences 1*. Editorial: SM.

Fragment D

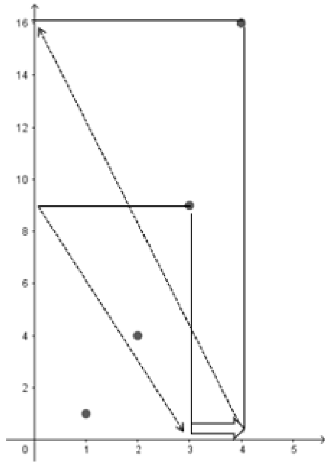
Let the sequence $a_n = \frac{n^2-3}{n}$. If it is represented graphically, it can be seen that the terms of the sequence grow indefinitely.



Bescós, E. and Pena, Z. (2001). *Applied mathematics in social sciences. Baccalaureate 1*. Publisher: Oxford.

Fragment E

Let the sequence $a_n = n^2$.



Given $H = 9$, the natural number ν exists, for example, $\nu=3$

When $n \geq \nu$, with the example n=4, we have

$a_4 = 16 > 9 = H$

$H = 16$, n = 5 $a_5 = 25 > 16$

$H = 25$, n = 6 $a_6 = 36 > 25$

…

$H = 100$, n = 11 $a_{11} = 121 > 100$

…

$H = 10\,000$, n = 101 $a_{101} = 10\,201 > 10\,000$

where $n(H) = \lfloor(\sqrt{H})\rfloor + 1$, if $H \geq 1$

As a consequence of the operations performed in the box on the right, the limit is positive infinity.

Elaborated by the authors.

Fragment F

The sequence $\{a_n\}$ has a 'positive infinity' limit if, for each element H from K, being K an ordered body, there is a natural number ν such that

$$a_n > H, \text{ for all } n \geq \nu.$$

The limit of the sequence $\{a_n\}$ is 'negative infinity', if for each element H from K, there is a natural number ν such that

$$a_n < H, \text{ for all } n \geq \nu.$$

Linés, E. (1983). *Principles of Mathematical Analysis*. Editorial: Reverté.

Fragment G

If n grows larger and larger, to what value do the terms of the sequence approach
$a_n = n^2 + 1$?
The terms get bigger and bigger, but in such a way that no matter how high the 'bar' is, you can find terms that exceed it. If we set a very high value, for example $K = 100{,}000{,}000$, then for any value of n greater than $n^* = 10{,}000$, the following terms are greater than the previously set value:

$$10\,000^2 + 1 = 100\,000\,001 > K$$

Vizmanos, J.R., & Anzola, M. (2002). *Algorithm. Mathematics applied to the social sciences 1.* Editorial: SM.

Fragment H

Let the sequence $a_n = n^2$.

n	4	5	6	7	8	...	93	...	9.993
a_n	16	25	36	49	64	...	8.649	...	99.860.049
H	10	17	28	39	50	...	8.469	...	99.840.086
v	3	4	5	6	7	...	92	...	9.992

Given $H = 9$, the natural number v exists, $v = 3$ such that $n \geq v$, $n = 4$, so $a_n = 16 > 9 = H$ where $n(H) = \lfloor(\sqrt{H})\rfloor + 1$ if $H \geq 0$.

$n(H)$ is obtained by solving the inequality $n^2 > H$. The whole part of $\sqrt{H}$ was used, so that n is a natural number and 1 was added to it to make it greater than the fixed v set.

As a result of these calculations, the limit of this sequence is positive infinity.

Elaborated by the authors.

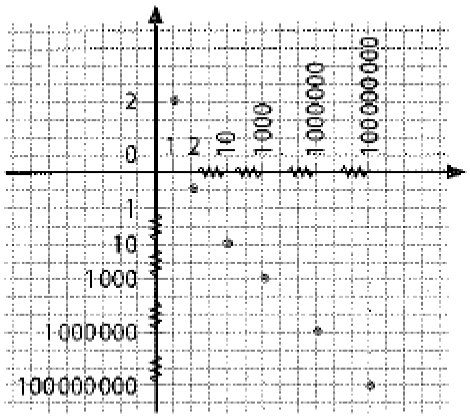
Fragment I

$-1, -4, -9, -16, -25, \ldots$ This sequence, whose general term is $a_n = -n^2$, tends to negative infinity, since its terms can be made as large in absolute values as desired, but as negative terms, in order to sufficiently advance the sequence.

By analogy to the one shown for the positive infinite limit. Martín, MA, Morán, M., Rey, JM, Reyes, M. (2001). *Mathematics. Baccalaureate 1. Science of nature and health*. Technology. Editorial: Bruño.

Fragment J

Given the sequence $a_n = -\frac{n^2-3}{n}$, as can be seen, the terms of this sequence increase in absolute value, but being negative, they are said to tend to $-\infty$.



Bescós, E., & Pena, Z. (2001). *Applied mathematics in social sciences. Baccalaureate 1*. Publisher: Oxford.

Fragment K

If n grows larger and larger, to what value do the terms of the sequence approximate
$a_n = n^2 + 1$?
The following table is obtained by giving increasing values to n:

n	1	10	100	1 000	10 000	…	tends to $+\infty$
a_n	2	101	10 001	1 000 001	100 000 001	…	tends to $+\infty$

Vizmanos, J.R., & Anzola, M. (2002). *Algorithm. Mathematics applied to the social sciences 1.* Editorial: SM.