Specialized Content Knowledge of pre-service teachers on the infinite limit of a sequence

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Abstract: This paper analyses how pre-service teachers approach the notion of the infinite limit of a sequence from two perspectives: Specialized Content Knowledge and Advanced Mathematical Thinking. The aim of this study is to identify the difficulties associated with this notion and to classify them. In order to achieve this, an exploratory qualitative approach was applied using a sample of 12 future teachers. Among the results, we can affirm that pre-service teachers mainly use algorithmic procedures to solve tasks in which this type of limit is implicit, although they would consider a resolution that specifically involves the notion with an intuitive approach if they had to explain it to their students.

INTRODUCTION

The limit is one of the main notions of Calculus, since its understanding is linked to most of its content (Morales et al., 2013). Consequently, the analysis of the limit and the difficulties associated with its teaching-learning process constitute lines of research in the area of Didactics of Mathematics. This notion, and in particular, the infinite limit, has been studied by many authors. Its difficulties constitute one of the lines of research in didactics of mathematics in recent decades (Dong-Joong, Sfard and Ferrini-Mundy, 2005, Jutter, 2006, Morales, Reyes and Hernández, 2013, Douglas, 2018). Many authors, however, have studied this notion without differentiating whether the limit was of a sequence or of a function (see Claros et al., 2013). In this paper, as already considered by Arnal-Palacián (2019) in a previous phenomenological study in the sense given by Freudenthal (1983), the learning of the infinite limit of a sequence is addressed in a particular way, since each type of limit has peculiarities that cannot be addressed together.

At the educational level, Galileo's and Cantor's notion of infinity appears very late in the mathematical education of students. Salat (2011) proposes, as early as possible, to get students to discuss infinity and its applications in terms of reflecting on their own mathematical thinking. On the other hand, Miranda et al. (2007) pointed out that students continue to experience difficulties associated with the intuitive and formal nature of the mathematical notion of infinity. Some of these difficulties are: a) failure of the link between geometry and arithmetic (epistemological), b) difficulties due to the very nature of the notion (didactic), c) the teaching methods used by teachers (didactic), d) eliminating the problem of infinity by taking as many terms as necessary (cognitive), among others. If we look at research on the limit and infinity from the point of view of the study of teachers, there is much less research on the limit and infinity from the point of view of teachers than that on students, although its content is very similar.

In view of the above, the main objective of this study is to analyze the difficulties that pre-service teachers have when working on tasks involving the infinite limit of a sequence.

The analysis of these difficulties will make it possible, firstly, to point out the difficulties associated with this notion when it is presented by pre-service teachers and, secondly, to classify these difficulties in two areas: when the trainee teacher carries out the proposed task or when the trainee teacher has to explain it to future students. We will go deeper into both aspects in order to anticipate such difficulties in a future didactic sequence that could be useful for the trainee teacher.

THEORETICAL FRAMEWORK

The theoretical framework on which this work is based is the difficulties of the notion of limit and Specialized Content Knowledge. We will also relate the latter to Advanced Mathematical Thinking.

The notion of infinite limit

The definition has a distinctive role in the technical use of a construct in contrast to its intuitive and colloquial uses (Vinner, 1991). In the particular case of the definition of the infinite limit of a sequence, as already observed in Arnal-Palacián (2019), it is placed in textbooks shortly before the introduction of the infinite limit of a function and consecutively to the finite limit of a sequence.

Taking into account the variety of definitions of the infinite limit of a sequence found, Arnal et al. (2017) consulted university professors and secondary education teachers with the aim of selecting a correct definition that is accepted by the mathematical community that teaches at different educational levels. The definition selected was the following:

"Let K be an ordered field, and $\{a_n\}$ a sequence of elements of K. The sequence $\{a_n\}$ has for limit 'more infinite', if for each element H of K, a natural number exists v, such that it is $a_n > H$, for all $n \geq v$." (Linés, 1983, p.29).

Although it was not part of the consultation described above, for the infinite limit of a sequence we present the definition provided by the same author:

"Let f: $X \to \mathbb{R}$, with $X \subset \mathbb{R}$ not upper bounded. It is said that f has limit $+\infty$, when it tends to $+\infty$, if for every real number H a real number exists K such that it is $f(x) > H$ for all $x \epsilon X$, that complies $x > K$. It is written $\lim_{x \to +\infty} f = +\infty$." (Linés 1983, p.201-202).

Among the notions that appear in the definition of the infinite limit of a sequence, we highlight the following: dependence, sufficiently large, infinite processes, bounding, types of infinity and divergence. All of them should be taken into account in the teaching and learning of the notion of the infinite limit of a sequence, specifically in the design of a didactic sequence that addresses the teaching of the infinite limit of a sequence. See Figure 1.

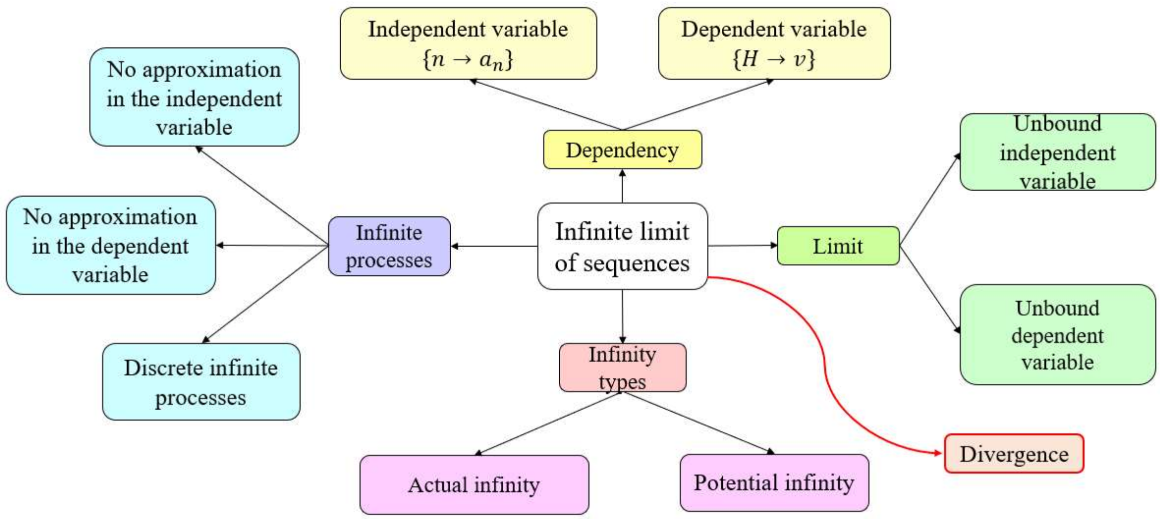


Figure 1. Mathematical notions to be used for the design of a didactic proposal.

Difficulties with the notion of limits

In order to study the notion of the infinite limit of a sequence, it is essential to be aware of the difficulties, obstacles and errors previously studied for the notion of limit. To this end, the works of Duval (1998), Medina and Rojas (2015), Morales et al. (2013), Vrancken et al. (2006), among others, will be taken into consideration.

One way of approaching problems arising from the notion of limit is to use different systems of representation, although there is currently a tendency to teach its calculation from an algorithmic and algebraic approach. Moreover, when the student uses only the algebraic representation, it is practically impossible to detect where the error lies. In many cases, graphical representation is not used because it is not considered as a support for these algebraic processes (Vrancken et al., 2006). In the analysis, a resolution or explanation that considers this algorithmic approach will be treated independently of when the handling of the notion itself is taken into account in a specific way.

In order to begin to understand a notion, as in this case the infinite limit of a sequence, students must identify it in different representations: graphical, numerical, algebraic and verbal. Moreover, it is necessary to coordinate the different systems of representation in order to obtain a comprehensive understanding of the notion (Duval, 1998).

Along the same lines, Morales et al. (2013) consider that activities related to the notion of the infinite limit using different representations make it possible to identify some difficulties such as: a) justification of the limit, b) application of the conditions of the definition of the notion, c) generalization and d) preference in the use of procedural methods as opposed to conceptual methods.

Regarding infinity, which is closely related to the notion of limit, Medina and Rojas (2015) carried out a review of the obstacles associated with it: non-acceptance of actual infinity and influence of potential infinity (Sierpinska, 1985); separation of the geometric and the numerical, i.e. the continuous from the discrete, since the successful solution of some problems through their geometric interpretation prevents the passage to the notion of numerical limit (Cornu, 1991); generalization of properties from the finite to the infinite, and Leibniz's principle of continuity.

We will take these difficulties into account in the analysis of the pre-service teachers' responses to the proposed task and indicate whether or not they are reproduced in the trainee teachers.

Specialized Content Knowledge

Mathematical as well as psychological and pedagogical content must be integrated into the university training of student teachers (Movshovitz and Hadass, 1990). Furthermore, teacher training, both initial and in-service, is a very important factor in the improvement of mathematics teaching and learning processes. This training must be focused on their professional development and for this it is essential that they put into practice a deep specialized knowledge of the content (Posadas & Godino, 2017).

In recent years, research in mathematics education has been nourished by models of descriptive analysis of the specialized knowledge of teachers (Shulman, 1986), who first included pedagogical content knowledge (PCK) as a differentiating element between the mathematical knowledge that a teacher should have and any other person who performs mathematical tasks. Shulman (1986) described the types of knowledge that a teacher should have in order to be competent in his or her job, initially including knowledge of the content to be taught, knowledge of the pedagogy needed to teach it, and knowledge of the curriculum. In the case of mathematics, Shulman's ideas were adapted by Ball, Thames and Phelps (2008) to give rise to what is known as the Mathematical Knowledge for Teaching (MKT) model. This model is divided into two domains: content knowledge (SMK) and pedagogical content knowledge (PCK). See Figure 2.

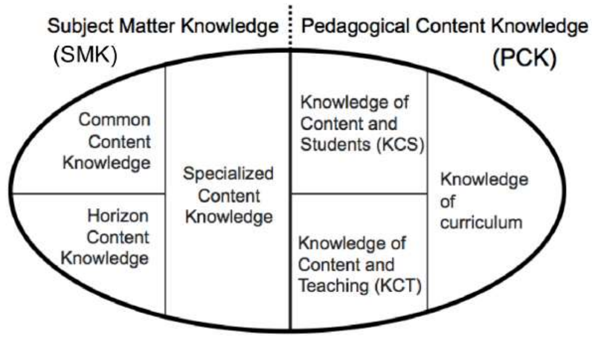


Figure 2: MKT Model (Ball, Thames & Phelps, 2008)

In this study, we will focus on the analysis of the Specialized Content Knowledge (SCK) associated with the infinite limit of a sequence, of the subdomain Subject Content Knowledge (SMK) that considers the mathematical content that a teacher needs from a teaching point of view (Ball et al. 2008). To specify what is and what is not SCK, Ball et al. (2008) list a number of activities that are: a) examining equivalences, b) recognizing what is involved in using a particular representation, c) relating representations to underlying ideas and to other representations, d) choosing and developing useful definitions, e) using mathematical notation and language and critiquing their use, f) connecting a topic being taught to topics from previous or future years, g) presenting mathematical ideas, among others.

SCK is the mathematical knowledge and skill that only teachers need to carry out their work. Mathematical knowledge that is not usually needed for purposes other than teaching.

Furthermore, we relate this mathematical content that a teacher needs to how people who are professionally engaged in mathematics think. The latter is called mathematical thinking and is understood as a spontaneous reflection carried out by mathematicians on the nature of the process of discovery and invention in mathematics, and also on advanced thinking processes, such as abstraction, justification, visualization, estimation or reasoning based on the justification of proposed hypotheses (Cantoral et al., 2000). In mathematical thinking we differentiate between Elementary Mathematical Thinking (EMT) and Advanced Mathematical Thinking (AMT). EMT is characterized by routine tasks in the classroom, using definitions only for the description of already known objects (Calvo, 2001). On the other hand, AMT is characterized by the intervention of the following processes: representation, abstraction, formalization, definition, among others (Garbin, 2015).

The justifications or explanations provided by the pre-service teacher where intuitive approaches are involved will be classified within EMT, while when this is formal it will be classified within AMT (Dreyfus, 1991; Tall, 1991).

METHOD

This study is based on an exploratory qualitative approach. By means of convenience sampling, i.e. the sample is made up of those trainee teachers to whom we have had access, we have analysed 12 students from a Spanish university who are studying for the Master's Degree in Teacher Training. In Spain, this Master's degree is a qualification without it is not possible to teach different subjects in Secondary Education (12-18 years), and its duration is one academic year, 600 hours. Previously, in Spain, students who take it have had to pass a Bachelor's Degree (8 semesters), related to the specialty to be taught. In the case of this study, the previous training of the sample members was as follows: Mathematics (9), Industrial Engineering (2) and Telecommunications Engineering (1). Their training in mathematical notions of calculus during their previous university degree, and in which the notion of limit appeared as it was taken up in the present study, can be found below (Table 1):

Degree	Subjects	Total hours of the subject	Content related to the infinite limit of sequences
Mathematics	Mathematical Analysis I	135	Sequences of real numbers. Convergence. Calculation of limits. Continuity. Limits of functions. Continuous functions. Properties. Weierstrass, Bolzano and Darboux theorems. Classification of discontinuities.
	Mathematical Analysis II	150	Functions of several real variables. Limits and continuity.
Physics	Mathematical Analysis	60	Sequences and limits. Cauchy sequences. Limit of functions. Continuity.
Industrial Engineering	Mathematics	60	Sequences and series of real numbers.
Telecommunications Engineering	Calculus	60	Real functions of a real variable: limits and continuity.

Table 1. University training in calculus.

At the time of the data collection for this study, they attend the last sessions of the subject Design of activities for learning Mathematics. Prior to this, the students have passed the subjects Disciplinary Contents of Mathematics and Curricular and Instructional Design of Mathematics, as well as having completed 100 hours of attendance at an educational centre. The elements of analysis have been the evidence of the work samples of the proposed task (Figure 3).

> Study the limit of the sequence a_n defined by $a_n = n^2 - 15$. Justify your answer. Then, indicate the explanation you would give to your students in the classroom.

Figure 3: Proposed task on the limit of a sequence.

For the analysis of the data, an inductive-deductive categorization process was carried out, including the categories expected from the theoretical review, described in the previous section, and also those arising from the analysis of the evidence collected from the trainee teachers' responses, so that all possible situations were included. The two variables taken into account were: task correctness and type of thinking involved. Both variables are used, on the one hand, for the resolution and justification of the resolution and, on the other hand, for the explanation provided to their future students. Therefore, the table we will use in the analysis (see Table 2) will be presented twice in the results section, the first time for the justification of the resolution and the second time for the explanation provided by the trainee teachers to their students.

<table>
<tr><th>Correction of the task</th><th colspan="2">Type of mathematical thinking involved</th></tr>
<tr><td>No resolution or explanation included</td><td colspan="2"></td></tr>
<tr><td rowspan="3">Performs incorrectly</td><td rowspan="2">EMT</td><td>Algorithmic</td></tr>
<tr><td>Involves the notion in a specific way</td></tr>
<tr><td colspan="2">AMT</td></tr>
<tr><td rowspan="3">Performs correctly</td><td rowspan="2">EMT</td><td>Algorithmic</td></tr>
<tr><td>Involves the notion in a specific way</td></tr>
<tr><td colspan="2">AMT</td></tr>
</table>

Table 2: Variables and categories considered for analysis.

a) In correcting the task we consider:
   - Does not include resolution or explanation: those answers in which the teacher does not respond to the proposed task, leaving the answer blank.
   - Performs incorrectly: the trainee teacher makes a mistake when solving the proposed task on the infinite limit of a sequence. This error may be in the use of one of the representation systems, in the understanding of the notion itself or algorithmic procedures, among others.
   - Performs correctly: the trainee teacher correctly solves the proposed task on the notion of the infinite limit of a sequence, both in the algorithmic processes and in the use of the different systems of representation that he/she decides to use. He/she also correctly uses the notation and vocabulary specific to this mathematical notion.

b) We differentiate the type of thinking developed by the trainee teacher: PME and PMA, described in the previous section, in relation to the type of response provided to each of the two tasks proposed, both for justification and explanation:
   - EMT. This category is subdivided into two:
     - Algorithmic: when the pre-service teacher uses properties of the limit that he/she uses routinely.
     - Involves the notion in a specific way: when intuitive approaches are involved, without the use of algorithmic procedures.
   - AMT: The pre-service teacher uses different systems of representation, abstraction, formalization or formal definition of the limit, i.e. when a formal approach is involved.

Afterwards, and given the relevance of the different systems of representation, an account will be made of them, both when they have to justify their answer and in the explanation to their students. This analysis takes into account both the ways of expressing and symbolizing when solving or explaining the limit of a sequence in the proposed task. Among the systems of representation, we consider: verbal, tabular, symbolic and graphic (Janvier, 1987).

In addition to the productions of future teachers, 35 textbooks published in Spain from 1936 to the present day have been considered, from the 1st and 2nd years of baccalaureate (16-18 years), both in Science and Social Sciences. Of these, we will look in depth at the 4 textbooks from 2010 to 2019, i.e. those that could be used by future teachers when mathematics at the educational level for which they are preparing. This analysis will take into account representation systems in which the infinite limit of a sequence is presented: verbal, tabular, graphical and symbolic; its format: definition and example; and the approach: intuitive and formal.

This analysis will serve to compare the way in which future teachers solve the proposed task in relation to the way in which it is presented in textbooks.

RESULTS

Procedures vs. notion involved

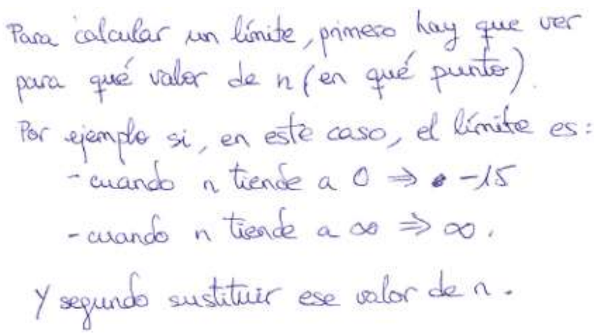
Given that the proposed task consists of two parts: the first in which the student would solve the limit, and the second in which he would explain it to his future students, we are going to approach the analysis independently. This analysis will allow us to identify the difficulties presented by the Master's student when solving the task and, on the other hand, to see what misconceptions, or not, the student would transmit when he or she becomes a teacher and has to explain the task to his or her students.

When the pre-service teachers are asked to solve and justify the limit, we obtain the following results (Table 3):

Correction of the task	**Type of mathematical thinking involved**		**Percentage**
No resolution or explanation included			0 %
Performs incorrectly	EMT	Algorithmic	0 %
		Involves the notion in a specific way	33.33 %
	AMT		
Performs correctly	EMT	Algorithmic	58.33 %
		Involves the notion in a specific way	0 %
	AMT		

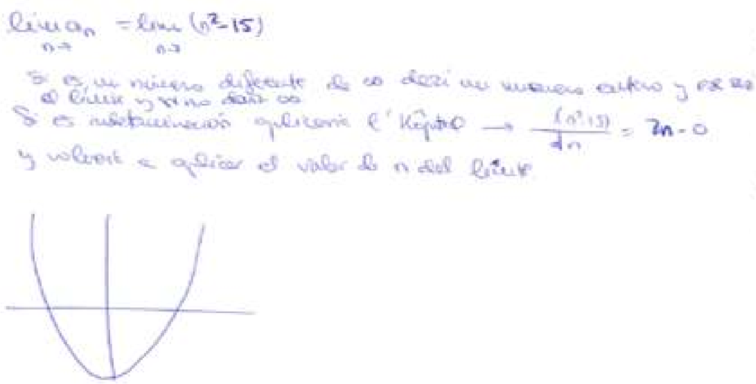
Table 3: Results of the answers to solving and justifying the limit of a sequence.

We found that 33.33 % of the pre-service teachers gave incorrect answers. In all cases, these errors occur when the future teacher involves the notion in a specific way from a PME. In all cases, elements appear that could appear in the limit of a function, but in no case are they attributed to the notion studied in this paper: the infinite limit of a sequence. Below, we show an example of an answer where the future teacher considers the limit at different points, as if it were a function (Figure 4) and another example where L'Hôpital is applied without indeterminacy and without the appearance of functions (Figure 5). In the latter case, we attribute the error not only to the lack of indeterminacy but also to the confusion between L'Hôpital, used for functions, and the Stolz Criterion, used for sequences.



To calculate a limit, we must first see
for what value of n (at what point).
For example, if, in this case, the limit is:
- when n tends to 0 → -15
- when n tends to ∞ → ∞
And second substitute that value of n.

Figure 4: Incorrect answer where invalid limits are contemplated and their translation.



If it is a number different from ∞
it will be an integer and that will be the limit,
otherwise, it will give ∞.
If it is indeterminacy, I will apply l'Hôpital
and I will apply again the value of n of the limit.

Figure 5: Incorrect answer where the future teacher tries to apply L'Hopital and its translation.

Most of the future teachers are able to solve the limit of the given sequence. Moreover, more than half of them prefer to solve it using algorithmic procedures, without actually using the mathematical notion of the infinite limit of a sequence in a particular way. In these algorithmic processes they employ equivalences, and use the correct mathematical notation and language. See Figure 6 and 7.

Figure 6: Correct answer of a student using an algorithmic procedure in a symbolic way.

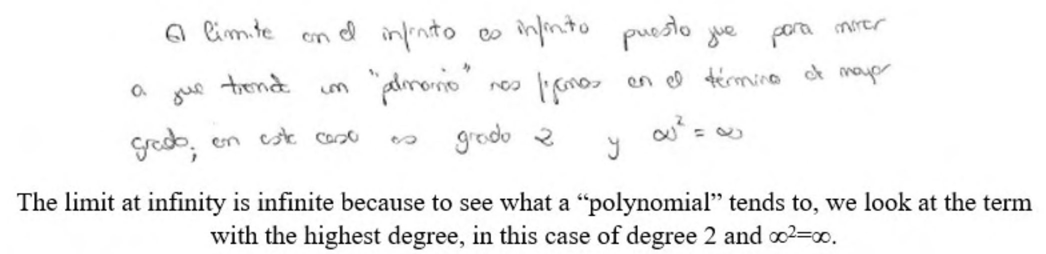


Figure 7: Correct answer of a student using an algorithmic procedure verbally and its translation.

Only one of the samples showed a development of the AMT. This future teacher not only accounts for the growth of the succession terms, but also relates them to the position they occupy, "As n grows a_n grows over n" (Figure 8).

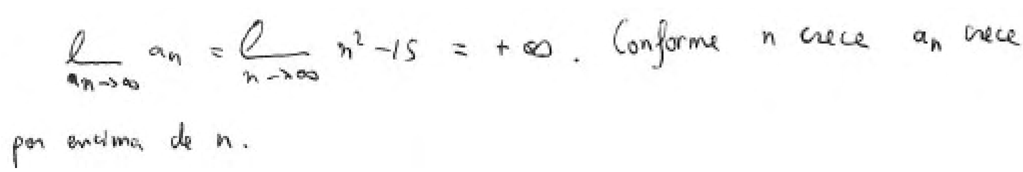


Figure 8: Correct response of a future teacher developing an AMT.

Subsequently, in the second part of the proposed task, when the trainee teachers were asked to explain what their intervention in the classroom would be like, we obtained the following results (Table 4):

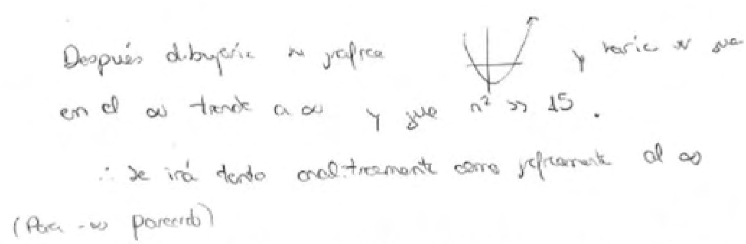
Correction of the task	**Type of mathematical thinking involved**		**Percentage**
No resolution or explanation included		No resolution or explanation included	33.33 %
Performs incorrectly	EMT	Performs incorrectly	0 %
		Involves the notion in a specific way	16.67 %
	AMT		0 %
Performs correctly	EMT	Performs correctly	0 %

	Involves the notion in a specific way	50 %
	AMT	0 %

Table 4: Results of the answers to the explanation of the limit of a sequence.

The same prospective teachers who performed poorly on the first proposed task, i.e. solving and justifying the limit of a sequence, did not include the explanation they would provide in the classroom of the notion involved.

On two of the occasions when prospective teachers had correctly justified the limit in the first part of the task, using an algorithmic procedure and without mentioning the notion they were using, they provided a wrong explanation in the second part of the task. This corresponds to 16.67 %. Given the presentation format of this second part of the task, which allowed for a more detailed explanation, these prospective teachers used the wrong graphical representation system. The notion represented was a function, which they had not previously used and which was not involved in the task. Moreover, they proposed to perform the limit for $n \to -\infty$, when this cannot be done for a sequence, but for a function. See Figure 9.



Then, I would draw its graph and show that at ∞ and that $n^2 \gg 15$.
It will go both analytically and graphically to ∞. (For similar $-\infty$)

Figure 9: Incorrect answer of a future teacher in the explanation of the notion of the limit of a sequence and its translation.

On the other hand, half of the future teachers analyzed used a development of the EMT in their explanation, involving the notion in a specific and correct way. Most of them are those who, in the resolution and justification, used a development of the EMT involving algorithmic procedure. We can observe very brief explanations, as well as a very intuitive view of the limit (Figure 10). We also find explanations where simpler examples are presented, such as $a_n = n^2$ in the tabular representation system (Figure 11).

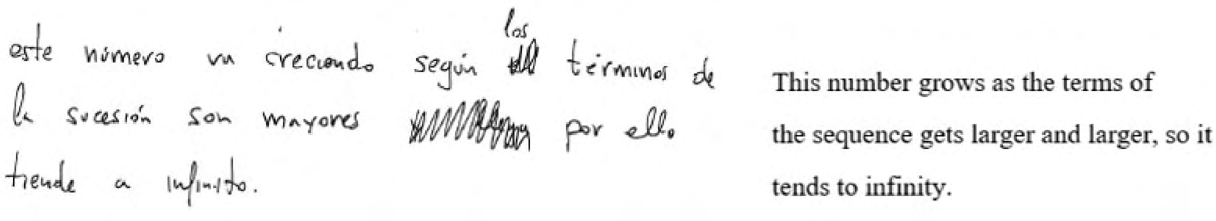


Figure 10: Example of a response with an intuitive view of the limit and its translation.

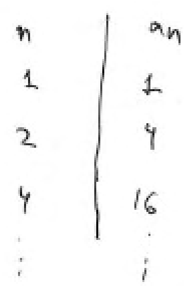


Figure 11: Example of a response using simpler examples.

Representation systems used

Given the relevance of the different systems of representation, which we have already mentioned, we have collected the ones used by trainee teachers by combining their complete response to the two proposed tasks. See Table 5.

	Verbal	Graphic	Tabular	Symbolic
Does not use representation	8.33 %	75.00 %	75.00 %	25.00 %
Uses an incorrect representation	33.33 %	25.00 %	0.00 %	25.00 %
Uses a correct representation	58.33 %	0.00 %	25.00 %	50.00 %

Table 5: Representation systems involved.

The representation system that was most frequently used correctly was verbal (58.33 %), where the future teachers, in addition to verbalizing the algorithmic procedure for the second part of the task, also used it correctly (58.33 %), "subtracting 15 is irrelevant against a power of n^2 when $n \to \infty$" (Figure 12) that they have followed in the first part of the task, specifically involve the notion with which they are working, "the limit tends to $+\infty$ because the numbers are getting bigger and bigger" (Figure 13).

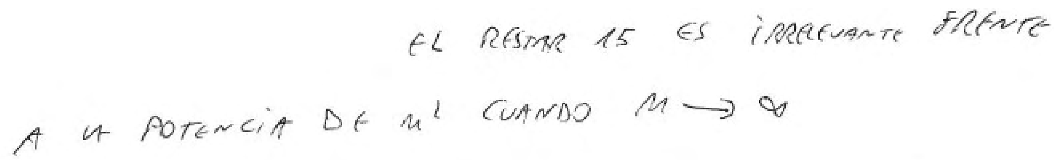


Figure 12: Use of the verbal representation system as an explanation of algorithmic procedure.

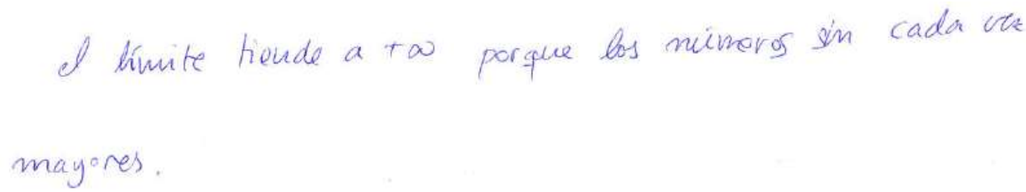


Figure 13: Use of the verbal representation system involving the notion of infinite limit.

At the other end of the spectrum is the graphic representation system, which, in addition to not being used by 75% of future teachers, those who do use it present it incorrectly, treating the notion as if it were a function instead of a succession (Figure 14).

Figure 14: Use of the graphical representation system.

The same percentage (75 %) of trainee teachers declines the use of the tabular representation system, although in this case, those who decide to use it use it correctly. See Figure 15.

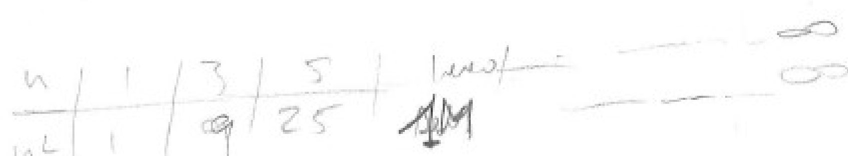


Figure 15: Use of the tabular representation system in the second part of the task.

The fourth of the representation systems, the symbolic one, used correctly by 50% of the future teachers, was used to a greater extent in the first part of the task, while in the second part it was relegated to very few cases. In both cases for the use of algorithmic procedures. See Figure 16 and Figure 17.

Figure 16: Incorrect answer using the symbolic representation system.

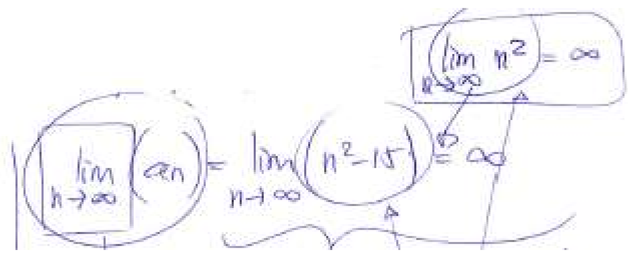


Figure 17: Correct answer using the symbolic representation system.

Textbook analysis

Given that the same number of books was not available for each of the decades, the average number per textbook was taken as a reference for each of the systems of representation: verbal (v), tabular (t), graphic (g) and symbolic (s), and formats: definition (d) and example (e). Also, on the one hand, fragments were collected that have to do with an intuitive approach, linked to the EMT; and on the other hand, fragments with a formal approach, linked to the AMT.

The historical evolution of the plus and minus infinite limit of a sequence is shown below (Figure 18).

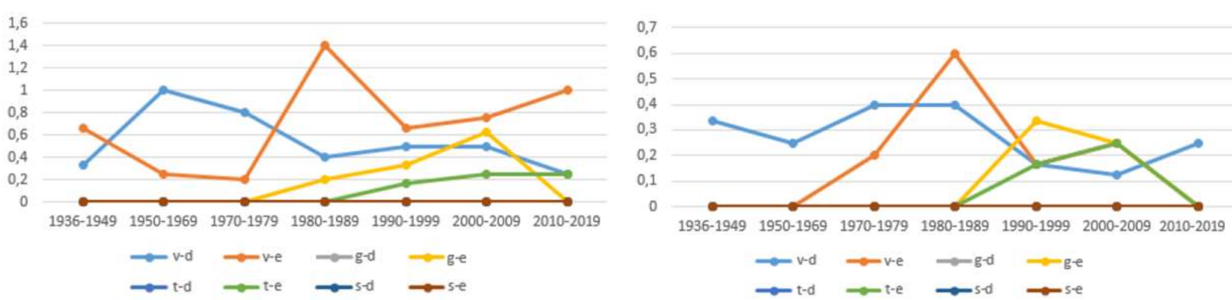


Figure 18. Historical evolution with an intuitive approach. More infinite (left) and less infinite (right).

Although the most infinite and least infinite limit have historically had a similar behaviour, it is important to highlight the presence of the most infinite limit higher than that of the least infinite limit, doubling its frequency in some instances.

The following image shows some of these fragments analysed (Figure 19 and Figure 20). In the first of them, we observe how the textbook presents an example of the most infinite limit of a sequence in the verbal representation system. While in the second one we observe the intuitive approach in the verbal representation system.

Veamos qué sucede con la sucesión $b_n = \dfrac{n^2}{n+1}$:

n	1	10	100	1 000	10 000	100 000	1 000 000
a_n	0,5	9,09	99	999	9 999	99 999	999 999

En esta sucesión los términos no se aproximan a ningún número, en cambio, a medida que n aumenta los términos se hacen cada vez mayores.

Let's see what happens for the sequence $b_n = \frac{n^2}{n+1}$.

In this sequence the terms do not approach any number, but as n increases the terms become larger and larger.

Figure 19. Fragment with intuitive approach (t-e) (Valverde et al., 2015)

Por otro lado, la sucesión $b_n = 2n$ verifica que sus términos se hacen cada vez mayores; es decir, tienden a $+\infty$: $b_1 = 2, b_2 = 4, b_3 = 6, b_4 = 8, b_5 = 10...$

Se dice que el límite de la sucesión es $+\infty$: $\lim_{n\to+\infty} b_n = +\infty$

On the other hand, the sequence $b_n = 2n$ verifies that its terms become greater and greater; that is, they tend to $+\infty$. $b_1 = 2, b_2 = 4, b_3 = 6, b_4 = 8, b_5 = 10...$

It is said that the limit of the sequence is $+\infty$: $\lim_{n\to\infty} b_n = +\infty$.

Figure 20. Fragment with intuitive approach (v-e) (Alcaide et al., 2016)

In a similar way, we present the infinite limit of a sequence from a formal approach. Given the process of generalisation and abstraction that the student has to carry out, on this occasion the difference between the plus and minus infinite limit has not been established. See Figure 21.

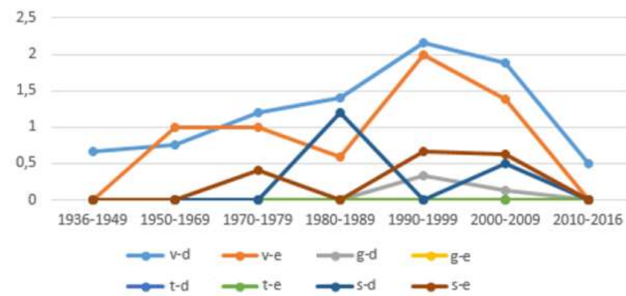


Figure 21. Historical evolution with a formal approach

As can be seen, the formal approach to the notion of limit has been decreasing its presence in Spanish textbooks in recent years. A sample of one of the last fragments found is the following (Figure 22) in the verbal representation system and definition format.

De forma teórica: $\lim_{n \to \infty} a_n = +\infty$ si para cualquier número, k, podemos encontrar un número natural h tal que:

Si $n > h$, se cumple que $a_n > k$.

Theoretically: $\lim_{n\to\infty} a_n = +\infty$ if for any number, k, we can find a natural number h such that:

If $n > k$, then $a_n > k$.

Figure 22. Fragment with formal approach (v-d) (Escoredo et al., 2009).

Analysis of future teachers vs. textbooks

The results obtained from the analysis of the answers given by the future teachers are related to those presented in the study of the textbooks we have carried out. In fact, prospective teachers use the verbal representation system more frequently and with a lower percentage of error. This corresponds to the fact that the verbal representation system is the most frequent in recent years in the textbooks when the infinite limit of a sequence has to be presented.

With respect to the graphical representation system, its frequency has declined in recent years and this could explain some of the errors that have arisen when future teachers use this system in their answers to justify their answers or to calculate a limit. To this difficulty we can also add the variety of limits of functions presented and the corresponding graphical representations of them, a fact which may lead us to think that there is a limit when n tends to $-\infty$ or when n tends to 0, as some future teachers point out in their answers. We attribute some of these difficulties or errors to what Fernández-Plaza and Simpson (2016) point out, where the symbols used for sequences and functions are almost the same, but connections must be established in the classroom that relate both concepts.

Furthermore, we can also attribute the development of the EMT in textbooks, to the detriment of the AMT, due to the boom that the intuitive approach to the notion studied seems to be experiencing in textbooks. This fact has meant that certain notions are not treated with the rigour and therefore a real understanding of them is not produced.

CONCLUSIONS

In the present study it has been possible to analyze the difficulties existing in pre-service teachers when performing tasks in which the infinite limit of a sequence is involved, both for their resolution and for their future explanation in the classroom. Given the professional development focus of the task, they have needed to put into practice specialized knowledge of the content (Posadas & Godino, 2017). In addition, we consider that this task has been appropriate for developing MKT, as it has allowed the use of multiple representations and has given the pre-service teachers the opportunity to propose mathematical practices for teaching the notion of the infinite limit of a sequence. In particular, and in relation to the SCK (Ball et al. 2008), during its resolution, we have been able to observe how future teachers use equivalences, notation and mathematical language to approach it. On the other hand, we note that they used different systems of representation, but they did not relate the underlying ideas, nor did they connect the notion with topics from previous and future years.

On the other hand, and in relation to the type of Mathematical Thinking, in the first part of the task in which the students had to solve the proposed limit, the correct answers are presented with a development of the EMT from an algorithmic approach and, on one occasion, with a development of the AMT. Incorrect answers are presented when students, from a EMT development, involve the notion of the infinite limit of a sequence. Likewise, we perceive a change in how future teachers will approach the explanation of the limit. They discard the use of the AMT, despite the fact that their future students are at an age when it can begin to be introduced, and when they develop the EMT it is always by involving the notion, without using algorithmic procedures and from a more intuitive point of view. We can therefore affirm that future teachers do not involve the processes of abstraction, formalization and definition (Garbin, 2015), which are characteristic of AMT, and opt for routine tasks in the classroom, which are characteristic of AMT (Calvo, 2001).

Despite Duval's (1998) indications that in order to understand a notion, different representations must be provided, not many future teachers have resorted to showing the notion of the infinite limit of a sequence in more than one system of representation. Moreover, in no case was coordination between them shown, which would provide a comprehensive understanding of the notion. On the other hand, as stated by Vrancken et al. (2006), future teachers are in favour of the current tendency to consider an algorithmic approach to solving a given limit. This has hardly made it possible to detect errors in the first part of the task. It was in the second part when, by using conceptual methods and different systems of representation (Morales et al. 2013), such as the graph, more difficulties were detected.

In view of the above, we can affirm that future teachers do not have an adequate knowledge of the notion of the infinite limit of a sequence and, consequently, this will undoubtedly be transmitted to their students.

As a future perspective, we consider the creation of a didactic sequence composed of different examples and definitions, developing both the EMT and the AMT, which will help to overcome the difficulties encountered.

Acknowledgements

Study developed in the group HUM-324 of the PAIDI of the Autonomous Community of Andalusia and in the group S60_20R-Research in Mathematics Education in the field of the Autonomous Community of Aragon for the period 2020-2022.

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