Nonlinear Schrödinger Equations with Rough Data

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Abstract

In this thesis we consider nonlinear Schrödinger equations with rough initial data. Roughness of the initial data in nonlinear Schrödinger equations can be understood as being of low regularity and as a lack of decay at infinity.

Firstly we prove low regularity a priori estimates for the derivative nonlinear Schrödinger equation in Besov spaces with positive regularity index. These a priori estimates are sharp at the level of regularity but are conditional upon small mass. The proof uses the operator determinant characterization of the transmission coefficient introduced by Killip–Vişan–Zhang.

Secondly we show global wellposedness for the tooth problem of defocusing nonlinear Schrödinger equations, that is the Cauchy problem with initial data in the space $H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})$. This result can be seen as an intermediate step between the wellposedness theory in the $L^2(\mathbb{T})$-based setting and more generic non-decaying behavior at infinity. In the case $s_1 = 1$ we obtain an at most exponentially growing energy, based on the Hamiltonian of the perturbed equation. For the cubic nonlinearity we may choose $s_2 > 3/2$ whereas for higher power nonlinearities our assumption is $s_2 > 5/2$.

Finally we investigate the question of wellposedness of nonlinear Schrödinger equations with initial data in modulation spaces. Modulation spaces $M^s_{p,q}(\mathbb{R}^n)$ encode both regularity (s and q) and decay (p) in their indices. By making use of multilinear interpolation we prove new local wellposedness results. The local wellposedness results we obtain are proven to be sharp with respect to the regularity index. Moreover we complement the local results by showing global wellposedness in several cases, including low regularity and very slow decay. This is done on the one hand by an extension of techniques developed by Oh–Wang to a broader range of modulation spaces, and on the other hand by applying calculations from Dodson–Soffer–Spencer to the modulation space setting.

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Contents

Chapter 0

Overview

This thesis is a contribution to the theory of nonlinear Schrödinger equations. Nonlinear Schrödinger equations appear in a number of physical applications and have also aroused great interest in the mathematical community.

One of their first appearances as a model equation in physics was in the 1960s when high-power lasers became available. Laser pulses are electromagnetic waves and as such their propagation in a medium is governed by Maxwell's equations. In an isotropic medium the highest order correction to the linear response is cubic, leading to the so called Kerr nonlinearity for the electric field equation. This term becomes physically relevant when the intensity of the laser beam is high enough and is of major interest in the research field of nonlinear optics and optical collapse. If one assumes an approximately linearly polarized laser beam, applies the paraxial approximation, and drops certain second order derivatives, the propagation of the electric field $\mathcal{E}$ in a medium is described by a two-dimensional cubic nonlinear Schrödinger equation [52, Section 1],

$$2ik_0\psi_z + \Delta_{x,y}\psi + k_0^2\frac{4n_2}{n_0}|\psi|^2\psi = 0, \qquad \psi = \psi(x,y,z). \tag{0.1}$$

Here the electric field is written as $\mathcal{E} = (\mathcal{E}_1, 0, 0)$ with $\mathcal{E}_1 = e^{i(k_0 z - \omega_0 t)}\psi$, z is the direction of propagation, x and y are the transverse coordinates, n_0 denotes the (linear) refractive index and n_2 the Kerr coefficient. The strength of this physical model is underlined by the fact that several mathematical predictions like self-similar collapse with a universal radial profile can be found experimentally as well [52, Section 14.5].

Equation (0.1) suggests that the electric field can be modeled by a one-dimensional nonlinear Schrödinger equation if one of the transverse coordinate directions can be ignored. This is the case when the input laser beam is highly elliptic or when the Kerr medium has a planar waveguide geometry, i.e. when one of the transverse directions is narrow while the other one is wide.

Another situation in which a one-dimensional nonlinear Schrödinger equation serves as a model equation is the propagation of laser pulses in optical fibres. In this situation the mathematical roles of time and space are reversed, and the equation reads

$$i\psi_z + \psi_{tt} + |\psi|^2\psi = 0,$$

see [52, Section 4]. Thus the initial condition at $z = 0$ is the signal which gets transmitted by the optical fibre. Mathematical questions such as the question of global existence for the initial value problem translate accordingly to the question whether or glass fibre cables can be arbitrarily long in this description.

Apart from their application in nonlinear optics nonlinear Schrödinger equations also appear in asymptotic equations of Klein-Gordon, KdV and water wave equations [1, Chapter 6]. They play a role in the form of the Gross-Pitaevskii equation in mathematical models of Bose-Einstein condensates [52, Section 4], and as derivative nonlinear Schrödinger equations in the description of Alfvén waves propagating along a magnetic field in plasma [123, Section 3.4].

Mathematical interest in nonlinear Schrödinger equations mainly began to sparkle with the birth of the inverse scattering method. This extremely powerful mathematical tool was shown to be applicable to the one-dimensional cubic nonlinear Schrödinger equation in a paper by Zakharov–Shabat [137]. Since then the research area of dispersive equations has grown steadily, and the class of nonlinear Schrödinger equations is only one of many that are being investigated.

This thesis is structured in the following way:

- Section 1 gives an introduction to nonlinear Schrödinger equations and some of their most important theoretical aspects. We introduce the notion of wellposedness, list some important local and global wellposedness results, and motivate why both low regularity and lack of decay can be understood as roughness of initial data.

- Section 2 focuses on completely integrable nonlinear Schrödinger equations. The transmission coefficient is introduced and we present two approaches of dealing with it analytically. We also show how low regularity almost conserved quantities can be derived from it. Some of the calculations done in this section are based on joint work in progress of the author with Prof. Dr. Herbert Koch (Universität Bonn) and Dr. Baoping Liu (Peking University) [80].

- Section 3 deals with low regularity almost conserved quantities for the derivative nonlinear Schrödinger equation. It is the result of a collaboration with Dr. Robert Schippa (KIT) which has been accepted but not yet published in Funkcialaj Ekvacioj. A manuscript is available online [82].[1] Only a few changes have been made to the manuscript to fit the work into this thesis.

- In Section 4 we consider the tooth problem for the nonlinear Schrödinger equation and prove global wellposedness for it. Sections 4.1 to 4.3 are based upon a joint work with apl. Prof. Dr. Peer Kunstmann (KIT)

[1] There is also an arXiv preprint available at `https://arxiv.org/abs/2007.13161`.

which has been published in J. Math. Anal. Appl [81]. The content of Section 4.4 and some parts of Section 4.2 are new. In Section 4.2 we include more estimates on products of periodic and non-periodic functions compared to [81]. In 4.4 we extend the wellposedness theory from [81] to initial data in $\mathcal{S}(\mathbb{R}) + C^\infty(\mathbb{T})$.

- The topic of Section 5 is the wellposedness of the nonlinear Schrödinger equation with initial data in a modulation space. It is based on a work of the author that has been sent to a peer-reviewed journal, with manuscript available online [79].[2] Few changes have been made to this manuscript in Sections 5.2 to 5.5. Section 5.1 extends the introduction from [79] to also include higher dimensions and gives more examples as well as a proof of the boundedness of the Schrödinger propagator in modulation spaces.

- Sections 1 to 5 are complemented by the Appendices A.1 to A.4. In Appendix A.1 we define the Fourier transform as well as Sobolev and Besov spaces. Appendix A.2 lists some often used inequalities. In Appendix A.3 we describe some results from operator theory, and in Appendix A.4 we give a short overview of the theory of Hamiltonian equations.

[2] There is also an arXiv preprint available at `https://arxiv.org/abs/2204.04957`

Chapter 1

Introduction to Nonlinear Schrödinger Equations

In this section we review some basic results on the initial value problem for the nonlinear Schrödinger equation (NLS). These results can be found for example in the standard textbooks on nonlinear dispersive equations [26, 90, 128, 84, 123]. We also want to mention the great overview article [27] and the lecture notes [88].

The initial value problem for the nonlinear Schrödinger equation is given by

$$\begin{cases} iu_t + \Delta u = \kappa |u|^{p-1} u, & p \in [1, \infty), \kappa \in \mathbb{R}, \\ u(0, x) = u_0(x). \end{cases} \tag{1.1}$$

Sometimes (1.1) is also called generalized nonlinear Schrödinger equation due to the power exponent p, and the case $p = 3$ is referred to as the NLS.

We search for a complex valued function $u = u(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ satisfying (1.1) classically, weakly or in the sense of an integral equation (i.e. mildly). The initial value function u_0 is assumed to lie in a function space X_0, and in most cases we want the solution to be an element of $C^0([0, T], X_0)$.

The parameter κ is a coupling constant and depending on its sign we call NLS either defocusing ($\kappa > 0$), or focusing ($\kappa < 0$). If $\kappa = 0$, (1.1) reduces to the linear Schrödinger equation. Note that only the sign, but not the actual value of κ is important for later discussions, as we can always define $u_\nu(t, x) = \nu u(t, x)$ which satisfies the same equation (1.1) with $\kappa_\nu = |\nu|^{-2}\kappa$. Thus in most cases, we will simply assume $\kappa \in \{\pm 1\}$. In the cubic, one-dimensional case $p = 3, n = 1$ though we will slightly differ from this and choose $\kappa = \pm 2$, which results in integer constants in the methods associated with the complete integrability of the equation.

The nonlinear Schrödinger equation (1.1) is a semilinear, dispersive PDE. "Semilinear" means that the equation can be seen as a perturbation of a linear equation by some lower order nonlinear term. "Dispersive" means that the phase velocity of a wave depends on its frequency, and that solutions of the linear equation with localized initial data will spread out converge pointwise in x to zero as $|t| \to \infty$ (see e.g. Example 1.2). Other equations in the family of semilinear, dispersive PDE include the (generalized) Korteweg-de Vries equation,

$$u_t = -u_{xxx} + \kappa u^{p-1} u_x, \tag{1.2}$$

and the nonlinear wave ($m = 0$) and nonlinear Klein-Gordon ($m \neq 0$) equation

$$u_{tt} = \Delta u - m^2 u + \kappa |u|^{p-1} u,$$

see [27]. One can generalize equation (1.1) and consider equations of the form

$$iu_t + \Delta u = f(t, x, u, \nabla u),$$

for a function $f : \mathbb{R}^2 \times \mathbb{C}^{1+n} \to \mathbb{C}$. Such equations are also called nonlinear Schrödinger equations. Examples of interesting nonlinearities of this type include combined power nonlinearities $f = \kappa_p |u|^{p-1} u + \kappa_q |u|^{q-1} u$ and spatial variable power nonlinearities $f = \kappa |x|^\alpha |u|^{p-1} u$. Moreover, the cubic one-dimensional derivative nonlinear Schrödinger equation

$$iu_t + u_{xx} = \kappa (|u|^2 u)_x, \tag{1.3}$$

is another example of a completely integrable nonlinear Schrödinger equation. For this equation, we will construct low regularity a priori estimates in Section 3.

There are several important symmetries for (1.1). Assume that $u(t, x)$ is a solution to (1.1). Then, also

- $u(t - t_0, x - x_0)$ is a solution for all $t_0, x_0 \in \mathbb{R}$ (time and space translation symmetry),
- $e^{i\gamma} u(t, x)$ is a solution for all $\gamma \in \mathbb{R}$ (phase rotation symmetry),
- $\bar{u}(-t, x)$ is a solution (time reversal symmetry),
- $e^{i(\eta \cdot x - |\eta|^2 t)} u(t, x - 2\eta t)$ is a solution for all $\eta \in \mathbb{R}^n$ (Galilean invariance),
- $u_\lambda(t, x) = \lambda^{-2/(p-1)} u(\lambda^{-2} t, \lambda^{-1} x)$ is a solution for all $\lambda > 0$ (scaling symmetry).

Moreover, if $u(t, x)$ is a solution of (1.1) and we consider the pseudo-conformal transformation by defining for $b \in \mathbb{R}$,

$$u_b(t, x) = e^{-i \frac{b|x|^2}{4(1-bt)}} (1 - bt)^{-\frac{n}{2}} u\Big(\frac{t}{1 - bt}, \frac{x}{1 - bt}\Big), \tag{1.4}$$

then u_b is again a solution of (1.1) if $p = 1 + 4/n$. For general p, u_b will solve a nonautonomous NLS,

$$iu_t + \Delta u = \kappa (1 - bt)^{\frac{n(p-1)-4}{2}} |u|^{p-1} u.$$

Closely tied to the symmetries by Noether's theorem (see [123, Section 2.2]) are the conservation laws of NLS. The quantities

$$\begin{aligned} M(u) &= \int_{\mathbb{R}} |u(t,x)|^2\,dx, \qquad \text{mass}, \\ P(u) &= \operatorname{Im} \int_{\mathbb{R}} \bar{u}(t,x)\nabla u(t,x)\,dx, \qquad \text{momentum}, \\ E(u) &= \int_{\mathbb{R}} \frac{1}{2}|\nabla u(t,x)|^2 + \frac{\kappa}{p+1}|u(t,x)|^{p+1}\,dx, \qquad \text{energy}, \end{aligned}$$

are all conserved for regular enough solutions $u(t,x)$. The pseudo-conformal transformation is related to the equality

$$\partial_t^2 \int_{\mathbb{R}} |x|^2|u(t,x)|^2\,dx = 16E(u),$$

when $p = 1 + \frac{4}{n}$.

The nonlinear Schrödinger equation (1.1) can also be seen as a Hamiltonian equation. More precisely, let p be an odd integer and consider

$$H(q,r) = \int_{\mathbb{R}} \nabla q \nabla r + \frac{2\kappa}{p+1}(qr)^{\frac{p+1}{2}}\,dx,$$

which is well-defined for complex valued functions $q(t,x), r(t,x)$ since $p+1$ is even. Define the functional derivatives $\frac{\delta H}{\delta q}$, $\frac{\delta H}{\delta \bar{r}}$ by the identity

$$\frac{d}{ds}\Big|_{s=0} H(q+s\phi, r+s\psi) = \int_{\mathbb{R}} \frac{\delta H}{\delta q}\phi + \frac{\delta H}{\delta r}\psi\,dx,$$

for all $\phi,\psi \in \mathcal{S}(\mathbb{R}^n)$. With these definitions, the Hamiltonian equations

$$q_t = \frac{1}{i}\frac{\delta H}{\delta r}, \qquad r_t = -\frac{1}{i}\frac{\delta H}{\delta q} \tag{1.5}$$

become

$$\begin{aligned} iq_t &= -\Delta r + \kappa q^{\frac{p+1}{2}} r^{\frac{p-1}{2}}, \\ ir_t &= \Delta q - \kappa q^{\frac{p-1}{2}} r^{\frac{p+1}{2}}. \end{aligned}$$

Setting $q = \bar{r} = u$ reduces both equations to (1.1). Since $H(u,\bar{u}) = 2E(u)$, the conservation of energy is a consequence of the Hamiltonian formalism. Indeed, define the Poisson bracket

$$\{F,G\} = \frac{1}{i}\int \frac{\delta F}{\delta q}\frac{\delta G}{\delta r} - \frac{\delta F}{\delta r}\frac{\delta G}{\delta q}\,dx,$$

with which (1.5) can be recast into the form

$$q_t = \{q,H\}, \qquad r_t = \{r,H\},$$

by setting

$$\frac{\delta r(x)}{\delta r(y)} = \delta(x-y) = \frac{\delta q(x)}{\delta q(y)}, \qquad \frac{\delta r(x)}{\delta q(y)} = 0 = \frac{\delta q(x)}{\delta r(y)}.$$

Then, in general, the time evolution of a quantity $F(t,q,r)$ can be recovered as

$$\frac{d}{dt}F(t,q(t,\cdot),r(t,\cdot)) = (\partial_t F)(t,q(t,\cdot),r(t,\cdot)) + \{F,H\}(q(t,\cdot),r(t,\cdot)). \tag{1.6}$$

In particular,

$$\frac{d}{dt}H(q(t),r(t)) = 0.$$

We refer to the Appendix A.4 for a more thorough overview of Hamiltonian methods.

With respect symmetries and conservation laws the cubic, one-dimensional case, $n=1, p=3$, is exceptional. It is completely integrable in the sense that there exists a canonical transformation into action-angle variables which turns (1.1) into a set of ODEs which are trivially solvable [50, Chapter III]. We will see in Section 2 that the cubic one-dimensional NLS also admits infinitely many conserved energies. This property is shared by the derivative NLS (1.3) and will be used in Section 3.

The dynamics of solutions to equation (1.1) in the focusing and defocusing case are different. In the focusing case $\kappa = -1$, there exist decaying in space, travelling wave solutions to (1.1), called solitary solutions, or for short solitons. Consider $Q \in H^1(\mathbb{R}^n)$ a solution to the nonlinear elliptic equation

$$-\Delta Q + \omega Q = -\kappa |Q|^{p-1} Q. \tag{1.7}$$

If $\kappa \geq 0$, non-trivial $H^1(\mathbb{R}^n)$ solutions to this equation do not exist (see [27]). For simplicity we set $\kappa = -1$. Given such a solution Q_ω, $u(t,x) = e^{i\omega t} Q_\omega(x)$ is a solution of (1.1). In one dimension $n=1$, and if $\omega = 1$, the solution (which is unique up to phase rotations and translations) Q is given explicitly by

$$Q(x) = \Big(\frac{p+1}{2}\operatorname{sech}^2\Big(\frac{p-1}{2}(x)\Big)\Big)^{\frac{1}{p-1}}.$$

Making use of the Galilean invariance and phase rotation invariance gives an equation for travelling wave solutions in one dimension,

$$u(t,x) = e^{i(t+\eta x - \eta^2 t + \theta)}\Big(\frac{p+1}{2}\operatorname{sech}^2\Big(\frac{p-1}{2}(x-2\eta t)\Big)\Big)^{\frac{1}{p-1}}, \tag{1.8}$$

where $\eta, \theta \in \mathbb{R}$ are arbitrary and give the velocity and initial phase of the soliton.

The existence of solitons is an effect that occurs only in the balance of the linear and the nonlinear part of the equation: while the linear part tries to disperse localized initial data (see e.g. Theorem 1.8), the nonlinearity tries to focus it. Mixing these effects creates the possibility of solitons.

After deciding on the notion of a solution (classic, weak or mild), some of the most important questions when dealing with initial value problems as (1.1) are:

- Do solutions exist?
- Are solutions unique?
- Do the solutions depend continuously on the initial data?

An initial value problem is called Hamadard wellposed [62] if all of these questions can be answered positively. We use the notion of local and global wellposedness [128, Definition 3.4] to approach these questions:

Definition 1.1. *We say that the problem* (1.1) *is locally wellposed (LWP) in the topological space X_0 if for any $u_0 \in X_0$ there exists a time $T > 0$, and an open neighborhood B of u_0 in X_0, and a subset X of $C^0([-T,T], X_0)$ such that for all $\tilde{u}_0 \in B$ there is a unique solution[1] $u \in X$ to* (1.1)*, and furthermore the map $\tilde{u}_0 \mapsto u$ is continuous from B to X (with respect to the $C^0([-T,T], X_0)$ topology).*

If we can take T arbitrarily large, we say that the problem (1.1) *is globally wellposed (GWP).*

From the time reversal symmetry we see that whenever X_0 is invariant under complex conjugation, we can restrict the wellposedness discussions to positive times $t \in [0,T]$. In most cases the space X_0 will have a more favorable structure than just being a topological space, for instance in Sections 4 and 5 it will be a Banach space, and often even a Hilbert space.

The inclusion of the space X in the definition of wellposedness comes from its role as an auxiliary space for closing fixed point arguments. An standard example is given by the Strichartz spaces (see Theorem 1.8 and Theorem 1.9). There are situations in which the space X is allowed to be a space which is not a subspace of $C^0([-T,T], X_0)$, though one may discuss whether the name "wellposedness in X_0" is justified in this case. Finally, global wellposedness is strictly weaker than being able to take $C^0_b(\mathbb{R}, X_0)$ in the definition of local wellposedness. Indeed, one could have global wellposedness even though

$$\limsup_{|t|\to\infty} \|u(t)\|_{X_0} = \infty.$$

The scaling symmetry is of particular interest for the question of wellposedness. It allows us to distinguish between the supercritical, the critical and the subcritical case. More precisely, given initial data in an L^2 based Sobolev space $u_0 \in H^s(\mathbb{R})$ and a solution $u(t,x)$ of NLS, the scaled solution $u_\lambda = \lambda^{-2/(p-1)} u(\lambda^{-2}t, \lambda^{-1}x)$ solves NLS with initial data $u_{0,\lambda} = u_\lambda(0,\cdot)$ which has Sobolev norm

$$\|u_{0,\lambda}\|_{H^s(\mathbb{R})} = \lambda^{-s+\frac{1}{2}-\frac{2}{p-1}} \|u_0\|_{H^s(\mathbb{R})}. \tag{1.9}$$

We define the critical exponent,

$$s_c = \frac{1}{2} - \frac{2}{p-1}, \tag{1.10}$$

[1] As mentioned earlier, the word "solution" can be understood in different ways and its definition changes depending on the situation.

as the exponent for which scaling the solution leaves the s_c-Sobolev norm invariant. Note that if $u(t,x)$ solves NLS on a maximal time interval of existence $[0, T_*)$, then likewise u_λ solves NLS on a maximal time interval of existence $[0, \lambda^2 T_*)$. Thus,

- for $s > s_c$ (subcritical case), we can scale down the initial data of the solution (λ large) making the maximal time of existence even larger. This shows that in most arguments, it is enough to consider small initial data, and this is often the most favorable case to prove local wellposedness.
- for $s = s_c$ (critical case), the size of the initial data does not change when scaling.
- for $s < s_c$ (supercritical case), decreasing the norm of the initial data also decreases the maximal time interval.

Similar arguments are applicable to other function spaces that have a given scaling, for example for Besov spaces $B^s_{p,q}$ or Fourier-Lebesgue spaces $\mathcal{F}L^p$.

As a general guideline, in the subcritical case one may rather expect wellposedness, in the supercritical case one may rather expect illposedness, and in the critical case, the situation is unclear. These are not strict guidelines though as the following two examples illustrate:

- For KdV, that is (1.2) with $p = 2$, the critical scaling exponent is $s_c = -\frac{3}{2}$, while wellposedness in the sense of uniformly continuous dependence of the solution on the initial data breaks down for $s < -\frac{3}{4}$ on $\mathbb{R}$ and $s < -\frac{1}{2}$ on $\mathbb{T}$ [36], and continuous dependence of the solution on the initial data breaks down for $s < -1$ [99, 100]. Wellposedness with merely continuous dependence on the initial data for $s \geq -1$ was shown on the torus $\mathbb{T}$ in [73], and on the line only recently in [76].
- For cubic NLS on the line $\mathbb{R}$, one can reach any regularity above the critical exponent $s_c = -\frac{1}{2}$ with continuous dependence on the initial data [64], while for $s < 0$ uniformly continuous dependence is not given, and illposedness in the sense of norm inflation holds for $s \leq s_c$ (see [37] for both statements).

We refer to Section 1.2 for a more detailed discussion on low regularity illposedness.

1.1 Wellposedness for Nonlinear Schrödinger Equations

As (1.1) is semilinear, it is instructive to first analyze the behavior of the linear equation,

$$\begin{cases} iu_t + \Delta u = 0, \\ u(0,x) = f(x), \end{cases}$$

and to handle the nonlinearity perturbatively. Recall the convention on the Fourier transform from Appendix A.1. We define the Schrödinger group $S(t) = e^{it\Delta}$ on the Fourier side via

$$\mathcal{F}(S(t)f)(\xi) = e^{-it|\xi|^2}\hat{f}(\xi). \tag{1.11}$$

The function $u(t,x) = (S(t)f)(x)$ is easily seen to solve the linear Schrödinger equation on the Fourier side

$$\begin{cases} \hat{u}_t = -i|\xi|^2\hat{u}, \\ \hat{u}(0) = \hat{f}. \end{cases}$$

Using the fact that multiplication is mapped to convolution we can rewrite for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$(S(t)f)(x) = (2\pi)^{-\frac{n}{2}}\left(\mathcal{F}^{-1}(e^{-it|\cdot|^2})\right) * f,$$

which by Lemma A.3 becomes,

$$(S(t)f)(x) = (4\pi i t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} f(y)\,dy, \tag{1.12}$$

and by density this formula extends to $f \in L^1(\mathbb{R}^n)$. In fact, since for $f \in \mathcal{S}(\mathbb{R}^n)$ we have $S(\cdot)f \in C(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$, the definition of $S(t)f$ can be extended by duality to $f \in \mathcal{S}'(\mathbb{R}^n)$, and $S(\cdot)f \in C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^n))$ in this case (see [26, Remark 3.2.3]).

From Young's convolution inequality (Theorem A.8) we infer

$$\|S(t)f\|_{L^\infty(\mathbb{R}^n)} \le (4\pi|t|)^{-\frac{n}{2}}\|f\|_{L^1(\mathbb{R}^n)}. \tag{1.13}$$

This estimate already shows a dispersive effect: as $|t| \to \infty$, and if $f \in L^1(\mathbb{R})$, $S(t)f$ will vanish uniformly in space. This is in contrast to the isometric property of $S(t)$ on L^2 based Sobolev spaces,

$$\|S(t)f\|_{H^s(\mathbb{R}^n)} = \|f\|_{H^s(\mathbb{R}^n)}, \qquad s, t \in \mathbb{R}, \tag{1.14}$$

which is immanent in (1.11). In particular, (1.13) tells us that mass will spread out over time, which is the effect usually referred to as dispersion. Using the Riesz-Thorin Theorem A.9 to interpolate between (1.13) and (1.14) for $s = 0$ gives

$$\|S(t)f\|_{L^p(\mathbb{R}^n)} \le (4\pi|t|)^{-n(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^{p'}}, \quad p \in [2,\infty]. \tag{1.15}$$

Equation (1.15) describes the effect that higher decay in space of the initial data f results in higher decay in time of $S(t)f$. A larger decay rate than $t^{-n/2}$ cannot be expected globally in space though, as the following example shows:

Example 1.2 (Example 4.1 in [90])**.** *If $u_0(x) = e^{-\pi|x|^2}$, then*

$$(S(t)u_0)(x) = (1 + 4\pi i t)^{-\frac{n}{2}} e^{-\frac{\pi|x|^2}{1+16\pi^2t^2}} e^{\frac{4\pi^2 it|x|^2}{1+16\pi^2t^2}}.$$

In particular, the initial bump function is spreading out as t gets large, that is, exponential decay is visible for $|x| > t$, and oscillations begin to arise in the region $|x| > t^{1/2}$. Moreover,

$$|(S(t)u_0)(x)| \ge ct^{-\frac{n}{2}}, \quad \textit{for} \quad |x| \le t.$$

We have seen in (1.12) that the Schrödinger group acts by integrating the function against an oscillating kernel. One way to deal with these integrals are stationary and non-stationary phase estimates.

We begin with the one-dimensional case and let $\phi \in C^\infty(a,b)$ be real valued and $\psi \in C^\infty(a,b)$ complex valued. Define for $\lambda \in \mathbb{R}$,

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx. \tag{1.16}$$

It turns out that the behavior of I for large λ is effected by the behavior of the derivatives of ϕ. If ϕ is not stationary, and if ψ is compactly supported we obtain large decay:

Lemma 1.3 (Proposition 1, p. 331 in [120])**.** *Assume $\phi'(x) \neq 0$ for all $x \in [a,b]$ and $\psi \in C_c^\infty((a,b))$. Then, for all $N \geq 0$ there exists a constant $c_N > 0$ such that*

$$|I(\lambda)| \leq c_N \lambda^{-N} \quad \textit{as} \quad \lambda \to \infty.$$

We need the compact support of ψ for this Lemma. Indeed, if for example $\phi(x) = x$ and $\psi(x) = 1$, direct integration shows

$$\int_a^b e^{i\lambda x}\,dx = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda},$$

which decays like λ^{-1} but not faster.

There is a nice generalization of this non-stationary phase estimate to higher dimensions. In this case let

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x)\,dx.$$

Then:

Lemma 1.4 (Proposition 4, p. 341 in [120])**.** *Assume $\psi \in C_c^\infty(\mathbb{R}^n)$, $\phi \in C^\infty(\mathbb{R}^n)$, where ϕ is real-valued with $\nabla\phi(x) \neq 0$ for all $x \in \operatorname{supp}(\psi)$. Then, for all $N \geq 0$ there exists a constant $c_N > 0$ such that*

$$|I(\lambda)| \leq c_N \lambda^{-N} \quad \textit{as} \quad \lambda \to \infty.$$

If ϕ has critical points, these points destroy the large decay. We begin again with the one-dimensional case. Firstly we can get rid of the compact support assumption for ψ since the possibility of large decay is excluded. The following result is known as van der Corput's Lemma:

Lemma 1.5 (van der Corput, cf. Proposition 2, p. 332 in [120])**.** *Assume $|\phi^{(k)}(x)| \geq 1$ for all $x \in (a,b)$. Then, there exists $c > 0$ such that*

$$I(\lambda) \leq c\lambda^{-\frac{1}{k}}(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}),$$

if either $k \geq 2$, or $k = 1$ and ϕ' is monotonic.

There is a result in higher dimensions as well:

Lemma 1.6 (Proposition 5, p. 342 in [120])**.** *Assume $\psi \in C_c^\infty(\mathbb{R}^n)$ has support in a unit ball, $\phi \in C^\infty(\mathbb{R}^n)$, where ϕ is real-valued and there exists $\alpha \in \mathbb{N}^n$ such that $|\partial^\alpha \phi(x)| \geq 1$ for all $x \in \operatorname{supp}(\psi)$. Then, with $k = |\alpha| > 0$,*

$$|I(\lambda)| \leq c_k(\phi)\lambda^{-\frac{1}{k}}(\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^1}),$$

where the constant $c_k(\phi)$ is independent of ψ and λ.

Note that this estimate in higher dimensions is often not sharp. One may consider Example 1.2 where one has decay $t^{-n/2}$ instead of $t^{-1/2}$.

As an application, let us consider a function f with compactly supported Fourier transform and see what we can say about $S(t)f$. This recovers some of the properties of Example 1.2 in the frequency-localized case and we will later use this estimate to show boundedness of the Schrödinger propagator in modulation spaces.

Lemma 1.7. *Assume $f \in \mathcal{S}(\mathbb{R})$ such that the support of $\hat{f}$ is contained in $(-R, R)^n$. Then*

$$|(S(t)f)(x)| \leq \begin{cases} c_N|x|^{-N} & \textit{if} \quad |x| \geq 4Rt, \\ ct^{-\frac{n}{2}} & \textit{if} \quad t \geq 1. \end{cases} \tag{1.17}$$

Proof. We write

$$(S(t)f)(x) = \mathcal{F}^{-1}\big(e^{-it\xi^2}\hat{f}(\xi)\big)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(x\xi - t\xi^2)}\hat{f}(\xi)\, d\xi.$$

Assume $|x| \geq 4Rt$ first. We want to take $\lambda = |x|$ as a parameter. In this case

$$\phi(\xi) = \frac{x}{|x|}\xi - \frac{t}{|x|}\xi^2, \quad \nabla\phi(\xi) = \frac{x}{|x|} - \frac{2t\xi}{|x|}.$$

If $|x| \geq 4Rt$ then $|\phi'(x)| \geq \frac{1}{2}$, and we can make use of Lemma 1.4, showing that

$$|(S(t)f)(x)| \leq c_N|x|^{-N} \quad \text{for all} \quad N \in \mathbb{N},$$

in this region.

For the second estimate we want to use $\lambda = t$ as a parameter. Then,

$$\phi = \frac{x}{t}\xi - \xi^2, \quad \nabla\phi = \frac{x}{t} - 2\xi, \quad \nabla^2\phi = -2\,\mathrm{id}\,.$$

The phase is critical at $\xi = \frac{x}{2t}$. In the one-dimensional case we can use Lemma 1.5 to find

$$|(S(t)f)(x)| \leq c|t|^{-\frac{1}{2}}.$$

In the higher-dimensional case Lemma 1.6 gives the same decay rate of $|t|^{-\frac{1}{2}}$. To get the better decay $|t|^{-\frac{n}{2}}$ we calculate directly, for $t \geq 1$,[2]

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{i(x\xi - t\xi^2)} \hat{f}(\xi)\, d\xi &= \int_{\mathbb{R}^n} e^{it(\frac{x}{t}\xi - \xi^2)} \hat{f}(\xi)\, d\xi \\
&= e^{i\frac{x^2}{4t}} \int_{\mathbb{R}^n} e^{-it(\frac{x}{2t} - \xi)^2} \hat{f}(\xi)\, d\xi \\
&= e^{i\frac{x^2}{4t}} \int_{\mathbb{R}^n} e^{-it\eta^2} \hat{f}\Big(\eta + \frac{x}{2t}\Big)\, d\eta, \\
&= e^{i\frac{x^2}{4t}} t^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\zeta^2} \hat{f}\Big(t^{-\frac{1}{2}}\zeta + \frac{x}{2t}\Big)\, d\zeta,
\end{aligned}$$

by completing the square, shifting and scaling. Thus to get the decay of $t^{-\frac{n}{2}}$ it is enough to show boundedness of the integral

$$\int_{\mathbb{R}^n} e^{-i\zeta^2} \hat{f}\Big(t^{-\frac{1}{2}}\zeta + \frac{x}{2t}\Big)\, d\zeta,$$

uniformly in $t \geq 1$. Since the size of the support of $\hat{f}$ grows with t this means that we want to put $\hat{f}$ and its derivatives in L^∞.

First we see that on the set $\{|\zeta| \leq 1\}$ we can estimate both factors in L^∞, hence we may restrict to $|\zeta| \geq 1$. In this case define the operator

$$A = \frac{i}{2|\zeta|^2} \zeta \cdot \nabla, \qquad \zeta \cdot \nabla = \sum_{j=1}^{n} \zeta_j \partial_{x_j},$$

which satisfies

$$Ae^{-i\zeta^2} = e^{-i\zeta^2}.$$

By partial integration we find

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-i\zeta^2} \hat{f}\Big(t^{-\frac{1}{2}}\zeta + \frac{x}{2t}\Big)\, d\zeta &= \int_{\mathbb{R}^n} \Big(Ae^{-i\zeta^2}\Big) \hat{f}\Big(t^{-\frac{1}{2}}\zeta + \frac{x}{2t}\Big)\, d\zeta \\
&= \int_{\mathbb{R}^n} e^{-i\zeta^2} \frac{i}{2} \operatorname{div}\Big(\frac{\zeta}{|\zeta|^2} \hat{f}\Big(t^{-\frac{1}{2}}\zeta + \frac{x}{2t}\Big)\Big)\, d\zeta + B,
\end{aligned}$$

where the boundary values B can be bounded via

$$|B| = \Big| \int_{\partial B_1} e^{-i\zeta^2} \frac{i}{2} \frac{\zeta}{|\zeta|^2} \hat{f}\Big(t^{-\frac{1}{2}}\zeta + \frac{x}{2t}\Big)\, d\zeta \Big| \lesssim \|\hat{f}\|_{L^\infty}.$$

If the divergence falls onto $\frac{\zeta}{|\zeta|^2}$ we are in the same situation but with (up to constants)

$$\frac{1}{|\zeta|^2} \hat{f}\Big(t^{-\frac{1}{2}}\zeta + \frac{x}{2t}\Big)$$

[2] This calculation may be compared to Theorem 1.16 in which a phase correction of $e^{-i\frac{x^2}{4t}}$ is needed.

as the integrated function. If the divergence falls as a gradient onto f, our integrated function becomes

$$\frac{t^{-\frac{1}{2}}\zeta}{|\zeta|^2}\nabla\hat{f}\Big(t^{-\frac{1}{2}}\zeta+\frac{x}{2t}\Big).$$

In both cases we gain decay of one power in ζ by paying with derivatives of $\hat{f}$. We repeat this procedure n more times and obtain sufficient decay in ζ to be able to put $\hat{f}$ and its derivatives in L^∞. Finally, we arrive at

$$\Big|\int_{\mathbb{R}^n} e^{-i\zeta^2}\hat{f}\Big(t^{-\frac{1}{2}}\zeta+\frac{x}{2t}\Big)\,d\zeta\Big| \lesssim \|\hat{f}\|_{L^\infty}+t^{-\frac{(n+1)}{2}}\|\hat{f}\|_{W^{n+1,\infty}},$$

which completes the desired estimate. □

Another elegant way to capture the dispersive effects of the Schrödinger group are Strichartz estimates. These estimates also play an important role in the wellposedness theory of several nonlinear equations, and we can apply them to solve the mass-subcritical NLS in $L^2(\mathbb{R}^n)$.

Theorem 1.8 (Strichartz estimates, Theorem 2.3 in [128])**.** *Fix a dimension $n \geq 1$. We call a pair (q,r) of exponents admissible if $2 \leq q,r \leq \infty$,*

$$\frac{2}{q}+\frac{n}{r}=\frac{n}{2},$$

and if $(q,r,n) \neq (2,\infty,2)$. Then for any admissible exponents (q,r) and $(\tilde{q},\tilde{r})$ we have the homogeneous Strichartz estimate

$$\|S(t)u_0\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim_{n,q,r} \|u_0\|_{L^2(\mathbb{R}^n)}, \tag{1.18}$$

and the inhomogeneous Strichartz estimate

$$\Big\|\int_0^t S(t-t')F(t',\cdot)\,dt'\Big\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim_{n,q,r,\tilde{q},\tilde{r}} \|F\|_{L^{\tilde{q}'}_tL^{\tilde{r}'}_x(\mathbb{R}\times\mathbb{R}^n)}. \tag{1.19}$$

If the power nonlinearity is not too large, and if $X_0 = L^2(\mathbb{R})$, Strichartz estimates allow to show wellposedness of the NLS (1.1). This works for all mass-subcritical exponents, i.e. for all $p > 1$ such that $s_c < 0$, see (1.10). To this end, we use the integral formulation of (1.1),

$$u(t) = S(t)u_0 - i\kappa\int_0^t S(t-s)(|u|^{p-1}u)(s)\,ds, \tag{1.20}$$

which follows from Duhamel's principle.

Theorem 1.9 (Theorem 5.2 in [90])**.** *Assume $1 < p < 1+\frac{4}{n}$. Then (1.1) is locally wellposed in $L^2(\mathbb{R})$ in the following sense: for each $u_0 \in L^2(\mathbb{R})$ there exists $T = T(\|u_0\|_{L^2(\mathbb{R})}, n, p) > 0$ and a unique solution u of (1.20) with*

$$u \in C([0,T],L^2(\mathbb{R}^n)) \cap L^q_tL^{p+1}_x([0,T]\times\mathbb{R}^n),$$

where q is such that the pair $(q, p+1)$ is admissible. Moreover, for all $T' < T$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^n)$ such that the data-to-solution map $\tilde{u}_0 \mapsto u(t,x)$ is Lipschitz continuous as a map

$$L^2(\mathbb{R}^n) \to C([0,T'], L^2(\mathbb{R}^n)) \cap L_t^q L_x^{p+1}([0,T'] \times \mathbb{R}^n).$$

From Strichartz estimates we see that the solution u constructed in Theorem 1.9 has more integrability than just $L_t^q L_x^{p+1}$. In fact, $u \in L_t^q L_x^r([0,T] \times \mathbb{R}^n)$ for all admissible (q,r), see [90, Corollary 5.1].

Making use of the mass conservation law allows to show that NLS is globally wellposed in $L^2(\mathbb{R}^n)$ in the mass sub-critical case $1 < p < 1 + \frac{4}{n}$ ([90, Theorem 6.1]).

In the mass-critical case $p = 1 + \frac{4}{n}$, a similar local result holds, the main difference being that the time of existence T now depends on the whole function u_0 instead of $\|u_0\|_{L^2(\mathbb{R}^n)}$. As a result, the conservation of mass is not sufficient any more to guarantee global wellposedness. On the other hand, when $\|u_0\|_{L^2(\mathbb{R}^n)} \leq \varepsilon_0$ is small enough, global wellposedness can be guaranteed (cf. Theorem 5.3 and Corollary 5.2 in [90]).

In fact, Dodson [43, 44, 45, 46] showed that both in the focusing and the defocusing case, when

$$\|u_0\|_{L^2(\mathbb{R}^n)} < \|Q\|_{L^2(\mathbb{R}^n)},$$

global wellposedness and scattering holds in the mass-critical case. Here Q is the ground state, that is, the unique positive radial solution of (1.7) with $\omega = 1$ (cf. [27]), and by scattering we mean that the solution u behaves like a solution of the linear equation, i.e. there exist $u_\pm \in L^2(\mathbb{R}^n)$ such that

$$\|S(t)u_\pm - u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \to 0, \quad \text{as} \quad t \to \pm\infty.$$

When $\|u_0\|_{L^2(\mathbb{R}^n)} \geq \|Q\|_{L^2(\mathbb{R}^n)}$, one has to distinguish between the focusing and the defocusing case: in the defocusing case, one still obtains global wellposedness and scattering [43, 45, 46]. In the focusing case though, there are special solutions which cause problems. Indeed, applying the pseudo-conformal transformation (1.4) with $b = 1$ to the solution $e^{it}Q(x)$, one arrives at

$$u(t,x) = (1-t)^{-\frac{n}{2}} e^{i\frac{4t-|x|^2}{4(1-t)}} Q\Big(\frac{x}{1-t}\Big).$$

This solution satisfies $u \in C((-\infty,1), L^2(\mathbb{R}^n))$, but $u(1) \notin L^2(\mathbb{R}^n)$. Moreover (see [27]), u blows up in any L^r space,

$$\|u(t,\cdot)\|_{L^r(\mathbb{R}^n)} = (1-t)^{-\frac{n}{2}(1-\frac{2}{r})} \|Q\|_{L^r(\mathbb{R}^n)},$$

and in $H^1(\mathbb{R}^n)$,

$$\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \sim (1-t)^{-1} \|\nabla Q\|_{L^2(\mathbb{R}^n)} \quad \text{as} \quad t \to 1.$$

This shows that in the focusing, mass-critical case, when the initial data is too large, i.e. $\|u_0\|_{L^2(\mathbb{R}^n)} \geq \|Q\|_{L^2(\mathbb{R}^n)}$, one cannot expect global wellposedness in $H^s(\mathbb{R}^n)$, $s \geq 0$.

Remark 1.10. *Note that by a Gauge transform, the derivative NLS equation* (1.3) *in one dimension is equivalent to a quintic NLS equation with a cubic nonlinearity with derivative (cf. Equation (1.29) in [63]),*

$$iu_t + u_{xx} = iu^2\bar{u}_x - \frac{1}{2}|u|^4u.$$

For this equation, $L^2(\mathbb{R})$ global wellposedness in the sense of continuous extension of the flow map from

$$[-T,T] \times \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$$

to $[-T,T] \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$, for arbitrarily large $T > 0$, was recently shown [63]. This is all the more surprising, because for the one-dimensional focusing quintic NLS,

$$iu_t + u_{xx} = -|u|^4u,$$

which is mass-critical, the soliton mass $\|Q\|_{L^2(\mathbb{R})}$ is a threshold for global wellposedness.

Concerning higher regularity, the following result was proven by Cazenave-Weissler [28] (see also [90, Theorem 5.8]):

Theorem 1.11. *Let $1 + \frac{4}{n} \leq p < \infty$ and*

$$\begin{aligned} &s > \frac{n}{2} - \frac{2}{p-1}, \\ &[s] < p - 1 \quad \text{if} \quad p \quad \text{is not an odd integer.} \end{aligned}$$

Then (1.1) *is wellposed in $H^s(\mathbb{R}^n)$ in the following sense: there exists a space $W_T^{s,n}$ such that for all $u_0 \in H^s(\mathbb{R}^n)$ there exist $T = T(\|u_0\|_{H^s(\mathbb{R}^n)}, s) > 0$ and a unique solution u of* (1.20) *in*

$$u \in C([-T,T], H^s(\mathbb{R}^n)) \cap W_T^{s,n}.$$

Moreover, for all $T' < T$, the data-to-solution map is locally Lipschitz continuous as a map

$$H^s(\mathbb{R}^n) \to C([-T',T'], H^s(\mathbb{R}^n)), \quad u_0 \mapsto u.$$

Some comments on the statement of Theorem 1.11 are in order. First of all, the space $W_T^{s,n}$ is either a Strichartz space $L^q([-T,T], W^{s,r}(\mathbb{R}^n))$ with s derivatives, when p is an odd integer, or a Strichartz space with some Besov refinement, $L^q([-T,T], B^s_{r,2}(\mathbb{R}^n))$, when p is not an odd integer. In both cases, the pair (q,r) is some particular admissible pair. The reason why for non-integer s one has to resort to a Besov space is because the nonlinearity $|u|^{p-1}u$ is not well behaved in the Sobolev spaces $W^{s,r}(\mathbb{R}^n)$.

Secondly, the regularity

$$s_c = \frac{n}{2} - \frac{2}{p-1}$$

is scaling critical, given p as in (1.1). For the range of p given by $1 + \frac{4}{n} \leq p < \infty$, the critical exponent s_c is non-negative. Thus, Theorem 1.11 shows local wellposedness for all subcritical NLS equations of type (1.1) in non-negative, subcritical regularity. The case of mass-subcritical $L^2(\mathbb{R}^n)$ wellposedness is covered in Theorem 1.9.

One can conclude that once the initial data is smooth enough (e.g. when working in subcritical and non-negative regularity) and has enough decay (in the sense that one works in $L^2(\mathbb{R})$ based spaces which are all decaying at $\pm\infty$), local wellposedness for (1.1) can be achieved. The fewer regularity and decay one has, the harder it is to obtain wellposedness results, and this is one direction of research in the field of dispersive equations: how much can we relax the assumptions and still able to define solutions and obtain some form of local wellposedness? We will look at this problem in two directions: low regularity, and lack of decay. Both can be understood as some form of "roughness" as we will see in the next two sections.

1.2 Rough Data: Low Regularity

Theorem 1.11 tells us that local wellposedness in Sobolev spaces $H^s(\mathbb{R})$ in the non-negative, subcritical range $s > \max(s_c, 0)$ is given. As we will see this is optimal in the sense that when $s = s_c$, continuous dependence in general fails. If we consider the focusing case, the result can be proven by considering soliton solutions:

Theorem 1.12 ([19], Theorem 5.9 in [90]). *Let $1 + \frac{4}{n} \leq p < \infty$ and $\kappa < 0$. Then* (1.1) *is illposed in $H^{s_c}(\mathbb{R}^n)$,*

$$s_c = \frac{n}{2} - \frac{2}{p-1},$$

in the following sense: there exists $c_0 > 0$ such that for any $\delta, t > 0$, there exist data $u_1, u_2 \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\|u_1\|_{H^{s_c}(\mathbb{R})} + \|u_2\|_{H^{s_c}(\mathbb{R})} \leq c_0, \qquad \|u_1 - u_2\|_{H^{s_c}(\mathbb{R})} \leq \delta,$$
$$\|u_1(t) - u_2(t)\|_{H^{s_c}(\mathbb{R})} > \frac{c_0}{2},$$

where $u_j(t)$ denotes the solution of (1.1) *with initial data u_j.*

Proof (Sketch). We restrict ourselves to $s_c > 0$ and $d = 1$. Consider the family of solutions given by

$$v_\mu(t,x) = e^{i\mu t} Q_\mu(x) = e^{i\mu t} \mu^{\frac{1}{p-1}} Q(\sqrt{\mu} x),$$

where Q is the ground state. Now from scaling criticality,

$$\begin{aligned}\|v_{\mu_1} - v_{\mu_2}\|^2_{\dot{H}^{s_c}} &= \|v_{\mu_1}\|^2_{\dot{H}^{s_c}} + \|v_{\mu_2}\|^2_{\dot{H}^{s_c}} - 2\operatorname{Re} e^{i(\mu_1 - \mu_2)t} \int_{\mathbb{R}} |\xi|^{2s_c} \hat{Q}_{\mu_1} \overline{\hat{Q}_{\mu_2}}\, d\xi \\ &= 2\|Q\|^2_{\dot{H}^{s_c}} - 2\operatorname{Re} e^{i(\mu_1 - \mu_2)t} \int_{\mathbb{R}} |\xi|^{2s_c} \hat{Q}_{\mu_1} \overline{\hat{Q}_{\mu_2}}\, d\xi\end{aligned}$$

We calculate for the last summand,

$$\begin{aligned}\int_{\mathbb{R}} |\xi|^{2s_c}\hat{Q}_{\mu_1}\overline{\hat{Q}_{\mu_2}}\,d\xi &= (\mu_1\mu_2)^{\frac{1}{p-1}-\frac{1}{2}}\int_{\mathbb{R}} |\xi|^{2s_c}\hat{Q}\Big(\frac{\xi}{\sqrt{\mu_1}}\Big)\overline{\hat{Q}}\Big(\frac{\xi}{\sqrt{\mu_2}}\Big)\,d\xi \\ &= \Big(\frac{\mu_1}{\mu_2}\Big)^{-\frac{1}{p-1}+\frac{1}{2}}\int_{\mathbb{R}} |\eta|^{2s_c}\hat{Q}(\eta)\overline{\hat{Q}}\Big(\sqrt{\frac{\mu_1}{\mu_2}}\eta\Big)\,d\eta.\end{aligned}$$

Now if we choose $\mu_1 = (N+1)^2$ and $\mu_2 = N^2$, we get $\mu_1 - \mu_2 > 2N$ and

$$\frac{\mu_1}{\mu_2} \to 1, \quad \text{as} \quad N \to \infty.$$

This shows for $t = 0$,

$$\lim_{N\to\infty} \|v_{\mu_1}(0) - v_{\mu_2}(0)\|^2_{\dot{H}^{s_c}} = 2\|Q\|^2_{\dot{H}^{s_c}} - 2\|Q\|^2_{\dot{H}^{s_c}} = 0.$$

On the other hand given $T > 0$ there exist $N \geq 1$, $t \in [0,T]$ such that

$$\operatorname{Re} e^{i(\mu_1-\mu_2)t}\int_{\mathbb{R}} |\xi|^{2s_c}\hat{Q}_{\mu_1}\overline{\hat{Q}_{\mu_2}}\,d\xi = 0.$$

This shows

$$\lim_{N\to\infty}\sup_{t\in[0,T]} \|v_{\mu_1} - v_{\mu_2}\|^2_{\dot{H}^{s_c}} \geq 2\|Q\|^2_{\dot{H}^{s_c}}.$$

From $s_c > 0$ we see that the L^2-part of $v_\mu(t,x)$ vanishes as $N \to \infty$, which shows the result. □

In the defocusing case there are similar results, but the proof is more involved due to the lack of explicit soliton solutions. Instead one may consider the explicit solution to (1.1) when the dispersion is set to zero. Using energy arguments, it can be seen that this solution is close to the solution to the NLS with small dispersion, for times that are sufficiently long if the dispersion is sufficiently small. After scaling and invoking the Galilean transformation, the result follows. We sketch the proof which can be found in [37].

Theorem 1.13. *Let $p > 1$ be an odd integer and $\kappa \in \{-1,1\}$. Let $s < \max(0,s_c)$. Then (1.1) is illposed in $H^s(\mathbb{R}^n)$ in the following sense: for any $0 < \delta, \varepsilon < 1$ and for any $t > 0$, there exist initial data $u_1, u_2 \in \mathcal{S}(\mathbb{R}^n)$ such that*

$$\begin{gathered}\|u_1\|_{H^s(\mathbb{R})} + \|u_2\|_{H^s(\mathbb{R})} \leq C\varepsilon, \qquad \|u_1 - u_2\|_{H^{s_c}(\mathbb{R})} < C\delta, \\ \|u_1(t) - u_2(t)\|_{H^s(\mathbb{R})} > c\varepsilon,\end{gathered}$$

where $u_j(t)$ denotes the solution of (1.1) with initial data u_j.

The same conclusion holds when $p > 1$ is not an odd integer if we additionally assume that $p \geq k+1$ for some integer $k > \frac{n}{2}$.

Proof (Sketch). By complex conjugation we can revert the time variable. Consider (1.1) with a variable dispersion,

$$\begin{cases} -i\phi_s + \nu^2 \Delta_y \phi = \kappa |\phi|^{p-1}\phi, \\ \phi(0) = \phi_0. \end{cases} \tag{1.21}$$

When $\nu \to 0$, the solution is expected to behave like the solution for $\nu = 0$,

$$\phi^{(0)}(s, y) = \phi_0(y) e^{i\kappa s |\phi_0(y)|^{p-1}}.$$

By making use of an energy argument, this is shown rigorously in [37, Lemma 2.1]: given $\phi_0 \in \mathcal{S}(\mathbb{R}^n)$ and $k > \frac{d}{2}$ there exists a solution ϕ to (1.21) which satisfies

$$\|\phi(s) - \phi^{(0)}(s)\|_{H^{k,k}(\mathbb{R}^n)} \leq C\nu \quad \text{if} \quad |s| \leq c|\log \nu|^c.$$

Here, $c > 0$ is some constant and

$$\|\phi\|_{H^{k,k}(\mathbb{R}^n)} = \sum_{j=0}^{k} \left\| \langle x \rangle^{k-j} \partial_x^j \phi \right\|_{L^2}.$$

Now consider the initial data $\phi_0(y) = aw(y)$ where $a \in [\frac{1}{2}, 1]$ and $w \in \mathcal{S}(\mathbb{R}^n)$. This gives rise to a family of solutions of (1.21), $\phi^{(a,\nu)}$. By making additional use of the scaling and Galilean transformations, a four parameter family is obtained, where $0 < \lambda \ll 1$ and $v \in \mathbb{R}^n$,

$$u^{(a,\nu,\lambda,v)}(t,x) := \lambda^{-2/(p-1)} e^{-iv\cdot x/2} e^{i|v|^2 t/4} \phi^{(a,\nu)}\left(\lambda^{-2}t, \lambda^{-1}\nu(x - vt)\right). \tag{1.22}$$

By making ν small, $|v|$ large, λ small and by comparing the solution for different a, a' close to each other, one can prove Theorem 1.13 in the case $s < 0$. For $0 < s < s_c$, Theorem 1.13 follows from Theorem 1.14. $\square$

Theorem 1.13 shows that the data-to-solution map cannot be uniformly continuous in $H^{s_c-}(\mathbb{R}^n)$. In the case of a positive critical exponent, or a regularity which is very low, a more extreme phenomenon called norm inflation may happen. This describes that in certain cases, norms may grow very rapidly. Again we refer to [37] for the full proof and only sketch the idea.

Theorem 1.14. *Let $p > 1$ be an odd integer and $\kappa \in \{-1, 1\}$. Let $0 < s < s_c$ or $s < -\frac{d}{2}$. Then (1.1) admits norm inflation in $H^s(\mathbb{R}^n)$ in the following sense: for any $\varepsilon > 0$ there exist initial data $u_0 \in \mathcal{S}(\mathbb{R}^n)$ and $0 < t < \varepsilon$ such that*

$$\|u_0\|_{H^s(\mathbb{R})} < \varepsilon, \qquad \|u(t)\|_{H^s(\mathbb{R})} > \varepsilon^{-1},$$

where $u(t)$ denotes the solution of (1.1) with initial data u_0.

The same conclusion holds when $p > 1$ is not an odd integer if we additionally assume that $p \geq k + 1$ respectively $-k < s < 0$.

Proof (Sketch). We consider the same family of functions as in the proof of Theorem 1.13 given in (1.22). Setting $v = 0$, and scaling via λ essentially reduces the question to analyzing $\phi^{(a,\nu)}$. Here one sees that when $0 < s < s_c$,

$$\|\phi^{(a,\nu)}\|_{H^s(\mathbb{R}^n)} \sim t^s$$

for large t, which is enough to conclude after scaling. The case $s < 0$ works slightly differently and relies on the transfer of energy from high to low modes, as opposed to the case $s > 0$. □

So far we have seen that initial data with low regularity can be seen as rough. In this thesis we will look at rough initial data as follows:

- In Section 3 we consider low regularity initial data for the cubic one-dimensional derivative NLS and prove a priori estimates in low regularity Sobolev spaces.
- In Section 5 take initial data from a modulation space $M^s_{p,q}(\mathbb{R})$. This includes the case of low regularity by choosing $s = 0$ and taking q as large as possible.

The proofs of Theorems 1.13 and 1.14 indicate that the interactions leading to illposedness of the Cauchy problem (1.1) in low regularity stem from the nonlinearity of the equation. This is also clear from the fact that the semigroup $S(t)$ is an isometry on all Sobolev spaces $H^s(\mathbb{R}^n), s \in \mathbb{R}$, and since the linear Schrödinger equation is globally wellposed in these spaces.

This point of view will be applied to the question of illposedness of (1.1) in modulation spaces of negative regularity index, which is considered in Section 5.5. There the inspection of the first nonlinear Picard iterate is enough to disprove some higher differentiability of the flow map.

1.3 Rough Data: Lack of Decay

In this section we want to obtain an understanding of what behavior we can expect from solutions of (1.1) when the initial data is only weakly decaying, or not decaying at all. In contrast to before, the problems will already be visible in the linear part of the equation.

Theorem 1.8 is maybe the first result in which the effect of higher spatial decay of f leading to better behaving $S(t)f$ is visible: if $f \in L^1(\mathbb{R}^n)$, then $S(t)f$ decays in time like $t^{-n/2}$, whereas for $f \in L^2(\mathbb{R}^n)$ Strichartz estimates yield no decay (though in different norms). Such an effect can also be seen in the differentiability properties of $S(t)f$. First of all, in the L^2 setting the Schrödinger group has a general local smoothing effect: Define the operators

$$D^{1/2}_{x_j} g(t,x) = \mathcal{F}^{-1}(|\xi_j|^{\frac{1}{2}} \hat{g}(t,\xi)),$$

and

$$D^{1/2}_{x} g(t,x) = \mathcal{F}^{-1}(|\xi|^{\frac{1}{2}} \hat{g}(t,\xi)).$$

Then:

Theorem 1.15 (Local smoothing, Theorem 4.3 and Corollary 4.2 in [90]). *If $n = 1$, then*

$$\sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} |D_x^{1/2} S(t)f(x)|^2 dt \lesssim \|f\|_{L^2(\mathbb{R})}^2, \tag{1.23}$$

and if $n \geq 2$, then for all $j \in \{1, \ldots, n\}$,

$$\sup_{x_j\in\mathbb{R}} \int_{\mathbb{R}^n} |D_{x_j}^{1/2} S(t)f(x)|^2 dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_n dt \lesssim \|f\|_{L^2(\mathbb{R}^n)}^2. \tag{1.24}$$

Moreover, for $n \geq 1$,

$$\sup_{R>0, x_0\in\mathbb{R}^n} \int_{\mathbb{R}} \int_{\{|x-x_0|\leq R\}} |D_x^{1/2} S(t)f(x)|^2 \, dxdt \lesssim R^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^n)}. \tag{1.25}$$

If more spatial decay is available, there is more smoothing of the Schrödinger group. The following result holds:

Theorem 1.16 (Corollary 3.3.2 in [26]). *Let $\phi \in L^2(\mathbb{R}^n)$ such that $\langle x\rangle^m \phi \in L^2(\mathbb{R})$ for some $m \in \mathbb{N}$. Then,*

$$e^{-i\frac{|x|^2}{4t}} S(t)\phi \in C(\mathbb{R} \setminus \{0\}, H^m(\mathbb{R}^n)),$$

with

$$\left\| \partial_x^\alpha \left(e^{-i\frac{|x|^2}{4t}} S(t)\phi \right) \right\|_{L^2} = (2|t|)^{-|\alpha|} \|x^\alpha \phi\|_{L^2(\mathbb{R}^n)},$$

for all $\alpha \in \mathbb{N}^n$, $|\alpha| \leq m$. Thus

$$S(\cdot)\phi \in \bigcap_{0\leq j\leq [n/2]} C^j(\mathbb{R} \setminus \{0\}, H_{loc}^{m-2j}(\mathbb{R}^n)).$$

In particular, if $\langle x\rangle^m \phi \in L^2(\mathbb{R})$ for all $m \in \mathbb{N}$, then

$$S(\cdot)\phi \in C_{t,x}^\infty(\mathbb{R} \setminus \{0\} \times \mathbb{R}^n).$$

Theorem 1.16 can be compared to the properties of the heat semigroup,

$$(e^{t\Delta} f)(x) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y) \, dy.$$

By (A.1), $e^{t\Delta} f \in C^\infty(\mathbb{R}^n)$ for $t > 0$, even if f is only a tempered distribution, $f \in \mathcal{S}'(\mathbb{R}^n)$. In fact, by taking time derivatives in t, it is not hard to see that $e^{t\Delta} f \in C_{t,x}^\infty(\mathbb{R} \setminus \{0\} \times \mathbb{R}^n)$. Thus, strong decay of the initial data gives nice behavior of the Schrödinger group, making it comparable to the heat semigroup.

In view of Theorem 1.16 and the time reversal symmetry $S(-t)f = \overline{S(t)f}$, one may wonder whether differentiability could in general also lead to decay. This is not the case, as the following result by Bona-Saut [21] shows:

Theorem 1.17 (Dispersive blow-up, Theorem 2.1 in [21] and Lemma 2.1 in [20]). *Let $(t^*, x^*) \in (0, \infty) \times \mathbb{R}^n$ be given. Then there exists a function $\phi \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $u(t, x) = (S(t)\phi)(x)$ satisfies*

1. $u \in C_b(\mathbb{R}_+, L^2(\mathbb{R}^n))$,

2. u *is continuous in* (t,x) *on* $(0,\infty) \setminus \{t^*\} \times \mathbb{R}^n$,

3. $u(t^*,\cdot)$ *is continuous in* x *on* $\mathbb{R}^n \setminus \{x^*\}$,

4. u *blows up at* (t^*,x^*):

$$\lim_{(t,x)\to(t^*,x^*)} |u(t,x)| = \infty.$$

Proof (Sketch). The proof of Theorem 1.17 is very explicit. Without loss of generality assume $t^* = 1/4, x^* = 0$. By choosing

$$\phi(x) = \langle x\rangle^{-2m} e^{-ix^2}, \tag{1.26}$$

one can use (1.12) to see that $u(t,x)$ formally becomes

$$u(1/4,0) = C\int_{\mathbb{R}^n} \langle y\rangle^{-2m} dy.$$

This integral is infinite when $m \leq n/2$, and if $m > n/4$ then $\phi \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Thus the function ϕ will satisfy the requirements. $\square$

We take a closer look at the blow-up example with most decay, that is $m = \frac{n}{2}$ in (1.26),

$$\phi(x) = \langle x\rangle^{-n} e^{-ix^2}.$$

It is not hard to see that not only $\phi \in L^\infty(\mathbb{R}^n)$, but also $\partial^\alpha\phi \in L^\infty(\mathbb{R}^n)$ for all $|\alpha| \leq n$. In particular the assumption $\phi \in C_b^n(\mathbb{R}^n)$ does not prevent pointwise blow-up.

A similar dispersive blow-up result can be shown for solutions of nonlinear Schrödinger equations (see [21, Theorem 2.2] and [20, Theorem 3.4]). The result holds for both focusing and defocusing nonlinearities:

Theorem 1.18. *Let* $(t^*,x^*) \in (0,\infty) \times \mathbb{R}^n, \kappa \in \{-1,1\}$ *be given. Assume* $2 \leq p < 4$, *or* $p \geq 4$ *and* $p \geq [\frac{n}{2}] + 1$ *when* p *is not an odd integer. Then there exist functions* $\phi \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ *with* $s = 0$ *if* $2 \leq p < 4$ *and* $s \in (\frac{n}{2} - \frac{1}{2(p-1)}, \frac{n}{2}]$ *in the case* $p \geq 4$, *such that*

1. *There exists* $T = T(\|u_0\|_{H^s(\mathbb{R}^n)}) > t^*$ *such that* (1.1) *has a unique local solution in a Strichartz space (see Theorem 1.9 if* $s = 0$ *and Theorem 1.11 for* $s > 0$*),*

2. u *is continuous in* (t,x) *on* $[0,T] \setminus \{t^*\} \times \mathbb{R}^n$,

3. $u(t^*,\cdot)$ *is continuous in* x *on* $\mathbb{R}^n \setminus \{x^*\}$,

4. u *blows up at* (t^*,x^*):

$$\lim_{(t,x)\to(t^*,x^*)} |u(t,x)| = \infty.$$

The results of Theorem 1.17 and Theorem 1.18 are a serious obstacle for the wellposedness theory of NLS equations in classes of functions without decay assumptions at infinity.

One solution would be to weaken the norm in which we want to control the solution, e.g. instead of trying to bound the solution u in L^∞, one could allow for growth in space as well. As the discussion at the beginning of Section 1.1 shows, we can define $S(t)$ continuously on $\mathcal{S}'(\mathbb{R}^n)$. On the other hand, from Section 1.2 we know that very low regularity will result in problems for the solvability of the equation with a nonlinearity. Moreover there is a result by Gonzales [54] (see below) which shows that any decay at infinity is already problematic.

Instead we can impose more restrictions on the initial data u_0. To this end we note that the proof of Theorem 1.17 suggests that strong oscillations of u_0 at infinity, which cancel with the oscillations from the dispersive semigroup, are causing the solution to blow up. And indeed, assuming $u_0 \in W^{2,\infty}(\mathbb{R}^n)$ is enough to prevent blow-up of $S(t)u_0$ in $L^\infty(\mathbb{R}^n)$ (see for example the work of Dodson-Soffer-Spencer [48], and the work of Mandel [93] who identified a different function space of weighted Sobolev norms in which blow-up can be excluded). This can be refined by considering functions in modulation spaces $M_{\infty,1}(\mathbb{R}^n)$. Such functions compose a superset of $W^{2,\infty}(\mathbb{R}^n)$, and also prevent norm inflation of $S(t)u_0$ by imposing restrictions on the oscillations. One major advantage of modulation spaces is that the Schrödinger propagator $S(t)$ defines a continuous operator on these spaces, making them very suitable for wellposedness theories for NLS. This ansatz will be pursued in Section 5.

We want to state two more interesting and related results. When looking at the continuity of $S(t)$ on $\mathcal{S}'(\mathbb{R}^n)$ one may wonder if one can allow for even more freedom for the asymptotics of u_0 at infinity. Unfortunately, we cannot expect uniqueness in this case:

Theorem 1.19 (Exercise 2.24 in [128])**.** *There exists a smooth function $u(t,x) \in C^\infty(\mathbb{R}^2)$ which solves $iu_t + u_{xx} = 0$, vanishes for $t \geq 0$ but is not identically zero.*

The idea is to consider an analytic function $f(z)$ in the upper right quadrant of the complex plane and to define $u(t,x) = \int_\gamma f(z)e^{i(xz+tz^2)}\,dz$, where γ is the boundary of said quadrant. If f is decaying fast enough on γ, u is a well-defined function satisfying the linear Schrödinger equation which by Cauchy's theorem is zero for $t \geq 0$ but is not identically zero.

Finally we want to mention the work of Gonzales [54] which shows that in general it is not even allowed for the initial data of the Cauchy problem (1.1) to grow like x^β for some $\beta > 0$ at infinity. More precisely, given an interval $I \subset \mathbb{R}$, define the space

$$S^\beta(I\times\mathbb{R}) = \Big\{ f \in C^\infty(I\times\mathbb{R}),\ f(t,x) \sim \sum_{k=0}^{\infty}(a_k^\pm(t) + ib_k^\pm(t))x^{\beta_k} \text{ as } x \to \pm\infty,$$
$$a_k^\pm, b_k^\pm \in C^\infty(I), \beta = \beta_0 > \beta_1 > \dots,\ \text{and}\ \lim_{k\to\infty}\beta_k = -\infty \Big\},$$

as the space of smooth functions with prescribed asymptotic expansion of highest order β at $\pm\infty$. Here we define the asymptotic relation $\sim$ as follows: for all compact intervals $J \subset I$ and integers $N, i, j \geq 0$ there exists a constant $C = C(J, N, i, j) > 0$ such that for all $\pm 1 \geq$ and $t \in J$,

$$\Big|\partial_t^i \partial_x^j \big(f(t,x) - \sum_{k=0}^{N} (a_k^{\pm}(t) + i b_k^{\pm}(t)) x^{\beta_k}\big)\Big| \leq C|x|^{\beta_{N+1}-j}.$$

In [54], Gonzales constructs (asymptotic) solutions of the cubic NLS in classes $S^{\beta}(I \times \mathbb{R})$ where $\beta \leq 0$. Moreover he gives a simple argument why formal solutions cannot exist for $\beta > 0$: plugging e.g. $\sum_{k=0}^{N} (a_k^+(t) + i b_k^+(t)) x^{\beta_k}$ into (1.1) with $p = 3$ formally yields

$$\begin{aligned} i \sum_{k=0}^{N} (\dot{a}_k^+(t) + i \dot{b}_k^+(t)) x^{\beta_k} + \sum_{k=0}^{N} \beta_k(\beta_k - 1)(a_k^+(t) + i b_k^+(t)) x^{\beta_k - 2} \\ = \kappa \Big(\sum_{k=0}^{N} (a_k^+(t) + i b_k^+(t)) x^{\beta_k} \Big)^2 \sum_{k=0}^{N} (a_k^+(t) - i b_k^+(t)) x^{\beta_k}. \end{aligned}$$

The largest exponent on the left-hand side is $\beta_0 = \beta$ while the largest exponent on the right-hand side is $3\beta_0 > \beta$, assuming $\beta > 0$. This shows that the coefficient in front of $x^{3\beta}$ must vanish, i.e.

$$\kappa (a_0^+ + i b_0^+)^2 (a_0^+ - i b_0^+) = 0.$$

Hence $a_0 = b_0 = 0$ which contradicts the maximality of β.

To sum things up, initial data with lack of decay can be considered 'rough', already for the linear problem associated to (1.1).

In this thesis we will consider initial data lacking decay in two different ways:

- In Section 4, we will investigate the global wellposedness problem for the NLS with initial data in a sum space $H^1(\mathbb{R}) + H^s(\mathbb{T})$. This ansatz can be seen as a toy model on the way towards initial data with less restrictive non-decaying behavior as $|x| \to \infty$, and it has the advantage that both in $H^s(\mathbb{R})$ and $H^s(\mathbb{T})$ when $s \geq 1$, the question of global wellposedness is well understood.
- In Section 5, we will look at the NLS with initial data in a modulation space $M_{p,q}^s(\mathbb{R}^n)$. It is known that the Schrödinger group $S(t)$ is a bounded operator on all modulation spaces $M_{p,q}^s(\mathbb{R}^n)$ (see Lemma 5.11). In addition, the space $M_{\infty,1}^0(\mathbb{R}^n)$, is an example of a subspace of $C_b^0(\mathbb{R}^n)$ which contains functions which do not decay and allow for more 'generic' non-decaying behavior as $|x| \to \pm\infty$.

Chapter 2

Complete Integrability and the Transmission Coefficient

In this section we focus on the cubic one-dimensional nonlinear Schrödinger equation, that is $p = 3$ and $n = 1$ in (1.1). We state the basic theory of complete integrability for this equation and introduce the transmission coefficient. This serves as a basis for the construction of almost conserved quantities for the cubic one-dimensional derivative NLS in Section 3.

It is based on the recent papers [83, 77, 64], on the classical books [50, 105], and on work in progress [80]. More precisely, the construction of Jost solutions and the transmission coefficient is a classical problem and was already covered in [50, 105]. Koch–Tataru used the same construction but provided a different functional analytic framework to make sense of the Jost solutions and the transmission coefficient even in low regularity. Killip–Visan and Harrop-Griffiths–Killip–Visan [77, 64] reintroduced a characterization of the transmission coefficient in terms of the Fredholm determinant of the Lax operator and also extended its definition to low regularity. The connection of these constructions will be spelled out in [80].

For later convenience relating to constants in the conserved quantities we choose $\kappa \in \{-2, 2\}$, where $\kappa = 2$ corresponds to the defocusing and $\kappa = -2$ to the focusing case. In this setting, the NLS

$$\begin{cases} iu_t + u_{xx} = \pm 2|u|^2 u, \\ u(0, x) = u_0(x). \end{cases} \tag{2.1}$$

is completely integrable.

Complete integrability is a property of some evolution equations with very special structure. It originates from the theory of finite-dimensional Hamiltonian systems, and can be defined as the ability to transform the Hamiltonian system into a system of ODEs which is explicitly solvable. The initial value problem (2.1) can be seen as a Hamiltonian equation, though in infinite dimensions. For the reader's convenience we sketch an introduction to Hamiltonian mechanics in infinite dimensions in Appendix A.4.

Integrability is connected to the existence of conserved quantities. In the finite-dimensional case it is intuitive that enough conserved quantities, that are in some way 'independent', enable to cancel enough variables in the equation to lead to trivial equations of motion. Without introducing the terminology we state the mathematical meaning of this intuition in terms of the following theorem (cf. [5, Chapter 10, Section 49]):

Theorem 2.1 (Liouville)**.** *Assume that $F_1, \dots, F_n$ are functions on a symplectic $2n$-dimensional manifold with pairwise vanishing Poisson brackets. Consider the level sets*

$$M_f = \{x : F_i(x) = f_i, i = 1, \dots, n\},$$

and assume that the 1-forms dF_i are linearly independent everywhere on M_f. Then:

1. *M_f is a smooth manifold, invariant under the phase flow with Hamiltonian $H = F_1$.*

2. *If the manifold M_f is compact and connected, then it is diffeomorphic to the n-dimensional torus $\mathbb{T}^n = \{(\phi_1, \dots, \phi_n) \bmod 2\pi\}$.*

3. *The phase flow with Hamiltonian H determines a conditionally periodic motion on M_f, i.e. in angular coordinates $\phi = (\phi_1, \dots, \phi_n)$ we have*

$$\frac{d}{dt}\phi = \omega, \qquad \omega = \omega(f).$$

4. *The canonical equations with Hamiltonian H can be integrated by quadratures.*

In the infinite-dimensional case, complete integrability is connected to the existence of infinitely many conserved quantities, or equivalently, infinitely many non-trivial commuting flows. For the cubic, one-dimensional NLS, there is one special conserved quantity which has particularly useful properties and gives rise to conserved quantities on the level of Sobolev regularities $H^s, s > -\frac{1}{2}$: the transmission coefficient $T(z, u)$.

The first central object on the way to define the transmission coefficient is the Lax operator associated to the NLS equation (2.1). Moreover, it is convenient to see (1.1) as the restriction of the AKNS system

$$\begin{aligned} iq_t &= -q_{xx} + 2q^2 r, \\ ir_t &= r_{xx} - 2r^2 q, \end{aligned} \tag{2.2}$$

to the set $\{q = \pm\bar{r} = u\}$, where the sign $\pm$ is the same as in (2.1). With this convention, both the focusing and defocusing NLS equation can be treated at the same time.

The Lax operator for the AKNS equations (2.2) which are named after their inventors Ablowitz–Kaup–Newell–Segur [2] is given by

$$L(z,q,r) = \begin{pmatrix} -iz - \partial & q \\ -r & -iz + \partial \end{pmatrix}. \tag{2.3}$$

With this definition, it can be checked that (2.1) formally becomes equivalent to the operator-valued equation

$$L_t = [P, L], \tag{2.4}$$

where the operator P is given by (cf [83, Section 2.1])

$$P = i \begin{pmatrix} 2\partial^2 - qr & -q\partial - \partial q \\ r\partial + \partial r & -2\partial^2 + qr \end{pmatrix}.$$

One of the consequences of (2.4) is that if $(q(t), r(t))$ solve (2.2), then the Lax operator at time t, $L(z, q(t), r(t))$, and the one at time zero, $L(z, q(0), r(0))$, are unitarily equivalent. This means that spectral information of the operator L will be preserved, and motivates to look at its spectral information.

2.1 The Classical Approach via Jost Solutions

To define the transmission coefficient we look at the spectral information of L given by the Jost solutions. These are special solutions to

$$L(z,q,r)\psi = 0, \tag{2.5}$$

which satisfy prescribed asymtptotics at $\pm\infty$. We assume that q and r decay fast, and $z = \xi \in \mathbb{R}$. There exist a fundamental system, ψ_{-+}, ψ_{--} normalized at $-\infty$,

$$\lim_{x\to-\infty} e^{i\xi x}\psi_{-+}(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \lim_{x\to-\infty} e^{-i\xi x}\psi_{--}(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and another fundamental system normalized at ∞,

$$\lim_{x\to\infty} e^{-i\xi x}\psi_{+-}(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \lim_{x\to\infty} e^{i\xi x}\psi_{++}(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The solution space of the problem (2.5) is a two-dimensional vector space. As a consequence these solutions are linearly independent and thus connected on the real line by

$$(\psi_{+-}, \psi_{-+}) = \begin{pmatrix} a_+(\xi) & b_+(\xi) \\ b_-(\xi) & a_-(\xi) \end{pmatrix} (\psi_{--}, \psi_{++})$$

There are simple alternative expression

$$a_+(\xi) = W(\psi_{-+}, \psi_{+-}) = \det(\psi_{-+}, \psi_{+-}), \quad a_-(\xi) = W(\psi_{++}, \psi_{--}).$$

Here W is the Wronskian, which is independent of x. The solutions ψ_{-+} and ψ_{+-} have a holomorphic extensions to the upper half plane $\{\operatorname{Im} z > 0\}$ and are called left and right Jost functions. We define the transmission coefficient

$$T(z)^{-1} = W(\psi_{-+}, \psi_{+-}) = \lim_{x\to\infty} e^{izx}\psi^1_{-+}(x) = \lim_{x\to-\infty} e^{-izx}\psi^2_{+-}(x).$$

for z in the upper half plane. Similarly, the solutions ψ_{--} and ψ_{++} can be used to define $T(z, u)$ in the lower half-plane.

To calculate the transmission coefficient, (2.5) needs to be solved. To this end define $\phi = e^{izx}\psi$. With this definition we can rewrite (2.5) as

$$\begin{aligned} \partial_x \phi_1 &= q\phi_2, \\ (\partial_x - 2iz)\phi_2 &= r\phi_1, \end{aligned} \tag{2.6}$$

By the variation of constants formula, and from the asymptotics of the left Jost solution at $-\infty$, we can transform (2.6) into

$$\begin{aligned} \phi_1(x) &= \int_{-\infty}^{x} q(y)\phi_2(y)\,dy + 1 \\ \phi_2(x) &= \int_{-\infty}^{x} e^{2iz(x-y)} r(y)\phi_1(y)\,dy, \end{aligned}$$

which in turn can be cast into a single equation[1] for ϕ_1,

$$\phi_1(x) = \int_{-\infty<x_1<y_1<x} e^{2iz(y_1-x_1)} q(y_1) r(x_1)\phi_1(x_1)\,dx_1\,dy_1 + 1. \tag{2.7}$$

We define the operator

$$(S(z,q,r)f)(x) = \int_{-\infty<x_1<y_1<x} e^{2iz(y_1-x_1)} q(y_1) r(x_1) f(x_1)\,dx_1\,dy_1. \tag{2.8}$$

Thus (2.7) has the form

$$\phi = S(z,q,r)\phi + 1,$$

and can hence be formally solved by

$$\phi = \sum_{n=0}^{\infty} S^n(z,q,r)1.$$

Note that in this notation the 1 corresponds to the constant function with value one. From this we obtain a series expansion

$$T^{-1}(z,q,r) = \sum_{n=0}^{\infty} T_{2n}(z,q,r), \tag{2.9}$$

[1] These considerations have been done already in the very first works on the direct scattering for NLS, see e.g. [3, Section 1.3]

where for $n \geq 1$,

$$T_{2n}(z,q,r) = \lim_{x\to\infty} \big(S^n(z,q,r)1\big)(x)$$
$$= \int_{x_1<y_1<x_2<\cdots<x_n<y_n} e^{2iz\sum_{j=1}^n (y_j-x_j)} q(y_1)r(x_1)\dots q(y_n)r(x_n)\,dxdy, \tag{2.10}$$

and $T_0 = 1$ respectively.

It is not hard to see that when $q \in L^1(\mathbb{R}), r \in L^\infty(\mathbb{R})$, and when $\operatorname{Im} z > 0$, the single operators

$$f \mapsto \int_{-\infty}^x q(y)f(y)\,dy \quad \text{respectively} \quad f \mapsto \int_{-\infty}^x e^{2iz(x-y)} r(y)f(y)\,dy,$$

are bounded as operators on $C_b^0(\mathbb{R})$ with norms

$$\|q\|_{L^1(\mathbb{R})} \quad \text{respectively} \quad (2\operatorname{Im} z)^{-1}\|r\|_{L^\infty(\mathbb{R})}.$$

This would be enough to cover the case $r = 1$ and $q \in L^1(\mathbb{R})$ which corresponds to the KdV equation. More generally, when looking at $S(z,q,r)$, we find that for all $q \in L^p(\mathbb{R}), r \in L^{p'}(\mathbb{R})$, by Young's inequality (Theorem A.8)

$$\begin{aligned}
&\|S(z,q,r)f\|_{L^\infty(\mathbb{R})} \\
&\qquad \leq \|f\|_{L^\infty(\mathbb{R})} \int_{-\infty<x_1<y_1<\infty} e^{-2\operatorname{Im} z(y_1-x_1)} |q(y_1)||r(x_1)|\,dx_1\,dy_1 \\
&\qquad = \|f\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \big(e^{-2\operatorname{Im} z\cdot}\chi_{\{\cdot>0\}} * |r|\big)(y_1)|q(y_1)|\,dy_1 \\
&\qquad \leq \|f\|_{L^\infty(\mathbb{R})}\|q\|_{L^p(\mathbb{R})}\big\|e^{-2\operatorname{Im} z\cdot}\chi_{\{\cdot>0\}} * |r|\big\|_{L^{p'}(\mathbb{R})} \\
&\qquad \leq \frac{1}{2\operatorname{Im} z}\|f\|_{L^\infty(\mathbb{R})}\|q\|_{L^p(\mathbb{R})}\|r\|_{L^{p'}(\mathbb{R})}.
\end{aligned}$$

Here the constant $(2\operatorname{Im} z)^{-1}$ appears by evaluating the L^1 norm of the function $e^{-2\operatorname{Im} z\cdot}\chi_{\{\cdot>0\}}$. Thus, the series in (2.9) can be guaranteed to converge in the L^2 based setting:

Lemma 2.2. *If $q, r \in L^2(\mathbb{R})$ are such that*

$$\|q\|_{L^2(\mathbb{R})}, \|r\|_{L^2(\mathbb{R})} < (2\operatorname{Im} z)^{-\frac{1}{2}},$$

then the series (2.9) *converges absolutely and*

$$|T_{2n}(z,q,r)| \leq (2\operatorname{Im} z)^{-n}\|q\|^n_{L^2(\mathbb{R})}\|r\|^n_{L^2(\mathbb{R})}.$$

In fact, Koch–Tataru [83] showed that T_{2n} can be defined in lower regularity as well. To this end, they define the spaces $l^2_\tau DU^2$ as a replacement of the spaces $H_\tau^{-1/2}(\mathbb{R})$, to which the theory does not extend (see Section 5.3.2 for a definition of the spaces U^2, V^2). For instance,

$$\|f\|_{l^2_\tau DU^2} \lesssim \tau^{-\frac{1}{2}+s}\Big(\int_{\mathbb{R}} \frac{|\hat{f}(\xi)|^2}{(\xi^2+\tau^2)^s}\,d\xi\Big)^{\frac{1}{2}}, \qquad 0 \le s < \frac{1}{2},$$

which shows that $H^{-1/2+}(\mathbb{R})$ embeds continuously into $l^2_\tau DU^2$. One of their results is the following estimate (cf. [83, Proposition 5.10]):

Lemma 2.3. *Let $z = \sigma + i\tau$. Assume $q, r \in l^2_1 DU^2$. There exists a constant $c > 0$ such that*

$$|T_{2n}(z,q,r)| \le \big(c\|e^{2i\sigma\cdot}q\|_{l^2_\tau DU^2}\|e^{-2i\sigma\cdot}r\|_{l^2_\tau DU^2}\big)^n.$$

In particular the series in (2.9) converges uniformly in small enough balls in $H^{-1/2+}(\mathbb{R})$, and the definition of the transmission coefficient can be extended to these spaces.

So far we have seen how to define the transmission coefficient $T(z,q,r)$. It turns out though that the logarithm of this object is even more useful. In the case $q = u = \pm\bar{r}$, $-\log T$ serves as a generating function for the so called NLS Hamiltonians,

$$\log T(z,u) \sim -i\sum_{n=1}^{\infty}(2z)^{-n}H_n^{\mathrm{NLS}}(u),$$

as an asymptotic series. In fact we will also use $-\log T$ to construct the low regularity almost conserved quantities in Section 2.3.

From the series expansion of $T^{-1}(z,q,r)$, we can derive a series expansion for $-\log T(z,q,r)$,

$$-\log T(z,q,r) = \sum_{n=1}^{\infty}\tilde{T}_{2n}(z,q,r), \tag{2.11}$$

where $\tilde{T}_{2n}(z,\lambda q,\nu r) = (\lambda\nu)^n\tilde{T}_{2n}(z,q,r)$. Indeed, from the Taylor expansion of $\log(1+x)$ we infer

$$\log\Big(1+\sum_{n=1}^{\infty}T_{2n}\Big) = \sum_{n=1}^{\infty}T_{2n} - \frac{1}{2}\Big(\sum_{n=1}^{\infty}T_{2n}\Big)^2 + \frac{1}{3}\Big(\sum_{n=1}^{\infty}T_{2n}\Big)^3 \pm \dots,$$

and hence,

$$\tilde{T}_2 = T_2, \qquad \tilde{T}_4 = T_4 - \frac{1}{2}T_2^2, \qquad \tilde{T}_6 = T_6 - T_2T_4 + \frac{1}{3}T_2^3, \qquad \dots$$

Intuitively the bounds of T_{2n} should carry over to bounds on $\tilde{T}_{2n}$. An indeed, bounds on $\tilde{T}_{2n}$ can be derived via subordination from Lemma 2.3 (cf. [83, Proposition 5.10]). One arrives at:

Lemma 2.4. *Let $z = \sigma + i\tau$. Assume $q, r \in l^2_1 DU^2$. There exists a constant $c > 0$ such that*

$$|\tilde{T}_{2n}(z,q,r)| \le \big(c\|e^{2i\sigma\cdot}q\|_{l^2_\tau DU^2}\|e^{-2i\sigma\cdot}r\|_{l^2_\tau DU^2}\big)^n.$$

It follows that the series in (2.11) is convergent as well, provided

$$\|e^{2i\sigma\cdot}q\|_{l^2_\tau DU^2}\|e^{-2i\sigma\cdot}r\|_{l^2_\tau DU^2} \ll 1.$$

This is true for example if $q, r \in H^s(\mathbb{R}), -\frac{1}{2} < s \leq 0$, $z = i\tau$, and if,

$$\|q\|_{H^s(\mathbb{R})}, \|r\|_{H^s(\mathbb{R})} \lesssim \tau^{\frac{1}{2}+s}.$$

In fact, if $q, r \in l^2_1 DU^2$, then $T^{-1}(z, q, r)$ is defined as a holomorphic function for z in the whole upper half-plane. This can be seen from the fact that the Jost functions can be constructed in the whole upper half-plane, and their dependence on z is holomorphic. Moreover, $T^{-1}(z, q, r)$ cannot have any zeroes in the region where (2.9) converges, though in the focusing case it can still have zeroes in the upper half plane away from the region of convergence (see Corollary 5.12 and the following discussion in [83]).

2.2 The Perturbation Determinant

We have seen how to use the classical approach via Jost solutions to define the transmission coefficient and its logarithm. A different approach to define $-\log T(z, q, r)$ for functions q, r with regularity below $L^2(\mathbb{R})$ was laid out by Killip-Visan-Zhang respectively Harrop-Griffiths-Killip-Visan in [77, 64]. To explain this approach, we take a fresh start and define for $\kappa \geq 1$,

$$A(\kappa, q, r) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \operatorname{tr}\big((\Lambda\Gamma)^n\big), \tag{2.12}$$

where

$$\Lambda = (\kappa - \partial)^{-\frac{1}{2}} q (\kappa + \partial)^{-\frac{1}{2}},$$
$$\Gamma = (\kappa + \partial)^{-\frac{1}{2}} r (\kappa - \partial)^{-\frac{1}{2}}.$$

Here the operators $R_\pm = (\kappa \pm \partial)^{-1}$ and their square-roots are defined via their Fourier transform, $\operatorname{tr}(A)$ denotes trace of an operator A, and is defined as in Appendix A.3. In particular,

$$\mathcal{F}((\kappa \pm \partial)^{-\frac{1}{2}} f)(\xi) = (\kappa \pm i\xi)^{-\frac{1}{2}} \hat{f}(\xi), \tag{2.13}$$

where we determine the complex square-root via $\sqrt{\kappa} > 0$ and continuity (see [77]). We also define

$$\alpha(\kappa, q, r) = \operatorname{Re} A(\kappa, q, r),$$

which up to signs coincides with the definition of [77, Chapter 4].[2] After cycling the trace, which is possible if we assume $q, r \in \mathcal{S}(\mathbb{R})$ one formally arrives at

$$\alpha(\kappa, q, r) = \operatorname{Re}\Big[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \operatorname{tr}\big((R_- q R_+ r)^n\big)\Big]. \tag{2.14}$$

[2] This sign difference stems from the fact that they consider cubic NLS with linear part $-i\partial_t + \partial_{xx}$, i.e. with reversed time direction.

At first sight the definition of A seems to appear out of nowhere. It can be motivated though by considering the so called perturbation determinant for the Lax operator L. If

$$L_0 = \begin{pmatrix} \kappa - \partial & 0 \\ 0 & \kappa + \partial \end{pmatrix}$$

denotes the Lax operator with $q = r = 0$ and $z = i\kappa$, then

$$\log \det(L_0^{-1} L) = \log \det \begin{pmatrix} 1 & R_- q \\ -R_+ r & 1 \end{pmatrix}$$

has the formal series expansion

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \operatorname{tr} \left((R_- q R_+ r)^n \right),$$

which up to cycling the trace and a sign is exactly the definition (2.12).

To make these definitions rigorous, all traces in (2.12) and (2.14) need to be well-defined, and the series need to converge. To this end, it is enough to show that Λ, Γ, $R_- q$ and $R_+ r$ are Hilbert-Schmidt operators with suitable estimates, since by Theorem A.27 their products will be trace class. This can be checked by calculating their kernels, and using Theorem A.28. For Λ, Γ one sees ([77, Lemma 4.1]

$$\|\Lambda\|_{\mathfrak{I}_2}^2 = \|\Gamma\|_{\mathfrak{I}_2}^2 \approx \int_{\mathbb{R}} \log\left(4 + \frac{\xi^2}{\kappa^2}\right) \frac{|\hat{q}(\xi)|^2}{\sqrt{\xi^2 + 4\kappa^2}} d\xi. \tag{2.15}$$

This shows for $-\frac{1}{2} < s < 0$ that

$$\|\Lambda\|_{\mathfrak{I}_2} = \|\Gamma\|_{\mathfrak{I}_2} \lesssim \kappa^{-\frac{1}{2}-s} \|q\|_{H^s(\mathbb{R})},$$

and convergence of the series in (2.12) is assured provided

$$\|q\|_{H^s(\mathbb{R})} \lesssim \kappa^{\frac{1}{2}+s}.$$

For $R_- q$ and $R_+ r$ one can use that the integral kernel of $(\kappa \pm \partial)^{-1}$ is given by

$$e^{\mp \kappa (x-y)} \chi_{\{\mp(x-y)<0\}}. \tag{2.16}$$

Using this we arrive at (see also the calculation in Lemma 3.4)

$$\|R_+ f\|_{\mathfrak{I}_2} = \|R_- f\|_{\mathfrak{I}_2} \approx \kappa^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R})}.$$

Correspondingly, the series written as in (2.14) can be defined provided

$$\|f\|_{L^2(\mathbb{R})} \lesssim \kappa^{\frac{1}{2}}.$$

The significance of $A(\kappa, q, r)$ is that it can be shown to be conserved under the AKNS flow (2.2). We give a proof, closely following [77, Proposition 4.3] but with the difference of including the more general case of two functions q and r. A different proof was given in [64].

Theorem 2.5. *Let $q, r \in C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}))$ be a solution of the AKNS system* (2.2). *Then for $\kappa \geq 1$ large enough,*

$$\partial_t A(\kappa, q(t), r(t)) = 0.$$

Proof. We rewrite

$$A = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \operatorname{tr} \big((R_- q R_+ r)^n \big)$$

by cycling the trace. Since

$$\partial_t A = \sum_{n=1}^{\infty} (-1)^n \operatorname{tr} \big((R_- q R_+ r)^{n-1} (R_-(-q_{xx}+2q^2 r) R_+ r + R_- q R_+ (r_{xx} - 2r^2 q)) \big),$$

it is enough to show that

$$\operatorname{tr} \big(- R_- q_{xx} R_+ r + R_- q R_+ r_{xx} \big) = 0, \tag{2.17}$$

and for $n \geq 1$,

$$\begin{aligned} 2 \operatorname{tr} \big((R_- q R_+ r)^{n-1} & (R_- q^2 r R_+ r - R_- q R_+ r^2 q) \big) \\ &= \operatorname{tr} \big((R_- q R_+ r)^n (-R_- q_{xx} R_+ r + R_- q R_+ r_{xx}) \big). \end{aligned} \tag{2.18}$$

To prove (2.17) we calculate (see Lemma 3.3)

$$\operatorname{tr} \big(R_- f R_+ g \big) = \big\langle (2\kappa - \partial)^{-1} f, \bar{g} \big\rangle$$

and use partial integration. We write the operator identities

$$\begin{aligned} q_{xx} &= q(\partial^2 - 2\kappa\partial - \kappa^2) + (\partial^2 + 2\kappa\partial - \kappa^2) q + 2(\kappa - \partial) q (\kappa + \partial), \\ r_{xx} &= (\partial^2 - 2\kappa\partial - \kappa^2) r + r(\partial^2 + 2\kappa\partial - \kappa^2) + 2(\kappa + \partial) r (\kappa - \partial), \end{aligned}$$

in the form

$$\begin{aligned} q_{xx} &= q A_1 + A_2 q + 2(\kappa - \partial) q (\kappa + \partial), \\ r_{xx} &= A_1 r + r A_2 + 2(\kappa + \partial) r (\kappa - \partial). \end{aligned}$$

The operators A_1 and A_2 commute with $R_\pm$. Hence, their contribution in the right-hand side of (2.18) cancels after cycling the trace, and it can be rewritten as

$$2 \operatorname{tr} \big((R_- q R_+ r)^n (-qr + R_- qr(\kappa - \partial)) \big),$$

which coincides with the left-hand side of (2.18) after cycling the trace once more. ∎

The conservation of A is not a coincidence, in fact

$$A(\kappa, q, r) = -\log T(i\kappa, q, r).$$

While this fact seems to be well-known, it is not easy to come up with a reference for it. Though not explicitly stated in terms of the transmission coefficient, the earliest reference for arguments leading to a formula for the KdV transmission coefficient seems to be [72]. A more modern approach can be found in [118, Proposition 5.7], though again the statement is only shown for KdV. The work [64] uses the result partially, in the sense that the convergence of the approximate Hamiltonian flows towards the mKdV and NLS flows gives the result for the first terms in the asymptotic expansion. Finally, the full statement can be derived from a result from [64] and an old calculation concerning the functional derivatives of the transmission coefficient, going back at least to [105, Section 10]. We give a proof by connecting these facts. Since it is based on the work in progress [80] we include proofs:

Theorem 2.6. *Let $\kappa \geq 1$ big enough and $q, r \in \mathcal{S}(\mathbb{R})$. Then,*

$$-\log T(i\kappa, q, r) = A(\kappa, q, r),$$

and in particular

$$\tilde{T}_{2n}(i\kappa, q, r) = \frac{(-1)^{n-1}}{n} \operatorname{tr}\left((R_- q R_+ r)^n\right).$$

Theorem 2.6 follows from the following results:

Lemma 2.7 (Proposition 3.1 and Lemma 4.1 in [64])**.** *Let $-\frac{1}{2} < s \leq 0$. There exists $\delta > 0$ such that for all $q, r \in H^s, \|q\|_{H^s} \leq \delta$, and all $\kappa \geq 1$, the Lax operator L is invertible on $L^2(\mathbb{R})$. Its inverse admits an integral kernel whose off-diagonal entries G_{12} and G_{21} are continuous. In particular, they admit a continuous restriction to $x = y$, $g_{12}(x, \kappa, q, r)$, respectively $g_{21}(x, \kappa, q, r)$. These functions satisfy*

$$\frac{\delta}{\delta q} A = g_{21}, \qquad \frac{\delta}{\delta r} A = -g_{12}. \tag{2.19}$$

Lemma 2.8 ([80])**.** *For z in the upper half-plane, the Green's function for the operator $L(z)$ is*

$$G(x,y,z) = T(z) \begin{cases} \begin{pmatrix} \psi^1_{-+}(x,z)\psi^2_{+-}(y,z) & \psi^1_{-+}(x,z)\psi^1_{+-}(y,z) \\ \psi^2_{-+}(x,z)\psi^2_{+-}(y,z) & \psi^2_{-+}(x,z)\psi^2_{+-}(y,z) \end{pmatrix} & \textit{if } x < y, \\[2ex] \begin{pmatrix} \psi^1_{+-}(x,z)\psi^2_{-+}(y,z) & \psi^1_{+-}(x,z)\psi^1_{-+}(y,z) \\ \psi^2_{+-}(x,z)\psi^2_{-+}(y,z) & \psi^2_{+-}(x,z)\psi^1_{-+}(y,z) \end{pmatrix} & \textit{if } y < x. \end{cases}$$

Proof. We observe that the columns considered as functions of x satisfy

$$L(z)G = 0$$

whenever $x \neq y$. It is the Green's function since, for x^+ being the limit from above and x_- being the limit from below,

$$G(x^+, x) - G(x^-, x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

which on the diagonal reduces to the fact that

$$T(z)^{-1} = W(\psi_{-+}, \psi_{+-}),$$

and shows that $L(z)G = \delta(x-y)$. $\square$

Lemma 2.9 ([80])**.** *For all $q, r \in \mathcal{S}(\mathbb{R})$, and all $\kappa \geq 1$ big enough,*

$$\frac{\delta}{\delta q} \log T = -g_{21}, \qquad \frac{\delta}{\delta r} \log T = g_{12}. \tag{2.20}$$

Proof. The equation

$$L(z)\psi = f$$

has a another fundamental solution $G(x, y; z)$ given by 0 if $x < y$ and otherwise (observe that $W(\psi_{++}, \psi_{+-}) = 1$)

$$-\begin{pmatrix} \psi^1_{++}(x)\psi^2_{+-}(y) - \psi^1_{+-}(x)\psi^2_{++}(y) & \psi^1_{++}(x)\psi^1_{+-}(y) - \psi^1_{+-}(x)\psi^1_{++}(y) \\ \psi^2_{++}(x)\psi^2_{+-}(y) - \psi^2_{+-}(x)\psi^2_{++}(y) & \psi^2_{++}(x)\psi^1_{+-}(y) - \psi^2_{+-}(x)\psi^1_{++}(y) \end{pmatrix}.$$

This can be seen by checking again that $L_x G(x, y; z) = 0$ away from the diagonal and by the jump condition

$$G(x^+, x) - G(x^-, x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

as this implies $L_x G(x, y) = \delta(x-y)$. To determine $\frac{\delta T}{\delta q}$ and $\frac{\delta T}{\delta r}$ resp. for $\operatorname{Im} z > 0$, recall

$$\frac{d}{dt} T(z; q + t\dot{q}, r + t\dot{r})|_{t=0} =: \int_{\mathbb{R}} \frac{\delta T}{\delta q}\dot{q} + \frac{\delta T}{\delta r}\dot{r}\, dy.$$

We differentiate the equation with respect to t (dots are t-derivatives) and consider $\dot{\psi} = \dot{\psi}_{-+}$ to arrive at

$$L(z)\dot{\psi} = \begin{pmatrix} -\dot{q}\psi^2_{-+} \\ \dot{r}\psi^1_{-+} \end{pmatrix}.$$

Hence

$$\begin{aligned} \dot{\psi}(x) &= \psi_{++}(x) \int_{-\infty}^{x} \psi^2_{+-}(y)\psi^2_{-+}(y)\dot{q}(y) - \psi^1_{+-}(y)\psi^1_{-+}(y)\dot{r}(y)\, dy \\ &\quad - \psi_{+-}(x) \int_{-\infty}^{x} \psi^2_{++}(y)\psi^2_{-+}(y)\dot{q}(y) - \psi^1_{+-}(y)\psi^1_{-+}\dot{r}\, dy, \end{aligned}$$

and thus

$$\frac{d}{dt} T^{-1}(q + t\dot{q}, r + \dot{r}) = \lim_{x\to\infty} e^{izx}\dot{\psi}^1(x) = \int_{\mathbb{R}} \psi^2_{+-}\psi^2_{-+}\dot{q}\, dy - \int_{\mathbb{R}} \psi^1_{+-}\psi^1_{-+}\dot{r}\, dy.$$

Here, the second summand vanishes due to the assumption $\operatorname{Im} z > 0$. Thus by Lemma 2.8,

$$\frac{\delta T^{-1}}{\delta q} = \psi^2_{+-}\psi^2_{-+} = T^{-1}g_{21}, \quad \frac{\delta T^{-1}}{\delta r} = -\psi^1_{+-}\psi^1_{-+} = -T^{-1}g_{12}.$$

The result for $\log T$ follows from this. $\square$

We end this section by giving the explicit form of the quartic term in the expansion. This will be needed in the construction of almost conserved quantities at the level of $M_{2,1}(\mathbb{R})$ regularity. For a proof we refer to [83, Section 8.1]. It uses from the construction of Jost functions laid out in Section 2.1.

Lemma 2.10. *The quartic term satisfies*

$$\tilde{T}_4(i\kappa, q, \bar{q}) = \frac{i}{2\pi}\int_{\xi_1+\xi_2=\eta_1+\eta_2} \frac{\operatorname{Re}(\overline{\hat{q}(\xi_1)\hat{q}(\xi_2)}\hat{q}(\eta_1)\hat{q}(\eta_2))}{(2i\kappa+\xi_1)(2i\kappa+\eta_1)(2i\kappa+\eta_2)}.$$

2.3 Low Regularity Conservation Laws for the NLS

The gist of the works [83, 77] is that the conserved quantity $-\log T(z, q, r)$ can be used to construct (almost) conservation laws for the NLS at the level of Sobolev regularities $H^s(\mathbb{R})$ for $s > -\frac{1}{2}$. This works for both defocusing and focusing NLS (i.e. $q = \pm\bar{r}$) which is related to the fact that we only consider the subcritical range.

The methods of Koch-Tataru [83] and Killip-Visan-Zhang [77] are somewhat different in their precise form. Koch-Tataru make use of a shuffle algebra structure in the series expansion of $-\log T$ to rewrite the term $\tilde{T}_{2n}$ as a finite sum of so called connected integrals. These connected integrals have nice decay properties and allow for L^p based estimates, which are needed to obtain sufficient decay in $\operatorname{Im} z$. On the other hand, the representation of $\tilde{T}_{2n}$ as a trace over operators used by Killip-Visan-Zhang (see Theorem 2.6) already allows for L^p based estimates. However, for larger s one has to make additional use of a growing number of cancellations to obtain the aforementioned decay in $\operatorname{Im} z$, which is more easily visible in the connected integrals than in the trace representation.

Both methods have their own advantages and disadvantages. In Section 3 we will make use of the Killip-Visan-Zhang ansatz, and we concentrate on this method in this section as well. The Koch-Tataru ansatz will play a role again in the construction of an almost conserved quantity adapted to the modulation space $M_{2,1}(\mathbb{R})$, but we will only use the explicit form of $\tilde{T}_4$ as a connected integral as stated in Lemma 2.10.

To construct the conserved energies, recall the definition (2.14) of $\alpha(\kappa, q, r)$. For simplicity we set $r = \bar{q}$, assuming that we are in the defocusing case. We write

$$\alpha_{2n}(\kappa, q) = \operatorname{Re}\Big[\frac{(-1)^{n-1}}{n}\operatorname{tr}\big((R_- q R_+ r)^n\big)\Big],$$

and thus

$$\alpha(\kappa, q) = \sum_{n=1}^{\infty} \alpha_{2n}(\kappa, q).$$

The main contribution in the construction of the almost conserved quantities is played by $\alpha_2(\kappa, q)$. Using the explicit integral kernels of $R_\pm$, it can be seen that (see also [77, Lemma 4.2])

$$\alpha_2(\kappa, q) = \int_{\mathbb{R}} \frac{2\kappa}{\xi^2 + 4\kappa^2} |\hat{q}(\xi)|^2 \, d\xi.$$

For small data the conservation of $\alpha(\kappa, q)$ (which follows from conservation of $A(\kappa, q)$) implies almost conservation of $\alpha_2(\kappa, q)$. By performing a weighted summation over α_2 in the parameter κ, one obtains a quantity which is near a Besov norm:

Definition 2.11. *Let $s \in \mathbb{R}$ and $r \in [1, \infty]$. We define the $B^s_{2,r}$ Besov norm as*

$$\|f\|_{B^s_{2,r}} = \begin{cases} \Big(\|\hat{f}\|^r_{L^2(|\xi| \le 1)} + \sum_{N \in 2^{\mathbb{N}}} N^{rs} \|\hat{f}\|^r_{L^2(N < |\xi| \le 2N)} \Big)^{\frac{1}{r}}, & \text{if } 1 \le r < \infty, \\ \max \Big(\|\hat{f}\|_{L^2(|\xi| \le 1)}, \sup_{N \in 2^{\mathbb{N}}} N^s \|\hat{f}\|_{L^2(N < |\xi| \le 2N)} \Big), & \text{if } r = \infty, \end{cases}$$

and the space $B^s_{2,r}$ as all those $f \in \mathcal{S}'(\mathbb{R})$ with $\|f\|_{B^s_{2,r}} < \infty$.

Note that since we are working in the L^2 based setting this Besov space definition coincides with the one from Definition A.6 which uses Littlewood-Paley projectors, and we have equivalence of norms. The following quantity is equivalent at negative regularity:

Lemma 2.12. *Let $-\frac{1}{2} < s < 0$ and $r \in [1, \infty]$. Given $\kappa_0 \in 2^{\mathbb{N}}$, define*

$$\|f\|_{Z_{\kappa_0}} = \begin{cases} \Big(\sum_{N \in 2^{\mathbb{N}}} N^{rs} \Big(\int \frac{2\kappa_0^2 N^2}{\xi^2 + 4\kappa_0^2 N^2} |\hat{f}(\xi)|^2 \, d\xi \Big)^{\frac{r}{2}} \Big)^{\frac{1}{r}}, & \text{if } 1 \le r < \infty, \\ \sup_{N \in 2^{\mathbb{N}}} N^s \Big(\int \frac{2\kappa_0^2 N^2}{\xi^2 + 4\kappa_0^2 N^2} |\hat{f}(\xi)|^2 \, d\xi \Big)^{\frac{1}{2}}, & \text{if } r = \infty. \end{cases}$$

Then,

$$\|f\|_{B^s_{2,r}} \lesssim \|f\|_{Z_{\kappa_0}} \lesssim \kappa_0^{|s|} \|f\|_{B^s_{2,r}}. \tag{2.21}$$

A proof of this Lemma can be found in [77, Lemma 3.2]. A closer look at the Z_{κ_0} norm reveals that

$$\|f\|_{Z_{\kappa_0}} = \Big\| N^s (\kappa_0 N)^{\frac{1}{2}} \alpha_2(\kappa_0 N, q)^{\frac{1}{2}} \Big\|_{\ell^r(N \in 2^{\mathbb{N}})}.$$

With this definition we are able to prove:

Theorem 2.13 (Theorem 4.5 in [77])**.** *Let $-\frac{1}{2} < s < 0$ and $r \in [1, \infty]$. Let $q \in C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}))$ be a Schwartz solution to the cubic one-dimensional NLS (2.1). Then,*

$$\|q(t)\|_{B^s_{2,r}} \lesssim \|q(0)\|_{B^s_{2,r}} \Big(1 + \|q(0)\|^2_{B^s_{2,r}} \Big)^{\frac{|s|}{1-2|s|}}.$$

Proof. Assume $\kappa \geq \kappa_0$. From (2.15), cutting to intervals in frequency space, and Hölder (using $r = \infty$), we see

$$\begin{aligned}\|\Lambda\|_{I_2}^2 &\lesssim \frac{1}{\kappa}\int_{|\xi|\leq\kappa_0} |\hat{q}(\xi)|^2\,d\xi + \sum_{N\in 2^{\mathbb{N}}} \frac{\log\left(2+\kappa_0^2N^2\kappa^{-2}\right)}{\kappa_0 N+\kappa}\int_{\kappa_0 N\leq|\xi|\leq 2\kappa_0 N} |\hat{q}(\xi)|^2\,d\xi\\ &\lesssim \left(\kappa_0\kappa^{-1}\right)^{1-2|s|}\kappa_0^{-1}\|q\|_{Z_{\kappa_0}}^2\\ &\lesssim \kappa^{2|s|-1}\|q\|_{B^s_{2,r}}^2.\end{aligned}$$

Thus by continuity of the norm in time, if

$$\kappa_0 \gtrsim (1+\|q(0)\|_{B^s_{2,r}}^2)^{\frac{1}{1-2|s|}}, \tag{2.22}$$

we obtain

$$\|\Lambda\|_{\mathfrak{I}_2} = \|\Gamma\|_{\mathfrak{I}_2}^2 \leq c < 1$$

for some small time interval I around $t=0$. This ensures convergence of the series expansion (2.14) for α. Moreover, by the geometric convergence and the estimate above, we also find for $t \in I$

$$|\alpha(\kappa, q(t)) - \alpha_2(\kappa, q(t))| \lesssim \kappa_0^{-4|s|}\kappa^{4|s|-2}\|q(t)\|_{Z_{\kappa_0}}^4.$$

We set $\kappa = \kappa_0 N$, raise this estimate to the power $r/2$ and sum over dyadic N to estimate

$$\Big(\sum_{N\in 2^{\mathbb{N}}} N^{rs}(\kappa_0 N)^{\frac{r}{2}}|\alpha(\kappa, q(t)) - \alpha_2(\kappa, q(t))|^{\frac{r}{2}}\Big)^{\frac{1}{r}} \lesssim \kappa_0^{-\frac{1}{2}}\|q(t)\|_{Z_{\kappa_0}}^2,$$

with convergence in the sum over N since $|s| < \frac{1}{2}$. This shows via conservation of α,

$$\begin{aligned}\kappa_0^{-\frac{1}{2}}\|q(t)\|_{Z_{\kappa_0}} &\leq \kappa_0^{-\frac{1}{2}}\Big\|N^s(\kappa_0 N)^{\frac{1}{2}}|\alpha(\kappa_0 N, q(t))|^{\frac{1}{2}}\Big\|_{\ell^r(N\in 2^{\mathbb{N}})}\\ &\qquad + \kappa_0^{-\frac{1}{2}}\Big\|N^s(\kappa_0 N)^{\frac{1}{2}}|\alpha(\kappa_0 N, q(t)) - \alpha_2(\kappa_0 N, q(t))|^{\frac{1}{2}}\Big\|_{\ell^r(N\in 2^{\mathbb{N}})}\\ &\leq \kappa_0^{-\frac{1}{2}}\Big\|N^s(\kappa_0 N)^{\frac{1}{2}}|\alpha(\kappa_0 N, q(0))|^{\frac{1}{2}}\Big\|_{\ell^r(N\in 2^{\mathbb{N}})} + O(\kappa_0^{-\frac{1}{2}}\|q(t)\|_{Z_{\kappa_0}})^2\\ &\leq \kappa_0^{-\frac{1}{2}}\|q(0)\|_{Z_{\kappa_0}} + O\big((\kappa_0^{-\frac{1}{2}}\|q(t)\|_{Z_{\kappa_0}})^2 + (\kappa_0^{-\frac{1}{2}}\|q(0)\|_{Z_{\kappa_0}})^2\big),\end{aligned}$$

where in the last line we traded α against α_2 once more. This estimate is amenable for a continuity argument: Assume that

$$\kappa_0^{-\frac{1}{2}}\|q(0)\|_{Z_{\kappa_0}} \leq \varepsilon \ll 1.$$

Let I denote the maximal time interval containing $t=0$ such that

$$\kappa_0^{-\frac{1}{2}}\|q(t)\|_{Z_{\kappa_0}} \leq C\varepsilon,$$

where $C > 0$ is fixed but chosen later. Due to the continuity of the $\|q(t)\|_{Z_{\kappa_0}}$ norm, I is closed. But moreover, it is non-empty (since $0 \in I$), and it is open: if $t \in I$, then by the above estimate for some $c > 0$,

$$\kappa_0^{-\frac{1}{2}} \|q(t)\|_{Z_{\kappa_0}} \leq c(\varepsilon + \varepsilon^2 + C^2\varepsilon^2) < \frac{C}{2}\varepsilon,$$

assuming $C > 4c$ and $\varepsilon(1 + C^2) < 1$. The latter condition can be assured by making κ_0 larger if needed, and (2.21). This shows

$$\sup_{t\in\mathbb{R}} \|q(t)\|_{Z_{\kappa_0}} \lesssim \|q(0)\|_{Z_{\kappa_0}}.$$

Combining (2.21) and (2.22) now yields the result. □

In Theorem 2.13 the decay of $\alpha_2(\kappa, q)$ in κ was sufficient for all the summations involved. If one wants to obtain higher regularity conservation laws, the decay is not sufficient anymore. The solution to this problem is to take a linear combination of α_2 with different spectral parameters, leading to more decay in κ. This was done in [77, Chapter 3] and will be used in Section 3.4 in the construction of almost conserved quantities for dNLS as well as in Section 5.4 in the construction of almost conserved quantities for NLS in modulation spaces $M_{2,q}(\mathbb{R})$ for $q \in [1, \infty)$.

Chapter 3

Low Regularity Conservation Laws for the Derivative Nonlinear Schrödinger Equation

3.1 Introduction

This section is based on the joint work [82] of the author with Robert Schippa. We closely follow the manuscript [82] and only changed some parts in Section 3.2 which have already been introduced in this thesis. We also added Remark 3.10 which explains how the work [75] was able to remove the smallness condition assumed in this section.

The following derivative nonlinear Schrödinger equation (*dNLS*) is considered

$$\begin{cases} i\partial_t q + \partial_{xx} q + i\partial_x(|q|^2 q) &= 0 \quad (t,x) \in \mathbb{R} \times \mathbb{K}, \\ q(0) &= q_0 \in H^s(\mathbb{K}), \end{cases} \tag{3.1}$$

where $\mathbb{K} \in \{\mathbb{R}, \mathbb{T} = (\mathbb{R}/(2\pi\mathbb{Z}))\}$. In the seventies (3.1) was proposed as a model in plasma physics in [112, 96, 97].

In the following let $\mathcal{S}(\mathbb{R})$ denote the Schwartz functions on the line and $\mathcal{S}(\mathbb{T})$ smooth functions on the circle. Here we prove a priori estimates

$$\sup_{t \in \mathbb{R}} \|q(t)\|_{H^s} \lesssim_s \|q_0\|_{H^s}, \quad 0 < s < \frac{1}{2},$$

where $q \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{K}))$ is a smooth global solution to (3.1), which is also rapidly decaying in the line case, *conditional upon small L^2-norm.* These estimates are the key to extend local solutions globally in time. Local wellposedness, i.e., existence, uniqueness and continuous dependence locally in time, in $H^{1/2}$ was proved by Takaoka [124] on the real line and Herr [67] on the circle. They proved local wellposedness via the contraction mapping principle, that is *perturbatively*. Furthermore, they showed that the data-to-solution mapping fails to be C^3 below $H^{1/2}$ in either geometry, respectively. Moreover, Biagioni–Linares [18] showed that the data-to-solution mapping even fails to be locally uniformly continuous on the real line below $H^{1/2}$. Thus, the results on local wellposedness in $H^{1/2}$ are the limit of proving local wellposedness via fixed point arguments. However, on the real line (3.1) admits the scaling symmetry

$$q(t,x) \to \lambda^{-1/2} q(\lambda^{-2}t, \lambda^{-1}x), \tag{3.2}$$

which distinguishes L^2 as scaling critical space. Hence, we still expect a milder form of local wellposedness in H^s for $0 \leq s < 1/2$. By short-time Fourier restriction, Guo [60] proved a priori estimates for $s > 1/4$ on the real line, which Schippa [115] extended to periodic boundary conditions.
Moreover, Grünrock [57] showed local wellposedness on the real line in Fourier Lebesgue spaces, which scale like H^s, $s > 0$. Deng *et al.* [41] recently extended this to periodic boundary conditions; see also the previous work [58].

Less is known about global wellposedness. Conserved quantities of the flow include the mass, i.e., the L^2-norm,

$$M[q] = \int_{\mathbb{K}} |q|^2 dx,$$

the momentum, related with the $H^{1/2}$-norm,

$$P[q] = \int_{\mathbb{K}} \operatorname{Im}(\bar{q} q_x) - \frac{1}{2}|q|^4 dx,$$

and the energy, related with the H^1-norm,

$$E[q] = \int_{\mathbb{K}} |q_x|^2 - \frac{3}{2}|q|^2 \operatorname{Im}(\bar{q} q_x) + \frac{1}{2}|q|^6 dx.$$

A local wellposedness result in L^2 seems to be very difficult due to the scaling criticality. On the other hand, it is not straight-forward to use the other quantities to prove a global result due to lack of definiteness. The remedy in previous works was to impose a smallness condition on the L^2-norm and use the sharp Gagliardo-Nirenberg inequality.

Wu [136] observed in the line case that combining several conserved quantities improves the L^2-threshold, which can be derived from the energy (cf. [135]). Mosincat–Oh carried out the corresponding argument on the torus [102]. Additionally making use of the I-method (cf. [38, 95]), Guo–Wu [61] proved global wellposedness in $H^{1/2}(\mathbb{R})$ for $\|u_0\|_{L^2}^2 < 4\pi$, and Mosincat [101] proved global wellposedness in $H^{1/2}(\mathbb{T})$ under the same L^2-smallness condition. Previously, Nahmod *et al.* [104] proved a probabilistic global wellposedness result in Fourier Lebesgue spaces scaling like $H^{1/2-\varepsilon}(\mathbb{T})$. On the half-line and intervals endowed with Dirichlet boundary conditions, Wu [135] and Tan [126] showed the existence of finite-time blow-up solutions.

The question of global wellposedness for arbitrary L^2-norm was still open at the time of the first submission of the work [82]. Afterwards, there were several new contributions to the global wellposedness of dNLS ([127, 7, 75, 68, 65, 63]). Among these, Bahouri–Perelman [7] showed global wellposedness in $H^{1/2}(\mathbb{R})$ without smallness assumption on the L^2-norm, and later Harrop-Griffiths–Killip–Ntekoume–Vişan[63] improved this result to global wellposedness in $L^2(\mathbb{R})$. The new works are discussed at the end of the Introduction.

Kaup–Newell [74] already observed shortly after the proposal of (3.1) that it admits a Lax pair with operator

$$L(t;q) = \begin{pmatrix} \partial + i\kappa^2 & -\kappa q \\ \kappa \bar{q} & \partial - i\kappa^2 \end{pmatrix}. \tag{3.3}$$

Consequently, there are infinitely many conserved quantities of the flow. However, to the best of the author's knowledge, there were no prior works using the complete integrability for solutions in *unweighted* L^2-based Sobolev space, i.e., without imposing additional spatial decay. In particular, there were no results for periodic boundary conditions making use of the complete integrability before the present ones at the time of publication.

Via inverse scattering, Lee [86, 87] proved global existence and uniqueness for certain initial data in $\mathcal{S}(\mathbb{R})$. Later, Liu [91] considered (3.1) with initial data in weighted Sobolev spaces $H^{2,2}(\mathbb{R})$ and proved global wellposedness via inverse scattering. See the subsequent works [71, 69] due to Jenkins *et al.* for results addressing soliton resolution in weighted Sobolev spaces and [70] for a recent survey. Recently, Pelinovsky–Shimabukuro [111] proved global well-posedness in $H^{1,1}(\mathbb{R}) \cap H^2(\mathbb{R})$ without L^2-smallness condition, but assumptions on the Kaup–Newell spectral problem; see also [110, 114].

There are major technical difficulties to apply inverse scattering techniques in unweighted Sobolev spaces, e.g., on the line the decay of the data is typically insufficient for classical arguments. For the nonlinear Schrödinger equation on the line, Koch–Tataru [83] could use the transmission coefficient to obtain almost conserved H^s-energies for all $s > -\frac{1}{2}$. Killip–Vişan–Zhang [77] pointed out a power series representation for the determinant

$$\log\det\Big(\begin{bmatrix} (-\partial + \tilde{\kappa})^{-1} & 0 \\ 0 & (-\partial - \tilde{\kappa})^{-1} \end{bmatrix} \begin{bmatrix} -\partial + \tilde{\kappa} & iq \\ \mp i\bar{q} & -\partial - \tilde{\kappa} \end{bmatrix} \Big),$$

given by

$$\sum_{l=1}^{\infty} \frac{(\mp 1)^{l-1}}{l} \operatorname{tr}\left\{ \Big[(\tilde{\kappa} - \partial)^{-1/2} q (\tilde{\kappa} + \partial)^{-1} \bar{q} (\tilde{\kappa} - \partial)^{-1/2} \Big]^l \right\},$$

which works in either geometry. Killip *et al.* [77] showed that it is conserved for NLS and mKdV by term-by-term differentiation. This led to low regularity conservation laws and corresponding a priori estimates in either geometry. Talbut [125] used the same approach to show low regularity conservation laws for the Benjamin-Ono equation.

Motivated by these results, we show that the determinant

$$\log\det\Big(\begin{bmatrix} (\partial + i\kappa^2)^{-1} & 0 \\ 0 & (\partial - i\kappa^2)^{-1} \end{bmatrix} \begin{bmatrix} \partial + i\kappa^2 & -\kappa q \\ \kappa \bar{q} & \partial - i\kappa^2 \end{bmatrix} \Big),$$

given by

$$\sum_{l=1}^{\infty} \frac{(-1)^l i^{l+1} \tilde{\kappa}^l}{l} \operatorname{tr} \left\{ \left[(\partial - \tilde{\kappa})^{-1} q (\partial + \tilde{\kappa})^{-1} \bar{q} \right]^l \right\}, \tag{3.4}$$

where we formally set $\tilde{\kappa} = -i\kappa^2$ (we drop the tilde later on), is conserved for solutions of (3.1). This yields the following theorem on the growth of Besov norms:

Theorem 3.1. *Let $q \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{K}))$ be a smooth solution to* (3.1). *For any $0 < s < 1/2$, $r \in [1, \infty]$, there is $c = c(s,r) < 1$ such that*

$$\|q(t)\|_{B^s_{2,r}} \lesssim \|q(0)\|_{B^s_{2,r}} \tag{3.5}$$

provided that $\|q(0)\|_2 \leq c$.

Remark 3.2. *We focus on regularities for which global results were previously unknown. It appears feasible to cover higher regularities following [77, Section 3]. This involves recombining $\alpha(\kappa; q)$ as defined below to gain more decay in κ and more growth in ξ on the Fourier side similar to what is done in Section 5.4.*

In follow-up works to [77], Killip–Vişan showed sharp global well-posedness for the KdV equation [76] and later on with Bringmann for the fifth order KdV equation [25]. Sharp global wellposedness for NLS and mKdV on the real line was shown by Harrop-Griffiths–Killip–Vişan [64]. In the first version of the article [82] (07/2020) we raised the question whether (3.1) is within the thrust of these works.

Indeed, in 12/2020, Tang–Xu [127] pointed out an underlying microscopic conservation law on the real line, which paralleled the results in [64] for mKdV and NLS, but did not prove wellposedness. On the real line, Bahouri–Perelman [7] (12/2020) showed global wellposedness in $H^{\frac{1}{2}}(\mathbb{R})$ without smallness assumptions on the L^2-norm, relying on profile decomposition, and also crucially on complete integrability. Moreover, Isom–Mantzavinos–Stefanov [68] (12/2020) showed that Sobolev norms $H^s(\mathbb{T})$, $s > 1$, of solutions to (3.1) are growing polynomially by using nonlinear smoothing and not relying on complete integrability. In [75] (01/2021) Killip–Ntekoume–Vişan showed global wellposedness of (3.1) for $q_0 \in H^s(\mathbb{K})$, $\frac{1}{6} \leq s < \frac{1}{2}$ but still under the restriction of small mass $\|q_0\|^2_{L^2} < 4\pi$. The small mass restriction was removed in [65] by Harrop-Griffiths–Killip–Vişan (06/2021), paving the way towards the final result of Harrop-Griffiths–Killip–Ntekoume-Vişan [63] who showed global wellposedness of dNLS in $L^2(\mathbb{R})$ without any mass restriction.

3.2 Preliminaries

We recall from Appendix A.1 our conventions on the Fourier transform is defined on $\mathbb{R}$. We will also work on the rescaled torus, for which we use the conventions from [107]. Given $\lambda \geq 1$, let $\mathbb{T}_\lambda = \mathbb{R}/(2\pi\lambda\mathbb{Z})$. The scalar product on $L^2(\mathbb{T}_\lambda)$ is given by

$$\langle f, g \rangle = \int_{\mathbb{T}_\lambda} f(x)\overline{g(x)}dx = \int_0^{2\pi\lambda} f(x)\overline{g(x)}dx.$$

We set

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi\lambda} f(x)e^{-ix\xi}dx \text{ and } f(x) = \frac{1}{\sqrt{2\pi}\lambda} \sum_{\xi \in \mathbb{Z}/\lambda} \hat{f}(\xi)e^{ix\xi}$$

for $f \in L^1(\mathbb{T}_\lambda, \mathbb{C})$, where $\xi \in \mathbb{Z}_\lambda = \lambda^{-1}\mathbb{Z}$. The guideline for the conventions is that Plancherel's theorem remains true:

$$\|f\|_{L^2(\mathbb{T}_\lambda)} = \|\hat{f}\|_{L^2(\mathbb{Z}_\lambda,(d\xi)_\lambda)},$$

where $(d\xi)_\lambda$ denotes the normalized counting measure on $\mathbb{Z}_\lambda$:

$$\int_{\mathbb{Z}_\lambda} f(\xi)(d\xi)_\lambda = \frac{1}{\lambda} \sum_{\xi \in \mathbb{Z}_\lambda} f(\xi).$$

For further basic Fourier analysis identities on $\mathbb{T}_\lambda$, we refer to [38, Section 2]. We turn to the definition of L^2-based Sobolev norms: For $s \in \mathbb{R}$, for $f \in \mathcal{S}(\mathbb{T}_\lambda) = C^\infty(\mathbb{T}_\lambda)$ we define

$$\|f\|_{H^s(\mathbb{T}_\lambda)} = \left(\int_{\mathbb{Z}_\lambda} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 (d\xi)_\lambda \right)^{\frac{1}{2}}.$$

Recall the definition of Besov norms. We consider a smooth partition of unity of the real line: Let $\beta_1 : \mathbb{R} \to [0, 1]$ denote a radially decreasing function $\beta_1(\xi) = 1$ for $\xi \in [-1, 1]$ and $\operatorname{supp} \beta_1 \subseteq [-2, 2]$. For $N \in 2^{\mathbb{N}}$ let $\beta_N(\xi) = \beta_1(\xi/N) - \beta_1(\xi/(N/2))$, and let P_N denote the Fourier multiplier on $\mathbb{R}$ or $\mathbb{T}_\lambda$:

$$(P_N f)\widehat{}(\xi) = \beta_N(\xi)\hat{f}(\xi).$$

We define the Besov norm of $f \in \mathcal{S}(\mathbb{R})$ or $f \in \mathcal{S}(\mathbb{T}_\lambda)$ for $1 \leq r < \infty$, $s \geq 0$ by

$$\|f\|_{B^s_{2,r}} = \big(\sum_{N \in 2^{\mathbb{N}_0}} N^{rs} \|P_N f\|^r_{L^2} \big)^{\frac{1}{r}}$$

and with the usual modification for $r = \infty$. Note that this gives an equivalent norm compared to the sharp cut-off in Definition 2.11.
For $\lambda \in 2^{\mathbb{N}_0}$, let $f_\lambda(x) = \lambda^{-\frac{1}{2}} f(\lambda^{-1}x)$. We record the following scaling of the Besov norms:

$$\lambda^{-s}\|f\|_{B^s_{2,r}} \lesssim \|f_\lambda\|_{B^s_{2,r}} \lesssim \|f\|_{L^2} + \lambda^{-s}\|f\|_{B^s_{2,r}}, \tag{3.6}$$

which follows from

$$\|P_1 f_\lambda\|_{L^2} \le \|f\|_{L^2}, \quad \|P_N f_\lambda\|_{L^2} = \|P_{\lambda N} f\|_{L^2} \qquad (N \in 2^{\mathbb{N}}).$$

Also recall the facts about trace class operators and Schatten norms from Appendix A.3, which will be used in the following. For $\kappa > 0$, $k \in \mathbb{Z}$, the mappings $(\partial \pm \kappa)^k : \mathcal{S}(\mathbb{K}) \to \mathcal{S}(\mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{T}_\lambda\}$, are defined as in (2.13) as Fourier multipliers:

$$((\partial \pm \kappa)^k f)\widehat{}(\xi) = (i\xi \pm \kappa)^k \hat{f}(\xi).$$

Bounds in L^2-based Sobolev spaces are immediate from Plancherel's theorem. We denote $R_\pm = (\partial \pm \kappa)^{-1}$, which have the kernels on the real line given by (2.16). On the circle, by the Poisson summation formula (cf. [107, Lemma 3.3]) we find the kernels of $R_\pm$ to be

$$k_-^\lambda(\kappa, x, y) = -\frac{e^{\kappa((x-y) - 2\pi\lambda \lceil \frac{x-y}{2\pi\lambda} \rceil)}}{1 - e^{-2\pi\lambda\kappa}}, \quad k_+^\lambda(\kappa, x, y) = \frac{e^{\kappa((y-x) - 2\pi\lambda \lceil \frac{y-x}{2\pi\lambda} \rceil)}}{1 - e^{-2\pi\lambda\kappa}}, \tag{3.7}$$

where $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ denotes the ceiling function given by $\lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}$. We note the following identity:

$$k_-^\lambda(\kappa, x, y)^2 = \frac{1 + e^{-2\pi\lambda\kappa}}{1 - e^{-2\pi\lambda\kappa}} k_-^\lambda(2\kappa, x, y). \tag{3.8}$$

3.3 The Perturbation Determinant for the dNLS

In this section we show conservation of the perturbation determinant

$$\alpha(\kappa; q) = \mathrm{Re} \sum_{l \geq 1} \frac{(-i)^{l+1} \kappa^l}{l} \mathrm{tr}((\partial - \kappa)^{-1} q (\partial + \kappa)^{-1} \bar{q})^l) = \sum_{l \geq 1} \alpha_l \tag{3.9}$$

through term-by-term differentiation. For the first term we note the following:

Lemma 3.3. *The following identities hold for $f, g \in \mathcal{S}$:*

$$\mathrm{tr}((\kappa - \partial)^{-1} f (\kappa + \partial)^{-1} g) = \begin{cases} \langle (2\kappa - \partial)^{-1} f, \bar{g} \rangle, & \text{if } \mathbb{K} = \mathbb{R}, \\ \frac{1 + e^{-2\pi\lambda\kappa}}{1 - e^{-2\pi\lambda\kappa}} \langle (2\kappa - \partial)^{-1} f, \bar{g} \rangle, & \text{if } \mathbb{K} = \mathbb{T}_\lambda. \end{cases} \tag{3.10}$$

Proof. We begin with the line case. Using the explicit kernels (2.16), we find

$$\begin{aligned} \mathrm{tr}((\kappa - \partial)^{-1} f (\kappa + \partial)^{-1} g) &= \iint_{\mathbb{R}^2 \cap \{x < y\}} e^{2\kappa(x-y)} f(y) g(x) \, dx dy \\ &= \langle (2\kappa - \partial)^{-1} f, \bar{g} \rangle, \end{aligned}$$

the last line using the $L^2(\mathbb{R})$ scalar product. In the circle case, (3.8) yields

$$\begin{aligned}\operatorname{tr}((\kappa-\partial)^{-1}f(\kappa+\partial)^{-1}g) &= \iint_{\mathbb{T}_\lambda^2} k_-^\lambda(\kappa,x,y)f(y)k_+^\lambda(\kappa,y,x)g(x)\,dxdy \\ &= \frac{1+e^{-2\pi\lambda\kappa}}{1-e^{-2\pi\lambda\kappa}}\iint_{\mathbb{T}_\lambda^2} k_-^\lambda(2\kappa,x,y)f(y)g(x)\,dxdy \\ &= \frac{1+e^{-2\pi\lambda\kappa}}{1-e^{-2\pi\lambda\kappa}}\langle(2\kappa-\partial)^{-1}f,\overline{g}\rangle.\end{aligned}$$

□

To ensure that we can differentiate α term by term, we show the following trace estimates leading to geometric convergence:

Lemma 3.4. *Let $q\in\mathcal{S}(\mathbb{R})$ or $q\in\mathcal{S}(\mathbb{T}_\lambda)$, $\lambda\geq 1$, $l\geq 2$, and $\kappa>0$. Then*

$$\left|\operatorname{tr}\{[\kappa(\kappa-\partial)^{-1}q(\kappa+\partial)^{-1}\overline{q}]^l\}\right| \lesssim \begin{cases}(\kappa^{-s}\|q\|_{H^s})^{2l} & (0\leq s<1/4),\\ (\kappa^{-1/4}\|q\|_{H^s})^{2l} & (s\geq 1/4).\end{cases} \tag{3.11}$$

Hence, for $\|q\|_{L^2}\leq c\ll 1$ small enough, or $\kappa\gg\|q\|_{H^s}^{1/a}$ for $s>0$ and $a=\min(1/4,s)$, α defined in (3.9) converges geometrically.

Proof. The L^2-estimate in (3.11) follows by

$$\|R_\pm q\|_{\mathfrak{I}_2}^2 \sim \frac{1}{\kappa}\|q\|_{L^2}^2, \tag{3.12}$$

To show the above display, we note that the kernel is given by $K(x,y)=k_\pm(x,y)q(y)$, and we compute by Fubini and Plancherel's theorem

$$\begin{aligned}\iint|K(x,y)|^2dxdy &= \int dy|q(y)|^2\int dx|k_\pm(x,y)|^2 \\ &= \int dy|q(y)|^2\int\frac{d\xi}{\kappa^2+\xi^2}\sim\frac{\|q\|_{L^2}^2}{\kappa}.\end{aligned}$$

Observe that the argument works in either geometry and in the periodic case gives a bound independent of the period length.

For the H^s-part, set $A=\kappa^{1/2}R_\pm q$. Firstly, we argue that we can gain powers of κ by estimating q in H^s-norms. Note that

$$\|A\|_{\mathfrak{I}_2}\lesssim\|q\|_2,\qquad \|A\|_{\mathfrak{I}_\infty}\lesssim\kappa^{-1/2}\|q\|_\infty. \tag{3.13}$$

The first estimate follows from (3.12) and the second follows from viewing q as a multiplication operator in L^2 and the bound $\|R_\pm\|_{L^2\to L^2}\lesssim\kappa^{-1}$, which is immediate from Plancherel's theorem. Interpolating the estimates in (3.13) by viewing A as a bounded operator from $L^p\to\mathfrak{I}_p$ (cf. [11, Proposition I.1]) and using Sobolev embedding, we find

$$\|A\|_{\mathfrak{I}_p}\lesssim\kappa^{-1/2+1/p}\|q\|_p\lesssim\kappa^{-s}\|q\|_{H^s}\text{ for }s=1/2-1/p,\quad 2\leq p<\infty.$$

Let $0 < s' < 1/4$ in the following and set $s' = \frac{1}{2} - \frac{1}{p_{s'}}$. By Hölder's inequality and embeddings for Schatten spaces, we find

$$|\alpha_2| \leq |\mathrm{tr}((A\bar{A})^2)| \leq \|A\|_{\mathfrak{I}_4}^4 \leq \|A\|_{\mathfrak{I}_{p_{s'}}}^4 \lesssim \kappa^{-4s'} \|q\|_{H^{s'}}^4.$$

Similarly for the higher order terms $l \geq 3$, we find

$$|\mathrm{tr}((A\bar{A})^l)| \leq \|A\|_{\mathfrak{I}_{2l}}^{2l} \leq \|A\|_{\mathfrak{I}_{p_{s'}}}^{2l} \lesssim \kappa^{-2ls'} \|q\|_{H^{s'}}^{2l}.$$

This is the H^s-estimate in (3.11) for $0 < s < 1/2$. Lastly, note that this implies that the series (3.9) converges for $q \in H^s$, $s > 0$ by choosing $\kappa \gg \|q\|_{H^s}^{1/a}$ for $a = \min(1/4, s)$ as claimed. $\square$

Next, we show that α is conserved by term-by-term differentiation.

Proposition 3.5. *Let $q \in C^\infty(\mathbb{R}; \mathcal{S})$ be a smooth global solution to* (3.1) *with $\|q(0)\|_2 \leq c \ll 1$. Then*

$$\frac{d}{dt}\alpha(\kappa; q) = 0.$$

Remark 3.6. *By Lemma 3.4 $\alpha(\kappa; q)$ converges* without smallness assumption *on the L^2-norm, but provided that κ is sufficiently large. However, we are not able to show bounds for the $B^s_{2,r}$-norm without L^2-smallness assumption.*

Proof. In the following we omit taking the real part in (3.4) and will thus show that both real and imaginary part are conserved. Consider

$$\sum_{l=1}^{\infty} \frac{(-i)^{l+1}\kappa^l}{l} \mathrm{tr}(\underbrace{(\partial - \kappa)^{-1}}_{R_-} q \underbrace{(\partial + \kappa)^{-1}}_{R_+} \bar{q})^l = \sum_{l \geq 1} \alpha_l.$$

We note similar to the considerations from [77, Section 4]:

$$\begin{aligned} (|q|^2 q)_x &= (\partial - \kappa)(|q|^2 q) - (|q|^2 q)(\partial + \kappa) + 2\kappa(|q|^2 q) \\ (|q|^2 \bar{q})_x &= (\partial + \kappa)(|q|^2 \bar{q}) - (|q|^2 \bar{q})(\partial - \kappa) - 2\kappa(|q|^2 \bar{q}), \end{aligned} \tag{3.14}$$

and furthermore,

$$\begin{aligned} q_{xx} &= q(\partial^2 - 2\kappa\partial - \kappa^2) + (\partial^2 + 2\kappa\partial - \kappa^2)q + 2(\kappa - \partial)q(\kappa + \partial), \\ \bar{q}_{xx} &= (\partial^2 - 2\kappa\partial - \kappa^2)\bar{q} + \bar{q}(\partial^2 + 2\kappa\partial - \kappa^2) + 2(\kappa + \partial)\bar{q}(\kappa - \partial). \end{aligned} \tag{3.15}$$

Differentiating term-by-term, we find two terms $\frac{d}{dt}\alpha_l = A_l + B_l$, which are given by

$$\begin{aligned} A_l &= -(-i)^{l+1}\kappa^l \mathrm{tr}((R_- q R_+ \bar{q})^{l-1}[R_-(|q|^2 q)_x R_+ \bar{q} + R_- q R_+ (|q|^2 \bar{q})_x] \\ B_l &= (-i)^{l+1}\kappa^l \mathrm{tr}((R_- q R_+ \bar{q})^{l-1}[R_- i q_{xx} R_+ \bar{q} - i R_- q R_+ \bar{q}_{xx}]). \end{aligned}$$

We show that $A_l + B_{l+1} = 0$ by substituting (3.14) and (3.15). However, with the substitutions introducing differential operators, we have to check that the single terms are well-defined and cycling the trace is admissible. Strictly speaking, already writing A_l and B_l as in the above display requires cycling the trace. Since $R_- q R_+ \bar{q}$, $R_-(|q|^2 q)_x R_+ \bar{q}$, and $R_- q_{xx} R_+ \bar{q}$ are of trace class, this is not an issue. We shall prove $A_l + B_{l+1} = 0$ for $l \geq 2$ by substitution and handle the terms A_1, B_1, and B_2 directly.

For A_l we find after substitution of (3.14):

$$\begin{aligned} A_l = -(-i)^{l+1}\kappa^l \mathrm{tr}((R_- q R_+ \bar{q})^{l-1}[&|q|^2 q R_+ \bar{q} - R_-|q|^4 + 2\kappa R_-|q|^2 q R_+ \bar{q} \\ &+ R_-|q|^4 - R_- q R_+ |q|^2 \bar{q} R_-^{-1} - 2\kappa R_- q R_+ |q|^2 \bar{q}]). \end{aligned} \tag{3.16}$$

With $R_- q R_+ \bar{q}$ being trace class, for $l \geq 2$ it is enough to check boundedness of the remaining six factors. With R_- and multiplication by q or $\bar{q}$ a bounded operator as $q \in \mathcal{S}$, it only remains to check boundedness of the fifth factor: $R_- q R_+ |q|^2 \bar{q} R_-^{-1}$. Now $R_-^{-1} : L^2 \to H^{-1}$ is bounded and so it is enough to see that multiplication with $|q|^2 \bar{q}$ is bounded in H^{-1} because R_+ is a bounded operator $H^{-1} \to L^2$. This follows from the estimate $\|fg\|_{H^{-1}} \lesssim \|f\|_{H^1} \|g\|_{H^{-1}}$, which is immediate by duality and the algebra property of $H^1(\mathbb{K})$. Hence, we can consider the traces of the single terms, and the second cancels the fourth term. For the fifth term, we compute by cycling the trace

$$\begin{aligned} &\mathrm{tr}(R_- q R_+ \bar{q}(R_- q R_+ \bar{q})^{l-2} R_- q R_+ |q|^2 \bar{q} R_-^{-1}) \\ &= \mathrm{tr}((R_- q R_+ \bar{q})^{l-2} R_- q R_+ |q|^2 \bar{q} R_-^{-1} R_- q R_+ \bar{q}) \\ &= \mathrm{tr}((R_- q R_+ \bar{q})^{l-2} R_- q R_+ |q|^4 R_+ \bar{q}), \end{aligned}$$

which thus cancels the first term. Hence, for $l \geq 2$, we have proved

$$A_l = -(-i)^{l+1}\kappa^l \mathrm{tr}((R_- q R_+ \bar{q})^{l-1}[2\kappa R_-|q|^2 q R_+ \bar{q} - 2\kappa R_- q R_+ |q|^2 \bar{q}]). \tag{3.17}$$

For B_{l+1}, $l \geq 1$, we find after substitution of (3.15):

$$\begin{aligned} B_{l+1} = (-i)^{l+1}\, \kappa^{l+1}\, \mathrm{tr}((R_- q R_+ \bar{q})^l [&R_- q(\partial^2 - 2\kappa\partial - \kappa^2) R_+ \bar{q} \\ &+ R_-(\partial^2 + 2\kappa\partial - \kappa^2) q R_+ \bar{q} - 2|q|^2 - R_- q R_+ (\partial^2 - 2\kappa\partial - \kappa^2)\bar{q} \\ &- R_- q R_+ \bar{q}(\partial^2 + 2\kappa\partial - \kappa^2) - 2R_-|q|^2(\kappa - \partial)]). \end{aligned} \tag{3.18}$$

We have to verify that the traces of the single terms are well-defined, for which it is again enough to see the boundedness of the six factors with $R_- q R_+ \bar{q}$ being trace class. This follows similarly to the above. Consider e.g. the first term $R_- q(\partial^2 - 2\kappa\partial + \kappa^2) R_+ \bar{q}$. With $(\partial^2 - 2\kappa\partial + \kappa^2) R_+ : L^2 \to H^{-1}$ bounded and multiplication with q or $\bar{q}$ bounded in H^s, $s \in \mathbb{R}$, we find that $q(\partial^2 - 2\kappa\partial + \kappa^2) R_+ \bar{q} : L^2 \to H^{-1}$ is bounded. Composition with R_- yields L^2-boundedness.

Next, observe that the first and fourth term cancel because constant coefficient differential operators are commuting. This also implies cancelling of the second and fifth term, after additionally cycling the trace:

$$\begin{aligned} &\mathrm{tr}((R_- q R_+ \bar{q})(R_- q R_+ \bar{q})^{l-1} R_- q R_+ \bar{q}(\partial^2 + 2\kappa\partial - \kappa^2)] \\ &= \mathrm{tr}((R_- q R_+ \bar{q})^l (\partial^2 + 2\kappa\partial - \kappa^2) R_- q R_+ \bar{q}). \end{aligned}$$

To summarize, we have found

$$B_{l+1} = (-i)^{l+1}\, 2\kappa^{l+1}\, \mathrm{tr}((R_- q R_+ \bar{q})^l [-|q|^2 + R_- |q|^2 R_-^{-1}]). \tag{3.19}$$

The first term in (3.19) is cancelled by the second term of (3.17), and the second term in (3.19) is cancelled by the first term of (3.17) after additionally cycling the trace.

It remains to prove $B_1 = 0$ and $A_1 + B_2 = 0$. The first claim was already shown in [77, Eq. (51)] and follows from similar considerations as below.

We turn to the second claim: The substitution (3.14) in A_1 cannot easily be justified[1]; for B_2 (3.19) remains correct. Hence, we resort to doing the integration by parts in A_1 directly. Recall

$$A_1 = \kappa \mathrm{tr}[R_-(|q|^2 q)_x R_+ \bar{q} + R_- q R_+ (|q|^2 \bar{q})_x].$$

By Lemma 3.3, we find

$$\begin{aligned}
\mathrm{tr}(R_-(|q|^2 q)_x R_+ \bar{q}) &= -\langle (2\kappa - \partial)^{-1} (|q|^2 q)_x, q \rangle \\
&= -\langle (2\kappa - \partial)^{-1} [(\partial - 2\kappa) + 2\kappa](|q|^2 q), q \rangle \\
&= \langle |q|^2 q, q \rangle - 2\kappa \langle (2\kappa - \partial)^{-1} (|q|^2 q), q \rangle \\
&= \int |q|^4 dx + 2\kappa \mathrm{tr}(R_- |q|^2 q R_+ \bar{q}).
\end{aligned}$$

In a similar spirit, we compute

$$\begin{aligned}
\mathrm{tr}(R_- q R_+ (|q|^2 \bar{q})_x) &= -\langle (2\kappa - \partial)^{-1} q, (|q|^2 q)_x \rangle \\
&= \langle (2\kappa - \partial)^{-1} q_x, |q|^2 q \rangle \\
&= \langle (2\kappa - \partial)^{-1} (\partial - 2\kappa + 2\kappa) q, |q|^2 q \rangle \\
&= -\langle q, |q|^2 q \rangle + 2\kappa \langle (2\kappa - \partial)^{-1} q, |q|^2 q \rangle \\
&= -\int |q|^4 dx - 2\kappa \mathrm{tr}(R_- q R_+ |q|^2 \bar{q}).
\end{aligned}$$

With the L^4-norms cancelling, we conclude

$$A_1 = 2\kappa^2 (\mathrm{tr}(R_- |q|^2 q R_+ \bar{q} - R_- q R_+ |q|^2 \bar{q})) = -B_2.$$

The proof is complete. $\square$

3.4 Conservation of Besov Norms with Positive Regularity Index

In the following, we want to construct Besov norms from the leading term of $\alpha(q;\kappa)$. Set

$$w(\xi, \kappa) = \frac{\kappa^2}{\xi^2 + 4\kappa^2} - \frac{(\kappa/2)^2}{\xi^2 + \kappa^2} = \frac{3\kappa^2 \xi^2}{4(\xi^2 + \kappa^2)(\xi^2 + 4\kappa^2)}$$

[1] The author thanks the referee of the article [82] for pointing this out.

and

$$\|f\|_{Z^s_r} = \Big(\sum_{N\in 2^{\mathbb{N}}} N^{rs} \langle f, w(-i\partial_x, N) f\rangle^{r/2}\Big)^{1/r}. \tag{3.20}$$

The Z^s_r–norm consists of homogeneous components, which can be linked to the perturbation determinant. We will use the identities [77, Eq. (40), (55)]:

$$\|f\|_{B^s_{2,r}} \lesssim_s \|f\|_{H^{-1}} + \|f\|_{Z^s_r}, \tag{3.21}$$

$$\|f\|_{Z^s_r} \lesssim \|f\|_{B^s_{2,r}}. \tag{3.22}$$

Consequently, it suffices to control the Z^s_r-norm to infer about the Besov norms.

Remark 3.7. *In the Z^s_r-quantities introduced in [77], there is an additional parameter κ_0. One might hope that this flexibility helps to obtain a result for arbitrary initial data. However, κ_0 enters with a positive exponent into the estimates. This reflects indeed the relation of κ_0 with rescaling and the L^2-criticality of* (3.1)*. To keep things simple, we choose $\kappa_0 = 1$.*

3.4.1 The Line Case

To analyze the growth of the Z^s_r-norm, we link the multiplier from above with the first term of α. We recall the following identity on the real line, which is immediate from Lemma 3.3:

Corollary 3.8. *For $\kappa > 0$ and $q \in \mathcal{S}$, we find*

$$\mathrm{Re}\big(\kappa \operatorname{tr}\left((\kappa-\partial)^{-1} q (\kappa+\partial)^{-1}\bar{q}\right)\big) = \int \frac{2\kappa^2 |\hat{q}(\xi)|^2}{\xi^2+4\kappa^2} d\xi.$$

This yields

$$\begin{aligned}\langle f, w(-i\partial_x, N) f\rangle &= \int \frac{N^2}{\xi^2+4N^2} |\hat{f}(\xi)|^2 d\xi - \int \frac{(N/2)^2}{\xi^2+N^2} |\hat{f}(\xi)|^2 d\xi \\ &= \frac{1}{2}[\alpha_1(N,f) - \alpha_1(N/2,f)].\end{aligned}$$

We can estimate $|\alpha - \alpha_1|$ favorably by (3.11)

$$\big|\sum_{l\geq 2} \alpha_l(\kappa, q(t))\big| \lesssim \kappa^{-4s'} \|q(t)\|^4_{H^{s'}} \tag{3.23}$$

provided that $0 < s' < 1/4$ and $\|q(t)\|_{H^{s'}} \leq d_{s'} \ll 1$. Let $D_{s,r}$ denote the constant such that

$$\|q(t)\|_{B^s_{2,r}} \leq D_{s,r}(\|q(0)\|_{L^2} + \|q(t)\|_{Z^s_r}) \tag{3.24}$$

by (3.21) and L^2-conservation.

(3.23) gives by the embedding $B^s_{r,2} \hookrightarrow H^{s'}$ for $s > s'$ and $r \in [1,\infty]$

$$\begin{aligned}&\langle q(t), w(-i\partial_x, N) q(t)\rangle \\ &\lesssim \langle q(0), w(-i\partial_x, N) q(0)\rangle + N^{-4s'} [\|q(t)\|^4_{H^{s'}} + \|q(0)\|^4_{H^{s'}}] \\ &\lesssim \langle q(0), w(-i\partial_x, N) q(0)\rangle + N^{-4s'} [\|q(t)\|^4_{B^s_{2,r}} + \|q(0)\|^4_{B^s_{2,r}}].\end{aligned}$$

Raising the estimate to the power $r/2$, multiplying with N^{rs}, and carrying out the dyadic sums over $N \in 2^{\mathbb{N}_0}$, we find

$$\|q(t)\|^r_{Z^s_r} \lesssim \|q(0)\|^r_{Z^s_r} + [\|q(t)\|^{2r}_{B^s_{2,r}} + \|q(0)\|^{2r}_{B^s_{2,r}}]$$

provided that we choose $s' < s < 2s'$. This can be satisfied for $0 < s < 1/2$. By (3.22) and L^2-conservation, we arrive at

$$\|q(t)\|_{Z^s_r} \leq C_{r,s} (\|q(0)\|_{Z^s_r} + \|q(0)\|^2_{L^2} + [\|q(t)\|^2_{Z^s_r} + \|q(0)\|^2_{Z^s_r}]) \tag{3.25}$$

with $C_{r,s} \geq 1$ provided that $\|q(t')\|_{B^s_{2,r}}$ is small enough for $t' \in [-t,t]$ such that (3.23) holds by $\|q(t')\|_{H^{s'}} \leq \|q(t')\|_{B^s_{2,r}}$.

(3.25) can be bootstrapped. Suppose that

$$\max(\|q(0)\|_{Z^s_r}, \|q(0)\|_{L^2}) \leq \varepsilon \ll 1,$$

where ε is chosen below as $\varepsilon = \varepsilon(C_{r,s}, D_{r,s})$. We prove that for any $t \in \mathbb{R}$

$$\|q(t)\|_{Z^s_r} \leq 2C_{r,s}\varepsilon. \tag{3.26}$$

For this purpose, let I denote the maximal interval containing the origin such that (3.26) holds for any $t \in I$. I is non-empty and closed due to continuity of $\|q(t)\|_{Z^s_r}$. Furthermore, I is open: For $t \in I$ (3.25) yields

$$\|q(t)\|_{Z^s_r} \leq C_{r,s}(\varepsilon + 2\varepsilon^2 + (2C_{r,s}\varepsilon^2)) \leq (3/2) C_{r,s}\varepsilon$$

by choosing $\varepsilon \leq (8C_{r,s})^{-1}$, but (3.25) hinges on veracity of (3.23). To guarantee this, we choose ε possibly smaller such that

$$\varepsilon D_{r,s}(1 + 2C_{r,s}) \leq c_{s'}.$$

By $\|q(t)\|_{H^{s'}} \leq \|q(t)\|_{B^s_{2,r}}$ and (3.24), this additionally shows that (3.25) is true. We conclude that $I = \mathbb{R}$.

This finishes the proof for initial data with small L^2- and Z^s_r-norm. The assumption $\|q(0)\|_{Z^s_r} \leq \varepsilon$ follows for initial data with smaller L^2-norm through rescaling $q_0(x) = \lambda^{-\frac{1}{2}} q_0(\lambda^{-1}x)$, $\lambda \in 2^{\mathbb{N}}$ by (3.22) and (3.6). The proof of Theorem 3.1 is complete in the line case.

3.4.2 The Circle Case

In this section we discuss the case of periodic boundary conditions. We shall rescale the circle, too, to accomplish smallness of the homogeneous norms. In [77], this was not necessary due to more freedom in the parameter κ. For the leading term in (3.4) we find from Lemma 3.3 (see also [107, Lemma 3.3]):

Corollary 3.9. *Let $\kappa \geq 1$ and $\lambda \geq 1$. Then we have*

$$\operatorname{Re}\operatorname{tr}\big(\kappa(\kappa-\partial)^{-1}q(\kappa+\partial)^{-1}\bar{q}\big) = \frac{1+e^{-2\pi\lambda\kappa}}{1-e^{-2\pi\lambda\kappa}} \int_{\mathbb{Z}_\lambda} \frac{2\kappa^2|\hat{q}(\xi)|^2}{4\kappa^2+\xi^2}(d\xi)_\lambda$$

for any smooth function q on $\mathbb{T}_\lambda$.

Set $C(\lambda, N) = (1+e^{-2\pi\lambda N})/(1-e^{-2\pi\lambda N})$. Clearly, $C(\lambda, N) \sim 1$ for $\lambda N \geq 1$. With w defined as above, we find

$$\langle f, w(-i\partial_x, N)f\rangle = \frac{1}{2}\Big[\frac{\alpha_1(N,f)}{C(\lambda,N)} - \frac{\alpha_1(N/2,f)}{C(\lambda,N/2)}\Big].$$

Suppose that $q \in C^\infty(\mathbb{R}\times\mathbb{T})$ is a solution to (3.1). Let q_λ denote the rescaled solution to (3.1):

$$q_\lambda : \mathbb{R}\times\mathbb{T}_\lambda \to \mathbb{C}, \quad q_\lambda(t,x) = \lambda^{-1/2}q(\lambda^{-2}t, \lambda^{-1}x).$$

With the conventions introduced above, the identities from Lemma 3.4 and Corollary 3.9 allow for the same error estimates as in the real line case uniformly for $\lambda \in 2^{\mathbb{N}_0}$. We arrive at

$$\|q_\lambda(t)\|_{Z^s_r} \lesssim_{r,s} \|q_\lambda(0)\|_{Z^s_r} + [\|q_\lambda(t)\|^2_{B^s_{2,r}} + \|q_\lambda(0)\|^2_{B^s_{2,r}}].$$

By L^2-conservation and estimating the $B^s_{r,2}$-norm in terms of the Z^s_r-norm:

$$\|q_\lambda(t)\|_{Z^s_r} \lesssim_{r,s} \|q_\lambda(0)\|_{Z^s_r} + \|q_\lambda(0)\|^2_{L^2_\lambda} + [\|q_\lambda(t)\|^2_{Z^s_r} + \|q_\lambda(0)\|^2_{Z^s_r}].$$

As in the real line case, smallness of the Z^s_r-norm can be achieved by taking $\lambda \to \infty$ provided that the L^2-norm of $q(0)$ is chosen small enough. Also, $\|q_\lambda(0)\|_{L^2(\mathbb{T}_\lambda)} = \|q(0)\|_{L^2}$. Hence, the continuity argument given in the line case proves global a priori estimates in the circle case for small L^2-norm of the initial data. The proof of Theorem 3.1 is complete.

Remark 3.10. *We shortly comment on how [75] were able to remove the smallness condition of the L^2 norm in Theorem 3.1. Write $q_N = P_N q$ where P_N is the projection onto the dyadic frequency N. Firstly they show*

$$\sum_{N\leq N_0} \|R_\pm q_N\|_{L^2\to L^2} \lesssim \kappa^{-1}\min(\sqrt{N}, \sqrt{\kappa})\|q\|_{L^2},$$

see [75, Lemma 2.4]. The proof of this statement works by combining (2.15) *with the operator estimate $\|R_\pm q_N\|_{L^2\to L^2} \lesssim \sqrt{N}\kappa^{-1}\|q_N\|_{L^2}$ which follows from Bernstein's inequality. Using this information and using κ as a frequency threshold they are able to show*

$$\sup_{q\in Q}\sqrt{\kappa}\|R_\pm q\|_{L^2\to L^2} \to 0 \quad \text{as} \quad \kappa\to\infty,$$

for every bounded and equicontinuous $Q \subset L^2$. In particular for every such subset Q they find $\kappa_0 \geq 1$ such that for $\kappa \geq \kappa_0$

$$\sup_{q \in Q} \sqrt{\kappa} \| R_\pm q \|_{L^2 \to L^2} \leq \frac{1}{2}.$$

This is applied to the series in (3.9) *and yields convergence independent of the size of the L^2 norm.*

Chapter 4

The Tooth Problem for the Nonlinear Schrödinger Equation

In this section we consider the wellposedness question for the defocusing nonlinear Schrödinger equation

$$\begin{aligned} iu_t + u_{xx} &= |u|^{p-1}u, \\ u(0) &= u_0 \in X \end{aligned} \tag{4.1}$$

with initial data in $X = H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})$. It is based on the joint work [81] of the author with Peer Kunstmann. Section 4.4 and some parts of Section 4.2 are new. In Section 4.2 we include more estimates on products of periodic and non-periodic functions compared to [81]. In 4.4 we extend the wellposedness theory from [81] to initial data in $\mathcal{S}(\mathbb{R}) + C^\infty(\mathbb{T})$.

The wellposedness problem for the NLS with non-decaying initial data has been an area of active research for many years now. One of its motivations is the propagation of signals in glass-fiber cables, where the cubic NLS is used as an approximate model equation [78]. In this model, the roles of space and time are reversed and the initial value u_0 describes the signal seen at a fixed point of the cable. Hence, periodic initial data can be understood as encoding, e.g., an infinite string of ones. Such a signal carries no information, and we want to consider signals where some of the ones have been overwritten by a zero. Following [31] this is done by adding a nonperiodic part $v_0 \in H^{s_1}(\mathbb{R})$ to the initial data $w_0 \in H^{s_2}(\mathbb{T})$. Global existence then translates to having no bound on the length of the cable.

From a mathematical point of view, there is a large number of directions by which NLS with non-decaying initial data has been approached, and we will only name some of them. The most classical one of them is purely periodic initial data, both in the general case [22] and even earlier in the integrable case $p = 3$

under assumptions on the spectral properties of the corresponding Lax operator [92]. A natural generalisation is to consider quasi-periodic, almost periodic and limit periodic initial data [50, 24, 106]. NLS with prescribed boundary value $\lim_{x\to\pm\infty}|u(x)| = 1$ is known as the Gross-Pitaevskii equation and describes Bose gases at zero temperature [78]. For these mentioned types of initial data, global results exist. Additionally, there are local results in the case of initial data lying in the modulation spaces $M^s_{\infty,2}(\mathbb{R})$ [13, 34] and the case of analytic initial data [47].

Our approach is to consider initial data which are the sum of a periodic and a decaying signal, i.e. $u_0 \in H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})$. We note that this type of data is more general than the periodic case, but is less general than for example $u_0 \in M^s_{\infty,2}(\mathbb{R})$. Local wellposedness results for this problem have been covered in [31, 33], and our main interest is to extend them to global solutions. These local solutions are constructed as follows: By writing $u = v + w \in H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})$ and using the fact that u satisfies (4.1), w is seen to also satisfy (4.1) but on the torus,

$$\begin{aligned} iw_t + w_{xx} &= |w|^{p-1}w, \\ w(0) &= w_0 \in H^{s_2}(\mathbb{T}), \end{aligned} \tag{4.2}$$

at least if u, v, w have suitable regularity in space and time. Hence v has to be the solution of the perturbed problem

$$\begin{aligned} iv_t + v_{xx} &= |v + w|^{p-1}(v + w) - |w|^{p-1}w, \\ v(0) &= v_0 \in H^{s_1}(\mathbb{R}), \end{aligned} \tag{4.3}$$

where $w \in C([0,\infty), H^{s_2}(\mathbb{T}))$ is the solution of (4.2). Equation (4.3) is a perturbation of NLS on the real line, and classical methods like Strichartz estimates can be used to establish local wellposedness [33]. This type of decomposition dates back at least to the works [23] and [132].

In order to extend local to global solutions, we only need to consider (4.3), because (4.2) is known to exhibit global solutions. The main problem here is that the conservation laws that exist both in the periodic and non-periodic case give rise to a conservation law for (4.3) with an L^1 part in it, for example we have formal conservation of

$$\int_{\mathbb{R}} |u|^2 - |w|^2 \, dx = \int_{\mathbb{R}} |v|^2 + 2\,\mathrm{Re}(v\bar{w}) \, dx,$$

but not of $\int |v|^2 \, dx$. As a consequence, these exact conservation laws are not applicable in the L^2-based setting. Instead, we want to make use of quantities for which we can control the growth rate. In this setting, the power of the nonlinearity plays a crucial role. Indeed, for the quadratic nonlinearity $p = 2$ [33] showed global wellposedness in low regularity by a Gronwall argument for $\int |v|^2 \, dx$. Such a straight forward calculation does not work anymore if $p > 2$.

To overcome this problem, we will work with higher regularity and assume

$v_0 \in H^1(\mathbb{R})$. Note that for odd integers p, the time-dependent Hamiltonian

$$H(q,r) = \int_{\mathbb{R}} q_x r_x + \frac{2}{p+1}\Big((q+w)^{\frac{p+1}{2}}(r+\bar{w})^{\frac{p+1}{2}} - |w|^{p+1}\Big) \\ -(q\bar{w}+rw)|w|^{p-1}\,dx, \tag{4.4}$$

gives rise to (4.3) as the Hamiltonian equation

$$iq_t = \frac{\delta}{\delta r}H(q,r)\Big|_{(q,r)=(v,\bar{v})}.$$

Our main idea is to make use of the formula

$$\frac{d}{dt}F(t,v(t)) = \{F,H\}(t,v(t)) + (\partial_t F)(t,v(t)), \tag{4.5}$$

which holds for Hamiltonian equations with Poisson bracket $\{\cdot,\cdot\}$ (see Appendix A.4 for more on the Hamiltonian formalism), and choosing F to be the Hamiltonian of the equation (4.3) itself. From (4.5) it follows that the time derivative of H with respect to the flow induced by itself is non-zero, but only contains time derivatives that fall on w, because this is the only explicitly time-dependent part of (4.4). We want to mention that these calculations were inspired by calculations performed in [48] and the described Hamiltonian formalism makes also transparent why they work there.

While the Hamiltonian structure lies implicit in all arguments used below, we choose not to use the Hamiltonian language in what follows. The reason for this is that the main difficulty in the calculation is not to calculate the formal time derivative, but rather to make sure that taking time derivatives is allowed. Still it may help the reader in understanding why the calculations work as they stand, as it helped the authors in doing so.

The section is organized as follows: in Section 4.1 we give a proof that global solutions on the torus exist, in Section 4.2 we prove that local solutions on the line exist, and in Section 4.3 we show that the local solutions on the line are global. Section 4.4 treats the case of smooth initial data in $\mathcal{S}(\mathbb{R}) + C^\infty(\mathbb{T})$. For the presentation we give simple and self-contained proofs. Compared to [81] in Section 4.2 we include more estimates on products of periodic and non-periodic functions, and Section 4.4 is new.

4.1 Global Periodic Solutions

The wellposedness theory of NLS with periodic data is a classical problem. When $2 \le p < 5$, Lebowitz, Rose and Speer [85] (refering to [53] for a proof) and later Bourgain [22] and Moyua and Vega [103] in lower regularity showed existence of local solutions. These solutions are global due to conservation of the $L^2(\mathbb{T})$-norm and the dependence of the guaranteed time of existence on $\|w_0\|_{L^2(\mathbb{T})}$.

Theorem 4.1 ([22],[103])**.** *If $2 \leq p \leq 3$, the Cauchy problem* (4.2) *is locally wellposed in $L^2(\mathbb{T})$ with uniqueness in $L^4([-T,T] \times \mathbb{T})$ for any $w_0 \in L^2(\mathbb{T})$.*

The guaranteed time of existence T depends only on $\|w_0\|_{L^2(\mathbb{R})}$.

The restriction $2 \leq p \leq 3$ comes from the fact that the estimate is only in L^4 and not in L^6 compared to the line (see Theorem 1.8). Similar wellposedness results in L^2 for $2 \leq p < 5$ can be obtained with the help of $X^{s,b}$ spaces, see for example [49, Section 3]. Using these techniques, the Chaichenets, Hundertmark, Kunstmann and Pattakos investigated the local wellposedness theory for $1 \leq p \leq 2$ [33]. Wellposedness in the mass-critical case $p = 5$ is an open problem [49, p. 93]. In general, there is wellposedness in $H^s(\mathbb{T}), s \geq 0$ if $2 \leq p < 1 + \frac{4}{1-2s}$ [22].

We only need wellposedness results in spaces $H^s(\mathbb{T})$, $s \geq 1$, which are far away from being critical for any p. To prove global existence in $H^s(\mathbb{T}), s > 1$, we first give the local result in $H^1(\mathbb{T})$ which due to conservation of energy becomes immediately global, and then argue by persistence of regularity.

Theorem 4.2 (Local wellposedness for w)**.** *Given $p \geq 2$, the Cauchy problem* (4.2) *is locally wellposed in $C^0([-T,T], H^1(\mathbb{T}))$ for any $w_0 \in H^1(\mathbb{T})$.*

The guaranteed time of existence T^* satisfies $T^* \gtrsim \|w_0\|_{H^1(\mathbb{T})}^{1-p}$.

Proof. Let $S(t) = e^{it\partial_x^2}$. The integral formulation of (4.2) is

$$w(t) = S(t)w_0 + i \int_0^t S(t-s)(|w|^{p-1}w)(s)ds. \tag{4.6}$$

We will show that the right-hand side is a contraction on

$$X_{R,T} = \{v \in C([0,T], H^1(\mathbb{T})) : \|v\|_{C([0,T],H^1(\mathbb{T}))} \leq R\}$$

where R, T will be chosen later. Note that

$$\partial_x\big(|f|^{p-1}f\big) = (p-1)|f|^{p-3}\operatorname{Re}(\bar{f}f_x)f + |f|^{p-1}f_x$$

Thus when $p \geq 1$, we find by Sobolev's embedding $H^1(\mathbb{T}) \subset L^\infty(\mathbb{T})$ that given $f \in H^1(\mathbb{T})$, $|f|^{p-1}f \in H^1(\mathbb{T})$ with $\||f|^{p-1}f\|_{H^1(\mathbb{T})} \lesssim \|f\|_{H^1(\mathbb{T})}^p$. Together with the fact that $S(t)$ is an isometry on $H^s(\mathbb{T})$ we can bound the $C([0,T], H^1(\mathbb{R}))$-norm of the right-hand side of (4.6) by

$$\|w_0\|_{H^1(\mathbb{T})} + cTR^p$$

for some constant $c > 0$ if $w \in X_{R,T}$. Choosing $R = 2\|w_0\|_{H^1(\mathbb{T})}$ deals with the first summand. For the second summand, we choose $T = (2c)^{-1}R^{1-p}$. This guarantees that the right-hand side defines a mapping on $X_{R,T}$. The contractive property is proven similarly: We have to estimate

$$\int_0^t S(t-s)\big(|w_1|^{p-1}w_1 - |w_2|^{p-1}w_2\big)\, ds \tag{4.7}$$

uniformly in $H^1(\mathbb{T})$. To this end, we bound

$$||w_1|^{p-1}w_1 - |w_2|^{p-1}w_2| \leq c|w_1 - w_2|\big(|w_1|^{p-1} + |w_2|^{p-2}\big)$$

and by adding a zero (see also Lemma 4.6 for a more general estimate)

$$\begin{aligned}|\partial_x(|w_1|^{p-1}w_1 - |w_2|^{p-1}w_2)| \leq c\big(&|w_1|^{p-1}|w_{1,x} - w_{2,x}| \\ &+ (|w_1|^{p-2} + |w_2|^{p-2})|w_1 - w_2||w_{2,x}|\big).\end{aligned}$$

This shows that we can estimate (4.7) in $C([0,T], H^1(\mathbb{R}))$ by

$$cTR^{p-1}\|w_1 - w_2\|_{H^1(\mathbb{T})}.$$

Hence the contractive property can be achieved by possibly making $T \sim R^{1-p}$ a bit smaller. $\square$

Combining Theorem 4.2 with conservation of the energy

$$E(w) = \int_{\mathbb{T}} \frac{1}{2}|w_x|^2 + \frac{1}{p+1}|w|^{p+1}\, dx \tag{4.8}$$

we obtain global wellposedness in $H^1(\mathbb{T})$. Indeed,

$$\begin{aligned}\|w_x(t)\|_{L^2}^2 - \|w_x(0)\|_{L^2}^2 &= \frac{2}{p+1}\big(\|w(0)\|_{L^{p+1}}^{p+1} - \|w(t)\|_{L^{p+1}}^{p+1}\big) \\ &\lesssim \|w(0)\|_{L^2}^{\frac{p+3}{2}}\|w_x(0)\|_{L^2}^{\frac{p-1}{2}} \leq \|w_0\|_{H^1}^{p+1}.\end{aligned}$$

Here, we are able to argue without any smallness condition on the L^2 norm, because we are considering the defocusing equation. This shows

$$\|w(t)\|_{H^1} \lesssim \|w_0\|_{H^1} + \|w_0\|_{H^1}^{\frac{p+1}{2}}. \tag{4.9}$$

We turn to higher regularity and will prove that the constructed solutions are in $C([-T,T], H^s(\mathbb{T}))$ if the initial data is in $H^s(\mathbb{T})$. As a byproduct, we obtain an exponential bound for these norms.

We need the following inequality, which was already used in the previous proof with $s = 1$. Recall the notation $[p] = \sup\{k \in \mathbb{Z}, k \leq p\}$. Then, if $0 \leq s \leq [p]$, respectively $0 \leq s < \infty$ when p is an odd integer, and $\mathbb{K} \in \{\mathbb{R}, \mathbb{T}\}$,

$$\||f|^{p-1}f\|_{H^s(\mathbb{K})} \lesssim \|f\|_{L^\infty(\mathbb{K})}^{p-1}\|f\|_{H^s(\mathbb{K})}. \tag{4.10}$$

The proof can be found in the Appendix (see Lemma A.14).

Theorem 4.3 (Global wellposedness for $w \in H^s(\mathbb{T})$)**.** *Let $p \geq 2$ and $w_0 \in H^s(\mathbb{T})$ for $1 \leq s \leq [p]$. Then the global solution of Theorem 4.2 is an $H^s(\mathbb{T})$-solution and satisfies*

$$\|w(t)\|_{H^s} \leq e^{c(\|w_0\|_{H^1(\mathbb{T})})t}\|w_0\|_{H^s(\mathbb{T})}, \tag{4.11}$$

for some constant $c(\|w_0\|_{H^1(\mathbb{T})})$. If p is an odd integer, the same holds for $1 \leq s < \infty$.

Proof. Let w satisfy the integral equation (4.6) on some time interval $[0,T]$. We take the $H^s(\mathbb{T})$ norm on both sides and estimate with the help of Lemma A.14

$$\begin{aligned}\|w(t)\|_{H^s} &\leq \|w_0\|_{H^s(\mathbb{T})} + \int_0^t \||w|^{p-1}w\|_{H^s(\mathbb{T})}\, dt' \\ &\leq \|w_0\|_{H^s(\mathbb{T})} + c\|w\|^{p-1}_{L^\infty([0,T]\times\mathbb{T})} \int_0^t \|w\|_{H^s(\mathbb{T})}\, dt'.\end{aligned}$$

(4.11) now follows from Gronwall's lemma, where we get by Sobolev's inequality and the bound from energy conservation (4.9),

$$\|w(t)\|^{p-1}_{L^\infty} \lesssim \|w(t)\|^{p-1}_{H^1} \lesssim \|w_0\|^{p-1}_{H^1(\mathbb{T}))} + \|w_0\|^{\frac{p^2-1}{2}}_{H^1(\mathbb{T}))} \sim c(\|w_0\|_{H^1(\mathbb{T})})$$

for the constant in the exponential. □

We end this section by noting that the bound on the $H^s(\mathbb{T})$ norm in Theorem 4.3 is not optimal, but sufficient in our case to prevent blow-up. For example in the integrable case $p = 3$, there are infinitely many Hamiltonians which give immediate control over the H^N norms for integer N.

4.2 Local Solutions on the Line

To show wellposedness of NLS-type equations on the line in $L^2(\mathbb{R})$, one usually makes use of Strichartz estimates. Indeed, the following result (both for the focusing and defocusing NLS) holds [33, Theorem 2]:

Theorem 4.4 ([33])**.** *If $1 \leq p < 5$, the Cauchy problem (4.3) is locally wellposed in $C([0,T], L^2(\mathbb{R})) \cap L^{\frac{4(p+1)}{p-1}}([0,T], L^{p+1}(\mathbb{R}))$ for any $v_0 \in L^2(\mathbb{R}), w_0 \in H^1(\mathbb{T})$.*

In the case $1 \leq p < 5$, the guaranteed time of existence T depends only on $\|v_0\|_{L^2(\mathbb{R})}$ and $\|w_0\|_{H^1(\mathbb{T})}$, whereas for $p = 5$ it depends on the profile of v_0 and $\|w_0\|_{H^1(\mathbb{T})}$.

The restriction $p \leq 5$ comes from the fact that the problem (4.1) is mass-supercritical when $p > 5$. On the other hand, we want to consider solutions in the energy space $H^1(\mathbb{R})$, and so we do not run into a supercritical range when making p large.

Both in [31] and [33], local wellposedness was shown under the assumption of w_0 being more regular than v_0. Indeed, Theorem 4.4 assumes $v_0 \in L^2(\mathbb{R})$ and $w_0 \in H^1(\mathbb{T})$, and it is clear that their techniques can also be used to obtain wellposedness if $v_0 \in H^1(\mathbb{R})$ and $w_0 \in H^2(\mathbb{T})$. This is due to the fact that the bound

$$\|vw\|_{H^s(\mathbb{R})} \lesssim \|v\|_{H^s(\mathbb{R})}\|w\|_{H^{s+1}(\mathbb{T})}$$

was used (see for example [31, Lemma 11]). A close inspection of the argument shows that a bound of the form

$$\|vw\|_{H^s(\mathbb{R})} \lesssim \|v\|_{H^s(\mathbb{R})}\|w\|_{H^{s+1/2+}(\mathbb{T})}$$

can be achieved by the same argument. On the other hand, the case $s = 1$ suggests that by localizing in space and using periodicity, one can also estimate with the same regularity s, because for example

$$\begin{aligned}\|vw'\|^2_{L^2(\mathbb{R})} &= \sum_{k\in\mathbb{Z}} \int_k^{k+1} |v|^2|w'|^2\,dx \le \|w'\|^2_{L^2(\mathbb{T})} \sum_{k\in\mathbb{Z}} \|v\|^2_{L^\infty([k,k+1))} \\ &\lesssim \|w\|_{H^1(\mathbb{T})} \sum_k \|v\|^2_{H^1([k,k+1))} = \|w\|_{H^1(\mathbb{T})}\|v\|_{H^1(\mathbb{T})}.\end{aligned}$$

This gives a strictly better bound than putting w' into $L^\infty(\mathbb{T}) = L^\infty(\mathbb{R})$ in the first place. With this remark at hand, and using the arguments from [31] and [33], we will obtain the local wellposedness results with initial data in $H^1(\mathbb{R}) + H^1(\mathbb{T})$.

First of all, we generalize this calculation to the case where s is not necessarily an integer. When we localize in space, working with the Fourier transform becomes cumbersome. Instead, we prove the bilinear estimate in two steps: first we prove it in high regularity via modulation spaces, and then we interpolate for the low regularity estimate.

Lemma 4.5. *Let v be a function and w be a periodic function. Let $s \ge 0$, then*

$$\|vw\|_{H^s(\mathbb{R})} \lesssim \begin{cases} \|v\|_{H^s(\mathbb{R})}\|w\|_{H^{\frac{1}{2}+}(\mathbb{T})}, & \text{if } 0 \le s \le 1/2, \\ \|v\|_{H^s(\mathbb{R})}\|w\|_{H^s(\mathbb{T})}, & \text{if } s > 1/2. \end{cases} \tag{4.12}$$

Proof. We begin with the case $s > 1/2$. Using the characterization of $H^s(\mathbb{R})$ as $M^s_{2,2}(\mathbb{R})$ and the Hölder-like inequality for modulation spaces (see Definition 5.4 and Lemma 5.10) shows

$$\|vw\|^2_{H^s(\mathbb{R})} \sim \|vw\|_{M^s_{2,2}(\mathbb{R})} \le \|v\|_{M^s_{2,2}(\mathbb{R})}\|w\|_{M_{\infty,1}(\mathbb{R})} + \|v\|_{M_{2,1}(\mathbb{R})}\|w\|_{M^s_{\infty,2}(\mathbb{R})},$$

and we reduced the problem to proving the estimates

$$\begin{aligned}\|w\|_{M_{\infty,1}(\mathbb{R})} &\lesssim \|w\|_{H^s(\mathbb{T})}, \\ \|w\|_{M^s_{\infty,2}(\mathbb{R})} &\lesssim \|w\|_{H^s(\mathbb{T})},\end{aligned}$$

for $s > 1/2$. By Theorem 5.5, the second estimate implies the first. Now if $w \in H^s(\mathbb{T})$ we can write

$$w(x) = \sum_{l\in\mathbb{Z}} a_l e^{ilx},$$

where

$$\sum_{l\in\mathbb{Z}} \langle l\rangle^{2s}|a_l|^2 < \infty.$$

Choose a family of isometric decomposition operators $\square_k$ as in Definition 5.3. Without loss of generality we may assume that the σ_k from (5.2) satisfy

$$\operatorname{supp}(\sigma_k) \subset \{|\xi - k| < 1\}.$$

Taking the full space Fourier transform, we obtain

$$\mathcal{F}(\Box_k w)(\xi) = \sigma(\xi - k) \sum_{l \in \mathbb{Z}} a_l \delta(\xi - l) = \sigma(\xi - k) a_k \delta(\xi - k) = c a_k \delta(\xi - k),$$

where $c = \sigma(0) > 0$. Thus,

$$\|w\|_{M^s_{\infty,2}} = \big\| \langle k \rangle^s \|\Box_k w\|_{L^\infty} \big\|_{l^2_k} \lesssim \big\| \langle k \rangle^s |a_k| \big\|_{l^2_k} = \|w\|_{H^s(\mathbb{T})}.$$

Now consider the case $s \leq 1/2$. Let $\varepsilon > \eta > 0$ be arbitrary. Then, the two bounds

$$\begin{aligned} \|vw\|_{L^2(\mathbb{R})} &\lesssim \|v\|_{L^2(\mathbb{R})} \|w\|_{H^{1/2+\varepsilon}(\mathbb{T})}, \\ \|vw\|_{H^{1/2+\eta}(\mathbb{R})} &\lesssim \|v\|_{H^{1/2+\eta}(\mathbb{R})} \|w\|_{H^{1/2+\varepsilon}(\mathbb{T})}, \end{aligned}$$

hold. Indeed, the first follows from Hölders inequality followed by Sobolev's inequality, and the second is the case $s > 1/2$. We read these two estimates as bounds on the linear operator

$$w \cdot : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \text{resp.} \quad H^{1/2+\eta}(\mathbb{R}) \to H^{1/2+\eta}(\mathbb{R}),$$

that acts by multiplication with the function w. Using the Riesz-Thorin interpolation theorem A.9 then yields

$$\|vw\|_{H^s(\mathbb{R})} \lesssim \|v\|_{H^s(\mathbb{R})} \|w\|_{H^{1/2+\varepsilon}(\mathbb{T})},$$

for all $s \leq 1/2$. Since ε was arbitrary, the claim follows. $\square$

There is also a way to keep being able to localize in space and still use fractional derivatives - that is the characterization of fractional Sobolev spaces via the Gagliardo semi-norm. For a great introduction to this characterization, see [42]. Using this, we can show a similar bound of the product that involves an ℓ^2-localized L^∞ norm of v. We believe that it is of independent interest and put it into the appendix, see Lemma A.15.

We need estimates on difference terms. This is done in the next lemma.

Lemma 4.6. *Let $p \geq 2$ and*

$$G(v_1, v_2, w) = |v_1 + w|^{p-1}(v_1 + w) - |v_2 + w|^{p-1}(v_2 + w).$$

There exists a constant $c > 0$ such that the following estimates hold:

$$|G(v_1, v_2, w)| \leq c|v_1 - v_2||(v_1, v_2, w)|^{p-1}, \tag{4.13}$$

$$\begin{aligned} |G(v_1, v_2, w)_x| \leq & c|v_{1,x} - v_{2,x}||(v_1, v_2, w)|^{p-1} + \\ & c|v_1 - v_2||(v_{1,x}, v_{2,x}, w_x)||(v_1, v_2, w)|^{p-2}. \end{aligned} \tag{4.14}$$

The same holds if we replace the function $|x|^{p-1}x$ by $|x|^p$ in all of the arguments.

Proof. Note that $\partial_s|f(s)|^p = p|f(s)|^{p-2}\operatorname{Re}(\bar{f}\partial_s f)$. By the fundamental theorem of calculus we can write with $v(s) = v_2 + s(v_1 - v_2) + w$

$$\begin{aligned} G(v_1, v_2, w) = (v_1 - v_2) &\int_0^1 |v(s)|^{p-1}\,ds \\ &+ (p-1)\int_0^1 |v(s)|^{p-3}v(s)\operatorname{Re}(v(s))\overline{(v_1 - v_2)})\,ds \end{aligned}$$

Since $|v_2 + s(v_1 - v_2) + w|^{p-1} \lesssim |v_1|^{p-1} + |v_2|^{p-1} + |w|^{p-1}$, this shows the first estimate. For the second estimate, the first summand takes care of when the derivative falls on $(v_1 - v_2)$. Moreover,

$$\left|\partial|v(s)|^{p-1}\right| \lesssim (|v_{1,x} + |v_{2,x}| + |w_x|)(|v_1|^{p-2} + |v_2|^{p-2} + |w|^{p-2})$$

which produces the second summand in the estimate coming from the derivative falling on the integrand. The case of $|x|^p$ instead of $|x|^{p-1}x$ is proven analogously. $\square$

Theorem 4.7. *Let $p \geq 2$. The Cauchy problem* (4.3) *is locally wellposed in $C([0,T], H^1(\mathbb{R}))$ for any $v_0 \in H^1(\mathbb{R}), w_0 \in H^s(\mathbb{T}), s \geq 1$.*

The guaranteed time of existence T^* depends only on the H^1 norms $\|v_0\|_{H^1(\mathbb{R})}$ and $\|w_0\|_{H^1(\mathbb{T})}$. More precisely,

$$T^* \gtrsim \min\left(\|v_0\|_{H^1(\mathbb{R})}^{1-p}, \|w_0\|_{H^1(\mathbb{T})}^{1-p}, \|w_0\|_{H^1(\mathbb{T})}^{-\frac{p^2-1}{2}}\right).$$

Proof. The proof is a Banach fixed point argument. If we let $S(t) = e^{it\partial_x^2}$, then the integral formulation of (4.3) is

$$v(t) = S(t)v_0 + i\int_0^t S(t-t')(|v+w|^{p-1}(v+w) - |w|^{p-1}w)\,dt'. \tag{4.15}$$

We will show that the right-hand side is a contraction on

$$X_{R,T} = \{v \in C([0,T], H^1(\mathbb{R})) : \|v\|_{C([0,T],H^1(\mathbb{R}))} \leq R\}$$

where R, T will be chosen later. We claim that

$$\||v+w|^{p-1}(v+w) - |w|^{p-1}w\|_{H^1(\mathbb{R})} \lesssim \|v\|_{H^1(\mathbb{R})}(\|w\|_{H^1(\mathbb{T})}^{p-1} + \|v\|_{H^1(\mathbb{R})}^{p-1}) \tag{4.16}$$

Indeed by Lemma 4.6 with $v_2 = 0$,

$$\left||v+w|^{p-1}(v+w) - |w|^{p-1}w\right| \lesssim |v|\Big(|v|^{p-1} + |w|^{p-1}\Big),$$

and

$$\begin{aligned} &\left|\partial\big(|v+w|^{p-1}(v+w) - |w|^{p-1}w\big)\right| \\ &\qquad \lesssim |v_x|\Big(|v|^{p-1} + |w|^{p-1}\Big) + |v||w_x|\Big(|v|^{p-2} + |w|^{p-2}\Big). \end{aligned}$$

The $L^2(\mathbb{R})$ norm of the first term and the first summand of the second term are seen to be bounded by the right-hand side of (4.16) simply by putting $w \in L^\infty(\mathbb{R})$ and Hölder. For the second summand, we localize in space and use Sobolev's inequality on the interval $[k, k+1]$ to find

$$\begin{aligned}
&\int_{\mathbb{R}} |v|^2 |w_x|^2 \left(|v|^{p-2} + |w|^{p-2}\right)^2 dx \\
&\qquad \leq \|w_x\|^2_{L^2(\mathbb{T})} \left(\|v\|^{p-2}_{L^\infty(\mathbb{R})} + \|w\|^{p-2}_{L^\infty(\mathbb{R})}\right)^2 \sum_k \|v\|^2_{L^\infty([k,k+1))} \\
&\qquad \lesssim \|w\|^2_{H^1(\mathbb{T})} \|v\|^2_{H^1(\mathbb{R})} \left(\|v\|^{p-2}_{H^1(\mathbb{R})} + \|w\|^{p-2}_{H^1(\mathbb{T})}\right)^2.
\end{aligned}$$

This proves (4.16)

For the $H^1(\mathbb{T})$ norm of w we have the energy bound (4.9), and together with the fact that $S(t)$ is an isometry on $H^1(\mathbb{R})$ we can bound the $C([0,T], H^1(\mathbb{R}))$-norm of the right-hand side of (4.15) by

$$\|v_0\|_{H^1(\mathbb{R})} + cTR\left(\|w_0\|^{p+1}_{H^1(\mathbb{T})} + \|w_0\|^{\frac{p^2-1}{2}}_{H^1(\mathbb{T})} + R^{p-1}\right)$$

if $v \in X_{R,T}$. Choosing $R = 2\|v_0\|_{H^1(\mathbb{R})}$ deals with the first summand, and for the second summand, we let

$$T \lesssim \min\left(\|w_0\|^{1-p}_{H^1(\mathbb{T})}, \|w_0\|^{-\frac{p^2-1}{2}}_{H^1(\mathbb{T})}, R^{1-p}\right)$$

be small enough. This guarantees that the right-hand side of (4.15) defines a mapping on $X_{R,T}$. The contractive property is proven similarly: If we keep the notation from Lemma 4.6 and use it, then

$$|G(v_1, v_2, w)| \lesssim |v_1 - v_2|(|v_1|^{p-1} + |v_2|^{p-1} + |w|^{p-1}),$$

and

$$\begin{aligned}
|G(v_1, v_2, w)_x| \lesssim\ & |v_{1,x} - v_{2,x}|(|v_1|^{p-1} + |v_2|^{p-1} + |w|^{p-1}) \\
& + |v_1 - v_2|(|v_{1,x}| + |v_{2,x}| + |w_x|)(|v_1|^{p-2} + |v_2|^{p-2} + |w|^{p-2})
\end{aligned}$$

When there is no derivative term falling on w, the $L^2(\mathbb{R})$ norm of G respectively G_x can be estimated by putting w in L^∞. The worst term is

$$\begin{aligned}
&\int |v_1 - v_2|^2 |w_x|^2 (|v_1|^{p-2} + |v_2|^{p-2} + |w|^{p-2})^2\, dx \\
&\quad \lesssim (\|v_1\|^{p-2}_{L^\infty} + \|v_2\|^{p-2}_{L^\infty} + \|w\|^{p-2}_{L^\infty})^2 \sum_k \|v_1 - v_2\|^2_{L^\infty([k,k+1))} \|w_x\|^2_{L^2(\mathbb{T})},
\end{aligned}$$

and we can estimate as before by Sobolev's embedding. Hence we find

$$\begin{aligned}
&\left\| \int_0^t S(t-t') G(v_1, v_2, w)\, dt' \right\|_{H^1(\mathbb{R})} \\
&\quad \lesssim \|v_1 - v_2\|_{L^\infty([0,T], H^1(\mathbb{R}))} T(R^{p-1} + \|w\|^{p-1}_{L^\infty([0,T], H^1(\mathbb{T}))}) \\
&\quad \lesssim \|v_1 - v_2\|_{L^\infty([0,T], H^1(\mathbb{R}))} T(R^{p-1} + \|w_0\|^{p-1}_{H^1(\mathbb{T})} + \|w_0\|^{\frac{p^2-1}{2}}_{H^1(\mathbb{T})})
\end{aligned}$$

In particular with the same relative smallness condition as before, we obtain a contraction on $X_{R,T}$. $\square$

As a byproduct of Theorem 4.7 we obtain a blow-up alternative for the solution v to (4.3): Denote by T^* the maximal time of existence. Then either $T^* < \infty$ and

$$\limsup_{t \to T^*} \|v(t)\|_{H^1(\mathbb{R})} = \infty, \tag{4.17}$$

or $T^* = \infty$.

Indeed, we see that if we had a maximal solution of (4.3) with the property $\limsup_{t \to T^*} \|v(t)\|_{H^1(\mathbb{R})} < \infty$, we could continue it to some time $T^* + \delta$ by Theorem 4.7, yielding a contradiction to its definition as a maximal solution.

4.3 Global Solutions on the Line

In this section, we will prove our main theorem. To this end, we define momentum M, energy E and Hamiltonian H as

$$M(v) = \int \frac{1}{2}|v|^2 \, dx, \qquad E(v) = \int \frac{1}{2}|v_x|^2 + \frac{1}{p+1}|v|^{p+1},$$

and

$$H(v) = \int \frac{1}{2}|v_x|^2 + \frac{1}{p+1}\big(|v+w|^{p+1} - |w|^{p+1} - (p+1)|w|^{p-1}\operatorname{Re}(v\bar{w})\big)\, dx.$$

Moreover, we introduce the notation $(f,g) = \operatorname{Re}\int_{\mathbb{R}} f(x)\bar{g}(x)\, dx$.

Theorem 4.8. *Let $v \in C^0([0,T), H^1(\mathbb{R}))$ be a solution of (4.3). Let $p \geq 3$ and $w_0 \in H^s(\mathbb{T})$ with*

- *$3/2 < s < \infty$, if $p = 3$,*
- *$5/2 < s < \infty$, if $p \geq 5$ is an odd integer and*
- *$5/2 < s \leq [p]$, else.*

Then there is a constant $C = C(T, \|v_0\|_{H^1}, \|w_0\|_{H^s}) > 0$ such that

$$\sup_{t\in[0,T)} M(v(t)) + E(v(t)) \leq C \tag{4.18}$$

In particular, (4.3) is globally wellposed.

The idea of the proof is that by (4.5), we know that there are no derivatives falling on v, and we essentially just have to count factors of v in the time derivative of H. Factors that depend on the L^∞ norm of w and its derivatives are allowed since by Theorem 4.3 it is bounded locally in time.

We will see that there are always more than two, but never too many factors of v. If there are too many factors of v, we are not able to estimate by $M^\alpha E^{1-\alpha}$

anymore, leading to a break down in the Gronwall argument. This is also the reason of the higher regularity assumption $w_0 \in H^{5/2+}(\mathbb{T})$ in the case $p > 3$, because the estimate

$$\int |v_x||v|^{p-1}\,dx \lesssim \|v_x\|_{L^2}^{2\alpha}\|v\|_{L^{p+1}}^{(1-\alpha)(p+1)}$$

only works for $p \leq 3$.

Before proving Theorem 4.8, we need additional estimates in the spirit of Lemma 4.6.

Lemma 4.9. *Let $p \geq 3$. There exists a constant $c = c(p) > 0$ such that the following estimates hold:*

$$\big||v+w|^{p-1} - |w|^{p-1} - (p-1)\operatorname{Re}(w\bar{v})|w|^{p-3}\big| \leq c|v|^2(|w|^{p-3} + |v|^{p-3}), \quad (4.19)$$

$$\begin{aligned} &\big||v+w|^{p+1} - |w|^{p+1} - (p+1)\operatorname{Re}(w\bar{v})|w|^{p-1} - |v|^{p+1}\big| \\ &\quad \leq c|v|^2|w|(|w|^{p-2} + |v|^{p-2}), \end{aligned} \quad (4.20)$$

$$\begin{aligned} &\big||v+w|^{p-1}(v+w) - |w|^{p-1}w - ((p-1)\operatorname{Re}(w\bar{v})|w|^{p-3}w + v|w|^{p-1})\big| \\ &\quad \leq c|v|^2(|w|^{p-2} + |v|^{p-2}). \end{aligned} \quad (4.21)$$

In particular, there exists a constant $c = c(p, \|w\|_{L^\infty(\mathbb{T})}) > 0$ such that

$$H \leq cM + E, \qquad E \leq cM + H. \quad (4.22)$$

Proof. We define $f(s,t) = |sv + tw|^q$. Note first that

$$\begin{aligned} \partial_s f(s,t) &= q|sv+tw|^{q-2}\operatorname{Re}((sv+tw)\bar{v}), \\ \partial_t f(s,t) &= q|sv+tw|^{q-2}\operatorname{Re}((sv+tw)\bar{w}), \\ \partial_s^2 f(s,t) &= q|sv+tw|^{q-2}|v|^2 + q(q-2)|sv+tw|^{q-4}\operatorname{Re}((sv+tw)\bar{v})^2, \\ \partial_s\partial_t f(s,t) &= q|sv+tw|^{q-2}\operatorname{Re}(v\bar{w}) \\ &\quad + q(q-2)|sv+tw|^{q-4}\operatorname{Re}((sv+tw)\bar{v})\operatorname{Re}((sv+tw)\bar{w}), \\ \partial_s^2\partial_t f(s,t) &= q(q-2)|sv+tw|^{q-4}\operatorname{Re}((sv+tw)\bar{v})\operatorname{Re}(v\bar{w}) \\ &\quad + q(q-2)|sv+tw|^{q-4}|v|^2\operatorname{Re}((sv+tw)\bar{w}), \\ &\quad + q(q-2)|sv+tw|^{q-4}\operatorname{Re}((sv+tw)\bar{v})\operatorname{Re}(v\bar{w}), \\ &\quad + q(q-2)(q-4)|sv+tw|^{q-6}\operatorname{Re}((sv+tw)\bar{v})^2\operatorname{Re}((sv+tw)\bar{w}). \end{aligned}$$

In particular, we see that for $0 \leq s,t \leq 1$,

$$\begin{aligned} |\partial_s^2 f(s,t)| &\lesssim |v|^2(|v|^{q-2} + |w|^{q-2}), \\ |\partial_s^2\partial_t f(s,t)| &\lesssim |v|^2|w|(|v|^{q-3} + |w|^{q-3}). \end{aligned}$$

Now the left-hand side of the first estimate is $f(1,1) - f(0,1) - \partial_s f(0,1)$ with $q = p-1$, and so the first estimate follows from the fundamental theorem of calculus,

$$\begin{aligned}|f(1,1)-f(0,1)-\partial_s f(0,1)| &= \Big|\int_0^1 \partial_s f(s,1)-\partial_s f(0,1)\,ds\Big|\\ &= \Big|\int_0^1\int_0^s \partial_s^2 f(s',1)\,ds'ds\Big| \lesssim |v|^2(|v|^{q-2}+|w|^{q-2}).\end{aligned}$$

For the second estimate we note that $f(0,0)=\partial_s f(0,0)=0$ and use the fundamental theorem of calculus three times to see

$$\begin{aligned}|f(1,1)-f(0,1)-f(1,0)-\partial_s f(0,1)| &= \Big|\int_0^1\int_0^1\int_0^s \partial_s^2\partial_t f(s',t)\,ds'dsdt\Big|\\ &\lesssim |v|^2|w|(|v|^{q-3}+|w|^{q-3}).\end{aligned}$$

The third estimate follows similarly by arguing with $g(s,t)=|sv+tw|^{p-1}(sv+tw)$. With these estimates, we use Hölder $\|v\|_{L^p}^p \le \|v\|_{L^2}^{2/(p-1)}\|v\|_{L^{p+1}}^{(p+1)(p-2)/(p-1)}$ and Young to see

$$\begin{aligned}|H(v)-E(v)| &\lesssim \int |v|^2|w|(|w|^{p-2}+|v|^{p-2})\,dx\\ &\lesssim \|w\|_{L^\infty(\mathbb{T})}^{p-1}M(v)+\|w\|_{L^\infty(\mathbb{T})}\|v\|_{L^p(\mathbb{R})}^p\\ &\lesssim \|w\|_{L^\infty(\mathbb{T})}^{p-1}M(v)+\|w\|_{L^\infty(\mathbb{T})}M(v)^{\frac{1}{p-1}}E(v)^{\frac{p-2}{p-1}}\\ &\lesssim \varepsilon E(v)+C(\varepsilon)M(v).\end{aligned}$$

Choosing ε small enough, we arrive at (4.22). □

In order to prove Theorem 4.8, we want to take time derivatives of E and M and hence of v. If $v_0\in H^1(\mathbb{R})$, then $v(t)\in H^1(\mathbb{R})$ and hence from (4.3) we see that $v_t\in H^{-1}(\mathbb{R})$ for all times. This is enough to rigorously calculate $\partial_t M=(v,v_t)$, by interpreting the involved integral as a dual pairing between H^1 and H^{-1}. For the bilinear part of the energy, $-(v_{xx},v_t)$, this does not suffice any more. Our solution is to employ a twisting trick (see for example [4] and [35]) and to work in the interaction picture, that is with the function $\psi(t)=e^{-it\partial_x^2}v(t)$.

Proof of Theorem 4.8. Fix $T>0$. We use Gronwall's lemma to obtain an exponential bound on $M+H$. By (4.22), this is enough to bound $M+E$ and hence the H^1 norm.

First of all, note that for any $2\le q\le p+1$, we have by Hölder and Young

$$\|v\|_{L^q}^q \le \|v\|_{L^2}^{2\frac{p-q+1}{p-1}}\|v\|_{L^{p+1}}^{(p+1)\frac{q-2}{p-1}} \le M(v)^{\frac{p-q+1}{p-1}}E(v)^{\frac{q-2}{p-1}} \le M(v)+E(v).$$

This shows that powers of v ranging from two to $p+1$ are allowed.

We begin with M and see, interpreting the integrals in the first line as a dual pairing between H^1 and H^{-1},

$$\begin{aligned}\partial_t M(v) &= (v, v_t) = (iv, -v_{xx} + |v+w|^{p-1}(v+w) - |w|^{p-1}w) \\ &= (iv, |v+w|^{p-1}(v+w) - |w|^{p-1}w) \\ &\lesssim E(v) + M(v) \lesssim H(v) + M(v).\end{aligned}$$

Here, the summand $(iv, -v_{xx})$ vanishes by partially integrating once and we used Lemma 4.6 in the last line. To calculate the time derivative of H, we start with a formal calculation for the bilinear part,

$$\partial_t \frac{1}{2} \int |v_x|^2\, dx = (v_t, -v_{xx}) = -\big(v_t, |v+w|^{p-1}(v+w) - |w|^{p-1}w\big),$$

using (4.3) to rewrite $-v_{xx}$ and $(v_t, iv_t) = 0$. As $v_t \in H^{-1}$ and $v_{xx} \in H^{-1}$, their product is not well defined and the middle step in the above calculation needs to be justified. To make it rigorous, we define $\psi(t) = e^{-it\partial_x^2} v(t) = S(-t)v(t)$. Recall that $v(t)$ satisfies (4.15), and hence $\psi(t)$ satisfies

$$\psi(t) = v_0 + \int_0^t S(-t')\big(|v+w|^{p-1}(v+w) - |w|^{p-1}w\big)\, dt'.$$

In particular, we see that

$$i\partial_t \psi = S(-t)(|v+w|^{p-1}(v+w) - |w|^{p-1}w)(t),$$

which shows $\psi \in C^1((0,T), H^1(\mathbb{R}))$. Since $S(-t)$ is an isometry on L^2 and commutes with derivatives, we calculate

$$\begin{aligned}&\partial_t \frac{1}{2} \int |v_x|^2 = -(\psi_{xx}, \psi_t) \\ &\quad = -\big(i\psi_{xx}, S(-t)(|v+w|^{p-1}(v+w) - |w|^{p-1}w)\big) \\ &\quad = -\big(iv_{xx}, |v+w|^{p-1}(v+w) - |w|^{p-1}w\big) \\ &\quad = -\big(v_t + i(|v+w|^{p-1}(v+w) - |w|^{p-1}w), |v+w|^{p-1}(v+w) - |w|^{p-1}w\big) \\ &\quad = -\big(v_t, |v+w|^{p-1}(v+w) - |w|^{p-1}w\big).\end{aligned}$$

which is well-defined as a dual pairing, because $|v+w|^{p-1}(v+w) - |w|^{p-1}w \in H^1(\mathbb{R})$ by Lemma 4.6.

For the nonlinear term of the Hamiltonian, we calculate

$$\begin{aligned}&\partial_t \frac{1}{p+1} \int |v+w|^{(p+1)} - |w|^{p+1} - (p+1)\operatorname{Re}(w\bar{v})|w|^{p-1}\, dx \\ &\qquad = \operatorname{Re} \int \big(|v+w|^{p-1}(v_t + w_t)(\bar{v} + \bar{w}) - |w|^{p-1}(w_t \bar{w}) \\ &\qquad\qquad - (w_t \bar{v}|w|^{p-1} + v_t \bar{w}|w|^{p-1} + (p-1) w\bar{v} \operatorname{Re}(w_t \bar{w})|w|^{p-3}\big) \\ &\qquad = \operatorname{Re} \int \big(|v+w|^{p-1}(\bar{v} + \bar{w})v_t - |w|^{p-1}\bar{w} v_t\big) + \int R,\end{aligned}$$

where the remainder R only carries time derivatives on w,

$$R = \operatorname{Re}\big(|v+w|^{p-1}(\bar{v}+\bar{w})w_t - |w|^{p-1}(\bar{w}+\bar{v})w_t\big) - (p-1)|w|^{p-3}\operatorname{Re}(w_t\bar{w})\operatorname{Re}(w\bar{v}).$$

Since the first summand cancels with the time derivative of the bilinear part, we arrive at $\partial_t H = \int R$ as predicted by (4.5). We conclude

$$\begin{aligned}\partial_t H &= \big(w_t, |v+w|^{p-1}(v+w) - |w|^{p-1}(v+w) - (p-1)\operatorname{Re}(w\bar{v})|w|^{p-3}w\big)\\ &= \big(w_t, v(|v+w|^{p-1} - |w|^{p-1})\big)\\ &\qquad + \big(w_t, w(|v+w|^{p-1} - |w|^{p-1} - (p-1)\operatorname{Re}(w\bar{v})|w|^{p-3})\big).\end{aligned}$$

We first argue how to handle this term in the case $p > 3$. In this case, we assumed $w_0 \in H^s(\mathbb{T})$, $s > 5/2+$ with corresponding upper bound depending on whether p is an odd integer or not. This means by Theorem 4.3 that we have local in time boundedness in L^∞ of w_{xx}, hence of w_t. Now using Lemma 4.9, this implies

$$\begin{aligned}\big|\partial_t H\big| &\lesssim \|w_t\|_{L^\infty}\int |v|^2(|v|^{p-2} + |w|^{p-2})\,dx\\ &\qquad + \|w_t\|_{L^\infty}\|w\|_{L^\infty}\int |v|^2(|v|^{p-3} + |w|^{p-3})\,dx\\ &\lesssim E(v) + M(v) \lesssim H(v) + M(v),\end{aligned}$$

with a bound depending on p, w_0, T. Gronwall gives

$$H(v(t)) + M(v(t)) \lesssim (H(v_0) + M(v_0))e^{Ct},$$

and proves the theorem for this case.

We turn to $p = 3$. By plugging in the equation (4.2) for w_t, we obtain two terms for each summand in $\partial_t H$, one with $|w|^2 w$ and one with w_{xx}. For the term with $|w|^2 w$, we estimate w in L^∞ and argue as above. The other two terms are

$$\begin{aligned}&\big(iw_{xx}, v(|v+w|^2 - |w|^2)\big) + \big(iw_{xx}, w(|v+w|^2 - |w|^2 - 2\operatorname{Re}(w\bar{v}))\big)\\ &\quad = \big(iw_{xx}, |v|^2 v + 2v\operatorname{Re}(v\bar{w}) + |v|^2 w\big).\end{aligned}$$

We integrate by parts once. This produces terms where the derivative falls on a copy of w, and terms where the derivative falls on a copy of v. The former case is handled just as above because $\|w_x\|_{L^\infty}$ is bounded locally in time by $w_0 \in H^{3/2+}$. In the latter case, we put v_x in L^2 and estimate

$$\begin{aligned}&\Big|\big(iw_x, (|v|^2 v)_x + 2v_x\operatorname{Re}(v\bar{w}) + 2v\operatorname{Re}(v_x\bar{w}) + (|v|^2)_x w\big)\Big|\\ &\qquad \lesssim \|w_x\|_{L^\infty}\big(\|v_x\|_{L^2}\|v\|_{L^4}^2 + \|w\|_{L^\infty}\|v_x\|_{L^2}\|v\|_{L^2}\big)\\ &\qquad \lesssim E(v) + E(v)^{\frac{1}{2}}M(v)^{\frac{1}{2}}\end{aligned}$$

which can be estimated by $E(v) + M(v)$ by Young. As before we conclude via Gronwall and finish the proof. $\square$

4.4 Smooth Data

In this section we expand the results of [81] and consider the case of smooth data,

$$u_0 = v_0 + w_0 \in \mathcal{S}(\mathbb{R}) + C^\infty(\mathbb{T}).$$

For simplicity we assume a cubic nonlinearity $p = 3$, though our results hold for general odd integer p. Our goal is to show that the solution u will be smooth as well. This is motivated by the fact that a similar result holds on the line [131], and it is not hard to see that if $w_0 \in C^\infty(\mathbb{T})$ the periodic part w satisfies

$$w \in C^\infty([0,\infty) \times \mathbb{T}) \tag{4.23}$$

as well. Indeed, by Theorem 4.3 we have $w \in L^\infty([0,T], H^k(\mathbb{T}))$ for all $T > 0, k \in \mathbb{N}$. Arguing as in the proof of Theorem 4.12, we can bootstrap this to obtain time regularity. Since the proof of Theorem 4.12 is more involved, we skip the proof of (4.23) here.

It remains to analyze v. To this end, our first step is to show persistence of regularity for v:

Theorem 4.10. *Let $s \in \mathbb{N}$, $s \geq 1$, $T > 0$. If $v_0 \in H^s(\mathbb{R})$ and if $w \in C^0([0,T], H^s(\mathbb{T}))$, then the solution to*

$$\begin{cases} iv_t + v_{xx} = |v+w|^2(v+w) - |w|^2 w, \\ v(0) = v_0. \end{cases} \tag{4.24}$$

satisfies $u \in C^0([0,T], H^s(\mathbb{R}))$.

Proof. By taking H^s norms in (4.15) and using that the Schrödinger propagator is an isometry on H^s we find

$$\|v(t)\|_{H^s(\mathbb{R})} \leq \|v_0\|_{H^s(\mathbb{R})} + 2\int_0^t \||v+w|^2(v+w) - |w|^2 w\|_{H^s(\mathbb{R})}\, ds.$$

We first use Lemma 4.5 to split off the periodic factors and then estimate

$$\|fg\|_{H^s(\mathbb{R})} \leq \|f\|_{H^s(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} + \|g\|_{H^s(\mathbb{R})}\|f\|_{L^\infty(\mathbb{R})},$$

for powers of v. This yields

$$\||v+w|^2(v+w) - |w|^2 w\|_{H^s(\mathbb{R})} \lesssim \|v\|_{H^s(\mathbb{R})}\big(\|v\|^2_{L^\infty(\mathbb{R})} + \|w\|^2_{H^s(\mathbb{T})}\big). \tag{4.25}$$

Hence,

$$\|v(t)\|_{H^s(\mathbb{R})} \leq \|v_0\|_{H^s(\mathbb{R})} + c\int_0^t \|v\|_{H^s(\mathbb{R})}\big(\|v\|^2_{L^\infty(\mathbb{R})} + \|w\|^2_{H^s(\mathbb{T})}\big)\, ds.$$

By Gronwall's Lemma A.7,

$$\|v(t)\|_{H^s(\mathbb{R})} \leq \|v_0\|_{H^s(\mathbb{R})}\big(1 + A(t)\big)e^{A(t)}$$

where

$$A(t) = \|v\|^2_{L^2([0,t],L^\infty(\mathbb{R}))} + \|w\|^2_{L^2([0,t],H^s(\mathbb{T}))}.$$

By the embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, Theorem 4.8 and our assumption on w, $A(t)$ is bounded on $[0,T]$. This shows $v \in L^\infty([0,T], H^s(\mathbb{R}))$.

It remains to show continuity. From (4.15) we find

$$v(t_1) - v(t_2) = (S(t_1) - S(t_2))v_0 - 2i \int_{t_2}^{t_1} (S(t_1 - s) - S(t_2 - s))G(v,w)\, ds,$$

where $G(v,w) = |v+w|^2(v+w) - |w|^2 w$. We take the H^s norm on both sides. For the first summand we see that

$$\|(S(t_1) - S(t_2))v_0\|_{H^s(\mathbb{R})} \to 0 \quad \text{as} \quad |t_1 - t_2| \to 0,$$

by the group property of $S(t)$ on $H^s(\mathbb{R})$. For the second summand, we estimate the $H^s(\mathbb{R})$ norm of the integrand as in (4.25) by

$$c\|v\|_{H^s(\mathbb{R})}\big(\|v\|^2_{L^\infty(\mathbb{R})} + \|w\|^2_{H^s(\mathbb{T})}\big).$$

Since $v \in L^\infty([0,T], H^s(\mathbb{R}))$ and $w \in C^0([0,T], H^s(\mathbb{R}))$ the integrand is integrable in t', and hence the integral vanishes as $|t_1 - t_2| \to 0$. $\square$

We turn to L^2 norms with spatial weights. Given $r, s \in \mathbb{R}$, and with the notation $\mathcal{F}(J^s v)(\xi) = \langle \xi \rangle^s v(\xi)$, we define

$$\begin{aligned} \|v\|_{H^s_r} &= \|\langle \cdot \rangle^r J^s v\|_{L^2}, \\ \|v\|_{S^s_r} &= \big(\|v\|^2_{H^s_0} + \|v\|^2_{H^0_r}\big)^{\frac{1}{2}}, \end{aligned} \tag{4.26}$$

and the corresponding spaces as the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norms. This notation follows the one given by Tsutsumi [131]. The proof of the following Theorem uses ideas which stem from a paper by Hayashi-Nakamitsu-Tsutsumi [66] and are standard by now.

Theorem 4.11. *Let $s, k \in \mathbb{N}$, $s \geq \max(k, 1)$, $T > 0$ arbitrary. If $v_0 \in S^s_k(\mathbb{R})$ and $w \in C^0([0,T], H^s(\mathbb{T}))$, then the solution to* (4.24) *satisfies $v \in C^0([0,T], S^s_k(\mathbb{R}))$.*

Proof. We have already seen that $u \in C^0([0,T], H^s_0)$ for all $T > 0$, $s \geq 0$ and it remains to show that $u \in C^0([0,T], H^0_k)$. Recall that on bounded time intervals, the H^s norm was shown to stay bounded,

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} \lesssim 1.$$

Let $P(t) = x + 2it\partial_x$. Then, $P(t)$ and $i\partial_t + \partial_x^2$ commute. Indeed,

$$[P(t), i\partial_t + \partial_x^2] = [x, \partial_x^2] + [2it\partial_x, i\partial_t] = -2\partial_x + 2\partial_x = 0.$$

This shows that if $\tilde{u}(t)$ is a solution to the homogeneous equation with initial data $\tilde{u}_0$, then $P(t)\tilde{u}(t)$ is a solution to the homogeneous equation with initial data $x\tilde{u}_0$. In formulas,

$$P(t)S(t) = S(t)x.$$

Taking adjoints reveals that

$$S(-s)P(s) = xS(-s),$$

and combining the two equations shows

$$P(t)S(t-s) = S(t)xS(-s) = S(t-s)P(s).$$

We apply this to the fixed-point equation (4.15),

$$P^k(t)v(t) = S(t)x^k v_0 - 2i\int_0^t S(t-s)P^k(s)(|v+w|^2(v+w)-|w|^2w)(s)\,ds. \tag{4.27}$$

First of all, we show that

$$\|x^k u\|_{L^2} \lesssim \|P^k(t)u\|_{L^2} + (1+t^k)\|u\|_{H^k}, \tag{4.28}$$

$$\|P^k(t)u\|_{L^2} \lesssim \|x^k u\|_{L^2} + (1+t^k)\|u\|_{H^k} \tag{4.29}$$

for all $u \in S_k^k$. Indeed, we can write

$$P^k(t) = (x+2it\partial_x)^k = x^k + \sum_{l=1}^{k-1}(2it)^l A_l + (2it)^k\partial_x^k,$$

where for example

$$A_1 = \sum_{l=0}^{k-1} x^{k-1-l}\partial_x x^l = \sum_{l=0}^{k-1} x^{k-1-l}(x^l\partial_x + lx^{l-1}) = kx^{k-1}\partial_x + \frac{(k-1)k}{2}x^{k-2},$$

$$A_2 = \sum_{l_1,l_2=0}^{k-1} x^{k-2-l_1-l_2}\partial_x x^{l_1}\partial_x x^{l_2} = c_2^0 x^{k-2}\partial_x^2 + c_2^1 x^{k-3}\partial_x + c_2^2 x^{k-4}.$$

In general there exist $c_l^m, 0 \le m \le \min(l, k-l)$ such that

$$A_l = \sum_{m=0}^{\min(l,k-l)} c_l^m x^{k-l-m}\partial_x^{l-m}.$$

Using Corollary A.13, Young's inequality with ε, and $x^{k-2m} \lesssim x^k + 1$, we estimate

$$\begin{aligned}
\|A_l u\|_{L^2} &\lesssim \sum_m \|\partial_x^{k-2m}u\|_{L^2}^{\frac{1}{k-2m}}\|x^{k-2m}u\|_{L^2}^{1-\frac{1}{k-2m}} \\
&\lesssim \varepsilon\sum_m \|x^{k-2m}u\|_{L^2} + C(\varepsilon)\sum_m \|\partial_x^{k-2m}u\|_{L^2} \\
&\lesssim \varepsilon\|x^k u\|_{L^2} + C(\varepsilon)\|u\|_{H^k}.
\end{aligned}$$

This shows

$$\|P^k(t)u - x^k u\|_{L^2} \lesssim \varepsilon \|x^k u\|_{L^2} + C(\varepsilon)(1+t^k)\|u\|_{H^k},$$

from which (4.28) and (4.29) can be deduced.

Taking L^2 norms in (4.27) and using (4.28) now shows

$$\|x^k v(t)\|_{L^2} \lesssim (1+t^k)\|v(t)\|_{H^k} + \|x^k v_0\|_{L^2} + \int_0^t \|P^k(s)G(v,w)(s)\|_{L^2}\,ds,$$

where $G(v,w) = |v+w|^2(v+w) - |w|^2 w$. By (4.29),

$$\begin{aligned}
&\|P^k(s)G(v,w)(s)\|_{L^2} \\
&\qquad \lesssim \|x^k G(v,w)(s)\|_{L^2} + (1+s^k)\|G(v,w)(s)\|_{H^k} \\
&\qquad \lesssim (\|v(s)\|_{L^\infty}^2 + \|w(s)\|_{H^s}^2)(\|x^k v(s)\|_{L^2} + (1+s^k)\|v(s)\|_{H^k}).
\end{aligned}$$

Using local in time boundedness of $\|v(s)\|_{L^\infty}^2$, $\|w(s)\|_{H^s}^2$, and $(1+s^k)\|v(s)\|_{H^k}$ shows $v \in L^\infty([0,T], H_k^0)$ via Gronwall.

To prove continuity, we infer from (4.27)

$$\begin{aligned}
P^k(t_1)v(t_1) - P^k(t_2)v(t_2) &= (S(t_1) - S(t_2))x^k v_0 \\
&\quad - 2i\int_{t_1}^{t_2} (S(t_1 - s) - S(t_2 - s))P^k(s)G(v,w)\,ds.
\end{aligned}$$

Since $x^k v_0 \in L^2$, the first summand vanishes by the group property of $S(t)$ on L^2 as $|t_1 - t_2| \to 0$. For the second summand we use that

$$P^k(s)G(v,w) \in L^\infty((0,T), L^2)$$

by the same estimate as before and since $v \in L^\infty([0,T], S_k^k)$. This shows that the L^2 norm of the integral vanishes as $|t_1 - t_2| \to 0$. Hence,

$$t \mapsto P^k(t)v(t) \in C^0([0,T], L^2).$$

Now to conclude, we write

$$P^k(t) - x^k = \sum_{l=1}^{k-1} (2it)^l A_l + (2it)^k \partial_x^k$$

as before. Arguing inductively, we may assume $t \mapsto A_l v(t) \in C^0([0,T], L^2)$ for all $l \le k-1$. Finally,

$$t \mapsto \partial_x^k v(t)$$

is known to be continuous into L^2 and we can conclude. □

Theorem 4.12. *If $v_0 \in \mathcal{S}(\mathbb{R})$ and $w \in C^\infty([0,\infty) \times \mathbb{T})$, then the solution to (4.24) satisfies $v \in C^\infty([0,\infty), \mathcal{S}(\mathbb{R}))$.*

Proof. By Theorem 4.11 the solution v is continuous as a map

$$[0,T] \to S_k^k$$

for all $k \in \mathbb{N}$, $T > 0$. In particular it is continuous as a map

$$[0,T] \to \mathcal{S}(\mathbb{R}),$$

for all $T > 0$. Moreover since continuity is a local property, we find $v \in C^0([0,\infty), \mathcal{S}(\mathbb{R}))$ and it remains to control the time derivatives. This is done by bootstrapping.

Consider the linear part first. Given a semigroup $T(t)$ with generator A, we have (defining $D(A^0) = X$)

$$t \mapsto T(t)x \in \bigcap_{j=0}^{k} C^j([0,\infty), D(A^j)) \quad \text{if} \quad x \in D(A^k),$$

see [109, Lemma 4.2]. In particular for all $k, m \in \mathbb{N}$,

$$t \mapsto T(t)x \in C^k([0,\infty), D(A^m)) \quad \text{if} \quad x \in D(A^\infty) = \cap_{n=0}^{\infty} D(A^n).$$

It follows that when $X = S_k^k$ for arbitrary k, and $v_0 \in \mathcal{S}(\mathbb{R})$,

$$t \mapsto S(t)v_0 \in C^\infty([0,\infty), X).$$

This shows the statement for the linear part of the equation.

Now if $v \in C^m([0,\infty), \mathcal{S}(\mathbb{R}))$, the integrand in the Duhamel term,

$$S(t-s)G(v(s), w(s)),$$

is C^m in $0 \le s \le t \le T$ into $S_k^s(\mathbb{R})$ for all $k \in \mathbb{N}$ and $T > 0$. Moreover, it is C^{m+1} in t into $S_k^{s-2}(\mathbb{R})$ by the properties of the linear operator. This implies that the map

$$(t_0, t) \mapsto \int_0^t S(t_0 - s)G(v(s), w(s))$$

is C^{m+1} in t_0 into S_k^{s-2} and C^{m+1} in t into $S_k^s(\mathbb{R})$ for all $s \ge k+2 \in \mathbb{N}, T > 0$. Hence its restriction to $t_0 = t$ satisfies

$$t \mapsto \int_0^t S(t-s)G(v(s), w(s)) \in C^{m+1}((0,T), S_k^{s-2}).$$

Since $s \ge k+2$ are chosen arbitrary, this shows that if $v \in C^m([0,\infty), \mathcal{S}(\mathbb{R}))$ then $v \in C^{m+1}([0,\infty), \mathcal{S}(\mathbb{R}))$. This concludes the proof. $\square$

Chapter 5

Wellposedness for the Nonlinear Schrödinger Equation in Modulation Spaces

In this section we investigate the Cauchy problem for the cubic NLS,

$$\begin{cases} iu_t + \Delta u = \pm 2|u|^2 u, \\ u(0,x) = u_0(x) \in X, \end{cases} \tag{5.1}$$

with initial data in a modulation space $X = M^s_{p,q}(\mathbb{R})$. It is based on the work [79] of the author. Section 5.1 is an extension of Section 2 from [79] and includes the analysis of modulation spaces in higher dimensions, as well as a proof of the boundedness of the Schrödinger propagator on modulation spaces.

Modulation spaces $M^s_{p,q}$ were introduced by Feichtinger [51] and have by now been used in the study of various different PDE, see also [113, 134]. One of the reasons why they serve as an interesting space of initial data is because the decay of functions in modulation spaces $M^s_{p,q}$ is comparable to the one of functions in L^p. In particular, the spaces with $p = \infty$ include non-decaying initial data and provide them with an elegant function space framework. In contrast to L^p and Besov spaces, the Schrödinger propagator is bounded on any modulation space $M^s_{p,q}$, and dispersive L^∞ blow-up phenomena as constructed in [20] can be ruled out. A major open problem in this context is whether global in time existence in $M^s_{\infty,q}$ can be guaranteed for certain s, q. Just to name one of the many consequences an affirmative answer would have, this would solve the question whether a local solution to

$$u_0(x) = \cos(x) + \cos(\sqrt{2}x),$$

can be continued globally. A unique local solution exists, e.g., by the work [47] in a space of analytic functions, or by Picard iteration in the space $M_{\infty,1}$. While we are not able to give an answer to this question, we are able to prove global results with arbitrarily large $p < \infty$. Among other results (see Theorem 5.1) we will show: *In the defocusing case, if* $1 \le p < \infty, 1 \le q \le \infty$ *and if* $s \ge 0$ *is large enough, there is a unique global solution of the cubic NLS in* $M^s_{p,q}(\mathbb{R})$.

Local wellposedness results for nonlinear Schrödinger equations with initial data in modulation spaces have first been proven in [133, 8, 13, 17]. These results rely on boundedness of the Schrödinger propagator and an algebra property which holds either when $s \ge 0, q = 1$, or when $s > 1 - 1/q$. Later, the works [59, 108, 32] increased the range of admissible p, q for $s = 0$ using refined trilinear estimates for $p = 2, 2 \le q < \infty$ and an infinite normal form reduction technique for $1 \le q \le 2, 2 \le p \le 10q'/(q' + 6)$, respectively. Using complete integrability of the cubic one-dimensional NLS, Oh-Wang [107] showed the solutions of [59] to be global. Global solutions for initial data in $M_{p,p'}$ with p sufficiently close to 2 were constructed in [30], though we note that these solutions were allowed to take value in a different space $M_{\tilde{p},\tilde{q}}$ for $t > 0$. Using decoupling techniques, Schippa [117] recently proved L^p smoothing estimates and extended the range of local wellposedness results for $p \in \{4, 6\}$ and also, inspired by the work [48], gave global results for $q = 2, 2 \le p < \infty, s > 3/2$. Finally we want to mention the preprint [116] in which Schippa very recently considered the energy-critical NLS with initial data in modulation spaces.

The goal of this section is twofold: On the one hand we want to give an overview of local wellposedness results and to unify the local results for $s = 0$. This is done by a Banach fixed point argument using multilinear interpolation on the estimates obtained in [59] and the trivial estimates for $q = 1$. From this we obtain local wellposedness in a range of (p, q), comprising all of the aforementioned range for $s = 0$ for which local wellposedness results were shown, except for the point $(p, q) = (4, 2)$ from [117]. The regularity $s = 0$ is sharp if we aim for analytic wellposedness by the considerations we provide in Section 5.5.

On the other hand, we aim to extend the range of (p, q) with global results, possibly assuming higher regularity of the initial data. To this end we first extend the almost conserved energies constructed in [107] to the range $p = 2, 1 \le q < 2$ and then use the principle of persistence of regularity to see that for a restricted range of $1 \le p, q \le 2$, the newly constructed local solutions are also global. Finally, we prove as in [117, 48] that in the defocusing case when we take $s \ge 1$, we obtain global solutions in $M^1_{p,1}$ for any $2 < p < \infty$. In fact, the same technique shows global wellposedness in $M^s_{p,q}$ for any $2 < p < \infty$ if $s > 2 - 1/q$ is large enough.

An overview of the wellposedness results achieved is given in Figure 5.1 and formulated in the following Theorem, also including the results described in Remarks 5.24 and 5.34:

Theorem 5.1. *For the cubic one-dimensional NLS* (5.1) *with initial data in a modulation space* $M^s_{p,q}(\mathbb{R})$, $s \in \mathbb{R}, 1 \le p, q \le \infty$ *we obtain*

1. *Local wellposedness in the sense of Definition 5.14 if* $s \geq 0$ *and at least one of the following condition holds,*
   - $s = 0, 1/q > |1 - 2/p|$,
   - $s = 0, p \geq 4$ *and* $1/q \geq 1 - 2/p$,
   - $s > 1 - 1/q$.
2. *Global wellposedness in the sense that the local solution exists for all times if* $s \geq 0$ *and at least one of the following condition holds,*
   - $s = 0, p = 2, 1 \leq q < \infty$,
   - $s = 0, 1/q \geq 1/p, 1 \leq p \leq 2$,
   - $s > 1 - 1/q, 1 \leq p \leq 2$,
   - $s = 1, q = 1, 2 \leq p < \infty$, *and* (5.1) *has a defocusing nonlinearity,*
   - $s > 2 - 1/q, 2 \leq p < \infty$, *and* (5.1) *has a defocusing nonlinearity.*
3. *Illposedness in the sense that the flow map cannot be* C^3 *at the origin if* $s < 0$.

Indeed, for $s = 0$, Theorem 5.23 and Remark 5.24 give the range of local wellposedness whereas global wellposedness is deduced from Theorem 5.27 and Lemma 5.29. If $s > 1 - 1/q$, local wellposedness in $M^s_{p,q}$ follows from the Banach algebra property of the space (see Theorem 5.5) and boundedness of the Schrödinger propagator (Lemma 5.11). Global wellposedness under the additional hypothesis $1 \leq p \leq 2$ then follows from Lemma 5.28, Theorem 5.5 and the almost conservation of the $M_{2,1}$ norm proven in Theorem 5.27. In the case of a defocusing nonlinearity Theorem 5.33 and Remark 5.34 give the remaining global wellposedness results. Illposedness is shown in Theorem 5.35.

This chapter is structured as follows: In Section 5.1 we state basic facts on modulation spaces, in Section 5.2 we introduce the notion of quantitative wellposedness which gives the analytical framework to obtain our local wellposedness results in Section 5.3. In Section 5.4 we prove the global results first for $p = 2$, then for $1 \leq p < 2$ and finally for $2 < p < \infty$. The wellposedness results are complemented by an illposedness result for $s < 0$ shown in Section 5.5.

5.1 Modulation Spaces

In this section we recall the definition of modulation spaces and state some results we need in later sections.

Modulation spaces were introduced by Feichtinger [51] in 1983 and have found growing interest in recent years. They can be introduced either via the short-time Fourier transform or equivalently via isometric decomposition on the Fourier side which also shows their close connection to Besov spaces. Modern introductions to modulation spaces are given in the books [56, 134], and we also

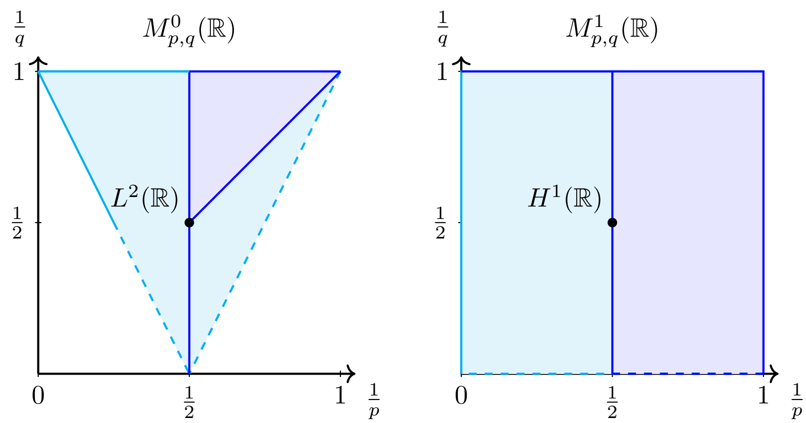

Figure 5.1: Wellposedness results for (1.1) with initial data in modulation spaces $M^s_{p,q}(\mathbb{R})$. The global results in $M^s_{p,q}(\mathbb{R})$ for $2 \leq p < \infty$ are restricted to the defocusing case. Blue: Global wellposedness, Cyan: Local wellposedness. A dashed line means that the boundary is not included.

want to mention the PhD thesis [29]. We will mostly refer to these for proofs of the following statements.

We begin by the classical definition of modulation spaces via a mixed $L^p - L^q$ norm of the short-time Fourier transform.

Definition 5.2. *The short-time Fourier transform of a function f with respect to the window function $g \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$ is defined as*

$$V_g f(x,\xi) = \int_{\mathbb{R}^n} e^{-iy\xi} f(y) \bar{g}(y-x)\, dy.$$

The modulation space norm of a function f is defined as

$$\|f\|^{\circ}_{M^s_{p,q}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \big|V_g f(x,\xi)\big|^p \, dx \Big)^{\frac{q}{p}} \langle \xi \rangle^{sq} \, d\xi \Big)^{\frac{1}{q}}.$$

With the usual modifications, this definition also includes $p, q = \infty$. Note that since the Fourier transform is defined for tempered distributions the same holds for the short-time Fourier transform. We define the modulation space $M^s_{p,q}(\mathbb{R}^n)$ as those distributions in $\mathcal{S}'(\mathbb{R}^n)$ which have finite modulation space norm. The modulation space norms for different window functions are equivalent, hence the space $M^s_{p,q}$ is independent of the window function.

One should think of the window function as something like a Gaussian or a smooth bump function. In this sense, the short-time Fourier transform of f at (x,ξ) can be understood as the usual Fourier transform at frequency ξ of the function f localized around x (where we ignore the constant $(2\pi)^{-\frac{n}{2}}$ to keep consistency with [56, 134, 29]).

The short-time Fourier transform in itself is a fascinating object and is used in different areas, including microlocal analysis and the analysis of pseudodifferential operators (see e.g. [129] for a short introductory overview). Instead of diving into its vast theory, we use an equivalent norm on modulation spaces related to a uniform partition in frequency.

Such a uniform partition of unify can be constructed as follows. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a smooth, symmetric bump function, that is $0 \le \rho \le 1$, $\rho(\xi) = 1$ if $|\xi| \le 1/2$, $\rho(\xi) = 0$ if $|\xi| \ge 1$. Let

$$\rho_k(\xi) = \rho(\xi - k), \quad k \in \mathbb{Z}^n.$$

Define $Q_0 = [-1/2, 1/2)^n$ and $Q_k = k + Q_0$ for $k \in \mathbb{Z}^n$. Define

$$\sigma_k(\xi) = \rho_k(\xi)\Big(\sum_{k\in\mathbb{Z}^n} \rho_k(\xi)\Big)^{-1}, \quad k \in \mathbb{Z}^n, \xi \in \mathbb{R}^n.$$

Then, for some $c > 0$, the functions σ_k satisfy

$$\begin{cases} |\sigma_k(\xi)| \ge c, \quad \forall \xi \in Q_k, \forall k \in \mathbb{Z}^n, \\ \operatorname{supp}(\sigma_k) \subset \{|\xi - k| \le \sqrt{n}\}, \quad \forall k \in \mathbb{Z}^n \\ \sum_{k\in\mathbb{Z}^n} \sigma_k(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n, \\ |D^\alpha \sigma_k(\xi)| \le C_m, \quad \forall \xi \in \mathbb{R}^n, |\alpha| \le m. \end{cases} \tag{5.2}$$

Definition 5.3. *Given a sequence of functions $(\sigma_k)_{k\in Z^n}$ satisfying (5.2), the sequence of operators*

$$\Box_k = \mathcal{F}^{-1}\sigma_k\mathcal{F}, \quad k \in \mathbb{Z}^n,$$

is called a family of isometric decomposition operators.

Definition 5.4. *Given $p, q \in [1, \infty]$, $s \in \mathbb{R}$ and $(\Box_k)_k$ a family of isometric decomposition operators. The modulation space norm with respect to $(\Box_k)_k$ is defined as*

$$\|f\|_{M^s_{p,q}(\mathbb{R}^n)} = \big\|\langle k\rangle^s \|\Box_k f\|_{L^p(\mathbb{R}^n)}\big\|_{\ell^q_k(\mathbb{Z}^n)}.$$

With these definitions at hand it can be shown that:

- different window functions g in Definition 5.2 lead to equivalent modulation space norms $\|\cdot\|_{M^s_{p,q}}$ and hence equivalent modulation spaces [29, Proposition 2.9],
- different families of isometric decomposition operators $\Box_k$ in Definition 5.3 lead to equivalent modulation space norms $\|\cdot\|^\circ_{M^s_{p,q}}$ [29, Proposition 2.24],
- on $S'(\mathbb{R}^n)$, the norms $\|\cdot\|^\circ_{M^s_{p,q}}$ and $\|\cdot\|_{M^s_{p,q}}$ are equivalent, hence $M^s_{p,q}$ can be equivalently characterized as those distributions in $\mathcal{S}'$ which have finite modulation space norm $\|\cdot\|_{M^s_{p,q}}$ [29, Proposition 2.26].

We use the notation $M_{p,q}(\mathbb{R}^n) = M^0_{p,q}(\mathbb{R}^n)$.

A useful fact is that the space of Schwartz functions $\mathcal{S}(\mathbb{R})$ is dense in $M^s_{p,q}$ for any $p, q \in [1, \infty)$ [29, Proposition 2.15]. If $p = \infty$, density fails. For instance we have continuous embeddings $C^2_b(\mathbb{R}) \subset M_{\infty,1} \subset C^0_b(\mathbb{R})$ (see Lemma 5.8). Moreover, duality works as expected in the sense that

$$\big(M^s_{p,q}(\mathbb{R}^n)\big)' = M^{-s}_{p',q'}(\mathbb{R}^n), \qquad 1 \leq p, q < \infty, s \in \mathbb{R}$$

see [29, Proposition 2.17].

The following theorem shows how modulation spaces are nested. The first inclusion describes that the lower p and q are, the stronger are the assumptions on a function f to be in a modulation space $M^s_{p,q}$. In particular in the graphs in Figure 5.1 the smallest space is in the upper right corner (i.e. $M^s_{1,1}$), and we have inclusions going to the left and downwards in the picture. The second inclusion shows that we can trade regularity for l^q summability. We give a short proof to emphasize the usefulness of the modulation space norm written in terms of isometric decomposition operators.

Theorem 5.5 (Embeddings, Proposition 2.5 in [133])**.** *The following embeddings hold,*

- $M^{s_1}_{p_1,q_1}(\mathbb{R}^n) \subset M^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ *if* $\quad p_1 \leq p_2, q_1 \leq q_2, s_1 \geq s_2,$
- $M^{s_1}_{p,q_1}(\mathbb{R}^n) \subset M^{s_2}_{p,q_2}(\mathbb{R}^n)$ *if* $\quad q_1 > q_2, s_1 > s_2, s_1 + \frac{n}{q_1} > s_2 + \frac{n}{q_2}.$

Proof. The first inclusion is a consequence of Bernstein's inequality and the embedding of ℓ^q spaces, whereas the second is a consequence of Hölder's inequality. More precisely, Bernstein's inequality

$$\|\Box_k f\|_{L^{p_2}} \lesssim \|\Box_k f\|_{L^{p_1}}, \quad p_1 \leq p_2,$$

follows from Young's inequality (Theorem A.8) and the fact that

$$\|\check{\sigma}_k\|_{L^r} = \|\check{\sigma}_0\|_{L^r} < \infty,$$

uniformly in $k \in \mathbb{Z}^n$. Together with the embedding $\ell^{q_1} \subset \ell^{q_2}$ if $q_1 \leq q_2$ this shows the first inclusion. For the second inclusion, write

$$\begin{aligned}\|f\|^{q_2}_{M^{s_2}_{p,q_2}} &= \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{s_2 q_2} \|\Box_k\|^{q_2}_{L^p} = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{s_1 q_2} \langle k\rangle^{(s_2-s_1)q_2} \|\Box_k\|^{q_2}_{L^p} \\ &\leq \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{s_1 q_1} \|\Box_k\|^{q_1}_{L^p}\Big)^{\frac{q_2}{q_1}} \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{(s_2-s_1)q_1 q_2/(q_1-q_2)}\Big)^{\frac{q_1-q_2}{q_1}},\end{aligned}$$

by Hölder's inequality. From comparing to the integrals and using radial coordinates,

$$\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{(s_2-s_1)q_1 q_2/(q_1-q_2)} \sim \sum_{j\in\mathbb{Z}} \langle k\rangle^{n-1+(s_2-s_1)q_1 q_2/(q_1-q_2)},$$

which is convergent if

$$n(q_1 - q_2) < (s_1 - s_2)q_1 q_2,$$

which translates into the condition of the second inclusion. This shows the statement. $\square$

In one dimension, the second inclusion gives for example $H^{1/2} \subset M_{2,1+}$ respectively $H^{1/2+} \subset M_{2,1}$. This is sharp since $M_{2,1} \subset L^\infty$ whereas $H^{1/2} \not\subset L^\infty$. On the other hand, ℓ^q summability does not gain regularity (see [133, Proposition 2.8]):

Lemma 5.6. *We have that* $M_{p,q} \not\subset B^{\varepsilon}_{p,r} \cup B^{\varepsilon}_{\infty,\infty}$ *for any* $0 < \varepsilon \ll 1, 1 \leq p, q, r \leq \infty$.

In particular an embedding of the form $M_{2,1} \subset H^s$ can never hold for positive regularity $s > 0$. The obstruction for this is $\ell^1 \not\subset \ell^2_s$ for $s > 0$ in the sense of weighted sequence spaces. Indeed, one can just consider the sequence $a_n = 1/k^2$ if $n = 2^k$ and $a_n = 0$ else, i.e. spreading out mass in ℓ^1 can be done without any problems - in contrast to ℓ^2_s.

There are some general results on relations between modulation spaces, Besov spaces and L^p spaces. The following inclusions give a hint of the most important ones:

Theorem 5.7. *The following continuous embeddings hold,*

1. $M^s_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ *with equivalence of norms,*
2. $M_{p,1}(\mathbb{R}^n) \subset C^0_b(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$*, if* $\quad 1 < p \leq \infty$,
3. $M_{p,p'}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$*, if* $\quad 2 \leq p \leq \infty$,
4. $M^\sigma_{p,q}(\mathbb{R}^n) \subset B_{p,q}(\mathbb{R}^n)$*, if* $\sigma = \max\Big(0, n\big(\frac{1}{\min(p,p')} - \frac{1}{q}\big)\Big)$,
5. $B^\tau_{p,q}(\mathbb{R}^n) \subset M_{p,q}(\mathbb{R}^n)$*, if* $\tau = \max\Big(0, n\big(\frac{1}{q} - \frac{1}{\max(p,p')}\big)\Big)$.

Proof. The first embedding follows from Plancharel's theorem. Up to the continuity, the second inclusion is shown in [133, Proposition 2.7], i.e. that

$$M_{p,1}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), \quad \text{if} \quad 1 \leq p \leq \infty.$$

Continuity on the other hand follows from density of $\mathcal{S}(\mathbb{R}^n)$ if $p > 1$ and the fact that $(C^0_b(\mathbb{R}^n), \|\cdot\|_{L^\infty})$ is a Banach space. The third, fourth and fifth embeddings are proven in [29, Proposition 2.34], [133, Theorem 2.12] and [133, Theorem 2.14], respectively. $\square$

We take a closer look at the consequences of Theorem 5.7. For example we see that in one dimension $B^{1/2}_{2,1} \subset M_{2,1} \subset L^\infty \cap L^2$. In this sense $M_{2,1}$ can be seen as a replacement for the space $H^{1/2}$, admitting an embedding into continuous functions and being less restrictive than $B^{1/2}_{2,1}$. On the other hand note that in many applications one wants to keep the same scaling with respect to the parameter s which is lost when going to modulation spaces (see Lemma 5.13).

On the other hand, Theorem 5.7 allows to understand better what kind of functions belong to $M_{\infty,1}(\mathbb{R}^n)$. We first give a small instructive lemma showing that functions in $M_{p,1}(\mathbb{R}^n)$ allow for exactly the same decay as functions in $L^p(\mathbb{R}^n)$, the only difference being their regularity:

Lemma 5.8. *Let $p \in [1,\infty]$. There is a continuous inclusion $W^{2,p}(\mathbb{R}) \subset M_{p,1}(\mathbb{R})$.*

Proof. Let σ_k be such that $\Box_k f = \mathcal{F}^{-1}(\sigma_k \hat{f})$. Define $\psi_k = \mathcal{F}^{-1}\sigma_k$ and $\psi = \psi_0$. Then, because the Fourier transform maps translation to modulation,

$$\Box_k f(x) = \int_{\mathbb{R}} f(x-y)\psi_k(y)\,dx = \int_{\mathbb{R}} f(x-y)e^{-iky}\psi(y)\,dy.$$

As we can bound $\|\Box_0 f\|_{L^p} \lesssim \|f\|_{L^p}$, we assume $k \neq 0$. Then by partial integration, and because $\psi, \psi', \psi'' \to 0$ as $|x| \to \infty$

$$\begin{aligned}\Box_k f(x) &= \int_{\mathbb{R}} f(x-y)e^{-iky}\psi(y)\,dy \\ &= \frac{1}{k^2}\int_{\mathbb{R}} e^{-iky}\big(f(x-y)\psi(y)\big)''\,dy \\ &= \frac{1}{k^2}\int_{\mathbb{R}} e^{-iky}\big(f(x-y)\psi''(y) - 2f'(x-y)\psi'(y) + f''(x-y)\psi(y)\big)\,dy.\end{aligned}$$

We estimate $f, f', f'' \in L^p$ and $\psi, \psi', \psi'' \in L^1$ via Young's inequality (Theorem A.8) and find for $k \geq 0$,

$$\|\Box_k f\|_{L^p} \lesssim_\sigma \langle k\rangle^{-2}\|f\|_{W^{2,p}},$$

which shows the statement. $\square$

In particular we can take any function in $L^p(\mathbb{R}^n)$, make it smooth via convolution without altering its decay, and obtain a function in $M_{p,1}(\mathbb{R}^n)$. It also means for example that the space $M_{\infty,1}(\mathbb{R}^n)$ is generic enough to include periodic functions, sums of periodic functions with irrational frequency relations, e.g.

$$\cos(x) + \cos(\sqrt{2}x),$$

or functions which are an infinite sum of bump functions with different amplitude, i.e.

$$f(x) = \sum_{l \in \mathbb{Z}^n} a_l h(x-l),$$

where $h \in \mathcal{S}(\mathbb{R}^n)$ and $(a_l)_l \in \ell^\infty$.

From Theorem 5.7 we also see that the Hölder space $C^{n+\varepsilon}(\mathbb{R}^n)$ embeds into $M_{\infty,1}(\mathbb{R}^n)$:

Lemma 5.9. *There is a continuous inclusion $C^{n+\varepsilon}(\mathbb{R}^n) \subset M_{\infty,1}(\mathbb{R}^n)$.*

Proof. We use the alternative characterization of Hölder space norm (see e.g. [55, Theorem 6.3.7]):

$$\|u\|_{C^{n+\varepsilon}(\mathbb{R}^n)} \sim \sup_{j\in\mathbb{N}} 2^{j(n+\varepsilon)} \|P_j u\|_{L^\infty(\mathbb{R}^n)} = \|u\|_{B^{n+\varepsilon}_{\infty,\infty}(\mathbb{R}^n)},$$

where P_N is the Littlewood-Paley projector supported at frequencies $2^{N-1} \leq |\xi| \leq 2^{N+1}$. Now by Theorem 5.7,

$$\|u\|_{M_{\infty,1}(\mathbb{R}^n)} \lesssim \|u\|_{B^n_{\infty,1}(\mathbb{R}^n)} = \sum_{j\in\mathbb{N}} 2^{j(n+\varepsilon)} 2^{-j\varepsilon} \|P_j u\|_{L^\infty(\mathbb{R}^n)} \lesssim \|u\|_{B^{n+\varepsilon}_{\infty,\infty}(\mathbb{R}^n)},$$

using Hölder's inequality in the last line. □

We remark that this embedding is sharp. Indeed, consider $n = 1$ dimension. Then there exists a function in $M_{\infty,1}(\mathbb{R})$ which does not lie in the Zygmund class of functions defined via

$$\|u\|_{B^1_{\infty,\infty}(\mathbb{R})} = \sup_{N\in\mathbb{N}} 2^N \|P_N u\| < \infty.$$

Indeed, we define

$$u(x) = \sum_{N\in\mathbb{N}} 2^{-N} e^{i2^N x},$$

and see that

$$\|u\|_{M_{\infty,1}} = \sum_{N\in\mathbb{N}} 2^{-N} < \infty,$$

but one can check that u is nowhere differentiable and hence not a Zygmund class function (see [55, Exercise 6.3.4]). On the other hand, there are functions which are continuous and bounded but which do not lie in $M_{\infty,1}$. Indeed, any function of the form

$$u(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx}$$

with $u \in M_{\infty,1}(\mathbb{R})$ must satisfy $a_k \in \ell^1_k$ and hence has absolutely convergent Fourier series. On the other hand, there are continuous functions whose Fourier series diverge, see for example [121, Section 2.2].

Next we analyze what happens when we multiply modulation space functions. As a consequence of Hölder's and Young's convolutional inequalities, we obtain bilinear bounds. These imply in particular that the spaces $M_{p,1}(\mathbb{R}^n)$ as well as $M_{p,q}(\mathbb{R}^n) \cap M_{\infty,1}(\mathbb{R}^n)$ are algebras under multiplication for all $p, q \in [1, \infty]$.

Lemma 5.10. *If $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$ and $m - 1 + \frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i}$ then*

$$\Big\| \prod_{i=1}^m f_i \Big\|_{M_{p,q}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i\|_{M_{p_i,q_i}(\mathbb{R}^n)}, \tag{5.3}$$

and if $s \geq 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2}$ *then*

$$\|fg\|_{M^s_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{M^s_{p_1,q_1}(\mathbb{R}^n)} \|g\|_{M_{p_2,q_2}(\mathbb{R}^n)} + \|f\|_{M_{p_1,r_1}(\mathbb{R}^n)} \|g\|_{M^s_{p_2,r_2}(\mathbb{R}^n)}. \quad (5.4)$$

Proof. We give a short proof since [29, Theorem 4.3] only proves a similar statement. If we use the notation $l_1 + l_2 \approx k$ for $l_1 + l_2 = k + \Lambda$, where

$$\Lambda = \{|\xi| \leq 2\sqrt{n}\} \cap \mathbb{Z}^n$$

Then,

$$\square_k(fg) = \square_k\big(\sum_{l_1 \in \mathbb{Z}^n} \square_{l_1} f\big)\big(\sum_{l_2 \in \mathbb{Z}^n} \square_{l_2} g\big) = \square_k \sum_{l_1 + l_2 \approx k} (\square_{l_1} f)(\square_{l_2} g).$$

By writing out the Fourier multiplication as a convolution and by Young's inequality (Theorem A.8) the operators $\square_k$ are bounded uniformly in k on L^{p_i}. Hence

$$\|\square_k(fg)\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{l_1 + l_2 \approx k} \|\square_{l_1} f\|_{L^{p_1}(\mathbb{R}^n)} \|\square_{l_2} f\|_{L^{p_2}(\mathbb{R}^n)}.$$

Consequently, (5.3) with $m = 2$ is obtained from Young's convolutional inequality using that the set Λ is finite. The case of general m follows by induction. For (5.4) we use Peetre's inequality to see

$$\begin{aligned} \langle k \rangle^s \|\square_k(fg)\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{l_1 + l_2 \approx k} & \langle l_1 \rangle^s \|\square_{l_1} f\|_{L^{p_1}(\mathbb{R}^n)} \|\square_{l_2} f\|_{L^{p_2}(\mathbb{R}^n)} \\ & + \|\square_{l_1} f\|_{L^{p_1}(\mathbb{R}^n)} \langle l_2 \rangle^s \|\square_{l_2} f\|_{L^{p_2}(\mathbb{R}^n)}, \end{aligned}$$

and we conclude using Young's inequality again. □

The bilinear bound from Lemma 5.10 allows to handle algebraic nonlinearities. More complicated nonlinearities on the other hand can cause problems. In [113] Ruzhansky-Sugimoto-Wang raised the question whether an inequality of the form

$$\||f|^\alpha f\|_{M^s_{p,q}} \lesssim \|f\|^{\alpha+1}_{M^s_{p,q}}$$

holds if $\alpha \in (0, \infty) \setminus 2\mathbb{N}$. A negative answer for $s = 0, q = 1$ is given by Bhimani-Ratnakumar in [17]. In fact, they proved the stronger result that if a function $F : \mathbb{R}^2 \to \mathbb{C}$ operates in $M_{p,1}$ for some $1 \leq p \leq \infty$, then F must be real analytic on $\mathbb{R}^2$. This also shows that in general, neither implication between $f \in M_{p,1}$ and $|f| \in M_{p,1}$ holds. On the other hand there is a result by Sugimoto–Tomita–Wang giving a positive result for $q = 2$ under the assumption that α is sufficiently large and $s > n/2$ [122].

We are interested in estimates for the Schrödinger propagator in modulation spaces. The following inequalities are optimal with respect to the time dependence of the constant. A first version of them are proven in [8] in the case $p = 2$ which [12] then extended for $p, q \in [1, \infty]$. Sharpness of the exponent for $p \in [1, 2]$ was proven in [39] and extended to $p \in [1, \infty]$ in [29, Theorem 3.4]. For the sake of completeness we give a proof using the isometric decomposition operators. It is similar to the one given in [133, Proposition 6.6] but uses a kernel estimate similar to the calculation in [117, Proposition 3.2].

Lemma 5.11. *Let $1 \le p \le \infty$ and $1 \le q \le \infty$. The following hold:*

$$\|S(t)f\|_{M_{2,q}} = \|f\|_{M_{2,q}}, \tag{5.5}$$

$$\|S(t)f\|_{M_{p,q}} \lesssim (1+|t|)^{n/2}\|f\|_{M_{p,q}}, \tag{5.6}$$

$$\|S(t)f\|_{M_{p,q}} \lesssim (1+|t|)^{-n(1/2-1/p)}\|f\|_{M_{p',q}}, \quad \textit{for} \quad p \ge 2, \tag{5.7}$$

$$\|S(t)f\|_{M_{p,q}} \lesssim (1+|t|)^{n|1/2-1/p|}\|f\|_{M_{p,q}}. \tag{5.8}$$

Proof. The equality (5.5) follows from Plancharel's theorem. Now consider (5.6). For small times we can estimate

$$\|S(t)f\|_{M_{p,q}} \le \|S(t)f\|_{M_{2,q}} = \|f\|_{M_{2,q}} \le \|f\|_{M_{p',q}},$$

due to the fact that $p \ge 2$, whereas for large times we use the estimate (1.15) on frequency localized functions to get

$$\|S(t)\Box_k f\|_{L^p} \lesssim |t|^{-n(1/2-1/p)}\|\Box_k f\|_{L^{p'}}.$$

This gives (5.7) for large times in view of commutativity of $S(t)$ and $\Box_k$ and after summing.

Since the estimate (5.8) is obtained by interpolating between (5.6) with $p = 1, \infty$ and (5.5), it is enough to prove (5.6). Clearly it is enough to show

$$\|S(t)\Box_k f\|_{L^p} \lesssim (1+|t|)^{\frac{n}{2}}\|\Box_k f\|_{L^p},$$

uniformly in k. We calculate the left-hand side,

$$\big\|\mathcal{F}^{-1}\big(e^{-it|\xi|^2}\sigma_k(\xi)\hat{f}(\xi)\big)\big\|_{L^p} = \big\|\mathcal{F}^{-1}\big(e^{-it|\xi+k|^2}\sigma_0(\xi)\hat{f}(\xi+k)\big)\big\|_{L^p}$$

Since $|e^{-it|k|^2}| = 1$ and since the the modulation $e^{-2it\xi k}$ is mapped to a translation via inverse Fourier transform, this is the same as

$$\big\|\mathcal{F}^{-1}\big(e^{-it|\xi|^2}\sigma_0(\xi)\hat{f}(\xi+k)\big)\big\|_{L^p}.$$

We write $\hat{f} = \sum_{l\in\mathbb{Z}^n}\sigma_l\hat{f}$ and apply the triangle inequality and Young's inequality (Theorem A.8)

$$\begin{aligned}\big\|\mathcal{F}^{-1}\big(e^{-it|\xi|^2}\sigma_0(\xi)\hat{g}\big)\big\|_{L^p} &\le \sum_{l\in\Sigma}\big\|\mathcal{F}^{-1}\big(e^{-it|\xi|^2}\sigma_0(\xi)\sigma_l\hat{g}\big)\big\|_{L^p} \\ &\le \sum_{l\in\Lambda}\big\|\mathcal{F}^{-1}\big(e^{-it|\xi|^2}\sigma_0(\xi)\big)\big\|_{L^1}\big\|\mathcal{F}^{-1}\big(e^{-it|\xi|^2}\sigma_l(\xi)\hat{g}\big)\big\|_{L^p}\end{aligned}$$

where $\hat{g}(\xi) = \hat{f}(\xi+k)$ and

$$\Lambda = \{|\xi| \le 2\sqrt{n}\} \cap \mathbb{Z}^n$$

is a finite set. Since $\|\Box_l g\|_{L^p} = \|\Box_{k+l} f\|_{L^p}$ it is enough to show

$$\big\|\mathcal{F}^{-1}\big(e^{-it|\xi|^2}\sigma_0(\xi)\big)\big\|_{L^1} \lesssim (1+|t|)^{\frac{n}{2}}. \tag{5.9}$$

If $K(t,x) = \big(\mathcal{F}^{-1}\big(e^{-it|\xi|^2}\sigma_0(\xi)\big)\big)(x)$, then we showed in Lemma 1.7 that

$$|K(t,x)| \leq \begin{cases} c|t|^{-\frac{n}{2}}, & \text{if} \quad |x| \lesssim |t|, \\ c_N|x|^{-N}, & \text{if} \quad |x| \gtrsim |t|. \end{cases}$$

From this (5.9) follows and the proof is finished. □

We remark that in view of the discussion after the blow-up result for the Schrödinger propagator in Theorem 1.17, and the embeddings $C_b^{n+\varepsilon}(\mathbb{R}^n) \subset M_{\infty,1}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, Lemma 5.11 shows a sharpness of the modulation space $M_{\infty,1}(\mathbb{R}^n)$ in terms of preventing blow-up for the Schrödinger propagator in spaces of functions without decay at infinity.

We need two more properties of modulation spaces. One describes the behavior under complex interpolation and the other one under scaling.

Theorem 5.12. *Let $p_0, p_1 \in [1,\infty]$ and $q_0, q_1 \in [1,\infty]$ such that $q_0 \neq \infty$ or $q_1 \neq \infty$. Let $s_0, s_1 \in \mathbb{R}$ and $\theta \in (0,1)$. Define*

$$\begin{aligned} &s = (1-\theta)s_0 + \theta s_1, \\ &\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \end{aligned}$$

with the usual convention in the extreme case $p_i, q_i = \infty$. Then

$$\left[M^{s_0}_{p_0,q_0}(\mathbb{R}^n), M^{s_1}_{p_1,q_1}(\mathbb{R}^n)\right]_\theta = M^s_{p,q}(\mathbb{R}^n), \tag{5.10}$$

in the sense of equality of spaces and equivalence of norms.

As we have seen the expectation of whether or not local wellposedness holds is often tied to scaling. Since the decomposition on the Fourier side is uniform in the modulation space norm, there is no neat scaling relation for modulation spaces. Estimates still hold (see Theorem 3.2. in [40]) and we list the ones for $p = 2$ and one dimension $n = 1$ here:

Lemma 5.13. *We have the scaling inequalities*

$$\|\psi(\lambda\cdot))\|_{M_{2,q}(\mathbb{R})} \lesssim \begin{cases} \lambda^{-1/2}\|\psi\|_{M_{2,q}(\mathbb{R})}, & \text{if} \quad 1 \leq q \leq 2 \\ \lambda^{1/q-1}\|\psi\|_{M_{2,q}(\mathbb{R})}, & \text{if} \quad 2 \leq q \leq \infty \end{cases}$$

and

$$\|\psi(\lambda\cdot))\|_{M_{2,q}(\mathbb{R})} \gtrsim \begin{cases} \lambda^{1/q-1}\|\psi\|_{M_{2,q}(\mathbb{R})}, & \text{if} \quad 1 \leq q \leq 2 \\ \lambda^{-1/2}\|\psi\|_{M_{2,q}(\mathbb{R})}, & \text{if} \quad 2 \leq q \leq \infty \end{cases}$$

for all $\lambda \leq 1$ and $\psi \in M_{2,q}$. Similarly,

$$\|\psi\|_{M_{2,q}(\mathbb{R})} \lesssim \begin{cases} \lambda^{1/2}\|\psi(\lambda\cdot))\|_{M_{2,q}(\mathbb{R})}, & \text{if} \quad 1 \leq q \leq 2 \\ \lambda^{1-1/q}\|\psi(\lambda\cdot))\|_{M_{2,q}(\mathbb{R})}, & \text{if} \quad 2 \leq q \leq \infty \end{cases}$$

and

$$\|\psi\|_{M_{2,q}(\mathbb{R})} \gtrsim \begin{cases} \lambda^{1-1/q}\|\psi(\lambda\cdot))\|_{M_{2,q}(\mathbb{R})}, & \textit{if} \quad 1 \le q \le 2 \\ \lambda^{1/2}\|\psi(\lambda\cdot))\|_{M_{2,q}(\mathbb{R})}, & \textit{if} \quad 2 \le q \le \infty \end{cases}$$

for all $\lambda \ge 1$ *and* $\psi \in M_{2,q}(\mathbb{R})$*.*

If u is a solution of cubic NLS (5.1), then so is $u_\lambda(x,t) = \lambda^{-1}u\left(\lambda^{-1}x, \lambda^{-2}t\right)$ for all $\lambda \in (0,\infty)$. Choosing $\lambda \ge 1$ we find that

$$\|u_\lambda(x,\lambda^2 t)\|_{M_{2,q}(\mathbb{R})} \lesssim \begin{cases} \lambda^{-\frac{1}{2}}\|u(x,t)\|_{M_{2,q}(\mathbb{R})}, & \text{if} \quad 1 \le q \le 2, \\ \lambda^{-\frac{1}{q}}\|u(x,t)\|_{M_{2,q}(\mathbb{R})}, & \text{if} \quad 2 \le q \le \infty \end{cases}$$

and

$$\|u_\lambda(x,\lambda^2 t)\|_{M_{2,q}(\mathbb{R})} \gtrsim \begin{cases} \lambda^{-\frac{1}{q}}\|u(x,t)\|_{M_{2,q}(\mathbb{R})}, & \text{if} \quad 1 \le q \le 2, \\ \lambda^{-\frac{1}{2}}\|u(x,t)\|_{M_{2,q}(\mathbb{R})}, & \text{if} \quad 2 \le q \le \infty. \end{cases}$$

In particular as long as $q < \infty$ we are in a subcritical range with respect to scaling.

5.2 Quantitative Wellposedness

Following [9] we quickly introduce the notion of quantitative wellposedness. While it is just a reformulation of the standard Picard iteration for homogeneous algebraic nonlinearities in a more quantitative fashion, it gives us the means to simply show linear and multilinear estimates and immediately obtain wellposedness. Our focus of application lies on the cubic NLS (5.1) on the real line,

$$\begin{cases} iu_t + u_{xx} = \pm 2|u|^2 u, \\ u(0) = f, \end{cases}$$

though the notion applies basically to any semilinear evolution equation with multilinear nonlinearity.

Definition 5.14. *Let L be a linear and N_k be a k-multilinear operator. The equation*

$$u = Lf + N_k(u,\ldots,u)$$

is called quantitatively wellposed in the spaces D, X if the two estimates

$$\|Lf\|_X \le C_1\|f\|_D, \tag{5.11}$$

$$\|N_k(u_1,\ldots,u_k)\|_X \le C_2 \prod_{i=1}^{k} \|u_i\|_X \tag{5.12}$$

hold for some constants $C_1, C_2 > 0$.

We will only consider norms which are invariant under complex conjugation, hence we may allow the nonlinearities here to be k-multilinear on real-valued functions, and complex conjugation in the nonlinear part as in 5.1 plays no major role. As a consequence of polarization identities for real symmetric multilinear operators [130], in order to show an estimate of the form (5.12), or more generally for some Banach space Y,

$$\|N_k(u_1, \ldots, u_k)\|_X \lesssim \prod_{i=1}^{k} \|u_i\|_Y,$$

it is enough to show the estimate

$$\|N_k(u, \ldots, u)\|_X \lesssim \|u\|_Y^k.$$

Indeed it is not hard to see via polarization that this implies

$$\|N_k(u_1, \ldots, u_k)\|_X \lesssim \sum_{i=1}^{k} \|u_i\|_Y^k,$$

and now putting $u_i = s_i \tilde{u}_i$ with $\prod s_i = 1$ and minimizing over s_i proves the claim. This shows that for symmetric multilinear nonlinearities and norms that are invariant under complex conjugation, the contraction property of the corresponding operator in the Banach fixed point argument usually follows from being a self-mapping. In a similar manner one proves that the estimate

$$\|N_k(u, \ldots, u)\|_X \lesssim \prod_{i=1}^{k} \|u\|_{Y_i}$$

implies the estimate

$$\|N_k(u_1, \ldots, u_k)\|_X \lesssim \sum_{\sigma \in S_k} \prod_{i=1}^{k} \|u_{\sigma(i)}\|_{Y_i},$$

where S_k denotes the permutation group of order k.

Denote by $B^X(R)$ the ball of radius R in the space X. The reason for speaking about wellposedness in Definition 5.14 is the following:

Theorem 5.15. *Let the equation*

$$u = Lf + N_k(u, \ldots, u) \tag{5.13}$$

be quantitatively wellposed. Then there exist $\epsilon > 0$ and $C_0 > 0$ such that for all $f \in B^D(\epsilon)$ there is a unique solution $u[f] \in B^X(C_0\epsilon)$ to (5.13). In particular, u can be written as an X-convergent power series for $f \in B^D(\epsilon)$,

$$u[f] = \sum_{n=1}^{\infty} A_n(f), \tag{5.14}$$

where A_n is defined recursively by

$$A_1(f) = Lf, \qquad A_n(f) = \sum_{n_1+\cdots+n_k=n} N_k(A_{n_1}(f), \ldots, A_{n_k}(f)),$$

and satisfies for some $C_1, C_2 > 0$,

$$\begin{cases} A_n(\lambda f) = \lambda^n A_n(f) \\ \|A_n(f) - A_n(g)\|_X \le C_1^n \|f-g\|_D (\|f\|_D + \|g\|_D)^{n-1}, \\ \|A_n(f)\|_X \le C_2^n \|f\|_D^n. \end{cases}$$

We will work in modulation spaces which do not admit homogeneous scaling, and are also above the scaling critical exponent for NLS. As a result, the bounds (5.11) and (5.12) will depend on the time variable T. This will show that a solution exists with guaranteed time of existence depending on $\|f\|_D$, and will result in a blow-up alternative later.

Lemma 5.16. *Let* (5.13) *be quantitatively wellposed in $D, X = X_T$, and assume that the constants in* (5.11) *respectively* (5.12) *are*

$$C_1 = c_1 \langle T \rangle^{\alpha_1}, \qquad C_2 = c_2 T^{\alpha_2} \langle T \rangle^{\alpha_3}.$$

Then we may choose

$$T \sim \min\left(\varepsilon^{-\beta_1}, \varepsilon^{-\beta_2}\right), \qquad \beta_1 = \frac{k-1}{(k-1)\alpha_1 + \alpha_2 + \alpha_3}, \qquad \beta_2 = \frac{k-1}{\alpha_2},$$

as a guaranteed time of existence.

Proof. If $\Phi(u) = Lf + N_k(u, \ldots, u)$, then (5.11) and (5.12) give

$$\|\Phi(u)\|_X \le C_1 \varepsilon + C_2 (C_0 \varepsilon)^k,$$

which has to be smaller than $C_0 \varepsilon$ for a contraction on $B^X(C_0\varepsilon)$. Taking $C_0 = 2C_1$ we need that

$$2C_2(2C_1\varepsilon)^{k-1} < 1,$$

which amounts to

$$T^{\alpha_2} \langle T \rangle^{\alpha_3 + \alpha_1(k-1)} \varepsilon^{k-1} \lesssim 1.$$

When ε is small, we can make T large and $T \sim \langle T \rangle$ so that β_1 is the relevant exponent for T. When ε is large, $\langle T \rangle \sim 1$ and we arrive at β_2. It is not hard to see that this also guarantees the Lipschitz bound to hold, and we obtain a contraction. $\square$

We apply this general setting to cubic NLS and obtain:

Definition 5.17. *Let D a Banach space of functions and let $S(t) = e^{it\partial_x^2}$. We call a function $u \in X_T \subset C^0([0,T], D)$ a (mild) solution of NLS with initial value u_0 if it solves the fixed point equation*

$$u = S(t)u_0 \mp 2i \int_0^t S(t-\tau)(|u|^2 u)(\tau)\, d\tau \tag{5.15}$$

in X_T. The supremum of all such T is called maximal time of existence and denoted by $T^* = T^*(u_0)$.

In the following we use the notation

$$N(u_1, u_2, u_3) = N_3(u_1, u_2, u_3) = 2i \int_0^t S(t-\tau)(u_1 \bar{u}_2 u_3)(\tau)\, d\tau,$$

and note that all local results we prove hold for both the focusing and the defocusing equation.

Corollary 5.18. *Consider the Cauchy problem* (5.1) *with initial data $f = u_0$ in a Banach space D. If the bounds*

$$\|S(t)u_0\|_{X_T} \lesssim \langle T \rangle^{\alpha_1} \|u_0\|_D, \tag{5.16}$$

$$\Big\| \int_0^t S(t-\tau)(u_1 \bar{u}_2 u_3)(\tau)\, d\tau \Big\|_{X_T} \lesssim T^{\alpha_2} \langle T \rangle^{\alpha_3} \prod_{i=1}^{3} \|u_i\|_{X_T}, \tag{5.17}$$

hold for some $\alpha_1, \alpha_2, \alpha_3 > 0$, then for all $R > 0$ and $u_0 \in B^D(R)$ there exists $T > 0$ such that for all $T' < T$ there exists a unique solution $u \in X_{T'}$ to (5.15)*. Moreover, the blowup-alternative*

$$T^* < \infty \quad \Rightarrow \quad \limsup_{t \nearrow T^*} \|u(\cdot, t)\|_D = \infty \tag{5.18}$$

holds.

Proof. The existence and uniqueness follow from Theorem 5.15. Assuming that $\|u(T)\|_D \leq C < \infty$ with T arbitrarily close to T^*, the assumptions from Lemma 5.16 are satisfied, hence there exists a small $\delta > 0$ such that (1.1) can be solved on $[T^*, T^* + \delta)$, which contradicts the maximality. $\square$

5.3 Local Wellposedness via Multilinear Interpolation

5.3.1 The Triangle $1/q \geq \max(1/p', 1/p)$

We recall the Strichartz estimates from Theorem 1.8 which lead to local wellposedness of (5.1) in $L^2(\mathbb{R})$: for all (q, p) and $(\tilde{q}, \tilde{p})$ admissible we have

$$\|S(t)f\|_{L_t^q L_x^p} \lesssim \|f\|_{L^2}, \tag{5.19}$$

$$\Big\| \int_0^t S(t-s)F(s, \cdot) \Big\|_{L_t^q L_x^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}'}}. \tag{5.20}$$

This allows to prove local (and due to L^2 conservation also global) wellposedness of cubic NLS in $L^2(\mathbb{R})$ by a fixed point argument: Let $X_T = L_t^\infty L_x^2([0, T] \times \mathbb{R}) \cap$

$L_t^4 L_x^\infty([0,T]\times\mathbb{R})$. Then from Hölder's inequality,

$$\begin{aligned}\|N(u_1,u_2,u_3)\|_{X_T} &= \|\int_0^t S(t-s)(u_1\bar{u}_2u_3)(s)ds\|_{X_T}\\ &\lesssim \|u_1\bar{u}_2u_3\|_{L_t^{8/7}L_x^{4/3}} \lesssim T^{1/2}\prod_{i=1}^3 \|u_i\|_{X_T}.\end{aligned}$$

Corollary 5.18 together with L^2-conservation then gives global wellposedness in $L^2(\mathbb{R})$. The space $L_t^\infty L_x^2([0,T]\times\mathbb{R})\cap L_t^8 L_x^4([0,T]\times\mathbb{R})$ would have been enough for the iteration of the trilinear term, too.

By Corollary 5.18 we obtain local wellposedness in $M_{p,1}$ for all $1\le p\le\infty$ with $X_T = C^0 M_{p,1}([0,T]\times\mathbb{R})$ due to the trivial estimate

$$\begin{aligned}\|N(u_1,u_2,u_3)\|_{X_T} &\lesssim \|\int_0^t S(t-s)(|u|^2u)(s)ds\|_{X_T}\\ &\lesssim T(1+T)^{|1/2-1/p|}\|u\|_{X_T}^3,\end{aligned}$$

which follows from the Banach algebra property of $M_{p,1}$.

Starting from the estimates for $M_{p,1}$ and $L^2 = M_{2,2}$ we use multilinear interpolation to obtain new local wellposedness results. The range of p,q that can be reached as line segments between points in $\{(1/p,1), p\in[1,\infty]\}$ and $(1/2,1/2)$ is exactly the triangle $1\le q\le 2$, $1/q\ge\max(1/p',1/p)$, and this is where this simple multilinear interpolation works.

Theorem 5.19. *Let $1\le p\le\infty$ and $1\le q\le 2$ such that $1/q\ge\max(1/p',1/p)$. Then for any initial data $u_0\in M^{p,q}$, there is a $T>0$ and a unique solution u to* (1.1) *in*

$$X_T^{p,q} = L_t^\infty M_{p,q}([0,T]\times\mathbb{R})\cap L_t^{8/\theta}[M_{\tilde{p},1},L^4]_\theta([0,T]\times\mathbb{R}). \tag{5.21}$$

Here, the numbers $\theta\in[0,1]$ and $\tilde{p}\in[1,\infty]$ are determined by $1/p=(1-\theta)/\tilde{p}+\theta/2$ and $1/q = 1-\theta/2$. Moreover, either the solution u exists globally in time, or there is $T^*<\infty$ such that

$$\limsup_{t\to T^*}\|u(t)\|_{M_{p,q}} = \infty.$$

Remark 5.20. *Note that due to $M_{\tilde{p},1}\subset L^\infty$ we have that $[M_{\tilde{p},1},L^4]_\theta\subset L^{4/\theta}$. This shows that the constructed solutions are also distributional.*

Proof of Theorem 5.19. Without loss of generality we assume $T\le 1$. The assumptions on θ and $\tilde{p}$ imply that $M_{p,q}=[M_{\tilde{p},1},L^2]_\theta$. We interpolate[1] the linear estimates

$$\begin{aligned}\|S(t)u_0\|_{L_t^\infty L_x^2\cap L_t^8L_x^4} &\lesssim \|u_0\|_{L^2},\\ \|S(t)u_0\|_{L_t^\infty M_{\tilde{p},1}} &\lesssim \|u_0\|_{M_{\tilde{p},1}},\end{aligned}$$

[1] Strictly speaking, instead of interpolating with the intersection we interpolate first on both spaces and then take the intersection. Interpolation of mixed-norm L^p spaces was shown to work in [10]. Since we can apply this to $\square_k f$ for each k the same works if we consider mixed-norm combinations of L^p and modulation spaces.

to obtain

$$\|S(t)u_0\|_{X_T^{p,q}} \lesssim \|u_0\|_{M^{p,q}}. \tag{5.22}$$

Moreover, the nonlinear estimates

$$\|N(u_1,u_2,u_3)\|_{L_t^\infty L_x^2 \cap L_t^8 L_x^4} \lesssim T^{1/2} \prod_{i=1}^{3} \|u_i\|_{L_t^8 L_x^4},$$

$$\|N(u_1,u_2,u_3)\|_{L_t^\infty M^{\tilde{p},1}} \lesssim T \prod_{i=1}^{3} \|u_i\|_{L_t^\infty M_{\tilde{p},1}},$$

give, by Theorem A.10,

$$\|N(u_1,u_2,u_3)\|_{X_T^{p,q}} \lesssim T^{1-\theta/2} \prod_{i=1}^{3} \|u_i\|_{X_T^{p,q}}. \tag{5.23}$$

The result now follows from Corollary 5.18. □

5.3.2 The Triangle $1/q > |1-2/p|$

Using Bourgain space techniques, Guo showed local wellposedness of cubic NLS in $M_{2,q}, 2 \leq q < \infty$ [59]. Since his results were also derived from a trilinear estimate of the form (5.12), we can use interpolation to get more wellposedness results in modulation spaces. The triangle $1/q > |1-2/p|$ is strictly larger than the triangle from Section 5.3.1 and can be obtained by means of interpolating between the three endpoints $M_{\infty,1}$, $M_{1,1}$ and $M_{2,\infty}$. Since the latter space contains the Dirac delta distribution and there is no local wellposedness theory for it, we have to exclude it and obtain wellposedness in a half-open triangle.

We introduce the U^p and V^p spaces in which wellposedness was achieved.

Definition 5.21. *A $U_t^p L_x^2((a,b)\times\mathbb{R})$ atom is a function $A:(a,b)\to L^2$ of the form*

$$A = \sum_{k=1}^{K} \chi_{[t_{k-1},t_k)} \phi_k,$$

where $a = t_0 < \cdots < t_K = b$ and $(\phi_1,\ldots,\phi_K) \in (L^2)^K$ which has unit norm in l^p, i.e. $\sum_i \|\phi_i\|_{L^2}^p = 1$. The space $U_t^p L_x^2$ is defined as the space of elements of the form $\sum_{j=1}^\infty \lambda_j A_j$, where $(\lambda_j) \in l^1$. It is equipped with the norm

$$\|u\|_{U^p} = \inf\{\|(\lambda_j)\|_{l^1} : u = \sum_{j=1}^{\infty} \lambda_j A_j \text{ for } A_j\ U^p \text{ atoms}\}. \tag{5.24}$$

The space U_Δ^p is defined as $S(\cdot)U_t^p L_x^2$ with norm

$$\|u\|_{U_\Delta^p} = \|S(-t)u(t)\|_{U_t^p L_x^2} \tag{5.25}$$

The spaces U_t^2 and its close cousin V_t^2 can be seen as refinements of Bourgain spaces in the case of $b = 1/2$, which satisfy $U_t^p \subset L_t^\infty$ for all $1 \le p < \infty$. Indeed, the $X^{s,b}$ space would be defined by the norm $\|u\|_{X^{s,b}} = \|S(-t)u(t)\|_{H_t^b H_x^s}$. The usual Strichartz spaces are connected to the U_Δ^p spaces via

$$\|v\|_{L_t^p L_x^q} \lesssim \|v\|_{U_\Delta^p}.$$

A proof of this can be found in [84, Chapter 4] and we refer to this book as a reference for an introduction to these spaces.

Theorem 5.22 ([59]). *Let $2 < q < \infty$ and let X_T^q denote the space of all tempered distributions u such that the norm $\|u\|_{X^q} = \|\|\Box_n u\|_{U_\Delta^2([0,T])}\|_{l^q}$ is finite. Then*

$$\|N(u_1, u_2, u_3)\|_{X_T^q} \lesssim (T^{1/2} + T^{1/4} + T^{1/q^+}) \prod_{i=1}^{3} \|u_i\|_{X_T^q}. \tag{5.26}$$

This estimate gives local wellposedness in $X_T^q \subset L_t^\infty M^{2,q}([0,T]\times\mathbb{R})$. Indeed, for the linear part the definition of U_Δ gives

$$\begin{aligned}\|S(t)u_0\|_{X_q} &= \|\|\Box_n S(t)u_0\|_{U_\Delta^2}\|_{l^q} = \|\|\Box_n u_0\|_{U_t^2 L_x^2}\|_{l^q} \\ &\lesssim \|\|\Box_n u_0\|_{L_x^2}\|_{l^q} = \|u_0\|_{M^{2,q}},\end{aligned} \tag{5.27}$$

Since this result was only shown for $2 < q < \infty$ for the sake of simplicity let us define $X_T^q = X_T^{2,q}$ if $1 \le q \le 2$, where $X_T^{p,q}$ is as in Theorem 5.19. Then we arrive at the following theorem which is proven analogously to Theorem 5.19:

Theorem 5.23. *Let $1/q > |1-2/p|$. Then for any initial data $u_0 \in M_{p,q}$, there is a $T > 0$ and a unique solution u to* (1.1) *in*

$$u \in Y_T^{p,q} = \begin{cases} [L_t^\infty M_{1,1}, X^{\tilde{q}}]_\theta, & \text{if } 1 < p < 2, \\ X_T^q, & \text{if } p = 2, \\ [L_t^\infty M_{\infty,1}, X^{\tilde{q}}]_\theta, & \text{if } 2 < p < \infty. \end{cases} \tag{5.28}$$

Here, $\tilde{q}$ is chosen such that

$$\frac{1}{q} = 1 - \theta + \frac{\theta}{\tilde{q}}, \quad \frac{1}{p} = \begin{cases} 1 - \frac{\theta}{2}, & \text{if } p < 2, \\ \frac{\theta}{2}, & \text{if } p > 2. \end{cases} \tag{5.29}$$

Moreover, either the solution u exists globally in time, or there is $T^* < \infty$ such that

$$\limsup_{t \to T^*} \|u(t)\|_{M_{p,q}} = \infty.$$

Remark 5.24. *Taking into account the wellposedness in $M_{4,2}$ from [117], these results can be slightly strengthened to include the line $1/q = 1-2/p$, $4 \le p \le \infty$. Indeed, in [117] the estimate*

$$\|S(t)f\|_{L^4([0,1]\times\mathbb{R})} \lesssim \|f\|_{M_{4,2}}$$

is shown to hold, which gives rise to an iteration in $L_t^\infty M_{4,2} \cap L_t^{\frac{24}{7}} L_x^4$. Interpolating the linear and the corresponding trilinear estimate with the estimates for $q = 1, p = \infty$ puts us into the setting of Corollary 5.18.

5.4 Global Wellposedness

5.4.1 Global Wellposedness if $p = 2$

If $p = 2$ and $2 < q < \infty$, Oh-Wang [107] showed the existence of almost conserved quantities that are equivalent to the norms in the spaces $M_{p,q}$. To this end they used the complete integrability of cubic NLS via techniques from Killip-Visan-Zhang [77] in combination with the Galilean transform. In this subsection, we extend these almost conserved quantities to the case $q \in [1, 2)$ by using a weight with more decay, as it was done in [77] for Besov spaces $B^s_{2,q}$.

We recall some of the basics from Section 2.2:

- The perturbation determinant

$$\alpha(\kappa, u) = \sum_{n=1}^{\infty} \alpha_{2n}(\kappa, u), \tag{5.30}$$

  where $\kappa > 0$, and

$$\alpha_{2n}(\kappa, u) = \frac{(-1)^{n-1}}{n} \operatorname{Re}\operatorname{tr} \big([(\kappa - \partial)^{-1/2} u(\kappa + \partial)^{-1} \bar{u}(\kappa - \partial)^{-1/2}]^n\big)$$

  is a conserved quantity for regular enough solutions of cubic NLS (5.1).

- The series (5.30) is absolutely convergent if $u \in H^{-1/2+}$ due to the estimate

$$|\alpha_{2n}(\kappa, u)| \lesssim \Big(\int_{\mathbb{R}} \frac{|\hat{u}(\xi)|^2}{(\xi^2 + 4\kappa^2)^{1/2-\delta}} \Big)^n. \tag{5.31}$$

- The leading order term satisfies

$$\alpha_2(\kappa, u) = \operatorname{Re}\operatorname{tr} \big((\kappa - \partial)^{-1} u(\kappa + \partial)^{-1} \bar{u}\big) = \int_{\mathbb{R}} \frac{2\kappa |\hat{u}(\xi)|^2}{\xi^2 + 4\kappa^2} \, d\xi. \tag{5.32}$$

In [107] the construction of the almost conserved quantity on the level of $M_{2,q}$ for $2 \leq q < \infty$ worked as follows: Combining the above facts and invariance of (5.1) under Galilean transformations, we obtain almost conservation of

$$\int_{\mathbb{R}} \frac{|\hat{u}(\xi)|^2}{(\xi - n)^2 + 1} \, d\xi,$$

uniformly in $n \in \mathbb{Z}$.

Moreover, considering $\langle \xi \rangle^{-1/2-}$ instead of a compactly supported bump function for the uniform decomposition on the Fourier side in the definition of the modulation space norm gives an equivalent norm for $2 \leq q \leq \infty$. More precisely, if one defines

$$\|f\|_{MH^{\theta,q}} = \big(\sum_{n \in \mathbb{Z}} \|\langle \xi - n \rangle^{\theta} \hat{f}(\xi)\|^q_{L^2_\xi} \big)^{\frac{1}{q}},$$

then for $\theta < -1/2$ and $2 \leq q \leq \infty$ one has

$$\|f\|_{MH^{\theta,q}} \sim \|f\|_{M_{2,q}}.$$

We follow the proof (see Lemma 1.2 in [107]) to motivate our next definition. The estimate "$\gtrsim$" is trivial since for σ as in Definition 5.3 we have $\sigma(\xi) \lesssim \langle\xi\rangle^{\theta}$. For the converse estimate, write $I_k = [k-1/2, k+1/2)$. Then,

$$\begin{aligned}
\|f\|_{MH^{\theta,q}} &= \Big(\sum_{n\in\mathbb{Z}} \Big(\int_{\mathbb{R}} \langle\xi - n\rangle^{2\theta} |\hat{f}(\xi)|^2 \, d\xi\Big)^{\frac{q}{2}}\Big)^{\frac{1}{q}} \\
&\sim \Big\| \sum_{k\in\mathbb{Z}} \langle k-n\rangle^{2\theta} \int_{I_k} |\hat{f}(\xi)|^2 \, d\xi \Big\|_{\ell_n^{q/2}}^{1/2} \\
&\lesssim \|\langle n\rangle^{2\theta}\|_{\ell_n^1}^{1/2} \Big\| \int_{I_n} |\hat{f}(\xi)|^2 \, d\xi \Big\|_{\ell_n^{q/2}}^{1/2} \\
&\lesssim \|f\|_{M_{2,q}}.
\end{aligned}$$

We see that both the restriction $q \geq 2$ and $\theta < -1/2$ enter in the third line when Young's convolution inequality is used. If we have more decay available, i.e. if $\theta < -1$, we can also use the triangle inequality to get the full range of q.

Lemma 5.25. *If $\theta < -1$ and $1 \leq q \leq \infty$, we have*

$$\|f\|_{MH^{\theta,q}} \sim \|f\|_{M_{2,q}}.$$

Proof. Again "$\gtrsim$" follows immediately from $\sigma_n \lesssim \langle\cdot\rangle^{\theta}$. Now for the converse statement write

$$\begin{aligned}
\Big(\int_{\mathbb{R}} \langle\xi - n\rangle^{2\theta} |\hat{f}(\xi)|^2 \, d\xi\Big)^{\frac{1}{2}} &\sim \Big(\int_{\mathbb{R}} \sum_{l\in\mathbb{Z}} \sigma_l^2(\xi) \langle l-n\rangle^{2\theta} |\hat{f}(\xi)|^2 \, d\xi\Big)^{\frac{1}{2}} \\
&\leq \sum_{l\in\mathbb{Z}} \Big(\int_{\mathbb{R}} \sigma_l^2(\xi) \langle l-n\rangle^{2\theta} |\hat{f}(\xi)|^2 \, d\xi\Big)^{\frac{1}{2}} \\
&= \sum_{l\in\mathbb{Z}} \langle l-n\rangle^{\theta} \Big(\int_{\mathbb{R}} \sigma_l^2(\xi) |\hat{f}(\xi)|^2 \, d\xi\Big)^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|u\|_{MH^{\theta,q}} &= \big(\sum_{n\in\mathbb{Z}} \|\langle\xi - n\rangle^{\theta} \hat{f}(\xi)\|_{L_\xi^2}^q\big)^{\frac{1}{q}} \\
&\lesssim \Big\| \sum_{l\in\mathbb{Z}} \langle l-n\rangle^{\theta} \|\square_l f\|_{L^2} \|_{\ell^q} \\
&\leq \|\langle n\rangle^{\theta}\|_{\ell_n^1} \|u\|_{M^{2,q}},
\end{aligned}$$

by Young's inequality in the last step. Since $\theta < -1$, $\|\langle n\rangle^{\theta}\|_{\ell_n^1} < \infty$. $\square$

From the form of α_2 in (5.32) we see that we will have $\theta = -1$ there. By recombining α_2 for different values of κ, we get more decay (see also Lemma 3.4 in [77]).

Definition 5.26. *Define the weight function $w(\xi, \kappa)$ as*

$$w(\xi, \kappa) = \frac{3\kappa^4}{(\xi^2 + \kappa^2)(\xi^2 + 4\kappa^2)}. \tag{5.33}$$

A short calculation reveals that

$$w(\xi, \kappa) = 4\frac{(\kappa/2)^2}{\xi^2 + \kappa^2} - \frac{\kappa^2}{\xi^2 + 4\kappa^2},$$

and hence

$$4\kappa\alpha_2\Big(\frac{\kappa}{2}, u\Big) - \frac{\kappa}{2}\alpha_2(\kappa, u) = \int w(\xi, \kappa)|\hat{u}(\xi)|^2 \, d\xi. \tag{5.34}$$

Correspondingly we define $\mathcal{F}(\tilde{\Box}_n u)(\xi) = w(\xi - n, 1)^{1/2}\hat{u}(\xi)$ and

$$\|u\|_{\tilde{M}^{2,q}} = \| \|\tilde{\Box}_n u\|_{L^2} \|_{l^q_n}.$$

With these preparations we can prove:

Theorem 5.27. *Let $q \in [1, \infty)$. There exists a constant $C = C(q)$ such that*

$$\|u(t)\|_{M_{2,q}} \leq \begin{cases} C(1 + \|u(0)\|_{M_{2,q}})^{\frac{2}{q}-1}\|u(0)\|_{M_{2,q}}, & \text{if} \quad 1 \leq q \leq 2, \\ C(1 + \|u(0)\|_{M_{2,q}})^{\frac{q}{2}-1}\|u(0)\|_{M_{2,q}}, & \text{if} \quad 2 \leq q < \infty \end{cases} \tag{5.35}$$

for all $u \in \mathcal{S}(\mathbb{R})$ solutions to the cubic NLS on $\mathbb{R}$.

Proof. The case $2 \leq q < \infty$ was treated in [107]. In what follows we slightly modify its argument when $1 \leq q < 2$. Consider the case of small initial data in $M_{2,q}$ first and assume

$$\|u(0)\|_{M_{2,q}} \leq \varepsilon \ll 1.$$

For $n \in \mathbb{Z}$, define $u_n(x, t) = e^{-inx + in^2 t}u(x - 2nt, t)$ which satisfies $|\hat{u}_n(\xi, t)| = |\hat{u}(\xi + n, t)|$ and is a solution to cubic NLS as well.
By Lemma (5.31) for any $\delta > 0$

$$\Big|\alpha\Big(u_n(t), \frac{1}{2}\Big) - \alpha_2\Big(u_n(t), \frac{1}{2}\Big)\Big| \lesssim \sum_{j=2}^{\infty} \bigg(\int_{\mathbb{R}} \frac{|\hat{u}(\xi, t)|^2}{(1 + (\xi - n)^2)^{1/2-\delta}}\bigg)^j.$$

Now for any $q \in (1, \infty)$ if δ is small enough,

$$\begin{aligned} \int_{\mathbb{R}} \frac{|\hat{u}(\xi, 0)|^2}{(1 + (\xi - n)^2)^{1/2-\delta}} &\sim \sum_k \frac{1}{(1 + (k - n)^2)^{1/2-\delta}} \int_{I_k} |\hat{u}(\xi, 0)|^2 d\xi \\ &\lesssim \|u(0)\|^2_{M_{2,q}}, \end{aligned}$$

uniformly in $n \in \mathbb{Z}$. Indeed, if $2 < q < \infty$ we can employ Hölder's inequality with exponent $q/2$ if $\delta > 0$ is small enough. The case $1 \leq q \leq 2$ follows from $q = 2$ because of the embedding $M_{2,q} \subset L^2$. This shows that at time $t = 0$ the series for α is convergent. By continuity in time we can then choose a small time interval $0 \in I$ such that the series stays convergent, and

$$\left|\alpha\Big(u_n(t), \frac{1}{2}\Big) - \alpha_2\Big(u_n(t), \frac{1}{2}\Big)\right| \lesssim \Big(\int_{\mathbb{R}} \frac{|\hat{u}(\xi,t)|^2}{(1+(\xi-n)^2)^{1/2-\delta}}\Big)^2,$$

for all $t \in I$. The same argument works for $\kappa = 1$ instead of $\kappa = 1/2$.

We calculate the difference of α and α_2 by first making use of the above estimate, then localizing in Fourier space and then using Young's convolution inequality, with $I_k = [k, k+1)$,

$$\begin{aligned}
\Big(\sum_{n\in\mathbb{Z}} \left|\alpha\Big(u_n(t), \frac{1}{2}\Big) - \alpha_2\Big(u_n(t), \frac{1}{2}\Big)\right|^{\frac{q}{2}}\Big)^{\frac{1}{q}} &\lesssim \Big(\sum_{n\in\mathbb{Z}}\Big(\int_{\mathbb{R}} \frac{|\hat{u}(\xi,t)|^2}{(1+(\xi-n)^2)^{1/2-\delta}}\Big)^q\Big)^{\frac{1}{q}} \\
&\sim \Big\|\sum_{k\in\mathbb{Z}} \langle k-n\rangle^{-1+2\delta} \int_{I_k} |\hat{u}(\xi)|^2\, d\xi\Big\|_{\ell^q} \\
&\lesssim \|\langle k\rangle^{-1+2\delta}\|_{\ell^{1+}} \|u\|^2_{M_{2,2q-}} \\
&\lesssim \|u\|^2_{M_{2,q}},
\end{aligned}$$

provided $\delta > 0$ is small enough such that we can choose $q < 2q-$, and $q > 1$.

We use the definition of $\tilde{M}_{2,q}$, the subadditivity of the square root, Minkowski's inequality, conservation of α, and the above estimate to find

$$\begin{aligned}
\|u(t)\|_{\tilde{M}_{2,q}} &= \big\|\|\tilde{\square}_n u(t)\|_{L^2}\big\|_{\ell^q_n} = \Big\|\Big(4\alpha_2\Big(\frac{1}{2}, u_n(t)\Big) - \frac{1}{2}\alpha_2(1, u_n(t))\Big)^{\frac{1}{2}}\Big\|_{\ell^q_n} \\
&\leq \Big\|\Big|4(\alpha_2-\alpha)\Big(\frac{1}{2}, u_n(t)\Big) - \frac{1}{2}(\alpha_2-\alpha)(1, u_n(t))\Big|^{\frac{1}{2}}\Big\|_{\ell^q_n} \\
&\quad + \Big\|\Big(4\alpha\Big(\frac{1}{2}, u_n(t)\Big) - \frac{1}{2}\alpha(1, u_n(t))\Big)^{\frac{1}{2}}\Big\|_{\ell^q_n} \\
&\leq 4\Big\|(\alpha_2-\alpha)\Big(\frac{1}{2}, u_n(t)\Big)\Big\|_{\ell^{\frac{q}{2}}_n} + \frac{1}{2}\Big\|(\alpha_2-\alpha)(1, u_n(t))\Big\|_{\ell^{\frac{q}{2}}_n} \\
&\quad + \Big\|\Big|4\alpha\Big(\frac{1}{2}, u_n(0)\Big) - \frac{1}{2}\alpha(1, u_n(0))\Big|^{\frac{1}{2}}\Big\|_{\ell^q_n} \\
&\leq \|u(0)\|_{\tilde{M}_{2,q}} + 4 \sum_{s\in\{0,t\},\kappa\in\{1/2,1\}} \|(\alpha_2-\alpha)(\kappa, u_n(s))\|^{\frac{1}{2}}_{\ell^{\frac{q}{2}}_n} \\
&\leq \|u(0)\|_{\tilde{M}_{2,q}} + C(\|u(0)\|^2_{\tilde{M}_{2,q}} + \|u(t)\|^2_{\tilde{M}_{2,q}}),
\end{aligned}$$

for some constant $C > 0$. Using a continuity argument gives

$$\|u(t)\|_{M_{2,q}} \lesssim \|u(0)\|_{M_{2,q}} \tag{5.36}$$

if $\|u(0)\|_{M_{2,q}} \leq \varepsilon$ with ε sufficiently small.
For general initial data, we apply Lemma 5.13 and the discussion thereafter.

Consider $u_\lambda(x,t) = \lambda^{-1}u\left(\lambda^{-1}x, \lambda^{-2}t\right)$, which is a solution to NLS for all $\lambda \geq 1$. Then for $1 < q \leq 2$, we have

$$\|u_\lambda(0)\|_{M_{2,q}} \lesssim \lambda^{-\frac{1}{2}}\|u(0)\|_{M_{2,q}} \leq \varepsilon \ll 1$$

if $\lambda \sim (1 + \|u(0)\|_{M_{2,q}})^2$. On the other hand,

$$\|u(t)\|_{M_{2,q}} \lesssim \lambda^{\frac{1}{q}} \left\|u_\lambda(\lambda^2 t)\right\|_{M_{2,q}},$$

and so

$$\|u(t)\|_{M_{2,q}} \lesssim \lambda^{\frac{1}{q}-\frac{1}{2}}\|u(0)\|_{M_{2,q}} \sim (1 + \|u(0)\|_{M_{2,q}})^{\frac{2}{q}-1}\|u(0)\|_{M_{2,q}},$$

which finishes the proof if $1 < q < 2$.

This proof does not extend yet to $q = 1$ because the estimate of the tail does not have enough decay in n. The problem here is the coefficient α_4 since for the tail of order homogeneity 6 and more we can estimate with Young's inequality

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |\alpha(u_n) - \alpha_2(u_n) - \alpha_4(u_n)|^{1/2} &\lesssim \sum_{n\in\mathbb{Z}} \Big(\int_{\mathbb{R}} \frac{|\hat{u}(\xi,t)|^2}{(1+(\xi-n)^2)^{1/2-\delta}}\Big)^{\frac{3}{2}} \\
&\sim \Big\| \sum_{k\in\mathbb{Z}} \langle k-n\rangle^{-1+2\delta} \int_{I_k} |\hat{u}(\xi)|^2\, d\xi \Big\|_{\ell^{\frac{3}{2}}}^{\frac{3}{2}} \\
&\lesssim \|\langle k\rangle^{-1+2\delta}\|_{\ell^{\frac{3}{2}}}^{\frac{3}{2}} \|u\|_{L^2}^3 \lesssim \|u\|_{L^2}^3,
\end{aligned}$$

as long as δ stays small enough. To handle the sum

$$\sum_{n\in\mathbb{Z}} |\alpha_4(u_n)|^{1/2},$$

we need to take a closer look at its structure. We use Lemma 2.10 and the fact that $\alpha_4 = \operatorname{Re}\tilde{T}_4(i\kappa)$. Thus

$$\tilde{T}_4(i\kappa) = \frac{i}{2\pi}\int_{\xi_1+\xi_2=\xi_3+\xi_4} \frac{\operatorname{Re}\left(\overline{\hat{u}(\xi_1)\hat{u}(\xi_2)}\hat{u}(\xi_3)\hat{u}(\xi_4)\right)}{(2i\kappa+\xi_1)(2i\kappa+\xi_3)(2i\kappa+\xi_4)},$$

which implies that α_4 can be written as

$$\frac{1}{2\pi}\int_{\xi_1+\xi_2=\xi_3+\xi_4} \frac{2\kappa(\xi_1\xi_3+\xi_1\xi_4+\xi_3\xi_4) - 8\kappa^3}{(4\kappa^2+\xi_1^2)(4\kappa^2+\xi_3^2)(4\kappa^2+\xi_4^2)} \operatorname{Re}\left(\overline{\hat{u}(\xi_1)\hat{u}(\xi_2)}\hat{u}(\xi_3)\hat{u}(\xi_4)\right).$$

We concentrate on the part where there are frequencies in the numerator because the other part is more easily estimated. Now for example,

$$\begin{aligned}
&\int_{\xi_1+\xi_2-\xi_3-\xi_4=0} \frac{|\xi_1\xi_3|}{(4\kappa^2+\xi_1^2)(4\kappa^2+\xi_3^2)(4\kappa^2+\xi_4^2)} |\hat{u}(\xi_1)||\hat{u}(\xi_2)||\hat{u}(\xi_3)||\hat{u}(\xi_4)| \\
&\quad \le \Big\| \frac{|\xi_1|\hat{u}}{4\kappa^2+\xi_1^2} * \frac{|\xi_3|\hat{u}}{4\kappa^2+\xi_3^2} * \frac{\hat{u}}{4\kappa^2+\xi_4^2} * \hat{u} \Big\|_{L^\infty} \\
&\quad \le \Big\| \frac{|\xi|\hat{u}}{4\kappa^2+\xi^2} \Big\|_{L^2}^2 \Big\| \frac{\hat{u}}{4\kappa^2+\xi^2} \Big\|_{L^1} \|\hat{u}\|_{L^1} \\
&\quad \lesssim \Big\| \frac{\hat{u}}{\sqrt{4\kappa^2+\xi^2}} \Big\|_{L^2}^2 \Big\| \frac{\hat{u}}{4\kappa^2+\xi^2} \Big\|_{L^1} \|u\|_{M_{2,1}}.
\end{aligned}$$

Here we used Young's convolution inequality and the fact that

$$\int_{\mathbb{R}} |\hat{u}(\xi)| d\xi = \sum_{k\in\mathbb{Z}} \int_{I_k} |\hat{u}(\xi)| d\xi \le \sum_{k\in\mathbb{Z}} \|\hat{u}\|_{L^2(I_k)} = \|u\|_{M_{2,1}}.$$

Thus to bound $\sum_{n\in\mathbb{Z}} |\alpha_4(\kappa,u_n)|^{1/2}$ we estimate

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \Big(\|u_n\|_{M_{2,1}} \int \frac{|\hat{u}_n(\xi)|^2}{4\kappa^2+\xi^2}\, d\xi \int \frac{|\hat{u}_n(\xi)|}{4\kappa^2+\xi^2}\, d\xi \Big)^{\frac12} \\
&\qquad \sim \sum_{n\in\mathbb{Z}} \|u\|_{M_{2,1}}^{\frac12} \Big(\sum_k \frac{\int_{I_k} |\hat{u}|^2\, d\xi}{4\kappa^2+(k-n)^2} \sum_l \frac{\int_{I_k} |\hat{u}|\, d\xi}{4\kappa^2+(l-n)^2} \Big)^{\frac12} \\
&\qquad \le \|u\|_{M_{2,1}}^{\frac12} \Big\| \Big(\sum_k \frac{\int_{I_k} |\hat{u}|^2\, d\xi}{4\kappa^2+(k-n)^2} \Big)^{\frac12} \Big\|_{\ell_n^2} \Big\| \Big(\sum_l \frac{\int_{I_k} |\hat{u}|\, d\xi}{4\kappa^2+(l-n)^2} \Big)^{\frac12} \Big\|_{\ell_n^2} \\
&\qquad \le \|u\|_{M_{2,1}}^{\frac12} \big(\|\hat{u}\|_{L^2}^2 \sum_k \frac{1}{4\kappa^2+k^2} \big)^{\frac12} \big(\|\hat{u}\|_{L^1} \sum_l \frac{1}{4\kappa^2+l^2} \big)^{\frac12} \\
&\qquad \lesssim \kappa^{-1} \|u\|_{M_{2,1}} \|u\|_{L^2}.
\end{aligned}$$

In the first line we estimated with the inequality from above, then we discretized in Fourier space, then we estimated via Hölder and Young's convolution inequality, and finally we used again that the L^1 norm of the Fourier transform is bounded by the $M_{2,1}$ norm and that the scaling behavior of the sums is $\kappa^{-1/2}$.

Arguing as before, we also obtain the case $q = 1$. □

5.4.2 Global Wellposedness if $p < 2$

If $p < 2$, the spaces $M_{p,q}$ are contained in $M_{2,q}$ and we expect an upgrade to a global result with the use of the principle of persistence of regularity (see e.g. [128]). We use Gronwall's Lemma A.7 to obtain the following blow-up alternative:

Lemma 5.28. *If for all $T > 0$,*

$$\sup_{t\in[0,T]} \|u(t)\|_{M_{\infty,1}} < \infty,$$

and if cubic NLS is locally wellposed in $M^s_{p,q}(\mathbb{R})$ for some $1 \leq p, q \leq \infty$, $s \geq 0$, then it is also globally wellposed in this space.

Proof. By Corollary 5.18 we have to show that the $M^s_{p,q}(\mathbb{R})$ norm cannot blow up. Now u solves

$$u(t) = S(t)u_0 + 2i \int_0^t S(t-s)|u|^2 u(s)ds, \tag{5.37}$$

and hence if $0 \leq t \leq T$, estimating with (5.4),

$$\|u(t)\|_{M^s_{p,q}} \lesssim_T \|u_0\|_{M^s_{p,q}} + \|u\|^2_{L^\infty([0,T],M_{\infty,1})} \int_0^t \|u(s)\|_{M^s_{p,q}} ds. \tag{5.38}$$

Using the assumption $\|u\|^2_{L^\infty([0,T],M_{\infty,1})} \leq C$ we can use Gronwall's inequality and conclude. $\square$

Lemma 5.28 tells us that the $M_{\infty,1}$ norm is a controlling norm in this setting. This shows that when $1 \leq p \leq 2$, $1 \leq q \leq \infty$ and s is high enough, not only the question of local but also of global wellposedness becomes trivial: From the embedding $H^{1/2+} \subset M_{2,1}$ and the construction of conserved quantities adapted to H^s for any $s > -1/2$ [83, 77] we find global in time bounds in $M_{\infty,1}$ if we just embed into $H^{1/2+}$. In the spaces $M_{p,1}$ with $1 \leq p \leq 2$ we also find global wellposedness due to Theorem 5.27. The case $p > 2$ is more complicated and treated below.

For $s = 0$ and general $1 < q < \infty$, we obtained the local wellposedness via interpolation. In the upper triangle $1/q \geq \max(1/p', 1/p)$ the Picard iteration space was

$$X^{p,q}_T = L^\infty_t M_{p,q}([0,T] \times \mathbb{R}) \cap L_t^{\frac{8}{\theta}} [M_{\tilde{p},1}, L^4]_\theta([0,T] \times \mathbb{R}).$$

Note that we could equally well have iterated in

$$\tilde{X}^{p,q}_T = L^\infty_t M_{p,q}([0,T] \times \mathbb{R}) \cap L_t^{\frac{4}{\theta}} [M_{\tilde{p},1}, L^\infty]_\theta([0,T] \times \mathbb{R}),$$

because the Strichartz estimates holds up to $L^4_t L^\infty_x$ in one dimension. With this at hand, we can prove:

Lemma 5.29. *Cubic NLS is globally wellposed in $M_{p,q}(\mathbb{R})$, $1 \leq p < 2$, $1/q \geq 1/p$.*

Proof. We interpolate the multilinear estimates

$$\|u_1 \bar{u}_2 u_3\|_{M_{\tilde{p},1}} \lesssim \|u_1\|_{M_{\infty,1}} \|u_2\|_{M_{\infty,1}} \|u_3\|_{M_{\tilde{p},1}},$$
$$\|u_1 \bar{u}_2 u_3\|_{L^2} \leq \|u_1\|_{L^\infty} \|u_2\|_{L^\infty} \|u_3\|_{L^2}$$

to obtain

$$\|u_1 \bar{u}_2 u_3\|_{M_{p,q}} \lesssim \|u_1\|_{[M_{\infty,1},L^\infty]_\theta} \|u_2\|_{[M_{\infty,1},L^\infty]_\theta} \|u_3\|_{M_{p,q}}, \tag{5.39}$$

where p, q, θ are exactly as in Theorem 5.19. This shows

$$\Big\| \int_0^t S(t-s)|u|^2 u \, ds \Big\|_{M_{p,q}} \lesssim \int_0^t \|u\|^2_{[M_{\infty,1}, L^\infty]_\theta} \|u\|_{M_{p,q}} \, ds,$$

and we can conclude as in Lemma 5.28 if we know that $\|u\|_{L^2([0,T],[M_{\infty,1},L^\infty]_\theta)}$ remains finite. Now with continuous inclusion with T-dependent constants,

$$\begin{aligned}[L^\infty([0,T], M_{2,1}), L^4([0,T], L^\infty)]_\theta &\subset [L^2([0,T], M_{2,1}), L^2([0,T], L^\infty)]_\theta \\ &= L^2([0,T], [M_{2,1}, L^\infty]_\theta) \\ &\subset L^2([0,T], [M_{\infty,1}, L^\infty]_\theta).\end{aligned}$$

Since we could have chosen the left-hand side as the iteration space in Theorem 5.19 we conclude that the solution has locally bounded norm in this space with estimate

$$\|u\|_{[L^\infty([0,1],M_{2,1}), L^4([0,1],L^\infty)]_\theta} \lesssim \|u_0\|_{M_{2,q}}.$$

Note that $p < 2$, hence $M_{p,q} \subset M_{2,q}$. The $M_{2,q}$ norm does not blow up by Theorem 5.27, hence the norm on the left-hand side does not blow up even if we replace $[0,1]$ by a time interval $[0,T]$ as we can just glue together solutions. □

5.4.3 Global Wellposedness if $p > 2$

In the case $u_0 \in M_{p,1}$ with $2 < p < \infty$, we want to use techniques inspired by [48]. Similar results were obtained for $p = 4$ and $p = 6$ in [117]. Note though that the spaces $M^s_{4,2}$ and $M^s_{6,2}$ with $s > 3/2$ embed into $M^1_{4,1}$ and $M^1_{6,1}$ in which we will prove global wellposedness. The goal is to make use of the fact that there is a number N such that for $n \geq N$, the nth Picard iterates will be in an L^2 based space. Indeed, if we keep the notation from Theorem 5.15, then by the multilinear estimate (5.3),

$$\|A_3(u_0)\|_{L^\infty([0,1],M_{2,1})} \lesssim \||S(t)u_0|^2 S(t)u_0\|_{L^\infty([0,1],M_{2,1})} \lesssim \|u_0\|^3_{M_{6,1}},$$

and similarly for each natural number of the form $4n+2, n \in \mathbb{N}_0$, we have

$$\|A_{2n+1}(u_0)\|_{L^\infty([0,1],M_{2,1})} \lesssim_n \|u_0\|^{2n+1}_{M_{4n+2,1}}. \tag{5.40}$$

More generally, we find:

Lemma 5.30. *Given odd natural numbers $k_1, k_2, k_3 \in \mathbb{N}$ and $2m+1 = k_1 + k_2 + k_3$, and $n \in \mathbb{N}$ with $m \geq n$, we have*

$$\|N(A_{k_1}, A_{k_2}, A_{k_3})\|_{L^\infty([0,T],M_{p,1})} \lesssim_m T^m \langle T \rangle^{m+1/2} \|u_0\|^{2m+1}_{M_{p(2m+1),1}} \tag{5.41}$$

$$\|A_{2n+1}\|_{L^\infty([0,T],M_{2,1})} \lesssim_n T^n \langle T \rangle^{n+1/2} \|u_0\|^{2n+1}_{M_{4n+2,1}} \tag{5.42}$$

$$\|A_{2m+1}\|_{L^\infty([0,T],M_{2,1})} \lesssim_m T^m \langle T \rangle^{m+1/2} \|u_0\|^{2n+1}_{M_{4n+2,1}} \|u_0\|^{2(m-n)}_{M_{\infty,1}}. \tag{5.43}$$

Proof. We use the estimate for $0 \le t \le T$

$$\begin{aligned}\|N(A_{k_1}, A_{k_2}, A_{k_3})\|_{M_{p,1}} &= \Big\| \int_0^t S(t-s) A_{k_1} \bar{A}_{k_2} A_{k_3}\, ds \Big\|_{M_{p,1}} \\ &\lesssim T \langle T \rangle^{1/2} \|A_{k_1}\|_{M_{p_1,1}} \|A_{k_2}\|_{M_{p_2,1}} \|A_{k_3}\|_{M_{p_3,1}},\end{aligned}$$

provided $\sum_i 1/p_i = 1/p$. Plugging in the definition of A_{k_i} from Theorem 5.15 iteratively shows that after m iterations we arrive at

$$\|A_{2m+1}(u_0)\|_{M_{p,1}} + \|N(A_{k_1}, A_{k_2}, A_{k_3})\|_{M_{p,1}} \lesssim_n T^m \langle T \rangle^{\frac{m}{2}} \|Lu_0\|^{2m+1}_{M_{(2m+1)p,1}},$$

if $k_1 + k_2 + k_3 = 2m+1$. Together with

$$\|Lu_0\|^{2m+1}_{M_{(2m+1)p,1}} \lesssim \langle T \rangle^{\frac{2m+1}{2}} \|u_0\|^{2m+1}_{M_{(2m+1)p,1}},$$

(5.41) and (5.42) follow. To prove (5.43) we additionally use

$$\|uvw\|_{M_{4n+2,1}} \lesssim \|u\|_{M_{\infty,1}} \|v\|_{M_{\infty,1}} \|w\|_{M_{4n+2,1}},$$

once we reached $p = 4n+2$ in the iteration. $\square$

As is shown for the usual Picard iteration (see for example Theorem 3 in [9]), and because there is no loss in the constant from Hölder's inequality (5.3), the constant in (5.40) grows at most exponentially in n meaning that we are able to sum the remainder term. This motivates that we will be able to construct a solution of NLS of the form

$$u(t) = \sum_{k=1}^{2n-1} A_k(u_0) + v = \tilde{u} + v, \tag{5.44}$$

where

$$\tilde{u} \in C^0([0,T], M_{4n+2,1}) \quad \text{and} \quad v \in C^0([0,T], M_{2,1}).$$

If u has the form (5.44) and solves NLS then v will solve the difference NLS

$$\begin{cases} iv_t + v_{xx} &= |u|^2 u - G(t), \\ v(0) &= 0 \end{cases} \tag{5.45}$$

where $G(t)$ is given by

$$G(t) = i\tilde{u}_t + \tilde{u}_{xx} = \sum_{k=3}^{2n-1} \sum_{k_1+k_2+k_3=k} A_{k_1}(u_0) \bar{A}_{k_2}(u_0) A_{k_3}(u_0).$$

As a fixed point equation this equation reads

$$v(t) = N\Big(v + \sum_{k=1}^{2n-1} A_k(u_0), v + \sum_{k=1}^{2n-1} A_k(u_0), v + \sum_{k=1}^{2n-1} A_k(u_0)\Big) - \sum_{k=3}^{2n-1} A_k(u_0). \tag{5.46}$$

The existence and uniqueness issue for v is covered in the following lemma.

Lemma 5.31. *Let $n \in \mathbb{N}$, $u_0 \in M_{4n+2,1}$. There exists $T > 0$ and a solution $v \in C^0([0,T], M_{2,1})$ of (5.46). The solution is unique in $L^\infty([0,T], M_{4n+2,1})$. If T^* denotes its maximal time of existence, then either $T^* = \infty$ or*

$$\limsup_{t \to T^*} \|v(t)\|_{M_{4n+2,1}} = \infty.$$

Proof. We ignore permutations of the arguments of N and rewrite (5.46) as

$$\begin{aligned} v(t) &= N\Big(v + \sum_{k=1}^{2n-1} A_k(u_0), v + \sum_{k=1}^{2n-1} A_k(u_0), v + \sum_{k=1}^{2n-1} A_k(u_0)\Big) - \sum_{k=3}^{2n-1} A_k(u_0) \\ &= N(v,v,v) + N\Big(v, v, \sum_{k=1}^{2n-1} A_k(u_0)\Big) + N\Big(v, \sum_{k=1}^{2n-1} A_k(u_0), \sum_{k=1}^{2n-1} A_k(u_0)\Big) \\ &\quad + N\Big(\sum_{k=1}^{2n-1} A_k(u_0), \sum_{k=1}^{2n-1} A_k(u_0), \sum_{k=1}^{2n-1} A_k(u_0)\Big) - \sum_{k=3}^{2n-1} A_k(u_0). \end{aligned}$$

If we define the function in the last line to be $F(t,x)$, then we can show

$$\|F\|_{L^\infty([0,T],M_{2,1})} \lesssim T^n \langle T \rangle^{n+1/2} \|u_0\|_{M_{4n+2,1}}^{2n+1} + T^{3n-2} \langle T \rangle^{3n-3/2} \|u_0\|_{M_{4n+2,1}}^{6n-3}. \tag{5.47}$$

Indeed, we rewrite

$$\begin{aligned} &N\Big(\sum_{k=1}^{2n-1} A_k(u_0), \sum_{k=1}^{2n-1} A_k(u_0), \sum_{k=1}^{2n-1} A_k(u_0)\Big) \\ &\qquad = \sum_{m=1}^{2n-1} \sum_{k_1+k_2+k_3=m} N(A_{k_1}(u_0), A_{k_2}(u_0), A_{k_3}(u_0)) + F(t,x) \\ &\qquad = \sum_{k=3}^{2n-1} A_k(u_0) + F(t,x), \end{aligned}$$

and use Lemma 5.30 to estimate. In the same fashion, we find

$$\Big\| \sum_{k=1}^{2n-1} A_k(u_0) \Big\|_{L^\infty([0,T],M_{\infty,1})} \lesssim \langle T \rangle^{1/2} \|u_0\|_{M_{4n+2,1}} + T^{n-1} \langle T \rangle^{n-1/2} \|u_0\|_{M_{4n+2,1}}^{2n-1}.$$

This shows that if $\Phi(v)$ is the right-hand side in (5.46), and if $\|v\|_{L^\infty([0,T],M_{2,1})} \leq R$, we have

$$\begin{aligned} &\|\Phi(v)\|_{L^\infty([0,T],M_{2,1})} \\ &\qquad \lesssim TR^3 + TR\langle T \rangle \|u_0\|_{M_{4n+2,1}}^2 + T^{2n-1} R \langle T \rangle^{2n-1} \|u_0\|_{M_{4n+2,1}}^{4n-2} \\ &\qquad\quad + T^n \langle T \rangle^{n+1/2} \|u_0\|_{M_{4n+2,1}}^{2n+1} + T^{3n-2} \langle T \rangle^{3n-3/2} \|u_0\|_{M_{4n+2,1}}^{6n-3}. \end{aligned}$$

Choosing $T \lesssim \min(1, \|u_0\|_{M_{4n+2,1}}^{-2})$ and $R \sim \|u_0\|_{M_{4n+2,1}}$ makes Φ into a mapping

$$\Phi : \{\|v\|_{L^\infty([0,T],M_{2,1})} \leq R\} \to \{\|v\|_{L^\infty([0,T],M_{2,1})} \leq R\}.$$

Since we can obtain a similar estimate on $\Phi(v_1) - \Phi(v_2)$ via polarization, this shows that we can employ the Banach fixed point argument to get a unique solution $v \in L^\infty([0,T], M_{2,1})$ of (5.46). Since we could have iterated in the space $C^0([0,T], M_{2,1})$ as well, we obtain continuity of v.

To prove the stronger blow-up criterion, if $\|v(T)\|_{M_{4n+2,1}}$ stays bounded close to T^*, we can use $\tilde{u}(T) + v(T) \in M_{4n+2,1}$ as new initial data for NLS. But then we transform this into an equation for v again and obtain a small $\delta > 0$ such that we can solve (5.46) on $[T, T+\delta]$ with $T + \delta > T^*$, yielding a contradiction to the maximality.

For the stronger uniqueness statement we note that we can also construct a unique solution u of NLS in $L^\infty([0,T], M_{4n+2,1})$ directly due to its algebra property. Since u and v only differ by finitely many terms which do not blow up in $M_{4n+2,1}$, the uniqueness from u transfers too. $\square$

To go from local to global wellposedness we need to bound a controlling norm for large times. Our controlling norm will be the H^1 norm and the way to bound it will be via estimating the derivative of the time-dependent Hamiltonian and using a Gronwall argument. Since we need the Hamiltonian to control the energy, the method only applies in the defocusing case. This method has also been used in Chapter 4 (respectively [81]) to prove global wellposedness of NLS equations in $H^1(\mathbb{R}) + H^s(\mathbb{T})$, as well as in [117], and it proves to be valuable here as well. More precisely, the difference NLS equation (5.45) is Hamiltonian with respect to

$$H(t,v) = \int \frac{1}{2}|v_x|^2 + \frac{1}{4}\big(|v + \tilde{u}(t)|^4 - |\tilde{u}(t)|^4 - 4\,\mathrm{Re}(\bar{v}G(t))\big)\,dx.$$

From the embedding $H^1 \subset M_{2,1} \subset M_{4n+2,1}$ and Lemma 5.31 we see that a bound on the H^1 norm suffices to upgrade our local to a global result. Arguing as in Lemma 5.28, we find that if we start with one more derivative, i.e. take $u_0 \in M^1_{4n+2,1}$, then the same holds for the solution u.

We first show that when adding the square of the L^2 norm, the Hamiltonian is strong enough to control the H^1 norm:

Lemma 5.32. *For all $T > 0$ and $u_0 \in M_{4n+2,1}$ there exists a constant $C > 0$ such that*

$$E(v) + \|v\|_{L^2}^2 \lesssim H(t,v) + \|v\|_{L^2}^2 + 1 \lesssim E(v) + \|v\|_{L^2}^2 + 1, \tag{5.48}$$

where

$$E(v) = \int \frac{1}{2}|v_x|^2 + \frac{1}{4}|v|^4\,dx.$$

The constant depends on n, $\|u_0\|_{M_{4n+2,1}}$ and T.

Proof. For $0 \le t \le T$,

$$\begin{aligned}\int |v+\tilde{u}|^4 - |v|^4 - |\tilde{u}|^4 - &4\operatorname{Re}(|\tilde{u}|^2\tilde{u}\bar{v})\,dx \\ &\le c\int |v|^2|\tilde{u}|(|v|+|\tilde{u}|)\,dx \\ &\le c(\|\tilde{u}\|_{L^\infty}^2\|v\|_{L^2}^2 + \|\tilde{u}\|_{L^\infty}\|v\|_{L^3}^3) \\ &\le c(\|\tilde{u}\|_{L^\infty}^2\|v\|_{L^2}^2 + \|\tilde{u}\|_{L^\infty}\|v\|_{L^2}\|v\|_{L^4}^2) \\ &\le (1+(C(\varepsilon))\|\tilde{u}\|_{L^\infty}^2\|v\|_{L^2}^2 + \varepsilon E(v).\end{aligned}$$

This term is fine due to the estimate $\|\tilde{u}\|_{L^\infty_{t,x}} \lesssim_T \|u_0\|_{M_{4n+2,1}}$. Knowing that

$$\int |\tilde{u}|^4\,dx \le C(\|u_0\|_{M_{4n+2,1}}, T),$$

it remains to show that $|\tilde{u}|^2\tilde{u} - G(t)$ can be estimated in L^2 if $u_0 \in M_{4n+2,1}$. Indeed, we rewrite it as

$$\begin{aligned}|\tilde{u}|^2\tilde{u} &= \sum_{k_1,k_2,k_3=1}^{2n-1} A_{k_1}(u_0)\bar{A}_{k_2}(u_0)A_{k_3}(u_0) \\ &= \sum_{k=1}^{2n-1}\sum_{k_1+k_2+k_3=k} A_{k_1}(u_0)\bar{A}_{k_2}(u_0)A_{k_3}(u_0) + R(t) = G(t) + R(t),\end{aligned}$$

where $R(t)$ has only terms of homogeneity $2n+1 \le k \le 6n-3$. Thus as in the proof of Lemma 5.31, for all $T > 0$,

$$\|R\|_{L^\infty([0,T],L^2)} \lesssim_T \|u_0\|_{M_{4n+2,1}}^{2n+1} + \|u_0\|_{M_{4n+2,1}}^{6n-3}.$$

Hence

$$\begin{aligned}\int \operatorname{Re}((|\tilde{u}|^2\tilde{u} - G(t))\bar{v})\,dx &\lesssim_T \|v\|_{L^2}(\|u_0\|_{M_{4n+2,1}}^{2n+1} + \|u_0\|_{M_{4n+2,1}}^{6n-3}) \\ &\le \|v\|_{L^2}^2 + C(\|u_0\|_{M_{4n+2,1}}),\end{aligned}$$

which implies (5.48). □

Theorem 5.33. *Let $2 < p < \infty$ and assume that $u_0 \in M_{p,1}^1$. Then the local solution from Lemma 5.31 exists for all times. In particular, there exists a unique global solution $u \in C^0([0,\infty), M_{p,1}^1)$ to the defocusing cubic NLS with initial data $u(0) = u_0$.*

Proof. Via scaling (see e.g. Theorem 3.2. in [40]) we reduce to consider small initial data. Moreover, there exists an $n \in \mathbb{N}_0$ such that $p \le 4n+2$, hence Lemma 5.31 is applicable and thus without loss of generality we may assume $p = 4n+2$. Fix some $T > 0$.

We look at the time derivatives of the L^2 norm and H and aim to use Gronwall. Now with the notation

$$(f,g) = \int \operatorname{Re}(f\bar{g})\,dx,$$

we calculate that for $0 \le t \le T$,

$$\begin{aligned}
\partial_t \frac{1}{2}\|v\|_{L^2}^2 &= (v, v_t) = \big(v, |v+\tilde{u}|^2(v+\tilde{u}) - G(t)\big) \\
&\lesssim \int |v|^2(|v|^2 + |\tilde{u}|^2) + |v|(|\tilde{u}|^2\tilde{u} - G(t))\,dx \\
&\lesssim E(v) + \|\tilde{u}\|_{L^\infty([0,T]\times\mathbb{R})}^2 \|v\|_{L^2}^2 + \|v\|_{L^2}\||\tilde{u}|^2\tilde{u} - G(t)\|_{L^\infty([0,T],L^2)} \\
&\lesssim E(v) + \|v\|_{L^2}^2 + 1.
\end{aligned}$$

The last inequality was proven in the proof of (5.48) and its constant depends both on T and $\|u_0\|_{M_{4n+2,1}}$. For the Hamiltonian, we argue as in Theorem 4.8 respectively [81, Theorem 4.1] to see that only time derivatives on terms with $\tilde{u}$ and G prevail,

$$\partial_t H = (\tilde{u}_t, |v|^2 v + |v|^2\tilde{u} + 2\operatorname{Re}(\bar{v}\tilde{u})v) + \big(v, \partial_t(|\tilde{u}|^2\tilde{u} - G)\big). \tag{5.49}$$

Indeed, for the bilinear part of H we calculate [2]

$$\partial_t \frac{1}{2}(v_x, v_x) = (v_t, -v_{xx}) = -(v_t, |v+\tilde{u}|(v+\tilde{u}) - G),$$

and for the remaining part,

$$\begin{aligned}
&\partial_t \int \frac{1}{4}\big(|v+\tilde{u}|^4 - |\tilde{u}|^4\big) - \operatorname{Re}(\bar{v}G)\,dx \\
&\quad = (v_t, |v+\tilde{u}|^2(v+\tilde{u}) - G) + (\tilde{u}_t, |v+\tilde{u}|^2(v+\tilde{u}) - |\tilde{u}|^2\tilde{u}) - (v, G_t),
\end{aligned}$$

from which (5.49) follows. We recall $\tilde{u}_t = -iG(t) + i\tilde{u}_{xx}$ and plug this into the first summand. The worst term is

$$(\tilde{u}_{xx}, |v|^2 v) = -(\tilde{u}_x, (|v|^2 v)_x) \lesssim \|\tilde{u}_x\|_{L^\infty_{t,x}} \|v\|_{L^4}^2 \|v_x\|_{L^2} \lesssim E(v),$$

since we are able to bound $\tilde{u}_x$ in L^∞ because $u_0 \in M_{4n+2,1}^1 \subset M_{\infty,1}^1$. Since G, $\tilde{u}$, and $\tilde{u}_x$ can be bounded in L^∞ uniformly in time, the other terms in the first summand of (5.49) are estimated more easily. It remains to estimate

$$\big(v, \partial_t(|\tilde{u}|^2\tilde{u} - G)\big) = (v, \partial_t R),$$

where with the notation from the proof of (5.48) we have

$$R = \sum_{k_i, k_1+k_2+k_3 \ge 2n+1}^{2n-1} A_{k_1}(u_0)\bar{A}_{k_2}(u_0)A_{k_3}(u_0).$$

[2] Strictly speaking this is only formal, the term $(v_t, -v_{xx})$ is not well-defined because both factors are only distributional. One can make this rigorous by going to the interaction picture in the calculation, see Theorem 4.8 respectively [81, Theorem 4.1] for details.

Now for each k,

$$i\partial_t A_k(u_0) + \partial_x^2 A_k(u_0) = \sum_{k_1+k_2+k_3=k} A_{k_1}(u_0)\bar{A}_{k_2}(u_0)A_{k_3}(u_0).$$

Again the worst term comes from the two derivatives. From partial integration,

$$\begin{aligned}&(v, (\partial_x^2 A_{k_1})\bar{A}_{k_2}A_{k_3})\\ &\quad = -(v_x, (\partial_x A_{k_1})\bar{A}_{k_2}A_{k_3}) - (v, (\partial_x A_{k_1})(\partial_x \bar{A}_{k_2})A_{k_3}) - (v, (\partial_x A_{k_1})\bar{A}_{k_2}\partial_x A_{k_3}).\end{aligned}$$

In order to use Cauchy Schwarz we have to be sure that the functions that are integrated against v or v_x are in L^2. But this holds since $k_1 + k_2 + k_3 \geq 2n + 1$ and since $u_0 \in M^1_{4n+2,1}$. All in all we find

$$\partial_t(H + C\|v\|^2_{L^2}) \lesssim H + C\|v\|^2_{L^2} + 1 \quad \text{for all} \quad 0 \leq t \leq T,$$

and hence by Gronwall's lemma

$$\sup_{t\in[0,T]} H(t,v) + C\|v\|^2_{L^2} < \infty,$$

which proves the theorem. □

Remark 5.34. *The same method applies to $M^s_{p,q}$ for $2 < p < \infty$ and $s > 2 - 1/q$. In this case by Theorem 5.5 an embedding $M^s_{p,q} \subset M^1_{p,1}$ holds so that the local wellposedness becomes trivial by the algebra property. See also [117] for $p = 4$ and $p = 6$, and the remark therein for general p and $q = 2$. This shows that for all spaces $M^s_{p,q}$ with $2 \leq p < \infty, 1 \leq q \leq \infty$ one has global wellposedness if s is large enough. Using Lemma 5.28 and Theorem 5.27 the same holds for $1 \leq p \leq 2$. It remains open whether a global result can be achieved in a space $M^s_{p,q}$ with $p = \infty$.*

5.5 Illposedness for Negative Regularity

We complement the wellposedness results and prove that the cubic NLS is not quantitatively wellposed in $M^s_{p,q}$ if $s < 0$. This includes the cases $p, q = \infty$ and extends considerations from the introduction of [117] where illposedness was shown using Galilean invariance. We want to remark that results on norm-inflation for nonlinear Schrödinger equations in modulation spaces have been proven in [15], though some of them rule out the cubic case due to the complete integrability. Norm inflation and infinite loss of regularity for fractional Hartree and cubic NLS equations have been investigated in [16]. The proof of our result is inspired by [98]. More precisely, we show that:

Theorem 5.35. *Let $1 \leq p, q \leq \infty$. When $s < 0$, there is no function space X_T which is continuously embedded into $C([0,T], M^s_{p,q}(\mathbb{R}))$ such that there exists a $C > 0$ with*

$$\|S(t)f\|_{X_T} \leq C\|f\|_{M^s_{p,q}}, \tag{5.50}$$

and

$$\| \int_0^t S(t-s)|u|^2 u(s)\, ds \|_{X_T} \le C \|u\|_{X_T}^3. \tag{5.51}$$

In particular, there is no $T > 0$ such that the flow map $f \mapsto u(t)$ mapping f to a unique local solution on the interval $[-T, T]$ is C^3 at $f = 0$ from $M^s_{p,q}$ to $M^s_{p,q}$.

Proof. We first prove that the failure of the above estimates implies that the data-to-solution map cannot be C^3. Indeed, we consider $f = \gamma u_0$ where $u_0 \in M^s_{p,q}$ is fixed, and denote by $u(\gamma, t, x)$ the unique solution of (5.15). Then

$$\begin{aligned}
u &= S(t)\gamma u_0 \mp 2i \int_0^t S(t-s)(|u|^2 u)\, ds, \\
\partial_\gamma u &= S(t) u_0 \mp 2i \int_0^t S(t-s)(2|u|^2 \partial_\gamma u + u^2 \partial_\gamma \bar{u})\, ds, \\
\partial_\gamma^2 u &= \mp 2i \int_0^t S(t-s)(2|u|^2 \partial_\gamma^2 u + u^2 \partial_\gamma^2 \bar{u} + 4|\partial_\gamma u|^2 u + 2(\partial_\gamma u)^2 \bar{u})\, ds, \\
\partial_\gamma^3 u &= \mp 2i \int_0^t S(t-s)(2|u|^2 \partial_\gamma^3 u + u^2 \partial_\gamma^3 \bar{u} + 6\partial_\gamma^2 u \partial_\gamma u \bar{u} + 6\partial_\gamma^2 u \partial_\gamma \bar{u} u + 6\partial_\gamma^2 \bar{u} \partial_\gamma u u \\
&\qquad + 6|\partial_\gamma u|^2 \partial_\gamma u)\, ds.
\end{aligned}$$

Putting $\gamma = 0$ will give $u = 0$, then $\partial_\gamma u = S(t)u_0$, then $\partial_\gamma^2 u = 0$ and,

$$\partial_\gamma^3 u(0, t, x) = \mp 12 i \int_0^t S(t-s)(|S(s)u_0|^2 S(s)u_0)\, ds.$$

If the flow is C^3, then this implies for any $t \in [0, T]$ the bound

$$\| \int_0^t S(t-s)(|S(s)u_0|^2 S(s)u_0)\, ds \|_{M^s_{p,q}} \lesssim \|u_0\|^3_{M^s_{p,q}}. \tag{5.52}$$

We will show below that (5.52) fails, which then gives the claim.

To show that there is no quantitative wellposedness, we show failure of (5.52) as well. Indeed, using the linear bound in the nonlinear bound would exactly imply (5.52).

To prove failure of (5.52), we look for a lower bound in $M^s_{p,q}$ of

$$g(t, x) = \int_0^t S(t-s)(|S(s)u_0|^2 S(s)u_0)\, ds.$$

Denote by $\hat{g}(t, \xi)$ the Fourier transform $x \mapsto \xi$ of g. We rewrite

$$\begin{aligned}
\hat{g}(t, \xi) &= \int_0^t e^{-i(t-s)\xi^2} \int_{\xi_1 - \xi_2 + \xi_3 = \xi} e^{-is(\xi_1^2 - \xi_2^2 + \xi_3^2)} \hat{u}_0(\xi_1) \overline{\hat{u}_0}(\xi_2) \hat{u}_0(\xi_3)\, d\xi_1 d\xi_3 ds \\
&= e^{-it\xi^2} \int_{\xi_1 - \xi_2 + \xi_3 = \xi} \hat{u}_0(\xi_1) \overline{\hat{u}_0}(\xi_2) \hat{u}_0(\xi_3) \frac{e^{it\chi} - 1}{i\chi}\, d\xi_1 d\xi_3,
\end{aligned}$$

where $\chi = \xi^2 - \xi_1^2 + \xi_2^2 - \xi_3^2$. We choose $\hat{u}_0(\xi) = \phi_{N,\alpha}$ a positive bump function compactly supported around N of width α, where $N \gg 1$, $\alpha \ll 1$. Then, $\hat{g}$ can only be nonzero when $\xi \in [N - \frac{3}{2}\alpha, N + \frac{3}{2}\alpha]$. Moreover, when $\xi = \xi_1 - \xi_2 + \xi_3$, we have the factorization

$$\chi = 2(\xi - \xi_1)(\xi - \xi_3), \tag{5.53}$$

which is of size α^2. In particular,

$$\frac{e^{it\chi} - 1}{i\chi} = t + O(t^2\alpha^2).$$

Now the modulation space norm in $M^s_{p,q}$ of u_0 is

$$\|u_0\|_{M^s_{p,q}} = N^s \|u_0\|_{L^p} \sim N^s \alpha^{1-\frac{1}{p}},$$

which can be seen from shifting and scaling on the Fourier side. Consider the case $1 \leq p \leq 2$ first. From the pointwise bound

$$|\hat{g}(t,\xi)| \gtrsim |t| \int_{\xi_1 - \xi_2 + \xi_3 = \xi} \hat{u}_0(\xi_1)\hat{u}_0(\xi_2)\hat{u}_0(\xi_3) d\xi_1 d\xi_2$$

if $|t|\alpha^2 \ll 1$, we infer

$$\|g(t,\cdot)\|_{M^s_{p,q}} \geq \|g(t,\cdot)\|_{M^s_{2,q}} \sim N^s \|\hat{g}(t,\cdot)\|_{L^2} \gtrsim N^s \alpha^{-\frac{1}{2}} \|\hat{g}(t,\cdot)\|_{L^1} \sim N^s t \alpha^{\frac{5}{2}}.$$

Here we used Hölder's inequality in the second last inequality and explicitly calculated the convolution $\|\hat{g}(t,\cdot)\|_{L^1} \sim \alpha^3$ for the last equality. This shows that in order for (5.52) to hold, we need to have

$$N^s |t| \alpha^{\frac{5}{2}} \lesssim N^{3s} \alpha^{3-\frac{3}{p}}.$$

Since t, α are fixed this gives a contradiction if $s < 0$ by letting $N \to \infty$.

We turn to the case $p \in (2, \infty]$. Write

$$\hat{g}(t,\xi) = \hat{g}_1(t,\xi) + \hat{g}_2(t,\xi),$$

where

$$\begin{aligned}
\hat{g}_1(t,\xi) &= t e^{-it\xi^2} \int_{\xi_1 - \xi_2 + \xi_3 = \xi} \hat{u}_0(\xi_1)\hat{u}_0(\xi_2)\hat{u}_0(\xi_3)\, d\xi_1 d\xi_3, \\
\hat{g}_2(t,\xi) &= t e^{-it\xi^2} \sum_{k=1}^{\infty} \int_{\xi_1 - \xi_2 + \xi_3 = \xi} \hat{u}_0(\xi_1)\hat{u}_0(\xi_2)\hat{u}_0(\xi_3) \frac{(it\chi)^k}{(k+1)!}\, d\xi_1 d\xi_3.
\end{aligned}$$

Write $\hat{g}_1(t,\xi) = t e^{-it\xi^2} \hat{G}(\xi)$, where $G = |u_0|^2 u_0$. Now, $\hat{g}_1(t,\xi)$ is still supported on an interval of size 3α around N, and

$$\begin{aligned}
\|g_1(t,\cdot)\|_{L^p} &= \sup_{\|h\|_{L^{p'}}=1} \langle g_1(t,\cdot), h\rangle = \sup_{\|h\|_{L^{p'}}=1} \int_{\mathbb{R}} t e^{-it\xi^2} \hat{G}(\xi) \overline{\hat{h}}(\xi)\, d\xi \\
&\geq t \int_{\mathbb{R}} \frac{|\hat{G}(\xi)|^2}{\|\mathcal{F}^{-1}(e^{-it\xi^2}\hat{G}(\xi))\|_{L^{p'}}}\, d\xi = t \frac{\|G\|^2_{L^2}}{\|S(t)G\|_{L^{p'}}},
\end{aligned}$$

by choosing an $L^{p'}$ normalized version of $\mathcal{F}^{-1}(e^{-it\xi^2}\hat{G}(\xi))$ for h. By Fourier localization of G and from Lemma **??** we see

$$\|S(t)G\|_{L^{p'}} \lesssim \langle t\rangle^{\frac{1}{2}}\|G\|_{L^{p'}} \leq \langle t\rangle^{\frac{1}{2}}\|u_0\|^3_{L^{3p'}} \sim \langle t\rangle^{\frac{1}{2}}\alpha^{2+\frac{1}{p}},$$

and hence, with $\|G\|_{L^2} \sim \alpha^{\frac{5}{2}}$,

$$\|g_1(t,\cdot)\|_{M^s_{p,q}} \gtrsim N^s t\langle t\rangle^{-\frac{1}{2}}\alpha^{3-\frac{1}{p}}.$$

On the other hand we can estimate $g_2(t,\cdot)$ by Lemma **??**, the Hausdorff-Young inequality, the triangle inequality and $|\chi| \sim \alpha^2$, and Young's convolution inequality,

$$\begin{aligned}
\|g_2(t,\cdot)\|_{L^p} &\leq |t|\sum_{k=1}^{\infty}\Big\|S(t)\mathcal{F}^{-1}\int_{\xi_1-\xi_2+\xi_3=\xi}\hat{u}_0(\xi_1)\hat{u}_0(\xi_2)\hat{u}_0(\xi_3)\frac{(it\chi)^k}{(k+1)!}\,d\xi_1 d\xi_3\Big\|_{L^p}\\
&\lesssim |t|\langle t\rangle^{\frac{1}{2}}\sum_{k=1}^{\infty}\frac{|t|^k}{(k+1)!}\Big\|\int_{\xi_1-\xi_2+\xi_3=\xi}\hat{u}_0(\xi_1)\hat{u}_0(\xi_2)\hat{u}_0(\xi_3)\chi^k\,d\xi_1 d\xi_3\Big\|_{L^{p'}_\xi}\\
&\leq |t|\langle t\rangle^{\frac{1}{2}}\sum_{k=1}^{\infty}\frac{|t|^k(c\alpha^2)^k}{(k+1)!}\Big\|\int_{\xi_1-\xi_2+\xi_3=\xi}\hat{u}_0(\xi_1)\hat{u}_0(\xi_2)\hat{u}_0(\xi_3)\,d\xi_1 d\xi_3\Big\|_{L^{p'}_\xi}\\
&\leq |t|\langle t\rangle^{\frac{1}{2}}\sum_{k=1}^{\infty}\frac{|t|^k(c\alpha^2)^k}{(k+1)!}\alpha^{3-\frac{1}{p}}\\
&= c\langle t\rangle^{\frac{1}{2}}|t|^2\alpha^{5-\frac{1}{p}}(1+O(t\alpha^2))
\end{aligned}$$

In particular assuming $|t| \leq 1$ and $\alpha \ll 1$ we see that

$$\|g_2(t,\cdot)\|_{M^s_{p,q}} \ll \|g_1(t,\cdot)\|_{M^s_{p,q}}.$$

Hence the bound (5.52) would imply

$$N^s t\langle t\rangle^{-\frac{1}{2}}\alpha^{3-\frac{1}{p}} \lesssim \|g(t,\cdot)\|_{M^s_{p,q}} \lesssim N^{3s}\alpha^{3-\frac{3}{p}},$$

which as before leads to a contradiction if $s < 0$. □

Appendix A

Appendix

In this section we introduce our notation and state some preliminary results and estimates that are used throughout the thesis.

A.1 Fourier Transform and Function Spaces

Proofs for the statements about the Fourier transform can be found in [90, Chapter 1 and 2]. Note that our definition of the Fourier transform involves different constants, but the proofs from [90, Chapter 1 and 2] remain valid.

For a function $f \in L^1(\mathbb{R}^n)$ we define its Fourier transform by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\, dx.$$

Here $x\xi = x \cdot \xi = x_1\xi_1 + \dots x_n\xi_n$. With this convention, the inverse Fourier transform of a function $g \in L^1(\mathbb{R}^n)$ becomes

$$\mathcal{F}^{-1}g(x) = \check{g}(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} g(\xi)\, d\xi.$$

If $f \in L^1(\mathbb{R}^n)$ and $\partial_{x_k} f \in L^1(\mathbb{R}^n)$, then

$$\mathcal{F}(\partial_{x_k} f)(\xi) = i\xi_k \hat{f}(\xi).$$

Given $f, g \in L^1(\mathbb{R}^n)$, multiplication is mapped to convolution in the sense that

$$\mathcal{F}(f * g)(\xi) = (2\pi)^{\frac{n}{2}} \hat{f}(\xi)\hat{g}(\xi), \qquad \mathcal{F}^{-1}(f * g)(x) = (2\pi)^{-\frac{n}{2}} \check{f}(\xi)\check{g}(\xi).$$

Moreover, both $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be extended to bounded linear operators on $L^2(\mathbb{R}^n)$ which have the isometry property

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\hat{f}\|_{L^2(\mathbb{R}^n)}.$$

The Fourier transform is unitary in the sense that if $f, g \in L^2(\mathbb{R}^n)$, we define

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx.$$

Then,

$$\langle \hat{f}, g \rangle = \langle f, \hat{g} \rangle.$$

For multiindices $\nu, \beta \in \mathbb{N}^n$ define the seminorm

$$\|f\|_{\nu,\beta} = \|x^\nu \partial_x^\beta\|_{L^\infty(\mathbb{R}^n)},$$

and the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ as

$$\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : \|\phi\|_{\nu,\beta} < \infty \text{ for any } \nu, \beta \in \mathbb{N}^n\}.$$

Convergence in $\mathcal{S}(\mathbb{R}^n)$ is defined as follows: we say $\phi_j \to 0$ as $j \to \infty$ if

$$\|\phi_j\|_{\nu,\beta} \to 0, \quad \text{as} \quad j \to \infty,$$

for all $\nu, \beta \in \mathbb{N}^n$. As derivatives are mapped to multiplication and vice versa, the Fourier transform maps Schwartz space into Schwartz space. Even more:

Theorem A.1. *The map $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isomorphism.*

The space of tempered distributions is defined as

$$\mathcal{S}'(\mathbb{R}^n) = \{\psi : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} : \psi \text{ is continuous and linear}\}.$$

Any locally integrable function $f \in L^1_{loc}(\mathbb{R}^n)$ with at most polynomial growth at infinity can be identified with a tempered distribution via

$$\phi \mapsto \psi_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)\,dx.$$

Note that the map $f \mapsto \psi_f$ is injective. Given a tempered distribution $\psi \in \mathcal{S}'(\mathbb{R}^n)$ we define its Fourier transform $\hat{\psi} \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\hat{\psi}(\phi) = \psi(\hat{\phi}) \quad \text{for all} \quad \phi \in \mathcal{S}(\mathbb{R}^n).$$

This definition is compatible with the identification $f \mapsto \psi_f$ because for $f \in L^1(\mathbb{R}^n)$,

$$\hat{\psi}_f = \psi_{\hat{f}}.$$

We can define the convolution of a Schwartz function $\phi \in \mathcal{S}(\mathbb{R})$ and a tempered distribution $\psi \in \mathcal{S}'(\mathbb{R})$ via

$$(\psi * \phi)(x) := \psi(\phi(x - \cdot)).$$

The result is a function which satisfies

$$\psi * \phi \in C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n). \tag{A.1}$$

With other words, convolution preserves the maximal differentiability, but also takes the minimal decay of its arguments. Moreover,

$$\mathcal{F}(\psi * \phi) = (2\pi)^{\frac{n}{2}} \hat{\psi}\hat{\phi} \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

in the sense that $\mathcal{F}(\psi*\phi)(\eta) = (2\pi)^{\frac{n}{2}} \hat{\psi}(\hat{\phi}\eta)$ for all $\eta \in \mathcal{S}(\mathbb{R}^n)$. We endow $\mathcal{S}'(\mathbb{R}^n)$ with the weak-* topology, which is to say that a sequence $\psi_j \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ if

$$\psi_j(\phi) \to 0 \quad \text{as} \quad j \to \infty,$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. With this definition we find:

Theorem A.2. *The map $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is an isomorphism.*

One of the most important examples for our purposes is the following (see [90, Example 1.11]):

Lemma A.3. *We have*

$$(\mathcal{F}e^{-it|\cdot|^2})(\xi) = (2it)^{-\frac{n}{2}} e^{i\frac{|\xi|^2}{4t}}.$$

Proof (Sketch). Consider

$$\phi_\varepsilon(x) = e^{-(\varepsilon+it)|x|^2}.$$

Due to the decay we gain by adding the ε in the exponent, we have $\phi_\varepsilon \in L^1(\mathbb{R}^n)$ for all $\varepsilon > 0$ and we can calculate directly

$$\hat{\phi}_\varepsilon(\xi) = (2(\varepsilon+it))^{-\frac{n}{2}} e^{-\frac{|\xi|^2}{4(\varepsilon+it)}}.$$

Taking $\varepsilon \to 0$, we see $\hat{\phi}_\varepsilon \to \hat{\phi}$ in $\mathcal{S}'(\mathbb{R}^n)$, which shows the statement. □

We turn to periodic functions. For a reference for these statements see [49, Chapter 1], and [121]. Denote $\mathbb{T}^n = (\mathbb{R}/(2\pi\mathbb{Z}))^n$. Then a continuous function $f : \mathbb{R}^n \to \mathbb{C}$ is periodic with period 2π in each coordinate direction if and only if it can be written as a continuous function $f : \mathbb{T}^n \to \mathbb{C}$. We write

$$\int_{\mathbb{T}^n} f(x)\,dx = \int_{[0,2\pi]^n} f(x)\,dx,$$

whenever the integral on the right-hand side is defined. In the periodic case the space of Schwartz functions $\mathcal{S}(\mathbb{T}^n)$ coincides with the space of smooth periodic functions $\mathbb{C}^\infty(\mathbb{T}^n)$. In the same way, tempered distributions are simply distributions, i.e. continuous linear functionals on $C^\infty(\mathbb{T}^n)$.

For an integrable, periodic function $f \in L^1(\mathbb{T}^n)$ we define its Fourier coefficients as

$$\mathcal{F}f(k) = \hat{f}(k) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{T}^n} f(x)e^{-ikx}\,dx, \qquad k \in \mathbb{Z}.$$

If $\hat{f} \in \ell^1(\mathbb{Z})$, the inverse Fourier transform becomes

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) = (2\pi)^{-\frac{n}{2}} \sum_{k\in\mathbb{Z}^n} \hat{f}(k)e^{ikx}, \qquad x \in \mathbb{T}.$$

The Cauchy problem for the (linear) Schrödinger equation is the initial value problem

$$\begin{cases} iu_t + \Delta u = 0, \\ u(0) = f. \end{cases}$$

By taking Fourier transforms on both sides, the equation formally becomes

$$\begin{cases} i\hat{u}_t - |\xi|^2\hat{u} = 0, \\ \hat{u}(0) = \hat{f}, \end{cases}$$

and can be solved by $\hat{u}(t,\xi) = e^{-it|\xi|^2}\hat{f}(\xi)$. This shows that

$$(S(t)f)(x) = (e^{it\Delta}f)(x) = \begin{cases} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi - it|\xi|^2}\hat{f}(\xi)\, d\xi, & x \in \mathbb{R}^n \\ (2\pi)^{-\frac{n}{2}} \sum_{k\in\mathbb{Z}^n} e^{ixk-it|k|^2}\hat{f}(k), & x \in \mathbb{T}^n, \end{cases}$$

solves the linear Schrödinger equation formally. In the real line case, Lemma A.3 can be used to write the integral in terms of a convolution.

In the following we define Sobolev and Besov spaces. We follow [55]. Let $\mathbb{K} \in \{\mathbb{T}, \mathbb{R}\}$.

Definition A.4. *A function $g \in L^1_{loc}(\mathbb{K}^n)$ is called jth weak derivative of $f \in L^1_{loc}(\mathbb{R}^n)$ if for all $\phi \in \mathcal{S}(\mathbb{K}^n)$*

$$\int_{\mathbb{K}^n} f(x)\partial_{x_j}\phi(x)\, dx = -\int_{\mathbb{K}^n} g(x)\phi(x)\, dx.$$

In this case write $g = \partial_{x_j} f$. Similarly one defines higher order weak derivatives.

We say that a function $f \in L^p(\mathbb{K}^n)$ is in the Sobolev space $f \in W^{k,p}(\mathbb{K}^n)$, $p \in [1,\infty], k \in \mathbb{N}$, if for all $\alpha \in \mathbb{N}^n$, $|\alpha| \le k$, there exist weak derivatives $\partial^\alpha f$ which satisfy

$$\partial^\alpha f \in L^p(\mathbb{K}^n).$$

The Sobolev space norm is defined as

$$\|f\|_{W^{k,p}(\mathbb{R}^n)} = \sum_{|\alpha|\le k} \|\partial^\alpha f\|_{L^p(\mathbb{K}^n)}.$$

With this norm, $W^{k,p}(\mathbb{K}^n)$ becomes a Banach space. In the case $p = 2$ there is an equivalent definition extending to fractional k:

Definition A.5. *The Sobolev space $H^s(\mathbb{K}^n)$, $s \in \mathbb{R}$, is defined as the set of all $f \in \mathcal{S}'(\mathbb{K}^n)$ with finite norm*

$$\|f\|_{H^s(\mathbb{K}^n)} = \left(\int_{\mathbb{K}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

where the integral is understood with respect to the counting measure in the case $\mathbb{K} = \mathbb{T}$, that is as a sum over $\xi \in \mathbb{Z}^n$.

In fact the definition for $s \in \mathbb{R}$ can be extended to L^p spaces with $1 < p < \infty$ in the same fashion: define

$$\|f\|_{L^p_s} = \big\|\mathcal{F}^{-1}\big(\langle\cdot\rangle^s \hat{f}(\cdot)\big)\big\|_{L^p},$$

and the space L^p_s correspondingly. Then this definition coincides with $W^{k,p}$ if $k = s$ and gives an equivalent norm.

To define Besov spaces we need Littlewood-Paley projections. Let $\beta_1 : \mathbb{R}^n \to [0,1]$ denote a radially decreasing function $\beta_1(\xi) = 1$ for $|\xi| \leq 1$ and $\operatorname{supp} \beta_1 \subseteq \{|\xi| \leq 2\}$. For $N \in 2^{\mathbb{N}}$ let $\beta_N(\xi) = \beta_1(\xi/N) - \beta_1(\xi/(N/2))$, and let P_N denote the following Fourier multiplier on $\mathbb{K}^n$:

$$(P_N f)\widehat{\ }(\xi) = \beta_N(\xi)\hat{f}(\xi).$$

Definition A.6. *Define the Besov norm of $f \in \mathcal{S}(\mathbb{K}^n)$ for $1 \leq p, q \leq \infty$, $s \geq 0$ by*

$$\|f\|_{B^s_{p,q}} = \Big(\sum_{N \in 2^{\mathbb{N}_0}} N^{qs} \|P_N f\|^q_{L^p}\Big)^{\frac{1}{q}}$$

and with the usual modification for $p, q = \infty$. The notation $N \in 2^{\mathbb{N}_0}$ means that we sum over $N = 2^j$, $j \in \mathbb{N}_0$. The Besov space is defined as the set of all $f \in \mathcal{S}'(\mathbb{K}^n)$ with finite Besov norm $\|f\|_{B^s_{p,q}} < \infty$.

A.2 Inequalities

The following inequalities will be used frequently in this thesis.

Lemma A.7 (Gronwall, Lemma 3.3 in [6])**.** *Let $u, \alpha, \beta : [a,b] \to \mathbb{R}$ be continuous with $\beta \geq 0$. Assume that for all $t \in [a,b]$,*

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)\, ds.$$

Then also

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_s^t \beta(s')\, ds'}\, ds.$$

Theorem A.8 (Young's convolution inequality, Theorem 2.2 in [90])**.** *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$. Then $f * g \in L^r(\mathbb{R}^n)$ with*

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)},$$

if

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Theorem A.9 (Riesz-Thorin, Theorem 2.1 in [90])**.** *Let $p_0 \neq p_1$, $q_0 \neq q_1$. Let T be a bounded linear operator from $L^{p_0}(X, \mathcal{A}, \mu)$ to $L^{q_0}(X, \mathcal{B}, \nu)$ with norm M_0 and from $L^{p_1}(X, \mathcal{A}, \mu)$ to $L^{q_1}(X, \mathcal{B}, \nu)$ with norm M_1. Then, T is bounded from $L^{p_\theta}(X, \mathcal{A}, \mu)$ to $L^{q_\theta}(X, \mathcal{B}, \nu)$ with norm M_θ, where*

$$M_\theta \leq M_0^{1-\theta} M_1^\theta,$$

and

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \theta \in (0,1).$$

Theorem A.10 ([14], 4.4.1)**.** *Let $(A_0^\nu, A_1^\nu)_{(\nu=1,\ldots,n)}$ and (B_0, B_1) be compatible Banach couples. Let $N : \sum_{1 \leq v \leq n}^{\oplus} A_0^\nu \cap A_1^\nu \to B_0 \cap B_1$ be multilinear such that*

$$\begin{aligned} \|N(a_1, \ldots, a_n)\|_{B_0} &\leqslant M_0 \textstyle\prod_{\nu=1}^n \|a_\nu\|_{A_0^\nu}, \\ \|N(a_1, \ldots, a_n)\|_{B_1} &\leqslant M_1 \textstyle\prod_{\nu=1}^n \|a_\nu\|_{A_1^\nu}. \end{aligned}$$

Then T can be uniquely extended to a multilinear mapping $\sum_{1 \leq v \leq n}^{\oplus} [A_0^\nu, A_1^\nu]_\theta \to [B_0, B_1]_\theta$ with norm at most $M_0^{1-\theta} M_1^\theta$.

Theorem A.11 (Sobolev embedding, Theorem 6.2.4 in [55])**.** *Let $1 < p < \infty$.*

- *Let $0 < s < n/p$. Then $L_s^p(\mathbb{R}^n)$ continuously embeds into $L^q(\mathbb{R}^n)$ if $1/p - 1/q = s/n$.*
- *Let $0 < s = n/p$. Then $L_s^p(\mathbb{R}^n)$ continuously embeds into $L^q(\mathbb{R}^n)$ for all $n/s < q < \infty$.*
- *Let $n/p < s < \infty$. Then $L_s^p(\mathbb{R}^n)$ continuously embeds into $C_b^0(\mathbb{R}^n)$ in the sense that every function $f \in L_s^p(\mathbb{R}^n)$ can be modified on a set of measure zero so that the resulting function is bounded and uniformly continuous.*

Theorem A.12 (Gagliardo-Nirenberg interpolation inequality with weights, [89])**.** *There is a constant $C > 0$ such that the following inequality holds for all $u \in C_c^\infty(\mathbb{R}^n)$*

$$\big\||x|^\gamma D^j u\big\|_{L^r} \leq C \big\||x|^\alpha D^m u\big\|_{L^p}^\theta \big\||x|^\beta u\big\|_{L^q}^{1-\theta}, \tag{A.2}$$

if and only if the following conditions hold:

$$\frac{1}{r} + \frac{\gamma - j}{n} = \theta\Big(\frac{1}{p} + \frac{\alpha - m}{n}\Big) + (1-\theta)\Big(\frac{1}{q} + \frac{\beta}{n}\Big), \tag{A.3}$$

$$\theta\alpha + (1-\theta)\beta \geq \gamma, \tag{A.4}$$

$$\theta(\alpha - m) + (1-\theta)\beta + j \leq \gamma \quad \text{if} \quad \frac{1}{q} + \frac{\beta}{n} = \frac{1}{p} + \frac{\alpha - m}{n}, \tag{A.5}$$

$$\theta\alpha + (1-\theta)\beta = \gamma \quad \text{if} \quad \theta = \frac{j}{m}. \tag{A.6}$$

Corollary A.13. *For all $0 \leq j \leq k$, there is a constant $C > 0$ such that the following inequality holds for all $u \in C_c^\infty(\mathbb{R})$*

$$\big\||x|^{k-j} D^j u\big\|_{L^2} \leq C \big\|D^k u\big\|_{L^2}^{\frac{j}{k}} \big\||x|^k u\big\|_{L^2}^{1-\frac{j}{k}}. \tag{A.7}$$

Proof. Set $\gamma = k - j$, $\alpha = 0$, $\beta = k$, and $r = p = q = 2$ in Theorem A.12 and check that all necessary conditions are fulfilled. $\square$

The following result is a standard result when s is an integer. The proof follows from [128, Lemma A.9] by following the explicit control on the constant in the estimate.

We introduce the notation $[p] = \sup\{k \in \mathbb{Z}, k \leq p\}$.

Lemma A.14. *Let $0 \leq s \leq [p]$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{T}\}$. If $f \in H^s(\mathbb{K}) \cap L^\infty(\mathbb{K})$, then also $|f|^{p-1}f \in H^s(\mathbb{K})$ with bound*

$$\||f|^{p-1}f\|_{H^s(\mathbb{K})} \lesssim \|f\|_{L^\infty(\mathbb{K})}^{p-1}\|f\|_{H^s(\mathbb{K})}. \tag{A.8}$$

If p is an odd integer, the same holds for all $0 \leq s < \infty$.

Proof. The case of p being an odd integer follows from the classical estimate

$$\|uv\|_{H^s(\mathbb{K})} \lesssim \|u\|_{L^\infty(\mathbb{K})}\|v\|_{H^s(\mathbb{K})} + \|u\|_{H^s(\mathbb{K})}\|v\|_{L^\infty(\mathbb{K})}.$$

Moreover, we may assume $0 < s < [p]$ because in the case $s = [p]$ the norm can be easily estimated by calculating the weak derivatives directly. We denote by P_N the usual Littlewood-Paley projection onto frequencies around the dyadic number N, which in the torus case is a projection onto a finite number of frequencies. Similarly, we define $P_{\geq N}$ and $P_{<N}$.

First of all we note that

$$\|P_{\leq 1}|f|^{p-1}f\|_{L^2} \leq \|f\|_{L^\infty}^{p-1}\|f\|_{L^2},$$

meaning that we can restrict to frequencies $N > 1$ in the sum

$$\||f|^{p-1}f\|_{H^s(\mathbb{K})}^2 \sim_s \|P_{\leq 1}|f|^{p-1}f\|_{L^2(\mathbb{K})}^2 + \sum_{N>1} N^{2s}\|P_N|f|^{p-1}f\|_{L^2}^2.$$

We write $f = P_{<N}f + P_{\geq N}f$ and observe that

$$\||f|^{p-1}f - |P_{<N}f|^{p-1}P_{<N}f\|_{L^2} \leq p\|P_{\geq N}f\|_{L^2}\|f\|_{L^\infty}^{p-1}$$

since $(|x|^{p-1}x)' = p|x|^{p-1}$ and by the fundamental theorem of calculus. The high-frequency contribution can now be estimated since

$$\sum_{N>1} N^{2s}\|P_{\geq N}f\|_{L^2} \leq \sum_{N' \geq N > 1} N^s(N')^s\|P_{N'}f\|_{L^2}^2 \lesssim \sum_{N'>1}(N')^{2s}\|P_{N'}f\|_{L^2}^2.$$

We turn to the low-frequency contribution and write for $k = [p]$,

$$\|P_N(|P_{<N}f|^{p-1}P_{<N}f)\|_{L^2}^2 \lesssim N^{-2k}\|\partial^k(|P_{<N}f|^{p-1}P_{<N}f)\|_{L^2}^2.$$

From $|\partial^l(|x|^{p-1}x)| \lesssim_l |x|^{p-l}$ for $l \leq k$ and repeatedly using the chain rule, we see that

$$|\partial^k(|P_{<N}f|^{p-1}P_{<N}f)| \lesssim_p \|f\|_{L^\infty}^{p-k} \sum_{r_1+\dots+r_k=k} |\partial^{r_1}P_{<N}f| \dots |\partial^{r_k}P_{<N}f|.$$

We can square this bound and estimate the squared sum on the right-hand side by the sum of the squares, losing a p-dependent factor. Because the number of such tuples depends only on p, we show the estimate for fixed $r_1, \dots, r_k$. By performing a Littlewood-Paley decomposition in each factor and giving up another combinatorical factor by ordering the frequencies, we see

$$\begin{aligned}
&\|\partial^k(|P_{<N}f|^{p-1}P_{<N}f)\|_{L^2}^2 \\
&\quad \lesssim_p \|f\|_{L^\infty}^{2p-2k} \sum_{N_1 \leq \dots \leq N_k < N} \|\partial^{r_1}P_{N_1}f\|_{L^\infty}^2 \dots \|\partial^{r_{k-1}}P_{N_{k-1}}f\|_{L^\infty}^2 \|\partial^{r_k}P_{N_k}f\|_{L^2}^2 \\
&\quad \lesssim \|f\|_{L^\infty}^{2p-2k} \sum_{N_1 \leq \dots \leq N_k < N} N_1^{2r_1} \dots N_k^{2r_k} \|P_{N_1}f\|_{L^\infty}^2 \dots \|P_{N_{k-1}}f\|_{L^\infty}^2 \|P_{N_k}f\|_{L^2}^2 \\
&\quad \lesssim \|f\|_{L^\infty}^{2p-2} \sum_{N' < N} (N')^{2k} \|P_{N'}f\|_{L^2}^2,
\end{aligned}$$

where from the third to the last line we estimated $\|P_{N_i}f\|_{L^\infty} \lesssim \|f\|_{L^\infty}$ to then do the summation first in N_1, then in N_2 and lastly in N_{k-1}, and rename $N_k = N'$. We conclude that

$$\begin{aligned}
&\sum_N N^{2s} \|P_N(|P_{<N}f|^{p-1}P_{<N}f)\|_{L^2}^2 \\
&\quad \lesssim \|f\|_{L^\infty}^{2(p-1)} \sum_N \sum_{N'<N} N^{2s-2k}(N')^{2k} \|P_{N'}f\|_{L^2}^2 \\
&\quad \lesssim \sum_{N'} (N')^{2s} \|P_{N'}f\|_{L^2}^2
\end{aligned}$$

by summing over N first. □

Lemma A.15. *Let $s \in (0,1)$. Then for some constant $c > 0$,*

$$\|vw\|_{H^s(\mathbb{R})} \leq c(\|v\|_{H^s(\mathbb{R})}\|w\|_{L^\infty(\mathbb{T})} + \|v\|_{l^2L^\infty}\|w\|_{H^s(\mathbb{T})}), \tag{A.9}$$

where the l^2L^∞ norm is

$$\|v\|_{l^2L^\infty} = \big(\sum_{k\in\mathbb{Z}} \|v\|_{L^\infty([k,k+1))}^2\big)^{\frac{1}{2}}.$$

In particular,

$$\|vw\|_{H^s(\mathbb{R})} \lesssim \|v\|_{H^s(\mathbb{R})}\|w\|_{H^s(\mathbb{T})}, \tag{A.10}$$

when $s > 1/2$.

Proof. For a set $I \subset \mathbb{R}$, we define

$$\|u\|_{\tilde{H}^s(I)}^2 = \int_I \int_I \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,dxdy + \|u\|_{L^2(I)}^2$$

If $I = \mathbb{R}$, we see that by a change of coordinates $z = x - y$, Plancherel, and another change of coordinates $\eta = \xi z$,

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}}\, dxdy &= \int_{\mathbb{R}} |z|^{-1-2s} \|u(\cdot + z) - u(\cdot)\|_{L^2}^2\, dz \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|e^{i\xi z} - 1|^2}{|z|^{1+2s}}\, dz\, |\hat{u}(\xi)|^2\, d\xi \\ &= C(s) \int_{\mathbb{R}} |\xi|^{2s} |\hat{u}(\xi)|^2\, d\xi,\end{aligned}$$

where $C(s) = \int_{\mathbb{R}} |e^{i\eta} - 1|^2 |\eta|^{-1-2s}\, d\eta$. This shows that $\|u\|_{\tilde{H}^s(\mathbb{R})} \sim \|u\|_{H^s(\mathbb{R})}^2$. In the case where u is a periodic function on $[0,2]$, a similar consideration shows that $\|u\|_{\tilde{H}^s([0,2])} \sim \|u\|_{H^s(\mathbb{T})}$.

The main contribution of the Gagliardo-seminorm comes from the diagonal $\{|x-y| < 1\}$. Indeed, we have the trivial estimate

$$\int_{\mathbb{R}} \int_{\{|x-y| \geq 1\}} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}}\, dxdy \lesssim \|u\|_{L^2(\mathbb{R})}^2.$$

Taking this into account, we can write the norm on $\mathbb{R}$ as an l^2 sum over localized norms,

$$\|u\|_{\tilde{H}^s(\mathbb{R})}^2 \sim \sum_{k \in \mathbb{Z}} \|u\|_{\tilde{H}^s([k,k+2])}^2 + \|u\|_{L^2}^2.$$

Note that the intervals $[k, k+2]$ and $[k-1, k+1]$ have to overlap, because otherwise not the whole region $\{|x-y| < 1\}$ near the diagonal is covered. Indeed, one estimate is trivial,

$$\sum_{k \in \mathbb{Z}} \|u\|_{\tilde{H}^s([k,k+2])}^2 \leq \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \int_k^{k+2} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}}\, dxdy \leq 2\|u\|_{\tilde{H}^s(\mathbb{R})}^2,$$

while for the other estimate we simply note that

$$\begin{aligned}\Big| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}}\, dxdy - \sum_{k \in \mathbb{Z}} \int_k^{k+2} \int_k^{k+2} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}}\, dxdy \Big| \\ \leq \int_{\mathbb{R}} \int_{\{|x-y| \geq 1\}} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}}\, dxdy \lesssim \|u\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

Lastly, we see that the estimate

$$\|vw\|_{\tilde{H}^s(I)} \leq \|v\|_{L^\infty(I)} \|w\|_{\tilde{H}^s(I)} + \|v\|_{\tilde{H}^s(I)} \|w\|_{L^\infty(I)}$$

holds, simply because we can write

$$v(x)w(x) - v(y)w(y) = v(x)(w(x) - w(y)) + (v(x) - v(y))w(y),$$

and by Hölder's inequality with $L^2(I)$ and $L^\infty(I)$.

We are ready to give a proof of (A.9). Without loss of generality, we may assume that w is 2-periodic, otherwise we have to take a slightly different localized sum. Then, by the above considerations and Minkowski,

$$\begin{aligned}
&\|vw\|_{H^s(\mathbb{R})} \sim \|vw\|_{\tilde{H}^s(\mathbb{R})} \sim \Big(\sum_{k\in\mathbb{Z}} \|vw\|^2_{\tilde{H}^s([k,k+2])}\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{k\in\mathbb{Z}} \|v\|^2_{L^\infty([k,k+2])}\|w\|^2_{\tilde{H}^s([k,k+2])}\Big)^{\frac{1}{2}} + \Big(\sum_{k\in\mathbb{Z}} \|v\|^2_{\tilde{H}^s([k,k+2])}\|w\|^2_{L^\infty([k,k+2])}\Big)^{\frac{1}{2}} \\
&\lesssim \|w\|_{H^s(\mathbb{T})}\big(\sum_{k\in\mathbb{Z}} \|v\|^2_{L^\infty([k,k+2])}\big)^{\frac{1}{2}} + \|w\|_{L^\infty(\mathbb{T})}\big(\sum_{k\in\mathbb{Z}} \|v\|^2_{\tilde{H}^s([k,k+2])}\big)^{\frac{1}{2}} \\
&\lesssim \|w\|_{H^s(\mathbb{T})}\|v\|_{l^2L^\infty} + \|w\|_{L^\infty(\mathbb{T})}\|v\|_{H^s(\mathbb{R})}.
\end{aligned}$$

This proves (A.9). Now (A.10) can be deduced with the help of the local Sobolev embedding [42, Theorem 8.2]

$$\|v\|_{L^\infty(I)} \lesssim \|w\|_{H^s(I)}, \tag{A.11}$$

when $s > 1/2$. □

A.3 Operator Theory

This section follows [119, Chapter 3]. We introduce traces and determinants of operators, and state trace class inequalities.

Consider a Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. We want to define trace class operators, which needs the concept of orthonormal bases.

Definition A.16. *Let $H, \langle\cdot,\cdot\rangle$ be a Hilbert space. A family of vectors $\{\phi_i\}_{i\in I}$ is called orthonormal basis if $\langle\phi_i,\phi_j\rangle = \delta_{ij}$ for all $i,j\in I$, and if the linear span of $\{\phi_i\}_{i\in I}$ is dense in H.*

Moreover, we need to be able to take absolute values of operators. Denote by $L(H)$ the linear, continuous operators on H.

Definition A.17. *Given $A \in L(H)$, define $|A| := \sqrt{A^*A}$ as the absolute value of A.*

Such a square root always exists and is unique, hence $|A|$ is well-defined (see [119, Theorem 2.4.4]). Assume that H is separable. Given $A \in L(H)$ and a countable orthonormal basis $\{\phi_n\}_{n=1}^\infty$, define

$$S(A;\{\phi_n\}_{n=1}^\infty) = \sum_{n=1}^\infty \langle\phi_n, A\phi_n\rangle, \tag{A.12}$$

$$S_2(A;\{\phi_n\}_{n=1}^\infty) = \sum_{n=1}^\infty \|A\phi_n\|^2. \tag{A.13}$$

Under the assumption that the summands are non-negative, these definitions do not depend on the choice of orthonormal basis:

Theorem A.18 (Theorems 3.6.1 and 3.6.2 in [119])**.** *Let $A \in L(H)$. Then S_2 is independent of the basis, and for all orthonormal bases $\{\phi_n\}_{n=1}^{\infty}$*

$$S_2(A; \{\phi_n\}_{n=1}^{\infty}) = S_2(A^*; \{\phi_n\}_{n=1}^{\infty}).$$

If moreover $A \geq 0$, then S is independent of the basis.

With these preparations, we can define trace class operators.

Definition A.19. *Let $A \in L(H)$. We say $A \in \mathfrak{I}_1$, the trace class, if and only if for all orthonormal bases $\{\phi_n\}_{n=1}^{\infty}$,*

$$S(|A|; \{\phi_n\}_{n=1}^{\infty}) < \infty.$$

One can show that all trace class operators are compact (see [119, Proposition 3.6.5]). For compact operators, we define singular values using the Hilbert-Schmidt theorem.

Theorem A.20 (Hilbert-Schmidt, Theorem 3.2.1 in [119])**.** *Let $A \in L(H)$ be positive and compact. Then there is an orthonormal basis of eigenvectors whose eigenvalues only accumulate at 0. More specifically, there exist two orthonormal sets, $\{\phi_n\}_{n=1}^{N}$, $\{\psi_m\}_{m=1}^{M}$ (possibly with $N, M = \infty$) whose union is an orthonormal basis of H so that*

$$\begin{gathered} A\phi_n = \lambda_n \phi_n, \qquad \lambda_1 \geq \lambda_2 \geq \cdots > 0, \\ \lambda_n \to 0 \qquad \text{if} \qquad N = \infty, \\ A\psi_m = 0. \end{gathered}$$

Definition A.21. *Given $A \in L(H)$ compact, the singular values $\{\mu_j(A)\}_{j=1}^{N(A)}$ (where $N(A) = rank(A)$) are the eigenvalues of $|A|$, written in decreasing order, counting multiplicities.*

In the case where A has finite rank, one often writes $\mu_j(A) = 0$ for all $j > N(A)$. Using the Hilbert-Schmidt Theorem and polar decomposition, the following canonical decomposition can be shown to exist for compact operators (see [119, Theorem 3.5.1]):

Theorem A.22. *For every compact operator $A \in L(H)$ there exists an expansion*

$$A = \sum_{j=1}^{N(A)} \mu_j(A) \langle \phi_j, \cdot \rangle \psi_j,$$

where $\mu_j(A)$ are the singular values of A, ϕ_j are orthonormal, and $A\phi_j = \mu_j(A)\psi_j$.

With these preparations, it can be shown that

$$S(|A|; \{\phi_n\}_{n=1}^{\infty}) = \sup_{\{\phi_n\},\{\psi_m\} \text{ orthonormal sets}} \sum_{n=1}^{\infty} |\langle \psi_n, A\phi_n \rangle|,$$

and if one of these quantities is finite, they coincide with

$$\|A\|_{\mathfrak{I}_1} := \sum_{n=1}^{\infty} \mu_n(A),$$

where $\mu_n(A)$ are the singular values of A (see [119, Proposition 3.6.5]). Moreover, for all $A \in \mathfrak{I}_1, B \in L(H)$,

$$\|A^*\|_{\mathfrak{I}_1} = \|A\|_{\mathfrak{I}_1}, \qquad \|BA\|_{\mathfrak{I}_1} \leq \|B\|\|A\|_{\mathfrak{I}_1}, \qquad \|AB\|_{\mathfrak{I}_1} \leq \|B\|\|A\|_{\mathfrak{I}_1}.$$

Correspondingly, for trace class operators, we can define the trace.

Theorem A.23 (Theorem 3.6.7 in [119])**.** *For any $A \in \mathfrak{I}_1$ and any orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$, define*

$$\operatorname{tr}(A) = \sum_{n=1}^{\infty} \langle \phi_n, A\phi_n \rangle. \tag{A.14}$$

Then this sum is absolutely convergent and independent of $\{\phi_n\}_{n=1}^{\infty}$. Moreover $\mathfrak{I}_1$ satisfies the operator ideal property, i.e. for all $B \in L(H)$, $AB, BA \in \mathfrak{I}_1$. Finally,

$$\operatorname{tr}(AB) = \operatorname{tr}(BA). \tag{A.15}$$

We will refer to (A.15) as 'cycling the trace'. One can generalize trace class operators similar to Lebesgue spaces:

Definition A.24. *Given $p \in [1, \infty]$, define $\mathfrak{I}_p$ as the set of all compact operators $A \in L(H)$ with*

$$\|A\|_{\mathfrak{I}_p} := \|\mu_n(A)\|_{\ell_n^p} < \infty.$$

This definition generalizes the trace class, and it has similar properties (see [119, Proposition 3.7.2]):

Proposition A.25. *Let $p \in (1, \infty)$ and $A \in L(H)$. Then, for all $A \in \mathfrak{I}_p$,*

$$\|A\|_{\mathfrak{I}_p}^p = \sup_{\{\phi_n\},\{\psi_m\} \ \textit{orthonormal sets}} \sum_{n=1}^{\infty} |\langle \psi_n, A\phi_n \rangle|^p, \tag{A.16}$$

and moreover $A \in L(H)$ is in $\mathfrak{I}_p$ if either side of (A.16) *is finite.*

The trace classes $\mathfrak{I}_p$ are Banach spaces with respect to $\|\cdot\|_{\mathfrak{I}_p}$, and for all $A \in \mathfrak{I}_p, B \in L(H)$, one has

$$\begin{aligned} \|A^*\|_{\mathfrak{I}_p} &= \|A\|_{\mathfrak{I}_p}, \\ \|BA\|_{\mathfrak{I}_p} &\leq \|B\|\|A\|_{\mathfrak{I}_p}, \quad \|AB\|_{\mathfrak{I}_p} \leq \|B\|\|A\|_{\mathfrak{I}_p}, \\ \|A+B\|_{\mathfrak{I}_p} &\leq \|A\|_{\mathfrak{I}_p} + \|B\|_{\mathfrak{I}_p}, \end{aligned}$$

see [119, Theorem 3.7.3]. In particular, $\mathfrak{I}_p$ are also operator ideals. Finally, the trace class norms inherit Hölder type inequalities ([119, Theorems 3.7.4 and 3.7.6]):

Theorem A.26. *Let $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $A \in \mathfrak{I}_p, B \in \mathfrak{I}_q$, we have $AB \in \mathfrak{I}_1$ and*

$$|\operatorname{tr}(AB)| \leq \|AB\|_{\mathfrak{I}_1} \leq \|A\|_{\mathfrak{I}_p} \|B\|_{\mathfrak{I}_q}.$$

Moreover,

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

More generally, if $A \in \mathfrak{I}_p, B \in \mathfrak{I}_q$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, where $p, q, r \in [1, \infty)$, then $AB \in \mathfrak{I}_r$ and

$$\|AB\|_{\mathfrak{I}_r} \leq \|A\|_{\mathfrak{I}_p} \|B\|_{\mathfrak{I}_q}.$$

The case $p = 2$ plays a special role, and operators $A \in \mathfrak{I}_2$ are called Hilbert-Schmidt operators. There is an elegant characterization in this case ([119, Theorem 3.8.3]):

Theorem A.27. *If $A \in \mathfrak{I}_2$, then $A^*A \in \mathfrak{I}_1$ and*

$$\operatorname{tr}(A^*A) = \|A\|_{\mathfrak{I}_2}^2.$$

Moreover, $\mathfrak{I}_2$ is a Hilbert space with inner product

$$\langle A, B \rangle_{\mathfrak{I}_2} = \operatorname{tr}(A^*B).$$

If $H = L^2(\Omega, d\mu)$ is the space of square-integrable functions, and if $A \in L(H)$ is an integral operator, one can characterize the Hilbert-Schmidt norm in terms of the integral kernel ([119, Theorem 3.8.5]):

Theorem A.28. *Let $H = L^2(\Omega, d\mu)$, where (Ω, Σ, μ) is a σ-finite measure space. Then $A \in \mathfrak{I}_2$ if and only if there is an integral kernel $K_A(x, y) \in L^2(\Omega \times \Omega, d\mu \otimes d\mu)$, such that*

$$(Af)(x) = \int_\Omega K_A(x, y) f(y) d\mu(y), \quad \textit{for all} \quad x \in \Omega.$$

Moreover,

$$\|A\|_{\mathfrak{I}_2} = \|K_A\|_{L^2(\Omega \times \Omega, d\mu \otimes d\mu)}.$$

Consider $A \in L(H)$ with integral kernel $K_A(x, y)$. Then, A^* has integral kernel $K_{A^*}(x, y) = \overline{K(y, x)}$, from which one derives

$$\operatorname{tr}(A^*A) = \|A\|_{\mathfrak{I}_2}^2 = \|K_A\|_{L^2(\Omega \times \Omega, d\mu \otimes d\mu)}^2 = \int_\Omega K_{A^*A}(x, x)\, d\mu(x),$$

where

$$K_{A^*A}(x, y) = \int_\Omega \overline{K(z, x)} K(z, y)\, d\mu(z),$$

is the integral kernel of the composition A^*A. From these formulas one could think that given $A \in L(H)$, which has integral kernel K_A such that $x \mapsto K_A(x, x)$ is continuous, then $A \in \mathfrak{I}_1$ if and only if

$$\int_\Omega |K_A(x, x)| d\mu(x) < \infty.$$

This equivalence is false in general though (see [119, Example 3.11.1]). Still, the following statement can be shown[1]:

Lemma A.29. *Let $A \in \mathfrak{I}_1$ be trace class operator on $L^2(\Omega, d\mu)$ with integral kernel $K_A(x,y)$. Then, $x \mapsto K_A(x,x) \in L^1(\Omega, d\mu)$, and*

$$\operatorname{tr}(A) = \int_\Omega K_A(x,x) d\mu(x).$$

Proof. Since A is compact, there is a canonical decomposition

$$A = \sum_{j=1}^{N(A)} \mu_j(A) \langle \phi_j, \cdot \rangle \psi_j,$$

where $N(A)$ is the rank of A, $\mu_1 \geq \mu_2 \geq \dots$, and $\{\phi_j\}_{j=1}^N, \{\psi_j\}_{j=1}^N$ are orthonormal families. Without loss of generality we assume $N(A) = \infty$, setting $\mu_n(A) = 0$ if $n > N(A)$. Note that

$$\|Af\|^2 = \sum_{n=1}^\infty \mu_n(A)^2 |\langle \phi_n, f\rangle|^2 \leq \|\mu_n(A)\|^2_{\ell^\infty_n} \|f\|^2,$$

and likewise one shows absolute convergence of the series. Now,

$$\begin{aligned}\int_\Omega K_A(x,y) f(y)\, d\mu(y) &= \sum_{j=1}^\infty \mu_j(A) \int_\Omega \overline{\phi_j(y)} f(y)\, d\mu(y) \psi_j(x) \\ &= \int_\Omega \sum_{j=1}^\infty \mu_j(A) \overline{\phi_j(y)} \psi_j(x) f(y)\, d\mu(y),\end{aligned}$$

where in order to exchange summation and integration in the last step, we have to show that the map $(j,y) \mapsto a_j(y) = \mu_j(A)\overline{\phi_j(y)}\psi_j(x) f(y)$, mapping from $\mathbb{N} \times \Omega \to L^2(\Omega, d\mu)$, is integrable in (j,y) with respect to the counting measure in j, and μ in y, in the sense of Bochner integrals. Indeed, since $A \in \mathfrak{I}_1$, and since ϕ_j, ψ_j are normed,

$$\sum_{j=1}^\infty \int_\Omega |\mu_j(A) \overline{\phi_j(y)} f(y)| \|\psi_j\|_{L^2(\Omega,d\mu)}\, d\mu(y) \leq \sum_{j=1}^\infty |\mu_j(A)| \|f\|_{L^2(\Omega,d\mu)} < \infty.$$

By the Schwartz kernel theorem this shows that for almost every $x, y \in \Omega$,

$$K_A(x,y) = \sum_{j=1}^\infty \mu_j(A) \overline{\phi_j(y)} \psi_j(x).$$

Now note that

$$\int_\Omega |K_A(x,x)| d\mu(x) \leq \int_\Omega \sum_{j=1}^\infty |\mu_j(A)| |\overline{\phi_j(x)} \psi_j(x)|\, d\mu(x) \leq \sum_{j=1}^\infty |\mu_j(A)|,$$

[1] The author, who was not able to find a reference for this lemma, wants to thank Dirk Hundertmark for showing him this proof.

since we can exchange integration and summation via Fubini for positive functions. This shows $x \mapsto K_A(x,x) \in L^1(\Omega, d\mu)$. Finally,

$$\begin{aligned}
\operatorname{tr}(A) &= \sum_{n=1}^{\infty} \langle \phi_n, \sum_{j=1}^{\infty} \mu_j(A) \langle \phi_j, \phi_n \rangle \psi_j \rangle = \sum_{n=1}^{\infty} \mu_n(A) \langle \phi_n, \psi_n \rangle \\
&= \sum_{n=1}^{\infty} \int_{\Omega} \mu_n(A) \overline{\phi_n(x)} \psi_n(x)\, d\mu(x) = \int_{\Omega} K_A(x,x)\, d\mu(x),
\end{aligned}$$

where in the last equality we have to use that the map $(n,x) \mapsto \mu_n(A)\overline{\phi_n(x)}\psi_n(x)$ as a function $\mathbb{N} \times \Omega \to \mathbb{C}$ is integrable with respect to the counting measure in n and μ in x. □

The definition of the trace given in Theorem A.23 does not involve eigenvalues. On the other hand we know that in the finite-dimensional case, the trace of a matrix is the sum over its eigenvalues. The same holds for trace class operators and this result is known as Lidskii's Theorem:

Theorem A.30 (Lidskii, Theorem 3.12.2 in [119])**.** *Let $A \in \mathfrak{I}_1$ and $\{\lambda_n\}_{n=1}^{N(A)}$ be its eigenvalues counted up to algebraic multiplicities. Then,*

$$\operatorname{tr}(A) = \sum_{n=1}^{N(A)} \lambda_n.$$

In finite dimensions the natural comrade of the trace is the determinant: instead of summing eigenvalues one multiplies eigenvalues. Clearly when $N(A) \to \infty$, the most interesting case is when then eigenvalues of the operator A approach one as $n \to \infty$, otherwise the infinite product will be either zero or not defined. This is why we are interested in the determinant of $1 + A$ where $A \in \mathfrak{I}_1$. We will not dig into the full story of defining the alternating algebras $\wedge^k(A)$ and just state that there is a definition of the determinant reading

$$\det(1 + A) = \sum_{n=1}^{\infty} \operatorname{tr}(\wedge^n(A)), \tag{A.17}$$

which generalizes the finite-dimensional theory. Indeed:

Theorem A.31 (Theorem 3.10.4 in [119])**.** *Let $A \in \mathfrak{I}_1$. Then the sum in* (A.17) *is absolutely convergent, and*

- $|\det(1 + A)| \leq \det(1 + |A|)$,
- $|\det(1 + A)| \leq \exp(\|A\|_{\mathfrak{I}_1})$,
- *The map $z \mapsto \det(1 + zA)$ is an entire function on $\mathbb{C}$.*

A.4 Hamiltonian Formalism

In this section we give an overview over the Hamiltonian mechanics in the infinite-dimensional case that we work with. We follow the book of Marsden–Ratiu [94]. Only the AKNS example and the last calculation connecting Lie bracket and Poisson bracket are not from this book, the latter being also part of the work in progress [80].

Let Z be a real Banach space, possibly infinite-dimensional. In classical mechanics one chooses $Z = \mathbb{R}^{2n}$, while for our purposes Z will always be some function space. Let $\Omega : Z \times Z \to \mathbb{R}$ be a continuous bilinear form. Ω is said to be (weakly) nondegenerate if $\Omega(z_1, z_2) = 0$ for all $z_2 \in Z$ implies $z_1 = 0$. Define the induced continuous linear mapping

$$\Omega^\flat : Z \to Z^*, \quad \Omega^\flat(z_1) = \Omega(z_1, \cdot).$$

Thus nondegeneracy of Ω is equivalent to injectivity of $\Omega^\flat$, and we call Ω strongly nondegenerate if $\Omega^\flat$ is an isomorphism. In this case, define $\Omega^\sharp = (\Omega^\flat)^{-1}$. Note when it exists, the open mapping theorem guarantees continuity of the inverse map $\Omega^\sharp$. Note also that in the finite-dimensional case, weak and strong degeneracy are equivalent, because Z and Z^* have the same dimension.

Definition A.32. *A nondegenerate skew-symmetric bilinear form Ω on Z is called a symplectic form. The tuple (Z, Ω) is called a symplectic vector space. If Ω is strongly nondegenerate, (Z, Ω) is called a strong symplectic vector space.*

In the finite-dimensional case, Z can only admit a symplectic form Ω if it is even-dimensional. Indeed, if $\Omega_{ij} = \Omega(e_i, e_j)$ for a basis (e_i) of Z, then the matrix of $\Omega^\flat$ relative to the basis (e_i) of Z and the corresponding dual basis (e^i) of Z^* is Ω^t. Injectivity is then equivalent to invertibility, and a skew-symmetric matrix of odd dimensions is never invertible.

We list some examples:

- If $Z = \mathbb{R}^{2n}$ and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where I_n is the n-dimensional identity matrix, and
$$\Omega(z_1, z_2) = z_1^t J z_2,$$
then (Z, Ω) is a strong symplectic space.

- If W is a vector space, let $Z = W \times W^*$. Then there is the canonical symplectic form
$$\Omega((w_1, \alpha_1), (w_2, \alpha_2)) = \alpha_2(w_1) - \alpha_1(w_2)$$
on Z. For example one can consider $W = H^s(\mathbb{R})$ or $W = \mathcal{S}(\mathbb{R})$.

- If W is a vector space with inner product $\langle \cdot, \cdot \rangle$, then
$$\Omega((w_1, w_2), (z_1, z_2)) = \langle z_2, w_1 \rangle - \langle z_1, w_2 \rangle$$

is a symplectic form on $Z = W \times W$. If we use the inner product to identify the dual space of W (for example when $W = L^2(\mathbb{R})$), this coincides with the last example.

- For $Z = \mathbb{C}^n$, define the Hermitian inner product as $\langle z, w \rangle = z^t \bar{w}$. If we identify $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$ in terms of writing $z_j = x_j + iy_j$, then $\mathrm{Re}\langle z, w \rangle$ is the real inner product and $-\mathrm{Im}\langle z, w \rangle$ is the symplectic form from the last example.

- Consider $Z = \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ and

$$\Omega((q_1, r_1), (q_2, r_2)) = \frac{1}{i} \int_{\mathbb{R}} q_2 r_1 - q_1 r_2 \, dx.$$

While Ω is not a form since it maps into the complex number, it is still weakly nondegenerate and symplectic and serves as the starting point to define the AKNS equations. We can extend this definition to $Z = L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

- If $Z = \mathcal{S}(\mathbb{R})$, then

$$\Omega(u_1, u_2) = \int_{\mathbb{R}} (\partial^{-1} u_1)(x) u_2(x) \, dx$$

is a weakly nondegenerate symplectic form. Here, the operator ∂^{-1} is defined as

$$\partial^{-1} u(x) = \int_{-\infty}^{x} u(y) \, dy.$$

Clearly this is a well-defined operator if $u \in L^1$, and $\partial^{-1} u$ will in general be a function in L^∞ which does not have decay any more. If we know that there exists $w \in \mathcal{S}(\mathbb{R})$ such that $w' = u$, then by the fundamental theorem of calculus $w = \partial^{-1} u$. Nondegeneracy follows by choosing $u_1 = u_2'$: if $u_2 \neq 0$, then $\Omega(u_1, u_2) = \|u_2\|_{L^2}^2 \neq 0$. If we want to define this symplectic form on L^2-based Sobolev spaces, then $\dot{H}^{-1/2}$ is the correct choice. This symplectic form is used to define the KdV equations.

Definition A.33. *A smooth map $f : Z \to Y$ between symplectic vector spaces (Z, Ω) and (Y, Θ) is called symplectic, or canonical if*

$$\Theta(Df(z)z_1, Df(z)z_2) = \Omega(z_1, z_2),$$

for all $z, z_1, z_2 \in Z$. Here, Df is the derivative of f.

At first sight, it may be confusing that the property of being symplectic is defined in terms of the derivative. This is due to the fact that we are working on vector spaces instead of manifolds. In the latter case, Df is the map on tangent spaces that needs to be considered.

Note that in the finite-dimensional case, when $\Omega(z_1, z_2) = z_1^t J z_2 = \Theta(z_1, z_2)$ the condition of being symplectic can be reformulated as

$$Df(z)^t J Df(z) = J,$$

for all $z \in Z$. This means that $Df : Z \to Sp(n, \mathbb{R})$, where $Sp(n, \mathbb{R})$ is exactly the subgroup of invertible matrices A such that $A^t J A = J$. It is called the symplectic group.

Definition A.34. *Let (Z, Ω) be a symplectic vector space. A vector field $X : Z \to Z$ is called Hamiltonian if there is a C^1 function $H : Z \to \mathbb{R}$ such that*

$$\Omega^\flat(X(z)) = DH(z),$$

for all $z \in Z$. In this case, we write $X = X_H$ and call H a Hamiltonian function or energy function for X.

In the AKNS example from above the definition gives

$$X_H(q, r) = \Big(\frac{1}{i} \frac{\delta H}{\delta r}(q, r), -\frac{1}{i} \frac{\delta H}{\delta q}(q, r) \Big),$$

where $\frac{\delta}{\delta \phi} H$ denotes the fractional derivative of H in the variable ϕ. In the KdV example, we find

$$\int_{\mathbb{R}} \partial^{-1} X_H(u_1) u_2 \, dx = DH(u_1) u_2 = \int \frac{\delta}{\delta u} H(u_1) u_2 \, dx.$$

Thus we see that

$$X_H(u) = \partial \frac{\delta}{\delta u} H(u)$$

satisfies the equation.

In the finite-dimensional case, strong non-degeneracy implies that given H, the vector field $X_H(z)$ exists for all $z \in Z$. In the infinite-dimensional case this is not guaranteed, but if it exists it is unique due to weak nondegeneracy.

Moreover, in the case where Z is infinite-dimensional, in many applications the vector fields X are not actually defined on all of Z but only on a certain subset. Take for example the KdV vector field

$$X(u) = -u_{xxx} + 6uu_x.$$

If $Z = \mathcal{S}(\mathbb{R})$, then this vector field indeed maps $Z \to Z$, but we can also consider it as a map $L^2 \to H^{-3}$, where some flexibility is needed in this definition.

Not all vector fields are Hamiltonian, but if we have a Hamiltonian vector field, we can (up to constants) recover its Hamiltonian by

$$H(z) - H(0) = \int_0^1 \frac{d}{dt} H(tz) \, dt = \int_0^1 \Omega(X_H(tz), z) \, dt.$$

In the KdV case for example,

$$H(u) - H(0) = \int_0^1 \int_{\mathbb{R}} (\partial^{-1}(-tu_{xxx} + 6t^2 uu_x))udt = \frac{1}{2}\int u_x^2 + 2u^3\, dx.$$

In a similar fashion it can be seen that

$$H(q,r) = \int_{\mathbb{R}} q_x r_x + q^2 r^2\, dx$$

is a Hamiltonian for the cubic NLS equation (2.1) if restricted to the subset $\{r = \pm\bar{q}\}$ of Z.

The natural question arises how to decide whether a given vector field is Hamiltonian. The following holds:

Proposition A.35. *Given a symplectic vector space (Z, Ω) a smooth vector field $X : Z \to Z$ is Hamiltonian if and only if for all $z, z_1, z_2 \in Z$,*

$$\Omega(z_1, DX(z)z_2) = -\Omega(DX(z)z_1, z_2).$$

Note that in the special case where X is a linear vector field, the Hamiltonian can be chosen to be $H(z) = \frac{1}{2}\Omega(Xz, z)$. Also note that due to being symplectic, the condition is equivalent to

$$\Omega(z_1, DX(z)z_2) = \Omega(z_2, DX(z)z_1),$$

for all $z, z_1, z_2 \in Z$.

Proof. We begin with the "if" part. Define $H(z)$ by the integral formula above, that is

$$H(z) = \int_0^1 \Omega(X(tz), z)\, dt.$$

We claim that $X = X_H$. Indeed,

$$\begin{aligned}
DH(z)v = \frac{d}{ds}\Big|_{s=0} H(z+sv) &= \int_0^1 \Omega(DX(tz)tv, z) + \Omega(X(tz), v)\, dt \\
&= \int_0^1 \Omega(DX(tz)tz, v) + \Omega(X(tz), v)\, dt \\
&= \Omega\Big(\int_0^1 DX(tz)tz + X(tz)\, dt, v\Big) = \Omega\Big(\int_0^1 \frac{d}{dt}(tX(tz))\, dt, v\Big) \\
&= \Omega(X(z), v).
\end{aligned}$$

We used the "if" condition in the transition from the first to the second line. Conversely, assume that $X = X_H$, that is

$$\Omega(X(z), v) = DH(z)v,$$

for all $z, v \in Z$. Differentiating with respect to z in the direction u gives

$$\Omega(DH(z)u, v) = D^2 H(v, u).$$

If H is smooth, then the second partial derivatives are symmetric and hence

$$\Omega(DH(z)u, v) = D^2H(v, u) = \Omega(DH(z)v, u) = -\Omega(u, DH(z)v),$$

finishing the proof. $\square$

Definition A.36. *Hamilton's equation for $H \in C^1(Z, \mathbb{R})$ is the (system of) differential equations given by*

$$\frac{d}{dt}c(t) = X_H(c(t)),$$

where a solution is a differentiable curve $c : (-T, T) \to Z$. The flow of X_H is the collection of maps $\phi_t : Z \to Z$ that satisfy Hamilton's equation for all $t \in (-T, T)$ where $T > 0$, with initial condition $\phi_0(z) = z$.

Proposition A.37. *The flow of a vector field X is Hamiltonian if and only if it consists of canonical transformations.*

Proof. We begin with the "only if" part. Let ϕ_t be the flow of a vector field. By the chain rule,

$$\frac{d}{dt}D\phi_t(z)v = D\frac{d}{dt}\phi_t(z)v = DX(\phi_t(z))(D\phi_t(z)v).$$

This shows

$$\begin{aligned}\frac{d}{dt}\Omega(D\phi_t(z)u, D\phi_t(z)v) = {}&\Omega(DX(\phi_t(z))(D\phi_t(z)u), D\phi_t(z)v) \\ &+ \Omega(D\phi_t(z)u, DX(\phi_t(z))(D\phi_t(z)v)).\end{aligned}$$

By Proposition A.35, if $X = X_H$ then $DX(z)$ is Ω-skew for all $z \in Z$, and hence the right-hand side vanishes. Thus for all $t \in (-T, T)$,

$$\Omega(D\phi_t(z)u, D\phi_t(z)v) = \Omega(D\phi_0(z)u, D\phi_0(z)v) = \Omega(u, v),$$

since $\phi_0(z) = z$. This shows that ϕ_t is canonical.

Conversely, let ϕ_t be canonical for all $t \in (-T, T)$. Then we see that $\Omega(D\phi_t(z)u, D\phi_t(z)v)$ is independent of time, and the same calculation shows that

$$0 = \Omega(DX(\phi_t(z))(D\phi_t(z)u), D\phi_t(z)v) + \Omega(D\phi_t(z)u, DX(\phi_t(z))(D\phi_t(z)v)),$$

hence for all $\tilde{z} = \phi_t(z), z_1 = D\phi_t(z)u, z_2 = D\phi_t(z)v$, we have

$$0 = \Omega(DX(\tilde{z})z_1, z_2) + \Omega(z_1, DX(\tilde{z})z_2).$$

Thus we are left to show that $\phi_t(z)$ is bijective and $D\phi_t(z)$ is invertible. But this follows easily $t = 0$ because for $t = 0$, $\phi_t(z) = z$ and $D\phi_t = id$. Thus by Proposition A.35 $X = X_H$ for some Hamiltonian H. $\square$

Definition A.38. *Given a symplectic vector space (Z, Ω) and two differentiable functions $F, G : Z \to \mathbb{R}$, the Poisson bracket $\{F, G\} : Z \to \mathbb{R}$ of F and G is defined as*

$$\{F, G\}(z) = \Omega(X_F(z), X_G(z)).$$

By the definition of the Hamiltonian vector fields X_F and X_G, this is equivalent to

$$\{F, G\}(z) = DF(z)X_G(z) = -DG(z)X_F(z).$$

In the AKNS example from before, this means

$$\{F, G\}(q, r) = \frac{1}{i} \int_{\mathbb{R}} \frac{\delta F}{\delta q} \frac{\delta G}{\delta r} - \frac{\delta G}{\delta q} \frac{\delta F}{\delta r} \, dx.$$

For KdV we find

$$\{F, G\}(u) = \int_{\mathbb{R}} \frac{\delta}{\delta u} F(u) \partial \frac{\delta}{\delta u} G(u),$$

and we recover the usual Gardner bracket.

Definition A.39. *Given a vector field $X : Z \to Z$, and a function $F : Z \to \mathbb{R}$ the directional derivative of F in the direction X is given by*

$$(\mathcal{L}_X F)(z) = DF(z)X(z),$$

for all $z \in Z$.

With this notation, the Poisson bracket can also be written as

$$\{F, G\} = \mathcal{L}_{X_G} F = -\mathcal{L}_{X_F} G$$

It is useful to note that the Lie derivative can be written as a derivative as

$$\mathcal{L}_X F(z) = \frac{d}{ds}\Big|_{s=0} F(z + sX(z)),$$

as long as $z + sX(z)$ doesn't leave the domain of definition for F.

Definition A.40. *The Jacobi-Lie bracket $[X, Y]$ of two vector fields X, Y is the vector field defined by the formula*

$$\mathcal{L}_{[X,Y]} F = \mathcal{L}_X \mathcal{L}_Y F - \mathcal{L}_Y \mathcal{L}_X F,$$

for all $F : Z \to \mathbb{R}$.

Rewriting this in terms of derivatives, we see that

$$\begin{aligned}
\mathcal{L}_{[X,Y]} F(z) &= \frac{d}{ds}\Big|_{s=0} DF(z + sX(z))Y(z + sX(z)) \\
&\quad - \frac{d}{ds}\Big|_{s=0} DF(z + sY(z))X(z + sY(z)) \\
&= DF(z)(DY(z)X(z) - DX(z)Y(z)),
\end{aligned}$$

which shows

$$[X, Y](z) = DY(z)X(z) - DX(z)Y(z).$$

The connection between the Lie bracket and the Poisson bracket in the case of Hamiltonian vector fields is given as follows:

Proposition A.41. *Let $F, G : Z \to \mathbb{R}$ be functionals. Then,*

$$X_{\{F,G\}} = -[X_F, X_G].$$

Proof. Since Ω is weakly nondegenerate, we can test the left hand side in Ω against an element $u \in Z$. Then,

$$\begin{aligned}\Omega(X_{\{F,G\}}(z), u) &= D\{F,G\}(z)u = D\Omega(X_F(z), X_G(z))u \\ &= \Omega(DX_F(z)u, X_G(z)) + \Omega(X_F(z), DX_G(z)u) \\ &= \Omega(DX_F(z)X_G(z), u) + \Omega(u, DX_G(z)X_F(z)) \\ &= \Omega(DX_F(z)X_G(z) - DX_G(z)X_F(z), u) \\ &= \Omega(-[X_F, X_G](z), u),\end{aligned}$$

and the result follows. ☐

Lemma A.42. *The flows of two smooth vector fields X, Y commute if $[X, Y] = 0$. If $X = X_F, Y = X_G$ then this is equivalent to $\{F, G\} = 0$.*

Proof. Denote by ϕ_t^X resp ϕ_t^Y the corresponding flows. Let $z \in Z$. We differentiate with respect to the time variable s and see

$$\frac{d}{ds}(\phi_s^X(\phi_t^Y(z)) - \phi_t^Y(\phi_s^X(z))) = X(\phi_s^X(\phi_t^Y(z)) - (D\phi_t^Y)(\phi_s^X(z))X(\phi_s^X(z)).$$

We claim

$$(D\phi_t^Y)(z)X(z) = X(\phi_t^Y(z)).$$

Assume the claim. Then the right hand side is

$$X(\phi_s^X(\phi_t^Y(z)) - X(\phi_t^Y(\phi_s^X(z))$$

and by Gronwall's inequality, the identity for $s = 0$, and local Lipschitz continuity of X, we find $\phi_s^X(\phi_t^Y(z)) = \phi_t^Y(\phi_s^X(z))$. It remains to prove the claim. To this end we differentiate with respect to t.

$$\begin{aligned}&\frac{d}{dt}(D\phi_t^Y(z)X(z) - X(\phi_t^Y(z))) \\ &= D(Y(\phi_t^Y(z)))X(z) - DX(\phi_t^Y(z))Y(\phi_t^Y(z)) \\ &= DY(\phi_t^Y(z))D\phi_t^Y(z)X(z) - DX(\phi_t^Y(z))Y(\phi_t^Y(z)) \\ &= DY(\phi_t^Y(z))(D\phi_t^Y(z)X(z) - X(\phi_t^Y(z))) + ((DY)X - (DX)Y)(\phi_t^Y(z)) \\ &= DY(\phi_t^Y(z))(D\phi_t^Y(z)X(z) - X(\phi_t^Y(z))) + [X, Y](\phi_t^Y(z))\end{aligned}$$

and the claim follows again by $[X, Y] = 0$, Gronwall's inequality, the identity for $s = 0$ and local Lipschitz continuity of DY. The Hamiltonian case follows from Proposition A.41. ☐

Very much connected to the above result is the following:

Lemma A.43. *If $[X, Y] = 0$, then*

$$\phi_t^X \circ \phi_t^Y = \phi_t^{X+Y}.$$

Proof. Calculate

$$\frac{d}{dt}\phi_t^X \circ \phi_t^Y(z) = X(\phi_t^X(\phi_t^Y(z)) + D\phi_t^X(\phi_t^Y(z))Y(\phi_t^Y(z))$$

We have seen that if $[X, Y] = 0$,

$$(D\phi_t^X)(z)Y(z) = Y(\phi_t^X(z)).$$

This shows

$$\frac{d}{dt}\phi_t^X \circ \phi_t^Y(z) = (X + Y)(\phi_t^X(\phi_t^Y(z)),$$

and the claim follows. □

We end this overview on Hamiltonian mechanics with a calculation connecting Lie bracket and Poisson bracket in a special case. Another example of a symplectic form on $Z = \mathcal{S}(\mathbb{R})$ is given by

$$\Omega(u_1, u_2) = \int_{\mathbb{R}} Ru_1(x)u_2(x)\,dx,$$

where R is again some operator mapping $Z \to Z^*$ that satisfies $R^* = -R$, hence generalizing the KdV example. Define $J = R^{-1}$. The vector fields for a given Hamiltonian H are again

$$X_H = J\frac{\delta}{\delta u}H(u) =: J\nabla H(u).$$

We claim that in this case,

$$-[J\nabla H_1, J\nabla H_2] = J\nabla\{H_1, H_2\}.$$

Indeed, let w, ϕ be test functions. Then, with $X = J\nabla H_1, Y = J\nabla H_2$,

$$\begin{aligned}
&\langle DX(w)Y(w) - DY(w)X(w), \phi\rangle \\
&\qquad = \frac{d}{ds}\Big|_0 \langle J\nabla H_1(w + sY(w)) - J\nabla H_2(w + sX(w)), \phi\rangle \\
&\qquad = -\frac{d}{ds}\Big|_0 \frac{d}{dt}\Big|_0 H_1(w + sY(w) + tJ\phi) - H_2(w + sX(w) + tJ\phi) \\
&\qquad = -\frac{d}{dt}\Big|_0 \langle\nabla H_1(w + tJ\phi), J\nabla H_2(w)\rangle - \langle\nabla H_2(w + tJ\phi), J\nabla H_1(w)\rangle \\
&\qquad = -\frac{d}{dt}\Big|_0 \langle\nabla H_1(w + tJ\phi), J\nabla H_2(w + tJ\phi)\rangle \\
&\qquad = \langle J\nabla\{H_1, H_2\}(w), \phi\rangle,
\end{aligned}$$

hence proving the claim. A similar calculation can be made for the AKNS example.

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