BIFUZZY TOPOLOGICAL SPACES

by

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A thesis submitted for the degree of Doctor of

Philosophy in Mathematics

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London

June 1993

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ACKNOWLEDGEMENT

I would like to express my sincere thanks and acknowledgement to my supervisors;Prof. M.W.Warner and Prof. A.A.Fora;for their help throughout the preparation of this thesis.Without their comments and suggestions,this work would not be in its present form. I thank them both for their effort .My thanks are also to my colleagues; Dr.Muhammad Abdulla;Chairman of Mathematics Department, University of Bahrain; Dr. Rashid Mahmud and Prof. Roshdi Khalil for their moral support and help.Thanks are also for the University of Bahrain for providing me with the facilities needed for typing my thesis.

DECLARATION

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ABSTRACT

Separation axioms for bifuzzy topological spaces namely : P-R_i, P-T_j,P-T_{jw} (i=1,2,j=0,1,2,2 1/2),P-regular and P-normal spaces are defined and many related results are proved such as a bfts (X,τ_1,τ_2) is P-normal iff for every τ_i-closed fuzzy set λ and τ_j-open fuzzy set μ such that $\lambda \subseteq \mu$, there exists a continuous function $f : (X,\tau_1,\tau_2)\to([0,1]_f,L,R)$ such that $\lambda(x)\leq f(x)(1-)\leq f(x)(0+)\leq\mu(x)$, for all $x\in X$.Bifuzzy connected topological spaces are defined such as S-connected,S_W-connected ,P-connected and P_W-connected .We have shown that connectedness is preserved under P-continuity and we have shown that the connectedness of (X,τ_1,τ_2) is not governed by the connectedness of (X,τ_1) and (X,τ_2).Many types of compactness were defined such as S-compact,P-compact,S-α-compact,S-weakly compact, S-α-weakly compact,P-weakly compact,P-α-weakly compact, S-C-compact,P-C-compact,S-C weakly compact,P-C-weakly compact,P-U-compact and P-S-compact .We have proved that P-S-compactness $\Rightarrow$ P-C-compactness $\Rightarrow$ P-U-compactness but P-U-compactness does not imply neither P-C-compactness nor P-S-compactness.Also we have shown that bifuzzy compactness is preserved under continuous surjection.Bifuzzy Lindelof spaces are also defined.We have shown that there are no analogous definitions of S-weakly compact and S-C-compact in Lindelof spaces.Finally we introduce induced and weakly induced bifuzzy topological spaces and prove that a P-Hausdorff compact bfts is P-weakly induced and a P-topological P-weakly induced bfts is P-induced.Lowen's goodness criterion is extended and then used to test the goodness of these definitions.We have proved that

(X,T_1,T_2) is P-T_i,P-T_{iw},P-regular and P-normal iff the bifuzzy topological space $(X,\omega(T_1),\omega(T_2))$ is P-T_i,P-T_{iw} (i=0,1,2,2 1/2), P-regular and P-normal respectively.We have shown that S-connectedness,P-connectedness are good extensions while S_w-connectedness and P_w-connectedness are not.Moreover we have also shown that S-α-compactness is a good extension of S-compactness if it is good for some $\alpha \in [0,1)$;while P-α-compactness is a good extension of P-compactness only for $\alpha=0$.Finally we prove a bitopological space (X,T_1,T_2) has P-f.p.p iff $(X,\omega(T_1),\omega(T_2))$ has P-f.p.p .

NOMENCLATURE

X	a non-empty set.
R	the real line.
N	the set of natural numbers {1,2,3,4,...}.
Q	the set of rational numbers.
I	the closed unit interval [0,1].
$[0,1]_f$	the fuzzy unit interval
φ	the empty set.
<	the relation "less than" on R.
≤	the relation "less than or equal to" on R.
>	the relation "greater than" on R.
≥	the relation "greater than or equal to" on R.
min	minimum.
max	maximum.
inf	infimum.
sup	supremum.

T	a topology on X.
τ	a fuzzy topology on X.
bts	bitopological space.
bfts	bifuzzy topological space.
T_{dis}	the discrete topology on X.
τ_{dis}	the discrete fuzzy topology on X.
T_{ind}	the indiscrete topology on X.
τ_{ind}	the indiscrete fuzzy topology on X.
T_u	the usual topology on the corresponding subset of R.
$T_{r.r}$	the right ray topology on the corresponding subset of R.i.e. $\{\phi,(a, \infty) : a\in R\}$.
$T_{l.r}$	the left ray topology on the corresponding subset of R.i.e. $\{\phi,(-\infty, a) : a\in R\}$.
$\tau_{r.r}$	the right ray fuzzy topology on the corresponding subset of R.i.e. $\{0,\chi_{(a, \infty)} : a\in R\}$
$\tau_{l.r}$	the left ray fuzzy topology on the corresponding subset of R.i.e. $\{0,\chi_{(-\infty,a)}: a\in R\}$
T_{cof}	the cofinite topology on X.

$<T_1,T_2>$	the least upper bound topology generated by T_1 and T_2.
$<\tau_1,\tau_2>$	the least upper bound fuzzy topology generated by τ_1 and τ_2.
cl_1	the closure on the first topology.
T/K	the subspace topology $\{K\cap U:U\in T\}$ on K.
V–U	the difference of V and U.
F^c	the complement of the set F.
Δ	an index set.
C_I	first countable.
C_{II}	second countable.
P	pairwise.
S	semi.
$x\in X$	x is an element of X.
$x\notin X$	x is not an element of X.
$\cup$	the union.
$\cap$	the intersection.
$\subseteq$	the relation "is contained in".
$\not\subseteq$	the relation "is not contained in".

$f:X \rightarrow Y$	a function f from X into Y.
$f(U), f^{-1}(V)$	the image of U and inverse image of V under the function f respectively.
fog	the composite function of f and g.
f.p.p	fixed point property.
χ_A	the characteristic function of A.
X/T	the collection $\{\chi_u : u \in T\}$.
λ/Y	the restriction function $\lambda \cap \chi_Y$.
l.s.c.	lower semi-continuous functions
u.s.c.	upper semi-continuous functions
$\omega(T)$	$\{f \mid f:(X,T) \rightarrow I \text{ is l.s.c.}\}$
τ_C^*	$\{A \subseteq X : \chi_A \in \tau\}$
$L(\tau)$	the initial topology of τ
I_{id}	the identity map

CHAPTER 0

INTRODUCTION

In (1963) Kelly introduced the notion of bitopological spaces which is the triple (X,T_1,T_2) where X is a nonempty set ,T_1 and T_2 are topologies on X.He also defined pairwise Hausdorff,pairwise regular and pairwise normal,then he obtained some generalization of several standard results such as Urysohn's lemma and Tietze's extension theorem.Later on ,several authors have studied these notions and other related concepts.Kim (1968) and Fletcher et al.(1969) introduced the definition of compactness of such spaces .Pervin (1967) initiated the study of connectedness and Birsan (1969),Reilly (1970) and Swart (1971) discussed various aspects of connectedness properties.

In (1965) Zadeh introduced the notion of a fuzzy set in X and other basic related notions such as fuzzy union,intersection and complement, all of which have become standard.Chang (1968) used them to introduce the concept of fuzzy topology as a collection of fuzzy sets on X,stable for arbitrary suprema and finite infima and containing the constant fuzzy sets 0 and 1.Then he studied a number of basic concepts,including fuzzy continuous maps and compactness.Lowen (1976) introduced a new fuzzy topology which moreover contains all constant fuzzy sets. The concept of induced fuzzy topological spaces was introduced by Weiss (1975).Later on,Lowen (1976,1977) introduced the so called "goodness criterion" defined by using lower semi-continuous functions to establish a relationship between fuzzy topological spaces and topological spaces and proved some deeper results on induced fuzzy topological spaces.This idea has been used in one way or another by several authors (see e.g. Martin (1980),Srivastava et al. (1981,1984), Bulbul(1984), Fora (1987,1989,1990) and Mohannadi and Warner (1988,1989)) .Since then an extensive work on fuzzy topological spaces has been carried out by many researchers.In fact most concepts and notions of general topology (which can be regarded as a special case of fuzzy topology where all membership functions in question take values 0 and 1 only) have been extended to fuzzy topology.

The notion of bifuzzy topology has so far remained almost untouched.In (1989) Abd El-Monsef and Ramadan introduced and studied the notion of pairwise α-almost compact in bifuzzy topological spaces.

The purpose of this study is to introduce the notion of bifuzzy topological spaces and concentrate on extending to bifuzzy those bitopological concepts that are most relevant to our purpose . However,most attention is paid to the extension of the separation axioms,compactness and connectedness.We hope that our study will be a further contribution to the development of fuzzy set theory into topological spaces and serve as a guide for the study of other bifuzzy extensions.

This thesis is divided into five main chapters.In the first chapter ,we present all relevant definitions and properties of all topics that will be used later in the thesis .Many of those properties have been proved as separate items by different authors .However we present them as a quick reference to the reader.Bitopological spaces and fuzzy topological spaces were presented .However we stress that all our work was carried according to the given definitions and concepts,for example we shall follow Wong(1974) for the definitions of fuzzy points , fuzzy membership,the direct and inverse image of a fuzzy set.With the exception of the first and second sections,each of the remaining sections can be considered as an introduction to each of the remaining chapters discussed in the thesis.

In chapter two we shall define separation axioms in bifuzzy topological spaces,namely $P\text{-}R_0$,$P\text{-}R_1$,P-Ti,$P\text{-}T_{iw}$(i=0,1,2,2 1/2),P-regular and P-normal.Moreover we have shown that all definitions we have presented are good extensions in the sense of Lowen's criterion.Many other results were proved,especially a version of Uryson's lemma in bifuzzy topological spaces.

Chapter three is reserved for connectedness in bifuzzzy topological spaces.We have defined several types of connectedness for a fuzzy set and show that connectedness is preserved under P-continuous functions.

In chapter four we discuss bifuzzy extensions to most of the existing definitions of fuzzy compactness in the literature.Moreover we shall discuss the goodness criterion and obtain many interesting results reflecting to a large extent parallel properties in classical topology.

The last chapter deals with induced and weakly induced bifuzzy topological spaces.Many important results regarding Hausdorff and compactness properties are presented.

Chapter I

SOME KNOWN RESULTS

In this chapter we present all relevant definitions and properties of bitopological spaces and fuzzy topological spaces that are used elsewhere in the study.For the sake of clarity we divide this part into six sections.The first contains the standard definitions and concepts related to bitopological spaces,the second is devoted to introduce the basic notion of fuzzy sets and fuzzy topology and the third deals with the induced fuzzy topological spaces .In section four we discuss the separation axioms in fuzzy topological spaces and section five is devoted to introduce connectedness in fuzzy topological spaces while section six deals with compactness in fuzzy topological spaces.

§ 1.1 Bitopological spaces.

Kelly (1963) introduced the notion of a bitopological space (bts for short) which is the triple (X,T_1,T_2) ,where X is a non-empty set and T_1,T_2 are two topologies on X .He also defined Pairwise Hausdorff,Pairwise regular and Pairwise normal spaces.

In this section we shall list the main bitopological definitions and results that will be used and investigated later in our work of the thesis.

Definition 1.1.1

A bitopological space (X,T_1,T_2) is called a:

a) P-R_0-space iff for all $x,y\in X, x\neq y$, whenever there exists $U\in T_i$ such that $x\in U$ and $y\notin U$, then there exists $V\in T_j$ such that $y\in V$ and $x\notin V (i,j=1,2\ i\neq j)$.

b) P-R_1-space iff for all $x,y\in X, x\neq y$, whenever there exists $U\in T_1\cup T_2$ such that $x\in U$ and $y\notin U$, then there exists $V\in T_i$ and $W\in T_j$ such that $x\in V$ and $y\in W$ and $V\cap W=\phi$ $(i,j=1,2\ i\neq j)$.

Definition 1.1.2 (Murdeshwar and Naimpally,1966)

A bitopological space (X,T_1,T_2) is called a:

a) P-T_0-space iff for every pair of distinct points, there exists a T_1-or a T_2-neighbourhood of one point not containing the other.

b) P-T_1-space iff for every pair of distinct points x,y, there exists a T_1-or a T_2-neighbourhood of x not containing y.

Definition 1.1.3 (Kelly,1963)

A bitopological space (X,T_1,T_2) is called a:

a) P-T_2-space iff for every pair of distinct points x,y, there exists a T_i-neighbourhood of x and a T_j-neighbourhood of y $(i\neq j)$ which are disjoint.

b) P-Regular space iff for every point $x\in X$ and a T_i-closed set A with $x\notin A$, there exists a T_j-neighbourhood U of A and a T_i-neighbourhood V of x such that $U\cap V=\phi$.

c) P-normal space iff for every T_i-closed set A and T_j-closed set B with $A \cap B = \phi$,there exists a T_j-neighbourhood U of A and a T_i-neighbourhood V of B such that $U \cap V = \phi$.

Definition 1.1.4 (Bourbaki.1966)

Let (X,T) be a topological space and $f: X \to I$.Then:

a) f is called a lower semi-continuous (in short l.s.c.) function iff $f^{-1}(a,1] \in T$ for all $a \in [0,1)$.

b) f is called an upper semi-continuous (in short u.s.c.) function iff $f^{-1}[0,b) \in T$ for all $b \in (0,1]$.

One of the best known characterizations of normal bitopological spaces is a version of Urysohn's Lemma,which is given by the following theorem.

Theorem 1.1.5 (Kelly.1963)

A bts (X,T_1,T_2) is P-normal iff for $i \neq j$ and for every pair of disjoint sets A,B which are respectively T_j-,T_i-closed,there exists a function $f: X \to [0,1]$ such that $f(A)=0, f(B)=1$ and f is T_i-upper semi-continuous and T_j-lower semi-continuous.

Definition 1.1.6 (Kelly.1963)

A bts (X,T_1,T_2) is said to satisfy the second axiom of countability iff (X,T_1) and (X,T_2) are second countable spaces.

Definition 1.1.7 (Weston,1957)

In a bts (X,T_1,T_2) we say T_1 is coupled to T_2 iff for all $G \in T_1$, $cl_1 G \subseteq cl_2 G$.

Lemma 1.1.8 (Weston,1957)

If a bts (X,T_1,T_2) is P-Hausdorff and T_1 is coupled to T_2 ,then (X,T_1) is Hausdorff.

The following result was noted by Kelly (1963) so we put it as a theorem and prove it .

Theorem 1.1.9

Let (X,T_1,T_2) be a P-regular space.

i) If T_2 is coupled to T_1, then $T_1 \subseteq T_2$.

ii) If T_i is coupled to T_j for each $i,j=1,2,i \neq j$.Then $T_1=T_2$.

Proof :

(i) Let (X,T_1,T_2) be a bts, $U \in T_1$ and $x \in U$.Since (X,T_1,T_2) is P-regular ,so there exists $G \in T_1$ such that $x \in G \subseteq cl_2 G \subseteq U$.Now $X-cl_2 G \in T_2$and T_2 is coupled to T_1,so $cl_2(cl_2 G)^c \subseteq cl_1(cl_2 G)^c$ .Since $x \in G$ and $G \in T_1$ and $G \cap (cl_2 G)^c = \phi$,then $x \notin cl_1(cl_2 G)^c$ and hence $x \notin cl_2(cl_2 G)^c$.Thus there exists $V \in T_2$ such that $x \in V$ and $V \cap (cl_2 G)^c = \phi$,i.e. $V \subseteq cl_2 G$ but $cl_2 G \subseteq U$.Hence $V \subseteq U$ and $x \in V \in T_2$. Consequently $T_1 \subseteq T_2$.

(ii) Immediate consequence of (i).

Definition 1.1.10 (Fletcher et al. ,1969)

Let (X,T_1,T_2) be a bts. A collection C of subsets of X is called T_1T_2-open (S-open) if $C \subseteq T_1 \cup T_2$.If in addition ,C contains at least one non-empty member of T_1 and at least one non-empty member of T_2,it is called P-open.

Pervin (1967) defined connectedness properties for bitopological spaces,Birsan (1969),Reilly (1970) and Swart (1971) discussed various aspects of connectedness properties.

Definition 1.1.11 (Pervin,1967)

A bts (X,T_1,T_2) is P-connected iff X can not be expressed as the union of two non-empty disjoint sets A and B such that $(A \cap cl_1 B) \cup (cl_2 A \cap B) = \phi$.

The following theorem is a characterization of P-connected bitopological spaces.

Theorem 1.1.12

Let (X,T_1,T_2) be a bts .Then the following are equivalent :

(i) (X,T_1,T_2) is P-connected.

(ii) X has no non-empty T_1-open,T_2-closed proper subset.

Proof :

(i)$\Rightarrow$(ii) Suppose there exists $A \in T_1$ such that $A \neq \phi, A \neq X$ and $A^c \in T_2$.Let $A^c = B$,then $A \cup B = X$.Moreover A is T_1-open and B is T_1-closed.That is $A \cap cl_1 B = \phi$.Similarly A is T_2-closed and B is T_2-

open.That is $cl_2A \cap B=\phi$.Hence (X,T_1,T_2) is P-disconnected which contradicts (i).

(ii)$\Rightarrow$(i) Suppose there exist non-empty disjoint subsets A,B of X such that $A \cup B=X, A \cap cl_1B=\phi$ and $B \cap cl_2A=\phi$.Now $A \cup B=X$ and $A \cap cl_1B=\phi$ implies that $cl_1B \subseteq B$.That is B is T_1-closed.Similarly $A \cup B=X$ and $B \cap cl_2A=\phi$ implies that $cl_2A \subseteq A$.That is A is T_2-closed.Since $B=A^c$, then B is a T_1-closed and a T_2-open subset of X,which contradicts (ii).

Definition 1.1.13 (Pervin.1967)

A subset K of (X,T_1,T_2) is P-connected if the bitopological space $(K,T_1/K,T_2/K)$ is P-connected.

Definition 1.1.14 (Fora and Al-Refa'ei.1987)

Let (X,T_1,T_2) be a bts.If there exist non empty sets $U,V \in T_1 \cup T_2$ such that $U \cap V=\phi$ and $U \cup V=X$ then (X,T_1,T_2) is called S-disconnected.A bts (X,T_1,T_2) is called S-connected if it is not S-disconnected.

It is clear that P-disconnectedness implies S-disconnectedness but the converse is not true in general.

Example 1.1.15

Let $X=\{1,2,3\}, T_1=\{\phi,X,\{3\},\{1,2\}\}$ and $T_2=\{\phi,X,\{2\}\}$ .Then (X,T_1,T_2) is S-disconnected but not P-disconnected.

The connectedness of a bts (X,T_1,T_2) is not governed by the connectedness of (X,T_1) and (X,T_2) as shown in the following examples.

Example 1.1.16

Let X={1,2} and T_1={ϕ,X,{1},{2}} and T_2={ϕ,X}.Then (X,T_1,T_2) is P-connected while (X,T_1) is not connected.

Example 1.1.17

Let (X,T_1,T_2) be a bts ,T_1={ϕ,X,U} and T_2={ϕ,X,U^c} where U is a non-empty proper subset of X.Then (X,T_1,T_2) is P-disconnected while (X,T_1) and (X,T_2) are connected.

Example 1.1.18

Let X={1,2,3},T_1={ϕ,X,{1,2},{3}} and {ϕ,X,{1,3},{2}}.Then the bts (X,T_1,T_2) is P-connected while (X,T_1) and (X,T_2) are disconnected.

Definition 1.1.19 (Fora and Al-Refa'ei,1987)

Let T_1 and T_2 be two topologies on X .Then $T_1 \cup T_2$ forms a subbase for some topology on X.This topology is called the least upper bound topology on X and is denoted by $<T_1,T_2>$.

Definition 1.1.20 (Fora and Al-Refa'ei,1987)

Consider a function f:(X,T_1,T_2)$\to$(Y,T_3,T_4),then f is said to be

1) continuous if f :(X,T_1)$\to$(Y,T_3) and f :(X,T_2)$\to$(Y,T_4) are continuous.

2) P-continuous iff for any $U \in T_3 \cup T_4$,$f^{-1}(U) \in T_1 \cup T_2$.

3) P-open iff for any $U \in T_1 \cup T_2$,$f(U) \in T_3 \cup T_4$.

Definition 1.1.21

A function $f:X \rightarrow X$ has a fixed point if there exists $t \in X$ such that $f(t)=t$.The point t is called a fixed point of f.

Definition 1.1.22

A topological space (X,T) has the fixed point property (f.p.p. for short) if every continuous function from X into itself has a fixed point.

Definition 1.1.23

Consider a bts (X,T_1,T_2),

i) if every continuous function from (X,T_1,T_2) into itself has a fixed point we say that (X,T_1,T_2) has the fixed point property.

ii) if every P-continuous function from (X,T_1,T_2) into itself has a fixed point we say that (X,T_1,T_2) has the pairwise fixed point property (P-f.p.p. for short).

Theorem 1.1.24 (Fora and Al-Refa'ei,1987)

If a bts (X,T_1,T_2) has the P-f.p.p. then (X,T_1,T_2) is S-connected and P-T_0

A set U in a topological space (X,T) is called weakly open if for any $x \in U$ there exists an open set V containing x such that V-U is a countable set .A set F is called weakly closed if X-F is weakly open.

If $A \subseteq X$ and $x \in X$,then x is called a weak interior point of A if there exists a weakly open set V containing x such that $x \in V \subseteq A$.The set of all weak interior points of a set A is denoted by Wint A.

Definition 1.1.25

A set U in a bts (X,T_1,T_2) is called weakly open iff U is weakly open with respect to T_1 and T_2.A set F is called weakly closed if F^c is weakly open.

The concept of pairwise compactness was introduced independently by Kim (1968) and Fletcher et al. (1969) and they also obtained that a pairwise compact pairwise Hausdorff bitopological space is pairwise normal.

In this part we recall some known definitions and theorems that will be used in our study.

Definition 1.1.26 (Fletcher et al. ,1969)

A bts (X,T_1,T_2) is called P-compact (P-Lindelof) if every P-open cover of the space X has a finite (countable) subcover.

Definition 1.1.27 (Datta,1972)

A bts (X,T_1,T_2) is called S-compact (S-Lindelof) if every S-open cover of the space has a finite (countable) subcover.

Theorem 1.1.28 (Fora and Hdeib,1983)

If a bts (X,T_1,T_2) is P-Lindelof and C is a weakly closed proper subset of (X,T_1),then C is T_2-Lindelof.

§ 1.2 Fuzzy topological spaces.

In this section we present some basic concepts of fuzzy sets and different definitions of fuzzy topology .The following definitions are due to Zadeh (1965).

A fuzzy set λ in X as defined by Zadeh (1965) is a function from X into I.The real number $\lambda(x)(x\in X)$ is called the grade of membership of x in λ.The fuzzy set λ is said to be contained in the fuzzy set μ (in symbols $\lambda\subseteq\mu$) iff $\lambda(x)\leq\mu(x)$ for all $x\in X$.Two fuzzy sets λ and μ are equal iff $\lambda\subseteq\mu$ and $\mu\subseteq\lambda$.The complement of the fuzzy set λ (denoted by λ^c) is defined by $\lambda^c(x)=1-\lambda(x)$,for all $x\in X$.

It is useful to identify a fuzzy constant function with its range .That is if λ is a constant function given by $\lambda(x)=c, c\in I$,then we write c to stand for λ.

A fuzzy set μ is said to be proper if $\mu\neq0$ and $\mu\neq1$.

Definition 1.2.1

If $\{\lambda_i:i\in\Delta]$ is a collection of fuzzy sets ,then the union and the intersection of fuzzy sets are defined as follows:

$(\cap\lambda_i)(x)=\inf\ \{\lambda_i(x):i\in\Delta\},x\in X$

and

$(\cup\lambda_i)(x)=\sup\ \{\lambda_i(x):i\in\Delta\},x\in X$.

If Δ is finite then inf and sup are replaced by min and max.

If λ and μ are two fuzzy sets on X then we define ,their sum $\lambda+\mu$ to be $(\lambda+\mu)(x)=\lambda(x)+\mu(x)$ while $\lambda+\mu=1$ means $\lambda(x)+\mu(x)=1$ for all $x\in X$.

In particular if $\lambda\cap\mu=0$ then the following are equivalent:

(i) $\lambda=\mu^c$ (ii) $\lambda\cup\mu=1$. (iii) $\lambda+\mu=1$

Definition 1.2.2 (Weiss,1975)

The support of a fuzzy set λ in X is denoted by supp λ and is defined by supp $\lambda=\lambda^{-1}(0,1]=\{x:\lambda(x)>0\}$.

Definition 1.2.3

A fuzzy set λ is called crisp iff $\lambda(x)\in\{0,1\}$ for all $x\in X$,i.e,λ is the characteristic function on X, given by $\lambda=\chi_A$ where $A=\lambda^{-1}(\{1\})$.If supp λ is a singleton subset of X,then λ is called a fuzzy crisp point.

The definitions of fuzzy point and fuzzy membership were proposed by Wong (1974).

Definition 1.2.4

A fuzzy point p in a set X is a fuzzy set in X given by

$p(x)=t$ for $x=x_p$ $(0< t <1)$

$p(x)=0$ for $x\neq x_p$

The supp p is often written as x_p and its value $p(x_p)\in(0,1)$.

Two fuzzy points p and q are said to be distinct iff $x_p\neq x_q$.

Definition 1.2.5

A fuzzy point p in X is said to belong to a fuzzy set λ in X (written as $p \in \lambda$) iff $p(x_p) < \lambda(x_p)$.

The following definition was introduced by Fora as a private communication.

Definition 1.2.6

A fuzzy point p with support $x_p \in X$ and value $p(x_p) \in (0,1)$ is called mature provided $p(x_p) \geq 1/2$ and is called immature provided $p(x_p) < 1/2$.

Definition 1.2.7 (Ghanim et al. ,1984)

A fuzzy set e is called a fuzzy singleton of X if it is zero everywhere except at one point of X (This point will be denoted by x_e).A fuzzy singleton e in X is said to belong to a fuzzy set λ in X (written as $e \in \lambda$) iff $e(x_e) \leq \lambda(x_e)$.

We note that every fuzzy point is a fuzzy singleton ,however the converse is not true .In fact a fuzzy singleton is either a fuzzy point or a fuzzy crisp point.Therefore every fuzzy set can be expressed as the union of all of its fuzzy singletons.

Definition 1.2.8

Let f be a function from X into Y , where X ,Y are arbitrary sets . Let λ be a fuzzy set in X and μ be a fuzzy set in Y.The direct image of λ under f is the fuzzy set $f(\lambda)$ in Y given by

$$f(\lambda)(y) = \sup \{0, \lambda(x) : x \in f^{-1}(\{y\})\} \quad , y \in Y.$$

The inverse image of μ under f is the fuzzy set $f^{-1}(\mu)$ in X given by

$$f^{-1}(\mu)(x) = \mu(f(x)) \quad , \ x \in X .$$

Lemma 1.2.9 .

Let $f : X \rightarrow Y$ be a function and μ a fuzzy set in Y.Then $(f^{-1}(\mu))^{-1}(A) = f^{-1}(\mu^{-1}(A))$ where $A \subseteq [0,1]$.

Proof :

Let $x \in (f^{-1}(\mu))^{-1}(A)$,then $(f^{-1}(\mu))(x)=\mu(f(x)) \in A$.This implies that $f(x) \in \mu^{-1}(A)$.That is $x \in f^{-1}(\mu^{-1}(A))$.Hence $(f^{-1}(\mu))^{-1}(A) \subseteq f^{-1}(\mu^{-1}(A))$. Conversely,let $y \in f^{-1}(\mu^{-1}(A))$,then $f(y) \in \mu^{-1}(A)$ which implies that $\mu(f(y)) \in A$.That is $(f^{-1}(\mu))(y) \in A$ which gives that $y \in (f^{-1}(\mu))^{-1}(A)$. Hence $f^{-1}(\mu^{-1}(A)) \subseteq (f^{-1}(\mu))^{-1}(A)$.

Different definitions of fuzzy topology have appeared since Chang (1968) introduced the concept .Lowen's definition which require that a fuzzy topology should have one more axiom,namely it includes the constant fuzzy sets was the most attractive one after Chang's definition.

In this thesis we shall adopt Chang's definition of fuzzy topology because Lowen's definition does not generalize topology.Any topology can be considered as a fuzzy topology which is called a crisp fuzzy topology and defined by $X/T=\{\chi_U : U \in T\}$ which is not a fuzzy topology in the sense of Lowen's definition.

Definition 1.2.10 (Chang,1968)

A fuzzy topology on a nonempty set X is a collection τ of fuzzy sets in X such that ;

i) $0, 1 \in \tau$.

ii) If λ , $\mu \in \tau$, then $\lambda \cap \mu \in \tau$.

iii) If $\lambda_i \in \tau$, $i \in \Delta$, then $\cup\lambda_i \in \tau$.

where 0,1 denote the fuzzy sets given by $0(x)=0$ and $1(x)=1, x\in X$. Members of τ are called τ-fuzzy open sets and the pair (X , τ) is called a fuzzy topological space (in short fts).Complements of open fuzzy sets are called fuzzy closed sets .

If λ is a fuzzy set of a fts (X,τ) ,then the interior and the closure of λ are defined by:

Int $\lambda=\cup\{\mu:\mu$ is an open fuzzy set and $\mu\subseteq\lambda\}$

cl $\lambda=\cap\{\mu:\mu$ is a closed fuzzy set and $\lambda\subseteq\mu\}$

A fuzzy set λ in (X,τ) is called a neighbourhood of a fuzzy point p (fuzzy singleton e) iff there exists $\mu\in\tau$ such that $p\in\mu\subseteq\lambda$ $(e\in\mu\subseteq\lambda)$.

Definition 1.2.11 (Wong.1974)

A subfamily B of τ is said to be a base for τ iff for each $\lambda\in\tau$ where $\lambda\neq0$,there is a subfamily B_λ of B such that $\lambda=\cup B_\lambda$.

Definition 1.2.12 (Wong.1974)

A subfamily D of τ is said to be a subbase for τ iff the family B of all finite intersections of members of D together with 1 forms a base for τ ,i.e, $B=\{\cap L:L$ is a finite subset of $D\}\cup\{1\}$ is a base for τ.

Definition 1.2.13

Let (X,τ) be a fuzzy topological space and $Y\subseteq X$.If $\lambda/Y=\lambda\cap\chi_Y$,then the family $\tau_Y=\{\lambda/Y:\lambda\in\tau\}$ is a fuzzy topology on Y.The fuzzy topological space (Y,τ_Y) is called a fuzzy subspace of (X,τ) with the underlying set Y.The subspace (Y,τ_Y) is called closed,open and proper if Y is closed,open and proper respectively.

A fuzzy set λ on Y may be considered as a fuzzy set on X in the sense that λ takes the value 0 on X-Y and,conversely, a fuzzy set on X taking value 0 on X-Y can be considered as a fuzzy set on Y.

Definition 1.2.14

Let f: $(X,\tau_1)\rightarrow(Y,\tau_2)$ be a function from a fts (X,τ_1) to a fts (Y,τ_2).The function f is called fuzzy continuous iff the inverse image of every τ_2-open fuzzy set in Y is τ_1- open fuzzy set in X.

Theorem 1.2.15 (Chang.1968)

Let $f:X\rightarrow Y$.Then $\lambda\subseteq f^{-1}(f(\lambda))$ for any fuzzy set λ in X.

Definition 1.2.16 (Fora.1990)

A fuzzy topological space (X,τ) is said to have the fixed point property (f.p.p for short) iff every fuzzy continuous function $f:(X,\tau)\rightarrow(X,\tau)$ has a fixed point.

§ 1.3 The induced fuzzy topological spaces

It is a well known fact that the family of all lower semi-continuous functions from a topological space (X,T) to the closed unit interval

[0,1](with the usual ordinary topology on [0,1]) forms a fuzzy topology on X.This fuzzy topology was given many names by different authors but the most famous one is given by Weiss (1975) . He called it the induced fuzzy topology.Lowen (1976) gave it the name "topologically generated" and denote it by $\omega(\mathbf{T})$ which has become now familiar.

<u>Definition 1.3.1</u>

The collection $\omega(\mathbf{T})=\{f \mid f:(X,T)\rightarrow I$ is l.s.c.$\}$ is called the **induced fuzzy topology** of (X,T).The pair $(X,\omega(T))$ is called the induced fuzzy topological space.

It is clear that if (X,T) is a topological space,then the characteristic function $\chi_u:(X,T)\rightarrow I$ is l.s.c iff $u\in T$.Therefore the collection $X/T=\{\chi_u:U\in T\}\subseteq\omega(T)$.

Lowen (1977) used the induced fuzzy topology as a guide in fuzzification of classical concepts and in (1978) he used it to invent the so called "goodness criterion" which is considered to be a big step in the development of fuzzy topology.

<u>Definition 1.3.2</u>

Let (X,τ) be a fts and p a fuzzy point .A subfamily B_p of τ is called a local base at p iff $p\in B$ for every B in B_p,and for every member λ of τ such that $p\in\lambda$ there exists a member B in B_p,such that $p\in B\subseteq\lambda$.

Proposition 1.3.3 (Mohannadi and Warner ,1989)

If $B=\{B_\alpha:\alpha\in\Delta\}$ is a base for the topological space (X,T),then a basis for the induced fts (X,ω(T)) is given by $\{A_{q\alpha}:A_{q\alpha}(x)=q$ for $x\in B_\alpha$ and $A_{q\alpha}(x)=0$ for $x\notin B_\alpha$,for all $\alpha\in\Delta$ and $q\in Q\cap(0,1]\}$.

Definition 1.3.4 (Wong.1974)

A fts (X,τ) is said to be fuzzy first countable (in short a C_I-space) iff every fuzzy point in X has a countable local base.

Definition 1.3.5 (Wong.1974)

A fts (X,τ) is said to be fuzzy second countable (in short a C_{II}-space) iff there exists a countable base B for τ.

Proposition 1.3.6

A topological space (X,T) is second countable (first countable) iff the induced fts (X,ω(T)) is second countable (first countable).

Theorem 1.3.7 (Fora.1990)

Let (X,T_1) and (Y,T_2) be two topological spaces.Then

i) $f:(X,T_1)\to(Y,T_2)$ is continuous if and only if $f:(X,\omega(T_1))\to(Y,\omega(T_2))$ is fuzzy continuous.

ii) $f:(X,T_1)\to(Y,T_2)$ is continuous if and only if $f:(X,X/T_1)\to(Y,Y/T_2)$ is fuzzy continuous.

Definition 1.3.8 (Lowen ,1976)

Let (X,τ) be a fts.A topology on X having the subbase $\{\lambda^{-1}(a,1],\lambda\in\tau, a\in[0,1)\}$ is called the **initial topology** of τ and denoted by $\mathbf{L(\tau)}$.

Definition 1.3.9 (Martin.1980)

Let (X,τ) be the fts ,then the topology $\tau_c{}^*$ on X is said to be the Martin topology of τ provided $\tau_c{}^*$ is defined by

$\tau_c{}^*=\{A\subseteq X:\chi_A\in\tau\}$.

We note that if τ_c **is the set of all characteristic maps in** τ then τ_c is a fuzzy topology on X .Moreover the fuzzy topological space (X,τ_c) is essentially the same as the topological space $(X,\tau_c{}^*)$.

Definition 1.3.10 (Martin.1980)

A fuzzy topological space (X,τ) is said to be an induced fuzzy space provided τ is the collection of all l.s.c. maps from $(X,\tau_c{}^*)\rightarrow I$.That is (X,τ) is induced iff $\tau=\omega(\tau_c{}^*)$.

Definition 1.3.11 (Martin.1980)

A fuzzy topological space (X,τ) is said to be a weakly induced fuzzy space provided whenever $\lambda\in\tau$ then λ: $(X,\tau_c{}^*)\rightarrow I$ is l.s.c. .That is (X,τ) is weakly induced iff $\tau\subseteq\omega(\tau_c{}^*)$.

Observe that every induced space is weakly induced but the converse is not true as we see in the following example.

Example 1.3.12

Let $X=[0,1]$,and $\tau=\{0,1\}$.Then $\tau_c{}^*=T_{ind}=\{\phi,X\}$ and $\lambda:(X,T_{ind})\to(I,T_{r.r})$ is continuous iff it is a constant.Therefore $\omega(T_{ind})=\{c:0\leq c\leq 1\}$.So $\tau=\{0,1\}\subseteq\omega(\tau_c{}^*)$ but $\tau\neq\omega(\tau_c{}^*)$.Hence τ is weakly induced but not induced.

Definition 1.3.13 (Lowen.1976)

A fts (X,τ) is said to be topological (topologically generated) provided τ is the collection of all l.s.c. maps from $(X,L(\tau))\to I$.That is (X,τ) is topological iff $\tau=\omega(L(\tau))$.

We note that for any fts (X,τ) we have $\tau\subseteq\omega(L(\tau))$ but the converse is not true in general as we see in the following example.

Example 1.3.14

Let $X=[0,1]$,and $\tau=\{0,1\}$.Then $L(\tau)=T_{ind}=\{\phi,X\}$ because $0,1:(X,T_{ind})\to(I,T_{r.r})$ are continuous.Therefore $\omega(L(\tau))=\omega(T_{ind})=\{c:0\leq c\leq 1\}$.Thus $\omega(L(\tau))$ is not a subset of τ.Hence τ is not topological.

Theorem 1.3.15 (Lowen.1976)

A fts (X,τ) is topologically generated iff for each continuous function $f:(I,T_{r.r})\to(I,T_{r.r})$ and for each $\nu\in\tau$ we have $f\circ\nu\in\tau$.

Definition 1.3.16 (Lowen.1978)

A property R_f of a fts is said to be a good extension of the property R in classical topology iff whenever the fts is topologically generated;

say by (X,T);then (X,ω(T)) has the property R_f iff (X,T) has the property R.

§ 1.4 Separation axioms in fuzzy topological spaces.

Several authors have introduced different definitions of separation properties for fuzzy topologies (see, e.g. Hutton (1975) and (1977), Pu and Liu (1980) and Srivastava, et al. (1981)). The Hausdorff axiom has had a hard life in fuzzy set theory since many authors have proposed different definitions (e.g. Pu and Liu (1980), Sarkar (1981) and Wong (1974)).

In this section we shall define some separation axioms which we need in our further study.

Definition 1.4.1 (Srivastava et al. ,1988)

A fts (X,τ) is said to be fuzzy R_0 iff for all $x,y\in X, x\neq y$, whenever there is $U\in\tau$ such that $U(x)=1$ and $U(y)=0$, there is also $V\in\tau$ such that $V(y)=1$ and $V(x)=0$.

Definition 1.4.2 (Srivastava et al. ,1987)

A fts (X,τ) is said to be fuzzy R_1 iff for all $x,y\in X, x\neq y$, whenever there is $U\in\tau$ with either ($U(x)=1$ and $U(y)=0$) or ($U(y)=1$ and $U(x)=0$) ,then for any pair of fuzzy points p and q with supports x and y there exist $V,W\in\tau$ with $p\in V, q\in W$ and $V\cap W=0$.

Definition 1.4.3 (Fora,1989)

A fuzzy topological space (X,τ) is said to be :

1) T_0 iff for any two distinct fuzzy points p,q in X , there exists an open fuzzy set μ such that ($p\in\mu$ and $\mu\cap q=0$) or ($q\in\mu$ and $\mu\cap p=0$).

2) T_{0w} iff for any two distinct fuzzy points p,q in X,there exists an open fuzzy set μ such that $p\in\mu\subseteq q^c$ or $q\in\mu\subseteq p^c$.

3) T_1 iff for any two distinct fuzzy points p,q in X , there exist open fuzzy sets μ_1 and μ_2 such that $p\in\mu_1$, $\mu_1\cap q=0$ and $q\in\mu_2$, $\mu_2\cap p=0$.

4)T_{1w} iff for any two distinct fuzzy points p,q in X,there exist open fuzzy sets μ_1 and μ_2 such that $p\in\mu_1\subseteq q^c$ and $q\in\mu_2\subseteq p^c$.

5)T_2 iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\in\mu_1$,$q\in\mu_2$ and $\mu_1\cap\mu_2=0$.

6)T_{2w} iff for any two distinct fuzzy points p,q in X,there exist open fuzzy sets $\mu_1$and μ_2 such that $p\in\mu_1$,$q\in\mu_2$ and $\mu_1\subseteq(\mu_2)^c$.

7)$T_{2\ 1/2}$ iff for any two distinct fuzzy points p,q in X,there exist open fuzzy sets μ_1 and μ_2 such that $p\in\mu_1$,$q\in\mu_2$ and $cl\mu_1 \cap cl\mu_2=0$.

8)$T_{2\ 1/2w}$ iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\in\mu_1$,$q\in\mu_2$ and $cl\mu_1 \subseteq(cl\mu_2)^c$.

Theorem 1.4.4 (Fora .1990)

Let $f : (X,\tau_1) \rightarrow (Y,\tau_2)$ be an injective fuzzy continuous map. If (Y,τ_2) is a T_i-space then (X,τ_1) is a T_i-space,$i\in\{0,1,2,2\ 1/2,$ $0w,1w,2w,2\ 1/2\,w\}$.

From the above definitions one can notice that the following chains of implications are true.

(i) $T_i \to T_{iw}$. for $i=0,1,2,2\ 1/2$

(ii) $T_{2\ 1/2} \to T_2 \to T_1 \to T_0$.

(iii) $T_{2\ 1/2w} \to T_{2w} \to T_{1w} \to T_{0w}$.

Definition 1.4.5 (Fora .1989)

A fuzzy topological space (X,τ) is said to be:

(i) regular iff for every fuzzy point p in X and every closed fuzzy set λ in X such that $p \in \lambda^c$, there exist open fuzzy sets μ_1 and μ_2 such that $p \in \mu_1, \lambda \subseteq \mu_2$ and $\mu_1 \subseteq (\mu_2)^c$.

(ii) normal iff for every pair of closed fuzzy sets λ_1, λ_2 such that $\lambda_1 \subseteq (\lambda_2)^c$, there exist open fuzzy sets μ_1, μ_2 such that $\lambda_1 \subseteq \mu_1 \subseteq (\mu_2)^c \subseteq (\lambda_2)^c$.

Definition 1.4.6 (Gantner et al. .1978)

A fts (X,τ) is Hausdorff if for any two distinct points x,y of X,then there exist $\lambda, \mu \in \tau$ such that $\lambda(x)=\mu(y)=1$ and $\lambda \cap \mu = 0$.

We note that definition 1.4.6 implies definition 1.4.3(5) but the converse is not true as was pointed by Liu (1982).

Definition 1.4.7

A property P is called hereditary (weakly hereditary, hereditary with respect to open subspaces), iff each subspace (closed subspace, open subspace) of a fts with property P also has the property P.

Definition 1.4.8 (Hutton ,1975)

Let $(L,\leq)$ be a completely distributive lattice with order reversing involution.The **fuzzy unit interval [0,1](L)** is the set of all monotonic decreasing maps λ: R→L satisfying:

$\lambda(t) = 1$ for $t<o$, $t\in R$,

$\lambda(t) = 0$ for $t>1$, $t\in R$;

after identification of $\lambda : R\to L$ and $\mu:R\to L$ iff $\lambda(t-)=\mu(t-)$ and $\lambda(t+)=\mu(t+)$ for every $t\in R$ (where $\lambda(t-)=\inf\{\lambda(s) :s<t\}$ and $\lambda(t+)=\sup\{\lambda(s):s>t\}$) where 0 and 1 are the top and bottom elements of L.

We define an L-fuzzy topology on [0,1](L) by taking as a subbasis $\{L_t,R_t:t\in R\}$ where we define $L_t(\lambda)=(\lambda(t-))^c$ and $R_t(\lambda)=\lambda(t+)$.This fuzzy topology is called the **usual topology for [0,1](L).**

Theorem 1.4.9 (Hutton ,1975)

A fts (X,τ) is normal iff for every closed fuzzy set F and open fuzzy set λ such that $F\subseteq\lambda$, there exists a fuzzy continuous function $f : (X,\tau) \to [0,1](L)$ such that for every $x\in X$, $F(x)\leq f(x)(1-)\leq f(x)(0+)\leq\lambda(x)$.

§ 1.5 Connectedness in fuzzy topological spaces.

Regarding connectedness,Pu and Liu (1980) have paid some attention to connectedness.They used the concepts of fuzzy subspace and a fuzzy closed set to define connectedness of a fuzzy set . Lowen (1981) had also defined an extension of connectedness in a restricted family of fuzzy topologies i.e., for a fuzzy set which is everywhere strictly positive.Fatteh and Bassan(1985) defined fuzzy connected subsets of a fuzzy topological space and studied their properties,Ajmal and Kohli(1989) had defined connected fuzzy sets and showed that fuzzy connectedness is preserved under fuzzy continuity.

We start this section with the following definition.

Definition 1.5.1

A fts (X,τ) is said to be fuzzy connected if it has no proper fuzzy clopen (closed and open) set.

The first characterization of fuzzy connectedness is given in the following theorem.

Theorem 1.5.2 (Fatteh and Bassan.1985)

A fuzzy topological space (X,τ) is fuzzy connected iff it has no non-zero fuzzy open sets λ and μ such that $\lambda(x)+\mu(x)=1$ for all $x\in X$.

Theorem 1.5.3

A fuzzy continuous image of a fuzzy connected space is fuzzy connected.

Proof :

Let X be a fuzzy connected space and let $f:X\rightarrow Y$ be a fuzzy continuous function from X onto a fuzzy space Y.

Suppose on the contrary that Y is not fuzzy connected,then it has a proper fuzzy clopen set λ.Since f is continuous,then $f^{-1}(\lambda)$ is a proper fuzzy clopen set in X.Hence X is not fuzzy connected,which is a contradiction.

The following definitions and results are due to Fatteh and Bassan (1985).

Definition 1.5.4

Let (X,τ) be a fts and $A \subseteq X$.Then A is said to be a fuzzy connected subset of X if A is a fuzzy connected space as a fuzzy subspace of X.

Theorem 1.5.5

Let (X,τ) be a fts , A be a fuzzy connected subset of X and λ, μ are non-zero fuzzy open sets in X such that $\lambda+\mu=1$,then either $\lambda/A=1$ or $\mu/A=1$.

Definition 1.5.6

Fuzzy sets λ and μ in a fts (X,τ) are said to be separated from each other if $cl\lambda+\mu\subseteq 1$ and $\lambda+cl\mu\subseteq 1$.

Theorem 1.5.7

Let $\{A_\alpha : \alpha\in\Delta\}$ be a family of fuzzy connected subsets of X such that for each α and β in Δ and $\alpha\neq\beta$,A_α,A_β are not separated from each other ,then $\cup\{A_\alpha : \alpha\in\Delta\}$ is a fuzzy connected subset of X.

Theorem 1.5.8

If A and B are subsets of a fts (X,τ) such that $\chi_A\leq\chi_B\leq cl\chi_A$ and A is a fuzzy connected subset of X,then B is also a fuzzy connected subset of X.

As we have noticed in the preceding results Fatteh and Bassan (1985) defined connectedness only for a crisp set of a fuzzy topological space.However Ajmal and Kohle (1989) have extended the notion of connectedness to an arbitrary fuzzy set.They considered all the variations of these definitions which are particularities of fuzzy setting.In a special case,when the fuzzy set λ is the whole fuzzy space (i.e., $\lambda=1$) all the conditions collapse to give us a condition more general than that of fuzzy connectedness studied by Fatteh and Bassan (1985).Moreover Ajmal and Kohle's definition is a good extension in the sense of Lowen's criterion while that of Fatteh and Bassan (1985) is not.

Definition 1.5.9 (Ajmal and Kohle,1989)

A fuzzy set σ in (X,τ) has a C_i-disconnection property if there exist open fuzzy sets λ and μ in X such that $\sigma\subseteq\lambda\cup\mu$ and :

C_1: $\lambda\cap\mu\subseteq 1-\sigma, \sigma\cap\lambda\neq 0\ \sigma\cap\mu\neq 0.$

C_2: $\sigma\cap\lambda\cap\mu=0, \sigma\cap\lambda\neq 0\ \sigma\cap\mu\neq 0.$

C_3: $\lambda\cap\mu\subseteq 1-\sigma, \lambda\not\subseteq 1-\sigma, \mu\not\subseteq 1-\sigma.$

C_4: $\sigma\cap\lambda\cap\mu=0, \lambda\not\subseteq 1-\sigma, \mu\not\subseteq 1-\sigma.$

σ is said to be C_i-connected if there does not exist any C_i-disconnection of σ in X $(i=1,2,3,4)$.

§ 1.6 Compactness in fuzzy topological spaces.

Compactness in fuzzy topological spaces was first introduced by Chang (1968).His definition was not good because a fuzzy space with one point fails to be compact. Kim (1968) and Fletcher et al. (1969),Goguen (1973),Wong (1974) and Weiss (1975) worked on compactness but their works were limited.Lowen (1976) gave a new definition of compactness while Gantner et al. (1978) introduced α-shading and α-compactness.

We start this section with the following definition.

Definition 1.6.1

Let (X,τ) be a fts.A collection $C=\{\lambda_\alpha:\alpha\in\Delta\}$ of fuzzy sets in X is called an open cover of X if $C\subseteq\tau$ and $\cup\{\lambda_\alpha:\alpha\in\Delta\}=1$.A subcollection L of C that is also a cover of X is called a subcover.If moreover L is finite (countable),we say L is a finite (countable) subcover of C.

Definition 1.6.2 (Chang,1968)

A fts (X,τ) is compact iff every open cover of X has a finite subcover. The above definition has a serious weak point ,that is a finite space needs not be compact .

Example 1.6.3

Let $X=\{x_0\}$ and $\tau=\{c:0\leq c\leq 1\}$.Then the collection $\{c:0\leq c<1\}$is an open cover for X which has no finite subcover.

Due to this weakness other definitions of compactness were proposed

Definition 1.6.4 (Gantner et al. ,1978)

Let (X,τ) be a fts and $0\leq\alpha<1$.A collection $U=\{\mu_i:i\in\Delta\}$ of fuzzy sets in X is called an α-shading of X if for each $x\in X$,there exists $\mu_i\in U$ with $\mu_i(x)>\alpha$.A subcollection V of U that is also an α-shading is called an α-subshading. If moreover V is finite (countable),we say V is a finite (countable) α-subshading of U.

Definition 1.6.5 (Gantner et al. ,1978)

A fts (X,τ) is called α-compact iff each α-shading of X has a finite α-subshading.

Definition 1.6.6 (Lowen,1978)

A fuzzy set f in a fts (X,τ) is said to be compact if for every family $G \subseteq \tau$ such that $\sup\{g : g \in G\} \geq f$ and for every $\varepsilon > 0$,there exists a finite subfamily $G_\varepsilon \subseteq G$ such that $\sup\{g : g \in G_\varepsilon] \geq f - \varepsilon$,where $(f-\varepsilon)(x) = f(x) - \varepsilon, x \in X$.

Definition 1.6.7 (Lowen,1978)

A fts (X,τ) is said to be:

(i) fuzzy compact provided all constant fuzzy sets in X are compact.

(ii) strong fuzzy compact iff it is α-compact for each $\alpha \in [0,1)$.

(iii) ultra fuzzy compact iff $(X,L(\tau))$ is compact.

Definition 1.6.8 (Lowen,1976)

A fts (X,τ) is weakly fuzzy compact iff for each family $B \subseteq \tau$ such that $\sup\{\mu : \mu \in B\} = 1$ and for each $\varepsilon > 0$ there exists a finite subfamily B_0 of B such that $\sup\{\mu : \mu \in B_0\} \geq 1 - \varepsilon$.

Definition 1.6.9 (Lowen,1976)

A fts (X,τ) is fuzzy compact iff for each $B \subseteq \tau$ and each $\alpha \in (0,1]$ such that $\sup\{\mu : \mu \in B\} \geq \alpha$ and for each $\zeta \in (0,1]$ there exists a finite subfamily B_0 of B such that $\sup\{\mu : \mu \in B_0\} \geq \alpha - \zeta$.

The following results can be found in Martin (1980).

Theorem 1.6.10

Every fuzzy compact (in the sense of definition 1.6.7(i)) Hausdorff space is a weakly induced space.

Theorem 1.6.11

If (X,τ) is fuzzy compact(in the sense of definition 1.6.7(i)) then $(X,\tau_c{}^*)$ is compact.

Definition 1.6.12 (Wong,1973)

A fts (X,τ) is Lindelof iff every open cover of X has a countable subcover .

Definition 1.6.13 (Gantner et al. ,1978)

Let X be a non -empty set and $\alpha \in I$.A collection $\wp$ of fuzzy sets in X is said to be α-centered if for all finite collections $\mu_i \in \wp$,i=1,2,...,n there exists $x \in X$ with $\mu_k(x) \geq 1-\alpha$ for all $k \in \{1,2,...,n\}$.

Chapter II

BIFUZZY SEPARATION AXIOMS

In this chapter we shall define separation axioms in bifuzzy topological spaces (which are the triples (X,τ_1,τ_2) where X is a nonempty set,τ_1 and τ_2 are fuzzy topologies on X) and then investigate the relation between them.Moreover we use Lowen's goodness criterion to show that all our definitions are good extensions .For the sake of clarity we divide this chapter into five sections.

In the first section we discuss bifuzzy P-R_0 and P-T_0 topological spaces and in the second section we discuss P-R_1 and P-T_1 topological spaces while in section three we discuss P-T_2 and P-$T_{2\ 1/2}$.In section four we discuss P-regular and P-normal bifuzzy topological spaces and in the last section we present different properties of separation axioms.

§ 2.1 Bifuzzy P- R_0 and P-T_0-topological spaces.

We start this section with the following definition

Definition 2.1.1

A bfts (X,τ_1,τ_2) is said to be P-R_0 iff for any distinct fuzzy points p and q in X,whenever there exists $\lambda\in\tau_i$ such that $p\in\lambda$ and $q\cap\lambda=0$,then there exists $\mu\in\tau_j$ such that $p\cap\mu=0$ and $q\in\mu(i,j=1,2,i\neq j)$.

Definition 2.1.2

A bfts (X,τ_1,τ_2) is said to be P-R_{0w} iff for any $x,y \in X, x \neq y$ whenever there exists $\lambda \in \tau_i$ such that $\lambda(x)=1$ and $\lambda(y)=0$,then there exists $\mu \in \tau_j$ such that $\mu(x)=0$ and $\mu(y)=1 (i,j=1,2,i \neq j)$.

The following theorem shows that P-R_0-space implies P-R_{0w}-space.

Theorem 2.1.3

If a bfts (X,τ_1,τ_2) is a P-R_0 space,then (X,τ_1,τ_2) is a P-R_{0w} space.

Proof :

Let $x,y \in X, x \neq y$ and $\lambda \in \tau_i$ such that $\lambda(x)=1$ and $\lambda(y)=0$.Let p be a fuzzy point with support x and $C=\{q_\alpha : \alpha \in \Delta\}$ be the collection of all fuzzy points with support y.Since p and q_α are distinct fuzzy points for all α and (X,τ_1,τ_2) is P-R_0,then for each α there exists $\mu_\alpha \in \tau_j$ such that $p \cap \mu_\alpha = 0$ and $q_\alpha \in \mu_\alpha$.Let $\mu = \sup\{\mu_\alpha : \alpha \in \Delta\}$.Then $\mu \in \tau_j$ and $\mu(y)=1$ and $\mu(x)=0$.Hence (X,τ_1,τ_2) is P-R_{0w}.

Example 2.1.4

There exists a bfts (X,τ_1,τ_2) which is P-R_{0w} but not P-R_0 .

To present our example,let $X=\{0,1\}, \tau_1=\{0,1,\mu\}$ and $\tau_2=\{0,1,\lambda\}$ where $\mu(0)=1/2$,$\mu(1)=0$, $\lambda(0)=0$ and $\lambda(1)=3/4$. Consider the fuzzy points p and q such that $p(0)=3/4$ and $q(1)=1/4$.Then it is clear that $q \in \lambda \in \tau_2, p \cap \lambda = 0$ and for any σ such that $p \in \sigma \in \tau_1, q \cap \sigma \neq 0$.Hence (X,τ_1,τ_2) is not P-R_0 although it is (vacuously) P-R_{0w} .

The following definition is an extension of Lowen's criterion for goodness in bifuzzy topological spaces.

Definition 2.1.5

A property R_f of a bfts is said to be a good extension of the property R in classical topology iff whenever the bfts is induced,say by (X,T_1,T_2),then $(X,\omega(T_1),\omega(T_2))$ has the property R_f iff (X,T_1,T_2) has the property R.

In the following theorem,we show that our definitions of P-R_0 and P-R_{0w} are good extensions .

Theorem 2.1.6

Let (X,T_1,T_2) be a bts ,then the following statements are equivalent:

i) (X, T_1,T_2) is a P-R_0 space.

ii) $(X,\omega(T_1),\omega(T_2))$ is a P-R_0 space.

iii) $(X,\omega(T_1),\omega(T_2))$ is a P-R_{0w} space.

Proof :

(i)⇒(ii) Let p and q be any two distinct fuzzy points with supports x_p and y_q respectively and $\lambda\in\omega(T_i)$ such that $p\in\lambda$ and $q\cap\lambda=0$.Since $\lambda:(X,T_i)\rightarrow I$ is l.s.c. ,then $\lambda^{-1}(0,1]\in T_i$ such that $x_p\in\lambda^{-1}(0,1]$ and $y_q\notin\lambda^{-1}(0,1]$.Since (X,T_1,T_2) is P-R_0,so there exists $V\in T_j$ such that $x_p\notin V$ and $y_q\in V$.Now it is clear that $p\cap\chi_V=0$ and $q\in\chi_V$ where $\chi_V\in\omega(T_j)$.Hence $(X,\omega(T_1),\omega(T_2))$ is a P-R_0 space.

(ii)⇒(iii) see theorem 2.1.3.

(iii)$\Rightarrow$(i)Let $x,y \in X, x\neq y$ and $U\in T_i$ such that $x\in U$ and $y\notin U$.Now if p and q are two fuzzy points with supports x and y respectively,then it is clear that $p\in \chi_u$ and $q\cap\chi_u=0$.That is $\chi_u(x)=1$ and $\chi_u(y)=0$.Since $(X,\omega(T_1),\omega(T_2))$ is P-R_{0w},then there exists $\lambda\in \omega(T_j)$ such that $\lambda(y)=1$ and $\lambda(x)=0$.Hence $x\notin \lambda^{-1}(0,1]$ and $y\in \lambda^{-1}(0,1]$ which shows that (X,T_1,T_2) is P-R_0.

Definition 2.1.7

A bfts (X,τ_1,τ_2) is said to be P-T_0 iff for any two distinct fuzzy points p,q in X,there exists a fuzzy set $\mu\in \tau_1\cup\tau_2$ such that $(p\in \mu$, $q\cap\mu=0)$ or $(q\in \mu ,p\cap\mu=0)$.

Definition 2.1.8

A bfts (X,τ_1,τ_2) is said to be P-T_{0w} iff for any two distinct fuzzy points p,q in X,there exists a fuzzy set $\mu\in \tau_1\cup\tau_2$ such that $(p\in \mu \subseteq q^c)$ or $(q\in \mu \subseteq p^c)$.

The following theorem shows that P-T_0-space implies P-T_{0w}-space.

Theorem 2.1.9

If a bfts (X,τ_1,τ_2) is a P-T_0 space,then (X,τ_1,τ_2) is a P-T_{0w} space.

Proof :

The proof is obvious because if $p\cap\mu=0$,then $\mu\subseteq p^c$.

Example 2.1.10

There exists a bfts (X,τ_1,τ_2) which is P-T_{0w} but not P-T_0 .

The following example illustrates our purpose.

Let $X=[0,1]$ and $\tau_1=\{0,\lambda:\lambda$ is fuzzy set on X for which $\lambda(x)>0$ for every $x\in X\}$ and $\tau_2=\{0,1,\chi_{[0,r)}: 0<r\leq 1/2\}$.Then (X,τ_1,τ_2) is a bifuzzy topological space.To prove X is a P-T_{0w} space ,let p,q be any two distinct fuzzy points in X with supports x_p,x_q respectively. Define the fuzzy sets λ_p,λ_q as follows:

$$\lambda_p(x)=\begin{cases} 1 & \text{if } x\neq x_q \\ q^c(x_q) & \text{if } x=x_q \end{cases} \quad , \quad \lambda_q(x)=\begin{cases} 1 & \text{if } x\neq x_p \\ p^c(x_p) & \text{if } x=x_p \end{cases}$$

then λ_p,λ_q are τ_1- open fuzzy sets and $p\in\lambda_p\subseteq q^c, q\in\lambda_q\subseteq p^c$.Hence (X,τ_1,τ_2) is a P-T_{0w} space. Moreover we notice that for any two distinct fuzzy points p,q such that $x_p>1/2$ and $x_q>1/2$,the only fuzzy set $\lambda\in\tau_2$ such that $p\in\lambda$ or $q\in\lambda$ must be $\lambda=1$.If $\lambda\in\tau_1$,then $\lambda(x)>0$ for all $x\in X$.Consequently $\lambda\in\tau_1\cup\tau_2$ implies that $\lambda\cap p\neq 0$ and $\lambda\cap q\neq 0$.Hence $(X,\tau_1,\tau_2,)$ is not a P-T_0 space.

In the following theorem,we show that our definitions of P-T_0 and P-T_{0w} are good extensions.

<u>**Theorem 2.1.11**</u>

Let (X,T_1,T_2) be a bitopological space ,then the following statements are equivalent:

(i) (X,T_1,T_2) is a P-T_0 space.

(ii) $(X,\omega(T_1),\omega(T_2))$ is a P-T_0 space.

(iii) $(X,\omega(T_1),\omega(T_2))$ is a P-T_{0w} space .

Proof:

(i)$\Rightarrow$(ii) : Let (X,T_1,T_2) be a P-T_0 space and let p,q be two distinct fuzzy points in X. Since (X,T_1,T_2) is a P-T_0 space and $x_p \neq x_q$ there exists $U \in T_i$ such that $x_p \in U, x_q \notin U$ or $x_q \in U, x_p \notin U$.It is clear that the function $\chi_u:(X,T_i) \to I$ is l.s.c. .Consequently $\chi_u \in \omega(T_1) \cup \omega(T_2)$. Moreover $p \in \chi_u, \chi_u \cap q=0$ or $q \in \chi_u$, $\chi_u \cap p=0$.Hence $(X,\omega(T_1),\omega(T_2))$ is a P-T_0 space.

(ii)$\Rightarrow$(iii) The proof is obvious because if $\mu \cap \lambda=0$ then $\mu \subseteq \lambda^c$.

(iii)$\Rightarrow$(i) Let $(X,\omega(T_1),\omega(T_2))$ be a P-T_{0w} space and let x,y be two distinct elements in X .Take p,q to be the fuzzy points in X for which $p(x)=q(y)=0.6$ .Then there exists $\lambda \in \omega(T_1) \cup \omega(T_2)$such that $p \in \lambda \subseteq q^c$ or $q \in \lambda \subseteq p^c$.

Consider the first case where $p \in \lambda \subseteq q^c$. In this case,$\lambda^{-1}((0.6,1])=U \in T_1 \cup T_2$ and contains x but not y.If $x \notin U$,then $\lambda(x) \notin (0.6,1]$ which gives $\lambda(x) \leq 0.6$.Since $p \in \lambda$,then $p(x)<\lambda(x) \leq 0.6$ and this implies $0.6<\lambda(x) \leq 0.6$ which is absurd.Therefore $x \in U$.

To show that $y \notin U$, suppose $y \in U$ .This implies that $\lambda(y) \in (0.6,1]$. Since $\lambda \subseteq q^c$,then $0.6=\lambda(y) \leq 1-q(y)=0.4$ which is absurd.Therefore $y \notin U$.

The other case can be treated similarly.

Theorem 2.1.12

If (X,τ_1) or (X,τ_2) is a fuzzy T_0-space (T_{0w}-space),then (X,τ_1,τ_2) is a P-T_0 space (P-T_{0w} space).

Proof :

Without loss of generality, we may assume that (X,τ_1) is a fuzzy T_0-space (T_{0w}-space).Let p,q be two distinct fuzzy points in X.Since (X,τ_1) is a fuzzy T_0-space (T_{0w}-space),there exists $\mu\in\tau_1\subseteq\tau_1\cup\tau_2$ such that $p\in\mu$, $\mu\cap q=0$ ($p\in\mu\subseteq q^c$) or $q\in\mu$, $\mu\cap p=0$ ($q\in\mu\subseteq p^c$).This completes the proof of our theorem.

Example 2.1.13

There exists a P-T_0 (P-T_{0w}) bfts (X,τ_1,τ_2) such that neither (X,τ_1) nor (X,τ_2) is a fuzzy T_0-space (T_{0w}-space).

Consider $X=[0,1]$,$\tau_1=\{0,1,\chi_{[0,r)} : 0<r\leq 1/2\}$ and $\tau_2=\{0,1,\chi_{(r,1]} : 1/2\leq r<1\}$.It is easy to show that neither (X,τ_1) nor (X,τ_2) is a fuzzy T_{0w}-space because if p is any fuzzy point with support greater (smaller) than 1/2 ,then 1 is the only fuzzy open set in $\tau_1(\tau_2)$ containing p.To show that (X,τ_1,τ_2) is a P-T_0 space (P-T_{0w}-space),let p,q be two distinct fuzzy points in X with supports x_p and x_q respectively.Without loss of generality we may assume that $x_p<x_q$.let $\xi\in(0,1)$ be such that $\xi\neq 1/2$ and $x_p<\xi<x_q$, then we have the following cases.

i) if $\xi<1/2$, then $\chi_{[0,\xi)}$ contains p and $\chi_{[0,\xi)}\cap q=0$ ($\chi_{[0,\xi)}\subseteq q^c$).

ii) if $\xi>1/2$,then $\chi_{(\xi,1]}$ contains q and $\chi_{(\xi,1]}\cap p=0$ ($\chi[\xi,1)\subseteq p^c$)

Therefore (X,τ_1,τ_2) is a P-T_0 space.

Definition 2.1.14

Let λ be a fuzzy set of a bfts (X,τ_1,τ_2).Then λ is called $\tau_1\tau_2$-open ($\tau_1\tau_2$-closed) iff $\lambda\in\tau_1\cup\tau_2$ ($\lambda^c\in\tau_1\cup\tau_2$).

Definition 2.1.15

Let τ_1 and τ_2 be two fuzzy topologies on X .Then $\tau_1\cup\tau_2$ forms a subbase for some fuzzy topology on X.This fuzzy topology is called the least upper bound topology on X and is denoted by $<\tau_1,\tau_2>$.

Now we have the following characterization of P-T_{0w} -spaces.

Theorem 2.1.16

Let (X,τ_1,τ_2) be a bifuzzy topological space,then the following are equivalent:

1) (X, τ_1,τ_2) is a P-T_{0w} space.

2) $(X,<\tau_1,\tau_2>)$ is a T_{0w} space

3) For any $0<\varepsilon<1$; $x_0,x_1\in X,x_0\neq x_1$,there exists $\lambda\in\tau_1\cup\tau_2$ such that $|\lambda(x_0)-\lambda(x_1)|>1-\varepsilon$

4) For any $0<\varepsilon< 1$; $x_0,x_1\in X,x_0\neq x_1$,there exists a fuzzy $\tau_1\tau_2$-closed set μ such that $|\mu(x_0)-\mu(x_1)|>1-\varepsilon$

Proof.:

The equivalence of (3) and (4) is clear by letting $\mu=\lambda^c$,we have $|1-\lambda(x_0)-1+\lambda(x_1)|>1-\varepsilon$.

(1)$\Rightarrow$(2): The proof is clear because $\tau_1 \cup \tau_2 \subseteq <\tau_1,\tau_2>$.

(2)$\Rightarrow$(3): Let $0<\varepsilon<1$ and $x_0,x_1 \in X, x_0 \neq x_1$.Let p_0,p_1 be the fuzzy points in X given by $p_i(x_i)=1-(1/2)\varepsilon$ (i=0, 1),then p_0,p_1 are distinct fuzzy points in the T_{0w}-space $(X,<\tau_1,\tau_2>)$.Thus there exists a fuzzy set $\lambda \in <\tau_1,\tau_2>$ such that $p_0 \in \lambda \subseteq p_1{}^c$ or $p_1 \in \lambda \subseteq p_0{}^c$ .Since $\lambda \in <\tau_1,\tau_2>$, there exist basic open sets $\mu \cap \sigma$ such that $p_0 \in \mu \cap \sigma \subseteq \lambda \subseteq p_1{}^c$ or $p_1 \in \mu \cap \sigma \subseteq \lambda \subseteq p_0{}^c$ where $\mu \in \tau_1$ and $\sigma \in \tau_2$.Since $p_i \in \mu \cap \sigma$, therefore $p_i(x_i)<(\mu \cap \sigma)(x_i)$ which implies $p_i(x_i)<\mu(x_i)$ and $p_i(x_i)<\sigma(x_i)$.That is $1-(1/2)\varepsilon<\mu(x_i)$ and $1-(1/2)\varepsilon<\sigma(x_i)$.Moreover $\mu \cap \sigma \subseteq (p_{1-i})^c$,therefore $\mu(x_{1-i}) \leq 1-p_{1-i}(x_{1-i})= (1/2)\varepsilon$ or $\sigma(x_{1-i}) \leq 1-p_{1-i}(x_{1-i})=(1/2)\varepsilon$.That is $-\mu(x_{1-i}) \geq -(1/2)\varepsilon$ or $-\sigma(x_{1-i}) \geq -(1/2)\varepsilon$.Hence we have, $|\mu(x_0)-\mu(x_1)|= \mu(x_i)-\mu(x_{1-i})>1-\varepsilon$ or $|\sigma(x_0)-\sigma(x_1)|=\sigma(x_i)-\sigma(x_{1-i})>1-\varepsilon$,which completes the proof.

(3)$\Rightarrow$(1): Let p_0,p_1 be any two distinct fuzzy points in X with supports x_0,x_1,respectively.Let $t=\min\{0.2,1-p_0(x_0),1-p_1(x_1)\}$. Applying (3) with $\varepsilon=(1/2)t, t \in (0, 1)$,there exists $\lambda \in \tau_1 \cup \tau_2$ such that $|\lambda(x_0)-\lambda(x_1)|> 1-(1/2)t$.We have two cases to consider:

Case 1: $\lambda(x_0)>\lambda(x_1)$. In this case $\lambda(x_0)-\lambda(x_1)>1-(1/2)t$ implies that $\lambda(x_0)>1-(1/2)t+\lambda(x_1) \geq 1-(1/2)t>1-t \geq p_0(x_0)$.That is $p_0 \in \lambda$.Notice that $\lambda(x_1)+p_1(x_1)<\lambda(x_0)-1+(1/2)t+1-t<1-(1/2)t$.That is $\lambda \subseteq p_1{}^c$.It follows that $\lambda \in \tau_1 \cup \tau_2$ satisfies the condition that $p_0 \in \lambda \subseteq p_1{}^c$.

Case 2: $\lambda(x_1)>\lambda(x_0)$.This case is similar to the above case and we conclude that there exists $\lambda \in \tau_1 \cup \tau_2$ satisfying the condition that $p_1 \in \lambda \subseteq p_0{}^c$.

Let us now present another characterization of P-T_{0w} spaces.

Theorem 2.1.17

Let (X,τ_1,τ_2) be a bifuzzy topological space. then the following are equivalent:

1) (X,τ_1,τ_2) is a P-T_{ow} space.

2) For any $0<\varepsilon<1$; $x_0,x_1\in X,x_0\neq x_1$,there exists $\lambda\in\tau_1\cup\tau_2$ such that $\lambda(x_i)=1$and $\lambda(x_{1-i})<\varepsilon$ for some $i\in\{1,0\}$.

3) For any $0<\varepsilon<1$; $x_0,x_1\in X,x_0\neq x_1$,there exists a fuzzy $\tau_1\tau_2$-closed set such that $\mu(x_i)=0$ and $\mu(x_{1-i})>\varepsilon$ for some $i\in\{1,0\}$.

Proof :

The equivalence of (2) and (3) is straightforward because if $\lambda\in\tau_1\cup\tau_2$ for which $\lambda(x_i)=1$and $\lambda(x_{1-i})<\varepsilon$ for some $i\in\{0, 1\}$,then $\mu=\lambda^c$ is a fuzzy $\tau_1\tau_2$-closed set for which $\mu(x_i)=0$ and $\mu(x_{1-i})> 1-\varepsilon=\varepsilon^c$.

$(1)\Rightarrow(2)$ Let $\varepsilon\in(0, 1)$ and $x_o,x_1\in X$ such that $x_0\neq x_1$.By theorem 2.1.16, for each natural number $n\in N$,there exists $\lambda_n\in\tau_1\cup\tau_2$ such that $|\lambda_n(x_0)-\lambda_n(x_1)|>1-(1/2n)$.Choose one such λ_n.Let

$A_1=\{n\in N:\lambda_n\in\tau_1$ and $\lambda_n(x_0)-\lambda_n(x_1)>0\}$,

$A_2=\{n\in N:\lambda_n\in\tau_2$ and $\lambda_n(x_0)-\lambda_n(x_1)>0\}$,

$A_3=\{n\in N:\lambda_n\in\tau_1$ and $\lambda_n(x_1)-\lambda_n(x_0)>0\}$ and

$A_4=\{n\in N:\lambda_n\in\tau_2$ and $\lambda_n(x_1)-\lambda_n(x_0)>0\}$.

Since $N=A_1\cup A_2\cup A_3\cup A_4$ is an infinite set, then there exist $i\in\{0,1\}$ and $j\in\{1,2\}$ such that the set $\{n\in N:\lambda_n\in\tau_j$ and $\lambda_n(x_i)-\lambda_n(x_{1-i})>0\}$ is an infinite set;call it $\{n_1,n_2,...\}$ where $n_1<n_2<...$ .Choose $n_k\in N$ such that $(1/n_k)<\varepsilon$ and take $\lambda=\bigcup_{m=K}^{\infty}\lambda_{n_m}\in\tau_j$.Moreover,for each $m\geq k$ we have $\lambda_{n_m}(x_i)-\lambda_{n_m}(x_{1-i})>1-1/(2n_m)$.Hence $\lambda_{n_m}(x_i) > \lambda_{n_m}(x_{1-i}) + 1-1/(2n_m)\geq 1-1/(2n_m)$. Consequently, we have $\lambda(x_i)=\bigcup_{m=K}^{\infty}\lambda_{n_m}(x_i)\geq 1$,i.e, $\lambda(x_i)=1$.On the other hand, since $\lambda_{n_m}(x_{1-i})<\lambda_{n_m}(x_i)-1+1/(2n_m)$ holds for all $m\geq k$,therefore $\lambda_{n_m}(x_{1-i})< 1/(2n_m)<(1/2)\varepsilon$ for $m\geq k$. Hence $\lambda(x_{1-i})= (\bigcup_{m=K}^{\infty}\lambda_{n_m})(x_{1-i})\leq(1/2)\varepsilon<\varepsilon$, and the proof is completed.

(2)$\Rightarrow$(1) Let p_0,p_1 be any two distinct fuzzy points in X with supports x_0,x_1 respectively. Applying (2) with $\varepsilon=\min\{1-p_0(x_0),1-p_1(x_1)\}$, there exists $\lambda\in\tau_1\cup\tau_2$ in X such that $\lambda(x_i)=1$ and $\lambda(x_{1-i})<\varepsilon$ for some $i\in\{0, 1\}$.It is clear to observe that $p_i\in\lambda\subseteq(p_{1-i})^c$ and this completes the proof.

Now we have the following characterization of P-T_0 spaces.

Theorem 2.1.18.

Let (X,τ_1,τ_2) be a bifuzzy topological space,then the following are equivalent;

1) (X, τ_1,τ_2) is a P-T_0 space.

2) $(X,<\tau_1,\tau_2>)$ is a T_0 space

3) For any $x_0,x_1\in X,x_0\neq x_1$,there exists $\lambda\in\tau_1\cup\tau_2$ such that $|\lambda(x_0)-\lambda(x_1)|=1$.

4) For any $x_0,x_1\in X,x_0\neq x_1$,there exists a fuzzy $\tau_1\tau_2$-closed set μ such that $|\mu(x_0)-\mu(x_1)|=1$.

Proof :

The equivalence of (3) and (4) is straightforward by letting $\mu=\lambda^c$.

(1)$\Rightarrow$(2): The proof is straightforward because $\tau_1 \cup \tau_2 \subseteq <\tau_1,\tau_2>$.

(2)$\Rightarrow$(3): Let $x_0,x_1 \in X$ be such that $x_0 \neq x_1$.For each natural number $n \in N$,let p_n,q_n be the fuzzy points in X defined by $p_n(x_0)=q_n(x_1)= 1-(1/2n)$.Then p_n,q_n are distinct fuzzy points in the T_0-space $(X,<\tau_1,\tau_2>)$.Thus there exists a fuzzy set $\lambda_n \in <\tau_1,\tau_2>$ such that ($p_n \in \lambda_n$ and $\lambda_n \cap q_n=0$) or ($q_n \in \lambda_n$ and $\lambda_n \cap p_n=0$).

Consider the case $p_n \in \lambda_n$ and $\lambda_n \cap q_n=0$.Since $\lambda_n \in <\tau_1,\tau_2>$,then there exists a basic open set $\mu_n \cap \sigma_n$ such that $p_n \in \mu_n \cap \sigma_n \subseteq \lambda_n$,where $\mu_n \in \tau_1$ and $\sigma_n \in \tau_2$.Now $p_n \in \mu_n \cap \sigma_n$ gives $p_n(x_0)<(\mu_n \cap \sigma_n)(x_0)$ which implies $p_n(x_0)<\mu_n(x_0)$ and $p_n(x_0)<\sigma_n(x_0)$.That is $1-(1/2n)<\mu_n(x_0)$ and $1-(1/2n)< \sigma_n(x_0)$.Since $\lambda_n \cap q_n=0$,therefore $(\mu_n \cap \sigma_n)\cap q_n=0$ which implies $(\mu_n \cap \sigma_n)(x_1)=0$.That is $\mu_n(x_1)=0$ or $\sigma_n(x_1)=0$.Let

$A_1=\{n \in N: \mu_n(x_1)=0\}$ and $A_2=\{n \in N: \sigma_n(x_1)=0\}$.

Since $N=A_1 \cup A_2$ is an infinite set,so either A_1 is infinite or A_2 is infinite. If A_1 is infinite, call it $\{n_1,n_2,.....\}$ where $n_1<n_2<....$ .Then by letting $\mu=\bigcup \mu_{n_j}$ we shall have $\mu \in \tau_1 \subseteq \tau_1 \cup \tau_2, \mu(x_0)=1$ and $\mu(x_1)=0$.If A_2 is infinite we shall get a similar result by taking $\sigma=\bigcup_{j=1}^{\infty} \sigma_{n_j}$ by which $\sigma \in \tau_2 \subseteq \tau_1 \cup \tau_2, \sigma(x_1)=0$ and $\sigma(x_0)=1$.

In the case we are dealing with,we get the existence of $\zeta \in \tau_1 \cup \tau_2$ for which $\zeta(x_1)=0$ and $\zeta(x_0)=1$.

In the case $q_n \in \lambda_n$ and $\lambda_n \cap p_n=0$; we treat this case as above to find $\zeta \in \tau_1 \cup \tau_2$,for which $\zeta(x_0)=0$ and $\zeta(x_1)=1$.

(3)⇒(1): The proof is straightforward by noticing that if $|\lambda(x_0)-\lambda(x_1)|=1$, then either ($\lambda(x_0)=1,\lambda(x_1)=0$) or ($\lambda(x_0)=0,\lambda(x_1)=1$).

§ 2.2 Bifuzzy P- R_1 and P-T_1-topological spaces.

Definition 2.2.1

A bfts (X,τ_1,τ_2) is P-R_1 iff for any distinct fuzzy points p and q in X, whenever there exists $\lambda\in\tau_1\cup\tau_2$ such that $p\in\lambda$ and $q\cap\lambda=0$, then there exist $\mu\in\tau_i$ and $\sigma\in\tau_j$ such that $p\in\mu$ and $q\in\sigma$ and $\mu\cap\sigma=0$ for some $i,j=1,2,i\neq j$.

Definition 2.2.2

A bfts (X,τ_1,τ_2) is P-R_{1w} iff for any distinct fuzzy points p and q in X, whenever there exists $\lambda\in\tau_1\cup\tau_2$ such that $p\in\lambda$ and $q\cap\lambda=0$, then there exist $\mu\in\tau_i$ and $\sigma\in\tau_j$ such that $p\in\mu$ and $q\in\sigma$ and $\mu\subseteq\sigma^c$ ($i,j=1,2,i\neq j$).

Theorem 2.2.3

If a bfts (X,τ_1,τ_2) is P-R_1 space, then (X,τ_1,τ_2) is a P-R_{1w} space.

Proof :

Clear because $\mu\cap\sigma=0$ implies $\mu\subseteq\sigma^c$.

Example 2.2.4

There exists a bfts (X,τ_1,τ_2) which is P-R_{1w} but not P-R_1.

To present our example let $X=\{0,1\}$, $\tau_1=\{0,1,\lambda_1,\lambda_2\}$ and $\tau_2=\{0,1,\mu_1,\mu_2\}$ where $\lambda_1(1)=1/4$, $\lambda_1(0)=0$, $\lambda_2(1)=1,\lambda_2(0)=3/4$,

$\mu_1=\lambda_2^c$ and $\mu_2=\lambda_1^c$.Then (X,τ_1,τ_2) is P-R_{1w} but not P-R_1 because $\lambda_2 \cap \mu_1 \neq 0$.

It is clear that if a bfts (X,τ_1,τ_2) is P-R_1 ,then it is P-R_0 but the converse is not true as we may observe in the following example.

Example 2.2.5

Let $X=I,\tau_1=\tau_2=X/T_{cof}$.Then (X,τ_1,τ_2) is a P-R_0-space but not P-R_1.

In the following theorem,we show that our definitions of P-R_1 and P-R_{1w} are good extensions .

Theorem 2.2.6

Let (X,T_1,T_2) be a bts ,then the following statements are equivalent:

i) (X, T_1,T_2) is a P-R_1 space.

ii) $(X,\omega(T_1),\omega(T_2))$ is a P-R_1 space.

iii)$(X,\omega(T_1),\omega(T_2))$ is a P-R_{1w} space.

Proof :

(i)$\Rightarrow$(ii) Let p and q be any two distinct fuzzy points with supports x and y respectively and $\lambda \in \omega(T_1) \cup \omega(T_2)$ be such that $p \in \lambda$ and $q \cap \lambda=0$.Since λ is l.s.c. then $\lambda^{-1}(0,1] \in T_1 \cup T_2$ such that $x \in \lambda^{-1}(0,1]$ and $y \notin \lambda^{-1}(0,1]$.Since (X,T_1,T_2) is P-R_1,then there exist $U \in T_i$ and $V \in T_j$ such that $x \in U$, $y \in V$ and $U \cap V=\phi$.Now it is clear that $p \in \chi_u$, $q \in \chi_v$ and $\chi_u \cap \chi_v=0$.Hence $(X,\omega(T_1),\omega(T_2))$ is P-R_1.

(ii)$\Rightarrow$(iii) see theorem 2.2.3 .

(iii)$\Rightarrow$(i) Let $x,y \in X, x\neq y$ and $U\in T_1 \cup T_2$ such that $x\in U$ and $y\notin U$. Now if p and q are two fuzzy points with supports x and y respectively and values $p(x)=q(y)=0.6$,then it is clear that $p\in \chi_u$ and $q\cap\chi_u=0$.Since $(X,\omega(T_1),\omega(T_2))$ is P-R_{1w},then there exist $\lambda\in \omega(T_i)$ and $\mu\in \omega(T_j)$ such that $p\in \lambda, q\in \mu$ and $\lambda\subseteq\mu^c$.Now it is clear that $x\in \lambda^{-1}(0.6,1],y\in \mu^{-1}(0.6,1]$ and $\lambda^{-1}(0.6,1]\cap\mu^{-1}(0.6,1]=\phi$.Hence (X,T_1,T_2) is P-R_1.

Definition 2.2.7

A bfts (X,τ_1,τ_2) is said to be P-T_1 iff for any two distinct fuzzy points p,q there exist $\mu_1\in \tau_1\cup\tau_2$ and $\mu_2\in \tau_1\cup\tau_2$ such that $p\in \mu_1, \mu_1\cap q=0$ and $q\in \mu_2$, $\mu_2\cap p=0$.

Definition 2.2.8

A bfts (X,τ_1,τ_2) is said to be P-T_{1w} iff for any two distinct fuzzy points p,q in X , there exist $\mu_1\in \tau_1\cup\tau_2$ and $\mu_2\in \tau_1\cup\tau_2$ such that $p\in \mu_1\subseteq q^c$ and $q\in \mu_2\subseteq p^c$.

Theorem 2.2.9

a) If a bfts (X,τ_1,τ_2) is a P-T_1 space,then (X,τ_1,τ_2) is a P-T_{1w} space.

b) There exists a bfts X which is P-T_{1w} but not P-T_1.

Proof:

a) The proof is obvious because if $p\cap\mu=0$ then $\mu\subseteq p^c$.

b) Example 2.1.10 serves our purpose here.

In the following theorem,we show that our definitions of P-T_1 and P-T_{1w} are good extensions.

Theorem 2.2.10

Let (X,T_1,T_2) be a bts.Then the following are equivalent:

i) (X,T_1,T_2) is a P-T_1 space.

ii) $(X,\omega(T_1),\omega(T_2))$ is a P-T_1 space.

iii) $(X,\omega(T_1),\omega(T_2))$ is a P-T_{1w} space.

Proof:

(i)$\Rightarrow$(ii): Let p,q be two distinct fuzzy points in X.Since (X,T_1,T_2) is a P-T_1 space and $x_p \neq x_q$,then there exist $u,v \in T_1 \cup T_2$ such that $(x_p \in u, x_q \notin u)$ and $(x_q \in v, x_p \notin v)$

Consider the first case,that is $x_p \in u, x_q \notin u$.Let $\chi_u : (X,T_i) \to I$ be the characteristic function of u then $\chi_u \in \omega(T_i)(i=1,2)$.Now $p \in \chi_u$ and $q \notin \chi_u$ because $x_p \in u, x_q \notin u$.

To show $q \cap \chi_u = 0$,suppose on the contrary,i.e. $q \cap \chi_u \neq 0$ .This implies $\chi_u(x_q) > 0$ which gives $x_q \in u$,a contradiction , therefore $q \cap \chi_u = 0$.

Similarly we treat the other case,that is $x_q \in v, x_p \notin v$.

(ii)$\Rightarrow$(iii) The proof is obvious because if $\lambda \cap q = 0$ then $\lambda \subseteq q^c$ where $\lambda \in \omega(T_i)$.

(iii)$\Rightarrow$(i) Let x,y be two distinct elements in X.Take p,q to be the fuzzy points in X for which p(x)=q(y)=0.6.Then there exist $\lambda, \upsilon \in \omega(T_1) \cup \omega(T_2)$ such that $p \in \lambda \subseteq q^c$ and $q \in \upsilon \subseteq p^c$.Let $u=\lambda^{-1}(0.6,1]$ and $v=\upsilon^{-1}(0.6,1]$.It is clear that $u,v \in T_1 \cup T_2$ (λ,υ are l.s.c. functions). Now we shall show that $x \in u, y \in v$.If $x \notin u$, then $\lambda(x) \notin (0.6,1]$ implies

$\lambda(x)\leq 0.6$. Since $p\in\lambda$, then $p(x)<\lambda(x)\leq 0.6$. That is $0.6<\lambda(x)\leq 0.6$, which is absurd. Therefore $x\in u$.

To show that $y\notin u$ suppose the contrary, that is $y\in u$. Then $\lambda(y)\in(0.6,1]$. Since $p\in\lambda\subseteq q^c$, then $\lambda(y)+q(y)\leq 1$. That is $\lambda(y)+0.6\leq 1$ which is not possible because $\lambda(y)>0.6$. Therefore $y\notin u$. Similarly we show $x\notin v$ and $y\in v$. Hence (X,T_1,T_2) is a P-T_0-space.

Theorem 2.2.11

A bfts (X,τ_1,τ_2) is P-T_1 (P-T_{1w}) if either (X,τ_1) or (X,τ_2) is a fuzzy T_1-space (T_{1w}-space).

Proof :

Without loss of generality we may assume that (X,τ_1) is a T_1-space (T_{1w} space). Let p,q be two distinct fuzzy points in X. Since (X,τ_1) is a T_1 space (T_{1w} space), there exist $\mu_1,\mu_2\in\tau_1\subseteq\tau_1\cup\tau_2$ such that $p\in\mu_1,\mu_1\cap q=0$ and $q\in\mu_2,\mu_2\cap p=0$ ($p\in\mu_1\subseteq q^c$ and $q\in\mu_2\subseteq p^c$). Consequently (X,τ_1,τ_2) is a P-T_1 space (P-T_{1w} space). This completes the proof of our theorem.

Example 2.2.12

There exists a P-T_1 bfts (X,τ_1,τ_2) in which neither (X,τ_1) nor (X,τ_2) is a T_{1w}-space.

To provide an example, let $X=[0,1]$. Then the space $(X,X/T_{l.r},X/T_{r.r})$ is P-T_1(P-T_{1w}) but neither $(X,X/T_{l.r})$ nor $(X,X/T_{r.r})$ is a T_{1w}-space.

Now we have the following result concerning the characterization of P-T_{1w} spaces.

Theorem 2.2.13

Let (X,τ_1,τ_2) be a bifuzzy topological space,then the following are equivalent:

1) (X, τ_1,τ_2) is a P-T_{1w} space.

2) $(X,<\tau_1,\tau_2>)$ is a T_{1w} space

3) For any $0<\varepsilon<1;x_0,x_1\in X,x_0\neq x_1$,there exists $\lambda_j\in \tau_1\cup\tau_2$ such that $|\lambda_j(x_i)-\lambda_j(x_{1-i})|>1-\varepsilon$ $i=0,1;j=1,2$.

4) For any $0<\varepsilon< 1$; $x_0,x_1\in X,x_0\neq x_1$,there exists a fuzzy $\tau_1\tau_2$-closed set μ_j such that $|\mu_j(x_i)-\mu_j(x_{1-i})|>1-\varepsilon$; $j=1,2$.

Proof :

The equivalence of (3) and (4) is clear by letting $\mu=\lambda^c$.

(1)$\Rightarrow$(2) The proof is clear because $\tau_1\cup\tau_2\subseteq<\tau_1,\tau_2>$.

(2)$\Rightarrow$(3) Let $0<\varepsilon<1$ and $x_0,x_1\in X$ such that $x_0\neq x_1$.Let p_0,p_1 be the fuzzy points in X given by $p_i(x_i)=1-(1/2)\varepsilon,i=0,1$.Then p_0,p_1 are distinct fuzzy points in the T_{1w}-space X.Thus there exists $\lambda_i\in<\tau_1,\tau_2>$ such that $p_i\in\lambda_i\subseteq(p_{1-i})^c,i=0,1$.Since $\lambda_i\in<\tau_1,\tau_2>$ there exist basic open sets $\mu_i\cap\sigma_i$ such that $p_i\in\mu_i\cap\sigma_i\subseteq\lambda_i\subseteq(p_{1-i})^c$,where $\mu_i\in\tau_1$ and $\sigma_i\in\tau_2,i=0,1$.Since $p_i\in\mu_i$ and $p_i\in\sigma_i$,then $p_i(x_i)<\mu_i(x_i)$ and $p_i(x_i)<\sigma_i(x_i)$.That is $\mu_i(x_i)>1-(1/2)\varepsilon$ and $\sigma_i(x_i)>1-(1/2)\varepsilon$.Now $\mu_i\cap\sigma_i\subseteq(p_{1-i})^c$ .Therefore $(\mu_i\cap\sigma_i)(x_{1-i})+p_{1-i}(x_{1-i})\leq 1$ which implies $\mu_i(x_{1-i})\leq 1-p_{1-i}(x_{1-i})=(1/2)\varepsilon$ or $\sigma_i(x_{1-i})\leq 1-p_{1-i}(x_{1-i})=(1/2)\varepsilon$.It follows that $\mu_i(x_i)-\mu_i(x_{1-i})>1-\varepsilon$ or $\sigma_i(x_i)-\sigma_i(x_{1-i})>1-\varepsilon$,$i=0,1$.Therefore there exists $\zeta_j\in\tau_1\cup\tau_2$ for which $|\zeta_j(x_i)-\zeta_j(x_{1-i})|>1-\varepsilon,i=0,1,j=1,2$.

(3)⇒(1) Similar to the proof of theorem 2.1.16.

Let us now present another characterization of P-T_{1w}- spaces.

Theorem 2.2.14

Let (X,τ_1,τ_2) be a bifuzzy topological space, then the following are equivalent:

1) (X,τ_1,τ_2) is a P-T_{1w} space.

2) For any $0<\varepsilon<1$; $x_0,x_1\in X.x_0\neq x_1$,there exist $\lambda_0,\lambda_1\in \tau_1 \cup \tau_2$ such that $\lambda_i(x_i)=1$ and $\lambda_i(x_{1-i})<\varepsilon$,i= 0,1.

3) For any $0<\varepsilon<1$; $x_0,x_1\in X,x_0\neq x_1$,there exist two fuzzy $\tau_1\tau_2$-closed sets μ_0,μ_1 such that $\mu_i(x_i)=0$ and $\mu_i(x_{1-i})>1-\varepsilon$,i=0,1.

Proof.:

The equivalence of (2) and (3) is straightforward.

(1)⇒(2): Let $0<\varepsilon< 1$ and $x_0,x_1\in X,x_0\neq x_1$.For each natural number $n\in N$,let p_n,q_n be the fuzzy points in X given by $q_n(x_1)= 1-(\varepsilon/4)$ and $p_n(x_0)= 1-(1/2n)$.Then p_n,q_n are distinct fuzzy points in the T_{1w}-space X.Therefore there exist $\tau_1\tau_2$-open fuzzy sets μ_n,σ_n such that $p_n\in \mu_n\subseteq(q_n)^c$ and $q_n\in \sigma_n\subseteq(p_n)^c$,$n\in N$.It follows that $\mu_n(x_0)>p_n(x_0)= 1-(1/2n)$ and $\mu_n(x_1)\leq 1-q_n(x_1)=\varepsilon/4$.Let $\lambda_0=\bigcup_{n=1}^{\infty}\mu_n$,then λ_0 is a $\tau_1\tau_2$-open fuzzy set in X satisfying $\lambda_0(x_0)= 1$ and $\lambda_0(x_1)\leq\varepsilon/4<\varepsilon$. Similarly, we can get a fuzzy $\tau_1\tau_2$-open set λ_1 such that $\lambda_1(x_1)= 1$ and $\lambda_1(x_0)<\varepsilon$.

(2)⇒(1) Let p_0, p_1 be any two distinct fuzzy points in X with supports x_0, x_1 respectively .Applying (2) with $\varepsilon = \min\{1-p_1(x_1), 1-p_0(x_0)\}$, then there exists $\lambda_0, \lambda_1 \in \tau_1 \cup \tau_2$ such that $\lambda_i(x_i)=1$ and $\lambda_i(x_{i-1})<\varepsilon$.,i=0,1. Consequently we have $p_i \in \lambda_i \subseteq (p_{i-1})^c$.

Now we introduce another characterization of P-T_1-spaces.

Theorem 2.2.15

Let (X, τ_1, τ_2) be a bifuzzy topological space,then the following are equivalent:

1) (X, τ_1, τ_2) is a P-T_1 space.

2) $(X, <\tau_1, \tau_2>)$ is a T_1 space.

3) For any $x_0, x_1 \in X, x_0 \neq x_1$,there exist $\tau_1\tau_2$-open fuzzy sets λ_0, λ_1 in X such that $\lambda_i(x_i)=1$ and $\lambda_i(x_{1-i})=0$,i=0,1.

4) For any $x_0, x_1 \in X, x_0 \neq x_1$,there exist $\tau_1\tau_2$-closed fuzzy sets μ_0, μ_1 in X such that $\mu_i(x_i)=0$ and $\mu_i(x_{1-i})=1$,i=0,1.

Proof :

The equivalence of (3) and (4) is straightforward.

(1)⇒(2) The proof is clear because $\tau_1 \cup \tau_2 \subseteq <\tau_1, \tau_2>$.

(2)⇒(3) Let $x_0, x_1 \in X, x_0 \neq x_1$.For each natural number $n \in N$,let p_n, q_n be the fuzzy points in X defined by $p_n(x_0)=q_n(x_1)=1-(1/2n)$.Then p_n, q_n are distinct fuzzy points in the T_1-space X.Thus there exist $\tau_1\tau_2$-open fuzzy sets μ_n and σ_n such that $p_n \in \mu_n, \mu_n \cap q_n = 0$ and $q_n \in \sigma_n, \sigma_n \cap p_n = 0$ ($n \in N$).This implies that for each $n \in N$ we have

$\mu_n(x_0)>p_n(x_0)=1-(1/2n),\mu_n(x_1)=0$ and $\sigma_n(x_1)>q_n(x_1)=1-(1/2n)$, $\sigma_n(x_o)=0$.If $\lambda_0=\bigcup_{n=1}^{\infty}\mu_n$ and $\lambda_1=\bigcup_{n=1}^{\infty}\sigma_n$,then λ_0,λ_1 are $\tau_1\tau_2$-open fuzzy sets with the properties that $\lambda_0(x_0)=1,\lambda_1(x_1)=0$ and $\lambda_1(x_1)=1$, $\lambda_1(x_0)=0$.

(3)⇒(1) Let p_0,p_1 be any two distinct fuzzy points in X with supports x_0,x_1 respectively .Applying (3) there exist $\tau_1\tau_2$-open fuzzy sets λ_0,λ_1 in X such that $\lambda_i(x_i)= 1$ and $\lambda_i(x_{1-i})=0,i=0,1$.It is clear that $p_i\in\lambda_i$ and $\lambda_i\cap p_{1-i}=0$ for i=0,1.

§ 2.3 Bifuzzy P- T_2 and P-T_2 1/2 topological spaces.

We start this section with the following definition.

Definition 2.3.1

A bfts (X,τ_1,τ_2) is said to be P-T_2 iff for any two distinct fuzzy points p,q in X,there exist fuzzy sets $\mu_1\in\tau_1$ and $\mu_2\in\tau_2$ such that $p\in\mu_1,q\in\mu_2,\mu_1\cap\mu_2=0$.

Definition 2.3.2

A bfts (X,τ_1,τ_2) is said to be P-T_{2w} iff for any two distinct fuzzy points p,q in X , there exist fuzzy sets $\mu_1\in\tau_1$ and $\mu_2\in\tau_2$ such that $p\in\mu_1$,$q\in\mu_2$, $\mu_1\subseteq(\mu_2)^c$.

Theorem 2.3.3

a) If a bfts (X,τ_1,τ_2) is a P-T_2 space,then (X,τ_1,τ_2) is a P-T_{2w} space.

b) There exists a bfts (X,τ_1,τ_2) which is P-T_{2w} but not P-T_2.

Proof :

a) The proof is obvious because if $p \cap \mu=0$,then $\mu \subseteq p^c$.

b) Let $X=I, \tau_1=\tau_2=\{0,\lambda:\lambda(x)>0$ for all $x \in X\}$.Let p and q be distinct fuzzy points in X and $\gamma=\min\{(1-p(x_p))/2,(1-q(x_q))/2\}$.Then we define λ and μ as follows:

$$\lambda(x)=\begin{cases}(p(x_p)+1)/2 & \text{if } x=x_p \\ \gamma/4 & \text{if } x\neq x_p\end{cases}, \quad \mu(x)=\begin{cases}(q(x_q)+1)/2 & \text{if } x=x_q \\ \gamma/4 & \text{if } x\neq x_q\end{cases}$$

It is clear that $p \in \lambda, q \in \mu$ and $\mu \subseteq \lambda^c$.Therefore (X,τ_1,τ_2) is a P-T_{2w} space. Moreover it is not a P-T_2 space because $\sigma_1 \cap \sigma_2 \neq 0$ for all $\sigma_1,\sigma_2 \in \tau_1-\{0\}=\tau_2-\{0\}$.

In the following theorem,we show that our definitions of P-T_2 and P-T_{2w} are good extensions .

Theorem 2.3.4

Let (X,T_1,T_2) be a bitopological space .Then the following statements are equivalent:

i) (X,T_1,T_2) is a P-T_2 space.

ii)$(X,\omega(T_1),\omega(T_2))$ is a P-T_2 space.

iii) $(X,\omega(T_1),\omega(T_2))$ is a P-T_{2w} space.

Proof :

(i)$\Rightarrow$(ii) Let p,q be two distinct fuzzy points in $(X,\omega(T_1),\omega(T_2))$ with supports x_p and x_q respectively. Since (X,T_1,T_2) is a P-T_2 space,then there exist $u_1,v_1 \in T_1$ and $u_2,v_2 \in T_2$ such that $x_p\in u_1,x_q\in u_2$, $u_1\cap u_2=\phi$ and $x_q\in v_1,x_p\in v_2$, $v_1\cap v_2=\phi$. Consider the first case .That is $x_p\in u_1,x_q\in u_2,u_1\cap u_2=\phi$.It is clear that the function $\chi_{u_1}:(X,T_1)\rightarrow I$ and $\chi_{u_2}:(X,T_2)\rightarrow I$ are l.s.c. .Therefore $\chi_{u_1}\in \omega(T_1)$ and $\chi_{u_2}\in \omega(T_2)$ which implies $p \in\chi_{u_1}$ and $q\in \chi_{u_2}$. We note that $\chi_{u_1}\cap\chi_{u_2}=\chi_{u_1\cap u_2}=\chi_{\phi}=0$.

Similarly we can prove the other case.

(ii)$\Rightarrow$(iii) The proof is obvious.

(iii)$\Rightarrow$(i) The proof is similar to the one proved in theorem 2.2.6

Theorem 2.3.5

If a bfts (X,τ_1,τ_2) is a P-T_2-space,then $(X,L(\tau_1),L(\tau_2))$ is a P-T_2-space,where $L(\tau_i)(i=1,2)$ is the coarsest topology on X making all fuzzy sets in τ_i lower semicontinuous.

Proof :

Let $x\neq y$ be two points of X .Define p,q to be fuzzy points in X such that $p(x)=q(y)=0.6$.Then p and q are distinct fuzzy points .Since (X,τ_1,τ_2) is a P-T_2-space,there exist a τ_i-open fuzzy set λ and a τ_j-open fuzzy set μ such that $p\in \lambda,q\in \mu$ and $\lambda\cap \mu=0$.Now let $U=\lambda^{-1}(0.6,1]$ and $V=\mu^{-1}(0.6,1]$.Then $U\in L(\tau_1)$ and $V\in L(\tau_2)$ (λ & μ are lower semicontinuous).To show that $U\cap V=\phi$,suppose that $U\cap V\neq\phi$,then there exists $z\in U\cap V$.It follows that $\lambda(z)>0.6$ and

$\mu(z)>0.6$. Consequently $(\lambda\cap\mu)(z)\neq 0$ which contradicts the fact that $\lambda\cap\mu=0$.

The following example shows that the converse of the above theorem is not true in general.

Example 2.3.6

Let X=I and $\tau_1=\tau_2=\{0,\lambda:\lambda(x)>0$ for all $x>0\}$.Then $L(\tau_1)=L(\tau_2)=T_{dis}$ and so the bts $(X,L(\tau_1),L(\tau_2))$ is P-T_2 while the bfts (X,τ_1,τ_2) is not P-T_2.

Theorem 2.3.7

A bfts (X,τ_1,τ_2) is P-T_{2w} iff for each mature fuzzy point p in X we have x_p=supp $\cap\{cl_1\mu_\alpha:\mu_\alpha$ is a τ_2-neighbourhood of p$\}$=supp $\cap\{cl_2\lambda_\alpha:\lambda_\alpha$ is a τ_1-neighbourhood of p$\}$.

Proof:

$\Rightarrow$ It is clear that $\{x_p\}\subseteq$supp $\cap\{cl_1\mu_\alpha:\mu_\alpha$ is a τ_2-NHD of p$\}$.

We have to show that $\bar{\omega}$=supp ω=supp $\cap\{cl_1\mu_\alpha:\mu_\alpha$ is a τ_2-NHD of p$\}\subseteq\{x_p\}$. Suppose not ,so there exists $y\in\bar{\omega}$ such that $y\notin\{x_p\}$.Let $\omega(y)=\zeta>0$.Define q as $q(y)=1-\xi/2>0$.Now p and q are distinct fuzzy points ,so there exists λ in τ_2 and μ in τ_1 such that $p\in\lambda,q\in\mu$,and $\lambda\subseteq\mu^c$.Then $\xi\leq(cl_1\lambda)(y)\leq(\mu^c)(y)$. Since $q(y)<\mu(y)$,then we have $\xi+q(y)=\xi+1-\xi/2=1+\xi/2<\mu(y)+(\mu^c)(y)=1$ which is a contradiction.

$\Leftarrow$ Let p_1 and q_1 be two distinct fuzzy points of X with supports x_p and x_q respectively.Define p and q as $p(x_p)=(1+p_1(x_p)/2$, $q(x_q)=(1+q_1(x_q))/2$ .Then p and q are mature fuzzy points. Therefore $\{x_p\}$=supp$\{\cap\{cl_1\mu_\alpha:\mu_\alpha$ is a τ_2-NHD of p$\}\}$.Since $x_p\neq x_q$ and $p\in\mu_\alpha$,then $q\in[\cap(cl_1\mu_\alpha)]^c$ which implies that $q\in\cup(cl_1\mu_\alpha)^c$ which is a τ_1-open.Since $q\in\cup(cl_1\mu_\alpha)^c$,then there exists μ_α such that $p\in\mu_\alpha$ and

$q\in(cl_1\mu_\alpha)^c$.Moreover $\mu_\alpha\subseteq[(cl_1\mu_\alpha)^c]^c=cl_1\mu_\alpha$ which completes the proof.

Theorem 2.3.8

Let (X,τ_1,τ_2) be a bfts ,then X is P-T_{2w} iff for any two distinct fuzzy points p,q in X,there exists $\lambda\in\tau_i$ such that $p\in\lambda$ and $q\in(cl_j\lambda)^c(i\neq j)$.

Proof :

$\Rightarrow$Let p and q be two distinct fuzzy points of a bft-P- T_{2w}-space (X,τ_1,τ_2).Then there exists λ in τ_1 and μ in τ_2 such that $p\in\lambda,q\in\mu$ and $\lambda\subseteq\mu^c$.Now $cl_2\lambda\subseteq cl_2\mu^c=\mu^c$ which implies that $\mu\subseteq(cl_2\lambda)^c$.i.e. $q\in(cl_2\lambda)^c$.

$\Leftarrow$ Let p and q be two distinct fuzzy points and $\lambda\in\tau_1$such that $p\in\lambda$and $q\in(cl_2\lambda)^c$.Since $\lambda\subseteq((cl_2\lambda)^c)^c=cl_2\lambda$,then $p\in\lambda,q\in(cl_2\lambda)^c$ and $\lambda\subseteq((cl_2\lambda)^c)^c=cl_2\lambda$.Hence (X,τ_1,τ_2) is P-T_{2w}-space.

Definition 2.3.9

A bfts (X,τ_1,τ_2) is said to be P-$T_{2\ 1/2}$ iff for any two distinct fuzzy points p,q in X, there exist fuzzy sets $\mu_1\in\tau_1$, $\mu_2\in\tau_2$ such that $p\in\mu_1,q\in\mu_2$ and $cl_2\mu_1\cap cl_1\mu_2=0$.

Definition 2.3.10

A bfts (X,τ_1,τ_2) is said to be P-$T_{2\ 1/2w}$ iff for any two distinct fuzzy points p,q in X , there exist fuzzy sets $\mu_1\in\tau_1,\mu_2\in\tau_2$ such that $p\in\mu_1,q\in\mu_2$ and $(cl_2\mu_1)\subseteq(cl_1\mu_2)^c$.

Theorem 2.3.11

(a)If a bfts (X,τ_1,τ_2) is a P-T_2 1/2 space,then (X,τ_1,τ_2) is a P-T_2 1/2w space.

(b) There exists a bfts (X,τ_1,τ_2) which is P-T_2 1/2w but not P-T_2 1/2.

Proof:

(a) The proof is obvious because if $cl_2\mu_1 \cap cl_1\mu_2=0$,then $cl_2\mu_1 \subseteq (cl_1\mu_2)^c$.

(b) Let $X=I,\tau_1=\tau_2=\{0,\lambda:\lambda(x)>0$ for every $x\in X\}$.Let p,q be any two distinct fuzzy points in X with supports x_p,x_q respectively .Define the fuzzy sets λ_p,λ_q in X by ,

$$\lambda_p(x)=\begin{cases}(1-q(x_q))/2 & \text{if } x\neq x_p\\ (1+p(x_p)/2 & \text{if } x=x_p.\end{cases} \quad \text{and} \quad \lambda_q(x)=\begin{cases}(1-p(x_p))/2 & \text{if } x\neq x_q\\ (1+q(x_q)/2 & \text{if } x=x_q.\end{cases}$$

Then λ_p,λ_q are simultaneously fuzzy $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed sets in X satisfying the conditions that $p\in\lambda_p, q\in\lambda_q$ and $cl_i\lambda_p=\lambda_p\subseteq\lambda_q{}^c=(cl_j\lambda_q)^c$ $(i,j=1,2,i\neq j)$. Hence (X,τ_1,τ_2) is a P-T_2 1/2w space but not P-T_2 1/2 because for every $\lambda\in\tau_1=\tau_2$ we have $\lambda(x)>0$ provided $\lambda\neq 0$.

In the following theorem,we show that our definitions of P-T_2 1/2 and P-T_2 21/2w are good extensions .

Theorem 2.3.12

Let $(X,T1,T2)$ be a bitopological space,then the following are equivalent.

(i) $(X,T1,T2)$ is a P-T2 1/2 space .

(ii) $(X,\omega(T1),\omega(T2))$ is a P-T2 1/2 space.

(iii)$(X,\omega(T1),\omega(T2))$ is a P-T2 1/2 w space.

Proof:

(i)⇒(ii) Let p,q be two distinct fuzzy points in X.Applying (i) there exist $u1,v1\in T1$ and $u2,v2\in T2$ such that $x_p\in u1, x_q\in u2$,$cl_2u1\cap cl_1u2=\phi$ and $x_q\in v1, x_p\in v2, cl_2v1\cap cl_1v2=\phi$.Consider the first part $x_p\in u1$, $x_q\in u2$, $cl_2u1\cap cl_1u2=\phi$.We note that $cl_2\chi_{u1}\cap cl_1\chi_{u2}=\chi_{cl_2u1\cap cl_1u2}=\chi_\phi=0$. Moreover $p\in cl_2\chi_{u1}$,$q\in cl_1\chi_{u2}$.The second part can be treated similarly.Therefore $(X,\omega(T1),\omega(T2))$ is a P-T2 1/2 space.

(ii)⇒(iii) The proof is obvious.

(iii)⇒(i) Let x,y be two distinct elements in X.Take p,q to be the fuzzy points in X for which $p(x)=q(y)=0.9$.By (iii) there exist $\mu 1,\nu 2,\in\omega(T1),\nu 1,\mu 2\in\omega(T2)$ such that $p\in\mu 1, q\in\mu 2$,$cl_2\mu 1\subseteq(cl_1\mu 2)^c$ and $q\in\nu 1, p\in\nu 2, cl_2\nu 1\subseteq(cl_1\nu 2)^c$.It is clear that $u=\mu 1^{-1}(0.9,1]\in T1$, $v=\mu 2^{-1}(0.9,1]\in T2, x\in u, y\in v$ and $cl_2u\cap cl_1v=\phi$. Indeed, if $z\in cl_2u\cap cl_1v$,then $cl_1\mu_1(z)\geq 0.9$ and $cl_2\mu 2(z)\geq 0.9$ which contradicts the fact that $cl_2\mu 1\subseteq(cl_1\mu 2)^c$.

Theorem 2.3.13

Let (X,T_1,T_2) be a bitopological space.then the following are equivalent:

(i) (X,T_1,T_2) is P-T_2 1/2.

(ii) $(X,X/T_1,X/T_2)$ is P-T_2 1/2.

(iii)$(X,X/T_1,X/T_2)$ is P-T_2 1/2w.

Proof:

(i)$\Rightarrow$(ii) Let p,q be two distinct fuzzy points with supports x_p,x_q respectively. Applying (i) there exist $u_1 \in T_1$ and $u_2 \in T_2$ such that $x_p \in u_1, x_q \in u_2$ and $cl_2u_1 \cap cl_1u_2=\phi$.It is clear that $p \in \chi_{u_1}, q \in \chi_{u_2}$ and $cl_2\chi_{u_1} \cap cl_1\chi_{u_2}= \chi_{cl_2u_1} \cap \chi_{cl_1u_2}= \chi_{\phi}=0$.

(ii)$\Rightarrow$(iii) Trivial.

(iii)$\Rightarrow$(i) Let x,y be two distinct points in X and p,q be fuzzy points in X with supports x,y respectively.Applying(iii),there exist $\chi_{u_1} \in X/T_1$ and $\chi_{u_2} \in X/T_2$ such that $p \in \chi_{u_1}, q \in \chi_{u_2}$ and $cl_2\chi_{u_1} \subseteq (cl_1\chi_{u_2})^c$. It is clear that $x \in u_1, y \in u_2$ and $cl_2u_1 \cap cl_1u_2=\phi$,because if $cl_2u_1 \cap cl_1u_2 \neq \phi$ then there exists $r \in X$ such that $\chi_{cl_2u_1}(r) \neq 0$ and $\chi_{cl_1u_2}(r) \neq 0$.Since $cl_2\chi_{u_1} \subseteq (cl_1\chi_{u_2})^c$ we have $cl_2\chi_{u_1}(r) \leq 1-cl_1\chi_{u_2}(r)$ which implies that $cl_2\chi_{u_1}(r) + cl_1\chi_{u_2}(r) \leq 1$.That is $1+1 \leq 1$ which is absurd.

We note that the above theorem is also true for P-T_i-spaces where $i=0,1,2$.

§ 2.4 Bifuzzy P-regular and P-normal topological spaces.

In this section we define P-regular and P-normal bifuzzy topological spaces and study some of their charaterizations.

Definition 2.4.1

A bfts (X,τ_1,τ_2) is said to be P-regular if for every fuzzy point p in X and each fuzzy τ_i-closed set λ such that $p \in \lambda^c$ there exist a τ_i-open fuzzy set μ and a τ_j-open fuzzy set υ such that $p \in \mu, \lambda \subseteq \upsilon$ and $\mu \subseteq \upsilon^c$; $i \neq j; i,j=1,2$.

An important and useful characterization of fuzzy regularity is given in the following theorem.

Theorem 2.4.2

Let (X,τ_1,τ_2) be a bfts,then the following are equivalent:

(i) (X,τ_1,τ_2) is P-regular.

(ii) For every fuzzy point p in X and for each τ_i -open fuzzy set λ such that $p \in \lambda$,there exists τ_i-open fuzzy set μ such that $p \in \mu \subseteq cl_j\mu \subseteq \lambda$.

(iii) For every τ_i-closed fuzzy set λ in X and for any fuzzy point $p \in \lambda^c$,there exists a τ_i-open fuzzy set μ_1 and a τ_j-open fuzzy set μ_2 such that $p \in \mu_1, \lambda \subseteq \mu_2$ and $cl_2\mu_1 \subseteq (cl_1\mu_2)^c$.

Proof:

(i) $\Rightarrow$(ii) Let p be any fuzzy point in X and let λ be a τ_i- open fuzzy set in X such that $p \in \lambda$.Since $p \in \lambda, p \in (\lambda^c)^c$ and λ^c is a τ_i-closed fuzzy set in X.By (i) there exist a τ_j - open fuzzy set v and a τ_i-open fuzzy set μ such that $p\in\mu, \lambda^c \subseteq \nu$ and $\mu \subseteq \nu^c$. Since $\lambda^c \subseteq \nu$ we have $\nu^c \subseteq \lambda$ and $\mu \subseteq \nu^c \subseteq \lambda$. Moreover $\mu \subseteq cl_j\mu \subseteq cl_j\nu^c = \nu^c \subseteq \lambda$. Hence $p\in\mu \subseteq cl_j\mu \subseteq \lambda$.

(ii)$\Rightarrow$(iii) Let λ be a τ_i-closed fuzzy set in X and let p be any fuzzy point in X such that $p\in\lambda^c$.Using (ii) there exists a τ_i-open fuzzy set σ_1 such that $p\in\sigma_1 \subseteq cl_j\sigma_1 \subseteq \lambda^c$ Applying (ii) again,there exists a τ_i-open fuzzy set σ_2 such that $p\in\sigma_2 \subseteq cl_j\sigma_2 \subseteq \sigma_1 \subseteq cl_j\sigma_1 \subseteq \lambda^c$.The proof is completed by taking $\mu_1=\sigma_2$ and $\mu_2=(cl_j\sigma_1)^c$ because $p\in\sigma_2, \lambda\subseteq\mu_2$ and $cl_j\sigma_2 \subseteq [cl_i(cl_j\sigma_1)^c]^c$.Indeed $\sigma_1 \subseteq cl_j\sigma_1$ implies $int_i\sigma_1 \subseteq int_i(cl_j\sigma_1)$.Thus we get a $\sigma_1 \subseteq [cl_i(cl_j\sigma_1)^c]^c$.Since $cl_j\sigma_2 \subseteq \sigma_1$,then $cl_j\sigma_2 \subseteq [cl_i(cl_j\sigma_1)^c]^c$. That is $cl_j\mu_1 \subseteq (cl_i\mu_2)^c$.

(iii)$\Rightarrow$(i) Let λ be a τ_i-closed fuzzy set in X and let p be any fuzzy point in X such that $p\in\lambda^c$.Using (iii) there exist a τ_i-open fuzzy set μ_1 and a τ_j-open fuzzy set μ_2 such that $p\in\mu_1, \lambda\subseteq\mu_2$ and $cl_j\mu_1 \subseteq (cl_i\mu_2)^c$. Since $\mu_1 \subseteq cl_j\mu_1$ and $\mu_2 \subseteq cl_i\mu_2$,then $(cl_i\mu_2)^c \subseteq (\mu_2)^c$ and so $\mu_1 \subseteq (\mu_2)^c$.

The following theorem shows that our definition is a good extension.

Theorem 2.4.3

Let (X,T_1,T_2) be a bts.Then the following are equivalent:

(i) (X,T_1,T_2) is P-regular.

(ii) $(X,\omega(T_1),\omega(T_2))$ is P-regular.

Proof:

(i)$\Rightarrow$(ii) Let (X,T_1,T_2) be a P-regular bitopological space.Let p be a fuzzy point in X and $\lambda\in\omega(T_i)(i=1,2)$ such that $p\in\lambda$.Let

$r=(1/2)[p(x_p)+\lambda(x_p)]$.Then $p(x_p)<r<\lambda(x_p)$ i.e. $x_p\in\lambda^{-1}(r,1]$.Using the P-regularity of (X,T_1,T_2), there exists $u\in T_i$ such that $x_p\in u\subseteq cl_ju\subseteq\lambda^{-1}(r,1]$.Since $[cl_j(r\chi_u)=r\chi_{cl_ju}]$, the following inclusions become clear $p\in r\chi_u\subseteq cl_j(r\chi_u)\subseteq r\chi_{\lambda^{-1}(r,1]}\subseteq\lambda$. The facts that $r\chi_u\in\omega(T_i)$ completes the proof of the first implication.

(ii)$\Rightarrow$(i) Let $(X,\omega(T_1),\omega(T_2))$ be a P-regular bifuzzy topological space.Let $U\in\tau_i$ and $x\in U$.Take p to be the fuzzy point in X for which $p(x)=0.6$. Then $p\in\chi_u$ and $\chi_u\in\omega(T_i)$ therefore there exists $\mu\in\omega(T_i)$ such that $p\in\mu\subseteq cl_j\mu\subseteq\chi_u$.Since $cl_j(\mu^{-1}(a,1])\subseteq(cl_j\mu)^{-1}[a,1]$ for all $a\in[0,1]$,we get $cl_j(\mu^{-1}(0.6,1])\subseteq(cl_j\mu)^{-1}[0.6,1]\subseteq(\chi_u)^{-1}[0.6,1]=U$.Let $V=\mu^{-1}(0.6,1]$, then $V\in T_i$ and $x\in V\subseteq cl_jV\subseteq U$.

Definition 2.4.4

A bfts (X,τ_1,τ_2) is said to be P- normal iff for any τ_i -closed fuzzy set σ_1 and a τ_j-closed fuzzy set σ_2 with $\sigma_1\subseteq(\sigma_2)^c$, there exist a τ_j-open fuzzy set μ and a τ_i-open fuzzy set λ such that $\sigma_1\subseteq\mu$, $\sigma_2\subseteq\lambda$ and $\mu\subseteq\lambda^c$.

An important and useful characterization of fuzzy normality is given in the following theorem.

Theorem 2.4.5

Let (X,τ_1,τ_2) be a bfts, then the following are equivalent:

(i) (X,τ_1,τ_2) is P-normal.

(ii) For every τ_i-closed fuzzy set σ and for every τ_j -open fuzzy set λ such that $\sigma\subseteq\lambda$, there exists a τ_j-open fuzzy set μ such that $\sigma\subseteq\mu\subseteq cl_i\mu\subseteq\lambda$.

(iii) For every τ_i-closed fuzzy set σ and for every τ_j -closed fuzzy set λ such that $\lambda \subseteq \sigma^c$, there exists a τ_i-open fuzzy set μ_1 and a τ_j-open fuzzy set μ_2 such that $\sigma \subseteq \mu_2, \lambda \subseteq \mu_1$ and $cl_j \mu_1 \subseteq (cl_i \mu_2)^c$.

(iv) For every τ_i-closed fuzzy set λ and for every τ_j -open fuzzy set μ such that $\lambda \subseteq \mu$,there exists a fuzzy set σ such that $\lambda \subseteq int_j \sigma \subseteq cl_i \sigma \subseteq \mu$.

Proof.

(i) $\Rightarrow$ (ii) Let σ be a τ_i-closed fuzzy set and λ be a τ_j-open fuzzy set such that $\sigma \subseteq \lambda$ .By (i) there exist a τ_i-open fuzzy set v and a τ_j- open fuzzy set μ such that $\sigma \subseteq \mu, \lambda^c \subseteq v$ and $\mu \subseteq v^c$.Since $\sigma \subseteq \mu \subseteq v^c \subseteq \lambda$ and $\mu \subseteq cl_i \mu \subseteq cl_i v^c = v^c$ then $\sigma \subseteq \mu \subseteq cl_i \mu \subseteq \lambda$.

(ii)$\Rightarrow$(iii) Let λ be a τ_i-closed fuzzy set in X and let μ be a τ_j-closed fuzzy set in X such that $\mu \subseteq \lambda^c$.Using (ii),there exists a τ_i-open fuzzy set σ_1 such that $\mu \subseteq \sigma_1 \subseteq cl_j \sigma_1 \subseteq \lambda^c$.Applying (ii) again,there exists a τ_i-open fuzzy set σ_2 such that $\mu \subseteq \sigma_2 \subseteq cl_j \sigma_2 \subseteq \sigma_1 \subseteq cl_j \sigma_1 \subseteq \lambda^c$.The proof is completed by taking $\mu_1 = \sigma_2$ and $\mu_2 = (cl_j \sigma_1)^c$ because $\mu \subseteq \sigma_2, \lambda \subseteq \mu_2$ and $cl_j \sigma_2 \subseteq [cl_i (cl_j \sigma_1)^c]^c$.Indeed $\sigma_1 \subseteq cl_j \sigma_1$ implies $int_i \sigma_1 \subseteq int_i (cl_j \sigma_1) = [cl_i (cl_j \sigma_1)^c]^c$.Thus we get $\sigma_1 \subseteq [cl_i (cl_j \sigma_1)^c]^c$ .Since $cl_j \sigma_2 \subseteq \sigma_1$ we have $cl_j \sigma_2 \subseteq [cl_i (cl_j \sigma_1)^c]^c$ .That is $cl_j \mu_1 \subseteq (cl_i \mu_2)^c$.

(iii)$\Rightarrow$(i) Let λ be a τ_i-closed fuzzy set in X and let μ be a τ_j-closed fuzzy set in X such that $\mu \subseteq \lambda^c$.Using (iii),there exist a τ_i-open fuzzy set μ_1 and a τ_j-open fuzzy set μ_2 such that $\lambda \subseteq \mu_2, \mu \subseteq \mu_1$ and $cl_j \mu_1 \subseteq (cl_i \mu_2)^c$ .Since $\mu_1 \subseteq cl_j \mu_1$ and $\mu_2 \subseteq cl_i \mu_2$ we have $\mu_1 \subseteq (\mu_2)^c$.

(ii)$\Rightarrow$(iv) is immediate because $int_j \sigma \subseteq \sigma$.

(iv)⇒(ii) Let μ be a τ_i-closed fuzzy set and λ be a τ_j-open fuzzy set such that $\mu \subseteq \lambda$.Using (iv),there exists a fuzzy set σ such that $\mu \subseteq int_j\sigma \subseteq cl_i\sigma \subseteq \lambda$.let $\nu = int_j\sigma$,then ν is a τ_j-open fuzzy set .Indeed $\nu = int_j\sigma \subseteq \sigma$ implies $cl_i\nu \subseteq cl_i\sigma \subseteq \lambda$.Consequently ν is a τ_j-open fuzzy set satisfying $\mu \subseteq \nu \subseteq cl_i\nu \subseteq \lambda$.

To show that our definition of fuzzy P-normality is a good extension we prove the following theorem.

Theorem 2.4.6

Let (X,T_1,T_2) be a bts.If $(X,\omega(T_1),\omega(T_2))$ is P-normal,then (X,T_1,T_2) is P-normal.

Proof:

Let $(X,\omega(T_1),\omega(T_2))$ be P-normal and A be a T_i-closed set and B a T_j-closed set in X such that $A \cap B = \phi$.Then χ_A is a $\omega(T_i)$-closed fuzzy set and χ_B is a $\omega(T_j)$-closed fuzzy set in $(X,\omega(T_1),\omega(T_2))$ such that $\chi_A \subseteq (\chi_B)^c$.Thus there exist a $\omega(T_j)$-open fuzzy set λ and a $\omega(T_i)$-open fuzzy set μ such that $\chi_A \subseteq \lambda, \chi_B \subseteq \mu$and $\lambda \subseteq \mu^c$.Let $U = \lambda^{-1}(0.6,1]$ and $V = \mu^{-1}(0.6,1]$.Then U is a T_j-open set and V is a T_i-open set in X and $U \cap V = \phi$.

Theorem 2.4.7

Let (X,T_1,T_2) be a bitopological space,then the following are equivalent:

(i) (X,T_1,T_2) is P-regular (P-normal)..

(ii) $(X,X/T_1,X/T_2)$ is P-regular (P-normal).

Proof:

Straightforward.

The following examples show that if a bfts (X,τ_1,τ_2) is P-regular (P-normal) then $(X,\tau_1)$and (X,τ_2) need not be fuzzy regular (fuzzy normal) and vice versa (i.e. ,our concepts of fuzzy P-regularity and P-normality are independent of the corresponding ones in single fuzzy topologies).

Example 2.4.8

Let $X=[0,1]$,$\tau_1=X/T_u$ and $\tau_2=X/T$ where $T=\{A\subseteq X:X-A$ is finite or $1/2\in A^c\}$,then (X,τ_1,τ_2) is not P-regular because of the τ_2-closed set χ_F where $F=\{0.6,0.7\}$ and the fuzzy point p with support $x_p=1/2$ and value 3/4.The bfts (X,τ_1,τ_2) is also not P-normal because of the τ_1-closed fuzzy set $\chi_{\{1/2\}}$ and the τ_2-closed set χ_F (where $F=\{0.6,0.7\}$). Notice that (X,τ_1) and (X,τ_2) are regular (normal) fuzzy topological spaces.

Example 2.4.9

Let $X=[0,1]$,$\tau_1=X/T_{cof}$ and $\tau_2=X/T$ where $T=\{\phi,u: 1/2\in u\}$ then (X,τ_1,τ_2) is P-normal while (X,τ_1) and (X,τ_2) are not normal.Moreover $(X,X/T_{l.r},X/T_{r.r})$ is P-regular while $(X,X/T_{l.r})$ and $(X,X/T_{r.r})$ are not regular.

The following example shows if (X,τ_1,τ_2) is P-normal and P-T_1,then (X,τ_1,τ_2) need not be P-regular.

Example 2.4.10

Let $X=[0,1]$, $T_1=T_{l.r}$ and $T_2=T_u$. Then the bfts $(X,\tau_1,\tau_2)=(X,X/T_1,X/T_2)$ is P-normal and P-T_1 but not P-regular because of the τ_2-closed fuzzy set $\chi_{[0.6,0.8]}$ and the fuzzy point p with support $x_p=0.3$ and value 1/2.

§ 2.5 Different properties of separation axioms.

In this section we show some relations among separation axioms.

Theorem 2.5.1

Let (X,τ_1,τ_2) be a bfts. Then the following are equivalent.

(i) (X,τ_1,τ_2) is P-T_2

(ii) (X,τ_1,τ_2) is P-T_0 and P-R_1.

Proof :

(i)⇒(ii) Clear since P-T_2 implies P-T_0 and P-T_2 implies P-R_1.

(ii)⇒(i) Let p and q be two distinct fuzzy points in X. Since X is P-T_0, so there exists $\lambda\in\tau_1\cup\tau_2$ such that ($p\in\lambda$, $q\cap\lambda=0$) or ($q\in\lambda$, $p\cap\lambda=0$). Since X itself is a P-R_1 space, so there exists a τ_i-open fuzzy set μ and a τ_j-open fuzzy set σ such that $p\in\mu$, $q\in\sigma$ and $\mu\cap\sigma=0$. Hence (X,τ_1,τ_2) is P-T_2

The following example shows that P-R_1 does not imply P-T_0.

Example 2.5.2

Let X=I, $\tau 1=\{0,1\}$ and $\tau 2=\{0,\lambda:\lambda(x)>0$ for all $x\in X\}$,then $(X,\tau 1,\tau 2)$ is (vacuously) P-R_1 but not P-T_0.

The following theorem reduces the fuzzy P-T_i-space (P-T_{iw}-space) to a fuzzy T_i-space (T_{iw}-space) where i=0,1.

Theorem 2.5.3

Let $(X,\tau 1,\tau 2)$ be a bfts.Then the following are equivalent.

(i) $(X,\tau 1,\tau 2)$ is P-T_i (P-T_{iw})

(ii) $(X,<\tau 1,\tau 2>)$ is T_i(T_{iw}- space)

where i=0,1.

We shall prove the theorem for i=0.The other cases can be treated similarly.

Proof :

(i)$\Rightarrow$(ii)Let p and q be any two distinct fuzzy points in X.Since $(X,\tau 1,\tau 2)$ is P-T_0,then there exists $\mu\in\tau 1\cup\tau 2\subseteq<\tau 1,\tau 2>$ such that $(p\in\mu,\mu\cap q=0)$ or $(q\in\mu,\mu\cap p=0)$ which completes the first part.

(ii)$\Rightarrow$(i) Let p and q be two distinct fuzzy points in X.Since $(X,<\tau 1,\tau 2>)$ is a T_0-space,then there exists $\mu\in<\tau 1,\tau 2>$ such that $(p\in\mu,\mu\cap q=0)$ or $(q\in\mu,\mu\cap p=0)$ .Since $\mu\in<\tau 1,\tau 2>$,so there exists a basic open set $\sigma 1\cap\sigma 2$ where $\sigma 1\in\tau 1$ and $\sigma 2\in\tau 2$ such that $\sigma 1\cap\sigma 2\subseteq\mu$. Now $\mu\cap q=0$ gives $(\sigma 1\cap\sigma 2)\cap q=0$ which implies $(\sigma 1\cap q)\cap(\sigma 2\cap q)=0$, i.e.$\sigma 1\cap q=0$ or $\sigma 2\cap q=0$.Hence $(X,\tau 1,\tau 2)$ is a P-T_0-space.

Theorem 2.5.4

Let (X,τ_1,τ_2) be a bfts,then we have the following:

(i) If (X,τ_1,τ_2) is a P-T_2 space,then $(X,<\tau_1,\tau_2>)$ is a T_2 space .

(ii) If (X,τ_1,τ_2) is a P-T_{2w} space,then $(X,<\tau_1,\tau_2>)$ is a T_{2w} space .

Proof :

(i)Let p,q be two distinct fuzzy points in X .By (i) there exists a τ_1-open fuzzy set λ and a τ_2 -open fuzzy set μ such that $p\in\lambda,q\in\mu$ and $\lambda\cap\mu=0$.Since $\lambda,\mu\in\tau_1\cup\tau_2\subseteq<\tau_1,\tau_2>$,therefore $(X,<\tau_1,\tau_2>)$ is a T_2-space .

(ii) Similar to the part (i).

Example 2.5.5

The converses of the above parts are not true in general.The following example illustrates our purpose.

Let $X=\{0,1\},\tau_1=\{0,1\},\tau_2=\{0,1,\lambda,\mu\}$ where $\lambda(0)=1,\lambda(1)=0$, $\mu(0)=0$ and $\mu(1)=1$.Now $<\tau_1,\tau_2>=\{0,1,\lambda,\mu\}=\tau_2$.It is clear that $(X,<\tau_1,\tau_2>)$ is a T_2-space but (X,τ_1,τ_2) is not a P-T_{2w} space because 1 is the only nonzero fuzzy open set in τ_1.

We notice that (X,τ_1,τ_2) is a P-T_i space if and only if $(X,<\tau_1,\tau_2>)$ is a T_i-space,where i=0,1,0w,1w but not for i=2 and 2w.This;of course; means that the new concept of pairwise separation axioms P-T_0,P-T_1, P-T_{0w},and P-T_{1w} may be reduced into a separation axiom for one single fuzzy topology.

Theorem 2.5.6

If a bfts (X,τ_1,τ_2) is P-T_2,then $(X,\tau_1)$and (X,τ_2) are T_1.

Proof :

Let p,q be two distinct fuzzy points in X.Then there exist $\mu_1\in\tau_1,\mu_2\in\tau_2$ such that $p\in\mu_1,q\in\mu_2,\mu_1\cap\mu_2=0$.To show that (X,τ_1) is T_1,it is sufficient to show that $\mu_1\cap q=0$.Suppose $\mu_1\cap q\neq 0$,this implies $\mu_1(x_q)>0$.Since $\mu_1\cap\mu_2=0$,we have $\mu_2(x_q)=0$.That is $q\notin\mu_2$, a contradiction. Hence $\mu_1\cap q=0$.

Example 2.5.7

There exists a bfts (X,τ_1,τ_2) such that $(X,\tau_1)$and (X,τ_2) are both T_2 fuzzy spaces but (X,τ_1,τ_2) is not P-T_2.

To provide an example let $X=[0,1],T_1=T_u$ and $T_2=\{A\subseteq X:X-A$ is finite or $1/2\in A^c\}$.Then (X,T_1,T_2) is not a P-T_2 space and so $(X,X/T_1,X/T_2)$ is not a P-T_2 space,while $(X,X/T_1)$ and $(X,X/T_2)$ are T_2-spaces.

Theorem 2.5.8

Every P -regular P-T_{0w} space is a P-$T_{2\ 1/2\ w}$ space .

Proof:

Let (X,τ_1,τ_2) be a P-regular bfts and let p,q be two distinct fuzzy points in X with supports $x_p\neq x_q$ and values r_p,r_q respectively.Let p_1,q_1 be the fuzzy points in X with supports x_p,x_q respectively and values $p_1(x_p)=(1/2)(1+r_p),q_1(x_q)=(1/2)(1+r_q)$.Then p_1,q_1 are two distinct fuzzy points in X .Using the fact that X is a P-T_{0w} space , there exists an open fuzzy set $\lambda \in \tau_i$ such that $p_1 \in \lambda \subseteq q_1^c$ or $q_1 \in \lambda \subseteq p_1^c$.Thus we have two cases to consider.

Case 1: $p_1 \in \lambda \subseteq q_1^c$.By P-regularity of (X,τ_1,τ_2),there exists a τ_i-open fuzzy set μ such that $p_1 \in \mu \subseteq cl_j\mu \subseteq \lambda \subseteq q_1^c$. Since $cl_j\mu \subseteq q_1^c$ we have $q_1 \subseteq (cl_j\mu)^c$.Noticing that $q(x_q)=r_q<(1/2)(1+r_q)=q_1(x_q)$ and $p(x_p)<p_1(x_p)$,therefore we get $q \in (cl_j\mu)^c$ and $p \in \mu$.Observe that $\sigma_1=\mu$ is τ_i-open and $\sigma_2=(cl_j\mu)^c$is a τ_j-open fuzzy set in X and $\mu \subseteq [(cl_j\mu)^c]^c$.Hence there exist a τ_i-open fuzzy set σ_1 and a τ_j-open fuzzy set σ_2 such that $p \in \sigma_1$and $q \in \sigma_2$and $\sigma_1 \subseteq \sigma_2^c$.By P-regularity of (X,τ_1,τ_2) there exist a τ_i-open fuzzy set σ_3 and a τ_j-open fuzzy set σ_4 such that $p \in \sigma_3 \subseteq cl_j\sigma_3 \subseteq \sigma_1$ and $q \in \sigma_4 \subseteq cl_i\sigma_4 \subseteq \sigma_2$ .It is clear now that $p \in \sigma_3, q \in \sigma_4$ and $cl_j\sigma_3 \subseteq \sigma_1 \subseteq \sigma_2^c \subseteq (cl_i\sigma_4)^c$.

Case 2: This case is similar to the above case .

Definition 2.5.9

Consider a function $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$,then f is said to be

1) continuous if $f:(X,\tau_1) \to (Y,\sigma_1)$ and $f:(X,\tau_2) \to (Y,\sigma_2)$ are continuous.

2) P-continuous iff for any $\mu \in \sigma_1 \cup \sigma_2$,$f^{-1}(\mu) \in \tau_1 \cup \tau_2$.

3) P-open iff for any $\mu \in \tau_1 \cup \tau_2$,$f(\mu) \in \sigma_1 \cup \sigma_2$

Clearly if f is continuous then it is P-continuous but the converse is not true in general.

Example 2.5.10

Let X=I and $f:(X,X/T_{l.r.},X/T_{r.r})\rightarrow(X,X/T_{r.r.},X/T_{l.r})$ be defined by f(x)=x,then f is P-continuous but not continuous.

To proceed to our next result, we need the following definition which is a special case of that in Hutton (1975).

Definition 2.5.11

The fuzzy unit interval $[0,1]_f$ is the set of all monotonic decreasing maps λ: $R\rightarrow I$ satisfying:

1) $\lambda(t) = 1$ for t<o, $t\in R$,

2) $\lambda(t) = 0$ for t>1, $t\in R$;

after identification of $\lambda : R\rightarrow I$ and $\mu:R\rightarrow I$ iff $\lambda(t-)=\mu(t-)$ and $\lambda(t+)=\mu(t+)$ for every $t\in R$ (where $\lambda(t-)=\inf\{\lambda(s) :s<t\}$ and $\lambda(t+)=\sup\{\lambda(s):s>t\}$).

We define a fuzzy topology on $[0,1]_f$ by taking as a subbasis $\{L_t,R_t:t\in R\}$;where we define $L_t(\lambda)=(\lambda(t-))^c$ and $R_t(\lambda)=\lambda(t+)$.This fuzzy topology is called the **usual topology for $[0,1]_f$.**

Note that our notation has not distinguished between the map $\lambda:R\rightarrow I$ and the equivalence class in $[0,1]_f$ containing λ.This causes no difficulty since we are only interested in the limit of the class at $t\in R$ which is exactly the same for each member of the class.

Since $L_a \cap L_b = L_{a\wedge b}$, $R_a \cap R_b = R_{a v b}$ ($a \wedge b = \inf\{a,b\}$, $a \vee b = \sup\{a,b\}$), $R_a = R_a \cap L_2$ and $L_b = L_b \cap R_{-1}$, it follows that $\{R_a \cap L_b : a,b \in R\}$ is a basis for the usual topology on $[0,1]_f$.

Moreover $L_0=0, L_2=1, R_1=0, R_{-1}=1$, so the collections

$$\mathcal{L}=\{L_t : t\in R\}= \{L_t : 0\le t\le 2\}, \mathcal{R}=\{R_t : t\in R\}=\{R_t : -1\le t\le 1\}$$

are bases for fuzzy topologies on $[0,1]_f$. In fact $\mathcal{L},\mathcal{R}$ are themselves fuzzy topologies on $[0,1]_f$. These fuzzy topologies are called **the left ray** and **the right ray** fuzzy topologies on the fuzzy unit interval $[0,1]_f$.

The following theorem is an extension of theorem 1.4.9 by Hutton (1975).

Theorem 2.5.12

A bfts (X,τ_1,τ_2) is P-normal iff for every τ_i-closed fuzzy set λ and τ_j-open fuzzy set μ such that $\lambda\subseteq\mu$, there exists a continuous function $f : (X,\tau_1,\tau_2)\to([0,1]_f,\mathcal{L},\mathcal{R})$ such that

$\lambda(x)\le f(x)(1-)\le f(x)(0+)\le\mu(x)$, for all $x\in X$.

Proof.

$\Leftarrow$ Since $\lambda(x)\le f(x)(1-)\le f(x)(0+)\le\mu(x)$ for all $x\in X$, then for any $t\in(0,1)$ we have $\lambda(x)\le f(x)(t+)\le f(x)(t-)\le\mu(x)$ because

$$\lambda(x)\le f(x)(1-)\le f(x)(t+)\le f(x)(t-)\le f(x)(0+)\le\mu(x) \quad \text{--------------(1)}$$

Since $f^{-1}(L_t^c)(x)= (L_t^c o f)(x)= L_t^c(f(x))= (f(x)(t-))$, $f^{-1}(R_t)(x)= (R_t o f)(x)= R_t(f(x))=f(x)(t+)$ and f is continuous, we have $f^{-1}(L_t^c)$ is

τ_i-closed and $f^{-1}(R_t)$ is τ_j-open.From inequality (1) we have $\lambda(x)\leq f^{-1}(R_t)(x)\leq f^{-1}(L_t^c)(x)\leq\mu(x)$ for all $t\in(0,1)$ and $x\in X$ i.e, $\lambda\subseteq f^{-1}(R_t)\subseteq f^{-1}(L_t^c)\subseteq\mu$. Therefore (X,τ_1,τ_2) is P-normal.

$\Rightarrow$ Let λ be a τ_i-closed fuzzy set and μ be a τ_j-open fuzzy set such that $\lambda\subseteq\mu$.Since (X,τ_1,τ_2) is P-normal, there exists a τ_j-open fuzzy set $v_{1/2}$ such that $\lambda\subseteq v_{1/2}\subseteq cl_i v_{1/2}\subseteq\mu$ .Since λ is τ_i-closed , $v_{1/2}$ is τ_j-open , $\lambda\subseteq v_{1/2}$ and $cl_i v_{1/2}$ is τ_i-closed ,μ is τ_j-open , $cl_i v_{1/2}\subseteq\mu$,there exist τ_j-open fuzzy sets $v_{1/4}$ and $v_{3/4}$ such that $\lambda\subseteq v_{1/4}\subseteq cl_i v_{1/4}\subseteq v_{1/2}\subseteq cl_i v_{1/2}\subseteq v_{3/4}\subseteq cl_i v_{3/4}\subseteq\mu$.Continuing this process, we construct a family $\{v_r: r\in(0,1)\}$of τ_j-open fuzzy sets such that $\lambda\subseteq v_r\subseteq\mu$ and $r<s$ implies $cl_i v_r\subseteq int_j v_s$.

Define $f : (X,\tau_1,\tau_2)\rightarrow([0,1]_f,L,R)$ by

$$f(x)(t)=\bigcup_{s\geq t} v_s(x) \quad \text{if} \quad 0<t<1 \text{ where s is dyadic rational.}$$

$$=1 \qquad \text{if} \quad t<0$$

$$=0 \qquad \text{if} \quad t>1$$

Since $\lambda\subseteq v_r\subseteq\mu$,we have $\lambda(x)\leq f(x)(1-)\leq f(x)(0+)\leq\mu(x)$,

To show that f is continuous we note that

$f^{-1}(L_t^c)(x)=(L_t^c o f)(x)=L_t^c(f(x))=f(x)(t-)=\inf\{f(x)(s):s<t\}=\bigcap_{s<t} v_s(x)=\bigcap_{s<t} cl_i v_s(x)$ is a τ_i-closed,i.e. ,f is upper semicontinuous

and

$f^{-1}(R_t)(x)=(R_t o f)(x)=R_t(f(x))=f(x)(t+)=\sup\{f(x)(s):s>t\}=\bigcup_{s>t} v_s(x)=\bigcup_{s>t} int_j v_s(x)$ is τ_j-open, i.e. , f is lower semicontinuous.Hence f is continuous.

Chapter III

CONNECTEDNESS

IN

BIFUZZY TOPOLOGICAL SPACES

The concept of fuzzy connectedness was studied by several authors (e.g ,Lowen (1976),Pu and Liu (1980),Fatteh and Bassan (1985),Zhao (1986),Saha (1987) and Ajmal and Kohli (1989)).Fatteh and Bassan (1985) defined connectedness only for a crisp fuzzy set of a fuzzy topological space while Ajmal and Kohli (1989) extended the notion of connectedness to an arbitrary fuzzy set.In this chaper we are going to extend the concept of connectedness to include bifuzzy topological spaces.We divide this chapter into four sections.In the first section we shall discuss bifuzzy connected topological spaces.In the second section we shall discuss the goodness of connectedness.In the third we show that connectedness is preserved under P-continuous functions .In the last section we deal with more results on bifuzzy connectedness.

§ 3.1 Bifuzzy Connectedness

The following definition is an extension of definition 1.5.9 which is due to Ajmal and Kohli (1989).

Definition 3.1.1

A fuzzy set σ in a bfts (X,τ_1,τ_2) is S-C_i-disconnected (P-C_i-disconnected) iff σ has S-C_i (P-C_i) disconnection(i=1,2,3,4).That is,there exist proper fuzzy sets $\lambda,\mu \in \tau_1 \cup \tau_2$ ($\lambda \in \tau_1, \mu \in \tau_2$) such that $\sigma \subseteq \lambda \cup \mu$ and

C_1: $\sigma \cap \lambda \cap \mu = 0, \lambda \not\subseteq 1-\sigma, \mu \not\subseteq 1-\sigma$.

C_2: $\lambda \cap \mu \subseteq 1-\sigma, \lambda \not\subseteq 1-\sigma, \mu \not\subseteq 1-\sigma$.

C_3: $\sigma \cap \lambda \cap \mu = 0, \sigma \cap \lambda \neq 0, \sigma \cap \mu \neq 0$.

C_4: $\lambda \cap \mu \subseteq 1-\sigma, \sigma \cap \lambda \neq 0, \sigma \cap \mu \neq 0$.

A fuzzy set σ in a bfts (X,τ_1,τ_2) is said to be S-C_i (P-C_i) connected if there does not exist an S-C_i(P-C_i) disconnection of σ in X (i=1,2,3,4).

Theorem 3.1.2

Let σ be a fuzzy set in a bfts (X,τ_1,τ_2),then we have the following:

i) If σ is S-C_1 (P-C_1) disconnected.Then σ is S-C_i (P-C_i) disconnected for i=2,3.

ii) If σ is S-C_2 (P-C_2) disconnected or S-C_3 (P-C_3) disconnected then σ is S-C_4 (P-C_4) disconnected.

iii) The converses of (i) and (ii) are not true in general.

Proof :

i) Let σ be S-C_1 (P-C_1) disconnected.Then there exist proper fuzzy sets $\mu,\nu \in \tau_1 \cup \tau_2$ ($\mu \in \tau_1, \nu \in \tau_2$) such that $\sigma \subseteq \lambda \cup \mu$,

$\sigma\cap\lambda\cap\mu=0$, $\lambda\not\subseteq 1-\sigma$ and $\mu\not\subseteq 1-\sigma$.Since $\sigma\cap\lambda\cap\mu=0$,then $\lambda\cap\mu\subseteq 1-\sigma$ and so σ is S-C_2 (P-C_2) disconnected.Similarly $\lambda\not\subseteq 1-\sigma$ implies $\sigma\cap\lambda\neq 0$,and $\mu\not\subseteq 1-\sigma$ implies $\sigma\cap\mu\neq 0$ .Therefore σ is S-C_3 (P-C_3) disconnected.

ii) Let σ be S-C_2 (P-C_2) disconnected.Then there exist proper fuzzy sets $\mu,\nu\in\tau_1\cup\tau_2$ ($\mu\in\tau_1,\nu\in\tau_2$) such that $\sigma\subseteq\lambda\cup\mu,\lambda\cap\mu\subseteq 1-\sigma,\lambda\not\subseteq 1-\sigma$ and $\mu\not\subseteq 1-\sigma$.The fact that $\lambda\not\subseteq 1-\sigma$ and $\mu\not\subseteq 1-\sigma$ implies $\sigma\cap\lambda\neq 0$ and $\sigma\cap\mu\neq 0$ shows that σ is S-C_4 (P-C_4) disconnected. Similarly if σ is S-C_3 (P-C_3) disconnected then σ is S-C_4 (P-C_4) disconnected.

The implications of the above parts on disconnectedness can be described by the following diagrams:

$$\begin{array}{ccc} S\text{-}C_1 & \Rightarrow & S\text{-}C_2 \\ \Downarrow & & \Downarrow \\ S\text{-}C_3 & \Rightarrow & S\text{-}C_4 \end{array} \qquad\qquad \begin{array}{ccc} P\text{-}C_1 & \Rightarrow & P\text{-}C_2 \\ \Downarrow & & \Downarrow \\ P\text{-}C_3 & \Rightarrow & P\text{-}C_4 \end{array}$$

iii) To show that all implications are not reversible we present the following examples.

Example 3.1.3

Let X=[0,1] and define fuzzy sets λ and μ as follows:

$$\lambda(x)=\begin{cases} 0 & \text{if } 1/3<x\leq 1 \\ 1/3 & \text{if } 0\leq x\leq 1/3 \end{cases}, \qquad \mu(x)=\begin{cases} 1/3 & \text{if } 1/3<x\leq 1 \\ 0 & \text{if } 0\leq x\leq 1/3 \end{cases}$$

Then $\tau_1=\{0,1,\lambda\}$ and $\tau_2=\{0,1,\mu\}$ are fuzzy topologies on X.

Consider the fuzzy set $\sigma(x)=1/4$,then σ is P-C_3-disconnected because $\sigma\subseteq\lambda\cup\mu=1/3,\sigma\cap\lambda\cap\mu=0,\sigma\cap\lambda\neq0$ $\sigma\cap\mu\neq0$ but σ is P-C_1-connected because $\sigma\subseteq\mu^c$ and $\sigma\subseteq\nu^c$.That is,P-C_3 does not imply P-C_1.Moreover σ is P-C_4-disconnected but P-C_2-connected.That is,P-C_4 does not imply P-C_2.

The same example shows that S-C_3 does not imply S-C_1 and S-C_4 does not imply S-C_2.

Example 3.1.4

Let X=[0,1].Define fuzzy sets λ and μ as follows:

$$\lambda(x)=\begin{cases} 3/4 & \text{if } 1/3<x\leq1 \\ 1 & \text{if } 0\leq x\leq1/3 \end{cases}, \qquad \mu(x)=\begin{cases} 1 & \text{if } 1/3<x\leq1 \\ 3/4 & \text{if } 0\leq x\leq1/3 \end{cases}$$

Then $\tau_1=\{0,1,\lambda\}$ and $\tau_2=\{0,1,\mu\}$ are fuzzy topologies on X.Consider the fuzzy set $\sigma(x)=1/6$.Then σ is P-C_2-disconnected.Hence P-C_4-disconnected;because $1/6=\sigma\subseteq\lambda\cup\mu=1,3/4=\lambda\cap\mu\subseteq1-\sigma=5/6,\lambda\not\subseteq\sigma^c$ $\mu\not\subseteq\sigma^c$. It is clear that σ is P-C_3-connected and P-C_1-connected because $\sigma\cap\lambda\cap\mu\neq0$.Hence P-$C_4$ does not imply P-C_3 and P-C_2 does not imply P-C_1.

The same example shows that S-C_4 does not imply S-C_3 and S-C_2 does not imply S-C_1.

Proposition 3.1.5

For $\sigma=1$,the above types of P-definitions (S-definitions) of disconnections are equivalent.

Proof :

Let $\sigma=1$ be S-C_4(P-C_4) disconnected.Then there exist proper fuzzy sets $\lambda,\mu\in\tau_1\cup\tau_2$ ($\mu\in\tau_1,\nu\in\tau_2$) such that $1\subseteq\lambda\cup\mu,\lambda\cap\mu\subseteq 0$, $1\cap\lambda\neq 0$ and $1\cap\mu\neq 0$.So we have $\lambda\cup\mu=1$, $1\cap\lambda\cap\mu=0,\lambda\not\subseteq 0$ and $\mu\not\subseteq 0$. Hence $\sigma=1$ is S-C_1(P-C_1) disconnected.

Definition 3.1.6

If σ is S-C_i(P-C_i) disconnected for all i=1,2,3,4 ,then σ will be called S-C (P-C) disconnected.In particular;proposition (3.1.5) shows that if 1 is S-C_i(P-C_i) disconnected for some $i\in\{1,2,3,4\}$, then (X,τ_1,τ_2) will be called **S-C (P-C) disconnected.**

The following result clarifies the relation between P-C_i and S-C_i disconnected fuzzy sets (i=1,2,3,4).

Theorem 3.1.7

Let (X,τ_1,τ_2) be a bfts and σ be a fuzzy set in X.

(i) If σ is P-C_i-disconnected,then it is S-C_i-disconnected for $i\in\{1,2,3,4\}$.

(ii) The converse of (i) is not true in general.

Proof :

(i) If σ is P-C_i-disconnected then there exist proper fuzzy sets $\lambda \in \tau_1$ and $\mu \in \tau_2$ satisfying the proper P-C_i disconnection. This implies that $\lambda, \mu \in \tau_1 \cup \tau_2$ and so σ is S-C_i disconnected.

(ii) To show that S-$C_i \Rightarrow$ P-C_i $(i=2,4)$ does not hold in general we present the following example:

Let $X=I$,$\tau_1=\{0,1,\lambda,\mu,1/2\}$ and $\tau_2=\{0,1\}$ where

$$\lambda(x)=\begin{cases} 1/2 & \text{if } 0 \leq x \leq 1/2 \\ 1 & \text{if } 1/2 < x \leq 1 \end{cases}, \quad \mu(x)=\begin{cases} 1 & \text{if } 0 \leq x \leq 1/2 \\ 1/2 & \text{if } 1/2 < x \leq 1 \end{cases}$$

Consider the bfts (X,τ_1,τ_2) and the fuzzy set $\sigma=1/4$.Then it is clear that σ is S-C_2-disconnected and so S-C_4-disconnected because $\sigma \subseteq \lambda \cup \mu$, $1/2 = \lambda \cap \mu \subseteq \sigma^c = 3/4$, $\lambda \not\subseteq \sigma^c$ and $\mu \not\subseteq \sigma^c$ but σ is P-C_i-connected for all $(i=1,2,3,4)$ because τ_2 has no proper fuzzy set.

The following example shows that S-$C_i \Rightarrow$P-C_i $(i=1,3)$ does not hold in general.

Let $X=I$,$\tau_1=\{0,1,\lambda,\mu,1/2\}$ and $\tau_2=\{0,1\}$ where

$$\lambda(x)=\begin{cases} 1 & \text{if } 0 \leq x \leq 1/2 \\ 0 & \text{if } 1/2 < x \leq 1 \end{cases}, \quad \mu(x)=\begin{cases} 0 & \text{if } 0 \leq x \leq 1/2 \\ 1 & \text{if } 1/2 < x \leq 1 \end{cases}$$

Consider the bfts (X,τ_1,τ_2) and the fuzzy set $\sigma=1/4$.Then it is clear that σ is S-C_1-disconnected and so S-C_3-disconnected because $\sigma\subseteq\lambda\cup\mu,\sigma\cap\lambda\cap\mu=0$,$\lambda\not\subseteq\sigma^c$ and $\mu\not\subseteq\sigma^c$ but σ is P-C_i-connected for all (i=1,2,3,4) because τ_2 has no proper fuzzy set .

Definition 3.1.8

A degenerate ordinary set is a set which contains at most one element.A degenerate fuzzy set is a fuzzy set whose support is a degenerate set.

The following result shows that S-C_i-connectedness for degenerate fuzzy sets agrees with the connectedness property for degenerate sets in ordinary bitopological spaces for i=1,2,3.However the same result shows that S-C_4-connectedness diverges from ordinary connectedness for these sets.

Theorem 3.1.9

Let (X,τ_1,τ_2) be a bfts .Then we have the following :

i) All fuzzy crisp points are S-C-connected.

ii) The zero fuzzy set is S-C-connected.

iii) All fuzzy points are S-C_i-connected,i=1,2,3.

vi) Let p,q be fuzzy points in X such that $x_p=x_q$ and $q(x_q)\geq p(x_p)$.If p is S-C_4-connected ,then q is also S-C_4-connected.

v) There exists a bfts (X,τ_1,τ_2) and a fuzzy point p in X which is S-C_4-disconnected.

__Proof :__

i) Let p_c be a fuzzy crisp point with support x_p.Suppose that p_c is S-C_4-disconnected,then there exist proper fuzzy sets $\mu,\nu\in\tau_1\cup\tau_2$ such that $p_c\subseteq\mu\cup\nu,\mu\cap\nu\subseteq p_c{}^c,\mu\cap p_c\neq 0$ and $\nu\cap p_c\neq 0$.Now

$\mu\cap\nu\subseteq p_c{}^c \Rightarrow (\mu\cap\nu)(x_p)\leq 1-1=0,$ (1)

$\mu\cap p_c\neq 0 \Rightarrow \mu(x_p)>0$ (2)

and $\nu\cap p_c\neq 0 \Rightarrow \nu(x_p)>0$ (3)

From (2) and (3) we have $(\mu\cap\nu)(x_p)>0$ which contradicts (1).Hence p_c is S-C_4-connected and so S-C-connected.

ii) Obvious.

iii) Let p be a fuzzy point with support x_p.Suppose that p is S-C_2-disconnected,then there exist proper fuzzy sets $\mu,\nu\in\tau_1\cup\tau_2$ such that $p\subseteq\mu\cup\nu,\mu\cap\nu\subseteq p^c,\mu\not\subseteq p^c$ and $\nu\not\subseteq p^c$.Now

$\mu\cap\nu\subseteq p^c \Rightarrow (\mu\cap\nu)(x_p)\leq 1-p(x_p),$ (1)

$\mu\not\subseteq p^c \Rightarrow \mu(x_p)>1-p(x_p)$ (2)

and $\nu\not\subseteq p^c \Rightarrow \nu(x_p)>1-p(x_p)$ (3)

From (2) and (3) we have $(\mu\cap\nu)(x_p)>1-p(x_p)$ which contradicts (1).Hence p is S-C_2-connected and so S-C_1-connected.

Suppose that p is S-C_3-disconnected,then there exist proper fuzzy sets $\mu,\nu\in\tau_1\cup\tau_2$ such that $p\subseteq\mu\cup\nu,\mu\cap\nu\cap p=0,\mu\cap p\neq 0$ and $\nu\cap p\neq 0$. Now

$\mu\cap\nu\cap p=0 \Rightarrow (\mu\cap\nu)(x_p)=0,$ (1)

$\mu \cap p \neq 0 \Rightarrow \mu(x_p)>0$ (2)

and $\nu \cap p \neq 0 \Rightarrow \nu(x_p)>0$ (3)

From (2) and (3) we have $(\mu \cap \nu)(x_p)>0$ which contradicts (1).Hence p is S-C_3-connected.

vi) Suppose that q is S-C_4-disconnected,then there exist proper fuzzy sets $\mu,\nu \in \tau_1 \cup \tau_2$ such that $q \subseteq \mu \cup \nu, \mu \cap \nu \subseteq q^c, \mu \cap q \neq 0$ and $\nu \cap q \neq 0$.Since $q(x_q) \geq p(x_p)$,then $1-q(x_q) \leq 1-p(x_p)$ and so we have $p \subseteq \mu \cup \nu$ and $\mu \cap \nu \subseteq p^c$.Moreover,$\mu \cap q \neq 0 \Rightarrow \mu(x_q)>0$ and $\nu \cap q \neq 0 \Rightarrow \nu(x_q)>0$.Since $p(x_p)>0$ and $x_p=x_q$,then $\mu \cap p \neq 0$ and $\nu \cap p \neq 0$.Hence p is S-C_4-disconnected which is a contradiction.Therefore q is S-C_4-connected.

v) Let X=I and p be a fuzzy point with support 1/2 and value 1/4.Define fuzzy sets λ and μ as follows:

$$\lambda(x)=\begin{cases} 0 & \text{if } 1/2<x\leq 1 \\ 1/3 & \text{if } 0\leq x\leq 1/2 \end{cases}, \qquad \mu(x)=\begin{cases} 1/4 & \text{if } 1/2\leq x\leq 1 \\ 0 & \text{if } 0\leq x<1/2 \end{cases}$$

Then $\tau_1=\{0,1,\lambda\}$ and $\tau_2=\{0,1,\mu\}$ are fuzzy topologies on X.It is clear that $p \subseteq \mu \cup \lambda$,$\mu \cap \lambda \subseteq p^c, p \cap \mu \neq 0$ and $p \cap \lambda \neq 0$.Hence p is S-C_4-disconnected (P-C_4-disconnected).

We observe from part (vi) of the above theorem that if we have a bfts (X,τ_1,τ_2),then with each $x \in X$,there exists a unique real number $r_x \geq 0$ such that all fuzzy points with support x and level greater than r_x are S-C_4-connected while all fuzzy points with support x and level less than r_x are S-C4-disconnected.Thus we have defined a function $d:X\to I$,$d(x)=r_x$.This fuzzy set d could be an important tool measuring the S-C4-connectedness of a bfts.

Definition 3.1.10

A bfts $(X,\tau 1,\tau 2)$ is S-disconnected iff there exist non zero fuzzy sets $\lambda,\mu\in\tau 1\cup\tau 2$ such that $\lambda+\mu=1$ and $\lambda\cap\mu=0$.A bfts $(X,\tau 1,\tau 2)$ is called S-connected if it is not S-disconnected.

Definition 3.1.11

A bfts $(X,\tau 1,\tau 2)$ is S_w-disconnected iff there exist non zero fuzzy sets $\lambda,\mu\in\tau 1\cup\tau 2$ such that $\lambda+\mu=1$.A bfts $(X,\tau 1,\tau 2)$ is called S_w-connected if it is not S_w-disconnected.

Definition 3.1.12

A bfts $(X,\tau 1,\tau 2)$ is P-disconnected iff there exist non zero fuzzy sets $\lambda\in\tau 1$ and $\mu\in\tau 2$ such that $\lambda+\mu=1$ and $\lambda\cap\mu=0$.A bfts $(X,\tau 1,\tau 2)$ is called P-connected if it is not P-disconnected.

Definition 3.1.13

A bfts $(X,\tau 1,\tau 2)$ is P_w-disconnected iff there exist non zero fuzzy sets $\lambda\in\tau 1$ and $\mu\in\tau 2$ such that $\lambda+\mu=1$ .A bfts $(X,\tau 1,\tau 2)$ is called P_w-connected if it is not P_w-disconnected.

The implications of the above types of bifuzzy disconnectedness can be described by the following diagram.

$$\begin{array}{ccc} P & \Rightarrow & P_w \\ \Downarrow & & \Downarrow \\ S & \Rightarrow & S_w \end{array}$$

To show that all implications are not reversible we present the following examples.

Example 3.1.14

Let $X=[0,1]$, $\tau1=\{0,1,\lambda\}$ and $\tau2=\{0,1,\mu\}$; where λ and μ are defined as follows:

$$\lambda(x)=\begin{cases} 2/3 & \text{if } 1/2\leq x<1 \\ 1 & \text{if } 0\leq x<1/2 \end{cases}, \qquad \mu(x)=\begin{cases} 1/3 & \text{if } 1/2\leq x\leq 1 \\ 0 & \text{if } 0\leq x<1/2 \end{cases}$$

It is clear that $\lambda+\mu=1$ and $\lambda\cap\mu\neq 0$. Hence $(X,\tau1,\tau2)$ is P_W-disconnected and S_W-disconnected but it is neither P-disconnected nor S-disconnected.

Example 3.1.15

Let $X=[0,1]$, $\tau1=\{0,1,\lambda,\mu\}$ and $\tau2=\{0,1\}$; where λ and μ are defined as follows: $\lambda=\chi_{[0.5,1]}$, $\mu=\chi_{[0,0.5)}$, i.e. ,

$$\lambda(x)=\begin{cases} 1 & \text{if } 1/2\leq x\leq 1 \\ 0 & \text{if } 0\leq x<1/2 \end{cases}, \qquad \mu(x)=\begin{cases} 0 & \text{if } 1/2\leq x\leq 1 \\ 1 & \text{if } 0\leq x<1/2 \end{cases}$$

It is clear that $\lambda+\mu=1$ and $\lambda\cap\mu=0$.Hence (X,τ_1,τ_2) is S-disconnected and S_w-disconnected but it is neither P-disconnected nor P_w-disconnected.

The following theorem shows that the category of S-C-disconnected bifuzzy topological spaces and the category of S-disconnected bifuzzy topological spaces are indeed identical.

Theorem 3.1.16

Let (X,τ_1,τ_2) be a bfts .Then the following are equivalent:

(i) (X,τ_1,τ_2) is S-disconnected.

(ii) (X,τ_1,τ_2) is S-C-disconnected.

Proof :

(i)$\Rightarrow$(ii) Let (X,τ_1,τ_2) be an S-disconnected bfts.Then there exist proper fuzzy sets $\lambda,\mu\in\tau_1\cup\tau_2$ such that $\lambda\cap\mu=0$ and $\lambda+\mu=1$.We note that $1\subseteq\lambda\cup\mu, 1\cap\lambda\cap\mu=0, \lambda\not\subseteq 0$ and $\mu\not\subseteq 0$.Therefore 1 is S-C_1-disconnected and so by proposition (3.1.5) 1 is S-C_i-disconnected for all $i\in\{1,2,3,4\}$.Hence (X,τ_1,τ_2) is S-C-disconnected.

(ii)$\Rightarrow$(i) Let (X,τ_1,τ_2) be an S-C-disconnected bfts.Then 1 is S-C_i-disconnected for all $i\in\{1,2,3,4\}$ and so there exist $\lambda,\mu\in\tau_1\cup\tau_2$ such that $1\subseteq\lambda\cup\mu, 1\cap\lambda\cap\mu=0, \lambda\not\subseteq 0$ and $\mu\not\subseteq 0$.It is clear that $\lambda\cap\mu=0$ and $\lambda+\mu=1$.Hence (X,τ_1,τ_2) is S-disconnected.

The following theorem shows that the category of P-C-disconnected bifuzzy topological spaces and the category of P-disconnected bifuzzy topological spaces are indeed identical.

Theorem 3.1.17

Let (X,τ_1,τ_2) be a bfts .Then the following are equivalent:

(i) (X,τ_1,τ_2) is P-disconnected.

(ii) (X,τ_1,τ_2) is P-C-disconnected.

Proof :

Similar to the proof of theorem 3.1.16.

§ 3.2 Goodness of connectedness.

Our next results show that the concept of S-connectedness and P-connectedness are in fact good extensions while S_w-connectedness and P_w-connectedness are not.We remind the reader of definition 1.1.15 that a bts (X,T_1,T_2) is called S-connected if there do not exist non empty disjoint sets $U,V\in T_1\cup T_2$ such that $U\cup V=X$.

Theorem 3.2.1

Let (X,T_1,T_2) be a bts .Then the following are equivalent:

(i) (X,T_1,T_2) is S-connected.

(ii) $(X,\omega(T_1),\omega(T_2))$ is S-connected.

Proof :

(i)$\Rightarrow$(ii) Let (X,T_1,T_2) be an S-connected bts .To prove that $(X,\omega(T_1),\omega(T_2))$ is S-connected,suppose not;then there exist non zero fuzzy sets $\lambda,\mu\in\omega(T_1)\cup\omega(T_2)$ such that $\lambda\cap\mu=0$ and $\lambda+\mu=1$.Let $A=\lambda^{-1}(0,1]$, $B=\mu^{-1}(0,1]$.Then $A,B\in T_1\cup T_2$ are non empty subsets of X because $\lambda \neq 0$ and $\mu \neq 0$. Now $A \cap B = \phi$ (for if not,then there exists $x \in X$ such that $x \in \lambda^{-1}(0,1]$ and $x \in \mu^{-1}(0,1]$ which implies that $\lambda(x) \in (0,1]$ and $\mu(x) \in (0,1]$; i.e.,$(\lambda \cap \mu)(x) \neq 0$ which contradicts $\lambda \cap \mu = 0$).Also $A \cup B = X$(for if not,then there exists $x \in X$ such that $x \notin A$ and $x \notin B$ and this implies that $\lambda(x)=0$ and $\mu(x)=0$;i.e.,$(\lambda+\mu)(x)=0$ which contradicts $\lambda+\mu=1$). Therefore (X,T_1,T_2) is S-disconnected which is a contradiction. Hence $(X,\omega(T_1),\omega(T_2))$ is S-connected.

(ii)$\Rightarrow$(i) Let $(X,\omega(T_1),\omega(T_2))$ be an S-connected bfts.To prove that (X,T_1,T_2) is S-connected,suppose not; then there exist non empty disjoint subsets $A,B \in T_1 \cup T_2$ such that $A \cup B = X$.Now $\chi_A,\chi_B \in \omega(T_1) \cup \omega(T_2)$.Moreover $\chi_A \neq 0$ and $\chi_B \neq 0$ because $A \neq \phi$ and $B \neq \phi$. We also have $\chi_A \cap \chi_B = \chi_{A \cap B} = \chi_\phi = 0$ and $\chi_A + \chi_B = \chi_{A \cup B} = \chi_X = 1$.Therefore $(X,\omega(T_1),\omega(T_2))$ is S-disconnected which is a contradiction.Hence (X,T_1,T_2) is S-connected.

Theorem 3.2.2

Let (X,T_1,T_2) be a bts .Then the following are equivalent;

(i) (X,T_1,T_2) is P-connected.

(ii) $(X,\omega(T_1),\omega(T_2))$ is P-connected.

Proof :

(i)$\Rightarrow$(ii) Similar to the proof of theorem 3.2.1.

The following theorem shows that definition (3.1.11) is not a good extension of S-connectedness for bitopological spaces.

Theorem 3.2.3

(i) A bts (X,T_1,T_2) is S-connected if the bfts $(X,\omega(T_1),\omega(T_2))$ is S_w-connected

(ii) The converse of (i) is not true in general.

Proof :

(i) Since $(X,\omega(T_1),\omega(T_2))$ is S_w-connected bfts,then $(X,\omega(T_1),\omega(T_2))$ is S-connected and so by theorem 3.2.1 the bts (X,T_1,T_2) is S-connected.

(ii) Let $X=I,T_1=T_{l.r}$ and $T_2=T_{r.r}$.Then (X,T_1,T_2) is S-connected because if $U\in T_1$ and $V\in T_2$ such that $U\cap V=\phi$,then $U\cup V\neq X$.Now define $\lambda:I\to I$ as follows:

$$\lambda(x)=\begin{cases} 2x & \text{if } 0\leq x\leq 1/2 \\ 1 & \text{if } 1/2<x\leq 1 \end{cases}$$

Then $\lambda\in\omega(T_2),\lambda^c\in\omega(T_1),\lambda\neq 0$ and $\lambda+\lambda^c=1$.Hence $(X,\omega(T_1),\omega(T_2))$ is S_w-disconnected.

The following theorem shows that definition (3.1.13) is not a good extension of P-connectedness for bitopological spaces.

Theorem 3.2.4

(i) A bts (X,T_1,T_2) is P-connected if the bfts $(X,\omega(T_1),\omega(T_2))$ is P_w-connected

(ii) The converse of (i) is not true in general.

Proof :

(i) Since $(X,\omega(T_1),\omega(T_2))$ is P_w-connected bfts,then $(X,\omega(T_1),\omega(T_2))$ is P-connected and so by theorem 3.2.2 the bts (X,T_1,T_2) is P-connected.

(ii) The example which was presented in Theorem (3.2.3)(ii) serves our purpose here.

§ 3.3 Connectedness and P-continuity.

In this section we show that images and inverse images of different types of connectedness are preserved under P-continuous functions.We start with the following definition which is an extension of definition 1.1.20 to bifuzzy topological spaces.

Theorem 3.3.1

If $f: (X,\tau_1,\tau_2)\rightarrow(Y,\gamma_1,\gamma_2)$ is a P-continuous and λ is an S-C_1-connected fuzzy set in X,then $f(\lambda)$ is an S-C_1-connected fuzzy set in Y.

Proof :

Suppose that $f(\lambda)$ is not S-C_1-connected ,then there exist proper fuzzy sets $\mu,\nu \in \gamma_1 \cup \gamma_2$ such that $f(\lambda)\subseteq\mu\cup\nu$, $(\mu\cap\nu)\cap f(\lambda)=0$,$f(\lambda)\not\subseteq\mu^c$ and $f(\lambda)\not\subseteq\nu^c$.Using theorem (1.2.15) ,since $\lambda\subseteq f^{-1}(f(\lambda))$ and $f^{-1}(f(\lambda)) \subseteq f^{-1}(\mu\cup\nu) = f^{-1}(\mu) \cup f^{-1}(\nu)$, then $\lambda\subseteq f^{-1}(\mu)\cup f^{-1}(\nu) \in \tau_1 \cup \tau_2$.Also $f^{-1}(\mu) \cap f^{-1}(\nu) \cap \lambda = f^{-1}(0)=0$. Since $f(\lambda)\not\subseteq \mu^c$ and $f(\lambda)\not\subseteq\nu^c$,so there exist y_1 and y_2 such that

$\mu(y_1)>1-f(\lambda)(y_1)$ ....(1)

and

$\nu(y_2)>1-f(\lambda)(y_2)$.............................(2)

From (1) and (2) we conclude $f(\lambda)(y_1)>0$ and $f(\lambda)(y_2)>0$ which gives that $f^{-1}(\{y_1\})$ and $f^{-1}(\{y_2\})$ are non-empty subsets of X.By inverse and direct image of f we have $f^{-1}(\mu)(x)=\mu(y_1)$ for every $x\in f^{-1}(\{y_1\})$ and $f(\lambda)(y_1)=\sup\{\lambda(x):x\in f^{-1}(\{y_1\})\}$.We claim that $f^{-1}(\mu)\not\subseteq\lambda^c$ and $f^{-1}(\nu)\not\subseteq\lambda^c$.Suppose $f^{-1}(\mu)\subseteq\lambda^c$,then $f^{-1}(\mu)(x)\leq 1-\lambda(x)$ for every $x\in f^{-1}(\{y_1\})$; i.e. ,$f(x)\in\{y_1\}$ and so we have $\mu(f(x))\leq 1-\lambda(x)$ which implies $\lambda(x)\leq 1-\mu(y_1)$;i.e. $\sup\{\lambda(x): x\in f^{-1}(\{y_1\})\}\leq 1-\mu(y_1)$ and so $f(\lambda)(y_1)\leq 1-\mu(y_1)$ which contradicts (1).Similarly $f^{-1}(\nu)\subseteq\lambda^c$ contradicts (2).Hence $f^{-1}(\mu)\not\subseteq\lambda^c$ and $f^{-1}(\nu)\not\subseteq\lambda^c$ and so λ is S-C_1-disconnected which is a contradiction.Therefore $f(\lambda)$ is S-C_1-connected.

Theorem 3.3.2

If f: $(X,\tau_1,\tau_2)\rightarrow(Y,\gamma_1,\gamma_2)$ is a P-continuous and λ is an S-C_2-connected fuzzy set in X,then $f(\lambda)$ is S-C_2-connected.

Proof :

Similar to the proof of Theorem 3.3.1.

Theorem 3.3.3

If f: $(X,\tau_1,\tau_2)\rightarrow(Y,\gamma_1,\gamma_2)$ is a P-continuous function and λ is an S-C_3-connected fuzzy set in X,then $f(\lambda)$ is S-C_3-connected.

Proof :

Suppose that $f(\lambda)$ is S-C_3-disconnected ,then there exist proper fuzzy sets $\mu,\nu \in \gamma_1 \cup \gamma_2$ such that $f(\lambda) \subseteq \mu \cup \nu$,$(\mu \cap \nu) \cap f(\lambda)=0$,$f(\lambda) \cap \mu \neq 0$ and $f(\lambda) \cap \nu \neq 0$.Since $\lambda \subseteq f^{-1}(f(\lambda))$ and $f^{-1}(f(\lambda)) \subseteq f^{-1}(\mu \cup \nu)=f^{-1}(\mu) \cup f^{-1}(\nu)$ then $\lambda \subseteq f^{-1}(\mu) \cup f^{-1}(\nu) \in \tau_1 \cup \tau_2$.Also $f^{-1}(\mu) \cap f^{-1}(\nu) \cap \lambda=f^{-1}(0)=0$.Since $f(\lambda) \cap \mu \neq 0$,so there exists $y_0 \in Y$ such that $f(\lambda)(y_0) \cap \mu(y_0) \neq 0$ which implies that $f(\lambda)(y_0)>0$ where $f(\lambda)(y_0)=\sup\{\lambda(x):x \in f^{-1}(\{y_o\})\}$ and gives that $f^{-1}(\{y_o\}) \neq \phi$.So there exists $x_0 \in X$ such that $x_0 \in f^{-1}(\{y_o\})$; i.e.,$f(x_0)=y_0$.Since $f(\lambda)(y_0)>0$,then there exists $x_1 \in X$ such that $x_1 \in f^{-1}(\{y_o\})$ and so $0<\lambda(x_1) \leq f(\lambda)(y_0)$.Now $f^{-1}(\mu)(x_1)=\mu(f(x_1))=\mu(y_0) \neq 0$ and $\lambda(x_1) \neq 0$.Hence $f^{-1}(\mu) \cap \lambda \neq 0$.Similarly we can show that $f^{-1}(\nu) \cap \lambda \neq 0$. This shows that λ is S-C_3-disconnected which is a contradiction.Hence $f(\lambda)$ is an S-C_3-connected fuzzy set in Y.

Theorem 3.3.4

If f: $(X,\tau_1,\tau_2) \to (Y,\gamma_1,\gamma_2)$ is a P-continuous function and λ is an S-C_4-connected fuzzy set in X,then $f(\lambda)$ is S-C_4-connected.

Proof :

Similar to the proof of Theorem 3.3.3.

Theorem 3.3.5

A P-continuous image of a bifuzzy S-connected space is S-connected.

Proof:

Let $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ be a P-continuous onto function and suppose on the contrary that Y is not S-connected.Then there exist non-zero fuzzy sets $\lambda,\mu\in\sigma_1\cup\sigma_2$ such that $\lambda+\mu=1$ and $\lambda\cap\mu=0$.Since f is P-continuous then $f^{-1}(\lambda),f^{-1}(\mu)\in\tau_1\cup\tau_2$.We claim that $f^{-1}(\lambda)+f^{-1}(\mu)=1$. To prove our claim ,suppose not .Then there exists $x\in X$ such that $f^{-1}(\lambda)(x)+f^{-1}(\mu)(x)\neq 1$ which implies that $\lambda(f(x))+\mu(f(x))\neq 1$ which contradicts $\lambda+\mu=1$.Hence $f^{-1}(\lambda)+f^{-1}(\mu)=1$.We claim that $f^{-1}(\lambda)\cap f^{-1}(\mu)=0$.To prove our claim,suppose not .Then there exists $x\in X$ such that $f^{-1}(\lambda)(x)\cap f^{-1}(\mu)(x)>0$ which implies that $\lambda(f(x))\cap\mu(f(x))>0$ which contradicts $\lambda\cap\mu=0$.Hence (X,τ_1,τ_2) is S-disconnected which is again a contradiction.Hence (Y,σ_1,σ_2) is S-connected.

Theorem 3.3.6

A P-continuous image of a P-connected bifuzzy topological space is P-connected.

Proof :

Similar to the proof of Theorem 3.3.5.

Definition 3.3.7

Consider a bfts (X,τ_1,τ_2)

i) If every continuous function from (X,τ_1,τ_2) into itself has a fixed point we say that X has f.p.p.

ii) If every P-continuous function from (X,τ_1,τ_2) into itself has a fixed point we say that X has P-f.p.p.

Theorem 1.1.24 asserts that a bts (X,T_1,T_2) which has the P-f.p.p must be S-connected and a P-T_0-space.The following example shows that Theorem 1.1.24 does not hold in bft spaces.

Example 3.3.8

There exists a bfts (X,τ_1,τ_2) which has P-f.p.p. and yet is neither S_w-connected nor a P-T_{0w}-space.

Let $X=\{0,1\}$ and $\tau_1=\tau_2=\{0,1,\lambda,\lambda^c\}$ where $\lambda(0)=1/2$ and $\lambda(1)=1/4$. Then it is clear that (X,τ_1,τ_2) is not S_w-connected.To show that $f:X\rightarrow X$ has the fixed point we note that if f is the identity or the constant map then it is P-continuous.It is left to show that f which is defined by $f(0)=1$ and $f(1)=0$ is not P-continuous because if f is P-continuous then it has no fixed point.Since $(f^{-1}(\lambda))(0)=\lambda(f(0))=\lambda(1)=1/4$ then $f^{-1}(\lambda)\notin \tau_1\cup\tau_2$ and so f is not P-continuous.Hence every P-continuous map from X into X has a fixed point.Moreover for the fuzzy points p,q in X given by $p(x_p)=0.8$ and $q(x_q)=0.8$ there does not exist $\mu\in\tau_1\cup\tau_2$ such that $p\in\mu\subseteq q^c$ or $q\in\mu\subseteq p^c$ .Hence (X,τ_1,τ_2) is neither S_w-connected nor a P-T_{0w}-space.

§ 3.4 More results on connectedness.

We start this section by extending some of the results obtained by Fatteh and Bassan (1985) to bifuzzy topological spaces.

Definition 3.4.1

Two non-zero fuzzy sets λ,μ in a bfts (X,τ_1,τ_2) are called P-separated iff $\lambda(x)+(cl_i\mu)(x)\leq 1$ and $\mu(x)+(cl_j\lambda)(x)\leq 1$ for all $x\in X$ and for some $i\neq j$. If in addition $\lambda+\mu=1$ then $\{\lambda,\mu\}$ is called a P-separation for X.

Lemma 3.4.2

If $\{\lambda,\mu\}$ is a P-separation of a bfts (X,τ_1,τ_2),then $\lambda\in\tau_i$ and $\mu\in\tau_j$ (for some $i\neq j;i,j=1,2$).

Proof :

Since $\{\lambda,\mu\}$ is a P-separation for (X,τ_1,τ_2),then $\lambda+\mu=1$, $(cl_j\lambda)(x)+\mu(x)\leq 1$ and $\lambda(x)+(cl_i\mu)(x)\leq 1$ for all $x\in X$ and for some $i\neq j$; $i,j=1,2$. Since $\mu\subseteq cl_i\mu,\lambda\subseteq cl_j\lambda$ and $\lambda+\mu=1$,then $cl_j\lambda+\mu=1$ and $\lambda+cl_i\mu=1$.So $\mu=1-cl_j\lambda$ and $\lambda=1-cl_i\mu$.Hence μ is τ_j-open and λ is τ_i-open;i.e.,$\mu\in\tau_j$ and $\lambda\in\tau_i$.

Theorem 3.4.3

A bfts (X,τ_1,τ_2) is P_W-disconnected iff it has a P-separation.

Proof :

$\Rightarrow$ Let (X,τ_1,τ_2) be a P_W-disconnected bfts.Then there exist non zero fuzzy sets $\lambda\in\tau_i$ and $\mu\in\tau_j$ (for some $i\neq j;i,j=1,2$) such that $\lambda+\mu=1$.Since $\lambda\subseteq\mu^c$ and $\mu\subseteq\lambda^c$ then $cl_j\lambda\subseteq\mu^c$ and $cl_i\mu\subseteq\lambda^c$ .Now $\lambda\subseteq cl_j\lambda\subseteq\mu^c$ and $\mu\subseteq cl_i\mu\subseteq\lambda^c$ and so we have $\lambda+\mu\subseteq\lambda^c+\mu^c=1$.Hence $cl_j\lambda+cl_i\mu\subseteq 1$ which gives that $cl_j\lambda+\mu\subseteq 1$ and $\lambda+cl_i\mu\subseteq 1$.That is $\{\lambda,\mu\}$ forms a P-separation for X.

$\Leftarrow$ Let $\{\lambda,\mu\}$ be a P-separation for X .Then by Lemma 3.4.2 ,$\mu\in\tau_j$ and $\lambda\in\tau_i$ where $\lambda+\mu=1$.Hence (X,τ_1,τ_2) is P_W-disconnected.

Definition 3.4.4

A subspace $(K,\tau_1/k,\tau_2/k)$ of a bfts (X,τ_1,τ_2) is S-connected, S_W-connected,P-connected and P_W-connected if the crisp fuzzy set χ_k is S-connected,S_W-connected,P-connected and P_W-connected respectively.

Theorem 3.4.5

Let $\{\lambda,\mu\}$ be a P-separation of a bfts (X,τ_1,τ_2).If K is a P_W-connected subset of X.Then either $\lambda/_k=1$ or $\mu/_k=1$.

Proof :

Suppose $\lambda/_k\neq 1$ and $\mu/_k\neq 1$.Then $\lambda+\mu=1$ implies that $\lambda/_k+\mu/_k=1$ which gives that $\lambda/_k\neq 0$ and $\mu/_k\neq 0$.Now by Lemma 3.4.2, $\lambda/_k\in \tau_i/k$ and $\mu/_k\in \tau_j/k$ (for some $i\neq j;i,j=1,2$) are non-zero fuzzy sets in $(K,\tau_1/k,\tau_2/k)$ such that $\lambda/_k+\mu/_k=1$.Hence χ_K is P_W-disconnected which is a contradiction.Hence $\lambda/_k=1$ or $\mu/_k=1$.

Theorem 3.4.6

Let $\{A_\alpha:\alpha\in\Delta\}$ be a family of P_W-connected subsets of a bfts (X,τ_1,τ_2) such that for each α ,β in Δ , $\alpha\neq\beta$;A_α,A_β are not separated from each other .Then $\cup A_\alpha$ is a P_W-connected subset of X.

Proof :

Let $Y=\cup\{A_\alpha:\alpha\in\Delta\}$.We shall prove that $(Y,\tau_1/_Y,\tau_2/Y)$ is fuzzy P_W-connected.To prove this suppose not ;i.e.,Y is P_W-disconnected. Then there exist non-zero fuzzy sets in Y say $\lambda/_Y\in\tau_i/Y$,$\mu/_Y\in\tau_j/Y$(for some $i\neq j;i,j=1,2$) such that $\lambda/_Y+\mu/_Y=1$.Since λ/A_α is a τ_i/A_α-open fuzzy set and μ/A_α is a τ_j/A_α-open fuzzy set then ($\lambda/A_\alpha=0$ or $\lambda/A_\alpha=1$) and ($\mu/A_\alpha=0$ or $\mu/A_\alpha=1$) because A_α is P_W-connected.If $\lambda/A_\alpha=0$ for each $\alpha\in\Delta$ then $\lambda/Y=0$ which contradicts the assumption. If $\lambda/A_\alpha=1$ for each $\alpha\in\Delta$ then $\lambda/Y=1$ which contradicts the assumption. Similarly ($\mu/A_\alpha=0$ for all $\alpha\in\Delta$ and $\mu/A_\alpha=1$ for all $\alpha\in\Delta$) are not possible for each $\alpha\in\Delta$.So there exist α_1 and α_2 such that $\lambda/A_{\alpha_1}=1$ and $\mu/A_{\alpha_2}=1$ which implies $\lambda/A_{\alpha 1}+\mu/A_{\alpha 2}=2>1$ which contradicts $\lambda/Y+\mu/Y=1$ because $\lambda/A_{\alpha_1}\subseteq Y$ and $\mu/A_{\alpha_2}\subseteq Y$.Hence $Y=\cup\{A_\alpha:\alpha\in\Delta\}$ is a fuzzy P_W-connected subset of X.

Theorem 3.4.7

Let A and B be two subsets of a bfts (X,τ_1,τ_2) such that $\chi_A\subseteq\chi_B\subseteq cl_i\chi_A$ (i=1,2). Then if A is a P_W-connected subset of X then B is also a P_W-connected subset of X.

Proof :

Suppose that B is P_W-disconnected.Therefore there exist non-zero fuzzy sets $\mu\in\tau_i$ and $\nu\in\tau_j$ such that $\mu/B\neq 0,\nu/B\neq 0$ and $\mu/B+\nu/B=1$.We claim that $\mu/A\neq 0$.To prove our claim,suppose not i.e.;$\mu/A=0$ then $(\mu/A)(x)+\chi_A(x)\leq 1$ for all $x\in X$ which implies that $(\mu/A)(x)+cl_i\chi_A(x)\leq 1$ for all $x\in X$.Hence $(\mu/B)(x)+cl_i\chi_A(x)\leq 1$ because $A\subseteq B$ and if $x\in B$-A,then $cl_i\chi_A(x)=cl_i0(x)=0$.So we have $\mu/B+\chi_B\subseteq 1$ because $\chi_B\subseteq cl_i\chi_A$. This implies that $\mu/B=0$ which is a contradiction..Therefore $\mu/A\neq 0$. Similarly we show that $\nu/A\neq 0$. Since $\mu/B+\nu/B=1$ and $A\subseteq B$ we

have $\mu/A+\nu/A=1$.Hence A is not P_W-connected which is a contradiction.Therefore B is P_W-connected.

Definition 3.4.8

Let (X,τ_1,τ_2) be a bfts and λ_1,λ_2 be two fuzzy sets in X.Then

1) λ_1 and λ_2 are said to be disjoint iff $\lambda_1 \cap \lambda_2=0$.

2) λ_1 and λ_2 are said to be intersecting iff $\lambda_1 \cap \lambda_2 \neq 0$.

3) λ_1 and λ_2 are said to be overlapping if there exists $x \in X$ such that $\lambda_1(x)>1-\lambda_2(x)$.In this case λ_1 and λ_2 are said to be overlapping at x.

Theorem 3.4.9

If λ_1 and λ_2 are intersecting S-C3-connected fuzzy sets in a bfts (X,τ_1,τ_2) then $\lambda_1 \cup \lambda_2$ is S-C3-connected .

Proof :

Suppose that $\lambda_1 \cup \lambda_2$ is S-C3-disconnected , then there exist $\mu,\nu \in \tau_1 \cup \tau_2$ such that $\lambda_1 \cup \lambda_2 \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap (\lambda_1 \cup \lambda_2)=0$, $\mu \cap (\lambda_1 \cup \lambda_2) \neq 0$ and $\nu \cap (\lambda_1 \cup \lambda_2) \neq 0$.Since $\lambda_1 \cup \lambda_2 \subseteq \mu \cup \nu$,then it is clear that $\lambda_1 \subseteq \mu \cup \nu$ and $\lambda_2 \subseteq \mu \cup \nu$ .Since $(\mu \cap \nu) \cap (\lambda_1 \cup \lambda_2)=0$,then we have $[(\mu \cap \nu) \cap \lambda_1] \cup [(\mu \cap \nu) \cap \lambda_2]=0$ which implies that $(\mu \cap \nu) \cap \lambda_1=0$ and $(\mu \cap \nu) \cap \lambda_2=0$. Since $\lambda_1$and λ_2 are S-C3-connected,then $(\mu \cap \lambda_1=0$ or $\nu \cap \lambda_1=0)$ and $(\mu \cap \lambda_2=0$ or $\nu \cap \lambda_2=0)$.Suppose $\mu \cap \lambda_1=0$. Since $\lambda_1$and λ_2 are intersecting ,then there exists $x \in X$ such that $(\lambda_1 \cap \lambda_2)(x) \neq 0$ which implies that $\lambda_1(x) \neq 0$ and $\lambda_2(x) \neq 0$ .We claim that $\nu \cap \lambda_2 \neq 0$. To prove our claim,suppose the contrary,i.e.,$\nu \cap \lambda_2=0$ .Then $(\nu \cap \lambda_2)(x)=0$ gives that $\nu(x)=0$ and so $(\mu \cup \nu)(x)=0$ which contradicts the fact that $(\lambda_1 \cup \lambda_2)(x) \leq (\mu \cup \nu)(x)$ because $(\lambda_1 \cap \lambda_2)(x) \neq 0$.Therefore $\nu \cap \lambda_2 \neq 0$ and so $\mu \cap \lambda_2=0$.Hence $\mu \cap (\lambda_1 \cup \lambda_2)=0$ which contradicts that $\mu \cap (\lambda_1 \cup \lambda_2) \neq 0$. Similarly if $\nu \cap \lambda_1=0$ we can show that $\mu \cap \lambda_2=0$ is not possible.Hence $\nu \cap \lambda_2 = 0$.Therefore $\nu \cap (\lambda_1 \cup \lambda_2) = 0$ which contradicts $\nu \cap (\lambda_1 \cup \lambda_2) \neq 0$.Hence $\lambda_1 \cup \lambda_2$ is S-C_3-connected .

Theorem 3.4.10

If λ_1 and λ_2 are intersecting S-C_4-connected fuzzy sets in a bfts (X,τ_1,τ_2) then $\lambda_1 \cup \lambda_2$ is S-C_4-connected .

Proof :

The fact that $\lambda_1 \cup \lambda_2 \subseteq \mu \cup \nu$ implies $\lambda_1 \subseteq \mu \cup \nu$ and $\lambda_2 \subseteq \mu \cup \nu$.Moreover, since $(\mu \cap \nu) \subseteq (\lambda_1 \cup \lambda_2)^c = \lambda_1{}^c \cap \lambda_2{}^c$ therefore we have $\mu \cap \nu \subseteq \lambda_1{}^c$ and $\mu \cap \nu \subseteq \lambda_2{}^c$.Now our proof will be completed by following the same steps as in Theorem 3.4.9.

The following example illustrates that the above theorems are not valid for disjoint (non-intersecting) fuzzy sets.

Example 3.4.11

Let X=[0,1] and define fuzzy sets μ and ν as follows:

$$\mu(x) = \begin{cases} 0 & \text{if } 2/3 < x \leq 1 \\ 2/3 & \text{if } 0 \leq x \leq 2/3 \end{cases} \quad , \quad \nu(x) = \begin{cases} 2/3 & \text{if } 2/3 < x \leq 1 \\ 0 & \text{if } 0 \leq x \leq 2/3 \end{cases}$$

Then $\tau_1 = \{0,1,\mu\}$ and $\tau_2 = \{0,1,\nu\}$ are fuzzy topologies on X.

Define fuzzy sets λ_1 and λ_2 as follows:

$$\lambda_1(x)=\begin{cases} 1/3 & \text{if } 2/3<x\leq 1 \\ 0 & \text{if } 0\leq x\leq 2/3 \end{cases}, \quad \lambda_2(x)=\begin{cases} 0 & \text{if } 2/3<x\leq 1 \\ 1/3 & \text{if } 0\leq x\leq 2/3 \end{cases}$$

It is clear that $\lambda_1 \cap \lambda_2=0$, λ_1 and λ_2 are S-C_3-connected (S-C_4-connected) because $\lambda_1 \cap \mu=0$ and $\lambda_2 \cap \nu=0$ but $\lambda_1 \cup \lambda_2=1/3$ is S-C_3-disconnected because $(\lambda_1 \cup \lambda_2) \subseteq \mu \cup \nu$, $(\lambda_1 \cup \lambda_2) \cap (\mu \cap \nu)=(1/3) \cap 0=0$, $(\lambda_1 \cup \lambda_2) \cap \mu \neq 0$ and $(\lambda_1 \cup \lambda_2) \cap \nu \neq 0$.

Theorem 3.4.12

Let $\{\lambda_i : i \in \Delta\}$ be a family of S-C_3-connected fuzzy sets in (X,τ_1,τ_2) such that for $i,j \in \Delta, i \neq j$, the fuzzy sets λ_i and λ_j are intersecting. Then $\cup\{\lambda_i : i \in \Delta\}$ is S-C_3-connected.

Proof :

Let $\lambda=\cup\{\lambda_i : i \in \Delta\}$, where λ_i is as stated in the above theorem $(i \in \Delta)$. To prove λ is S-C_3-connected, suppose not. Then there exist $\mu,\nu \in \tau_1 \cup \tau_2$ such that $\lambda \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap \lambda=0$ $\mu \cap \lambda \neq 0$ and $\nu \cap \lambda \neq 0$. Now fix $k \in \Delta$. Since λ_k is S-C_3-connected and we have clearly $\lambda_k \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap \lambda_k=0$, therefore $\mu \cap \lambda_k=0$ or $\nu \cap \lambda_k=0$. We shall deal with the first case only because the second case can be treated similarly. So we may assume that $\mu \cap \lambda_k=0$. We claim that $\nu \cap \lambda_i \neq 0$ for all $i \in \Delta-\{k\}$. To prove our claim, suppose not; i.e., $\nu \cap \lambda_i=0$ for some $i \in \Delta-\{k\}$. Let $\Delta_1=\{i \in \Delta-\{k\} : \nu \cap \lambda_i=0\}$, then $\Delta_1 \neq \phi$. Now let $i \in \Delta_1$. Then $\lambda_k \cap \lambda_i \neq 0$. So there exists $x \in X$ such that $(\lambda_k \cap \lambda_i)(x) \neq 0$. This implies that $\lambda_k(x) \neq 0$ and $\lambda_i(x) \neq 0$. Since $\mu \cap \lambda_k=0$, therefore $\mu(x)=0$. Since $\nu \cap \lambda_i=0$ and $\lambda_i(x) \neq 0$, therefore $\nu(x)=0$. Consequently $(\mu \cup \nu)(x)=0$. The fact that

$\lambda \subseteq \mu \cup \nu$ implies that $\lambda(x)=0$ and this implies that $\lambda_i(x)=0$ for all $i \in \Delta$ which is a contradiction .This completes the proof of our claim that $\nu \cap \lambda_i \neq 0$ for all $i \in \Delta-\{k\}$.For $i \in \Delta-\{k\}$ we have $\lambda_i \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap \lambda_i=0$ and λ_i is S-C_3-connected.This implies that $\mu \cap \lambda_i=0$ or $\nu \cap \lambda_i=0$.Combining this result with the above claim we conclude that $\mu \cap \lambda_i=0$ for all $i \in \Delta-\{k\}$.But we know that $\mu \cap \lambda_k=0$.This implies $\mu \cap \lambda_i=0$ for all $i \in \Delta$.Consequently $\mu \cap \lambda=0$ and this is absurd.Hence $\lambda=\cup\{\lambda_i : i \in \Delta\}$ is S-C_3-connected .

Theorem 3.4.13

Let $\{\lambda_i : i \in \Delta\}$ be a family of S-C_4-connected fuzzy sets in (X,τ_1,τ_2) such that for $i,j \in \Delta, i \neq j$,the fuzzy sets λ_i and λ_j are intersecting .Then $\cup\{\lambda_i : i \in \Delta\}$ is S-C_4-connected .

Proof :

The proof follows the same steps as in Theorem 3.4.12.

Corollary 3.4.14

If $\{\lambda_i : i \in \Delta\}$ is a family of S-C_3-connected (S-C_4-connected) fuzzy sets in (X,τ_1,τ_2) and $\cap \{\lambda_i : i \in \Delta\} \neq 0$,then $\cup \lambda_i$ is S-C_3-connected (S-C_4-connected).

Proof :

The proof follows easily by applying Theorem 3.4.12 and the fact that $\cap\{\lambda_i : i \in \Delta\} \neq 0$ implies $\lambda_i \cap \lambda_j \neq 0$ for all $i \neq j$, i.e. λ_i and λ_j are intersecting for $i \neq j$.

The following example shows that Theorem 3.4.9 fails for S-C_2-connectedness .

Example 3.4.15

Let X=[0,1] and define fuzzy sets μ and ν as follows:

$$\mu(x)=\begin{cases} 6/7 & \text{if } 2/3<x\leq 1 \\ 2/7 & \text{if } 0\leq x\leq 2/3 \end{cases}, \qquad \nu(x)=\begin{cases} 2/7 & \text{if } 2/3<x\leq 1 \\ 6/7 & \text{if } 0\leq x\leq 2/3 \end{cases}$$

Then $\tau_1=\{0,1,\mu\}$ and $\tau_2=\{0,1,\nu\}$ are fuzzy topologies on X.

Define fuzzy sets λ_1 and λ_2 as follows:

$$\lambda_1(x)=\begin{cases} 1/7 & \text{if } 2/3<x\leq 1 \\ 2/7 & \text{if } 0\leq x\leq 2/3 \end{cases}, \qquad \lambda_2(x)=\begin{cases} 2/7 & \text{if } 2/3<x\leq 1 \\ 1/7 & \text{if } 0\leq x\leq 2/3 \end{cases}$$

It is clear that $\lambda_1\cap\lambda_2\neq 0$, λ_1 and λ_2 are S-C_2-connected because $\mu\subseteq\lambda_1{}^c$ and $\nu\subseteq\lambda_2{}^c$ but $\lambda_1\cup\lambda_2$ is S-C_2-disconnected because $2/7=(\lambda_1\cup\lambda_2)\subseteq(\mu\cup\nu)=6/7$, $2/7=(\mu\cap\nu)\subseteq(\lambda_1\cup\lambda_2)^c)=5/7$, $\mu\not\subseteq(\lambda_1\cup\lambda_2)^c$ and $\nu\not\subseteq(\lambda_1\cup\lambda_2)^c$

The following example shows that Theorem 3.4.9 fails for S-C_1-connectedness .

Example 3.4.16

Let X=[0,1] and define fuzzy sets μ and ν as follows:

$$\mu(x)=\begin{cases} 6/7 & \text{if } 2/3<x\leq 1 \\ 0 & \text{if } 0\leq x\leq 2/3 \end{cases}, \qquad \nu(x)=\begin{cases} 0 & \text{if } 2/3<x\leq 1 \\ 6/7 & \text{if } 0\leq x\leq 2/3 \end{cases}$$

Then $\tau_1=\{0,1,\mu\}$ and $\tau_2=\{0,1,\nu\}$ are fuzzy topologies on X.

Define fuzzy sets λ_1 and λ_2 as follows:

$$\lambda_1(x)=\begin{cases} 1/7 & \text{if } 2/3<x\leq 1 \\ 2/7 & \text{if } 0\leq x\leq 2/3 \end{cases}, \qquad \lambda_2(x)=\begin{cases} 2/7 & \text{if } 2/3<x\leq 1 \\ 1/7 & \text{if } 0\leq x\leq 2/3 \end{cases}$$

It is clear that $\lambda_1\cap\lambda_2\neq 0$, λ_1 and λ_2 are S-C_1-connected because $\mu\subseteq\lambda_1{}^c$ and $\nu\subseteq\lambda_2{}^c$ but $\lambda_1\cup\lambda_2$ is S-C_1-disconnected because $(\lambda_1\cup\lambda_2)\cap\mu\cap\nu=0$, $(\mu\cap\nu)\subseteq(\lambda_1\cup\lambda_2)^c)=5/7$, $\mu\not\subseteq(\lambda_1\cup\lambda_2)^c$ and $\nu\not\subseteq(\lambda_1\cup\lambda_2)^c$

Theorem 3.4.17

If λ_1 and λ_2 are overlapping S-C_1-connected fuzzy sets in a bfts (X,τ_1,τ_2) then $\lambda_1\cup\lambda_2$ is S-C_1-connected .

Proof :

Suppose that $\lambda_1\cup\lambda_2$ is S-C_1-disconnected,then there exist proper fuzzy sets $\mu,\nu\in\tau_1\cup\tau_2$ such that $(\mu\cap\nu)\cap(\lambda_1\cup\lambda_2)=0$, $(\lambda_1\cup\lambda_2)\subseteq(\mu\cup\nu)$, $\mu\not\subseteq(\lambda_1\cup\lambda_2)^c$ and $\nu\not\subseteq(\lambda_1\cup\lambda_2)^c$. (1)

Since $\lambda_1 \cup \lambda_2 \subseteq \mu \cup \nu$, then it is clear that $\lambda_1 \subseteq \mu \cup \nu$ and $\lambda_2 \subseteq \mu \cup \nu$. Since $(\mu \cap \nu) \cap (\lambda_1 \cup \lambda_2) = 0$, then we have $[\mu \cap \nu \cap \lambda_1] \cup [\mu \cap \nu \cap \lambda_2] = 0$ which implies that $\mu \cap \nu \cap \lambda_1 = 0$ and $\mu \cap \nu \cap \lambda_2 = 0$. Since λ_1 and λ_2 are S-C_1-connected then ($\lambda_1 \subseteq \mu^c$ or $\lambda_1 \subseteq \nu^c$) and ($\lambda_2 \subseteq \mu^c$ or $\lambda_2 \subseteq \nu^c$).Since λ_1 and λ_2 are overlapping ,then there exists $y \in X$ such that

$$\lambda_1(y) > 1 - \lambda_2(y). \quad (2)$$

Now consider the following cases:

Case I.Suppose $\lambda_1 \subseteq \mu^c$,then by (2) we have

$$\mu(y) \leq 1 - \lambda_1(y) < \lambda_2(y), \quad (3)$$

We claim that $\lambda_2 \not\subseteq \nu^c$.To prove our claim,suppose not,i.e.,$\lambda_2 \subseteq \nu^c$. This yields $\nu(y) \leq 1 - \lambda_2(y) < \lambda_1(y)$ (4)

Now by (3) and (4),$(\mu \cup \nu)(y) < (\lambda_1 \cup \lambda_2)(y)$ which implies that $(\lambda_1 \cup \lambda_2) \not\subseteq (\mu \cup \nu)$,this contradicts (1).Hence our claim is valid,i.e.,$\lambda_2 \not\subseteq \nu^c$ and so $\lambda_2 \subseteq \mu^c$. Therefore $\mu \subseteq \lambda_1{}^c \cap \lambda_2{}^c = (\lambda_1 \cup \lambda_2)^c$ which contradicts (1)

Case II.Suppose $\lambda_1 \subseteq \nu^c$.Here we can show as in case I that $\mu \not\subseteq (\lambda_2)^c$. Therefore $\nu \subseteq (\lambda_2)^c$.Hence $\nu \subseteq \lambda_1{}^c \cap \lambda_2{}^c = (\lambda_1 \cup \lambda_2)^c$ .Therefore $\nu \subseteq (\lambda_1 \cup \lambda_2)^c$.This contradicts (1).

Hence $\lambda_1 \cup \lambda_2$ is S-C_1-connected .

Theorem 3.4.18

If λ_1 and λ_2 are overlapping S-C_2-connected fuzzy sets in a bfts (X,τ_1,τ_2) then $\lambda_1 \cup \lambda_2$ is S-C_2-connected .

Proof :

The fact that $\lambda_1 \cup \lambda_2 \subseteq \mu \cup \nu$ implies $\lambda_1 \subseteq \mu \cup \nu$ and $\lambda_2 \subseteq \mu \cup \nu$.Moreover since $(\mu \cap \nu) \subseteq (\lambda_1 \cup \lambda_2)^c = \lambda_1{}^c \cap \lambda_2{}^c$,therefore $\mu \cap \nu \subseteq \lambda_1{}^c$ and $\mu \cap \nu \subseteq \lambda_2{}^c$. Then the proof follows by using similar steps as in Theorem 3.4.17.

Theorem 3.4.19

Let $\{\lambda_i : i \in \Delta\}$ be a family of S-C_1-connected fuzzy sets in X such that for $i,j \in \Delta, i \neq j$,the fuzzy sets λ_i and λ_j are overlapping .Then $\cup\{\lambda_i : i \in \Delta\}$ is S-C_1-connected .

Proof :

Let $\lambda = \cup\{\lambda_i : i \in \Delta\}$ and let λ_k be any fuzzy set of the given family $(k \in \Delta)$.Suppose that λ is S-C_1-disconnected.Then there exist proper fuzzy sets $\mu, \nu \in \tau_1 \cup \tau_2$ such that

$\lambda \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap \lambda = 0$,$\mu \not\subseteq \lambda^c$ and $\nu \not\subseteq \lambda^c$. (1)

Since $\lambda \subseteq \mu \cup \nu, (\mu \cap \nu) \cap \lambda = 0$,then $\lambda_k \subseteq \mu \cup \nu$, $(\mu \cap \nu) \cap \lambda_k = 0$ .Since λ_k is S-C_1-connected,then $\mu \subseteq (\lambda_k)^c$ or $\nu \subseteq (\lambda_k)^c$.Since λ_k and λ_i are overlapping, therefore there exists $y \in X$ such that

$\lambda_k(y) > 1 - \lambda_i(y)$ for all $i \neq k$ (2)

Now consider the following cases :

Case I.Suppose $\lambda_k \subseteq \mu^c$,then by (2) we have

$\mu(y) \leq 1 - \lambda_k(y) < \lambda_i(y)$, (3) We claim that $\lambda_i \not\subseteq \nu^c$.To prove our claim ,suppose on the contrary,that is $\lambda_i \subseteq \nu^c$ .This yields $\nu(y) \leq 1-\lambda_i(y) < \lambda_k(y)$ (4)

Now by (3) and (4),$(\mu \cup \nu)(y) < (\lambda_k \cup \lambda_i)(y)$ which implies that $(\lambda_k \cup \lambda_i) \not\subseteq (\mu \cup \nu)$,this contradicts (1).Hence $\lambda_i \not\subseteq \nu^c$ and so $\lambda_i \subseteq \mu^c$ for all i. Therefore $\mu \subseteq \cap \lambda_i{}^c = \lambda^c$ which again contradicts (1).

Case II.Suppose $\lambda_k \subseteq \nu^c$.Here we can show as in case I that $\mu \not\subseteq (\lambda_i)^c$. Therefore $\nu \subseteq (\lambda_i)^c$.Hence $\nu \subseteq \cap \lambda_i{}^c = \lambda^c$. This contradicts (1).

Hence λ is S-C_1-connected .

Theorem 3.4.20

Let $\{\lambda_i : i \in \Delta\}$ be a family of S-C_2-connected fuzzy sets in X such that for $i,j \in \Delta, i \neq j$,the fuzzy sets λ_i and λ_j are overlapping .Then $\cup\{\lambda_i : i \in \Delta\}$ is S-C_2-connected .

Proof :

The proof follows the same steps as in Theorem 3.4.19.

Corollary 3.4.21

If $\{\lambda_i : i \in \Delta\}$ is a family of S-C_1-connected (S-C_2-connected) fuzzy sets in (X,τ_1,τ_2) and p is a fuzzy point with support x and value 1/2 such that $p \in \cap\{\lambda_i : i \in \Delta\}$,then $\cup\{\lambda_i : i \in \Delta\}$ is an S-C_1-connected (S-C_2-connected) fuzzy set in X..

Proof :

Since $p \in \cap\{\lambda_i : i \in \Delta\}$,then λ_i and λ_j are overlapping for all $i,j \in \Delta$.

it ts important to note that all the results from theorem 3.4.9 till corollary 3.4.21 are also valid if we replace S by P.

Theorem 3.4.22

(i) If a bfts (X,τ_1,τ_2) is S_w-connected then (X,τ_1) and (X,τ_2) are connected .

(ii) The converse of (i) is not true in general .

Proof :

(i) Suppose that (X,τ_i) is disconnected for some $i\in\{1,2\}$ .Then there exist proper fuzzy sets $\mu,\lambda\in\tau_i$ such that $\mu+\nu=1$ and $\mu\cap\nu=0$ but this gives $\mu,\nu\in\tau_1\cup\tau_2$ and so (X,τ_1,τ_2) is S-disconnected (S_w-disconnected) which is a contradiction. Hence (X,τ_i) is connected $i\in\{1,2\}$..

(ii) Let $X=[0,1],\tau_1=\{0,1,\lambda\}$ and $\tau_2=\{0,1,\lambda^c\}$ where λ is defined as follows:

$$\lambda(x)=\begin{cases} 1/2 & \text{if } 0\le x<1/2 \\ 0 & \text{if } 1/2\le x\le 1 \end{cases}$$

It is clear that (X,τ_1,τ_2) is S_w-disconnected but (X,τ_1) and (X,τ_2) are connected.

We also notice that P_w-connectedness of a bfts (X,τ_1,τ_2) are not governed by the connectedness of each fuzzy topological space (X,τ_1) and (X,τ_2) as shown in the following examples.

Example 3.4.23

Let (X,τ_1,τ_2) be a bfts where $X=[0,1]$,$\tau_1=\tau_{dis}$ and $\tau_2=\tau_{ind}$.Then (X,τ_1,τ_2) is P_w-connected (and hence P-connected) while (X,τ_1) is not fuzzy connected.

Example 3.4.24

Let $X=[0,1]$,$\lambda(x)=x$,$\mu(x)=1-x$,$x\in X$.Let (X,τ_1,τ_2) be a bfts ,where $\tau_1=\{0,1,\lambda\}$ and $\tau_2=\{0,1,\mu\}$ .Since $\lambda+\mu=1$,the space (X,τ_1,τ_2) is P_w-disconnected while (X,τ_1) and (X,τ_2) are fuzzy connected.

Example 3.4.25

Let $X=[0,1]$,$\tau_1=\{0,1,1/3,2/3\}$ and $\tau_2=\{0,1,1/4,3/4\}$.Then the bfts (X,τ_1,τ_2) is P_w-connected while (X,τ_1) and (X,τ_2) are fuzzy disconnected.

Chapter IV

COVERING PROPERTIES

IN

BIFUZZY TOPOLOGICAL SPACES

Compactness is one of the most important notions in topology and thoroughly investigated in general topology.In literature, different kinds of fuzzy compactness notions have been introduced and studied.Chang (1968) was the first to introduce compactness in fuzzy topological spaces .His definition was not so interesting because a fuzzy space with one point fails to be compact .Research about this notion was then carried out by many authors,but their works were not so effective.Gantner et al.(1978) introduced the notion of α-compactness in fuzzy topological spaces .The concepts of almost compactness and near compactness in fuzzy topological spaces were introduced by Concilio and Gerla (1984) while the α-almost compactness was introduced by Abd El-Monsef and Ramadan (1989).

In this thesis we shall discuss the bifuzzy extension to most of the existing definitions of fuzzy compactness in the literature.Moreover we shall discuss the goodness criterion and obtain many interesting results of bifuzzy compactness ,reflecting to a large extent,parallel properties in classical general topology.For the sake of clarity we divide this chapter into four sections .In the first section we shall discuss different types of compactness in bifuzzy topological spaces. The goodness of bifuzzy extensions will be discussed in the second section.In the third section we are going to discuss bifuzzy Lindelof spaces while in the fourth section we shall discuss countability properties in bifuzzy topological spaces.

§ 4.1 Different types of compactness

We start this section with the following definition.

Definition 4.1.1

A collection $U \subseteq I^X$ is called S-open (P-open) if $U \subseteq \tau_1 \cup \tau_2$ ($U \subseteq \tau_1 \cup \tau_2$ and U contains a non-zero τ_1 open set and a non -zero τ_2-open set).A collection C is called S-closed (P-closed)iff $U=\{\lambda^c : \lambda \in C\}$ is S-open (P-open).

Definition 4.1.2

Let $\alpha \in [0,1)$,then a collection $U \subseteq \tau_1 \cup \tau_2$ is called an S-α-shading (P-α-shading) for X iff for each $x \in X$,there exists $\mu \in U$ such that $\mu(x) > \alpha$ where U is S-open (P-open).

Definition 4.1.3

A collection $C=\{\lambda_\alpha : \alpha \in \Delta\} \subseteq \tau_1 \cup \tau_2$ is called an S-open cover (P-open cover) for X iff $\cup\{\lambda_\alpha : \alpha \in \Delta\}=1$ and C is S-open (P-open).

In (1968) Chang gave a definition of compactness in fuzzy topological spaces which formally is the same one as in topological spaces.This definition has also been used in Goguen (1973) and Wong (1974).We use the idea of his definition to give a similar version in bifuzzy topological spaces.

Definition 4.1.4

A bfts (X,τ_1,τ_2) is called S-compact (P-compact) if and only if every S-open (P-open)cover of X has a finite subcover.

The above definition has a serious weak point ,that is a finite space needs not be S-compact (P-compact) as the following example shows.

Example 4.1.5

(a) Let $X=\{x_0\}$ and $\tau_1=\tau_2=\{c:0\leq c\leq 1\}$.Then the collection $\{c:0\leq c<1\}$ is a P-open cover for X which has no finite subcover.

(b) If X is a non-empty set and T_1 and T_2 are any two topologies on X,then $(X,\omega(T_1),\omega(T_2))$ is never an S-compact (P-compact) space because the P-open cover $C=\{c:0<c<1\}$ for X has no finite subcover.

In (1978) Gantner et al. introduced the concept of α-compactness in fuzzy topological spaces (see definition 1.6.5).The following definition is an extension of α-compactness in bifuzzy topological spaces.

Definition 4.1.6

A bfts (X,τ_1,τ_2) is called S-α-compact (P-α-compact) if and only if every S-α-shading (P-α-shading) for X has a finite α-subshading.

Clearly S-α-compact implies P-α-compact but the converse is not true in general as we see in the following example.

Example 4.1.7

There exists a P-α-compact bfts which is not S-α-compact.

The bfts $(X, X/T_{l.r}, X/T_{r.r})$,with X=R is P-$\alpha$-compact but not S-$\alpha$-compact for $0 \leq \alpha < 1$.Indeed the collection $\{\chi_{(-\infty,n)}: n \in N\}$ is an S-α-shading of X which has no finite α-subshading for each $0 \leq \alpha < 1$.Hence $(X, X/T_{l.r}, X/T_{r.r})$ is not S-α-compact.

To show that $(X, X/T_{l.r}, X/T_{r.r})$ is P-α-compact ,let $C = \{\chi_{u_\alpha}: \alpha \in \Delta\} \cup \{\chi_{v_\beta}: \beta \in \gamma\}$ be a P-α - shading for X;where $u_\alpha \in T_{l.r}$ and $v_\beta \in T_{r.r.}$.Let $U = \cup\{u_\alpha: \alpha \in \Delta\}$and $V = \cup\{v_\beta: \beta \in \gamma\}$,then $U \in T_{l.r}$ and $V \in T_{r.r}$. If $U \cap V = \phi$,then there exists $x \in X$ such that $x \in X \backslash U \cup V$,i.e,$x \notin U$ and $x \notin V$.Hence $\chi_{u_\alpha}(x) = 0$ for all $\alpha \in \Delta$ and $\chi_{v_\beta}(x) = 0$ for all $\beta \in \gamma$ which implies that C is not a P-α-shading.Consequently,we have $U \cap V \neq \phi$.Let $x \in U \cap V$,then $x \in u_\alpha$ for some α and $x \in v_\beta$ for some β.The collection $\{\chi_{u_\alpha}, \chi_{v_\beta}\}$ is a finite α-subshading of C for X.Hence $(X, \tau_{l.r.}, \tau_{r.r.})$ is P-α-compact.

We remind the reader that a collection $\wp$ of fuzzy sets in a bfts (fts) X is said to be α-centered if for all finite collections $\mu_i \in \wp$,i=1,2,...,n there exists $x \in X$ with $\mu_k(x) \geq 1-\alpha$ for all $k \in \{1,2,...,n\}$.

Theorem 4.1.8

Let (X, τ_1, τ_2) be a bfts ,then the following are equivalent:

(i) (X, τ_1, τ_2) is S-α-compact.

(ii) for every α-centered system $\wp$ of S-closed fuzzy sets in X there exists $x \in X$ such that $\lambda(x) \geq 1-\alpha$ for all $\lambda \in \wp$.

Proof :

(i)⇒(ii) Let $\wp$ be an α-centered collection of S-closed fuzzy sets in X.Assume that for each $x\in X$ there exists $\lambda_x \in \wp$ such that $\lambda_x(x)<1-\alpha$. Then $\hat{U}=\{(\lambda_x)^c : x\in X\}$ is an S-α-shading of X.Consequently there is a finite α-subshading $\{(\lambda_{xi})^c : i=1,2,...,n\}$ for X.Now,$\{\lambda_{xi} : i=1,2,...,n\}$ is finite.Hence there exists x_0 such that $\lambda_{xi}(x_0)\geq 1-\alpha$ for all i.This contradicts that $\{(\lambda_{xi})^c : i=1,2,...,n\}$is α-subshading.That is,for each $x\in X$ there is $(\lambda_{xi})^c$ such that $(\lambda_{xi})^c(x)>\alpha$ and so $\lambda_{xi}(x_0)<1-\alpha$ which is a contradiction.

(ii)⇒(i) Let $\hat{U}$ be an S-α-shading for X such that no finite subcollection of $\hat{U}$ is an α-subshading for X.Hence for every finite subcollection V of $\hat{U}$ there is a point $x_0\in X$ such that $\lambda(x_0)\leq\alpha$ for all $\lambda\in V$.Then $\{\lambda^c : \lambda\in \hat{U}\}$is S-closed α -centered.Therefore there exists a point $x_0\in X$ such that $\lambda^c(x_0)\geq 1-\alpha$ for all $\lambda\in \hat{U}$.Thus for all $\lambda\in \hat{U}$ we have $\lambda(x_0)\leq\alpha$ and this contradicts that $\hat{U}$ is S-α-shading for X.

Whenever we say that a bfts (X,τ_1,τ_2) has a property Q we mean both spaces (X,τ_1) and (X,τ_2) have Q.For example (X,τ_1,τ_2) is a T_2-space provided both (X,τ_1) and (X,τ_2) are T_2-spaces.

Theorem 4.1.9

Let (X,τ_1,τ_2) be a bfts ,then the following are equivalent:

(i) (X,τ_1,τ_2) is S-α-compact.

(ii)(X,τ_1,τ_2) is P-α-compact and α-compact.

Proof :

(i)$\Rightarrow$(ii) If $\hat{U}$ be a P-α-shading or α-shading for X.Then $\hat{U}$ is an S-α-shading for (X,τ_1,τ_2) and so it has a finite α -subshading.

(ii)$\Rightarrow$(i) Let $\hat{U}$ be an S-α-shading for X.Then either $\hat{U}\subseteq\tau_1$ or $\hat{U}\subseteq\tau_2$ or $\hat{U}$ is a P-α-shading for X.In either case $\hat{U}$ has a finite α-subshading.

Definition 4.1.10

A bfts (X,τ_1,τ_2) is called S-weakly compact iff for each S-open cover for X and for each $\varepsilon>0$ there exists a finite subcover of $1-\varepsilon$.

Definition 4.1.11

A bfts (X,τ_1,τ_2) is called S-α-weakly compact iff for each $\varepsilon>0$ and for all S-$(\alpha+\varepsilon)$-shading for X there exists a finite α-subshading ($\alpha\in[0,1)$).

Example 4.1.12

Example 4.1.5 (a) is not S-compact but it is S-weakly compact.

Definition 4.1.13

A bfts (X,τ_1,τ_2) is called P-weakly compact iff for each P-open cover for X and for each $\varepsilon>0$ there exists a finite subcover of $1-\varepsilon$.

Definition 4.1.14

A bfts (X,τ_1,τ_2) is called P-α-weakly compact iff for each $\varepsilon>0$ and for all P-$(\alpha+\varepsilon)$-shading for X there exists a finite α-subshading ($\alpha\in[0,1)$).

The following theorem shows the relation between P-α-compact and P-α-weakly compact.

Theorem 4.1.15

If a bfts $(X,\tau 1,\tau 2)$ is P-α-compact then $(X,\tau 1,\tau 2)$ is P-α-weakly compact.

Proof:

Clear .

Example 4.1.16

There exists a P-α-weakly compact bfts $(X,\tau 1,\tau 2)$ which is not P-α-compact.

Let $X=I, 0<\alpha<1$ and for each $0<p<1$.

Define $$U^{\alpha}{}_{p}(x)=\begin{cases}(1-\alpha)(x-1)+1 & \text{if } x=p\\ 0 & \text{if } x\neq p\end{cases}$$

The collection $\{U^{\alpha}{}_{p}:0<p<1\}$ is a subbase for some fuzzy topology $\tau 1$ on X.

Define $\tau 2=\{0,1,\chi_{\{0,1\}}\}$.

Then the bfts $(X,\tau 1,\tau 2)$ is not P-α-compact because the collection $\{U^{\alpha}{}_{p},\chi_{\{0,1\}}\}$ is a P-α-shading for X that has no finite α-subshading.Notice that $(X,\tau 1,\tau 2)$ is P-α-weakly compact because any P-$(\alpha+\varepsilon)$-shading for X must contain 1.To see that,let $x\in X$ be such that

$0<x<\varepsilon/(1-\alpha)$. Then there is no fuzzy open set in τ_1 or in τ_2 other than 1 covers x.That is,there exists no P-$(\alpha+\varepsilon)$-shading for X that does not contain 1.If it contains 1,then {1} is a finite α-subshading for X and therefore X is P-α–weakly compact.

Now we have the following version of definition 1.6.6 in bifuzzy topological spaces.

Definition 4.1.17

A fuzzy set f in a bfts (X,τ_1,τ_2) is said to be S-compact (P-compact) iff for every family G of S-open (P-open) fuzzy sets such that $\sup\{g:g\in G\}\geq f$ and for every $\varepsilon>0$,there exists a finite subfamily $G_\varepsilon\subseteq G$ such that $\sup\{g:g\in G_\varepsilon\}\geq f-\varepsilon$.

Definition 4.1.18

A bfts (X,τ_1,τ_2) is S-C-compact (P-C-compact) provided that each constant map from X into I is S-compact (P-compact) .

Definition 4.1.19

A bfts (X,τ_1,τ_2) is S-C-weakly compact (P-C-weakly compact) iff the constant fuzzy set 1 is S-compact (P-compact),(in the sense of definition 4.1.17).

From the next definition and onwards the symble $\tau^*{}_{ci}$ means $(\tau_i)^*{}_c$,i=1,2.

Definition 4.1.20

A bfts (X,τ_1,τ_2) is P-U-compact (S-U-compact) iff $(X,\tau^*{}_{c1},\tau^*{}_{c2})$ is P-compact (S-compact).

Definition 4.1.21

A bfts (X,τ_1,τ_2) is P-S-compact (S-S-compact) iff for each $0\leq\alpha<1$,the space (X,τ_1,τ_2) is P-α-compact.(S-α-compact) ,

Theorem 4.1.22

If a bfts (X,τ_1,τ_2) is P-C-weakly compact ,then $(X,\tau^*_{c1},\tau^*_{c2})$ is P-compact.

Proof :

Let (X,τ_1,τ_2) be P-C-weakly compact.Let $\mathfrak{R}=\{G_\alpha:\alpha\in\Delta\}$ be a P-open cover for $(X,\tau^*_{c1},\tau^*_{c2})$.For each $\alpha\in\Delta$,let g_α denote $\chi_{G\alpha}$.Then it is clear that $\sup\{g_\alpha:\alpha\in\Delta\}=1$.Since 1 is P-compact,therefore there exists a finite set $\Delta_1\subseteq\Delta$ such that $\sup\{g_\alpha:\alpha\in\Delta_1\}\geq1/2$.It is obvious that the collection $\{G_\alpha:\alpha\in\Delta_1\}$ is a finite subcover of $\mathfrak{R}$.

Example 4.1.23

There exists a bfts (X,τ_1,τ_2) which is not P-C-weakly compact but $(X,\tau^*_{c1},\tau^*_{c2})$ is P-compact.

Let $X=I,\tau_1=\tau_2=\{0,\lambda:\lambda(x)>0$ for all $x\in X\}$.Then $\tau^*_{c1}=\tau^*_{c2}=\{\phi,X\}$. Thus $(X,\tau^*_{c1},\tau^*_{c2})$ is P-compact.

Define $\quad U_p(p)=1,\ U_p(x)=1/2$ for $x\neq p$.

Then the collection $\{U_p:p\in I\}$ is a P-open cover for 1 which has no finite subcover for 0.9.Hence (X,τ_1,τ_2) is not P-C-weakly compact.

We may write Theorem 4.1.22 in a different way as the following version shows.

Theorem 4.1.24

A bfts (X,τ_1,τ_2) is P-U-compact if (X,τ_1,τ_2) is P-C-weakly compact.

Theorem 4.1.25

Let (X,τ_1,τ_2) be a bfts.Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii),where

(i) (X,τ_1,τ_2) is P-S-compact.

(ii)(X,τ_1,τ_2) is P-C-compact.

(iii)(X,τ_1,τ_2) is P-U-compact .

Proof :

(i)$\Rightarrow$(ii) Let c be a constant fuzzy set in X.We have to show that c is compact.To do this,let G be a P-open cover for c.That is, $\sup\{g : g\in G\}\geq c$ .Let $\varepsilon>0$.Now $\{g : g\in G\}$is a c-ε-shading for X.So there exists $\{g_i : i=1,2,...,n\}$ finite c-ε-subshading.This implies that $\sup\{g_i : i=1,2,...,n\}\geq c-\varepsilon$.Hence c is compact.That is,(X,τ_1,τ_2) is P-C-compact.

(ii)$\Rightarrow$(iii) Since P-C-compact implies P-C-weakly compact.Then Theorem 4.1.24 completes the proof of our implication.

It is important to note that example 4.1.23 is P-U-compact (see definition 4.1.20) and not P-C-compact because it is not P-C-weakly compact.Also it is not P-S-compact because if $\alpha=3/4$,then the collection $\{U_p : p\in I\}$ is a P-α-shading for α which does not have a finite α-subshading.Therefore (iii) does not imply (i) and does not imply (ii).

Theorem 4.1.26

Let (X,τ_1,τ_2) be a bfts.Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii),where

(i) (X,τ_1,τ_2) is S-S-compact.

(ii)(X,τ_1,τ_2) is S-C-compact.

(iii)(X,τ_1,τ_2) is S-U-compact .

Proof :

Similar to the proof of theorem 4.1.25.

In the following Theorem we show that all well behaved compactness concepts that we have introduced are preserved under continuous surjections.

Theorem 4.1.27

Let X,Y be two bft spaces and $f:X\rightarrow Y$ be a continuous surjection .

(i) If X is S-α-compact then Y is S-α-compact.

(ii) If X is P-α-compact then Y is P-α-compact.

(iii) If X is S-compact then Y is S-compact.

(iv) If X is P-compact then Y is P-compact.

(v) If X is S-C-compact then Y is S-C-compact.

(vi) If X is P-C-compact then Y is P-C-compact.

Proof :

We shall prove (ii) and (v) only .The other cases can be treated similarly.

(ii) Let $\hat{U}$ be a P-α-shading for Y.Since f is continuous then $f^{-1}(\hat{U})=\{f^{-1}(\lambda):\lambda\in\hat{U}\}$is a P-$\alpha$-shading for X.Indeed if $x\in X$ then $f(x)\in Y$,so there exists $\lambda\in\hat{U}$ such that $\lambda(f(x))>\alpha$.That is $f^{-1}(\lambda)(x)>\alpha$.

Hence $\{f^{-1}(\lambda):\lambda\in\hat{U}\}$ has a finite α-subshading $\{f^{-1}(\lambda_i): i=1,2,...,n\}$.Now $\{\lambda_i:i=1,2,...,n\}$ is a finite α -subshading of $\hat{U}$.Indeed, if $y\in Y$ then $y=f(x)$ for some $x\in X$.Thus there exists j such that $f^{-1}(\lambda_j)(x)>\alpha$.This implies that $\lambda_j(f(x))=\lambda_j(y)>\alpha$.

(v) Let h be a constant fuzzy set in Y and G be a family of S-open fuzzy sets in Y such that $\sup\{g:g\in G\}\geq h$.We have to show that for any $\varepsilon>0$ there exists a finite subfamily $G_\varepsilon\subseteq G$ such that $\sup\{g:g\in G_\varepsilon\}\geq h-\varepsilon$. Since f is continuous then $f^{-1}(G)=\{f^{-1}(g) :g\in G\}$ is a family of S-open fuzzy sets in X.Therefore $\sup\{f^{-1}(g):g\in G\}\geq h$ because $f^{-1}(g)(x)=g(f(x))$. Since X is S-C-compact then there exists a finite subfamily $\{f^{-1}(g_i):i=1,2,...,n\}$ of $f^{-1}(G)$ such that $\sup\{f^{-1}(g_i): i=1,2,...,n\}\geq h-\varepsilon$. This implies $\sup\{g_i: i=1,2,...,n\}\geq h-\varepsilon$.Consequently Y is S-C-compact.

§ 4.2 Goodness of bifuzzy extension

In this section we prove that our definitions of P-0-compactness,S-α-compactness ($\alpha\in[0,1)$) and S-C-compactness are good extensions while the definition of S-compact (P-compact) is not (see example 4.1.5 (b)).

Theorem 4.2.1

Let (X,T_1,T_2) be a bitopological space ,then the following are equivalent:

(i) (X,T_1,T_2) is S-compact.

(ii) $(X,\omega(T_1),\omega(T_2))$ is S-α-compact for each $0\leq\alpha<1$.

(iii)$(X,\omega(T_1),\omega(T_2))$ is S-α-compact for some $0\leq\alpha<1$.

Proof :

(i)$\Rightarrow$(ii) Let $\hat{U}$ be an S-α-shading for $(X,\omega(T_1),\omega(T_2))$.Then $U=\{\lambda^{-1}(\alpha,1]:\lambda\in\hat{U}\}$ is an S-open cover for (X,T_1,T_2).Indeed,for each $x\in X$ there exists $\lambda\in\hat{U}$ such that $\lambda(x)>\alpha$.This implies that $x\in\lambda^{-1}(\alpha,1]$. Since (X,T_1,T_2) is S-compact,U has a finite subcover $\{\lambda_i^{-1}(\alpha,1] : i=1,2,...,n\}$.Now the set $\{\lambda_i : i=1,2,...,n\}$ is an α-subshading for $(X,\omega(T_1),\omega(T_2))$.Indeed if $x\in X$ then there exists $i\in\{1,2,...,n\}$ such that $x\in\lambda_i^{-1}(\alpha,1]$.This implies $\lambda_i(x)>\alpha$.

(ii)$\Rightarrow$(iii) Trivial.

(iii)$\Rightarrow$(i) Let V be an S-open cover of (X,T_1,T_2),where $(X,\omega(T_1),\omega(T_2))$ is S-α-compact for some $\alpha\in[0,1)$.Then $\hat{U}=\{\chi_v:v\in V\}$ is an S-α-shading for $(X,\omega(T_1),\omega(T_2))$.Indeed if $x\in X$ then there exists $v\in V$ such that $\chi_v(x)>\alpha$.So there exists a finite α-subshading, say $\{\chi_{vi} : v_i\in V,i=1,2,...,n\}$.Now the collection $\{v_i :i=1,2,...,n\}$ is a finite subcover of V.Indeed if $x\in X$ then there exists $i\in\{1,2,...,n\}$ such that $\chi_{vi}(x)>\alpha$.This implies $x\in v_i$.

The above result shows that S-α-compactness is a good extension of S-compactness.

In the following theorem and example we show that P-α-compactness is a good extension of P-compactness only if α=0.

Theorem 4.2.2

Let (X,T_1,T_2) be a bitopological space.Then the following are equivalent:

(i) (X,T_1,T_2) is P-compact.

(ii)$(X,\omega(T_1),\omega(T_2))$ is P-0-compact .

Proof :

(i)$\Rightarrow$(ii) Let $\hat{U}$ be a P-0-shading of $(X,\omega(T_1),\omega(T_2))$.Then $U=\{\lambda^{-1}(0,1]:\lambda\in\hat{U}\}$ is a P-open cover for (X,T_1,T_2) because for each $x\in X$ there exists $\lambda\in\hat{U}$ such that $\lambda(x)>0$.This implies that $x\in\lambda^{-1}(0,1]$. Since (X,T_1,T_2) is P-compact,U has a finite subcover $\{\lambda_i^{-1}(0,1] : i=1,2,...,n\}$. Now the set $\{\lambda_i : i=1,2,...,n\}$ is a 0-subshading for $(X,\omega(T_1),\omega(T_2))$ because if $x\in X$ then there exists $i\in\{1,2,...,n\}$ such that $x\in\lambda_i^{-1}(0,1]$ which implies $\lambda_i(x)>0$.

(ii)$\Rightarrow$(i) Let V be a P-open cover for (X,T_1,T_2),where $(X,\omega(T_1),\omega(T_2))$ is P-0-compact.Then $\hat{U}=\{\chi_v:v\in V\}$ is a P-0-shading for $(X,\omega(T_1),\omega(T_2))$ because if $x\in X$ then there exists $v\in V$ such that $\chi_v(x)>0$.So there exists a finite 0-subshading,say $\{\chi_{v_i} : v_i\in V,i=1,2,...,n\}$.Now the collection $\{v_i :i=1,2,...,n\}$ is a finite subcover of V.Indeed,if $x\in X$ then there exists $i\in\{1,2,...,n\}$ such that $\chi_{v_i}(x)>0$ which implies $x\in v_i$.

The following example shows that P-α-compactness is not a good extension of P-compactness if $\alpha>0$.

Example 4.2.3

There exists a bitopological space (X,T_1,T_2) such that (X,T_1,T_2) is P-compact but $(X,\omega(T_1),\omega(T_2))$ is not P-α-compact for all $0<\alpha<1$.

Let $X=R, T_1=T_{l.r.}$ and $T_2=T_{r.r.}$. Then the bts (X,T_1,T_2) is P-compact but $(X,\omega(T_1),\omega(T_2))$ is not P-α-compact for all $0<\alpha<1$. To prove our assertion, suppose not. Then the P-α-shading $C=\{\chi_{(-\infty,n)}, n\in N\}\cup\{(\alpha/2)\chi_{(0,\infty)}\}$ for $(X,\omega(T_1),\omega(T_2))$ has a finite α-subshading, say $\{\chi_{(-\infty,ni)}, i=1,2,...,p\}$. Let $m=\max\{n_i: i=1,2,...,p\}$ and $x=m+1$. Then $\chi_{(-\infty,ni)}(x)=0$ for all $i=1,2,...,p$ and $(\alpha/2)\chi_{(0,\infty)}(x)\leq(\alpha/2)<\alpha$. This is a contradiction. Hence $(X,\omega(T_1),\omega(T_2))$ is not P-α-compact.

Theorem 4.2.4

Let (X,T_1,T_2) be a bitopological space, then the following are equivalent:

(i) (X,T_1,T_2) is S-compact.

(ii) There exists c : $0<c\leq1$ and c is S-compact in $(X,\omega(T_1),\omega(T_2))$.

(iii) $(X,\omega(T_1),\omega(T_2))$ is S-C-compact .

Proof :

(i)$\Rightarrow$(ii) Let f be a fixed constant fuzzy set in X and G be a family of S-open fuzzy sets for X in $(X,\omega(T_1),\omega(T_2))$ such that $\sup\{g:g\in G\}\geq f$ and let $\varepsilon>0$. Then the family $U=\{u=g^{-1}(f-\varepsilon,1]:g\in G\}$ is an S-open cover for X.Indeed if $x \in X$,then there exists $g \in G$ such that $g(x)>f-\varepsilon$.That is $g(x) \in (f-\varepsilon,1]$ or $x \in g^{-1}(f-\varepsilon,1]$.Since X is S-compact,U has a finite subcover $V=\{g_i^{-1}(f-\varepsilon,1] : i=1,2,...,n\}$ Now $\sup\{g_i : i=1,2,...,n\} \geq f-\varepsilon$,which shows that f is compact .

(ii)⇒ (iii) Let f be a constant fuzzy set in X and G be a family of S-open fuzzy sets in $(X,\omega(T_1),\omega(T_2))$ such that $\sup\{g : g \in G\} \geq f$.Let $\varepsilon>0$ and c be an S-compact fuzzy set in X.If Since $\sup\{g : g \in G\} \geq f$,then $\sup\{g-f+c : g \in G\} \geq c$. Now $\{g-f+c : g \in G\}$ is S-open and c is S-compact. Therefore there exists a finite subcover$\{g_i-f+c : i=1,2,...,n\}$ of $c-\varepsilon$.That is, $\sup\{g_i-f+c : i=1,2,...,n\} \geq c-\varepsilon$. This implies that $\sup\{g_i-f+c+f-c : i=1,2,...,n\} \geq c-\varepsilon+f-c=f-\varepsilon$. Consequently f is S-compact.Hence $(X,\omega(T_1),\omega(T_2))$ is S-C-compact.

(iii)⇒ (i) Let U be an S-open cover for (X,T_1,T_2).Then $G=\{\chi_u : u \in U\}$is a family of S-open fuzzy sets such that $\sup\{\chi_u : u \in U\} \geq 0.5$.Since 0.5 is compact ,then there exists a finite subfamily $\{\chi_{u_i} : u_i \in U, i=1,2,...,n\} \subseteq G$ such that $\sup\{\chi_{u_i} : u_i \in U, i=1,2,...,n\} \geq 0.4$.Now $\{u_i : i=1,2,...,n\}$ is a finite subcover of U for X.Indeed if $x \in X$ then there exists $i \in \{1,2,...,n\}$ such that $\chi_{u_i}(x)=1 \geq 0.5$ for some i.That is,$x \in \chi^{-1}_{u_i}(0.4,1]=u_i$.

The above result shows that S-C-compactness is a good extension of S-compactness.Now we present the following result concerning the goodness of P-compactness in bifuzzy topological spaces.

Theorem 4.2.5

If there exists c :$0<c\leq 1$ and c is P-compact in $(X,\omega(T_1),\omega(T_2))$ then (X,T_1,T_2) is P-compact.

Proof :

Let U be a P-open cover for (X,T_1,T_2),then $G=\{\chi_u:u\in U\}$is a family of P-open fuzzy sets such that $\sup\{\chi_u:u\in U\}\geq c$.Since c is compact ,then there exists a finite subfamily $\{\chi_{u_i}:u_i\in U,i=1,2,...,n\}\subseteq G$ such that $\sup\{\chi_{u_i}:u_i\in U,i=1,2,...,n\}\geq c/2$.Now $\{u_i:i=1,2,...,n\}$ is a finite subcover of U for X because if $x\in X$ then there exists $i\in\{1,2,...,n\}$ such that $\chi_{u_i}(x)=1\geq c$ for some i.That is , $x\in\chi^{-1}{u_i}(c/2,1]=u_i$.

Example 4.2.6

There exists a P-compact bitopological space (X,T_1,T_2) such that for no c;$0<c\leq 1$,c is P-compact in $(X,\omega(T_1),\omega(T_2))$.

Let $X=R$,$T_1=T_{l.r.}$and $T_2=T_{r.r.}$.In example 4.2.3 we have noticed that (X,T_1,T_2) is P-compact.Now we claim that for any $0<c\leq 1$,c is not P-compact .To prove our claim ,we notice that $\{\chi(-\infty,n)$ $,n\in N\}\cup\{(c/2)\chi(0,\infty)\}$ is a P-open cover for c (in fact P-open cover for 1) and admits no finite subcover for c-ε where ε=c/4.

The following is an immediate corollary of Theorem 4.2.5.

Corollary 4.2.7

If a bfts $(X,\omega(T_1),\omega(T_2))$ is P-C-compact then (X,T_1,T_2) is P-compact..

§ 4.3 Bifuzzy Lindelof spaces.

In this section we extend the concept of bifuzzy compactness to a wider class of bifuzzy topological spaces ,called bifuzzy Lindelof spaces and prove some results similar to those known bitopological ones.

Definition 4.3.1

A bfts (X,τ_1,τ_2) is said to be S-Lindelof (P-Lindelof) iff every S-open (P-open) cover for X has a countable subcover.

Definition 4.3.2

A bfts (X,τ_1,τ_2) is said to be S-α-Lindelof (P-α-Lindelof) iff every S-α–shading (P-α-shading) for X has a countable α-subshading.

Theorem 4.3.3

Every countable (i.e.,X is a countable set) bfts (X,τ_1,τ_2) is S-Lindelof.

Proof :

Let X be a countable set,i.e.,$X=\{x_i:i\in N\}$.

Let $C=\{V\alpha:\alpha\in\Delta\}$ be an open cover for X.For $i\in N,n\in N$ there exists $V(\alpha i,n)\in C$ such that $V_{(\alpha i,n)}(x_i)>1-(1/n)$.Since the countable union of countable sets is countable,therefore $\{V_{(\alpha i,n)}:i,n\in N\}$ is a countable subcover of C for X.

Theorem 4.3.4

A bfts (X,τ_1,τ_2) is S-Lindelof iff (X,τ_1,τ_2) is P-Lindelof and Lindelof.

Proof :

Similar to the proof of Theorem 4.1.9.

Definition 4.3.5

A fuzzy set f in a bfts (X,τ_1,τ_2) is said to be S-Lindelof (P-Lindelof) iff for every family G of S-open (P-open) fuzzy sets such that $\sup\{g:g\in G\} \geq f$ and for every $\varepsilon>0$,there exists a countable subfamily $G_\varepsilon \subseteq G$ such that $\sup\{g:g\in G_\varepsilon\} \geq f-\varepsilon$.

Now we have the following characterization of S-Lindelof fuzzy sets.

Theorem 4.3.6

A fuzzy set λ in a bfts (X,τ_1,τ_2) is S-Lindelof in the sense of definition 4.3.5 iff every S-open cover of λ has a countable subcover.

Proof :

$\Rightarrow$ Let λ be an S-Lindelof fuzzy set and let $G=\{V_\alpha:\alpha\in\Delta\}$ be an S-open cover for λ . Since λ is S-Lindelof ,then for each $n\in N$ there exists $Gn\subseteq G$ which is countable such that $\sup\{g:g\in G_n\}\geq\lambda-(1/n)$.It is clear that $\cup\{G_n:n=1,2,...\}$ is a countable subcover for λ.

$\Leftarrow$ Let $\{V_\alpha:\alpha\in\Delta\}$ be an open cover for λ.Then;by assumption; there exists a countable subcover $\{V_{\alpha i}:i=1,2,...\}$ for λ.Now for any $\varepsilon>0$,$\sup\{V_{\alpha i}:i=1,2,...\}\geq\lambda-\varepsilon$.Hence λ is S-Lindelof.

The following example shows that the above theorem does not hold in case of compactness.

Example 4.3.7

Let $X=I,\tau_1=\tau_2=\{c:0\leq c\leq 1\}$.The fuzzy set c=0.5 is S-compact in the sense of definition 4.1.17.But we note that the collection {c:c<0.5} is an open cover for c which has no finite subcover. Therefore there are no analogous definitions of S-weakly compact and S-C-compact in Lindelof spaces.

Definition 4.3.8

A bfts (X,τ_1,τ_2) is S-C-Lindelof (P-C-Lindelof) iff each constant fuzzy set in X is S-Lindelof (P-Lindelof).

Theorem 4.3.9

A bts $(X,\tau^*_{c1},\tau^*_{c2})$ is S-Lindelof if (X,τ_1,τ_2) is S-Lindelof.

Proof :

Let $M=\{A_\alpha:\alpha\in\Delta\}$ be an S- open cover for $(X,\tau^*_{c1},\tau^*_{c2})$.Then the family $G=\{\chi_{A_\alpha}:\alpha\in\Delta\}$ is a fuzzy S-open cover for (X,τ_1,τ_2).Since (X,τ_1,τ_2) is S-Lindelof.Then there exists a countable subfamily $\{\chi_{A_{\alpha i}}:\alpha\in\Delta,i=1,2,...\}$ of G that covers X.This implies that $\{A_{\alpha i}:i=1,2,..\}$ is a countable subcover of M for $(X,\tau^*_{c1},\tau^*_{c2})$.Hence $(X,\tau^*_{c1},\tau^*_{c2})$ is S-Lindelof.

The converse of the above theorem is not true in general as we see in the following example.

Example 4.3.10

There exists a non S-Lindelof bfts (X,τ_1,τ_2) such that $(X,\tau^*_{c1},\tau^*_{c2})$ is S-Lindelof.

Let $X=I,\tau_1=\tau_2=\{0,\lambda:\lambda(x)>0$ for all $x\in X\}$.Then $\tau^*_{c1}=\tau^*_{c2}=\{\phi,X\}$. That is, $(X,\tau^*_{c1},\tau^*_{c2})$ is S-Lindelof.

Define the fuzzy set U_p ;$p\in I$;as follows:

$U_p(p)=1$, $U_p(x)=0.5$ if $x\neq p$

Then the collection $\{U_p:p\in I\}$is a P-open cover for (X,τ_1,τ_2) which has no countable subcover.Hence (X,τ_1,τ_2) is not S-Lindelof.

Definition 4.3.11

A bfts (X,τ_1,τ_2) is hereditary Lindelof if each crisp fuzzy set is τ_1-Lindelof and a τ_2-Lindelof.

Example 4.3.12

Let $X=I,\tau_1=\{0,1\}$ and $\tau_2=\{0,\lambda:\lambda(x)>0$ for all $x\in X\}$.Then the bfts (X,τ_1,τ_2) is P-Lindelof but not hereditary Lindelof because if $\mu=\chi_{[0,0.5]}$, then the collection $\{\lambda_t:t\in I\}$; where $\lambda_t(x)=1$ if $x=t$ and $\lambda_t(x)=0.5$ if $x\neq t$: is a τ_2-open cover for μ which does not have a countable subcover.

Theorem 4.3.13

If a bfts (X,τ_1,τ_2) is hereditary Lindelof ,then it is S-Lindelof.

Proof :

Let $C=\{V_\alpha:\alpha\in\Delta\}$ be an S-open cover of X.We have two cases to consider.The first case is whenever C is a τ_1 or a τ_2-open cover for X.Then C is an open cover of the crisp constant fuzzy set 1.Since X is τ_1-Lindelof and τ_2-Lindelof,hence C has a countable subcover.The second case is if C is a P-open cover for X,then $C=\{\lambda_\alpha:\alpha\in\Delta\}\cup\{\mu_\alpha:\alpha\in\Delta\}$,where $\lambda_\alpha\in\tau_1$ and $\mu_\alpha\in\tau_2$. let $A=\{x\in X:\cup\lambda_\alpha(x)=1\}$ and

$B=\{x\in X:\cup\mu_\alpha(x)=1\}$ .Since $C_1=\{\lambda_\alpha:\alpha\in\Delta\}$ is a τ_1-open cover for χ_A and $C_2=\{\mu_\beta:\beta\in\Delta\}$ is a τ_2-open cover for χ_B and X is hereditary Lindelof, then there exist countable subcovers C_1' of C_1 and C_2' of C_2 .It follows that $C_1'\cup C_2'$ is a countable subcover for $\chi_A\cup\chi_B=\chi_{A\cup B}=\chi_X=1$.

Definition 4.3.14

A fuzzy set λ in a fts (X,τ) is called weakly open if for each $p\in\lambda$,there exists $\mu\in\tau$ such that $p\in\mu$ and $\mu(x)\leq\lambda(x)$ for all $x\in X$ except for countably many $x\in X$.A fuzzy set F is called weakly closed if F^c is weakly open.

Definition 4.3.15

A fuzzy set λ in a bfts (X,τ_1,τ_2) is called weakly open iff λ is weakly open with respect to τ_1 and with respect to τ_2 simultaneously.A fuzzy set F is called weakly closed if F^c is weakly open.

Theorem 4.3.16

Let (X,τ_1,τ_2) be a bft P-Lindelof space and F a weakly closed proper crisp fuzzy set in (X,τ_1).Then F is τ_2-Lindelof.

Proof :

Let $\{V_\alpha:\alpha\in\Delta\}$ be a τ_2-open cover of F.Since F is τ_1-weakly closed, then F^c is τ_1-weakly open and therefore for each $p\in F^c$ there exists μ_p in τ_1 such that $p\in\mu_p$ and $\mu_p(x)\leq F^c(x)$ except for countably many $x\in X$.Let $C=\{V_\alpha:\alpha\in\Delta\}\cup\{\mu_p:p\in F^c\}$.Then C is a P-open cover for X.Since X is P-Lindelof,C has a countable subcover ,say $C_1=\{V_{\alpha_n}:\alpha_n\in\Delta\}\cup\{\mu_{p_k}:p_k\in F^c\}$.Since $\mu_{p_k}(x)=0$ for all $x\in F$ except countably many,let $\{x_1,x_2,...\}$ be a collection of points in F such that $\mu_{p_k}(x_i)>0$ for all i=1,2,... .Now fix x_i in F.Since $\{V_\alpha:\alpha\in\Delta\}$ is a family of open cover,there exists a countable subfamily $\{V_{\alpha_{kji}}\}$ such that sup $V_{\alpha_{kji}}(x_i)=1$.Then for every x_i (i=1,2,...) there exists a countable subfamily $\{V_{\alpha_{kji}}\}$ such that sup $V_{\alpha_{kji}}(x_i)=1$ Since μ_{p_k} is countable and the union of countable sets is countable,therefore the collection $\{V_{\alpha_{kji}}\}$ of V_α together with $\{V_{\alpha_n}\}$ is countable and so F is τ_2-lindelof.

Corollary 4.3.17

A τ_i-weakly closed proper crisp fuzzy set of a P-Lindelof space is τ_j -Lindelof ($i\neq j$,i,j=1,2.).

It is important to notice that the word "proper" in the above theorem can not be removed.

Example 4.3.18

Let X=I,$\tau_1=\{0,1\}$ and $\tau_2=\{0,\lambda:\lambda(x)>0$ for all $x\in X\}$.Then (X,τ_1,τ_2) is P-Lindelof having 1 as a τ_1-closed fuzzy set that is not τ_2-Lindelof.

Definition 4.3.19

A bfts (X,τ_1,τ_2) in which every countable intersection of τ_1-open (τ_2-open) fuzzy sets is in $\tau_1(\tau_2)$ is called a P-space .

Example 4.3.20

Let X=I,$\tau_1=\{0,1\}$ and $\tau_2=\{0,\lambda:\lambda(x)\geq0.5$ for all $x\in X\}$.Then (X,τ_1,τ_2) is a bfts which is a P-space.

Theorem 4.3.21

If a bfts (X,τ_1,τ_2) is P-T_{2w} and P-space then every τ_i-Lindelof crisp fuzzy set is τ_j-closed $(i\neq j;i,j=1,2)$.

Proof :

Let F be a τ_i-Lindelof crisp fuzzy set of X and $p\in F^c$,where p is a mature fuzzy point.Then x_p=supp $\cap\{cl_i\mu_\alpha:\mu_\alpha$ is a τ_j-neighborhood of $p\}$ because X is P-T_{2w} (see theorem 2.3.6) .Now $F\subseteq p^c$,therefore $C=\{(cl_i\mu_\alpha)^c:\alpha\in\Delta\}$ is a τ_i-open cover of F .Since F is τ_i-Lindelof then there exists $\Delta_1\subseteq\Delta$ such that $C'=\{(cl_i\mu_\alpha)^c:\alpha\in\Delta_1\}$ is countable and $F\subseteq\cup C'$.Let $U=\cap\mu_\alpha$.Then U is a τ_j-open fuzzy set containing the fuzzy point p and $U\subseteq F^c$.That is, $p\in U\subseteq 1-F$.Hence F^c is a τ_j-open fuzzy set which implies that F is τ_j-closed.

Definition 4.3.22

If A is a fuzzy set of a bfts (X,τ_1,τ_2) and p is a fuzzy point ,then p is called a weak-interior point of A if there exists a weakly open fuzzy set $B\in\tau_1\cup\tau_2$ containing p such that $B\subseteq A$.The set of all weak -interior points of a set A is denoted by w-int A.

Theorem 4.3.23

Let (X,τ_1,τ_2) be a P-Lindelof bfts and let F be a τ_1-weakly-closed fuzzy set such that w-$int_2(F^c)\neq\phi$.Then F is τ_1-Lindelof.

Proof:

Let $q\in$ w-$int_2(F^c)$.Then there exists a τ_2- open fuzzy set G such that $q\in G$ and $G\subseteq F^c$ except for countably many $x\in X$.Let $C=\{c_\alpha:\alpha\in\Delta\}$ be a τ_1-open cover for F.For each $p\in F^c$,there exists a τ_1-open fuzzy set μ_p such that $p\in\mu_p$ and $\mu_p\subseteq F^c$ except for countably many x's in X because F^c is a τ_1-weakly open..Since $\{c_\alpha:\alpha\in\Delta\}\cup\{G\}\cup\ \{\mu_p:p\in F^c\}$ is a P-open cover for the P-Lindelof space X,there exist two countable sets $\Delta_1\subseteq\Delta$ and $\Delta_2=\{p_1,p_2,...\}$ such that $\{c_\alpha:\alpha\in\Delta_1\}\cup\{G\}\cup$ $\{\mu_p:p\in\Delta_2\}$ is a countable subcover for X.Let $\{x_1,x_2,...\}$,$\{z_1,z_2,...\}$ be a collection of points in F such that $\mu_p(x_i)>0$ and $G(z_i)>0$ for all i=1,2,... .Now fix x_i and z_i.Since $\{c_\alpha:\alpha\in\Delta\}$ is a family of a τ_1-open cover for F,there exist countable subfamilies $\{c_{\alpha_j}\}$ and$\{c_{\beta_j}\}$ of $\{c_\alpha:\alpha\in\Delta_1\}$ such that sup $c_{\alpha_j}(x_i)=1$ and sup $c_{\beta_j}(z_i)=1$ .Since Δ_2 is countable and the union of countable sets is countable,then $\{c_\alpha:\alpha\in\Delta_1\}\cup\{c_{\alpha j}:j\in N\}\cup\{c_{\beta_j}:j\in N\}$ is a countable subcover of C.Hence F is τ_1-Lindelof.

Theorem 4.3.24

Let X,Y be two bft spaces and $f:X\rightarrow Y$ be a P-continuous surjection.

(i) If X is S-Lindelof then Y is S-Lindelof.

(ii) If X is S-α- Lindelof then Y is S-α- Lindelof.

Proof :

Because (i) is similar to (ii),we shall prove only (ii).Let C be an S-α-shading for Y.Since if $x\in X$ then $f(x)\in Y$, there exists $\lambda\in C$ such that $\lambda(f(x))>\alpha$,that is $f^{-1}(\lambda)(x)>\alpha$,and hence $f^{-1}(C)=\{f^{-1}(\lambda):\lambda\in C\}$ is an S-α-shading for X. Consequently $\{f^{-1}(\lambda):\lambda\in C\}$ has a countable α-subshading $\{f^{-1}(\lambda_i):\ i=1,2,...\}$.Now $\{\lambda_i:i=1,2,...\}$ is a countable α -subshading of C because if $y\in Y$ then $y=f(x)$ for some $x\in X$.Thus there exists j such that $f^{-1}(\lambda_j)(x)> \alpha$.This implies that $\lambda_j(f(x))=\lambda_j(y)>\alpha$.

Theorem 4.3.25

Let X,Y be two bft spaces and $f:X\to Y$ be a continuous surjection. Then:

(i) If X is P-Lindelof then Y is P-Lindelof.

(ii) If X is P-α- Lindelof then Y is P-α- Lindelof.

Proof :

We shall prove (i) only because (ii) is similar to (i).

Let $C=\{V_\alpha:\alpha\in\Delta\}$ be a P-open cover for Y.Then $f^{-1}(C)=\{f^{-1}(V_\alpha): V_\alpha\in C\}$ is a P-open cover for X.Indeed, if $x\in X$ then $f(x)\in Y$,so there exists a subfamily $V_{\alpha i}$ of C such that sup $V_{\alpha i}(f(x))=1$.Hence $\{f^{-1}(V_\alpha):V_\alpha\in C\}$ has a countable open cover $\{f^{-1}(V_i): i=1,2,...\}$.Now $\{V_i:i=1,2,...\}$ is a countable subcover of C.Indeed if $y\in Y$ then $y=f(x)$ for some $x\in X$.Thus there exists a subfamily $f^{-1}(V_j)$ of $f^{-1}(C)$ such that sup $f^{-1}(V_j)(x)=1$.

We note that if f is P-continuous then the above theorem does not hold.This is because $f^{-1}(C)=\{f^{-1}(V_\alpha):V_\alpha\in C\}$ is not necessarily a P-open cover of X.It is in fact S-open cover but not necessarily a P-open cover.The following example clarifies our claim.

Example 4.3.26

Let $I_d:(R,\tau_{dis},\tau_{ind}) \to(R,\tau_{dis},\tau_{dis})$ denote the identity map.Then I_d is P-continuous and not continuous surjection.In fact $(R,\tau_{dis},\tau_{ind})$ is P-

Lindelof and P-α-Lindelof while $(R,\tau_{dis},\tau_{dis})$ is neither P-Lindelof nor P-α-Lindelof.Moreover if C denotes the collection of all fuzzy points in Y ,then C is a P-open cover for Y but $f^{-1}(C)$ is an S-open cover for X which is not a P-open cover because $1\notin f^{-1}(C)$.

Theorem 4.3.27

(i) Every $\tau_1\tau_2$-closed subspace of an S-Lindelof bfts is S-Lindelof.

(ii)Every $\tau_1\tau_2$-closed subspace of a P-Lindelof bifuzzy space is P-Lindelof.

Proof :

(i) Let A be a $\tau_1\tau_2$-closed subspace of the S-Lindelof bfts (X,τ_1,τ_2).Let$\{\lambda_\alpha:\alpha\in\Delta\}$ be a $\tau_{1A}\tau_{2A}$-open cover of A.Then for each λ_i, $i\in I$ there exists a $\mu_i\in\tau_1\cup\tau_2$ such that $\lambda_i=\mu_i\cap\chi_A$.Since χ_A is a $\tau_1\tau_2$-closed fuzzy set of X,then $\zeta=\{\mu_i:i\in I\}\cup\{\chi_A{}^c\}$ is an S-open cover of X.Since X is S-Lindelof ,then ξ has a countable subcover say $\{\mu_{ik}:k\in N\}\cup\{\chi_A{}^c\}$.Consequently $\{\lambda_{ik}:k\in N\}$ is a countable subcover for A.Therefore (A,τ_{1A},τ_{2A}) is S-Lindelof.

(ii) Proof : Similar to (i).

§ 4.4 Countability in bifuzzy topological spaces.

Definition 4.4.1

Let (X,T_1,T_2) be a bts and $x\in X$.A subfamily B_x of S-open (P-open) sets is called an S-local base (P-local base) at x iff

1) $x\in U$ for every $U\in B_x$.

2) For every $V \in T_1 \cup T_2$ such that $x \in V$ there exists $W \in B_x$ such that $x \in W \subseteq V$.

Definition 4.4.2

Let (X,τ_1,τ_2) be a bfts and p a fuzzy point in X.A subfamily B_p of S-open (P-open) fuzzy sets is called an S-local base (P-local base) at p iff

(1) $p \in \mu$ for every $\mu \in B_p$.

(2) for every $\lambda \in \tau_1 \cup \tau_2$ such that $p \in \lambda$ there exists $\mu \in B_p$ such that $p \in \mu \subseteq \lambda$.

Definition 4.4.3

A bts (X,T_1,T_2) is called an S-C_I-space (P-C_I-space) iff every $x \in X$ has a countable S-Local base (P-local base).

Definition 4.4.4

A bfts (X,τ_1,τ_2) is called an S-C_I-space (P-C_I-space) iff every fuzzy point in X has a countable S-local base (P-local base).

Definition 4.4.5

Let (X,τ_1,τ_2) be a bfts. A subfamily B of S-open (P-open) fuzzy sets is called an S-base (P-base) for $\tau_1 \cup \tau_2$ iff for each fuzzy point p in X and for each $\lambda \in \tau_1 \cup \tau_2$ such that $p \in \lambda$ there exists a member $\mu \in B$ such that $p \in \mu \subseteq \lambda$.

Remark 4.4.6

If B_i is a base for τ_i,then $B_1 \cup B_2$ is an S-base for $\tau_1 \cup \tau_2$.

Theorem 4.4.7

If $C \subseteq \tau_1 \cup \tau_2$ and B_i (i=1,2) is a collection of fuzzy sets such that $C \cap B_1$ is a base for τ_1 and $C \cap B_2$ is a base for τ_2.Then C is an S- base for $\tau_1 \cup \tau_2$.

Proof:

Let $0 \neq \lambda \in \tau_1 \cup \tau_2$ and p be a fuzzy point such that $p \in \lambda$.Now $\lambda \in \tau_1$ or $\lambda \in \tau_2$ .If $\lambda \in \tau_1$ then there exists $O_1 \in C \cap B_1$ such that $p \in O_1 \subseteq \lambda$. Similarly if $\lambda \in \tau_2$ then there exists $O_2 \in C \cap B_2$ such that $p \in O_2 \subseteq \lambda$.Hence C is an S-base for $\tau_1 \cup \tau_2$.

Remark 4.4.8

The converse of the above theorem is not true in general as we see in the following example,

let $X=R, \tau_1=\{0,1,\chi_{(-\infty,0)}\}=B_1$ and $\tau_2=\{0,1,\chi_{[0,\infty)}\}=B_2$.

Now $\{\chi_{(-\infty,0)}, \chi_{[0,\infty)}\}$ is an S-base for $\tau_1 \cup \tau_2$ but neither $\{\chi_{(-\infty,0)}\}$ nor $\{\chi_{[0,\infty)}\}$ is a base for τ_1 or τ_2 .

Definition 4.4.9

A bfts (X,τ_1,τ_2) is called an S-C_{II}-space (P-C_{II}-space) iff (X,τ_1,τ_2) has a countable S-base (P-base) for $\tau_1 \cup \tau_2$.

Theorem 4.4.10

A bfts (X,τ_1,τ_2) is an S-C_{II}-space if (X,τ_1) and (X,τ_2) are C_{II}-space.

Proof:

If B_1 and B_2 are countable bases for τ_1 and τ_2 respectively, then $B_1 \cup B_2$ is a countable S-base for $\tau_1 \cup \tau_2$.

Remark 4.4.11

The converse of the above theorem is not true in general as we see in the following example.

let $X=R\setminus\{0\}, B_1=\{\chi_A, A\subseteq(-\infty,0):(-\infty,0)-A$ is finite$\}\cup\{\chi_{(a,b)}, a,b>0\}$ and $B_2=\{\chi_{(a,b)}:a,b<0\}$ $\cup$ $\{\chi_B, B\subseteq(0,\infty)$: $(0,\infty)-B$ is finite$\}$,then $(X,\tau_1(B_1))$ and $(X,\tau_2(B_2))$ are not C_{II}-spaces,but $(X,\tau_1(B_1),\tau_2(B_2))$ is an S-C_{II}-space because $B=\{\chi_{(a,b)}$:a,b rationals,ab>0$\}$ is an S-base for $\tau_1(B_1)\cup\tau_2(B_2)$.

Definition 4.4.12.

Let (X,τ_1,τ_2) be a bfts. A subfamily L of S-open (P-open) fuzzy sets is called an S- subbase (P-subbase) for $\tau_1\cup\tau_2$ iff the family $C=\{\cap B$: B is a finite subfamily of L$\}\cup\{1\}$ is an S-base (P-base) for $\tau_1\cup\tau_2$.

Definition 4.4.13

Let D be a collection of fuzzy points in X.Then D is dense in (X,τ_1,τ_2) if for every $0\neq\lambda\in\tau_1\cup\tau_2$,there exists $p\in D$ such that $p\in\lambda$.

Definition 4.4.14.

A bfts (X,τ_1,τ_2) is called bifuzzy separable if it contains a countable dense set .

Theorem 4.4.15

Let $f:(X,\tau 1,\tau 2)\to(Y,\sigma 1,\sigma 2)$ be a P-continuous P-open surjection.Then:

1) If X is S-C_I then Y is S-C_I.

2) If X is S-C_{II} then Y is S-C_{II}.

Proof:

1) Let $(X,\tau 1,\tau 2)$ be a bifuzzy S-C_I-space and q_1 be a fuzzy point in Y with support x_q and value y_1.Let q be a fuzzy point with support x_q and value $y=(y_1+1)/2$.Then $f^{-1}(q)$ is a fuzzy set in X;not necessarily a fuzzy point; such that

$$f^{-1}(q)(x)=\begin{cases} q(x_q)=y & \text{for all} \quad x\in f^{-1}(x_q) \\ 0 & \text{otherwise.} \end{cases}$$

Define a fuzzy point p in X as follows:

$$p(x)=\begin{cases} y & \text{if } x=x0 \\ 0 & \text{otherwise} \end{cases}$$

where $x0$ is a fixed point in $f^{-1}(x_q)$.Clearly $p\in f^{-1}(q)$ and $f(p)=q$.By assumption,p has a countable S-Local base in $(X,\tau 1,\tau 2)$,say B_p.Let $V_q=\{f(\lambda):\lambda\in B_p\}$.Then V_q forms a countable S-local base at q because $f(\lambda)\in\sigma_1\cup\sigma 2$ for all $\lambda\in B_p$ If $\mu\in\sigma_1\cup\sigma 2$ such that $q\in\mu$,then $f^{-1}(\mu)\in\tau_1\cup\tau 2$ and $p\in f^{-1}(\mu)$.Thus there exists $\lambda\in B_p$ such that $p \in \lambda \subseteq f^{-1}(\mu)$.Consequently $q \in f(\lambda) \subseteq \mu$.But $q_1 \in q$,therefore $q_1 \in f(\lambda) \subseteq \mu$ .That is,V_q forms a countable S-local base at q_1.Hence (Y,σ_1,σ_2) is a bifuzzy S-C_I-space.

2) Let B be a countable S-base for $\tau_1 \cup \tau_2$.Since f is P-open therefore $f(\mu) \in \sigma_1 \cup \sigma_2$ whenever $\mu \in B$.Consequently the family $\{f(\mu):\mu \in B\}$ forms a countable S-base for $\sigma_1 \cup \sigma_2$ .To see this let $\lambda \in \sigma_1 \cup \sigma_2$ and $p \in \lambda$.Then $f^{-1}(\lambda) \in \tau_1 \cup \tau_2$ because f is P-continuous.If q is a fuzzy point in X such that f(q)=p,then $q \in f^{-1}(\lambda)$ and so there exists $\mu \in B$ such that $q \in \mu \subseteq f^{-1}(\lambda)$ which gives $f(q) \in f(\mu) \subseteq \lambda$.That is $p \in f(\mu) \subseteq \lambda$.Hence Y is a bifuzzy S-$C_{II}$-space.

Theorem 4.4.16

Let $f:(X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ be a P-continuous surjection and X a bifuzzy separable space.Then Y is bifuzzy separable.

Proof :

Let $D=\{p_i:i=1,2,...\}$ be a countable dense subset of (X,τ_1,τ_2).Now the set $\{f(p_i):i=1,2,...\}$ forms a countable set of fuzzy points in Y.Let $0 \neq \mu \in \sigma_1 \cup \sigma_2$.Hence there exists a fuzzy point p_i such that $p_i \in f^{-1}(\mu)$. Notice that $f^{-1}(\mu) \in \tau_1 \cup \tau_2$ because f is P-continuous.Consequently $f(p_i) \in \mu$.Therefore (Y,σ_1,σ_2) is bifuzzy separable.

Theorem 4.4.17

Every bifuzzy S-C_{II}-space is bifuzzy S-C_I.

Proof:

Let (X,τ_1,τ_2) be a bft S-C_{II}-space and let p be a fuzzy point in X.By assumption,$\tau_1 \cup \tau_2$ has a countable S-base B.Let B_0 be a subfamily of B defined by $B_0=\{\mu:\mu\in B, p\in\mu\}$.Then B_0 is countable Let $\lambda\in\tau_1\cup\tau_2$ such that $p\in\lambda$.Then there exists an element μ of B such that $p\in\mu\subseteq\lambda$ because B is S-base.By definition of B_p we have μ is an element of B_p.Hence B_p is a countable S-local base for $\tau_1\cup\tau_2$ at p.Therefore (X,τ_1,τ_2) is S-C_I-space.

Theorem 4.4.18

Every S-C_{II}-space is bifuzzy separable .

Proof:

Let (X,τ_1,τ_2) be a bft S-C_{II}-space.So it has a countable S-base ,say $B=\{\mu_i:i=1,2,...\}$.For $\mu_i\neq 0$,there exists a point $x_i\in X$ such that $\mu_i(x_i)>0$.Define a fuzzy point p_i as follows:

$$p_i(x)=\begin{cases}(1/2)\mu_i(x_i) & \text{for } x=x_i\\ 0 & \text{otherwise}\end{cases}$$

Clearly $p_i\in\mu_i$.The countable set $\{p_i:i=1,2,...\}$ is the required set for (X,τ_1,τ_2) to be bifuzzy separable because any member $\lambda\neq 0$ of $\tau_1\cup\tau_2$ will contain a member of B,say $\mu_i\subseteq\lambda$.Consequently $p_i\in\lambda$.Therefore (X,τ_1,τ_2) is bifuzzy separable.

Definition 4.4.19

Let (X,τ_1,τ_2) be a bfts and A be an arbitrary subset of X.We have defined the relative topologies for A by $\tau_{iA}=\{\chi_A \cap \lambda : \lambda \in \tau_i\}$ (i=1,2).A subspace (A,τ_{1A},τ_{2A}) is called a $\tau_1\tau_2$-open ($\tau_1\tau_2$-closed) subspace iff the fuzzy set χ_A is $\tau_1\tau_2$-open ($\tau_1\tau_2$-closed).

Theorem 4.4.20

(i) Every subspace of S-C_{II} space is an S-C_{II} space.

(ii)Every subspace of S-C_I space is an S-C_I space.

Proof :

(i) Let A be a subset of the bifuzzy S-C_{II} space (X,τ_1,τ_2).Let $B=\{\mu_i : i\in N\}$ be a countable S-base for $\tau_1 \cup \tau_2$.Then it is clear that $B'=\{\chi_A \cap \mu_i : i\in N\}$ is a countable S-base for the fuzzy subspace (A,τ_{1A},τ_{2A}).Hence (A,τ_{1A},τ_{2A}) is C_{II}-space.

(ii)Let A be a subset of the bifuzzy S-C_I space (X,τ_1,τ_2) and let p be a fuzzy point such that $p\in\chi_A$.Let $B=\{\mu_i : i\in N\}$ be a countable S-local base at p.Then it is clear that $B'=\{\chi_A \cap \mu_i : i\in N\}$ is a countable S-local base for the bifuzzy subspace (A,τ_{1A},τ_{2A}).Hence (A,τ_{1A},τ_{2A}) is C_I-space.

Theorem 4.4.21

Every open subspace of a bifuzzy separable space is bifuzzy separable.

Proof :

Let A be any $\tau_1\tau_2$-open subspace of a bifuzzy separable space (X,τ_1,τ_2).Let $\{p_i:i\in N\}$ be a set of fuzzy points such that for every non-zero fuzzy set $\mu\in\tau_1\cup\tau_2$ there exists some p_i which is a proper subset of μ.Since A is a $\tau_1\tau_2$-open subspace,every $\tau_{1A}\tau_{2A}$-open fuzzy set λ is $\tau_1\tau_2$-open.Consequently for every $\lambda\in\tau_{1A}\cup\tau_{2A}$ there is some $i\in N$ such that $p_i\in A$.Hence $\{p_i:i\in N\}$ is a countable set of fuzzy points in A such that for every non-zero $\tau_{1A}\tau_{2A}$-open bifuzzy set λ there is an i for which $p_i\in\lambda$ holds.

In the following theorems we show that our definitions of S-C_I and S-C_{II} bifuzzy topological spaces are good extensions.

Theorem 4.4.22

A bts (X,T_1,T_2) is S-C_{II} iff the induced bfts $(X,\omega(T_1),\omega(T_2))$ is S-C_{II}.

Proof:

$\Rightarrow$ Let $B=\{u_1,u_2,...\}$ be a countable S-base for $T_1\cup T_2$.Then the collection $A=\{A_{q_i}: q_i\in Q\cap(0,1],i=1,2,...\}$;where

$$A_{q_i}(x)=\begin{cases} q_i & \text{if } x\in u_i\in B \\ 0 & \text{if } x\notin u_i\in B \end{cases}$$

is a countable S-base for $(X,\omega(T_1),\omega(T_2))$ .To show this we note first that A is a collection of lower semi-continuous functions and so

$A \subseteq \omega(T_1) \cup \omega(T_2)$.Moreover let p be a fuzzy point and $\lambda \in \omega(T_1) \cup \omega(T_2)$ such that $p \in \lambda$.So there exists $q \in Q \cap (0,1]$ such that $p(x_p) < q < \lambda(x_p)$.Now $\lambda^{-1}(q,1] \in T_1 \cup T_2$ and so there exists a basic open set $u_i \in B$ such that $x_p \in u_i \subseteq \lambda^{-1}(q,1]$.Hence there exists $A_{qi} \in A$ such that $p \in A_{qi} \subseteq \lambda$ for some i.

$\Leftarrow$ Let $A = \{\lambda_n : n \in N\}$ be a countable fuzzy S-base for $(X, \omega(T_1), \omega(T_2))$.Then the collection $B = \{(\lambda_n)^{-1}(q,1]: q \in Q \cap (0,1], n \in N\}$ is a countable S-base for $T_1 \cup T_2$ .Let $u \in T_1 \cup T_2$ and $x \in u$.So χ_u is a l.s.c function.Hence $\chi_u \in \omega(T_1) \cup \omega(T_2)$.Hence there exists a basic open set λ such that $\lambda \subseteq \chi_u$.Thus $x \in \lambda^{-1}(q,1] \subseteq u$.

Theorem 4.4.23

A bts (X,T_1,T_2) is S-C_I iff the induced bfts $(X,\omega(T_1),\omega(T_2))$ is S-C_I.

Proof :

$\Rightarrow$ Let p be a fuzzy point in X with support x_p.Let $B=\{u_n : n \in N\}$be a countable S-local base at x_p.Since $\chi_{u_n} \in \omega(T_1) \cup \omega(T_2)$ for all n,then $\{\varepsilon\chi_{u_n} : 0<\varepsilon<1, \varepsilon \in Q$ and $u_n \in B\}$ forms a countable S-local base at p because if $G \in \omega(T_1) \cup \omega(T_2)$ such that $p \in G$.Then $p(x_p) < \varepsilon < G(x_p) \leq 1$ and hence $x_p \in G^{-1}(\varepsilon,1] \in T_1 \cup T_2$.Thus there exists U_n such that $x_p \in u_n \subseteq G^{-1}(\varepsilon,1]$.Hence $p \in \varepsilon\chi_{u_n} \subseteq G$.

$\Leftarrow$ Let $x \in X$ and p be a fuzzy point with support x. Let $C=\{\lambda_n : n \in N\}$ be a countable local base at p .Hence $B=\{\lambda_n^{-1}(q,1] : q \in Q \cap (0,1]\}$ is a countable S-local base at $x \in X$,where $q<p(x)$.

Chapter V

INDUCED AND WEAKLY INDUCED BIFUZZY TOPOLOGICAL SPACES

The concept of induced fuzzy topological spaces was introduced by Weiss (1975).Lowen (1976,1977) established the relation between fuzzy topological spaces and topological spaces by introducing the two functors ω:TOP→FTOP and L: FTOP→TOP,where TOP and FTOP denote the categories of topological spaces and fuzzy topological spaces respectively.Martin (1980) defined and studied the concept of weakly induced fuzzy topological spaces .Since then several authors continued the investigation of such spaces .

In this chapter we shall introduce the concept of induced and weakly induced bifuzzy topological spaces and study some important results regarding Hausdorff and compactness properties .For the sake of clarity we divide this chapter into two sections .In the first section we shall discuss induced and weakly induced bifuzzy topological spaces while in the second we prove some important results.

§ 5.1 Induced and weakly induced bifuzzy topological spaces

To start this section with the first result in fixed point theory,we need the following proposition.

Proposition 5.1.1

If f is a P-continuous function from (X,τ_1,τ_2) into (Y,σ_1,σ_2),then f is continuous as a function from $(X,<\tau_1,\tau_2>)$ into $(Y,<\sigma_1,\sigma_2>)$.

Proof:

Let μ be a subbasic open fuzzy set in $(Y,<\sigma_1,\sigma_2>)$ then $\mu\in\sigma_1\cup\sigma_2$ and so $f^{-1}(\mu)\in\tau_1\cup\tau_2$ but $\tau_1\cup\tau_2\subseteq <\tau_1,\tau_2>$,therefore f is a continuous function from $(X,<\tau_1,\tau_2>)$ into $(Y,<\sigma_1,\sigma_2>)$.

Theorem 5.1.2

If (X,τ_1,τ_2) is a bfts such that $(X,<\tau_1,\tau_2>)$ has the f.p.p ,then (X,τ_1,τ_2) has the P-f.p.p.

Proof:

Let $f:(X,\tau_1,\tau_2)\rightarrow(X,\tau_1,\tau_2)$ be a P-continuous function ,then from proposition 5.1.1 $f:(X,<\tau_1,\tau_2>)\rightarrow(X,<\tau_1,\tau_2>)$ is continuous and so f has a fixed point .Hence (X,τ_1,τ_2) has the P-f.p.p.

Proposition 5.1.3

Let $f:X\rightarrow X$ be a function .Then the following are equivalent:

(i) f: $(X,T_1,T_2)\rightarrow(X,T_1,T_2)$ is P-continuous.

(ii) f: $(X,\omega(T_1),\omega(T_2))\rightarrow(X,\omega(T_1),\omega(T_2))$ is P-continuous.

(iii) $f: (X,X/T_1,X/T_2)\rightarrow(X,X/T_1,X/T_2)$ is P-continuous.

Proof :

(i)$\Rightarrow$(ii) Let $\mu\in\omega(T_1)\cup\omega(T_2)$ .We are going to show that $f^{-1}(\mu)\in\omega(T_1)\cup\omega(T_2)$.Now $\mu:(X,T_i)\rightarrow(I,T_{r.r})$ is continuous and so $\mu^{-1}(0,1]\in T_1\cup T_2$ and so we have by (i) $f^{-1}(\mu^{-1}(0,1])\in T_1\cup T_2$ but $f^{-1}(\mu^{-1}(0,1])=[f^{-1}(\mu)]^{-1}(0,1])\in T_1\cup T_2$.Hence $f^{-1}(\mu):(X,T_i)\rightarrow(I,T_{r.r})$ is continuous and so $f^{-1}(\mu)\in\omega(T_1)\cup\omega(T_2)$ which completes the proof.

(ii)$\Rightarrow$(iii) Let $\chi_u\in X/T_1\cup X/T_2$.Then $\chi_u\in\omega(T_1)\cup\omega(T_2)$ and so by (ii) $f^{-1}(\chi_u)\in\omega(T_1)\cup\omega(T_2)$.That is $f^{-1}(\chi_u):(X,T_i)\rightarrow(I,T_{r.r})$ is continuous. Therefore $[f^{-1}(\chi_u)]^{-1}(0,1]\in T_1\cup T_2$ but $[f^{-1}(\chi_u)]^{-1}(0,1]=[\chi_{f-1(u)}]^{-1}(0,1]\in T_1\cup T_2$ which implies that $\chi_{f-1(u)}:(X,T_i)\rightarrow(I,T_{r.r})$ is continuous.Hence $\chi_{f-1(u)}\in X/T_1\cup X/T_2$.That is $f^{-1}(\chi_u)\in X/T_1\cup X/T_2$.

(iii)$\Rightarrow$(i) Let $u\in T_1\cup T_2$.Then $\chi_u\in X/T_1\cup X/T_2$ and so $f^{-1}(\chi_u)\in X/T_1\cup X/T_2$ but $f^{-1}(\chi_u)=\chi_{f^{-1}(u)}\in X/T_1\cup X/T_2$ which implies that $f^{-1}(u)\in T_1\cup T_2$.Hence f is P-continuous.

Theorem 5.1.4

Let (X,T_1,T_2) be a bts .Then

(1) (X,T_1,T_2) has P-f.p.p iff $(X,\omega(T_1),\omega(T_2))$ has the P-f.p.p .

(2) (X,T_1,T_2) has P-f.p.p iff $(X,X/T_1,X/T_2)$ has the P-f.p.p .

Proof :

(1) Let f: $(X,\omega(T_1),\omega(T_2))\rightarrow(X,\omega(T_1),\omega(T_2))$ be P-continuous.We are going to prove that f has a fixed point .Using proposition 5.1.3, f: $(X,T_1,T_2)\rightarrow(X,T_1,T_2)$ is P-continuous and hence f has a fixed point. Similarly we treat the other implication.

(2) The proof is similar to (1).

Before we procced to the next definition we refer the reader to the definitions 1.3.8 and 1.3.9 regarding $L(\tau)$ and $\tau_c{}^*$ which are due to Lowen (1976) and Martin (1980) respectively.

Definition 5.1.5

A bfts (X,τ_1,τ_2) is said to be P-induced iff $\tau_i=\omega(\tau^*{}_{cj})$,i,j=1,2 ($i\neq j$).

Definition 5.1.6

A bfts (X,τ_1,τ_2) is said to be P-weakly induced iff $\tau_i\subseteq\omega(\tau^*{}_{cj})$,i,j=1,2 ($i\neq j$).

Example 5.1.7

If X=I and $\tau_1=\{0,1\},\tau_2=\{0,1,0.5\}$ then $\tau^*{}_{c1}=\tau^*{}_{c2}=\{\phi,X\}$ and $\omega(\tau^*{}_{c1})=\omega(\tau^*{}_{c2})=\{c:0\leq c\leq 1\}$ which gives that (X,τ_1,τ_2) is weakly induced and P-weakly induced but neither induced nor P-induced.

Theorem 5.1.8

If (X,T_1,T_2) is a bitopological space such that the bfts $(X,X/T_1,X/T_2)$ is P-weakly induced,then $T_1=T_2$.

Proof :

Let $u \in T_1$, then $\chi_u \in X/T_1$. Since $(X, X/T_1, X/T_2)$ is P-weakly induced, then $X/T_1 \subseteq \omega((X/T_2)^*{}_c)=\omega(T_2)$ which implies $(\chi_u)^{-1}(0.5,1] \in T_2$. That is $u \in T_2$. Consequently $T_1 \subseteq T_2$. Similarly we can show $T_2 \subseteq T_1$.

Theorem 5.1.9

If a bfts (X,τ_1,τ_2) is P-weakly induced then $\omega(\tau^*{}_{c1})=\omega(\tau^*{}_{c2})$ but the converse is not true in general .

Proof :

Let $\lambda \in \omega(\tau^*{}_{c1})$ and let $0 \leq a < 1$. Then $A=\lambda^{-1}(a,1] \in \tau^*{}_{c1}$ which implies $\chi_A \in \tau_{c1} \subseteq \tau_1 \subseteq \omega(\tau^*{}_{c2})$. That is $(\chi_A)^{-1}(a,1] \in \tau^*{}_{c2}$ which shows $\lambda^{-1}(a,1] \in \tau^*{}_{c2}$ and therefore $\lambda \in \omega(\tau^*{}_{c2})$. That is $\omega(\tau^*{}_{c1}) \subseteq \omega(\tau^*{}_{c2})$. Similarly we can show that $\omega(\tau^*{}_{c2}) \subseteq \omega(\tau^*{}_{c1})$.

To show that the converse of the theorem is not true in general,we present the following example.Let $X=I, \tau_1=\{0,1\}$ and $\tau_2=\{0,\lambda:\lambda(x)>0$ for all $x \in I\}$ .Then $\tau^*{}_{c1} = \tau^*{}_{c2} = \{\phi, X\}$ and $\omega(\tau^*{}_{c1}) = \omega(\tau^*{}_{c2}) = \{c: 0 \leq c \leq 1\}$ but τ_2 is not a subset of $\omega(\tau^*{}_{c1})$ which gives that (X,τ_1,τ_2) is not P-weakly induced.

Corollary 5.1.10

If a bfts (X,τ_1,τ_2) is P-weakly induced then it is weakly induced but the converse is not true in general.

Proof :

Since (X,τ_1,τ_2) is P-weakly induced then Theorem 5.1.9 implies that $\tau_2 \subseteq \omega(\tau^*_{c1})=\omega(\tau^*_{c2})$ and $\tau_1 \subseteq \omega(\tau^*_{c2})=\omega(\tau^*_{c1})$.To discuss the other implication,we provide the following example.

Let $X=I,\tau_1=\{0,1\}$ and $\tau_2=\{\chi_A : A\subseteq X\}$.Then it is clear that $\tau_1 \subseteq \omega(\tau^*_{c1})$ and $\tau_2 \subseteq \omega(\tau^*_{c2})$;i.e,$(X,\tau_1,\tau_2)$ is weakly induced;but τ_2 is not a subset of $\omega(\tau^*_{c1})$ i.e. (X,τ_1,τ_2) is not P-weakly induced.

§ 5.2 Main results

Before we present the first main result of this section,we have the following definition.

Definition 5.2.1

A bfts (X,τ_1,τ_2) is P-Hausdorff if for any two distinct points x,y of X there exist $\lambda\in\tau_1$ and $\mu\in\tau_2$ such that $\lambda(x)=\mu(y)=1$ and $\lambda\cap\mu=0$.

Theorem 5.2.2

If a bfts (X,τ_1,τ_2) is P-C-compact and (X,τ_1) is Hausdorff,then $\tau_2 \subseteq \omega(\tau^*_{c1})$.

Proof :

Let $f\in\tau_2$ and $0\leq a<1$.We are going to prove $f^{-1}(a,1]= A\in\tau^*_{c1}$.That is $\chi_A\in\tau_{c1}$.

Let x be any point of A ,choose $\varepsilon>0$ such that $a_1=a+\varepsilon<f(x)$.Let $A_1=f^{-1}(a_1,1]$. For any $y\in A_1{}^c$, x,y are distinct points.Since (X,τ_1) is Hausdorff, there exist τ_1-open fuzzy sets λ_y,μ_y such that

$\lambda_y(x)=\mu_y(y)=1$ and $\lambda_y \cap \mu_y=0$.Now the family $Ç=\{\mu_y:y\in A_1{}^c\}\cup\{f\}$ is P-open satisfying the inequality $\sup\{\{\mu_y:y\in A_1{}^c\}\cup\{f\}\}\geq a_1$.Since (X,τ_1,τ_2) is fuzzy P-C-compact ,then each constant fuzzy set in X is fuzzy P-compact.Hence $c(x)=a_1$ is fuzzy P-compact .Therefore the family Ç has a finite subfamily $\{\mu_{y_i}:i=1,2,...,n\}$ such that $\sup\{\{\mu_{y_i}:i=1,2,...,n\}\cup\{f\}\}\geq a_1-\varepsilon/2$.Let $\lambda_x=\inf\{\lambda_{y_i}:i=1,2,...,n\}$.Then $\lambda_x(x)=1$.If $y\in A^c$,then $f(y)\leq a$ so that $\mu_{y_i}(y)>a$ for some $1\leq i\leq n$.Since $\lambda_{y_i}\cap\mu_{y_i}=0$ and $\mu_{y_i}(y)>0$,we have $\lambda_{y_i}(y)=0$ where $\lambda_x(y)=0$ for every $y\in A^c$.We have shown that for each $x\in A$ there exists $\lambda_x\in\tau_1$ such that $\lambda_x(x)=1$ and $\lambda_x(y)=0$ if $y\in A^c$.Define $\lambda=\sup\{\lambda_x:x\in A\}$.Then $\lambda\in\tau_1$ and $\lambda=\chi_A$, i.e. $A\in\tau^*{}_{c1}$ .Hence $f\in\omega(\tau^*{}_{c1})$.

Corollary 5.2.3

If (X,τ_1,τ_2) is a Hausdorff P-C- compact space,then (X,τ_1,τ_2) is P-weakly induced .

Theorem 5.2.4

A bfts (X,τ_1,τ_2) is a P-induced space iff (X,τ_1,τ_2) is a P-weakly-induced space in which every constant map from X into I belongs to each τ_i (i=1,2).

Proof :

$\Rightarrow$ Since every P-induced space is P-weakly induced and every constant map is continuous,so the first implication is clear.

$\Leftarrow$ To show that (X,τ_1,τ_2) is a P-induced space,let $f\in\omega(\tau^*{}_{ci})$.We are going to prove that $f\in\tau_j$. Let $z\in X$ such that $f(z)=b>0$.For any arbitrary $\varepsilon>0$,there exists $a>0$ such that $a<b$ and $b-a<\varepsilon$.Let g be defined by $g(x)=a$ for all $x\in X$ and $G=f^{-1}(a,1]$,then $G\in\tau^*_{ci}$ which gives $\chi_G\in\tau_{ci}\subseteq\tau_i\subseteq\ \omega(\tau^*_{cj})$.,i.e,$G\in\tau^*_{cj}$ which implies that $\chi_G\in\tau_{cj}\subseteq\tau_j$. Now if $h_z=\chi_G\cap g$ then $h_z\in\tau_j$, $h_z\subseteq f$ and $f=\cup\{h_z:z\in X\}$ which implies $f\in\tau_j$.

Corollary 5.2.5

If a bfts (X,τ_1,τ_2) is a P-C-compact Hausdorff space in which the constant fuzzy sets are fuzzy P-open then (X,τ_1,τ_2) is a P-induced space.

Theorem 5.2.6

If a bfts (X,τ_1,τ_2) is S-C-compact and P- Hausdorff then it is P-weakly induced.

Proof :

Similar to the proof of Theorem 5.2.2

Theorem 5.2.7

If a bfts (X,τ_1,τ_2) is P-C compact and P- Hausdorff then X is weakly induced.

Proof :

Similar to the proof of Theorem 5.2.2

Definition 5.2.8

A bfts (X,τ_1,τ_2) is said to be P-topological iff $\tau_i=\omega(L(\tau_j))$, $i,j=1,2$ $(i\neq j)$.

The following theorem is a generalization of Theorem 1.3.15 by Lowen (1976).

Theorem 5.2.9

A bfts (X,τ_1,τ_2) is P-topological iff for each continuous function $f:(I,T_{r.r})\to(I,T_{r.r})$ and for each $\nu\in\tau_i$, we have $fo\nu\in\tau_j$.

Proof:

$\Rightarrow$ Let $\nu\in\tau_i$ and $f:(I,T_{r.r})\to(I,T_{r.r})$ be continuous. Since $\nu\in\tau_i\subseteq\omega(L(\tau_i))$, then ν is continuous and so $fo\nu$ is continuous because $\nu^{-1}(f^{-1}(a,1])=(fo\nu)^{-1}(a,1]\in L(\tau_i)$. That is $fo\nu\in\omega(L(\tau_i))=\tau_j$.

$\Leftarrow$ We have to show that $\tau_i=\omega(L(\tau_j))$. To do this, let $\mu\in\tau_i$. Then $\mu\in\tau_i\subseteq\omega(L(\tau_i))$ and so $\mu:(X,L(\tau_i))\to(I,T_{r.r})$ is continuous. Now $\mu=I_{id}o\mu\in\tau_j\subseteq\omega(L(\tau_j))$. Hence $\tau_i\subseteq\omega(L(\tau_j))$.

Now let $\mu\in\omega(L(\tau_j))$, we have to show that $\mu\in\tau_i$.

Since a base for $L(\tau_j)$ is provided by the finite intersection $\cap\nu_i^{-1}(\varepsilon_i,1]$ where $\nu_i\in\tau_j$ and $\varepsilon_i\in I$. This is equivalent to saying $\forall\varepsilon\in I,\forall\, x\in\mu^{-1}(\varepsilon,1]$ there exists a finite set $I_{\varepsilon,x}$ such that $x\in\bigcap_{i\in I_{\varepsilon,x}}\nu_i^{-1}(\varepsilon_i,1]\subseteq\mu^{-1}(\varepsilon,1]$. Now fix x and let $\mu(x)=K_x$. Then for all $\varepsilon<K_x$, there exists I_ε finite such that $x\in\bigcap_{i\in I_\varepsilon}\nu_i^{-1}(\varepsilon_i,1]\subseteq\mu^{-1}(\varepsilon,1]$. For all $\varepsilon<K_x$ and $\forall\ i\in I_\varepsilon$ put $\mu_{i,\varepsilon}=\varepsilon\chi_{(\varepsilon_i,1]}\, o\nu_i\in\tau_i$.

Then
$$\mu_{i,\varepsilon}(y)=\begin{cases}\varepsilon & \text{iff } \nu_i(y)>\varepsilon_i.\\ 0 & \text{iff } \nu_i(y)\le\varepsilon_i..\end{cases}$$

Put $\nu_\varepsilon^x = \inf_{i \in I_\varepsilon} \mu_{i,\varepsilon} \in \tau_i$.

Then

$$\nu_\varepsilon^x(y) = \begin{cases} \varepsilon & \text{iff } \nu_i(y) > \varepsilon_i \quad \forall i \in I \\ 0 & \text{iff } \exists\, r \in I_\varepsilon, \nu_r(y) \leq \varepsilon_i. \end{cases}$$

Thus $\nu_\varepsilon^x(y) = \varepsilon$ implies $\mu(y) > \varepsilon$ and $\forall\, \varepsilon < K_x$ we have $\nu_\varepsilon^x \leq \mu$. Hence $\mu = \sup \sup \nu_\varepsilon^x \in \tau_i$ .So $\omega(L(\tau_j)) \subseteq \tau_i$. Hence the proof is completed.

To prove the next result we use the fact that if τ is a fuzzy topology on X,then $\tau \subseteq \omega(L(\tau))$.

Theorem 5.2.10

If a bfts (X, τ_1, τ_2) is P-topological,then $\omega(L(\tau_1)) = \omega(L(\tau_2))$.

Proof :

Since (X, τ_1, τ_2) is P-topological then $\tau_1 = \omega(L(\tau_2))$ and $\tau_2 = \omega(L(\tau_1))$. Moreover $\tau_1 \subseteq \omega(L(\tau_1))$ and $\tau_2 \subseteq \omega(L(\tau_2))$ which gives that $\omega(L(\tau_2)) = \tau_1 \subseteq \omega(L(\tau_1))$ and $\omega(L(\tau_1)) = \tau_2 \subseteq \omega(L(\tau_2))$.Hence we have $\omega(L(\tau_1)) = \omega(L(\tau_2))$.

Theorem 5.2.11

If a bfts (X, τ_1, τ_2) is P-topological,then $\omega(\tau^*_{cj}) \subseteq \tau_i$, $i,j=1,2, i \neq j$.

Proof :

Let $\mu \in \omega(\tau^*_{cj})$.Then $\mu:(X, \tau^*_{cj}) \to (I, T_{r.r})$ is continuous which implies that $A_a = \mu^{-1}(a,1] \in \tau^*_{cj}$ for all $a \in I$ and so $\chi_{A_a} \in \tau_j$ which is continuous. That is $\chi_{A_a}:(X, L(\tau_j)) \to (I, T_{r.r})$ is continuous and so $A_a \in L(\tau_j)$.Hence

$\mu:(X,L(\tau_j))\rightarrow(I,T_{r.r})$ is continuous and so $\mu\in\omega(L(\tau_j))$. That is $\omega(\tau^*_{cj})\subseteq\omega(L(\tau_j))=\tau_i$. i.e $\omega(\tau^*_{cj})\subseteq\tau_i$.

<u>**Corollary 5.2.12**</u>

If a bfts (X,τ_1,τ_2) is P-topological and P-weakly induced then (X,τ_1,τ_2) is P-induced.

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