# Introduction to the papers of TWG14: University mathematics education

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## Introduction

TWG14, *University Mathematics Education* (UME), was launched in CERME7 (Nardi et al., 2011) acknowledging the fast growth of research in UME, as well as some specificities proper to UME. Some of these specificities are: the abstract, formal nature of a significant portion of the mathematical content; the absence of national curriculum guidelines, leading to great variations in organization and practices across institutions; the general lack of systematic preparation for teaching; and, the volume of content to learn in a short period of time and the degree of autonomy expected from students. The consolidation of this research area, both outside and inside CERME, is visible through the number of UME-related activities in the last years: the publication in 2014 of a special issue about theoretical frameworks being used in UME research; the creation in 2015 of the *International Journal of Research in Undergraduate Mathematics Education*; the creation in 2015 of the *International Network for Didactic Research in University Mathematics* (INDRUM), which since 2016 has launched the biannual INDRUM conferences (an ERME Topic Conference) with associated special issues in journals (IJRUME and IJMEST) and the book *Research and Development in University Mathematics Education* (Durand-Guerrier et al. 2021); and, the organization in 2019 of the first *Calculus in upper secondary and beginning university mathematics* conference, to cite just a few.

In 2021 and 2022 we celebrated important anniversaries for the UME field: in 2021, the 10th anniversary of the creation of TWG14, the 30th anniversary of the publication of *Advanced Mathematical Thinking* (Tall, 1991), and the 20th anniversary of the publication of the ICMI study *The Teaching and Learning of Mathematics at University Level* (Holton, 2001); in 2022, the 15th anniversary of the emblematic handbook chapter *Mathematics Thinking and Learning at Post-secondary Level* (Artigue et al., 2007) with the follow-up chapter 10 years later, *Post-Calculus*

*Research in Undergraduate Mathematics Education* (Rasmussen & Wawro, 2017). For all these reasons, CERME12 provided an opportunity to celebrate UME research and to discuss its future.

This year, we received 41 paper and 9 poster submissions, with 27 papers and 14 posters presented at the conference and published in the proceedings. This number of presentations led to the decision to have parallel session in two groups (TWG14A and TWG14B) addressing six themes (see below). We also held some common sessions to discuss transversal issues and recent achievements and challenges in UME research. This introductory paper summarizes the works presented in both groups organized according to the six themes, as well as the common discussions.

As in previous CERMEs, a significant number of papers focused on *students' learning of mathematical topics and practices* (9 papers), such as Calculus, reasoning, and proof. Compared to CERME11, we received less papers in total (27 vs. 35) with some variation in the distribution in each theme: the number of papers addressing *teaching and teachers* decreased from four to three; the number of papers addressing *students' identity and experience* went from three to five; the number of papers addressing the *use of mathematics by non-specialists* remained the same (two); and, the number of papers in *interventions* decreased from seven to two. The CERME11 theme about resources and curriculum (five papers) became the new theme on *teaching and learning with digital resources* (six papers). Finally, studies on transition (a theme in CERME11) were discussed within the six themes above. In the next section, we briefly present the six themes with examples from paper contributions. While many papers could fit in more than one theme, this classification helped structure our work at the conference and the presentation below.

## Themes and contributions

### Students' learning of mathematical topics and practices

Ten papers (Baldino & Cabral; Borji et al.; Karavi & Mali; Körtling & Eichler; Noah-Sella et al.; Rogovchenko & Rogovchenko; Spratte; Utsch; Wallach et al.) and six posters (Beran; Fuchs; Hanke; Oldenburg et al.; Piroi; Vincenzi) were classified under this theme.

These studies investigated different mathematical areas, with Calculus being the dominant (Baldino & Cabral; Fuchs; Körtling & Eichler; Noah-Sella et al.; Oldenburg et al.), including advanced Calculus topics such as multivariable functions (Borji et al.), convergence of sequences (Utsch) and differential equations (Rogovchenko & Rogovchenko). There were also studies focusing on Linear Algebra (Beran; Piroi; Wallach et al.), Complex Analysis (Hanke; Karavi & Mali), and Abstract Algebra (Beran). Finally, one paper was on students' proof reading (Spratte) and one poster on incommensurability in regular polygons (Vincenzi).

Some of the papers discussed students' learning (Körtling & Eichler; Spratte; Utsch) focusing on students' definitions (Körtling & Eichler), their intentions when reading proofs (Spratte), and students' connections between their concept images and definitions (Utsch). There were also studies that discussed teaching innovations (Baldino & Cabral; Beran; Borji et al.; Fuchs; Piroi) focusing on visualization in Linear Algebra (Piroi); the connection between Calculus and modeling techniques in Physics (Fuchs); the introduction of the Fundamental Theorem of Calculus using discrete graphs (COVID-19 graphs) and then moving to continuous graphs (Baldino & Cabral); the use of

mathematical structures (Beran); and, the introduction of activities designed using APOS theory (Borji et al.). Other studies investigated experienced and less experienced learners (Noah-Sella et al.). Finally, there were studies focusing explicitly on teaching resources and lecturers' practices (Hanke; Karavi & Mali; Rogovchenko & Rogovchenko; Wallach et al.). These dealt with definitions in textbooks (Hanke), the potential of Linear Algebra tasks to assist in transitions between multiple discourses (Wallach et al.), an investigation on differential equations tasks in terms of assessing students' conceptual understanding (Rogovchenko & Rogovchenko), and the proving routines used by a lecturer in Complex Analysis (Karavi & Mali).

**Teaching and learning with digital resources**

Six papers (Albano; Broley et al.; Davies et al.; Donevska-Todorova & Turgut; Gueudet et al.; Przybilla et al.) and one poster (Thoma & Iannone) were discussed in this theme.

Some authors considered the use of *traditional* technologies: Broley et al. explored the issue of learning programming for mathematical investigations with the Anthropological Theory of the Didactic (ATD – Bosch et al., 2020). They established an epistemological reference model and investigated a student's praxeological equipment. Gueudet et al. studied a similar issue with the instrumental approach (Gueudet, 2017), focusing on the social aspects in the schemes developed by students and associated to the programming artefact.

Other authors considered relatively *unexplored* technologies: digital assessment and its use by university teachers (Davies et al.) or an automated theorem prover and its impact on students' reasoning (Thoma & Iannone). Digital maps also seem to be a promising new tool, with different intended uses, such as fostering students' collaborative work and their conceptualization processes in Linear Algebra by connecting different representations (Donevska-Todorova & Turgut). Experts can design digital mathematical maps to evidence connections between secondary school and university mathematics (more precisely Geometry, in the study by Przybilla et al.).

As we mentioned above, the theme of digital resources was new in TWG14. One of the reasons for the emergence of this new theme was the COVID-19 pandemic, which was evoked in several papers and played a central role in the study by Albano around the new orchestrations required in the context of hybrid teaching. The generalized use of digital platforms in university courses and its consequences were discussed in TWG14 and identified as directions requiring further research.

**Students' identity and experience**

Five papers (Gandell; Göller; Kontorovich & Greenwood; Mullen & Cronin; Rasmussen et al.) and one poster (Nardi) were presented and discussed in this theme.

Three of the contributions to this theme focused on students' in-class experiences. Gandell investigated students' spontaneous mathematical thinking in movement, illustrating how this approach offers new insights into students' mathematical knowing. Kontorovich and Greenwood investigated student experiences with proof in a Topology course, where students were provided with opportunities to prove the same mathematical statement in different social situations. Rasmussen et al. analyzed the individual and collective mathematical progress of one small group of four students in an inquiry-oriented differential equations classroom as they reinvented Euler's method.

The other three contributions focused on students' out of class experiences and identity. Göller investigated first-year mathematics students' everyday coping strategies while dealing with the challenges they face transitioning from secondary to university mathematics. Mullen and Cronin described a suite of online and in-person mathematics supports designed for in-coming first-year university students. Finally, Nardi took up the question of how well undergraduates understand the actual work that mathematicians do and argued that more needs to be done to make this work more visible and salient for undergraduate mathematics students.

**Teaching and teachers**

Three papers (Nseanpa & González-Martín; Tabchi & Sabra; Viirman) were in this theme.

The first two papers paid particular attention to the use of resources in teaching, building on the documentational approach (Gueudet, 2017). Tabchi and Sabra investigated the connection between the teaching practices of a lecturer at a Lebanese university and her activity as a researcher in Graph Theory. The lecturer herself perceived little connection between the teaching and researching aspects of her work, but analysis of her teaching practice and use of resources revealed such connections, for instance, regarding the use of generic examples. Meanwhile, Nseanpa and González-Martín, in a study of the teaching of derivatives at the pre-university level in Cameroon, focused on how strong institutional constraints shaped teachers' practices and use of resources. These constraints included a prescribed textbook, official teaching guidelines and a high-stake national examination. Findings indicate that national examinations strongly influenced the didactical choices of teachers concerning the teaching of the derivative, shaping, for instance, the way derivatives are introduced. Calculus was also the topic of the paper by Viirman, which otherwise has quite a different focus from the first two papers in this theme. Viirman analyzed a set of 14 national accounts, written by experts in the field, of the teaching of Calculus in secondary education, at university and in teacher education. Differences and similarities between the accounts were highlighted, and findings were used to discuss how Klein's second discontinuity plays out in different countries around the world.

**Interventions**

Two papers (Albano et al.; Markulin et al.) and three posters (Akrouti; Dreyfus et al.; Vourenpää et al.) were presented under this theme.

The research work presented by Albano et al. addressed the development of the problem-solving competence at the university level. In particular, they presented the design, implementation, and analysis of an activity in Topology. Markulin et al. discussed the use of Study and Research Paths (SRP) in statistics courses. We return to this paper in the next section.

Regarding the posters in this theme, they discussed activities designed for the teaching of integrals (Akrouti), the use of flipped classroom formats (Vourenpää et al.), and the development of a methodological approach for characterizing the interplay of mathematical progress across individuals, small groups, and the whole class (Dreyfus et al.).

### The use of mathematics by non-specialists

Two papers (Florensa et al.; Hitier & González-Martín) and two posters (Feil & Strauer; Schmitz et al.) were presented under this theme.

The two papers of this theme and one in the previous theme (Markulin et al.) used ATD as theoretical framework to address issues related to mathematics for students who did not chose mathematics as their main subject. Markulin et al. presented an analysis of the conditions and constraints affecting the implementation of SRPs in statistics courses at university level. Florensa et al. drew on ATD to analyse the discontinuities of the mathematical education of engineers. Finally, Hitier and González-Martín conducted a praxeological analysis of the use of the derivative by students in post-secondary institutions concurrently following Calculus and Mechanics courses. The use of gaps in worked-out examples (Feil & Strauer) and of application examples (Schmitz et al.) were also discussed.

## Transversal issues addressed in plenary discussions

### Resources (including digital) and interventions

The focus of this discussion was on interventions and digital resources and was organized around four thematic areas: design and sustainability of interventions; nature and impact of interventions; design of digital resources and use by teachers; and, digital resources and students' learning.

In relation to the design and sustainability of interventions, collaboration of mathematics education researchers with teachers with mathematics or non-mathematics specialty (e.g., physicists, biologists, etc.) is pertinent and necessary. However, practice highlights tensions in such collaborations, which calls for systematic studies of what makes those collaborations work effectively. Additionally, more evidence is needed on the sustainability of interventions. In this sense, it seems that having a community already working together helps the initiation and stability of changes. Our discussions also suggested that institutional and socio-cultural perspectives have the potential to capture the development of such changes. Moreover, there is an overall agreement that the pandemic has triggered significant changes to UM teaching and learning practice. Digital resources obviously played a significant role in those changes; however, doubts were expressed about whether and which of those changes will remain. Other questions we discussed are: Have online practices changed practices and interaction with mathematical content? If writing mathematics by hand is important, how can online platforms support mathematical communication effectively?

In relation to the nature and the impact of interventions, it seems that there is a variation of models regarding design and expectations. Several interventions aim towards inquiry-based learning and more student-centred approaches. However, the nature of those interventions is influenced by institutional characteristics. Consideration of such institutional characteristics should be critical in future investigations. Furthermore, clarity on the aims of proposed interventions is essential. Such clarity can assist the evaluation of the impact of interventions. Such impact is discussed in recent studies also in relation to specific student demographic profiles (gender, socio-economic status, etc.), a discussion that opens new opportunities for research on equity and access issues at UM.

Regarding the design and use of digital resources by teachers, some papers addressed new types of technology, such as theorem provers or digital assessment by indicating the pertinence of research

into innovative technologies, in particular artificial intelligence, for future works. Other studies that are necessary concern teachers' use of digital resources (and how we could support productive uses and orchestrations), theoretical approaches and methods that can be appropriate to analyse the design and use of technology at UM (and the possible differences with other educative levels), and the short- and long-term impact of the COVID-19 pandemic on practices related to the use of technology (e.g., teachers' use of flipped classroom approaches during the pandemic may continue).

Finally, regarding the use of digital resources and students' learning, we discussed the importance of identifying some consequences of technology use on learning, as well as on the epistemological level of the mathematics taught and learned. We also discussed the importance of theoretical and methodological approaches to address these issues, and of assessing the short- and long-term impact of the use of technology during the COVID-19 pandemic.

**Students' identity, experience and learning (including non-specialists)**

Regarding these issues, we saw some innovative contributions at the tertiary level: the analysis of data about movement and its role in mathematical activity, the analysis of the individual-collective dynamic, models to study different transitions for non-specialists (pre-university to university; university to workplace; between mathematics courses and non-mathematics courses), studies about advanced topics (such as Topology and Complex Analysis), and studies about students' appreciation of mathematics (for instance, in support centers, or with non-specialists). We also discussed the methodological challenges for some of these studies and how to upscale them.

Among the main challenges for the future, we discussed those related to remote teaching (for instance, how to assess, how to consider communication) and its impact on students' learning and experience of mathematics. Another issue of interest is that, to better understand the experience of students in programs for non-specialists, we need to better understand how mathematics is used (or not) in other disciplines. These studies can also open new perspectives about students' appreciation of mathematics, their identity and experiences. Such studies have the potential to move beyond a decontextualized investigation of students' learning of specific topics.

We also discussed some potential challenges on research findings dissemination, such as the communication of research data related to innovative topics (e.g., students' movement, or how to share the large amounts of data that can be collected during remote teaching). Another challenge concerns the replication of studies conducted in other education levels with consideration of the cultural and institutional characteristics of UME. Also, some of the issues discussed in our group are not UME specific. These observations call for more interaction with other CERME groups.

**Supporting university mathematics teachers and teaching**

Research on UM teachers and their teaching has been gaining attention in recent years. However, research related to UM teacher education and professional development is rather scarce (Winsløw et al., 2021). Overall, there is a variation of practices and approaches on the preparation of UM teachers and the support they need for their profession. With this in mind, we opened the discussion around two questions: "What support for teaching do mathematics teachers have access to at your institution?" and "What would you like to see in research on UM teachers' support in the next years?"

Regarding the availability of support, practices shared by participants in the group confirm the variability mentioned above. Very often teacher education and support is non-mathematics-specific, with attention to general skills such as use of digital resources, organising lessons, engaging with educational literature and developing pedagogical practices. Mathematics-specific support often relies on 'local' projects supported by individuals or small teams. The proximity between mathematics and mathematics education departments seems to influence the interaction between mathematics teaching and research on mathematics teaching. Sustainable models of mathematics-specific teaching support seem to be those that are institutionally embedded and maintain mutual participation of mathematicians and mathematics educators. Institutionalized acknowledgment of and support for teachers and teaching-developmental activities (e.g., release of time, accreditation, funding) is also mentioned as a factor facilitating interventions and other teaching-related projects.

Education and support for UM teachers is an emerging area of research with need for further studies. Suggestions for future research proposed in our group include the study of the role of institutional structures on the support for mathematics teaching; the preparation of new or graduate teachers; the investigation around support for non-mathematics specialists who teach mathematics; and, the identification of the characteristics of productive collaborations between mathematics education and mathematics communities (see a recent publication by Goméz-Chacón et al., 2021). Furthermore, developmental research projects in this area should be more attentive to designing, implementing and evaluating pedagogical interventions towards institutional change, including the development of appropriate resources (design principles and implementation) for UM teaching.

## Reflections and ways forward

We can see that many notable contributions of this year, as well as challenges for the future, are in line with the overview on CERME research in UME identified by Winsløw et al. (2018): (a) *what is it?*, namely research into current practices of UME (with no direct intervention), such as: mathematical content; methods and resources; transition phenomena; student experiences; and, teaching non-mathematics specialists; and (b) *what could it be?*, namely developmental or experimental research, that includes an intervention design as part of the research project (e.g., research on, and for innovation in UME; i.e., interventions in specific courses or programs) and professionalization of UME practice (preparation of mathematics teachers).

In our discussions, the different impacts of the pandemic on many aspects – such as teaching and learning, conducting research or collecting data – appeared as an important point for the research agenda. This impact may lead to more studies considering technological issues in UME research in the coming years. Other important topics are in line with those identified in CERME11 (González-Martín et al., 2019): 1) the establishment of different types of collaborations and the development of theoretical tools to study them; 2) the study of complex phenomena, and the networking of theories (or the use of theories from other fields); 3) the need for large-scale studies and replications to consolidate results; 4) the development and testing of innovative research methods and data collection procedures; 5) the identification of the cultural, institutional, and local characteristics in some studies, and how changes in these factors may influence studies and results in other contexts; and, 6) studies on new or understudied topics, such as equity, access, and inclusion in UME.

We believe the first ten years of existence of TWG14 have brought many advances in our field, and we predict more important contributions in the years to come. We are confident that the coming CERME conferences will allow us to pursue research on the areas and questions discussed this year, to address and propose implementations for practice, and to open new areas for investigation.

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