Essays on the economics of networks

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Abstract

Networks (collections of nodes or vertices and graphs capturing their linkages) are a common object of study across a range of fields including economics, statistics and computer science. Network analysis is often based around capturing the overall structure of the network by some reduced set of parameters. Canonically, this has focused on the notion of centrality. There are many measures of centrality, mostly based around statistical analysis of the linkages between nodes on the network. However, another common approach has been through the use of eigenfunction analysis of the centrality matrix. My thesis focuses on eigencentrality as a property, paying particular focus to equilibrium behaviour when the network structure is fixed. This occurs when nodes are either passive, such as for web-searches or queueing models or when they represent active optimizing agents in network games. The major contribution of my thesis is in the application of relatively recent innovations in matrix derivatives to centrality measurements and equilibria within games that are function of those measurements. I present a series of new results on the stability of eigencentrality measures and provide some examples of applications to a number of real world examples.

Declaration

I, Xiangyu Wu, hereby declare that this is entirely my own work unless referenced to the contrary in the text. No part of this thesis has previously been submitted else where for any other degree or qualification in this or any other university.

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Last but not least, thanks also to my friends for their encouragement.

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Nomenclature

Symbol	Description
$\boldsymbol{A}$	a matrix in $\mathbb{R}^{n\times n}$
	a concave, nonempty, compact and bounded set
$\boldsymbol{A}^D$	the Drazin/Group Inverse of matrix $\boldsymbol{A}$
$\boldsymbol{A}^\ddagger$	the Moore-Penrose Inverse
$\boldsymbol{A}_G$	a $n \times n$ adjacency matrix of G
$\bar{A}$	a compact set of action profile, $A \in \bar{A}$
A	an action space of players' strategies, $A \subset \mathbb{R} \subset \mathbb{E}$
	an independent set relative to a network (N, g), $A \subset N$, no two nodes are adjacent in A
	a maximal independent set if $A' \neq A, A \subset A' \subset N$
	an dominating set relative to a network (N, g), $A \subset N$
	every node in A is connected to every other node in A via a path that involves only nodes in A
$A(D)$	the set of arcs/directed path of a digraph
$A(X, Y)$	a set of arcs of D whose tails lie in X and whose heads lie in Y
$\boldsymbol{a}$	vectors in the adjacency matrix $\boldsymbol{A}_G$
$\boldsymbol{a_{-i}}$	other players' actions except player i's action
$\boldsymbol{a}^*$	a vector of Nash Equilibrium
$\bar{(}a)$	a node on the trajectory $a(t)$
	an equilibrium node for $r = \bar{r}$
a	a profile of actions, $a \in A = A_1 \times \ldots A_n$
	a node in N player concave game network
	a vector of a player's action/strategy
	a nonnegative parameter
	a discrete value of a player's best reply
	a temporary variable
a^*	an Equilibrium set, $a^* \in \mathbb{R}$
$a(D)$	the number of arcs/directed path in a directed graph
$a(X, Y)$	the number of arcs of D whose tails lie in X and whose heads lie in Y
$a(t)$	the deviation of a^*, $a(t) = a^* + \boldsymbol{\varepsilon}$, $\lim_{s\to\infty} a(t+s) \to a^*$
	a trajectory of nodes $a \in \bar{A} \subset \mathbb{E}$
a'	some nodes on the trajectory $a(t)$
BR	Best responses of a game
b	a discrete value of a player's best reply
	a temporary variable
b_i	a temporary variable
$\boldsymbol{C}$	a $m \times m$ constant matrix
$\mathcal{C}$	Degree
$\mathcal{C}_O$	Out Degree
$\mathcal{C}_I$	In Degree
$\mathcal{C}_\gamma$	Closeness
$\mathcal{C}_\omega$	Out Closeness
$\mathcal{C}_\iota$	In Closeness
$\mathcal{C}_B$	Betweenness
$\mathcal{C}_P$	PageRank
$\mathcal{C}_H$	Hubs
$\mathcal{C}_A$	Authorities
c	the cost of taking action $\mathbf{1}$
$\boldsymbol{D}$	the diagonal positive definite matrix

Symbol	Description
D	the directed graph, or digraph
d	the directed network
$d_G(v)$	the *degree* of a vertex v in a graph G
$\mathbb{E}^{\gtrdot\sqsupset}$	the Euclidian space, $\mathbb{E}^{\gtrdot\nVdash} \times \mathbb{E}^{\gtrdot\nvDash} \times \cdots \times \mathbb{E}^{\gtrdot\ltimes}$ and $m = \sum_i^n m_i$
$d + ij$ or $d - ij$	a shorthand notation for the directed network obtained by adding(+) or deleting(−) a link ij
$E(G)$	a set of edges of a graph
$e, e(G)$	an edge of a graph
$F_\gamma(.)$	a monotone function, such that $F : \mathbb{R}^n \to \mathbb{R}^n$ returns the optimal strategy
F_1, F_2	functions in the proof process of uniqueness of a Nash equilibrium point
G	a graph
$dist_G(v_0, v_1)$	The length of a shortest walk between vertices v_0 and v_1
g	a network
g_{ij}^k	the coefficient in the (i, j) cell of $g^k = g \times \overset{k\,\times}{\cdots} g$
$H(a; g)$	an $m \times k$ matrix whose jth column is $\nabla h_j(a_j; g)$
H^*	a set for $h(a; g) \leq 0$
$h(a; g)$	the mapping function for action spaces
h_i	a Hub score of i component
$\boldsymbol{I}$	the identity matrix
I	a set of players in a game
i	a player in a game, $i \in I > 1$
	a temporary variable
ij	a link between two nodes i and j, $ij \in g$
$J(a, r; g)$	a Jacobian determinant of $\nabla u(a, r; g)$
j	a player in a game
k	an exponent of a weighted Katz-Bonacich centrality function
	a parameter in the mapping function $h(a; g)$ for defining action spaces $mathbbE^k$
$\bar{k}$	a parameter for defining action spaces $\mathbb{E}^{\bar{k}}$, $\bar{k} \leq k$
$\mathfrak{L}[.]$	a linear mapping operator that is determined by the choice of normalization for the eigenvector/value pair
L	some finite upper bounded for players' actions spaces
l	a parameter equals to $*$ or $*1$
$\boldsymbol{M}$	a square, nonnegative and irreducible matrix
$\boldsymbol{M}_G$	the incidence matrix of a network
m_{ve}	the number of times that vertex v and edge e are incident
m	a parameter
N	a finite set of players/nodes in a network
$N_D^+(v)$	the set of vertices' out-neighbours
$N_D^-(v)$	the set of vertices' in-neighbours
(N, g)	a network
(N', g')	a component/subnetwork of a network (N, g), $N' \subset N, g' \subset g$
(N, d)	a directed network
$N_i(g)$	the neighbours of a node i in a network (N, g), $N_i(g) = \{j \mid ij \in g\}$
NP_k	a set of non-providers of the public good, k is the step in the algorithm
n	a positive integer which denotes a player in a game
P_k	a set of the providers of the public good, k is the step in the algorithm
$\boldsymbol{P}$	a permutation matrix
$\boldsymbol{p}$	a matrix of diagonal left Perron vector
$\boldsymbol{p}^{max}$	left Perron vector, the associated eigenvector of the largest eigenvalue $\boldsymbol{\lambda}^{max}$
p	the probability of i player' s each neighbor take action $\mathbf{1}$ independently
$\boldsymbol{Q}$	a matrix of eigenvectors
Q	a convex subset of the positive orthant of $\mathbb{E}^n$
$\boldsymbol{q}$	a matrix of diagonal right Perron vector
$\boldsymbol{q}^{max}$	right Perron vector, the associated eigenvector of the largest eigenvalue $\boldsymbol{\lambda}^{max}$
$\mathbb{R}$	every node not in A is linked to at least one member of A

Symbol	Description
r	a fixed parameter for a gradients function
S	a bounded subset of the positive orthogonal constraint set of action space $\mathbb{E}$
s	a parameter in deviation of a^* that $\lim_{s\to\infty} a(t+s) \to a^*$
t	a threshold value of a player i's best reply
u	a vertice in a graph
$u_i/u_i(a)$	a player i's payoff function
$u(a_i, g)$	a player i's payoff function which is quasiconcave in a_i
$\mathscr{V}_{\max}[g]$	Eigenvector centrality/Eigencentrality
$V(G)$	a set of vertices of a network
$(V(G), E(G))$	a graph
$(V(D), E(D))$	a directed graph, or digraph
$v, v(G)$	a vertex of a network
$(v_0, v_1) - walk$	a walk in a graph G in which vertex v_0 and vertex v_1 are the ends of edge e_1
$v(a, r; g)$	the pseudograndient of $r_i \nabla u_i(a, g)$
$\boldsymbol{W}$	an adjacency matrix of weighted graph, $\boldsymbol{W} \in \mathbb{R}^{n\times n}$
$\bar{W}$	a mean value of the Jacobian of $w(a, \lambda, r; g)$
$\tilde{W}$	a weighting matrix
W	a weighted graph
	a walk of a graph
$\boldsymbol{w}$	a vector of weighted Katz-Bonacich centralities
$w(a, \lambda, r; g)$	a mapping function
w_i	a parameter in a generalized payoff function
$w_i(a_i, \lambda, \bar{r})$	a diagonally strictly concave for $A \in \mathbb{R}$ and fixed $\lambda_i \geq 0$
w_{ij}	a weight w_{ij}, is the weight of the linkage between node i and node j, for $i, j \in \{1, \ldots, K\}$
X	a set of vertices of a digraph
$\lvert\partial^+(X)\rvert$	a $out-degree$ of a set of vertices of a digraph X
$\lvert\partial^-(X)\rvert$	a $in-degree$ of a set of vertices of a digraph X
$d^+(X)$	the quantity of $out-degree$ of a set of vertices of a digraph X
$d^-(X)$	the quantity of $in-degree$ of a set of vertices of a digraph X
x	a nonnegative parameter
	a initial and terminal vertex of a directed walk Z
$x(t)$	deviation of optimal and rational decision
Y	a set of vertices of a digraph
y	a initial and terminal vertex of a directed walk Z
Z	a directed walk in digraph D
γ	a function for conveniently define the equilibrium situation
Φ_G	an incidence function of a network/graph
Φ_D	an incidence function of a directed graph
$\phi(a, r; g)$	a weighted nonnegative sum of payoff functions $u_i(a_i; g)$
δ	a payoff impact parameter $\delta \in [-1, 1]$
	a nonnegative Lagrange multiplier vector $\lambda \in \mathbb{E}$
$\delta^+(X)$	the set of out-cut of directed graph
$\delta^-(X)$	the set of in-cut of directed graph
$\lvert\delta^+(X)\rvert$	the out-degree of directed graph
$\lvert\delta^+(X)\rvert$	the in-degree of directed graph
$\rho(\boldsymbol{M})$	the spectral radius of matrix $\boldsymbol{M}$
ϵ	the constant in deviation of optimal and rational decision
θ	a random parameter, $1 \leq \theta \leq 0$
$\boldsymbol{\Lambda}$	a matrix of diagonal eigenvalues
λ	a Lagrange multiplier vector $\lambda \in \mathbb{E}$
$\bar{\lambda}$	a Lagrange multiplier vector
$\boldsymbol{\lambda}, \boldsymbol{\mu}$	eigenvalues
λ^{min}	the lowest eigenvalue of a symmetric matrix
$\lambda^{min}(\boldsymbol{M})$	the lowest eigenvalue of matrix $\boldsymbol{M}$
λ^{max}	the highest eigenvalue of a symmetric matrix
$\lambda^{max}(\boldsymbol{M})$	the highest eigenvalue of matrix $\boldsymbol{M}$
σ	a fixed positive constant of first entry of Perron vector
∇	a symbol of gradients
τ	the step length to be selected

Chapter 1

Background and Introduction

1.1 Economics of Networks

Connectivity and interaction are key features of economics. However, heterogeneity of type and heterogeneity of interaction bring complexity to the mathematical representation of human interactions that make the tractable analysis of models difficult. Indeed, the clean predictions on how different types of behaviours, endowments and interactions affect welfare are muddied by the inherent complexity in economic systems.

Networks and the associated mathematical theory of graphs offers a pathway to collecting and categorising complex interactions. Providing the type of insight that economics wants on complex systems that are closer to the reality of the systems of interest. This is a relatively recent area of research from the perspective of applied theory and applied empirical analysis. However, the theory of networks and the theory of graphs has a long track record.

1.1.1 Thesis Modelling Assumptions

The objective of this chapter is to establish a baseline set of assumptions that will continue throughout this thesis. These assumptions are somewhat restrictive, but allow for tractable analysis and modelling. By moving beyond these assumptions, the ability of models to tell stories about the underpinning architecture of the network is very much limited.

1. Fixed network structure, characterised by a strongly connected graph.
2. Existence and uniqueness of equilibrium actions.
3. Concavity of the welfare or payoff function and stability of the equilibrium.
4. Equilibrium is a function of centrality and in particular Eigenvector centrality.

That is I consider models where, for a given network structure, there is an equilibrium set of actions $a^* = [a_1, \ldots, a_i]'$ for $i \in I > 1$ agents is a) exists and b) stable. By existence, I am presuming that $a^* \in \mathbb{R}$ and by stable I am referring to the notion that for any deviation $a(t) = a^* + \boldsymbol{\varepsilon}$ that the optimal, rational decision making by the set of I agents results in $\lim_{s \to \infty} a(t+s) \to a^*$, hence the system state converges back to equilibrium after a finite number of steps.

I will introduce in the next chapter and fully explain how I will define the network structure and how this definition fits into the straddles the definitions used in graph theory and more recently in the economics of networks. My basic object of interest will be a strongly connected graph, characterising interactions between welfare optimising agents, making continuous scalar choices. At first these assumptions might seem restrictive however, as I progress through my first chapter I will then proceed to illustrate some of the rich set of network structures that can be described by such a set of graphs.

After describing this set of graphs I will then describe the relevant set of theoretical results from the theory of multi-player games under diagonal strict concavity and the condition that describe a stable and unique equilibrium. Finally, I will use a case study to demonstrate how the notion of centrality and the notion of equilibrium are intertwined and this will form the basis of the first model I will develop in the final chapter of this review. Here, I develop a fully featured concave model and decompose the solution to characterise a) the equilibrium and b) the comparative statics through the use of measures of centrality and in particular the Perron-Frobenius form of the Katz/Pagerank and eigencentrality approach.

1.2 Open Problems in Network Analysis

This current version of this chapter presents the preliminary work needed to analyse the variation in equilibrium with point entry of the network structure. I have recast the set of games covered in Jackson and Zenou [2015] using the foundational approaches documented in Rosen [1965] that address stability of the equilibrium choice of strategies on the network.

The next step will be to establish the main result and is intending to address the following assumption, for games where the solution is of the form:

$$\mathbf{a}^* = \mathbf{F}_\gamma(\mathscr{V}_{\max}[\mathbf{g}])$$

where $\mathscr{V}_{\max}[g]$ is an operator that returns the eigenvector corresponding to the largest eigenvalue and $F_\gamma(.)$ is a monotone function, such that $F : \mathbb{R}^n \to \mathbb{R}^n$ returns the optimal strategy. My interest is in the comparative statics using the fact, derived in my next chapter, that:

$$\frac{\partial \mathscr{V}_{\max}[\mathbf{A}]}{\partial a_{ij}} = \mathfrak{L}[\mathbf{I} - \mathbf{A}\mathbf{A}^{\mathbf{D}}]$$

where $\mathbf{A}^{\mathbf{D}}$ is the 'Drazin' or 'group' inverse of $\mathbf{A}$ and $\mathfrak{L}[.]$ is a linear mapping operator that is determined by the choice of normalization for the eigenvector/-value pair. This will form the basis of my first derived esult for my thesis and my central theorem.

1.3 Findings and Contributions

The thesis contributes to the development of game theory and network theory which provides the mathematical foundation of analysis of complex interactions with strongly connected networks. In chapter 4, I rewrite and reprove a series of classic theorem from Rosen [1965] in network definitions, which presents the existence, uniqueness and global stability of equilibrium in a concave N-player game with the diagonal strict concavity of its payoff functions. In my network analysis, this equilibria is the stability of a network with strongly connected network structure.

Moreover, I build my own simple concave model which contain the characteristics of concave N-player game and strongly connected network structure. Based on Perron-Frobenius Theorem 5.1.2, I relate my model to eigenvector centrality and after comparative statics of the model, I achieve my first theorem 4.97 which creatively present sensitivity parameters of changes of strategic changes on stability of such network.

Furthermore, to improve computability of theorem 4.97, Drazin inverse as a new technology for network research makes a contribution to comparative statics of theorem 4.97 and produce a new result, second theorem 5.29 which presents a computable sensitivity equation for further practical research in the future.

1.4 Structure of the Thesis

There are six chapters in the thesis. Chapter 1 briefly introduces the background and modelling assumptions of the research. It also address an open problem in current network research which inspire me to design my own research process and my research questions.

In chapter 2, the fundamental knowledge of graphs, networks and centralities are well introduced which provide the definitions and basic understanding of related concepts and theorems to my research topic. In chapter 3, I review recent literatures on networks.

In chapter 4 introduces the research of concave games following six steps general research process. In step one, step two and step three, essential definitions and models of games should be introduced which includes definition of payoff function of an individual agent a_i, a statistical model of the actions of all other agents for each agent a_{-i} and a model of the reaction functions of each agent to the expected actions of the others $u_i(a_i, a_{-i}; \delta, g)$. These knowledge are in section 4.1. Based on introductions of essential definitions of games, step 4 is to choose solution approach, Nash Equilibrium which is a rational expectations averagely in section 4.2 and find the equilibrium in step 5 in section subsec:existence2 Rosen [1965]. In the last step, it is to derive the change in the equilibrium with respect to the underlying parameters which is determined by the network. To achieve this purpose, I build my own simple concave network model and apply comparative statics into it, then I achieve my first theorem 4.97 which provide a result of sensitivity about how changes of player's strategy affect the stability of the strongly connected network. To improve the calculability in a quantitative database in further study, I apply the Drazin Inverse into the comparative statics of theorem 4.97 and achieve the second theorem 5.29 which provide a more computable equation for network analysis.

All chapters are self-contained and can be read independently of each other.

Chapter 2

Review of Graphs, Networks and Centrality

As noted by Bramoullé et al. [2016] networks typify a fundamental departure from standard models for describing economic systems.

In the 1990s, the development of digital data availability and efficient search algorithms resolved methodological issues and provided new opportunities on network research, thus social networks were viewed as largely outside the realm of economics Oh and Monge [2016].

Based on the growth of game theory, new applications of social networks and innovative studies of social networks in computer science, physics and sociology, economists tended to use theorems of network as radical methods to analyze economics issues. Over last decade, the literature has grown significantly and on both the extensive and intensive margins. Economics of networks became a field of research in its own right with dedicated JEL codes Glasserman and Young [2016], massive online courses Baez [2014], workshops and conferences, and bestselling textbooks Mitchell [2006] and Bramoullé et al. [2016]. Recently, network study now reached a third phase. The field of network economics has matured into an established discipline both within the community of economists, and at the boundaries of several interdisciplinary efforts Bramoullé et al. [2016].

Recent economic research on networks has focused on two distinct areas: first, empirical evaluation of network properties, such as centrality and cyclicity; second, the theory of decision making in the presence of fixed network structures with implications for dynamic network formation. My interest is in typifying the types of games played on fixed network structures with a view to developing new and innovative analytical techniques to identify equilibrium properties and the comparative statics of such equilibrium properties. My primary tool for understanding the presence and stability of equilibria will be using the result from multiplayer concave games and most notably the dynamics of equilibrium adjustments for players seeking to maximize their payoffs.

My research questions are: 1. Can we derive a fully analytically tractable network model with a diagonal concave solution for the equilibrium? 2. Can this model then be utilized to make theoretical predictions and specify new classes of estimators?

2.1 Introduction to Graphs

Graphs are efficient methods to describe events or situations which have occurred in real world. For example, events/objects are indicated by a set of vertices and relations between these events/objects are showed as edges between pairs of these vertices. A diagram which is consisted of these vertices and edges is a graph.

In history, the earliest form of graphs (vertices linked by edges) can be traced back to Ancient Egypt Mill gameboards and family trees from the Middle Ages Kruja et al. [2001]. The modern research of graph almost started in 1736 when Leonhard Euler solved the famous topology problem on königsberg bridges Euler [1741], Euler [1766] and Gross et al. [2013]. Euler's work led to the concept of an eulerian graph Biggs et al. [1986] and Alexanderson [2006]. Gross et al. [2013] reported that the first appearance of Euler's formula was in a letter from Euler to C. Goldbach in 14th November 1750 and valid proofs of this formula was proved later by A.M. Legendre in 1794 Hon and Goldstein [2005] and A.L. Cauchy in 1813 Maunder [1996]. S.A.J. Lhuilier extended Euler's polyhedron formula and contributed the Euler characteristic Caparrini [2002]. Euler's polyhedron formula and Euler characteristics provided the theoretical foundation of topological graph research Biggs et al. [1986] and Gross et al. [2013]. The first appearance of word 'graph' applied into the graphic research was in 1878 Sylvester [1878] and Gross et al. [2013].

In the next two centuries, research of graphic theorems developed. To begin with the research of cycles on polyhedra, there were two important contributions which are diagram-tracing puzzles from Kirkman [1856] and hamiltonian graphs from Hamilton [1856]. In addition on the research of connected graph without cycles, a graph theoretical idea of 'tree' was applied into practical calculation. For instance, Kirchhoff [1847] applied a graphic thinking of tree into accounting currents in electrical networks. With the gradual deepening of the study, the enumeration of different types of trees involved into research of calculus certain and chemical molecules, such as counting trees Cayley [1857], Pólya [1937], Otter [1948], Harary [1955] and Read [1963], chemical trees Cayley [1874],Sylvester

[1878] and Lunn and Senior [1929] and labelling tree problem Cayley [1897].

Euler's polyhedron formula inspired the research topology. In planar graph, the main obstructions can be displayed as two graphs that the complete graph K_5 and the complete bipartite graph $K_{3,3}$ (also known as the utility graph) below Gross et al. [2013].

Focusing on these problems, Kuratowski [1930] reported that there is a homeomorphic subgraph as K_5 or $K_{3,3}$ in every nonplanar graph Bang-Jensen and Gutin [2008]. H. Whitney defined a dual of a planar graph by purely combinatorial methods in 1931 Whitney [1992a], and also Whitney [1992b] reported a concept of a matroid to describe the similarities of independence in graphs and vector space in 1935. In higher surfaces, Heawood [1890] posed a question of embedding the complete graph K_7 in a torus and presented a Heawood Conjecture (formula) on the minimum number (*chromatic number*) for the genus of a surface Gross et al. [2013]. Heffter [1891] proved the accuracy of Heawood's formula in orientable surfaces of low genus. Tietze [1910] extended Heawood Conjecture to certain non-orientable surface. The completed proof of Heawood formula for general non-orientable surfaces was reported by Ringel and Youngs [1968]. Moreover, with the development of depth on mathematics in 1980s, Robertson and Seymour [1985] generalized Kuratowski's theorem from planar to other surfaces, and proved that for each orientable genus there exist a finite set of graphs which are not homeomorphic (forbidden subgraph) Gross et al. [2013]. For non-orientable surfaces, Glover et al. [1979] focused on the real projective plane and listed a set of 103 forbidden subgraph.

The graph coloring problem is also one of popular topics in graphical theory research. One outstanding question posed by Francis Guthrie in 1852 was whether four colors can color every map (plane or sphere) so that no two neighbouring countries are of the same color Gross et al. [2013]. This four-color problem was studied over 100 years Heawood [1890], Birkhoff [1912] and Franklin [1922]. However, the complete proof was not reported until Appel et al. [1976]'s paper. Arising from studies of four-color problems, other graph problems relating to coloring vertices or edges were explored as well Kempe [1879], Brooks [1941] and Vizing [1964].

The research of Graph theory algorithms can track back to the 19th century such as Fleury's algorithm and Hierholzer's algorithm for Eulerian trails. In the 20th century, several algorithmic results of graphical problems have been published involving traveling salesman problem Dantzig et al. [1954] and Padberg and Rinaldi [1987], minimum connector problems Prim [1957] and Kruskal [1956], maximum flows in networks Fulkerson and Dantzig [1955] and Gomory and Hu [1961], finding the longest path and shortest path Dijkstra et al. [1959] and Chinese postman problem Guan [1962]. Graphical algorithms were also applied into operational research, such as marriage theorem in matching and assignment problems Hall [1935] and Halmos and Vaughan [1950]. In the last thirty decades, researchers have made contributions in polynomial-time algorithm questions Edmonds [1965b], Edmonds [1965a], Toda [1991], Kolter and Ng [2009], Bogdanowicz et al. [2012] and Ancona et al. [2019], and developed the concept of NP-completeness in which NP is nondeterministic polynomial-time Cook [1971], Karp [1972]. The set of tractable decision problems are all in the polynomial-time class P, such as graph connectivity Even and Tarjan [1975], Kapron et al. [2013], primality testing Bressoud [2012], maximum matching Giel and Wegener [2003], remoteness testing Farmakis [2018], linear programming Karmarkar [1984], van

Dooren [2018], while the traveling salesman Archetti et al. [2003] and Hamiltonian cycle Alekseev et al. [2007] belongs to NP-hard problems. The one million dollars prize question concerns whether $P = NP$ is still unsolved.

2.1.1 Definitions of Graphs

Following definitions from Freeman [1978] and Bondy and Murty [2008], a *graph* G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* and a set of $E(G)$, disjoint from $V(G)$, of *edges*, together with an incidence function Φ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G.

$$G = (V(G), E(G)) \tag{2.1}$$

where

$$V(G) = \{v_0, v_1, v_2, \ldots, v_n\}, n \in \mathbb{R} \tag{2.2}$$

$$E(G) = \{e_0, e_1, e_2, \ldots, e_n\}, n \in \mathbb{R} \tag{2.3}$$

If e is an edge and (v_0, v_1) are vertices such that $\Phi_G(e) = (v_0, v_1)$, then e is said to join v_0 and v_1, and the vertices v_0 and v_1 are called the ends of e. The numbers of vertices and edges in G by $v(G)$ and $e(G)$ are called the *order* and *size* of G, respectively. If edge e with identical ends is called a *loop/circle* Bondy and Murty [2008].

For applying mathematical methods to study graphs, there are two matrices associated with a graph that its incidence matrix and its adjacency matrix. When two vertices are directly connected by an edge, they are *adjacent* Freeman [1978].

The adjacency matrix of G is the $n \times n$ matrix $\boldsymbol{A}_G := (a_{v_0 v_1})$, where $a_{v_0 v_1}$ is the number of edges joining vertices v_0 and v_1. In a graph $G = (V(G), E(G))$, The incidence matrix is the $n \times m$ matrix $\boldsymbol{M}_G := (m_{ve})$, where m_{ve} is the number of times $(0, 1, or\ 2)$ that vertex v and edge e are incident Bondy and Murty [2008].

A *path* is a simple graph whose each vertex is reachable from others in a linear sequence of one or more edges Bondy and Murty [2008]. Similarly, a *cycle* on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise Bondy and Murty [2008]..

A *walk* in a graph $G = (V(G), E(G))$ is a finite or infinite sequence of edges $W := \{v_0, e_1, v_1, \ldots, v_{i-1}, e_i, v_i\}$, whose terms are alternately vertices and edges of G, such that v_0 and v_1 are the ends of e_1, denoted by $(v_0, v_1) - walk$ Bondy and Murty [2008]..

A *trail* is an open *walk* $W := \{v_0, e_1, v_1, \ldots, v_{i-1}, e_i, v_i\}$ in which no edge edges is repeated but vertex can be repeated Bondy and Murty [2008].

A *tour* of a connected graph G is a closed *walk*that traverses each edge of G at least once, and an *Euler tour* one that traverses each edge exactly once (in other words, a closed Euler trail) Bondy and Murty [2008].

The length of a shortest walk (geodesics) between vertices v_0 and v_1 is called *distance*, denoted by $dist_G(v_0, v_1)$. If there is no path connecting v_0 and v_1, then the distance here can be sat as $dist_G(v_0, v_1) := \infty$.

A *degree* of a vertex v in a graph G, is the number of edges of G incident with v and denoted by $d_G(v)$. if G is a simple graph, $d_G(v)$ is the number of neighbours who v links with in G. Each loop counts as two edges.

2.1.2 Directed Graph

In a graph G, if its each link has an assigned orientation, this type of graphs is called *directed graphs* or *digraph*. Formally, a *digraph* D is an ordered pair $(V(D), E(D))$ consisting of a set $V := V(D)$ of vertices and a set $A := A(D)$, disjoint from $V(D)$, of $arcs$(directed path), together with an $incidence\ function$ Φ_D that associates with each arc of D an ordered pair of (not necessarily distinct) vertices of D Bondy and Murty [2008]. If a is an arc and $\Phi_D(a) = (v_0, v_1)$, then a is said to join v_0 to v_1; it also call that v_0 dominates v_1. The number of arcs in D is denoted by $a(D)$. The vertices which dominate a vertex v are its $in-neighbours$, those which are dominated by the vertex its $out - neighbours$. These sets are denoted by $N_D^-(v)$ and $N_D^+(v)$, respectively.

2.1.3 Strongly Connected Graphs

In strongly connected graphs, it is possible to reach any vertices starting from any other vertices by traversing edges in the direction(s) in which they point Brualdi et al. [1991]. Before introducing the formal definition of this type of graphs, I need to define what are *edge cuts* in directed graphs.

Let X and Y be sets of vertices of a digraph $D = (V, A)$. I denote a set of arcs of D whose tails lie in X and whose heads lie in Y by $A(X, Y)$, and their number by $a(X, Y)$. This set of arcs is denoted by $A(X)$ when $Y = X$, and their number by $a(X)$. When $Y = V/X$, the set $A(X, Y)$ is called the $out - cut$ of D associated with X, and denoted by $\partial^+(X)$. Analogously, the set $A(Y, X)$ is called the $in - cut$ of D associated with X, and denoted by $\partial^-(X)$. Observe that $\partial^+(X = \partial^-(V/X))$. Note $\partial(X) = \partial^+(X) \cup \partial^-(X)$. In the case of loopless digraphs, I refer to $|\partial^+(X)|$ and $|\partial^-(X)|$ as the $out-degree$ and $in-degree$ of X, and denote these quantities by $d^+(X)$ and $d^-(X)$, respectively. A digraph D is called strongly connected or strong if $\partial^+(X) \neq \emptyset$ for every nonempty proper subset X of V (and thus $\partial^+(X) \neq \emptyset$ for every nonempty proper subset X of V, too).

A directed walk in digraph D is an alternating sequence of vertices and arcs

$$Z := \{v_0, a_1, v_1, \ldots, v_{l-1}, a_l, v_l\} \tag{2.4}$$

such that v_{i-1} and v_i are the tail and head of a_i, respectively, $1 \leq i \leq l$. If x and y are initial and terminal vertices of Z, we refer to Z as a $directed$ $(x, y) - walk$. Directed walks, trails, tours, paths and cycle in digraphs are defined analogously. Thus a vertex y is reachable from a vertex x if there is a directed $(x, y) - path$. The property of reachability can be expressed in terms of outcuts, as follows: Let x and y be two vertices of a digraph D. Then y is reachable from x in D if and only if $\partial^+(X) \neq \emptyset$ for every subset X of V which contains x but not y.

In a digraph D, two vertices x and y are strongly connected if there is a directed $(x, y) - walk$ and also a directed $(y, x) - walk$ (that is, if each of x and y is reachable from the other). Strong connection is an equivalence relation on the vertex set of a digraph.

Irreducibility is another characteristic in a digraph D. Let $\boldsymbol{A} = [a_{ij}], (i, j, = 1, 2, ..., n)$ be a matrix of order n consisting of real or complex numbers. To $\boldsymbol{A}$ there corresponds a digraph $D = D(A)$ of order n as follows. The vertex set is the n-set $V = a_1, a_2, ..., a_n$. There is an arc $\alpha = (a_i, a_j)$ from a_i to a_j if and only if $a_{ij} \neq 0$,$(i, j = 1, 2, ..., n)$.

If a_{ij} is a nonzero weight attached to the arc α. In the event that $\boldsymbol{A}$ is a matrix of nonnegative integers, the weight a_ij of a can be regarded as the multiplicity $m(\alpha)$ of a. Then $\boldsymbol{D}$ is a general digraph and $\boldsymbol{A}$ is its adjacency matrix. However, unless specified to the contrary, $\boldsymbol{D}$ is the unweighted digraph as defined above.

The matrix $\boldsymbol{A}$ of order n is called reducible if by simultaneous permutations of its lines we can obtain a matrix of the form

$$\begin{bmatrix} A_1 & O \\ A_21 & A_2 \end{bmatrix} \tag{2.5}$$

where $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are square matrices of order at least one. If $\boldsymbol{A}$ is not reducible, then $\boldsymbol{A}$ is called irreducible. Notice that a matrix of order 1 is irreducible. Irreducibility has a direct interpretation in terms of the digraph $\boldsymbol{G}$ of $\boldsymbol{A}$. Thus, let $\boldsymbol{A}$ be a matrix of order n. Then $\boldsymbol{A}$ is irreducible if and only if its digraph $\boldsymbol{G}$ is strongly connected.

Let $\boldsymbol{A}$ be a matrix of order n. Then there exists a permutation matrix $\boldsymbol{P}$ of order n and an integer $t \geq 1$ such that

$$PAP^T = \begin{bmatrix} A_1 & A_2 & \dots & A_{1t} \\ O & A_2 & \dots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & A_t \end{bmatrix} \tag{2.6}$$

where $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_t$ are square irreducible matrices. The matrices $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_t$ that occur as diagonal blocks uniquely determined to within simultaneous permutation of their lines, but their ordering is not necessarily unique.

Let $\boldsymbol{A}$ be a matrix of order n. There exists a permutation matrix $\boldsymbol{Q}$ of order n such that $\boldsymbol{AQ}$ is an irreducible matrix if and only if $\boldsymbol{A}$ has at least one nonzero element in each line.

2.2 Introduction to Networks

Theoretically, network theory is a part of graph theory. In network theory, the vertices denoted objects are called nodes, and the edges which connect vertices are called links. More different definitions of networks will be introduced in the following subsection below.

2.2.1 Definitions of Networks

Network theory is part of graph theory. Thus definitions and characteristics of a network are similar or even same as a graph. To present the denotations of networks in further chapters more exactly, I introduce the different notations and symbols of networks from graphs.

In a network, let's consider a finite set of players $N = \{1, \ldots, n\}$ who are linked together. A network is a pair (N, g) in which g is a network of the set of nodes N. The interaction structure in a game or network is that one given player's payoff will be affected by neighbours' actions Jackson and Zenou [2015]. The *neighbors* of a node i in a network (N, g) are denoted by $N_i(g)$ Jackson and Zenou [2015].

There are two standard representations for the network g. One is denoting network g by adjacency matrix whose definition is similar as the graph in previous section. Let $\boldsymbol{A}_g$ be a $n \times n$ adjacency matrix of a network g, if entry g_{ij} is in

$\{0, 1\}$ in which i, j are nodes $i, j \in N$, then it denotes the connections that node i and node j are linked, and also the intensity of the connection Jackson and Zenou [2015]. The other is listing the pairs of connected nodes such as $ij \in g$ which indicates node i links to node j.

Definitions of *walk*, *path*, *circle/loop* and *distance* in networks are the same as in graphs which have introduced in previous section. A *component* of a network (N, g) is a subnetwork (N', g') in which $N' \subset N$ and $g' \subset g$. In a subnetwork (N', g'), there exist a path in g' from node $i \in N'$ to node $j \in N'$, $i \neq j$ and $ij \in g'$ Jackson and Zenou [2015].

If a network (N, g) is undirected, then g is required to be symmetric. Links between two nodes i and j are necessarily reciprocal and bidirectional that $g_{ij} = g_{ji}$ for all i and j, $ij \in g$ Jackson and Zenou [2015]. However, in a directed network (N, d), links can be unidirectional. Respectively, adding a link ij to an existing directed network d is denoted by $d + ij$, and deleting a link ij from an existing directed network d is denoted $d - ij$. A directed network is also strongly connected and Irreducible.

2.3 Examples of Various Network Structures

Figures 2.1 to 2.4 illustrate a variety of different network structures using a plotting technique, such that the distance between nodes is based on their *Katz centrality* relative to the mass of nodes within the matrix. In these cases the matrix is weighted and each edge of a graph has an associated numerical value, called a weight and these weights show the range of connection between nodes. The weight of an edge is often referred to as the "cost" of the edge. In these four

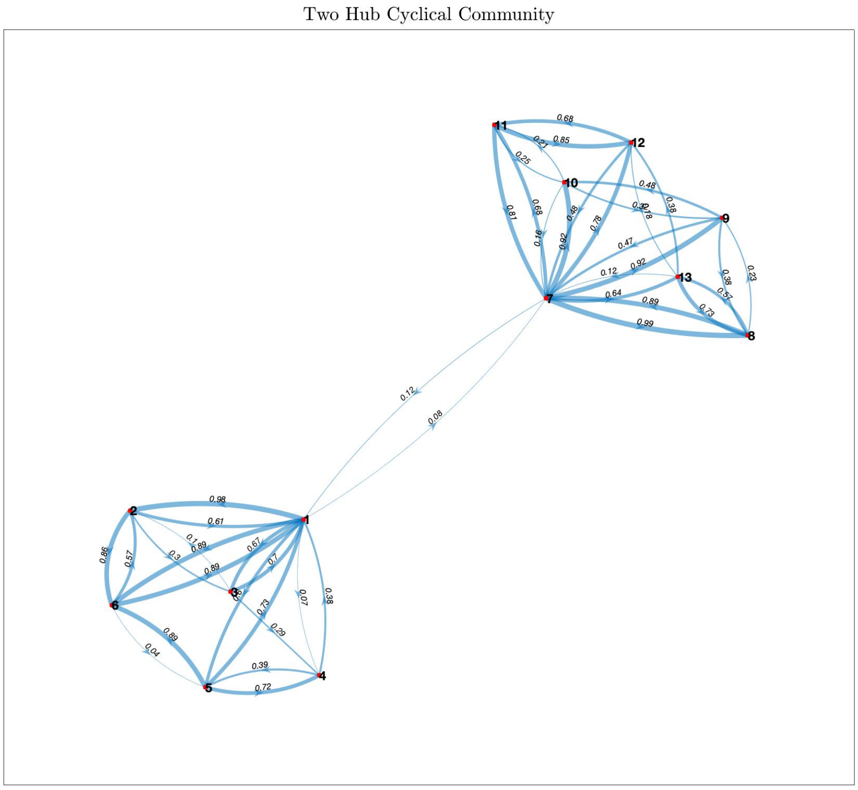

Figure 2.1: Ilustration of a two way direct graph where the bold numbers are the nodes and the connection weights are labelled in addition to determining the width of the connecting edge.

plots, vertices are red points and named as font-weight number. Edges are blue lines between red nodes with different degrees of thickness which shows weights of connection in edges.

In the first graph of two hub cyclical community, there are two pathway vertices No.1 and No. 7 which are centres of each cycle. In a cycle, vertices can connect with each other but only central vertices (No.1 and No. 7) can connect two cycles directly. If noncentral vertices in different cycles want to connect each

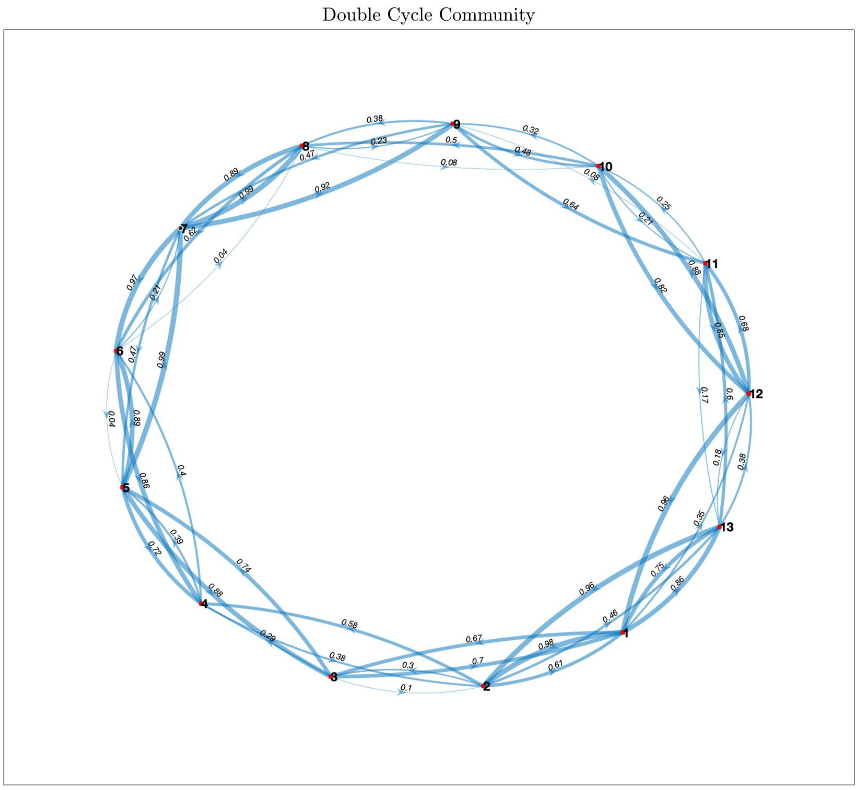

Figure 2.2: Ilustration of a double cycle network graph.

other such as No.6 wants to connect No.10, they must walk through No.1 and No.7. Two hub cycle community has been explored in various situations. For example, in a company the connection between two departments is similar as this graph. Each leader (node No.1 or No.7) has connections with all colleagues in his or her department. If two departments need to cooperate with each other, two leaders have communications and transfer information to their colleagues.

If vertices can only be directly connected with neighbours next them, the graph is like a perimeter having one cycle. However, if vertices can have direct

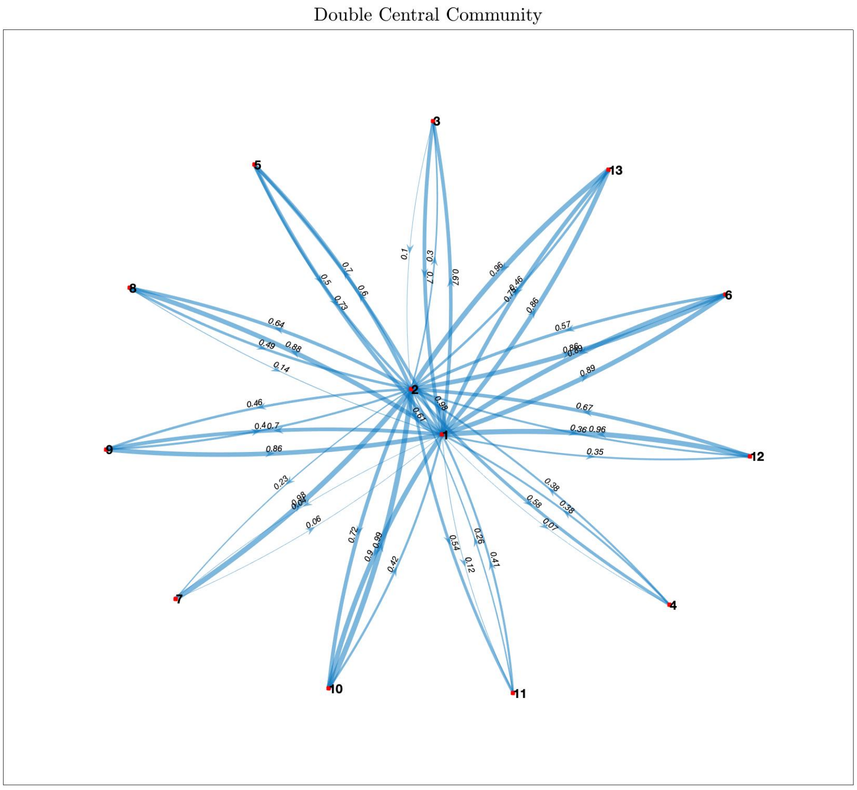

Figure 2.3: Ilustration of a double central network graph.

connection with vertices next next from them (node No.1 can directly connect node No. 3 and node No. 12), it is the graph of double cycle community. This graph is fairly common in social life. In a community, people only know their next door and next two doors neighbours, but they can always find a person through other people's introduction.

The graph of double central community shows connections that both central vertices can directly connect all vertices but noncentral vertices can only directly connect central vertices. In a simple financial market, these two central vertices

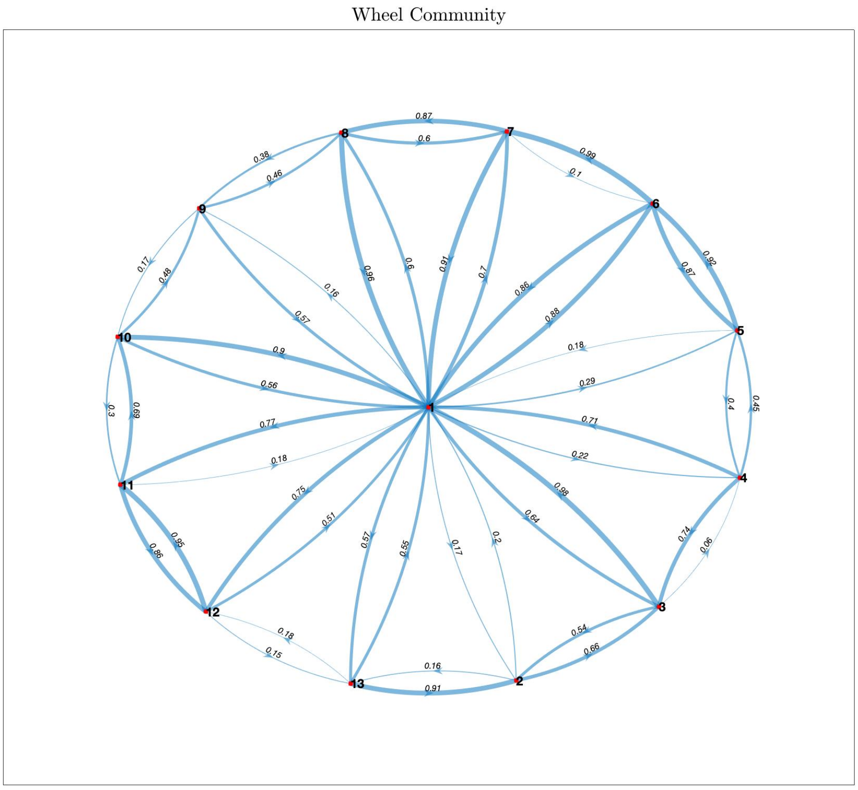

Figure 2.4: Ilustration of a wheel/hub-perimeter network graph.

can be big banks and other nodes can be depositors. Depositors only save or draw their deposit from these two banks.

Figure 2.4 looks like a wheel of a vehicle, so it is called wheel community. In the network, each vertices can directly connect with central node No.1 and their next neighbours. In the graph, if a node No.12 wants to connect with other node No.7 which is not in its direct walk path, there are three options. The first and second options are following the perimeter cycle in the clockwise or anticlockwise direction walking through direct and indirect paths and finally reaching No.7.

The third option is connecting No.7 only through the central node No.1 directly. Obviously, the third option costs less than other two options. In job market, the central vertices can be a job agency or consultant company and all these noncentral nodes can be employers and employees. In some situations, going through a job agency is a more effective way to find a good job.

In Figure 2.5 I present an illustration of a game played between two players for which the pay-off is concave, a) for each player and b) jointly for each player (it is supermodular). The stable equilibrium is denoted by the green dot (this is the final resting place for a sequence of updated strategies). In this case, the players actions are drawn at a few random points (the red dots) and the joint iteration of the game, within the constraint space is iterated to a solution; that is each player is given a random solution to the game, then updates based on the other players initial random choice, the players then respond to each other in rounds until there is no change in the choice within the action space. In this case the game is diagonal concave for both players hence for any viable random starting point the game will always iterate to the attractor that describes a fixed point that solves the game, which is the Nash equilibrium.[1]

In this case the game is of the 2×2 type and easy to illustrate, we know that if only a single attractor point exists then this is the unique Nash equilibrium of the game. Extending this type of example to multiple player networks involves describing the impact of player i's action on all of the other players in the game and sequentially solving the updates to illustrate whether an equilibrium a) exists and b) is stable, as in this case. The objective of my chapter will be to combine

[1]In this case each player has a payoff $U_i = L_i \exp(-a_i x_i + b_i x_j) - c_i(x_i + x_j)$, for $i, j \in \{1, 2\}$ and $i \neq j$, with non negative coefficients L_i, a_i, b_i and c_i.

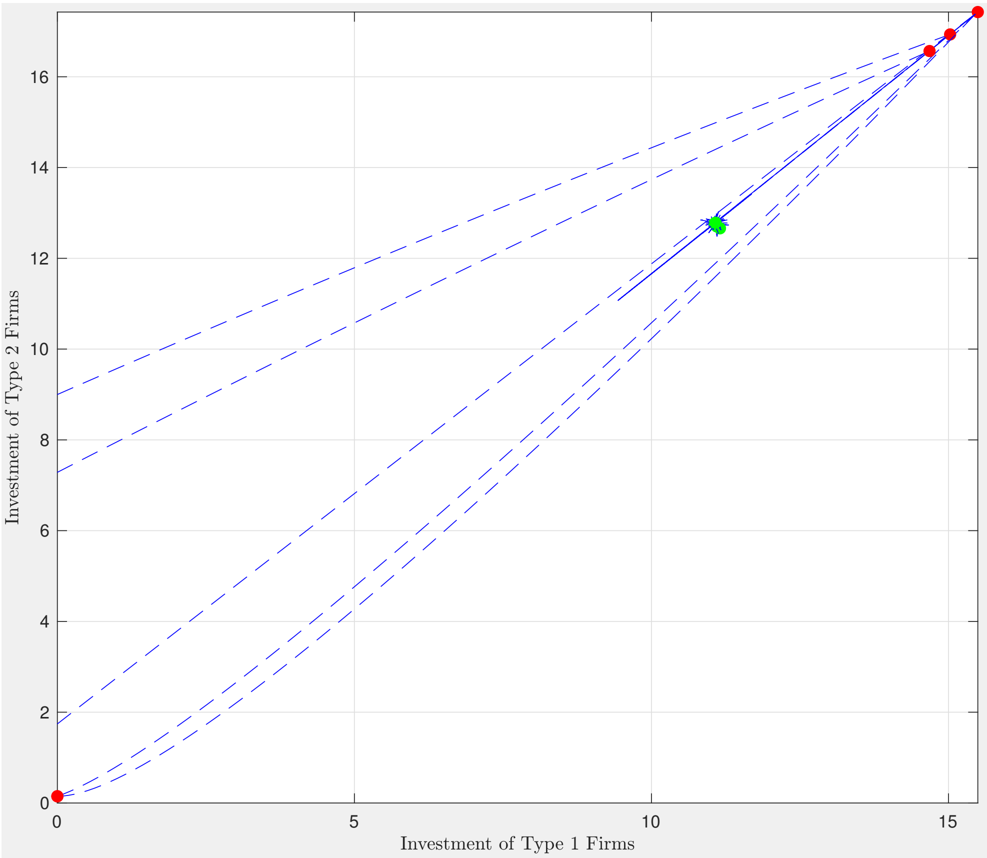

Figure 2.5: A simple two player concave game.

the network structure literature and the equilibrium stability literature to build a series of general results in regard to the stability of games.

2.4 Centrality and Eigencentrality

Identification of central vertices, edges or paths is one of the main components of analysis of networks. Of core issue is the orientation of the node and vertex structure. For any given set of nodes and vertices an inversion in the translation or ordering to the adjacency matrix changes the way in which the network structure is represented either graphically or through the statistical properties of linkages.

Centrality has a number of definitions and numerous mathematical descriptions of different centrality measures have been proposed over time. Generally, network centrality is assumed to produce power or 'importance' of vertices within networks. Clearly, ‘importance’ is not a strictly mathematically defined concept. Whereas centrality indicates some degree of central importance. The first approaches to measurement have focused on countiung the number of connections and how many connections any given node is away from another. This counting framework has formed the basis of network measurement. In contrast to the counting based approaches eigencentrality uses eigenfunction based matrix decomposition of the adjacency matrix to deliver metrics that indicate the degree of explanatory power of specific nodes across the network. The approach draws parallels from principal component analysis, PCA, where a non-negative definite matrix is decomposed by $\boldsymbol{M} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}$, where $\boldsymbol{V}^{-1}$ is the standard inverse of $\boldsymbol{V}$. Here $\boldsymbol{D}$ is a diagonal matrix. In the case of networks, we have potentially negative matrices (such as those where nodes do not connect with any weight to themselves), and potentially enclaves disconnected from the rest of the network. However, by assumption we presume that we deal with fully connected networks (i.e. every node is accessible by a finite number of steps) even if the adjacency matrix is notably sparse.

Consider a small sample from the data set of Kumar et al. [2016] and Kumar et al. [2018], which looks at trusted bitcoin transactions. Bitcoin networks is a anonymously trading online system for cryptocurrency exchange. Due to the characteristics of anonymity, there exists counterparty risks which lead to some Bitcoin trading platforms allowing Bitcoin users to rate the level of trust they have in others Moore and Christin [2013]. In this sample there is a weighted

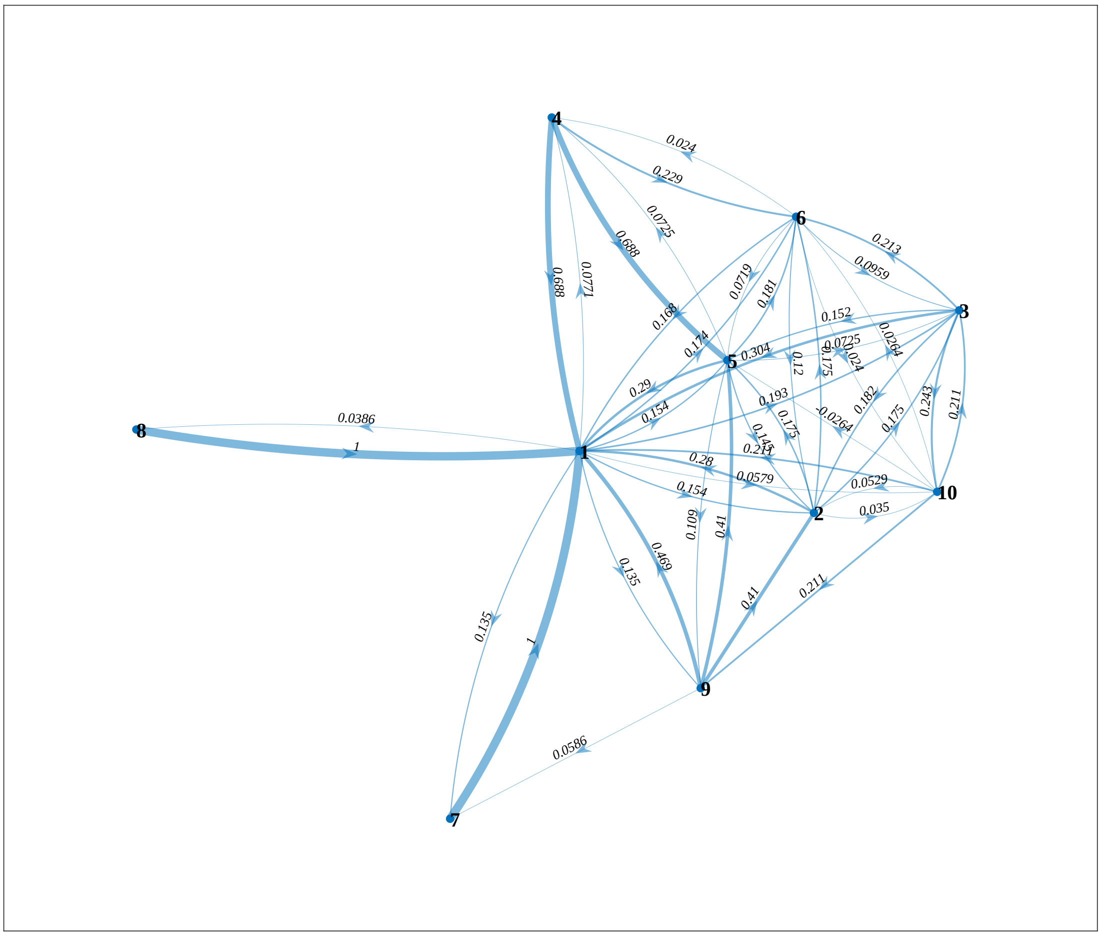

Figure 2.6: Sample of Bitcoin trusted users from Kumar et al. [2016].

network of 10 bitcoin traders, with trust ratings between them, see Fig. 2.6 for the network diagram. In Table 2.1 I present the weight adjacency matrix $\boldsymbol{A}_G$ for the first ten traders in the bitcoin network.

From Fig. 2.6, the highest ratings 1 are from $user8$ and $user7$ to $user1$. However, $user1$ does not have the same high level trust to these two users, that 0.0386 ($user1$ to $user8$) and 0.135($user1$ to $user7$). Similar examples are also showed between $user4$ and $user5$, $user1$ and $user4$. The lowest rating -0.0264 is from $user10$ to $user5$, but there is no direct connection from $user5$ to $user10$.

Table 2.1: Adjacency Matrix

0.0000	0.1543	0.1928	0.0771	0.1543	0.1736	0.1350	0.0386	0.1350	0.0579
0.2804	0.0000	0.1752	0.0000	0.1752	0.1752	0.0000	0.0000	0.0000	0.0350
0.3036	0.1822	0.0000	0.0000	0.1518	0.2125	0.0000	0.0000	0.0000	0.2429
0.6882	0.0000	0.0000	0.0000	0.6882	0.2294	0.0000	0.0000	0.0000	0.0000
0.2902	0.1451	0.0725	0.0725	0.0000	0.1814	0.0000	0.0000	0.1088	0.0000
0.1678	0.1198	0.0959	0.0240	0.0719	0.0000	0.0000	0.0000	0.0000	0.0240
1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4690	0.4103	0.0000	0.0000	0.4103	0.0000	0.0586	0.0000	0.0000	0.0000
0.2115	0.0529	0.2115	0.0000	-0.026	0.0264	0.0000	0.0000	0.2115	0.0000

Consider the nine measures of centrality outlined in Table 2.2. Here we have the four simplest measures of centrality (Out degree, In Degree, Out Closeness, In Closeness) and then five measures that are variations on eigenvector centrality (Betweenness, PageRank, Hubs, Authorities and Eigenvector centrality). From this table, It is obvious that $user1$ is the most important user in the whole Bitcoin network. $User1$ has the highest rate in all four centralities. For instance, rates of Out and In Degrees of $user1$ are both 9 which means there are other 9 users directly connect to or be directly connected by $user1$. For Closeness Centrality, $user1$ receives 0.111 which means $user1$ is the most convenient nodes to directly connect with other users. Betweenness-centrality of $user1$ is 33.583 which means $user1$ have most connections with other users. On the contrary, Betweenness-centrality rates of $user4$, $user7$ and $user8$ are 0 because they have loops. For Hubs, Authorities, PageRank and Eigenvector Centrality, $user1$ is the most influenced nodes in the network because $user1$ has most connections to other high-influenced nodes/users like $user5$, $user6$, $user8$ and $user10$.

The first research application of centrality is introduced by Bavelas [1948]. Here centrality is deemed to be related to group influence in human communication. Bavelas [1950] and Leavitt [1951] apply centrality ideas into the analysis of

Table 2.2: Centrality Measures

$\mathcal{C}_O$	$\mathcal{C}_I$	$\mathcal{C}_\omega$	$\mathcal{C}_\iota$	$\mathcal{C}_B$	$\mathcal{C}_P$	$\mathcal{C}_H$	$\mathcal{C}_A$	$\mathscr{V}_{\max}[g]$
9.0000	9.0000	0.1110	0.1110	33.5830	0.2070	0.1530	0.1530	0.6040
5.0000	6.0000	0.0770	0.0830	1.4170	0.1160	0.1160	0.1350	0.2630
5.0000	5.0000	0.0770	0.0770	0.2500	0.1040	0.1190	0.1190	0.2440
3.0000	3.0000	0.0670	0.0670	0.0000	0.0720	0.0780	0.0730	0.3140
6.0000	7.0000	0.0830	0.0910	4.3330	0.1370	0.1240	0.1480	0.3410
6.0000	6.0000	0.0830	0.0830	2.4170	0.1240	0.1300	0.1310	0.2150
1.0000	2.0000	0.0590	0.0630	0.0000	0.0490	0.0280	0.0430	0.2840
1.0000	1.0000	0.0590	0.0590	0.0000	0.0350	0.0280	0.0280	0.2540
4.0000	3.0000	0.0710	0.0670	1.0000	0.0670	0.0870	0.0750	0.2820
6.0000	4.0000	0.0830	0.0710	1.0000	0.0900	0.1370	0.0940	0.1450

Notes: Out Degree $\mathcal{C}_O$, In Degree $\mathcal{C}_I$, Out Closeness $\mathcal{C}_\omega$, In Closeness $\mathcal{C}_\iota$, Betweenness-centrality $\mathcal{C}_B$, PageRank $\mathcal{C}_P$, Hubs $\mathcal{C}_H$, Authorities $\mathcal{C}_A$, Eigenvector Centrality $\mathscr{V}_{\max}[g]$

communication patterns and their performance in small groups, and conclude the relationship between centrality and group efficiency, especially in perception of leadership and the personal satisfaction of participants. Centrality has been used in a great amount of experimental studies such as political integration in Indian social life Cohn and Marriott [1958], design of organizations Beauchamp [1965] and Mackenzie [1966] explanation of the diffusion of a technological innovation in the steel industry Czepiel [1974] and advantage in exchange network Marsden and Lin [1982] and Cook et al. [1983]. After reviewing a number of published measures, Freeman [1978] reduces these methods to three basic concepts which are degree, betweenness and closeness, and he produces conceptual foundations of centrality, both point-centrality and graph-centrality. Here is a brief review of definitions of these centrality measures.

Following definitions presented in previous chapter, a network G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set of $E(G)$ of edges. The adjacency matrix of G is the $n \times n$ matrix $\boldsymbol{A}_G := (a_{v_iv_j})$, where $a_{v_iv_j}$ is the number of edges joining vertices v_i and v_j, each loop counting as two edges. $a_{v_iv_j} = 1$ if vertices v_0 and v_1 are connected by an edge and $a_{v_iv_j} = 0$ if they are not. The distance is the length of the shortest walk between vertices, denoted by $dist_G(v_i, v_j)$ $(v_i, v_j \in V)$. $d_G(v)$ is the degree of vertex v which is the amount number of neighbours who v links with in G.

The fundamental quantity description of centrality measures is *Degree* $\mathcal{C}(v_i) = \sum_{i=1}^{n} a(v_i, v_j) = d(v_i), v_i \in V$, which take into account the number of edges (direct relations) for a vertex Nieminen [1974].

This method is more useful for explaining immediate effects instead of long-term influences in network systems Freeman [1978]. *Betweenness* is denoted by $\mathcal{C}_B(v_i) = \sum_{k=1}^{n} \sum_{j:j>k}^{n} [g_{kj}(v_i)/g_{kj}], v_i \in V$ where g_{kj} is the number of geodesics between v_k and v_j, and $g_{kj}(v_i)$ is the number of such geodesics containing v_i Technical Report BN9/71 [1971] and Ruhnau [2000]. It measures the frequency that a shortest or geodesic path between vertices v_k and v_j travels through a vertex v_i whose centrality is being measured. *Closeness* of vertex v_i is defined as the reciprocal of the sum of the length of the shortest distance to its linked vertices, denoted by $\mathcal{C}_\gamma(v_i) = 1/\sum_{j=1}^{n} dist(v_i, v_j), v_i \in V$ Sabidussi [1966], Freeman [1978] and Ruhnau [2000]. This measure is only valid for connected networks because the distance between unlinked vertices is undefined.

Another important centrality measure, which is among the most popular centrality measures and I use in my research, is Eigenvector centrality, also called eigencentrality. This measure uses principal or dominant eigenvector of an adjacency matrices in a network Bonacich [1972] to rank/weight connections according to their centralities. In other words, a vector has higher eigencentrality if its connected vertices are themselves central. This is quite different from Degree-centrality which considers every connection equally. And also, comparing with Betweenness-centrality and Closeness-centrality, it is no restriction of geodesics for eigencentrality. It means that eigencentrality can weightedly sum the both directed and indirected connections of every length Bonacich [2007].

In mathematical notions, eigencentrality is calculated by the eigenvector of the largest eigenvalue of an adjacency matrix to analysis the centrality of a vertex Hubbell [1965]. Bonacich [1972] define eigencentrality $\mathscr{V}_{\max}[g](v_i)$ of a vertex v_i as a matrix equation and as a positive multiple of the sum of adjacent centralities. Consider λ^{max} is the largest eigenvalue of $\boldsymbol{A}_G$ and n is the number of vertices:

Definition 1.

$$\boldsymbol{A}_G \mathscr{V}_{max}[g](v_i) = \lambda^{max} \mathscr{V}_{max}[g](v_i) \tag{2.7}$$

$$\lambda^{max} \mathscr{V}_{max}[g](v_i) = \sum_{j=1}^{n} a_{ij} \mathscr{V}_{max}[g](v_i)(v_j), \quad i = 1, \ldots, n \tag{2.8}$$

The matrix $\boldsymbol{A}_G := (a_{v_i v_j})$ is symmetric and all its eigenvalues are real, so it is diagonalizable and its eigenvectors are orthogonal Golub and Van Loan [2013]. There also exists $-\lambda^{min}(\boldsymbol{A}_G) = \lambda^{max}(-\boldsymbol{A}_G)$ in which λ_{min} presents the lowest eigenvalue Bramoullé and Kranton [2015]. The first eigenvector of an adjacency matrix is seen as the eigenvector associated with the largest eigenvalue to evaluate the actors' positions in networks Katz [1953] and Bonacich [1972]. Multiple eigenvectors (the second, third and subsequent eigenvectors) with relatively large eigenvalues are discussed frequently since the early time of centrality analysis Comrey [1962] and Romney et al. [1986]. These additional eigenvectors may provide more comprehensive understanding of modest- or even large-scale networks Iacobucci et al. [2017].

Eigencentrality provide efficient measures to identify and analysis the importance of individuals and their interactions in real-world networks. Among many centrality indices applied to determine structures of players' importance in networks is making use of the eigenvector with largest eigenvalue of an adjacency. For taking one of the main jobs of centrality measures, eigencentrality has made a great contribution on identify of the most influential players who are the highly active to affect their neighbours actions in different network systems. In mobile ad hoc systems (MANETs) eigencentrality is applied to identify dissemination power of nodes Atsan and Ozkasap [2007], and in a disconnected mobile network, Carreras et al. [2007] extend the use of eigencentrality to connectivity matrix and analyze spreading power of nodes in a highly partitioned mobile networks. Ding and He [2010] have a discussion of biological significations of top 10 metabolites (ranked by eigencentrality) in 20 different metabolic networks and provide some new principles for drug target identification and therapy design.

In social networks, Maharani et al. [2014] use eigencentrality measures to identify the most influential users in small and medium enterprise (SME) twitter database to improve effectiveness of social media marketing. Bihari and Pandia [2015] use it to find out who is the prominent author in research professionals' relationship network. Parand et al. [2016] propose that using eigenvector centrality to calculate the post influential player in a combined algorithm of fuzzy inference system. Li et al. [2016] introduce a conductance eigenvector centrality (CEC) model in multiplex network system to determine influential peers. Taylor et al. [2017] develop eigencentrality measure to quantify the influences of nodes in time-dependent networks. Agryzkov et al. [2019] focus on urban layout and use eigencentrality to locate the most active areas in geo-located data and urban street spatial networks.

The other importance applications of eigencentrality is making improvement in the efficiency of network systems. In order to improve observability of the whole power system, Hurtgen et al. [2008] anaylze the Energy Management Systems(EMS) and present a better approach for the placement of measurement devices. Katsirelos and Simon [2012] use eigencentrality to identify an essential structural property of industrial instances, and have better understanding of behavior of Conflict Driven Clause Learning algorithms (CDCL) such as modern satisfiability problem (SAT) solvers. In data mining, New Frontiers in Mining Complex Patterns in Conjunction with ECML/PKDD [2016] present a low time cost but still secure method of features selection drawn from symmetric matrices that ranking features based on their eigencentrality values, but in data set with asymmetric matrices matrices singular values are regular choice Wang and Sukthankar [2014]. Ditsworth and Ruths [2019] provide better indicators than conventional method for community detection by leveraging localization of eigencentrality against the robustness of Katz centrality in sufficiently modular networks. Cheung et al. [2020] analyze the world container shipping network connectivity in which nodes are ports and edges are sailings between ports, and present a novel max-min integer optimization model to locate the potentially better ship routes by computing the eigencentrality value of links.

Moreover, eigencentrality has been widely applied in research of brain network connectivity. As a graph analytical technique, eigencentrality measure has characteristics of alternative assumption-free and parameter-free. Lohmann et al. [2010] take this advantage and present applications of voxel-wise eigencentrality mapping (ECM) on capturing intrinsic neural architecture on a voxel-wise level in task-absent conditions in functional magnetic resonance imaging (fMRI) times series. Wink et al. [2012] propose a fast eigencentrality mapping (fECM) measure which is higher performance to calculate the voxel-wise centralities directly from fMRI data rather than explicitly storing the brain connectivity matrix. Based on real practical data of patients' brain networks, ECM measure efficiently identify the changes in patients' brain network hierarchy caused by Alzheimer's disease (AD) Binnewijzend et al. [2014] and Major depressive disorder (MDD) Song et al. [2016].

Recent years the interests of network research are towards complex system, also known as multiplex networks. Comparing with classical mono-layer network presentation, multiplex network can more accurately map interactions of vertices through multilayers of edges. In order to calculate centralities in multiplex networks, several researchers have made contribution to generalize centrality measures in mono-layer network to the frame of multilayered systems. As one of the most popular centrality measures in network science, eigencentrality measures have been developed to apply in multiplex networks in two main ideas. Firstly, it is the matrix-based centrality indices. Solá et al. [2013] introduce an influence matrix which is nonnegative and define the local and finally the global heterogeneous eigenvector-like centrality matrix of the multiplex which is a matrix of eigenvector with leading eigenvalue. They, of course, proof the existence and uniqueness of such eigencentrality measures. Secondly, it is a forth-order tensor-based centrality index. De Domenico et al. [2013], De Domenico et al. [2013] and De Domenico et al. [2015] propose multilayer adjacency tensor and produce a tensorial equation to extend the mathematical formulation of eigencentrality measures to multiplex networks. Based on their works, Tudisco et al. [2018] propose a new eigencentrality measure in multiplex networks relies on the Perron eigenvector of a multi-homogeneous map. Pedroche et al. [2019] extend eigencentrality measures for single layer networks to multilayer networks and Benson [2019] extend the concept of such centrality measure to uniform hypergraphs and propose three tensor eigencentralities.

However, eigencentrality measures are not always perfectly valid in real-world complex systems. For instance, eigencentrality displays a localization transition that assigning large weights to hub nodes and comparatively smaller weights to their neighbouring nodes but ignoring (negligible weights) other nodes even whose degrees are higher than hub nodes' neighbours. In such situation, eigencentrality is no longer play a role that efficiently distinguishing the importance of nodes. Martin et al. [2014] provide an alternative centrality measure based on the non-backtracking matrix to avoid localization problems. Pradhan et al. [2020] suggest that degree centrality is a better choice for ranking nodes in networks which have delocalized localized principal eigenvector (PEVs).

Chapter 3

Literature Review

3.1 Literature Review on Networks

Network analysis from the perspective of strategic economic agents is a relatively new area of research, with most key contributions arising since 2000, with the main contributions after 2010. Economic analysis on networks has developed from more general analysis of graphs and graph theory within the realm of complex analysis. I have broken down this review of the canonical literature into the following areas, underlying mathematical results, basic theory, applied theory and various empirical applications.

The majority of research in the area of multiplayer games on networks follows from key results on multi-dimensional fixed point theorems starting with Tarski [1955] for the core theory on concave functions and single crossing of the function within a fixed point. As I will carefully demonstrate later in this chapter Rosen [1965] utilized the results given in Tarski [1955] to construct a general theory of multiplayer games when the joint payoff structure is diagonally concave. In this case when strict diagonal concavity occurs, the Nash equilibrium is unique and a stable attractor. That is for any random starting configuration within the valid set of strategies, iterative adjustment in strategy based on the reaction to other players choices always results in convergence on a single point, this attractor is the unique Nash equilibrium and is a stable equilibrium to which all players will eventually converge.

The Tarski [1955]/Rosen [1965] analysis presumes no real network structure, modularity in pay-off is driven by the pseudo gradient of the game, that is the Jacobian matrix that describes the relative changes in welfare of player i to all other players, usually denoted $-i$, in the game. However, in many cases whilst this matrix describes interactions, it needs not to have a trivial functional structure. For instance, if there are cliques, such as the example in Bonacich [1972], the matrix will have a lumpy structure, with internal communication within a clique and no external communication other than by key nodes within the matrix. This results in an eigenvalue-eigenvector pairing for the largest eigenvalue that contains sudden changes in the magnitude of the coefficients. Bonacich [1972] illustrates that such eigendecompositions are not necessarily the optimal approach for describing the network centrality structure.

More recent innovations in the complex analysis literature have provided results on graphs and in particular, strongly connected graphs, that permit a more detailed analysis of the derivatives of eigenvalue-eigenvector pairs for the largest eigenvalue following the Frobenius-Perron system. Most importantly, Deutsch and Neumann [1985] presents a technique that uses a novel inverse to describe the dertivatives of the eigensystem for the largest eigenvalue when the graph is strongly connected. This inverse, known as the group or Drazen inverse is extremely useful for the analysis of the sensitivity of networks to changes in the pointwise elements of the graph. It is to this theory of derivatives of the Perron root of strongly connected graphs that we will appeal, when analyzing strategic choice on n-player games when the graph describing the players interactions is strongly connected.

Using results from graph theory to describe n-player games has a history going back prior to the standard network literature. Diamond and Dybvig [1983] and Hirshleifer [1983] present games on, respectively, bank runs and weakest links and best shots that capture similar properties to those described by Rosen [1965] and implicitly, but not explicitly, use the Tarski [1955] fixed point theorem to provide a set of solution.

However, it is in the more recent literature that multiple uses for such analysis have become commonplace. Pre the financial crisis of 2008/9 Allen and Gale [2000], Freixas et al. [2000] Morris [2000], Eisenberg and Noe [2001], Brusco and Castiglionesi [2007] and Castiglionesi and Navarro [2008] presented models of financial interdependency that exploited the network structure to illustrate various types of network dependency, much of which could be classed as a Rosen [1965] type game.

Using certain stylized facts on centrality and equilibrium these works presented the early face of the network literature and laid the foundation for work in the post crisis period. Indeed, the nature of this early work should not be underestimated, Allen and Gale [2000] hypothesized that with a more densely interconnected financial network, the losses of a distressed bank are divided among more creditors, reducing the impact of negative shocks to individual institutions on the rest of the system. Indeed, a view that bank risk could be diversified away by increasing connectivity through securitization permeated the early work in this area.

Post 2008/9 a new set of literature appeared that built on the existing models, but took the precise topology of the network more seriously. For instance Acharya [2009] presents a new theory of systemic risk within the design of prudential bank regulation. Whilst Duffie et al. [2009], Elsinger [2011] and Ibragimov et al. [2011] present different views on how to organize markets and institutions to reduce vulnerability to systemic shocks through cascading failures. Similarly, Upper [2011], Wagner [2011], Battiston et al. [2012], Bimpikis and Tahbaz-Salehi [2012], Chen et al. [2013], Chen et al. [2014] Caballero and Simsek [2013], Sachs [2014] and Elliott et al. [2014] present various different models for measuring the effect of complex interactions on the global financial system.

The mix of theory and actual measures of interbank lending have been of interest more recently to Rogers and Veraart [2013], Summer [2013] and Zawadowski [2013] all of whom utilize a mic of theory models mapped onto measures of network structure and centrality to describe financial risk.

Moving away from the purely financial interdependency framing of the economic analysis of networks Bramoullé et al. [2014] present a framework for describing the equilibrium strategies of agents on networks when the payoff function is linear in your won action and concave across the game. This analysis explicitly utilizies the results of Rosen [1965] and Tarski [1955] to determine the uniqueness of equilibrium when the network structure is fixed. A more general theoretical framework is presented in Acemoglu et al. [2015] who fully exploits the results from Rosen [1965] to demonstrate the conditions under which games on networks generate stable equilibrium outcomes.

The literature suggests that an increasing number of researchers have used graph theory to analyze how the stability of financial systems or network structures are affected by different factors. Anderson and Moore [2006] report that difference of network topologies can strongly impact the conflict dynamics especially the robustness properties with respect to different attacks. In symmetric networks, there exists a clear positive externality in security investments Acemoglu et al. [2016] that a player/agent with failure of self-protection will increase the probability of self-infection and contagions. Based on this intuition, Goyal and Vigier [2010], Larson [2011] and Bachrach et al. [2012] comment that there is under-invested in financial security systems. However, there are very limited situations of symmetric networks in real networks. Thus Acemoglu et al. [2016] study the influence/contagion of structures of networks on security in a more generalized asymmetric network, and characterize the infection probability of different agents in different locations in the network with small amount of security investments. Overinvestment can also cause forces such as a robust force in security. De Meza and Gould [1992] report that there exists negative externalities when one player/agent takes preventive actions to move the risk of attacks to others. Of course, the amount of investment in security is one of the elements which affect the stability of financial networks. There are also a large number of literature focusing on other important elements like spreads of infections (contagions) Molloy and Reed [1998] Molloy et al. [2011] Newman et al. [2001] and Chung and Lu [2002]. In the early age of such research, Sanders [1971] and Sethi [1974] pay attention to the control of spreads, and Brito et al. [1991], Geoffard and Philipson [1997], Goldman and Lightwood [1996], Toxvaerd [2009] and Galeotti and Rogers [2013] investigate certain aspects of precautionary or vaccination behavior in different situations. In recent researches, Bachrach et al. [2012], Goyal and Vigier [2014] and Larson [2011] analyze that if infections have already existed in the network how the endogenous formation of networks reacts on upon security decision-makings in symmetric networks. Jackson and Wolinsky [1996], Bala and Goyal [2000] and Blume et al. [2013] are also interested in similar questions but make contribution to provide bounds on the inefficiency of equilibria situation. There are also related researches on strategic attacks where attacks are shifted from one player to another by his/her precautionary behaviors without consideration of influence of network structures De Meza and Gould [1992], Baccara and Bar-Isaac [2008], Bachrach et al. [2012], Goyal and Vigier [2010], Kovenock and Roberson [2018], Bier et al. [2007] and Hoyer and Jaegher [2016]. Moreover, literature on different interactions of networks are also worth to review. Ballester et al. [2006], Bramoullé and Kranton [2007a], Bramoullé and Kranton [2007b], Calvó-Armengol et al. [2009], Galeotti et al. [2010], Bramoullé et al. [2014] and Allouch [2015] report the relationship between equilibrium strategies and centrality measures in networks such as using eigencentrality in a linear-quadratic structures. These researches are closely related to my works. There are also different measurements of spreads of shocks over networks which more focus on the interactions of gatekeeper nodes Acemoglu et al. [2015], Golub and Jackson [2012], and Goyal and Vigier [2013].

Before the financial crisis in 2008/2009, some researchers considered liquidity of banks as an important index to 'forestall' the contagious failures. For instance, Cifuentes et al. [2005] present that liquidity requirements on institutions may be more effective in a shock such as financial crisis in 2007 because it can internalize some of the externalities. However, Allen and Carletti [2008] report that assets' future earning power is more effective in assessing financial institution's insolvency rather than liquidity when a shock happen in the insurance sector. Thus, they suggest using the the market value of bank's assets should be calculated on the basis of historical cost accounting rather than market-to-market accounting. Moreover, there are other two factors such as structure of the financial system and information contagion affecting the stability as well. Nier et al. [2007] consider the influence of structure of the financial system which includes its capitalisation, the degree of connection between banks and the concentration of the system, and the size of interbank exposures and the degree of concentration of the system. Acharya and Yorulmazer [2008] and Cabrales et al. [2015] discuss how the various aspects of information contagion to affect the systemic risk in different level markets.

In the context of the current crisis, researchers were not satisfied with finding the factors affecting or leading to the contagious failures, and engaged in an alternative or deeper study of the control of system risk. For example, Caballero and Krishnamurthy [2008] built a model which incorporates Knightian uncertainty to explain the crisis regularities. Gertler et al. [2010] present a canonical framework to discuss the credit market frictions and aggregate economic activity. Alternatively, Gai and Kapadia [2010], Gai et al. [2011], Allen et al. [2012], Georg [2013], Brunnermeier and Sannikov [2014], Caccioli et al. [2014] and Alvarez and Barlevy [2015] develop different network models of contagion to explain how the bank interaction works and provide some future suggestions concerning effective tools and policy measures.

More recently, many researchers have offered further new contributions toward the interconnection among financial institutions to system risk. Glasserman and

Young [2015], Erol [2016], Babus [2016] and Hurd [2016], for example, analyze the amplification of shocks of interconnections to the financial system. Babus [2016] suggest a modelling of bank's decisions that involves sharing this risk through bilateral agreements. In further research, some analytical expressions for systemic risk assessment in financial network have been developed. For example, Amini et al. [2016] derive rigorous asymptotic results for the magnitude of contagion in a large counterparty network, and Gandy and Veraart [2016] develop a Bayesian methodology. Staum et al. [2016] use the Shapley and Aumann-Shapley values to attribute the systemic risk in a network model. Cabrales et al. [2017] use the socially optimal design of financial networks to tackle the trade-off between risk sharing and contagion. Another ideas to discuss the interaction between different financial institutions is based on agency such as Erol [2016] and Dang et al. [2017].

Moving away from purely theoretical research into financial networks, there are a large number of previous empirical researches on contagions and assets spillover of banks, macro or micro economics, and other applications. Mostly, the data driven analysis of financial network structure and interbank contagion uses centrality measures. In the1990s, Sheldon and Maurer [1998] use an entropy maximization method to analyse the credit risk from the structure of interbank loans in Switzerland. He find that from 1987 to 1995 there was a quite low possibility of bank crisis spreading though the network within the banking system. However, this early research is considered to contain a number of limitations. Similarly, Van Lelyveld and Liedorp [2004] explore the interlinkages and contagion risks through the maximum-entropy and minimum-density approach. They use the balance sheet data and large exposures reports from De Nederlandsche Bank (DNB) and find an adequate approximation of the actual linkages between banks.

Moreover, Upper and Worms [2004], Wells [2004], Elsinger et al. [2006], Degryse et al. [2007], Santos and Cont [2010], Cont et al. [2010], Mistrulli [2011], Hałaj and Kok [2013], Alter et al. [2014] and Anand et al. [2015] estimate two main sources of system risk such as correlated credit exposures and interbank connectivity through a matrix of bilateral exposures and money-centre model for banking systems in Germany, United Kingdom, Austria, Belgian, Brazil and Italy. However, other methodologies can be used to analyse the contagiousness and vulnerability in different interbank market as well. For instance, Puhr et al. [2012] use a panel model approach to explore the defaults (dependent variables) to network generated by balance sheet indicators (independent variables). More generally, Chen et al. [2016] study the interconnection in financial institutions which is affected by the network channel and the liquidity channel. Anand et al. [2018] reconstruct the he structures of links and exposures in network in financial system. Both of these researches contribute to the effectiveness of certain policy intervention and micro- and macro-prudential policy.

In the interbank network, assets spillover is another hot topic which explore the correlation and dependency literacy in asset pricing. There are various methods to estimate the spillover. Craig and Von Peter [2014] analyze how the systemic risk is affected by interbank lending channel, and present empirical evidence on spillover effects between banks' probabilities of distress and the financial profiles of connected peers. Bonaldi et al. [2015] and Duarte and Eisenbach [2018] construct new systemic risk measure of the 'systemicness' (Duarte and Eisenbach [2018]) and vulnerability to quantify the spillovers between funding costs of individual banks and the vulnerability to fire sale spillovers.

Other researchers have conducted more general studies into the stability of the network structures in financial system. Inaoka et al. [2004], Soramäki et al. [2007], Shin [2010], Battiston et al. [2012], Gabrieli [2012], Craig and Von Peter [2014] and Blasques et al. [2018] develop some novel measures in which they treat the network structure of financial transactions between commercial banks as elements, and evaluate the financial stability. Drehmann and Tarashev [2013] and Allahrakha et al. [2015] empirically present the systemic importance of interconnection between banks. Bassett et al. [2014], Gabrieli and Georg [2014], Martinez-Jaramillo et al. [2014], Bennett and Unal [2015] and Fricke and Lux [2015b] discuss the influence of macro and economic effects of financial crisis (or credit shocks) on the stability of financial systems based on explorations into the supply of loans and interbank exposures, the payments system networks, liability concentration, resolution costs, the liquidity allocation and distribution of credit links between institutions.

In the micro research of networks, Furfine [1999], Boss et al. [2004], Bech and Atalay [2010], Gofman [2011], Haldane and May [2011], Diebold and Yılmaz [2014] and Fricke and Lux [2015a] study the microstructure of network of some typical markets such as the Federal Funds Market and Interbank Market, and present some empirical findings of how the network topology affect the efficiency and stability of the financial system. Other researchers have focused on the exact events of successful or failure financial events. For instance, Lucas Jr [1993] find the growth miracles of East Asia in 1990s is because of the accumulation of human capital of knowledge in these areas. James [1991], Khandani and Lo [2011], Rothman [2007], Afonso et al. [2011], Commission and Commission [2011], Fleming and Sarkar [2014] and System [2015]survey different events of bank failures and financial crisis. They sought to understand what and why these events happened and, based on their findings, offered some effective suggestions for regulators and banks toward keeping the financial system being stable in the future based on their findings. On the other hand, in the research of macroeconomy, Adrian and Boyarchenko [2012],Billio et al. [2012], Hansen [2012], Hüser [2015] and Acemoglu et al. [2016] present econometric approaches, involving principal-components analysis and Granger-causality networks, and conducted some empirical investigations into the influence of different financial shocks for the systemic risks in finance.

Furthermore, network research not only focuses on the financial context, but is widely used in different areas. Keener [1993] use the Perron-Frobenius theorem of game theory to rank football teams in uneven paired competition. Zhang et al. [2007] focus on the expertise networks in web-based communities and explore the difference of the structure and algorithms in these communities. Van Rijnsoever et al. [2015] conclude that it is helpful to design "smart" innovation policy instruments based on an exploration of the relationship between social network and innovation system on the creation diversity of an emerging technology.

Finally, there are some foundational books from Allen and Babus [2009], Gorton [2010], Newman [2010], Angelides et al. [2011], Duffie [2012], Bramoullé et al. [2016] that offer a comprehensive introduction to and summary of earlier networks research, theories, models and research methodologies.

3.2 Case Study

In Figure 3.1 I illustrate a variety of different network structures using a plotting technique, such that the distance between nodes is based on their *Katz centrality*

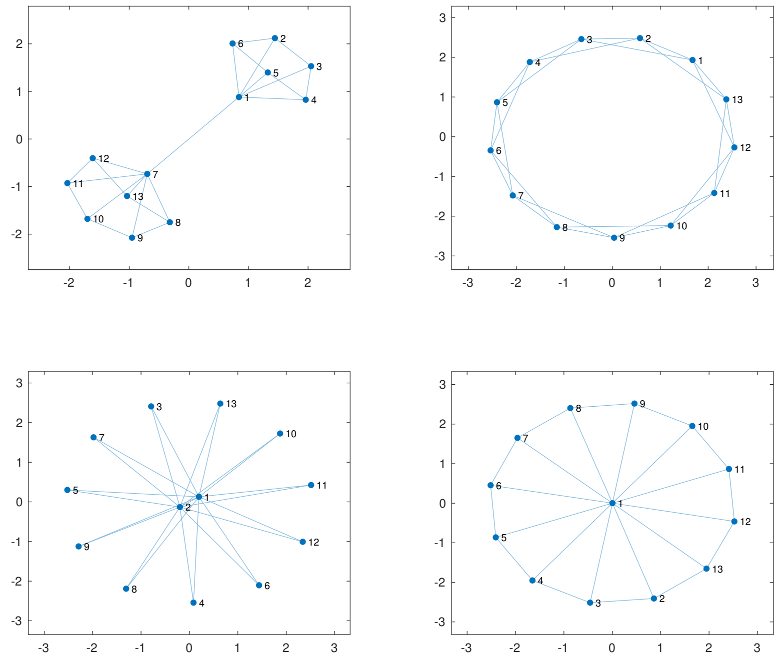

Figure 3.1: A variety of simple network structures.

relative to the mass of nodes within the matrix. In these cases the matrix is unweighted, hence a connection between two nodes is reported as a one in a matrix representing the graph of the network. However, in general I will be looking at cases where the network is weighted, hence each element of the graph reports either a zero for no connection or a positive number to indicate the weight of the connection between nodes. A further restriction on my work is that I am exclusively dealing with networks whereby the number of steps to go from any node to any other node in the network is finite. This is referred to as a strongly connected graph and this will form the basis of my analytical work.

3.2.1 Example: Centrality and a simple network

Let $G(a, l, x) = [.]$ be a matrix representing a graph, with nonnegative parameters a, l and x. Gandy and Veraart [2016] illustrate their analysis of financial networks using a simplification of the following structure for the graph:

$$G(a, l, x) = \begin{pmatrix} 0 & a - l & -a - l + x \\ a - l & 0 & a - l \\ -a - l + x & a - l & 0 \end{pmatrix}, \tag{3.1}$$

where $x > 0, a > 0, l > 0, a > l$ and $a + l < x$. We can see this is a strongly connected graph with all nodes having a positive value, in this case the values can be interpreted as transfer deposits for interbank rates, but any network interaction is valid.

Several authors, including Gandy and Veraart [2016], show that for networks such as this, the eigenvector associated with the largest eigenvalue provides information on the solution to a game with some concave payoff $U_i(a, l, x)$ for $i \in \{1, 2, 3\}$, such that the Jacobian matrix described by $\partial U(a, l, x)/\partial z$ for $z \in \{a, l, x\}$ is negative definite. It is to this class of pay-off functions that my first chapter is mostly applicable.

Setting $G(a, l, x) = \boldsymbol{V}\text{diag}[\boldsymbol{d}]\boldsymbol{V}'$ to be the eigensystem decomposition of $G(a, l, x)$ where $\boldsymbol{V}$ is the 3×3 matrix of eigenvectors (by column) and $\boldsymbol{d}$ is a 3×3 with the eigenvalues of $G(a, l, x)$ on its diagonal, using the Lanczos algorithm the special case for the eigensystem of this particular 3×3 matrix can be written as follows:

$$\boldsymbol{d} = \begin{pmatrix} a+l-x \\ \frac{1}{2}\left(-\sqrt{9a^2-2x(a+l)-14al+9l^2+x^2}-a-l+x\right) \\ \frac{1}{2}\left(\sqrt{9a^2-2x(a+l)-14al+9l^2+x^2}-a-l+x\right) \end{pmatrix} \tag{3.2}$$

with corresponding eigenvectors:

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & \frac{a+l-x-\sqrt{9a^2-14la+9l^2+x^2-2(a+l)x}}{2(a-l)} & 1 \\ 1 & \frac{a+l-x+\sqrt{9a^2-14la+9l^2+x^2-2(a+l)x}}{2(a-l)} & 1 \end{pmatrix} \tag{3.3}$$

Picking the non-trivial eigenvalue, eigenvector pair, we have:

$$d_2 = \frac{1}{2}\left(-\sqrt{9a^2-2x(a+l)-14al+9l^2+x^2}-a-l+x\right), \quad \text{eigenvalue} \tag{3.4}$$

$$\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ \frac{a+l-x-\sqrt{9a^2-14la+9l^2+x^2-2(a+l)x}}{2(a-l)} \\ \frac{a+l-x+\sqrt{9a^2-14la+9l^2+x^2-2(a+l)x}}{2(a-l)} \end{pmatrix}, \quad \text{eigenvector} \tag{3.5}$$

This eigencentrality measure is useful, in of itself, however, when combined with a suitable payoff function, we can even describe the location, existence and stability of a Nash equilibrium, given a set of action variables for each player on the network, which I will describe in §(4.1). Clearly, given the simple algebraic structure of the eigenvalues and eigenvectors, for this special case we can write down the derivatives of $\partial \boldsymbol{v}_2/\partial \boldsymbol{z}$, for $\boldsymbol{z} = (a, l, x)'$ for the structure of entries in

$G(a, l, x)$

$$\partial \boldsymbol{v}_2 / \partial \boldsymbol{z} = \begin{pmatrix} 0 & \frac{(2l-x)(a+l-x-)}{2(a-l)^2 \mathscr{A}} & -\frac{(2l-x)(a+l-x+\mathscr{A})}{2(a-l)^2 \mathscr{A}} \\ 0 & \frac{\frac{a+l-x}{\mathscr{A}}-1}{2(a-l)} & \frac{\frac{-a-l+x}{\mathscr{A}}-1}{2(a-l)} \\ 0 & \frac{(2a-x)(-a-l+x+\mathscr{A})}{2(a-l)^2 \mathscr{A}} & \frac{(2a-x)(a+l-x+\mathscr{A})}{2(a-l)^2} \end{pmatrix}, \quad \text{where} \tag{3.6}$$

$$\mathscr{A} = \sqrt{9a^2 - 14la + 9l^2 + x^2 - 2(a+l)x} \tag{3.7}$$

which permits a whole host of opportunities for analyzing the sensitivity of the centrality measure to the structure $G(a, l, x)$. However, this is only available for this specific case and there are very few others for which an algebraically tractable set of derivatives is possible. The objective of this thesis is to present a general methodology for determining the derivatives of the eigencentrality of of a graph $G(.)$ with respect to the pointwise entries of $G(.)$ for any dimension when the only assumption is that $G(.)$ is a strongly connected graph and illustrate the types of games for which this analysis provides a useful characterisation of the Nash equilibrium and measuring the sensitivity of centrality to changes in the underlying network structure.

Chapter 4

Model Description and Comparative Statics

This chapter focuses on strategic games played on networks. There are three objectives. The first one is to establish a common analytical framework. It starts from defining a class of canonical and widely applicable games with fixed network of interactions such as definitions of games (players, links and payoffs), best response and Nash equilibrium, and class of games and restrictions on the strategy space. Based on these definitions, there is a review of two basic strategic interactions classified as games of strategic complements and games of strategic substitutes. In each type of game, I will introduce typical examples of games, such as the majority game in games of strategic complements and best-shot games in games of strategic substitutes. Moreover, existence of equilibrium of each type of games will be defined and proved.

The second objective is to study characterizations of equilibrium. In my research, I am interested in peer effect which is known as the dependence of individual outcomes on group behavior Ballester et al. [2006]. In standard peer effects games, each player's activity is homogeneous in group but heterogeneous across groups. Ballester et al. [2006] first generalized the pattern of bilateral influence, and applied Bonacich network centrality Bonacich [1987] into analysis of the Nash equilibrium situations. Thus, I introduce the existence of equilibria in a concave multiplayer games following Kakutani et al. [1941], and prove the existence of unique Nash Equilibrium nodes, global stability and determination of the equilibrium node in such game by rewriting Rosen [1965]'s results in graph theory. This work builds a bridge between game theory and graph theory that the existence, uniqueness and globally stable Nash Equilibrium in N-player concave game Rosen [1965] describe the condition of global stability of a weighted network with strongly connected network structure. It is one of my great contributions in the thesis.

As the main centrality measure in my research, Bonacich centrality will be defined base on a benchmark linear-quadratic payoffs model. The relationship between Nash equilibrium and the Bonacich centrality will be introduced as well. Finally, I introduce the general network comparative statics which is an exercise for analyzing the interaction between equilibrium strategies and network topology by the Bonacich measure.

Base on the background introduction of networks, games, Nash equilibrium and Bonacich centrality measure, I will present my first concave network model in the third section. This model describes a simple network externality whereby an agents payoff depends on a function of the actions of the other agents to which the agent is connected. The game is inherently supermodular which ensure the concavity, and the network structure is fixed and strongly connected in which the equilibrium is a positive vector of actions for all agents. After the first derivative of the exponential payoff function by Jacobian matrix, I get a functional form of equilibrium solution in the game. Then, after connecting the equilibrium result with centrality measure, I exercise the comparative statics and present the result.

Contributions of this chapter are first I critically rewrite definitions of strategic games, Nash equilibrium and Bonacich centrality in my own notations. Moreover, I review games of strategic complements and strategic substitutes and reprove the existence of equilibrium of both games. Furthermore, I develop a new simple concave network model. I link the Bonacich centrality (Eigencentrality) with Nash equilibrium and present the equations of comparative statics of equilibrium.

4.1 Games on Network

This section is for introducing the background of basic strategic games played on networks. There are three overarching objectives. The first is to establish notations of strategic games which are also consistent with the previous definitions of networks. The second objective is to review characteristics of games such as strategic complements and strategic substitutes. The final objective is to review games according to their characteristics.

4.1.1 Strategic Games on Networks

A strategic game is a model of a situation in which each player choose an action without information of the other players' actions, and all players take actions simultaneously Osborne et al. [2004]. There are three elements of this game that a set of players, each player's actions and each player's payoffs/preferences.

According to the definitions in previous chapter, N is a finite set of players, $N = 1, \ldots, n$, and g is a network which presents the interconnections of players. There are n players taking simultaneous actions in a strategic game, $n \in N$. Let A be an action space of players' strategies and a be a profile of actions, $a \in A \subset \mathbb{R}$. Let $\boldsymbol{a_{-i}}$ be an action profile of other players' actions except player i's action. For player i and player j, their actions are respectively expressed as a_i and a_j. If both players' actions are embedded in a fixed-structure network, it is denoted by g_{ij}, $g_{ij} \in \boldsymbol{g} \subset \mathbf{R}$. Bramoullé and Kranton [2015] report a *payoff impact parameter* δ to sign and analyze the magnitude of the influence of players' choices on their neighbors, $\delta \in [-1, 1]$. Thus, a player i's payoff function u_i can be presented as $u_i(a_i, \boldsymbol{a_{-i}}; \delta, \boldsymbol{g})$.

For any square matrix $\boldsymbol{M}$ which is also nonnegative, symmetric and irreducible, the lowest eigenvalue is denoted by $\lambda^{min}(\boldsymbol{M})$ and the highest eigenvalue is denoted by $\lambda^{max}(\boldsymbol{M})$, and $\lambda^{min}(-\boldsymbol{M}) = -\lambda^{max}(\boldsymbol{M})$.

Equilibrium is a cooperative result of players' best available actions in a game. A *Nash equilibrium* is an action profile, denoted as a^*, in a strategic game with the property that no player can have better payoffs by choosing action differently from the current one (*best response*) Osborne et al. [2004]. There are two assumptions underlying the analysis of equilibrium. First, each player makes rational decisions based on his/her past experiences which give him/her the correct belief of the other players' choices. Second, each player takes actions individually. Players' do not exactly know each other's action and each individual's choice does not affect his/her neighbours' future behaviors Osborne et al. [2004]. Based on these assumptions, each player makes his/her *best response* to other players' actions in a pure-strategy game which is denoted below Bramoullé and Kranton [2015].

$$f_i(\boldsymbol{a_{-i}}; \delta, \boldsymbol{g}) = \arg\max_{a_i}\{u_i(a_i, \boldsymbol{a_{-i}}; \delta, \boldsymbol{g})\} \tag{4.1}$$

The system of all best response are as follows Bramoullé and Kranton [2015].

$$\begin{aligned} a_1 &= f_1(\boldsymbol{a_{-1}}; \delta, \boldsymbol{g}) \\ &\vdots \\ a_n &= f_n(\boldsymbol{a_{-n}}; \delta, \boldsymbol{g}). \end{aligned}$$

Thus, a Nash Equilibrium which is a profile of players' best responses can be denoted by a vector $\boldsymbol{a}^* = (a_1, \ldots, a_n)$, and $a^* \in \boldsymbol{a}^*$. Both full set and subset of Nash Equilibriums are considered to be stable which means an equilibria is robust to small changes in players' actions Bramoullé and Kranton [2015]. It is worth noting that the notation of stability for binary games is different from continuous games. A stochastic stability based on asynchronous best-reply dynamics and payoffs is applied into binary action games Young [2020]. For continuous action games, the Nash Equilibrium $\boldsymbol{a}^*$ is asymptotically stable when converges to a following any small enough perturbation Bramoullé and Kranton [2015].

Bramoullé and Kranton [2015] report a generalized payoff function as follows:

$$u_i(a_i, \boldsymbol{a_{-i}}; \delta, \boldsymbol{g}) = v_i(a_i - a_i{}^0 + \delta \sum_j g_{ij}a_j) + w_i(\boldsymbol{a_{-i}}) \tag{4.2}$$

where v_i is increasing on $(-\infty, 0]$, decreasing on $[0, +\infty)$ and symmetric around 0, so that 0 is the unique maximum of v_i, and w_i can take any shape. The individual parameter $a_i{}^0$ denotes player i's optimal action without interactions

($\delta = 0$ and/or $g_{ij} = 0$). A higher ${a_i}^0$ means player i receives greater benefit or pays lower cost. If $|\delta|$ increases, it means that the payoff externalities of players' actions are getting globally stronger.

For each player i, $a_i \in A \subset \mathbb{R}$, best replies are linear in other players' actions with the payoffs function in basic cases 4.2:

$$f_i(\boldsymbol{a}_{-i}) = {a_i}^0 - \delta \sum_j g_{ij} a_j. \tag{4.3}$$

where ${a_i}^0$ denotes the play's autarkic optimum, and $\delta \sum_j g_{ij} a_j$ denotes the weighted sum of this player's neighbors' actions. The best reply for the player is the difference between the two.

While in principle, a player's action can take any real value. However, according to the real-world situations, there are different restrictions on players' actions in different games. For instance, the natural upper bounds of a day is no more than twenty-four hours. For action spaces, the first restriction is that players' actions are non-negative, so for each player i, $a_i \in [0, \infty)$. Thus the corresponding best reply is Bramoullé and Kranton [2015]

$$f_i(\boldsymbol{a_{-i}}) = max(0, ({a_i}^0 - \delta \sum_j g_{ij} x_j)) \tag{4.4}$$

In the second restriction of action spaces, players' actions must not be below zero nor be above some finite upper bound L: for each player i, $a_i \in A_i = [0, L]$ with $0 < L < \infty$. The corresponding reply is Bramoullé and Kranton [2015]

$$f_i(\boldsymbol{a_{-i}}) = min(max(0, ({a_i}^0 - \delta \sum_j g_{ij} x_j)), L) \tag{4.5}$$

In both cases a player's best reply is that the difference between ${a_i}^0$, and the weighted sum $\delta \sum_j g_{ij} x_j$.

Finally, players must choose between two discrete values: $a_i \in A_i = a, b$with $a \leq b$. Player i's best reply can be presented as a threshold value $t_i = {a_i}^0 - \frac{1}{2}(a+b)$. If the weighted sum of neighbors' actions is above the threshold, i's best response is a; if the weighted sum is below the threshold, agent i's best response is b; if the sum is equal to the threshold, i is indifferent between a and b Bramoullé and Kranton [2015].

$$\begin{aligned}
f_i(\boldsymbol{a_{-i}}) &= a \quad if \quad \delta \sum_j g_{ij} a_j > t_i; \\
f_i(\boldsymbol{a_{-i}}) &= b \quad if \quad \delta \sum_j g_{ij} a_j < t_i; \\
f_i(\boldsymbol{a_{-i}}) &= a, b \quad if \delta \sum_j g_{ij} a_j = t_i.
\end{aligned} \tag{4.6}$$

The best replies for the constrained actions, 4.4, 4.5, 4.6 can all be obtained from 4.3.

Let $\widehat{a}_i(\boldsymbol{a_{-i}}) \equiv {a_i}^0 - \delta \sum_j g_{ij} x_j$ denote the *unconstrained optimum*. When action spaces are constrained, player i's best reply is the closet value to $\widehat{a}_i(\boldsymbol{a_{-i}})$ within the restricted space.

According to the equivalence of the best replies, it is worth to produce an extensive use of payoffs in quadratic form which is also satisfied conditions of 4.2.

$$U_i(a_i, \boldsymbol{a_{-i}}; \delta, \boldsymbol{g}) = -\frac{1}{2}(a_i - {a_i}^0 - \delta \sum_j g_{ij} x_j)^2 + w_i(\boldsymbol{a_{-i}}). \tag{4.7}$$

4.1.2 Games of Strategic Complements

There are two basic strategic interactions classified as games of strategic complements and games of strategic substitutes. In games of strategic complements, a given player's relatively higher/lower payoffs is based on his or her higher actions following other players' increasing/decreasing actions. In other words, other players' increasing actions leads the given player's higher actions to have relatively higher payoffs compared to that player's lower actions. These games are also called coordination games. For player i and player j, if relationships between their actions are strategic complement, then there exists $\delta g_{ij} < 0$ Bramoullé and Kranton [2015]. For pure complements in a game, there exists $\delta < 0$ and $\forall_{i,j,g_{ij}} \geq 0$ Bramoullé and Kranton [2015].

Games of strategic complements are well-behaved in various ways. In such games, the set of actions A is finite (or compact) and payoffs are continuous. There exists an equilibrium in pure strategies directly in terms of the actions without requiring any additional randomizations, and the set of equilibria form a (nonempty) complete lattice which is well-ordered and easy to find the maximal and minimal equilibria Jackson and Zenou [2015].

It is noticed that in games of strategic complement, the equilibria exists in pure strategies which is based on the actions A directly without considering any additional randomizations, and also dynamics that iterate on best response dynamics will generally converge to equilibrium nodes in such games.

Following notations, ith player's action a_i and their neighbours' actions a_j are in set $A = A_1 \times \ldots A_n \in \mathbb{R}$ for i in N, and the network on the set of nodes N is g which is a $n \times n$ adjacency matrix. The payoff function of such games is $u_i(a_i, g)$

which depends only on $a_i \in A$ and a_j, $j \in N_i(g)$.

The Majority Game is a typical game of strategic complements. In such game, players only have two choices such as 1 or 0 and the action space $A_i = \{0, 1\}$. The strategy of decision making is if more than one half of your neighbours prefer action 1 (or 0) , it is best for you to choose the other one 0 (or 1). The better payoff to a player from taking one action depends on the neighbours' choices. Thus, the equilibrium will be all players taking the same action either 1 or 0. There are multiple equilibria in such games. The relation of majority games can be showed as following

$$u_i(1, a_{N_i(g)}) > u_i(0, a_{N_i(g)}) \quad if \quad \frac{\sum_{j \in N_i(g)} a_j}{|N_i(g)|} > \frac{1}{2}, \tag{4.8}$$

and

$$u_i(1, a_{N_i(g)}) < u_i(0, a_{N_i(g)}) \quad if \quad \frac{\sum_{j \in N_i(g)} a_j}{|N_i(g)|} < \frac{1}{2}. \tag{4.9}$$

As mentioned before, finding the maximal or minimal node is the way of proving the existence of an equilibrium node in games of strategic complements. There is an advantage in such games that they form a lattice. It is easy algorithm to find the equilibria. Jackson and Zenou [2015] report a ***Theorem*** as follows: Consider a game of strategic complements such that: for every player i, and specification of strategies of the other players, $a_{-i} \in A_{-i}$, player i has a nonempty set of best responses $BR_i(a_{-i})$ that is a closed sublattice of the complete lattice A_i, and for every player i, if $a'_{-i} \geq a_{-i}$, then $sup_i BR_i(a'_{-i}) \geq_i sup_i BR_i(a_{-i})$ and $inf_i BR_i(a'_{-i}) \geq_i inf_i BR_i(a_{-i})$. An equilibrium exists and the set of equilibria form a (nonempty) complete lattice.

To prove this theorem, it needs to find at least one maximal or minimal equilibrium node in a game of strategic complements. Assume the action set A is finite and compact, and the payoffs are continuous. The maximal action which all players play is $a^0 = \bar{a}$, $a \in A$. Let $a_i^1 = sup_i(BR_i(a_{-i}^0))$ for each i, where $sup_i(BR_i(a_{-i})) \in BR_i(a_{-i})$. Iteratively, let $a_i^k = sup_i(BR_i(a_{-i}^{k-1}))$, and a node such as $a^k = a^{k-1}$ will be found which is the maximal equilibria. Because the set A of strategies is finite, there must be a number as the equilibria here after iterations. Analogously, if $a^0 = \bar{a}$ is the minimal action which all players play, after iterating upward, there exists a minimal equilibrium node in the finite set A as well.

4.1.3 Games of Strategic Substitutes

In games of strategic substitutes there is an opposite true that an increase in other players' actions leads to relatively lower payoffs to higher actions of a given player. This type of games is called incoordination games. For player i and player j, if relationships between their actions are strategic complement, then there exists $\delta g_{ij} > 0$ Bramoullé and Kranton [2015]. For pure substitutes, there exists $\delta > 0$ and $\forall_{i,j,g_{ij}} \geq 0$ Bramoullé and Kranton [2015].

There is a good example on games of strategic substitutes called *Best-shot Public Goods Games*. Assume that the action set is $A = \{1, 0\}$ and the maximum payoffs is 1. Taking action 1 is a costly choice with payment c for for $1 > c > 0$, while taking action 0 is no cost. However, if no neighbor j takes action 1 when the given player i takes action 0, there is no payoff for any players. Thus, there are three situations in such games. If none of jth neighbours take any actions, ith player taking actions 1 will get payoffs of $1 - c$. If jth neighbours take action

0 when ith player takes action 1, the ith player can get the maximum payoff 1. If none of players choose action 0, there is no utility to any of them. Thus, players prefer neighbors taking action 1 rather than taking it by themselves because of the cost, while taking action 1 by themselves is better than nobody takes it. These relations can be showed by functions as follows:

$$u_i(a,g) = \begin{cases} u_i(1, a_{N_i(g)}) = 1 - c, \quad if \quad a_i = 1, \quad 1 > c > 0 \\ u_i(1, a_{N_i(g)}) = 1, \quad if \quad a_i = 0, a_j = 1, \quad for\ some \quad j \in N_i(g) \\ u_i(1, a_{N_i(g)}) = 0, \quad if \quad a_i = 0, a_j = 0, \quad for\ all \quad j \in N_i(g) \end{cases} \tag{4.10}$$

Moreover, the other good example of strategic substitutes games is *Weakest-Link Public Goods Game* Hirshleifer [1983], in which some equilibria are easy to find.

Each player chooses some level of public good contribution ($A_i = \mathbb{R}_+$) and the payoff to a player is the minimum action taken by any player in his or her neighbourhood (in contract to the maximum, as in the best-shot game).

$$u_i(a_i, a_{N_i(g)}) = \min_{j \in N_i(g) \cup i} \{a_j\} - c(a_i)$$

where c is an increasing, convex and differentiable cost function.

If there is a smallest a^* such that $c'(a^*) \geq 1$, and each player has at least one neighbor in the network g, then any profile of actions where every player chooses the same contribution $a_i = a^*$ is an equilibrium of this game. Note that in a network in which every player has at least one neighbor, everyone playing $a_i = 0$ is also an equilibrium (or any common $a \leq a^*$), and so the game will have multiple equilibria when it is nondegenerate.

Furthermore, in general games of strategic substitutes, if the games with best-replies are linear, there are more results can be explored both in terms of characterization of equilibria and comparative statics. Following Bramoullé et al. [2014], if the best-reply function is linear in a game of strategic substitutes, each individual player's best response to action a is linear. And also, players try to obtain new information and take advantages of their neighbors' experimentation in such games. Let the best-reply linear function denoted as $u_i(a, g)$. It can be presented as below.

$$u_i(a, g) = v(a_i + \phi \sum_{j=1}^{n} g_{ij} a_j) - c a_i \tag{4.11}$$

where a is the action profile, and g is the underlying network. $v(\cdot)$ is an increasing, differentiable and strictly concave function on $\mathbb{R}_+$, and $c > 0$ is the constant marginal cost of own action. Following Bramoullé et al. [2014], when $\phi = 1$, based on the mathematics property of the concave $v(\cdot)$, the second derivative of $v(\cdot)$ is less than 0.

$$\frac{\partial u_i(a, g)}{\partial a_i \partial a_j} = v''(a_i + \sum_{j=1}^{n} g_{ij} a_j) < 0.$$

Moreover, assume a^* as the action level of player who experiments by him/her-self, so a^* can be denoted in the following way

$$v'(a^*) - c = 0$$
$$v'(a^*) = c$$
$$a^* = v'^{-1}(c)$$

Then, the best response to a_{-i} is given by:

$$a_i^* = \begin{cases} a^* - \sum_{j=1}^{n} g_{ij}a_j & if \quad a^* > \sum_{j=1}^{n} g_{ij}a_j \\ 0 & if \quad a^* \leqslant \sum_{j=1}^{n} g_{ij}a_j \end{cases}$$

In games of strategic substitutes, although there is lack of lattice structure, finding one maximal or minimal independent set is still possible. An algorithm can be used to find equilibrium can be found such nodes. Consider the best-shot public goods games, let P_k is a set of the providers of the public good which is the eventual maximal independent set of nodes. Let NP_k is a set of non-providers of the public good which will not be in final P_k, where k is the step in the algorithm. In best-shot games, players taking action 1 are listed in final P_k, and players who take action 0 are listed in final NP_k. The algorithm is as following:

Step 1: Pick some node i and let $P_1 = \{i\}$ and $NP_1 = N_i(g)$.

Step k: Iterate by picking one of the players j who is not yet assigned to sets P_{k_1} or NP_{k_1}. Let $P_k = P_{k1} \cup j$and $NP_k = NP_{k1} \cup N_j(g)$.

End: Stop when $P_k \cup NP_k = N$.

This algorithm proves the possibility of finding equilibriums in general games of strategic substitutes, but it is still quite difficult to characterize. However, if the games with best-replies are linear, there are more results can be explored both in terms of characterization of equilibria and comparative statics. Bramoullé et al. [2014] demonstrate that is if the best-reply function is linear in a game of strategic substitutes, each individual player's best response to action a is linear. And also, players try to obtain new information and take advantages of their neighbors' experimentation in such games. Let the best-reply linear function denoted as $u_i(a, g)$. It can be built as:

$$u_i(a, g) = v(a_i + \phi \sum_{j=1}^{n} g_{ij} a_j) - c a_i \tag{4.12}$$

where a is the action profile, and g is the underlying network. $v(\cdot)$ is an increasing, differentiable and strictly concave function on $\mathbb{R}_+$, and $c > 0$ is the constant marginal cost of own action. When $\phi = 1$, based on the mathematics property of the concave $v(\cdot)$, the second derivative of $v(\cdot)$ is less than 0.

$$\frac{\partial u_i(a, g)}{\partial a_i \partial a_j} = v''(a_i + \sum_{j=1}^{n} g_{ij} a_j) < 0.$$

Moreover, assume a^* as the action level of player who experiments by him/her-self, so a^* can be denoted in the following way

$$v'(a^*) - c = 0$$
$$v'(a^*) = c$$
$$a^* = v'^{-1}(c)$$

Then, the best response to a_{-i} is given by:

$$a_i^* = \begin{cases} a^* - \sum_{j=1}^{n} g_{ij}a_j & if \quad a^* > \sum_{j=1}^{n} g_{ij}a_j \\ 0 & if \quad a^* \leqslant \sum_{j=1}^{n} g_{ij}a_j \end{cases}$$

These two nodes are two types of equilibria. An action profile a is specialized if players actions are such that $a_i = 0$ or $a_i = a^*$ for every i. A player for which $a_i = a^*$ is a *specialist*. An action profile a is distributed when all players choose a positive action less than the individually optimal action level: $0 < a_i < a^*, \forall i \in N$. Hybrid equilibria are other than these extremes.

Because actions are strategic substitutes, maximal independent sets are a natural notion in this model. Indeed, in equilibrium, no two specialists can be linked. Hence, specialized equilibria are characterized by this structural property of a net- work, i.e. the specialists are equal to a maximal independent set of the network. A result from Bramoullé et al. [2014] can be stated as follows: A specialized profile is a Nash equilibrium of the above game if and only if its set of specialists is a maximal independent set of the structure g. Since for every g there exists a maximal independent set, there always exists a specialized Nash equilibrium.

4.1.3.1 Games of Incomplete Information on Network

Comparing with in complete information games, players who play incomplete games are now unsure about the network that will be in place in the future, but have some idea of the number of interactions that they will have. To fix ideas, think of choosing whether to adopt a new software program that is only useful in interactions with other players who adopt the software as well, but without being sure of with whom one will interact in the future.

In particular, the set of i players $N, i \in N$ is fixed, but the network (N, g) is unknown when players choose their actions. A player i knows his or her own degree d_i, when choosing an action, but does not yet know the realized network.

Players' action set $A = \{0, 1\}$. If players choose action 0, their payoffs will be 0, and so effectively consider the difference in payoffs between choosing action 0 and 1. Player i has a cost of choosing action 1, denoted c_i. Player i's payoff from action 1 when i has d_i neighbors and expects them each independently to choose 1 with a probability p is

$$u^*(d_i, p) - c_i, \tag{4.13}$$

and so action 1 is a best response for player i if and only if $c_i \leq u^*(d_i, p)$.

It is easy to see how this incorporates some of the games we considered earlier. For instance, in the case of a Best-Shot Public Goods game of Example:

$$u^*(d_i, p) = (1 - p)^{d_i} \tag{4.14}$$

In the case of our coordination game, the payoff is

$$u^*(d_i, p) = \sum_{m=0}^{d_i} B_{d_i}(m, p)[mb - (d_i - m)c] \tag{4.15}$$

where $B_{d_i}(m, p)$ is the binomial probability of having exactly m neighbors out of d_i play action 1 when they independently choose action 1 with probability p.

4.2 Existence of Equilibrium

The equilibrium node of a game is defined as a node that at this node, no player can increase his or her payoff by change strategy individually. In mathematics, the question of finding an equilibrium node in game theory can be transformed as finding an extreme node in optimization theory.

In 1950s, Nash et al. [1950], Nash [1951] creatively introduce and prove the existence of an equilibrium node for an N-player games under restrictions that players' action spaces are simplex and payoff functions are bilinear. There are certain generalizations of Nash's results Arrow and Debreu [1954], McKenzie [1959]. What I am interested is Rosen's work Rosen [1965] in which the existence and uniqueness of an equilibrium node have been proved for a concave N-player game. In a concave game, each player's space is convex, closed and bounded in $\mathbb{R}$ and his/her payoff function u_i, $i = 1, \ldots, n$ is concave in the player's individual strategy. To prove the existence of an equilibrium node, Rosen [1965] used the theorem of Kakutani fixed node Kakutani et al. [1941] which is generalized from Brouwer's fixed node theorems Wallace [1941]. To prove the uniqueness of such node, Rosen [1965] sets the orthogonal constraint for payoff functions so they can satisfy the diagonal strict concavity requirement. Moreover, Rosen [1965] analyze the existence and uniqueness of such equilibrium node in a dynamic model of N-player concave game, and compute the node by a gradient method which suits for a concave mathematical programming problems. In this section, I will explain Rosen [1965]'s measurements in my own notations.

In a concave N-player game, player i's action is denoted by a_i, $a_i \in A \subset \mathbb{R} \subset \mathbb{E}^{\gtrdot\beth}$, $i = 1, \ldots, n$. $\mathbb{E}^{mi}$ is the Euclidian space, $\mathbb{E}^{m1} \times \mathbb{E}^{m2} \times \cdots \times \mathbb{E}^{mn}$ and $m = \sum_i^n m_i$. The action space is a convex, closed and bounded set. All players' strategies are homogeneous in the game. The payoff function for player i is presented as $u_i(a; g) = u_i(a_1, \ldots, a_i, \ldots, a_n; g)$, where $u_i(a; g)$ is continuous in all action a and concave in a_i for each fixed value of $(a_1, \ldots, a_i, \ldots, a_n)$, and g is the graph/network of the game. The equilibrium node of this concave game is denoted by $a^* \in \mathbb{R}$ such that

$$u_i(a^*; g) = \max_{b_i}\{u_i({a_1}^*, \ldots, b_i, \ldots, {a_n}^*)|({a_1}^*, \ldots, b_i, \ldots, {a_n}^*; g) \in \mathbb{R}\}. \tag{4.16}$$

For conveniently define $(a, b; g) \in \mathbb{R}$, there is a function $\gamma(a, b; g)$ that $\gamma(a, b; g) = \sum_{i=1}^n u_i(a_i, \ldots, b_i, \ldots, a_n; g)$, $i = 1, \ldots, n$. This function is continuous in a and b and is concave in b for every fixed a. Rosen [1965] reports Theorem 1 below.

Theorem 1: *An equilibrium node exists for every concave N-player game.*

In this theorem, Rosen [1965] defines an equilibrium node in a concave N-player game as a maximum value of payoffs. Due to the characteristics of a concave game, there exists a extreme value (maximum value) in such game, so there exists at least one equilibrium node in such game.

To prove the theorem, assume the node-to-set mapping $a \in \mathbb{R} \to \Gamma(a) \subset \mathbb{R}$, it is

$$\Gamma a = \{b | \gamma(a, b) = \max_{c \in \mathbb{R}} \gamma(a, c)\} \tag{4.17}$$

It follows from the continuity of $\gamma(a, c)$ and the concavity in c of $\gamma(a, c)$ for fixed a that Γ is an upper semi-continuous mapping that maps each node of convex, compact set $\mathbb{R}$ in to a closed convex subset $\mathbb{R}$ Rosen [1965]. Then following Kakutani Fixed Point Theorem Kakutani et al. [1941], there exists a node $a^* \in \mathbb{R}$ such that $a^* \in \Gamma a^*$ or $\gamma(a^*, a^*) = \max_{c \in \mathbb{R}} \gamma(a^*, c)$. The fixed point a^* is an equilibrium node satisfying . Suppose it were not true. Then for $i = j$, there would be a node $a_j = \bar{a_j}$ such that $\bar{a} = (a_i{}^*, \ldots, \bar{a_j}, \ldots, a_n{}^* \in \mathbb{R}$ and $u_j(\bar{a}) > u_j(a^*)$. Then it is contradiction that $\gamma(a^*, \bar{a}) > \gamma(a^*, a^*)$.

4.2.1 Existence of the Unique Equilibrium Node

Rosen [1965] have proved the uniqueness of an equilibrium node in a concave N-player game when the payoff functions $u_i, i = 1, \ldots, n$ satisfy the diagonal strict concavity in terms of certain Hessian matrix of the u_i. In this section, I will explain Rosen [1965]'s idea and reprove this process.

Before discussing the existence of unique equilibria, Rosen [1965] redefines the convex action space explicitly and try to satisfy the sufficient condition for Karush-Kuhn-Tucker (KKT) constraint qualification Kuhn [1951] and Arrow et al. [1961]. KKT conditions and Lagrange Multiplier are the classic method to solve the optimization problema. An optimal solution of the optimization is a pure strategy equilibrium in strategic form games.

For the general coupled constraint set where $A \subset \mathbb{R} \subset \mathbb{E}^{mi}$, A is described by means of the mapping $h(a;g)$ of $\mathbb{E}^m \to \mathbb{E}^k$. $h(a;g)$ is a concave function of each action a, and each component is denoted by $h_j(a_j;g)$, $j = 1, \ldots, k$. Assume $A = \{a|h(a;g) \geq 0\}$ is nonvoid, bounded and convex. For the special example of the orthogonal constraint set $A = S = A_1 \times A_2 \times \cdots \times A_n$, the nonvoid and bounded sets are $A_i = \{a_i|h_i(\bar{a_i};g) \geq 0\}$, $i = 1, \ldots, n$, where each $h_{ij}(a_i;g)$ of $h_i(\bar{a_i};g)$ is concave. Thus, $A_i \subset \mathbb{E}^{mi}$ is convex, closed and bounded. If there exist a node which is strictly interior to every nonlinear constraint, $\exists \bar{A} \in A$, such that $h_j(\bar{a};g) > 0$ for every nonlinear constraint $h_j(a_j;g) \geq 0$. Thus the sufficient condition for Karush-Kuhn-Tucker (KKT) constraint qualification is satisfied Rosen [1965], Arrow et al. [1961].

To use the differential form of the necessary and sufficient Kuhn-Tucker conditions for a constrained maximum Kuhn [1951], an additional assumption for $h_j(a_j;g)$ is added by Rosen [1965] that there exists continuous first derivatives for $h_j(a_j;g)$, $a_j \in A$ and for the payoff function $u_i(a_i;g)$, $a_i \in A$. For any scalar function $u_i(a_i;g)$, the gradient is presented as $\nabla_i u_i(a_i;g)$, $\nabla_i u_i(a_i;g) \in \mathbb{E}^{mi}$. Here, **gradient** is a vector-valued function which shows a multi-variable generalization of the derivative. The gradient nodes in the direction of the greatest rate of increase of the function, and its magnitude is the slope of the graph in that direction. The components of the gradient in coordinates are the coefficients of the variables in the equation of the tangent space to the graph.

In my research, the gradient vector of payoff function u is noted as following

$$\nabla u(a_i;g) = [\frac{\partial u(a_i;g)}{\partial a_i^1}, \ldots, \frac{\partial u(a_i;g)}{\partial a_i^n}]^T \tag{4.18}$$

$$= [\nabla_1 u_1(a;g), \ldots, \nabla_n u_n(a;g)]^T. \tag{4.19}$$

Consider the optimization

$$maximize \quad u(a_i; g) \tag{4.20}$$

$$subject\ to \quad h_i(a_i; g) \geq 0, \quad i = 1, \ldots, n, \tag{4.21}$$

where the cost/payoff function $u(a_i; g)$: $\mathbb{E}^n \to \mathbb{E}$ and the constraint functions $h_i(a_i; g)$: $\mathbb{E}^n \to \mathbb{E}$ are continuously concave functions.

The KKT conditions equivalent to can be presented as $h(a^*; g) \geq 0$, for $i = 1, \ldots, 0$. And for $i = 1, \ldots, n$, $\exists \lambda_i \geq 0$, $\lambda_i \in \mathbb{E}^k$, there exists

$$\lambda_i h(a^*; g) = 0 \tag{4.22}$$

$$u_i({a_i}^*; g) \geq u_i({a_1}^*, \ldots, b_i, \ldots, {a_n}^*; g) + \lambda_i h({a_1}^*, \ldots, b_i, {a_n}^*; g) \tag{4.23}$$

Since $u_i(a_i; g)$ and $h_j(a_j; g)$ are concave and differentiable, the inequality is presented as follows:

$$\nabla_i u_i({a_i}^*; g) + \sum_{j=1}^{k} {\lambda_{ij}}^* \nabla_i h_j({a_j}^*; g) = 0 \tag{4.24}$$

$$\lambda_i h_i(a_i, g) = 0, \quad i = 1, \ldots, n.$$

Following the result of the concavity of $h_j(a_j; g)$, for every a^*, $a^{*1} \in A$, there exists

$$h_j(a^{*1}; g) - h_j(a^*; g) \leq (a^{*1} - a^*)' \nabla h_j(a^*; g) = \sum_{i=1}^{n} (a_i^{*1} - a^*)' \nabla_i h_j(a^*; g) \tag{4.25}$$

A weighted nonnegative sum of the functions $u_i(a_i; g)$ can be denoted as following

$$\phi(a, r; g) = \sum_{i=1}^{n} r_i u_i(a_i; g), \quad r_i \geq 0 \tag{4.26}$$

For each fixed r, there exists a matrix of the gradients $\nabla u_i(a_i; g)$ that

$$v_i(a_i, r; g) = \begin{bmatrix} r_1 \nabla_1 u_1(a_1; g) \\ r_2 \nabla_2 u_2(a_2; g) \\ \vdots \\ r_n \nabla_n u_n(a_n; g) \end{bmatrix} \tag{4.27}$$

which has the unique answer. $v_i(a_i, r; g)$ is called pseudograndient of $\phi(a_i, r; g)$.

Thus, the key condition for uniqueness of a pure strategy Nash equilibrium can be showed in a definition below Rosen [1965]: the function $\phi(a_i, r; g)$ is *diagonally strictly concave* for $a \in A \subset \mathbb{E}$, if for every $a^*, a^{*1} \in \mathbb{E}$. There is a function that

$$(a^{*1} - a^*)^T \nabla v(a^*, r; g) + (a^* - a^{*1})^T \nabla v(a^{*1}, r; g) > 0 \tag{4.28}$$

For the pseudograndient $v_i(a_i, r; g)$, the Jacobian determinant is denoted as $J(a, r; g)$. Generally, Jacobian determinant is a generalized gradient for vector-valued functions of variables and differentiable maps between Euclidean spaces $\mathbb{E}$ Burmeister [1968]. Suppose $f : \mathbb{E}^n \to \mathbb{E}^m$ is a function which takes as input the vector $a \in \mathbb{E}^n$ and produces as output the vector $f(a) \in \mathbb{E}^m$. Then the Jacobian matrix J of f is an $m \times n$ matrix defined as follows:

$$J = [\frac{\partial f}{\partial a_1} \cdots \frac{\partial f}{\partial a_n}] = \begin{bmatrix} \frac{\partial f_1}{\partial a_1} & \cdots & \frac{\partial f_1}{\partial a_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial a_1} & \cdots & \frac{\partial f_m}{\partial a_n} \end{bmatrix} \tag{4.29}$$

There is a sufficient condition that the symmetric matrix $[J(a_i, r; g) + J'(a_i, g, r)]$ be negative definite for $a_i \in A$ if $\phi(a, r; g)$ is diagonally strictly concave.

Furthermore, Rosen [1965] extended the existence of the unique equilibrium node in a concave N-play game based on different constraints. If there is $A \in \mathbb{E}^n$ in orthogonal constraint sets, an important theorem is given by the following:

Theorem 2: *If the functions $\phi(a, r; g)$ are diagonally strictly concave for $r > 0$, then there exists a unique pure strategy Nash equilibrium node a^* which satisfies 4.2.1.*

In this theorem, Rosen [1965] restrict the game to be a diagonally strictly concave game, then there definitely exists a unique extreme value in such game. This unique extreme value is the unique Nash equilibria in such game.

To prove this theorem, assume that there are two distinct pure strategy Nash equilibrium nodes, a^* and a^{*1}, by the necessiTy of KKT conditions, there exists nonnegative vectors $h_i(\bar{a_i}^l; g) \geq 0$ for $l = {}^*or^*1$ and $i = 1, \ldots, 0$. For $\exists \lambda_i{}^l \geq 0$, $\lambda_i{}^l \in \mathbf{E}^{ki}$, there exists

$$\lambda_i{}^{l'} h_i(\bar{a_i}^l; g) = 0 \tag{4.30}$$

$$\nabla_i u_i(a_i^l; g) + \sum_{j=1}^{ki} \lambda_{ij}{}^l \lambda_i h_{ij}(a_i{}^l; g) = 0 \tag{4.31}$$

For $l = *$, there exists $\bar{r}_i({a_i}^{*1} - {a_i}^{*})^T$. For $l = {}^{*}1$, there exists $\bar{r}_i({a_i}^{*} - {a_i}^{*1})^T$, and $i = 1, \ldots, n$. If I multiply these two functions together, there exists

$$F_1 + F_2 = 0 \tag{4.32}$$

$$F_1 = (a^{*1} - a^{*})^T v(a^*, \bar{r}) + (a^* - a^{*1})^T v(a^{*1}, \bar{r}) \tag{4.33}$$

$$F_2 = \sum_{i=1}^{n} \sum_{j=1}^{ki} \bar{r}_i \{ {\lambda_{ij}}^{*} ({a_i}^{*1} - {a_i}^{*})^T \nabla h_{ij}({a_i}^{*}) + {\lambda_{ij}}^{*1} ({a_i}^{*} - {a_i}^{*1})^T \nabla_i h_{ij}({a_i}^{l}) \} \tag{4.34}$$

$$\geq \sum_{i=1}^{n} \sum_{j=1}^{ki} \bar{r}_i [h_{ij}({a_i}^{*1}) - h_{ij}({a_i}^{*})] + {\lambda_{ij}}^{*1} [{a_i}^{*}) - h_{ij}({a_i}^{*1})] \tag{4.35}$$

$$= \sum_{i=1}^{n} \bar{r} \{ {\lambda_i}^{*T} \bar{h}_i ({a_i}^{*1}) + {\lambda_i}^{*1T} \bar{h}_i ({a_i}^{*}) \}. \tag{4.36}$$

In F_2, the inequality is from and the concavity of $h_{ij}(a)$ and the last equation is from ${\lambda_i}^{l'} h_i(\bar{a_i}^l; g) = 0$. Then $F_2 \geq 0$ is from $h_i(\bar{a_i}^l; g) \geq 0$. Since $\phi(a_i, r; g)$ is diagonally strictly concave from , then $F_1 > 0$. This result is contradict that of the preceding function $F_1 + F_2 = 0$. Thus, there exists a unique equilibrium node a^*.

Consider the $A = \{a | h(a; g) \geq 0\} \subset \mathbb{E}$ is nonvoid, bounded and convex Rosen [1965], and the values of the nonnegative multipliers $\lambda_i, i = 1, \ldots, n$ are generally independent from KKT conditions at an equilibrium node. Thus, a special case of equilibrium node can be presented as follows Rosen [1965]:

$${\lambda_i}^{*} = \lambda^{*} / r_i \tag{4.37}$$

where $i = 1, \ldots, n$, some $r > 0$ and $\lambda > 0$. This special node is a *normalized equilibrium node*.

Rosen [1965] reports the third theorem as follows.

Theorem 3: *There exists a normalized equilibrium node to a concave N-player games for every specified $r > 0$.*

In this theorem, Rosen [1965] apply KKT conditions at the unique Nash equilibrium node as fixed point in diagonally strictly concave games and normalize this node in such games.

To prove this theorem, consider a fixed value $r = \bar{r} > 0$ is in payoff functions as follows

$$\gamma(a, b, \bar{r}; g) = \sum_{i=1}^{n} \bar{r}_i u_i(a_1, \ldots, b_i, \ldots, a_n; g) \tag{4.38}$$

Following the Kakutani Fixed Point Theorem, there exists a node a^* such that

$$\gamma(a^*, a^*, \bar{r}; g) = \max_b \{\gamma(a^*, b, \bar{r}) | h(b) \geq 0\} \tag{4.39}$$

Then based on the necessity of the KKT conditions as $h(a^*; g) \geq 0$ and $\exists \lambda^* \geq 0$, there exist $\lambda^* h(a^*; g) = 0$ and

$$\bar{r}_i \nabla_i u_i({a_i}^*; g) + \sum_{j=1}^{k} {\lambda_j}^* \nabla_i h_j({a_j}^*; g) = 0, \quad i, j = 1, \ldots, 0 \tag{4.40}$$

The KKT conditions $\lambda_i j^* = {\lambda_j}^* / \bar{r}_i$ can insure that a^* satisfies . Thus, $a*$ is a normalized equilibrium node for the specified value of $r = \bar{r}$ Rosen [1965]:

Theorem 4: *Consider Q as a convex subset of the positive orthant of $\mathbb{E}^n$. For each $r \in Q$ from the diagonally strictly concave $\phi(a, r; g)$, there is a unique normalized equilibrium node.*

In this theorem, Rosen [1965] extend the existence of unique normalized equilibrium node to a convex game.

To prove this theorem, assume that there exists two normalized equlibrium nodes a^* and a^{*1} for some $r = \bar{r} \in Q$. For $l = {}^*or{}^*1$ and each $i = 1, \ldots, n$, both equilibrium nodes can be an optimal for an optimization problem. There exists $h(a^l) \geq 0$. For $\exists \lambda^l \geq 0$, $\lambda^l \in \mathbb{E}^k$, there exists

$$\lambda^{l'} h(a^l) = 0 \tag{4.41}$$

$$\bar{r}_i \nabla_i u_i({a_i}^l; g) + \sum_{j=1}^{k} {\lambda_j}^l \nabla_i h_j({a_j}^l; g) = 0 \tag{4.42}$$

For $l = {}^*$, there exists $\bar{r}_i({a_i}^{*1} - {a_i}^*)^T$. For $l = {}^*1$, there exists $\bar{r}_i({a_i}^* - {a_i}^{*1})$, and $i = 1, \ldots, n$. If I multiply these two function together, there exists

$$F_1 + F_2 = 0 \tag{4.43}$$

$$F_1 = (a^{*1} - a^*)^T v(a^*, \bar{r}) + (a^* - a^{*1})^T v(a^{*1}, \bar{r}) \tag{4.44}$$

$$F_2 = \sum_{j=1}^{k} \sum_{i=1}^{n} \{ {\lambda_j}^* ({a_i}^{*1} - {a_i}^*)^T \nabla_i h_j({a_i}^*) + {\lambda_j}^{*1} ({a_i}^* - {a_i}^{*1})^T \nabla_i h_j({a_i}^{*1}) \} \tag{4.45}$$

$$\geq \lambda^{*'} [h({a_i}^{*1}) - h({a_i}^*)] + \lambda^{*1'} [h({a_i}^*) - h({a_i}^{*1})] \tag{4.46}$$

$$= \lambda^{*'} h({a_i}^{*1}) + \lambda^{*1'} h({a_i}^*) \geq 0 \tag{4.47}$$

Since $\phi(a_i, r; g)$ is diagonally strictly concave from , then $F_1 > 0$. This result is contradict that of the preceding function $F_1 + F_2 = 0$. Thus, there exists a unique normalized equilibrium node Rosen [1965].

Based on previous two theorem that uniqueness of a equilibrium node and existence of a normalized equilibrium, the next consideration is about the dependence of the normalized equilibrium node on the value of r for the general case where $A \in \mathbb{E}$ is a coupled constraint set Rosen [1965]. Following the theorem of uniqueness of a equilibrium node a^*, if $\phi(a, r; g)$ is diagonally strictly concave for $r = \bar{r} > 0$, the relationship between a^* and r is independent. However, Rosen [1965] intuitional find certain example that the equilibrium value is related to r, so the next theorem is about that the equilibrium value is function monotonicity of r_i:

Theorem 5: *Following the previous theorem, $\phi(a, r; g)$ is diagonally strictly concave for $r \in Q$. Consider $r^*, r^{*1} \in Q$ with ${r_i}^{*1} = r_i^*$, $i \neq q$ and ${r_q}^{1*} > {r_q}^*$.*

In this theorem, Rosen [1965] present that there exists characteristics of monotone increasing for normalized equilibrium nodes in diagonally strictly concave games.

Consider a^* and a^{*1} with $a^* \neq a^{*1}$ as the corresponding unique normalized equilibrium nodes. Then there exists a positive directional derivative of $u_q(a^*, r; g)$ along the ray $({a_q}^{*1} - {a_q}^*)$.

These theorem present that the equilibrium value of payoff function $u(a, g; r)$ is a monotone increasing function of r Rosen [1965].

To prove this theorem, consider λ^* and λ^{*1} be the multiplier corresponding to two normalized equilibrium nodes a^* and a^{*1} for some $\bar{r_i} = r_i^0$. If $l = {}^*$, then $i \neq q$. If $l = {}^{*1}$, then $i = 1, \ldots, n$. For $\bar{r_i} = {r_i}^*$, there exists

$$
\begin{aligned}
& h(a^l) \geq 0 \\
& {\lambda^l}' h(a^l) = 0, \exists \lambda^l \geq 0, \lambda^l \in \mathbb{E}^k \\
& \bar{r_i} \nabla_i u_i({a_i}^l; g) + \sum_{j=1}^{k} {\lambda_j}^l \nabla_i h_j({a_j}^l; g) = 0
\end{aligned}
$$

For $l = {}^*$ and $i = q$, there exists

$$({r_q}^* - {r_q}^{*1})\nabla_q u_q(a^*; g) + {r_q}^{*1}\nabla_q u_q(a^*; g) + \sum_{j=1}^{k} {\lambda_j}^* \nabla_q h_j(a^*) = 0. \tag{4.48}$$

Multiplying by $({a_i}^{1*} - a^*)^T$ for $l = {}^*$ and by $({a_i}^* - {a_i}^{*1})^T$ for $l = {}^{*1}$, and summing, there exists

$$({r_q}^* - {r_q}^{*1})({a_q}^{*1} - {a_q}^*)^T \nabla_q u_q(a^*; g) < 0, \tag{4.49}$$

or, since ${r_q}^{*1} > {r_q}^*$

$$({a_q}^{*1} - {a_q}^*)^T \nabla_q u_q(a^*) > 0. \tag{4.50}$$

The last theorem in this chapter is about a sufficient condition on the function $u_i(a_i, r; g)$. This sufficient condition insures that the $v(a_i, r; g)$ has the property of diagonal strict concavity in terms of certain strategies.

Assume $J(a, r; g)$ is a $m \times m$ Jacobian of $v(a, r; g) = [r_i \nabla_1 u_1(a; g), \ldots, r_n \nabla_n u_n(a; g)]^T$, for fix $r > 0$. The jth column of $J(a, r; g)$ is $\partial v(a, r; g)/\partial a_j$, $j = 1, \ldots, n$.

Theorem 6: *If there exists a sufficient condition that $\phi(a, r; g)$ is diagonally strictly concave for $a_i \in A$ and $r = \bar{r} > 0$, then there exists*

$$[J(a, \bar{r}; g) + J(a, \bar{r}; g)^T] < 0, \quad \forall a \in A. \tag{4.51}$$

In this theorem, Rosen [1965] reports a sufficient condition for diagonally strictly concave games that symmetric Jacobin matrix of gradients of payoffs $[J(a, \bar{r}; g) + J(a, \bar{r}; g)^T]$ is negative. Moreover, this sufficient condition confirms that every eigenvalue of adjacency matrix of payoffs in such has a negative real part. This condition provides an important characteristics of eigenvalue and it will be applied in my own model in further sections.

To prove this theorem, assume $a^*, a^{*1} \in A$ are two distinct nodes. There exists θ which satisfies $a(\theta) = \theta a^{*1} + (1-\theta)a^*$ for $1 \leq \theta \leq 0$. Then there exists

$$\frac{\partial v(a(\theta), \bar{r}; g)}{\partial \theta} = J(a(\theta), \bar{r}; g)(a^* - a^{*1}) \tag{4.52}$$

or

$$v(a^{*1}, \bar{r}; g) - v(a^*, \bar{r}; g) = \int_0^1 J(a(\theta), \bar{r}; g)(a^{*1} - a^*)d\theta \tag{4.53}$$

Multiplying the preceding function by $(a^* - a^{*1})^T$, there exists

$$\begin{aligned}
&(a^* - a^{*1})^T v(a^{*1}, \bar{r}; g) + (a^{*1} - a^*)^T v(a^*, \bar{r}; g) \\
&= -\frac{1}{2}\int_0^1 (a^{*1} - a^*)^T [J(a(\theta), \bar{r}; g) + J^T(a(\theta), \bar{r}; g](a^{*1} - a^*)d\theta \\
&> 0
\end{aligned}$$

In this function, for getting the strict inequality, an assumption should be sat that the symmetric matrix $[J(a, \bar{r}; g) + J^T(a, \bar{r}; g)]$ is negative for all $a \in A$.

If the payoff function $u_i(a_i; g)$ is bilinear in the strategies a_j, there is a considerable relationship between the sufficient condition and a stability matrix Rosen [1965]. Consider

$$u_i(a_i; g) = \sum_{j=1}^{n} [{e_{ij}}^T + {a_i}^T \boldsymbol{C_{ij}}]a_j \tag{4.54}$$

in which $a_i, a_j \in A \subset \mathbb{E}$, $i, j = 1, \ldots, n$ denote player i and player j's action respectively and g is the underlying network. Moreover, e_{ij} is a constant vector in $\mathbb{E}^{mj}$. $\boldsymbol{C_{ij}}$ is a $m_i \times m_j$ constant matrix.

Assume $n = 2$, $e_{ij} = 0$, $\boldsymbol{C_{11}} = \boldsymbol{C_{22}} = 0$ and $\boldsymbol{C_{12}} \neq 0$, $\boldsymbol{C_{21}} \neq 0$, it is a special example of a two-player nonzero-sum game (also called bimatrix game) Mangasarian and Stone [1964] for . If $\boldsymbol{C_{21}} = -\boldsymbol{C_{12}}^T$, then it is a two-player zero-sum game.

Following definitions of gradients $v_i(a_i, r; g)$ in and Jacobian Matrix $J(a, r; g)$ in , there is

$$J(a, r; g) = \boldsymbol{DC} \tag{4.55}$$

where $\boldsymbol{D}$ is the diagonal positive definite matrix $\boldsymbol{D} = diag\{r_i\}$, and C is the $m \times m$ matrix that

$$\boldsymbol{C} = \begin{bmatrix} 2\boldsymbol{C_{11}} & \boldsymbol{C_{12}} & \dots & \boldsymbol{C_{1n}} \\ \boldsymbol{C_{21}} & 2\boldsymbol{C_{22}} & & \\ \vdots & & & \\ \boldsymbol{C_{n1}} & & & 2\boldsymbol{C_{nn}} \end{bmatrix} \tag{4.56}$$

Based on Theorems of uniqueness of the equilibrium node and of the sufficient condition of payoffs, if there are some $\bar{r} > 0$, there exists

$$\bar{\boldsymbol{D}}\boldsymbol{C} + \boldsymbol{C}^T\bar{\boldsymbol{D}} = -\boldsymbol{I} \tag{4.57}$$

in which $\bar{\boldsymbol{D}} = diag\{\bar{r}_i\}$ and $\boldsymbol{I}$ is the identity matrix. This function provides a sufficient condition for every eigenvalue of matrix $\boldsymbol{C}$ that it has a negative real part. Thus the same condition which ensure uniqueness also implies that $\boldsymbol{C}$ is a stability matrix Rosen [1965].

For two-player zero-sum game, the generalization of such game actually is a

N-player game called *skew-symmetric game*. In such game, there exists $\boldsymbol{C_{ji}} = -\boldsymbol{C_{ij}^T}, \quad i, j = 1, \ldots, n$. Moreover, there exists $[\boldsymbol{C} + \boldsymbol{C}^T]$ negative definite when $[\boldsymbol{C_{ii}} + \boldsymbol{C_{ii}^T}]$ is negative definite for $i = 1, \ldots, n$ Rosen [1965].

4.2.2 Global Stability of Equilibrium Node

Rosen [1965] reports that in a dynamic model of a concave N-player game, player i's choice $a_i \in A \subset \mathbb{E}$ is at a rate proportional to the gradient of its payoff function $u_i(a_i; g)$, subject to the constraints. In other words, if player i changes his/her individual action $a_i \in A$ and in the meantime other players still held their current action without changes, then player i's payoff $u_i(a_i; g)$ will increase.

Assume the proportionality constant for ith player is r_i, so differential equations for strategy a_i is as follows Rosen [1965] :

$$\frac{\partial a_i}{\partial_t} = \dot{a_i} = r_i \nabla_i u_i(a_i; g) + \sum_{j=1}^{k} \lambda_j \nabla_i h_j(a_i; g), \quad i = 1, \ldots, n, \tag{4.58}$$

where the vector λ is in $S(a_i; g)$ which is a bounded subset of the positive orthogonal constraint set of action space $\mathbb{E}^k$ and t is a threshold value that $a(t)$ is trajectory of action node. The right side of this function is the projection of the pseudogradient on the manifold formed by the constraints that any $a \in A$.

Assume that there exists an $m \times k$ matrix $H(a; g)$ whose jth column is $\nabla h_j(a_j; g)$, so there exists Rosen [1965]

$$H(a; g) = [\nabla h_1(a; g) \quad \nabla h_2(a; g) \quad \ldots \quad \nabla h_k(a; g)] \tag{4.59}$$

Following definition of the pseudogradient $v(a, r; g)$ in 4.2.1, the new mapping function as $w(a, \lambda, r; g)$ of $\mathbb{E}^{m+k} \rightarrow \mathbb{E}^m$ for each fixed $\bar{r} > 0$ can be presented as below Rosen [1965] :

$$w(a, \lambda, \bar{r}; g) = v(a, \bar{r}; g) + \lambda H(a; g), \quad \lambda \in S(a, g). \tag{4.60}$$

Then $\dot{a}$ in 4.58 can be presented as

$$\dot{a} = w(a, \lambda, \bar{r}; g), \quad \lambda \in S(a). \tag{4.61}$$

And the set $S(a; g) \subset \mathbb{E}^k$ can be presented as

$$S(a; g) = \{\lambda | \| w(a, \lambda, \bar{r}; g) \| = \min_{v_j \geq 0, j \in H^*, v_j = 0, j \notin H^*} \| w(a, \lambda^*, \bar{r}; g) \| \} \tag{4.62}$$

in which $H^* = H(a; g)^* = j | h_j(a_j; g) \leq 0$. As an interior node $a \in A$, there exists $H(a; g)^* = \emptyset$ and $S(a; g) = 0$. Then there is $w(a, \lambda, \bar{r}; g) = v(a, \bar{r}; g)$ for all $a \in A$.

For each node/action a, $v(a, r; g)$ and $H(a)$ are both continuous in $a \in \bar{A}$. The set $\bar{A} \in \mathbb{E}$ is a compact set that each $a \in A$ is interior to $\bar{A}$, $A \subset \bar{A}$. Thus, there is a new theorem that

Theorem 7: *Starting at any node $a \in \mathbb{E}$, a continuous solution $a_i(t)$ to $w(a_i, \lambda, \bar{r}; g)$, $\lambda \in S(a_i; g)$ exists, such that $a_i(t)$ remains in A for all $t > 0$.*

In this theorem, Rosen [1965] reports that in a dynamic model of a concave N-player game, player's changes their actions/strategies will continuously affect the equilibria of such game.

To prove this theorem, I need to introduce the well-known Carathéodory

Existence Theorem first. Following the definition from Bárány [1982], given a convex set $\bar{A} \subset \mathbb{E}^n$ and a node $a \in \bar{A}$, there exists a convex subset $A \in \bar{A}$ that $|A| \leq n + 1$ and $a \in A$. In the Rosen's theorem, a is continuous in $a \in \bar{A}$. Assume λ is measurable in t, it is satisfied Carathéodory Existence Theorem that a continuous solution $a(t)$ exists with $a(t) \in \bar{A}$. For completely satisfying , if there exists some nodes $a' \in \bar{A}$ and $h_m(a'; g) < 0$, there must be an earlier point $\bar{a}$ on the trajectory $a(t)$ because of the continuity in a, so $h_m(\bar{a}; g) = 0$ and $h_m(\bar{a}; g) < 0$. However, based on , there is Rosen [1965]

$$h_m(\bar{a}; g) = \nabla h_m^T(\bar{a}; g) w(a, \lambda, \bar{r}; g) < 0. \tag{4.63}$$

If the corresponding value of λ is $\bar{\lambda} \in S(\bar{a}; g)$, then

$$\begin{aligned} \| w(a, \lambda, \bar{r}; g) \|^2 =& v(a, \bar{r}; g)^T v(a_i, \bar{r}; g) \\ &+ 2\bar{\lambda}^T H(a; g)^T v(a, \bar{r}; g) \\ &+ \bar{\lambda}^T H(a; g)^T H(a; g)\bar{\lambda} \end{aligned}$$

or

$$\begin{aligned} \frac{\partial \| w(a, \lambda, \bar{r}; g) \|^2}{\partial \lambda_m} &= 2\nabla h_m^T(\bar{a}; g)[v(a, \bar{r}; g) + H(a; g)\bar{\lambda}] \\ &= 2\nabla h_m^T(\bar{a}; g) w(a, \lambda, \bar{r}; g) \\ &< 0 \end{aligned}$$

Thus, the norm $\| w(a, \lambda, \bar{r}; g) \|$ can be decreased by increasing $\bar{\lambda}_m > 0$. However, as mentioned before that $h_m(\bar{\lambda}) = 0$ and $m \in H(\bar{a}; g)$, the $\bar{\lambda}$ cannot satisfy the $S(a;g)$ as $\bar{\lambda} \notin S(\bar{a};g)$. Thus the contradiction presents that there is no node a' on the trajectory for $h_i(a';g) < 0$ for $i = 1, \ldots, n$. The theorem has been proved.

Moreover, using KKT conditions for the constrained minimization issue in $S(a,g)$, there is a new solution as follows Rosen [1965]:

Lemma: *the nonzero elements of every vector* $\lambda \in S(a;g)$ *are given by a vector* $\bar{\lambda} \in \mathbb{E}^{\bar{k}}, \bar{k} \leq k$, *where*

$$\bar{\lambda} = -(\bar{H}^T(a;g)\bar{H}(a,g))^{-1}\bar{H}(a;g)^T v(a,\bar{r};g) \geq 0. \tag{4.64}$$

The $n \times \bar{k}$ matrix $\bar{H} = \bar{H}(a;g)$ consists of $\bar{k}$ linearly independent column of $H(a;g)$ selected from $\nabla h_j(a;g)$ for $j \in H$.

Furthermore, the stability of the dynamic system is important to study. Based on , there is a fixed node that $r = \bar{r}$ in the network. Consider a nodes $\bar{a}_i$ as an equilibrium node of $w(a, \lambda, \bar{r}; g)$ if

$$w(\bar{a}, \lambda, \bar{r}; g) = 0, \quad \lambda \in S(\bar{a}; g) \tag{4.65}$$

Then the network of $w(a, \lambda, \bar{r}; g)$ is asymptotically stable in $\mathbb{R}$ if for each initial node $a_i \in A \subset \mathbb{E}$, the solution of $w(a, \lambda, \bar{r}; g)$ coverages to an equilibrium node $\bar{a}_i \in \mathbb{R}$ as $t \to \infty$ Rosen [1965]. Thus, a new theorem can be showed as follows:

Theorem 8: *If* $A = \{a | h(a;g) \geq 0\}$ *and* $[J(a,r;g) + J(a,r;g)^T]$ *is negative definite for all* $a \in A$, *where* $J(a,\bar{r};g)$ *is the Jacobian of* $v(a,\bar{r};g)$, *then the system* $w(a, \lambda, \bar{r}; g), \lambda \in S(a_i, g)$ *is asymptotically stable in* A.

This theorem applies the sufficient condition of utility function $u_i(a, r; g)$ from

Theorem 6, Rosen [1965] present a new result that there exist a globally stabled equilibria which asymptotically converge to one λ.

For a and λ satisfying $w(a_i, \lambda, \bar{r})$, there is a negative the rate of change of $\|w(a, \lambda, \bar{r}; g)\|^2$ for $w(a, \lambda, \bar{r}; g) \neq 0$. If the selection of columns in $\bar{H}(a_i, g)$ remains unchanged, then all λ are zero except which given by $\bar{\lambda} \geq 0$ Rosen [1965]. Thus,

$$w(a, g, \bar{r}; g) = v(a, \bar{r}; g) + \bar{H}(a; g)\bar{\lambda} = v(a, \bar{r}; g) + \sum \bar{\lambda}_j \nabla h_j(a_j; g) \tag{4.66}$$

and

$$\dot{w}(a, \lambda, \bar{r}; g) = J(a, r; g)\dot{a} + \sum \bar{\lambda}_j J_j(a, r; g)\dot{a} + \bar{H}(a, g)\dot{\bar{\lambda}}, \tag{4.67}$$

where $J_j(a, r; g)$ is the Jacobian of $\nabla h(a; g)$ and is therefore negative semidefinite from the concavity of $h(a; g)$. Then, there exists Rosen [1965]

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w(a, \lambda, \bar{r}; g)\|^2 &= \frac{1}{2}\frac{d}{dt}(w^T(a, \lambda, \bar{r}; g)w(a, \lambda, \bar{r}; g)) \\
&= w^T(a, \lambda, \bar{r}; g)J(a, r; g)w(a, \lambda, \bar{r}; g) \\
&+ \sum \bar{\lambda}_j w^T(a, \lambda, \bar{r}; g)J(a; g)w(a, \lambda, \bar{r}; g) \\
&+ w^T(a, \lambda, \bar{r}; g)\bar{H}(a; g)\dot{\bar{\lambda}}.
\end{aligned}$$

Consider using 4.64, 4.66 into the last term, then the result is

$$\begin{aligned}
&w^T(a, \lambda, \bar{r}; g)\bar{H}(a; g)\dot{\bar{\lambda}} \\
&= [v^T(a, \bar{r}; g)\bar{H}(a_i, g) + \bar{\lambda}^T \bar{H}^T(a_i, g)\bar{H}(a_i, g)]\dot{\bar{\lambda}} \\
&= [v^T(a_i, \bar{r}; g)\bar{H}(a; g) - v^T(a, g, \bar{r}; g)\bar{H}(a; g)]\dot{\bar{\lambda}} \\
&= 0
\end{aligned}$$

Then because of $[J(a,r;g)+J(a,r;g)^T]$ is negative definite and $J_j(a;g)$ are negative semidefinite, so there exists Rosen [1965]

$$\frac{1}{2}\frac{d}{dt}\|w(a,\lambda,\bar{r};g)\|^2$$
$$=\frac{1}{2}w^T(a,\lambda,\bar{r};g)[J(a,r;g)+U(a,r;g)^T]^{w(a,\lambda,\bar{r};g)}+\sum\bar{\lambda_j}w^T(a,\lambda,\bar{r};g)J_j(a;g)w(a,\lambda,\bar{r};g)$$
$$\leq -\delta\|w(a_i,\lambda,\bar{r})\|^2$$

where the $\delta > 0$.

A change in the columns selected for $\bar{H}(a_i,g)$ can never increase the value of $\|w(a,\lambda,\bar{r};g)\|$ since the selection as mentioned by $S(a;g)$ will always minimize $\|w(a,\lambda,\bar{r};g)\|$. Thus, there exists $lim_{t\to\infty}\|w(a,g,\bar{r};g)\| = 0$, so that $a(t)\to\bar{a}$ is an equilibrium node satisfy $w(\bar{a},\lambda,\bar{r};g)=0$ for $\lambda\in S(\bar{a},g)$. Thus, when $\bar{a}in\mathbb{E}$, $w(a,\lambda,\bar{r};g),\lambda\in S(\bar{a};g)$ is asymptotically stable in A Rosen [1965].

If the question has been studied more generally, an equilibrium node of $a^*\in A$ can be found as the globally asymptotically stable in A if for every starting node $a\in A$ the solution $a(t)$ to $wa,\lambda,\bar{r};g$ coverages to a^*. Thus, with the appropriate concavity conditions the unique equilibrium node a^* of $u_i(a_i,a_{-i};g)\geq u_i(a_i{'},a_{-i};g)$ is also globally asymptotically stable in A Rosen [1965]. Thus, there is a new theorem as follows:

Theorem 9: *let $A=\{a|h(a;g)\geq 0\}$ and $J(a,r;g)$ be the Jacobian of $v(a,r;g)$ for some fixed $r=\bar{r}>0$. Then if $[J(a,r;g)+J(a,r;g)^T]$ is negative definite for $a\in A$, the normalized equilibrium node $a^*(\bar{r})$ is globally asymptotically stable in A.*

In this theorem, Rosen [1965] extend the global existence of equilibria to a normalized global equilibria. It means there exists a normalized equilibrium node that which presents the globally stability of the whole system.

As mentioned before, the $[J(a, r; g) + J(a, r; g)^T]$ is negative definite, and $\phi(a, r; g)$ is diagonally strictly concave in Theorem . Then in Theorem , there is a unique normalized equilibrium node $a^* = a_i{}^*(\bar{r})$ satisfies the

$$
\begin{aligned}
& h_i(a_i; g) \geq 0, \quad i = 1, \ldots, n, \\
& a^* \nabla u(a_i{}^*; g) + \bar{\lambda}_i \nabla h_i(a_i{}^*; g) = 0, \\
& \bar{\lambda}_i h_i(a_i{}^*; g) = 0, \quad i = 1, \ldots, n.
\end{aligned}
$$

However, an equilibrium node $a_i{}^*$ of $w(a, \lambda, \bar{r}; g)$ also satisfies these three relations. The first relation is satisfied since $\bar{a}_i \in A$, while $w(a_i{}^*, \lambda, \bar{r}; g) = 0$ is equivalent to the second and third relations. Therefore, there must be $a_i{}^* = a_i{}^*(\bar{r})$. Moreover, $w(a_i{}^*, \lambda, \bar{r}; g) = 0$ is asymptotically stable in A. Since $a_i{}^* = a_i{}^*(\bar{r})$ is unique, the solution to $w(a_i, \lambda, \bar{r}; g)$ will converge to $a_i{}^*$ from every starting node in A, and the system is globally asymptotically stable.

4.2.3 Determination of Equilibrium node

In previous section, Rosen [1965] proves the uniqueness of equilibrium node in the global stability situation. This section deals with the existence of unique equilibrium node as a maximization problem through gradient methods for a concave game, KKT conditions. However, the only difference between Rosen's methods Rosen [1965] and a true maximization problem is that how much step length should be chosen in the latter case.

Assume the finite difference approximation to $w(a_i, \lambda, \bar{r})$ is as following

$$a^{j+1} = a^j + \tau^j w(a^j, \lambda^j, \bar{r}; g), \quad \lambda^j \in S(a^j) \tag{4.68}$$

where τ^i is the step length to be selected.

To solve this problem, Rosen [1965] reports a new theorem of method as follows:

Theorem 10: *If $A = \{a | h(a; g) \geq 0\}$ and $[J(a, r; g) + J(a, r; g)^T]$ is negative definite for all $a \in A$, where $J(a, \bar{r}; g)$ is the Jacobian of $v(a, \bar{r}; g)$, then a finite step length τ^i can be chosen so that*

$$\|w^{j+1}(a^{j+1}, \lambda^{j+1}, \bar{r}^{j+1}; g)\| < \|w^j(a^j, \lambda^j, \bar{r}^j; g)\| \tag{4.69}$$

for $w^j(a^j, \lambda^j, \bar{r}^j; g) \neq 0$.

In this theorem, for $\lambda = \lambda^j$, there is

$$\bar{w}(a, \lambda, \bar{r}; g)^{j+1} = w(a^{j+1}, \lambda^j, \bar{r}; g) = w^j(a, \lambda, barr; g) + \bar{W}(a^{j+1} - a^j), \tag{4.70}$$

where $\bar{W}$ is a mean value of the Jacobian of $w(a, \lambda, \bar{r}; g)$, so there is

$$w^T(a, \lambda, \bar{r}; g)\bar{W}(a)w(a, \lambda, \bar{r}; g) < 0 \tag{4.71}$$

for $w(a, \lambda, \bar{r}; g) \neq 0$. Then

$$\bar{w}^{j+1}(a, \lambda, \bar{r}; g) = (\boldsymbol{I} + \tau^j \bar{W})w(a, \lambda, \bar{r}; g)^j. \tag{4.72}$$

The norm of $\bar{w}^{j+1}(a, \lambda, \bar{r}; g)$ as $\|\bar{w}^{j+1}(a, \lambda, \bar{r}; g)\|$ is minimized by the choice

$$\tau^j = -\frac{{w^j}^T(a^j, \lambda^j, \bar{r}; g)\bar{W} w^j(a^j, \lambda^j, \bar{r}; g)}{\|\bar{W} w^j(a^j, \lambda^j, \bar{r}; g)\|^2} > 0, \tag{4.73}$$

which gives

$$\begin{aligned}\|\bar{w}^{j+1}(a^{j+1}, \lambda^{j+1}, \bar{r}; g)\|^2 &= \|w^j(a^j, \lambda^j, \bar{r}; g)\|^2 + \tau^j {w^j}^T(a^j, \lambda^j, \bar{r}; g)\bar{W} w^j(a^j, \lambda^j, \bar{r}; g) \\ &< \|w^j(a, \lambda, \bar{r}; g)\|^2\end{aligned}$$

Finally, there is $\lambda^{j+1} \in S(a^{j+1}; g)$ for $w^{j+1}(a^{j+1}, \lambda^{j+1}, \bar{r}; g)$. Then there exists

$$\|w^{j+1}(a^{j+1}, \lambda^{j+1}, barr; g)\| \leq \|\bar{w}^{j+1}(a^{j+1}, \lambda^{j+1}, \bar{r}; g)\| < \|w^j(a, \lambda, \bar{r}; g)\|^2. \tag{4.74}$$

The convergence of this finite difference procedure to the unique equilibrium node $\bar{a}$ can be shown as in previous theorem.

4.3 The Benchmark Linear-Quadratic Payoffs Model

Linear quadratic payoffs are commonly used to represent a variety of games with constrained continuous actions, such as peer effects, oligopoly and consumption externalities. Different games has different specifications of the action spaces A, $a_i \in A \subset \mathbb{R}$, networks g_{ij} for player i and player j, the payoff impact parameter $\delta \in [-1, 1]$ and various restrictions on the strategy space.

Consider a game in which each player i ($i = 1, \ldots, n$) decides an effort $a_i \geq 0$ to exert in some actions, and a payoffs model to player i as a function of the action profile and network is presented below Bramoullé and Kranton [2015]. It is actually a special case of the generalized payoffs in quadratic form.

$$u_i(a_i, \boldsymbol{a_{-i}}; \delta, \boldsymbol{g}) = {a_i}^0 a_i - \frac{1}{2}{a_i}^2 + \delta \sum_{i \neq j}^{n} g_{ij} a_i a_j, \tag{4.75}$$

In such payoffs model, players are ex ante homogeneous which means all players have the same ${a_i}^0$ of course and δ, and the difference of their locations in the network cause their heterogeneity stems. The first two terms of this function, ${a_i}^0 a_i - \frac{1}{2}{a_i}^2$, give the benefits and costs of taking action a_i. The last term $\delta \sum_{j \neq i} g_{ij} a_i a_j$ which is square matrix reflects cross-effects between player's own actions and neighbours' actions. If network links are positive $g_{ij} = g_{ji} \geq 0$ and the payoff parameter is positive $\delta \geq 0$, these payoffs reflect strategic complementarity in efforts, such as peer effects. For pure substitutes, network links are still positive $g_{ij} = g_{ji} \geq 0$ while the payoff parameter is negative $\delta \leq 0$, such as a Cournot game. When player i and player j are directly linked $g_{ij} = g_{ji} = 1$, the model of δ provides the overall extent of substitutability among goods Bramoullé and Kranton [2015].

Ballester et al. [2006] present determinations of its unique Nash equilibrium in pure strategies. To maximize the payoff of each player i's action, the first-order necessary condition is as following.

$$\frac{\partial u_i(a_i, \boldsymbol{a_{-i}}; \delta, \boldsymbol{g})}{\partial a_i} = {a_i}^0 - a_i + \delta \sum_{i \neq j}^{n} g_{ij} a_j = 0,$$

which leads to:

$${a_i}^* = {a_i}^0 + \delta \sum_{i \neq j}^{n} g_{ij} {a_j}^* \tag{4.76}$$

In matrix form:

$$\boldsymbol{a}^* = {a_i}^0\mathbf{1} + \delta\boldsymbol{g}\boldsymbol{a}^*$$

in which $\mathbf{1}$ is the column vector of 1 and $\boldsymbol{g}$ is the adjacency matrix. After solving, the function leads to

$$\boldsymbol{a}^* = {a_i}^0(\boldsymbol{I} - \delta\boldsymbol{g})^{-1}\mathbf{1} \tag{4.77}$$

where $\boldsymbol{I}$ is the identity matrix.

4.3.1 Katz-Bonacich Network Centrality and Strategic Behavior

Measures of network centrality are for identifying structurally important players and influences of player's action in networks. Ballester et al. [2006] focus on the peer effects on group behavior and use Katz-Bonacich centrality measurement (Katz [1953], Bonacich [1987]) to analyze the influence of individual network positions on individual actions.

In Katz-Bonacich centrality model, individuals are homogenous but for their network position (x0i = x0 for all i). For a network $\boldsymbol{g}$ and a scalar δ such that $(\boldsymbol{I} - \delta\boldsymbol{g})$ is invertible, $b(\boldsymbol{g}, \delta) = (\boldsymbol{I} - \delta\boldsymbol{g})^{-1}\boldsymbol{g}\mathbf{1}$ is the vector of Bonacich centralities. Given a scalar $\delta \geq 0$ and a network $\boldsymbol{g}$, a matrix is defined by Katz [1953] and Bonacich [1987] as follows Ballester et al. [2006] :

$$M(\boldsymbol{g}, \delta) = (\boldsymbol{I} - \delta\boldsymbol{g})^{-1} = \sum_{k=0}^{+\infty} \delta^k \boldsymbol{g}^k \tag{4.78}$$

Consider $\boldsymbol{w} \in \mathbb{R}^n_+$ as a weight for Katz-Bonacich centralities, the vector of weighted Katz-Bonacich centralities relative to a network $\boldsymbol{g}$ is

$$b_{\boldsymbol{w}}(\boldsymbol{g}, \delta) = M(\boldsymbol{g}, \delta)\boldsymbol{w} = (\boldsymbol{I} - \delta\boldsymbol{g})^{-1}\boldsymbol{w} = \sum_{k=0}^{+\infty} \delta^k \boldsymbol{g}^k \boldsymbol{w}. \tag{4.79}$$

When $\boldsymbol{w} = 1$, the unweighted Katz-Bonacich centrality of node i is

$$b_{1,i}(\boldsymbol{g}, \delta) = \sum_{j=1}^{n} M_{ij}(\boldsymbol{g}, \delta) \tag{4.80}$$

and keeps a count of the total number of walks in $\boldsymbol{g}$ starting from i, discounted exponentially by δ. It is the sum of all loops $M_{ii}(\boldsymbol{g}, \delta)$ from i to i itself, and of all the outer walks $\sum_{j \neq i}(\boldsymbol{g}, \delta)$ from i to every other player $j \neq i$, that is:

$$b_{1,i}(\boldsymbol{g}, \delta) = M_{ii}(\boldsymbol{g}, \delta) + \sum_{j \neq i}(\boldsymbol{g}, \delta). \tag{4.81}$$

By definition, $M_{ii}(\boldsymbol{g}, \delta) \geq 1$, and thus $b_i(\boldsymbol{g}, \delta) \geq 1$, with equality when $\delta \neq 0$.

4.3.2 Supermodular Games

Supermodular games are finite noncooperative games, also characterized by strategic complementarities, in which the marginal returns of to increasing one player's partially ordered strategy rise with increases in the competitors' strategies, and this marginal returns to any one component of one player's strategy rise with increases in the other components if the player's strategy set is multidimensional Topkis [1979], Vives [1989], Milgrom and Roberts [1990], Milgrom and Shannon [1994], Topkis [2011]. Supermodularity is applied for capturing economic phenomena that affect the behavior of optimizing agents; but optimization is guided by ordinal, not cardinal, properties. For example, comparative statics are determined by the ordinal properties of the objective function being optimized.

In economics, the comparative statics is a method to compare the difference of economic outcomes/equilibrium sates before and after a process of adjustment under various exogenous conditions Mas-Colell et al. [1995], for instance analyzing changes in supply and demand in a single market. The most common analytic process is comparative statics which applys the implicit function theorem to first-order conditions or on exploiting the identities of duality theory Milgrom and Shannon [1994]. I will review monotone comparative statics below.

Let n is a positive integer, $n \in N$ and $\boldsymbol{A}$ is a set that $\boldsymbol{A}^n$ denote cartesian product. The n-dimensional Euclidean space is $\mathbb{R}^{n\times n}$ and $\boldsymbol{A} \subset \mathbb{R}^{n\times n}$.

For $a, b \in \mathbb{R}^{n\times n}$, let $a \leq b$ if $a_i \leq b_i$ for $i \in N$. That $a < b$ if $a \leq b$ and $a \neq b$. Finally that $a \ll g$ if $a_i < b_i$ for all $i \in N$. The symbols $\geq, >$ and $\gg$ have the obvious meaning.

Let n is a positive integer, $n \in N$ and $\boldsymbol{A}$ is a set that $\boldsymbol{A}^n$ denote cartesian product. The n-dimensional Euclidean space is $\mathbb{R}^{n\times n}$ and $\boldsymbol{A} \subset \mathbb{R}^{n\times n}$.

For $a, b \in \mathbb{R}^{n\times n}$, let $a \leq b$ if $a_i \leq b_i$ for $i \in N$. That $a < b$ if $a \leq b$ and $a \neq b$. Finally that $a \ll g$ if $a_i < b_i$ for all $i \in N$. For a measure space (Θ, T), let $\triangle(\Theta, T)$ be the set of all probability measurements on (Θ, T). For $\Theta \subseteq \mathbb{R}^{n\times n}$, the probability distribution is measured by a cumulative distribution function. For a pair $(\boldsymbol{A}, \leq)$, $\boldsymbol{A}$ is a set and $\leq$ is a partial order on $\boldsymbol{A}$, which is called partially ordered set.

Let $(\boldsymbol{A}, \leq)$ be a lattice and $(T, \leq')$ be a partially ordered set.

A function $f : \boldsymbol{A} \times T \to \mathbb{R}$ has **increasing differences** if, for all $a, a' \in \boldsymbol{A}$

and $t, t' \in T$ with $a < a'$ and $t < t'$

$$f(a', t) - f(a, t) \leq f(a', t') - f(a, t'). \tag{4.82}$$

In this case, If f has increasing differences in (a, t), then the incremental gain is greater with pursuing a' $(a' > a)$ while t is higher. There is nondecreasing in t when $f(a', t) - f(a, t)$. Moreover, the increasing differences is symmetric that $f(a, t') - f(a, t)$ is nondecreasing in a when $a' > a$.

A function $f : \boldsymbol{A} \times T \to \mathbb{R}$ has ***strictly* increasing differences** if for all $a, a' \in \boldsymbol{A}$ and $t, t' \in T$ with $a < a'$ and $t < t'$

$$f(a', t) - f(a, t) < f(a', t') - f(a, t'). \tag{4.83}$$

Definition A function $f : \boldsymbol{A} \times T \to \mathbb{R}$ is ***supermodular*** if for all $a, a' \in \boldsymbol{A}$,

$$f(a) + f(a') \leq f(a \vee a') + f(a \wedge a'). \tag{4.84}$$

Supermodularity requires that

$$f(a) - f(a \vee a') \leq f(a \wedge a') - f(a'). \tag{4.85}$$

Thus, it is obvious that supermodularity has the same characteristics on increasing differences in each dimension.

Following Topkis [1979]'s definition, let $\boldsymbol{A}_i$ be a lattive for all $i \in N$. Let $\boldsymbol{A}$ be a sublattice of $\Pi_{i \in N}$. If $A \to \mathbf{R}$ is super modular, then f has increasing differences. Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be lattice and $f : \boldsymbol{A}_1 \times \boldsymbol{A}_2 \to \mathbf{R}$. If (i) $a_1 \mapsto f(a_1, a_2)$

is supermodular for all $a_2 \in \boldsymbol{A}_2$ and (ii) $a_2 \mapsto f(a_1, a_2)$ is supermodular for all $a_1 \in \boldsymbol{A}_1$ and (iii) f has increasing difference, then f is a supermodular game.

Let $(\boldsymbol{A}, \leq)$ be a lattice, $(T, \leq')$ be a partially ordered set, and $f : \boldsymbol{A} \times T \to \mathbb{R}$ a function. A main question in monotone comparative statics is that there is non-decreasing in t, when

$$a(t) = \arg\max_{a \in \boldsymbol{A}} f(a, t) \tag{4.86}$$

Due to Topkis [1979], Vives [1989], Milgrom and Roberts [1990]Milgrom and Shannon [1994], ***Monotonicity Theorem*** is as follows.

Let $\boldsymbol{A} \subset \mathbb{R}$ be compact and T a partially ordered set. Assume $f : \boldsymbol{A} \times T \to \mathbb{R}$ has increasing difference in (a, t), and is upper semi-continuous in a. Then (i) $a \mapsto f(a, t)$ is quasi-supermodular; (ii) f satisfies the single crossing property in (a, t) that if $t' \leq t$, then $a(t') \leq a(t)$, $\bar{a}(t') \leq \bar{a}(t)$ and $\underline{a}(t') \leq \underline{a}(t)$.

4.4 A Simple Concave Network Model

In this section, I present my first concave network game, which is a new type of game that provides a useful tractable set of solutions. The model presents a simple network externality whereby an agents payoff depends on a function of the actions of the other agents to which the agent is connected. The game is inherently supermodular as the payoff for any given agent is strictly increasing with increasing action of another agent and the form is structured to ensure concavity. Hence, the equilibrium is a positive vector of actions for all agents when the graph describing the network is strongly connected.

4.4.1 Defending My Modelling Assumptions

From the perspective of the remainder of the thesis, I will just spend a moment to review my four primary assumptions.

1. Fixed network structure, characterised by a strongly connected graph.

The first assumption reduces the set of objects of interest in two ways, one obvious, one subtle. First, that the network that is of interest can be described as a mathematical object. Clearly, there are networks of phenomena that are either not mathematical in nature, or cannot be described by a standard network object or approximated in some convenient manner. I will always restrict myself to cases where data can be constructed into a digraph of some description. I am also mostly interested in digraph as higher dimensional objects, whilst interesting are no particularly tractable for the types of models I am interested in, mostly game theoretic in nature. The second part of the assumption, that is I only study networks described by strongly connected graphs, is more restrictive, but necessary for mathematical tractability. Strongly connected networks have invertible adjacency matrices.

My focus on eigencentrality as the core theme of the thesis, means that networks with isolated sets of points, have to be considered as separate networks as the nodes cannot influence optimizing behaviour by agents assumed to be those nodes or vertices. Hence a fully partitioned adjacency matrix where one collection or nodes is completely disconnected from another is treated as separate network problems in my set-up.

2. Existence and uniqueness of equilibrium actions.

I look at a fixed network structure. However, I look at several cases where the network is the structure that links agents in non-cooperative games. Of course the set of all games on any given fixed network, will, in all likelihood be dominated by edge cases where agents are unable to engage in a range of strategies because of the constraints on behaviour. However, I am interested almost exclusively in games where the linear and non-linear constraints are not binding, that is agents have degrees of freedom to optimise behaviour against the observed collection of other agents within the network. That is my interest is always in the non-trivial decision cases.

3. Concavity of the welfare or payoff function and stability of the equilibrium.

Following from the second assumption, I will restrict myself to cases where the payoff functions for agents are marginally concave with respect to the action set of the individual and all other agents on the network and globally diagonally concave as a whole. To secure the concavity of payoffs in a game, I will choose exponential function for payoffs because its graph is naturally continuous, derivable and concave. It is simpler to find the result of first-derivative solution of an exponential payoffs and explore its extreme value and monotonicity for equilibria in a N-player game. A good example of such game with exponential function is how financial arrangements of security in different international airports affect the possibility of a terrorist attack in an airport. A payoff function of such game is $u = a^{-x}$ in which u presents the possibility of being attacked and x means the security arrangements in an airport. The better the security arrangements are, the lower (asymptotically reach 0) the possibility of an attack. To find the best financial arrangement, it needs to find the first-derivative of $u = a^{-x}$ and consider the extraneous solutions.

Indeed, this assumption imposes assumption two by construction. However, edge cases, as noted in (2) are possible.

4. Equilibrium is a function of centrality and in particular eigencentrality.

For concave games, equilibrium is the solution to the minimax problem of optimizing individual behaviour versus the entirety of the observed actions on the network. However, a continuum of such solutions exist. In this thesis I generally restrict myself to games where the equilibrium action space is some non-stochastic continuous function of the eigencentrality. This allows me to target a number of interesting cases globally that can then be modelled and directly parameterized.

4.4.2 A Model of Weighted Networks

Consider a weighted graph W with adjacency matrix $\boldsymbol{W} \in \mathbb{R}^{n \times n}$, were w_{ij} is the weight of the linkage between node i and node j, for $i, j \in \{1, \ldots, I\}$. Set $g_{ij} = \mathbb{1}_{w_{ij}>0}$ to be a binary matrix reporting the existence of a link between node i and node j. Let $0 \leq x_i < \infty$ be a vector of strictly positive actions our base case is to evaluate the games with the following exponential payoff structure:

$$u_i = -a_i \exp\left(-\sum\nolimits_{j=1}^{K} w_{i,j} x_j\right) - \sum\nolimits_{j=1} b_{ij} x_j \tag{4.87}$$

which in matrix form this is equivalent to the following payoff

$$\boldsymbol{u} = -\boldsymbol{A}e^{-\boldsymbol{W}\boldsymbol{x}} - \boldsymbol{B}\boldsymbol{x} \tag{4.88}$$

where $\text{diag}\left[\boldsymbol{A}\right] = \boldsymbol{a} \in \mathbb{R}_{+}^{n}$ and $\text{diag}\left[\boldsymbol{B}\right] = \boldsymbol{b} \in \mathbb{R}_{+}^{n}$ are diagonal matrices of strictly positive coefficients.

Setting the Jacobian matrix of the payoffs with respect to each action x_j as:

$$\boldsymbol{\nabla}_{\boldsymbol{x}}[\boldsymbol{u}] = [\partial u_i/\partial x_j] = -(\boldsymbol{B} - \boldsymbol{A}\text{diag}(\exp(-\boldsymbol{W}\boldsymbol{x}))\boldsymbol{W}) \tag{4.89}$$

decomposing the weighted graph into a matrix of eigenvector $\boldsymbol{Q}$ and eigenvalues $\boldsymbol{\Lambda} = \text{diag}\left[\boldsymbol{\lambda}\right]$, where $\boldsymbol{\lambda}$ is the column vector of eigenvalues of $\boldsymbol{W}$, hence $\boldsymbol{W} = \boldsymbol{Q}^{-1}\boldsymbol{\Lambda}\boldsymbol{Q}$, then rearranging we recover the following derivative

$$\boldsymbol{\nabla}_{\boldsymbol{x}}[\boldsymbol{u}] = [\partial u_i/\partial x_j] = -(\boldsymbol{B} - \boldsymbol{A}\text{diag}(\exp(-\boldsymbol{Q}^{-1}\boldsymbol{\Lambda}\boldsymbol{Q}\boldsymbol{x}))\boldsymbol{Q}^{-1}\boldsymbol{\Lambda}\boldsymbol{Q}) \tag{4.90}$$

the conditions for equilibrium are given by:

$$\text{diag}\left[\boldsymbol{\nabla}_{\boldsymbol{x}}[\boldsymbol{u}]\right] = \text{diag}\left[-(\boldsymbol{B} - \boldsymbol{A}\text{diag}(\exp(-\boldsymbol{Q}^{-1}\boldsymbol{\Lambda}\boldsymbol{Q}\boldsymbol{x}))\boldsymbol{Q}^{-1}\boldsymbol{\Lambda}\boldsymbol{Q})\right] = \boldsymbol{0}_N \tag{4.91}$$

expanding out the terms in the $\text{diag}\left[.\right]$ operator:

$$-\text{diag}\left[\boldsymbol{B}\right] + \text{diag}\left[\boldsymbol{A}\text{diag}\left[\exp(-\boldsymbol{Q}^{-1}\boldsymbol{\Lambda}\boldsymbol{Q}\boldsymbol{x})\right]\boldsymbol{Q}^{-1}\boldsymbol{\Lambda}\boldsymbol{Q}\right] = \boldsymbol{0}_N \tag{4.92}$$

setting $\boldsymbol{A} = \text{diag}\left[\boldsymbol{a}\right]$ and $\boldsymbol{b} = \text{diag}\left[\boldsymbol{B}\right]$, we recover

$$\text{diag}\left[\text{diag}\left[\boldsymbol{a}\right]\text{diag}\left[\exp(-\boldsymbol{Q}^{-1}\boldsymbol{\Lambda}\boldsymbol{Q}\boldsymbol{x})\right]\boldsymbol{Q}^{-1}\boldsymbol{\Lambda}\boldsymbol{Q}\right] = \boldsymbol{b} \tag{4.93}$$

Rearranging this equation yields my first theorem:

Theorem 11: *Working the diag operators through yields the solution of the game as having the following functional form:*

$$\boldsymbol{x}^* = \boldsymbol{Q}\mathrm{diag}\,[\boldsymbol{\lambda}]^{-1}\,\boldsymbol{Q}^{-1}\log(\mathrm{diag}\left[\mathrm{diag}\,[\boldsymbol{b}]\,\mathrm{diag}\,[\boldsymbol{a}]^{-1}\,\boldsymbol{Q}\mathrm{diag}\,[\boldsymbol{\lambda}]^{-1}\,\boldsymbol{Q}^{-1}\right]) \tag{4.94}$$

It is worth noting that this game, similarly to the quadratic games commonly analyzed in previous section, has a fully tractable solution to the equilibrium and hence the main node of interest is the analysis of the graph that underpins the game.

Proof. Proof of Theorem 4.4.2 follows directly from the preceding statements.

4.4.3 The Relationship to Centrality

Inspection of (4.94) in Theorem 4.4.2 provides quite specific insight on the nature of centrality and the relationship to the game. Recall the basic definition of centrality on a graph using the eigen decomposition approach. Let $\boldsymbol{\lambda}^{max} = \max[\lambda_1, \ldots, \lambda_I]$ be the largest eigenvalue of the weighting matrix and $\boldsymbol{q}^{max}$ be the associated eigenvector. First, when W is strongly connected, the Perron-Frobenius theorem shows us that the elements of $\boldsymbol{q}^{max} = [q_1^{max}, \ldots, q_I^{max}]$ are strictly positive.

The ranking centrality is then determined by the magnitude of the element of $\boldsymbol{q}^{max}$ associated with an individual agents actions. More central agents have a greater impact on other agents, hence their actions are more critical in determining the equilibrium. We see this in terms of the structure of Equation 4.94.

4.4.4 Comparative Statics

Theorem 11 presents an equation that how eigencentrality changes affect the stability of a weighted network with strongly connected network structure, and the next step is solving the equation for practically applications. However, when I start doing inversions and eigenvalues polynomial and fractional polynomial operations on each element, there is no algebraic solution for the equation if any game with N-players in which $N \geq 5$. **Abel and Ruffini Theorem** Żoladek [2000] well explained the reason that *A general algebraic equation of degree* ≥ 5 *cannot be solved in radicals. This means that there does not exist any formula which would express the roots of such equation as functions of the coefficients by means of the algebraic operations and roots of natural degrees.*

In game theory, there are some classic examples based on different numbers of players in a game. For instance, if there is only one player, the game is trivial. Prisoner's Dilemma presents all situations of two-player games. Three and four players games are classic public goods games. If players are five or more than five $N \geq 5$, classical methods cannot provide a simple formulas and statements for such game. The method I applied in **Theorem 11:** was making the Jacobian matrix trivial. The Jacobian Matrix of all of the utility functions of each player action is affected by all others. Network and the Jacobian are intertlinked linkage depends on how the game is set up. Most existing economic and security games are linear or quadratic.

Based on Rosen [1965]'s results from previous subchapters that there exists a unique normalized globally stable equilibria in a diagonally strictly concave game. In my simple concave model, such equilibria action is presented as the equation 4.94 from **Theorem 11** in a weighted network with strongly connected network structure. To analyze the sensitivity of that equilibirum to the parameters/networks of the game, comparative statics on the equation 4.94 is an efficient method. To identify the characteristics of parameters/networks, I decomposed the network into its eigenvalues and eigenvectors and found the largest eigenvalue of the weighting matrix with its associated eigenvector to value the eigencentrality of the network. Thus, the results of comparative statics actually present a sensitivity that how eigencentrality changes affect the stability of the network. The equations underpinning the comparative statics are given as follows:

$$
\begin{aligned}
\nabla \boldsymbol{\xi}_\lambda &= \frac{\partial}{\partial \boldsymbol{\lambda}} \boldsymbol{\xi}_{\boldsymbol{\lambda}} = \frac{\partial}{\partial \boldsymbol{\lambda}} \boldsymbol{Q} \mathrm{diag}\left[\boldsymbol{\lambda}\right]^{-1} \boldsymbol{Q}^{-1} \mathbf{1} \\
&= \mathrm{diag}_N \left[\frac{\partial}{\partial \mathrm{diag}\left[\boldsymbol{\lambda}\right]} \boldsymbol{Q} \mathrm{diag}\left[\boldsymbol{\lambda}\right]^{-1} \boldsymbol{Q}^{-1} \right] \\
&= \mathrm{diag}_N \left[-(\mathrm{diag}\left[\boldsymbol{\lambda}\right]^{-1} \boldsymbol{Q}^{-1}) \otimes (\boldsymbol{Q} \mathrm{diag}\left[\boldsymbol{\lambda}\right]^{-1}) \right] && (4.95) \\
\nabla \boldsymbol{\zeta}_\lambda &= \frac{\partial}{\partial \boldsymbol{\lambda}} \boldsymbol{\xi}_{\boldsymbol{\lambda}} = \frac{\partial}{\partial \boldsymbol{\lambda}} \mathrm{diag}\left[\mathbf{b}\right] \mathrm{diag}\left[\mathbf{a}\right]^{-1} \boldsymbol{Q} \mathrm{diag}\left[\boldsymbol{\lambda}\right]^{-1} \boldsymbol{Q}^{-1} \mathbf{1} \\
&= \mathrm{diag}_N \left[\frac{\partial}{\partial \mathrm{diag}\left[\boldsymbol{\lambda}\right]} \mathrm{diag}\left[\mathbf{b}\right] \mathrm{diag}\left[\mathbf{a}\right]^{-1} \boldsymbol{Q} \mathrm{diag}\left[\boldsymbol{\lambda}\right]^{-1} \boldsymbol{Q}^{-1} \right] \\
&= \mathrm{diag}_N \left[-(\mathrm{diag}\left[\boldsymbol{\lambda}\right]^{-1} \mathbf{Q}^{-1})' \otimes (\mathrm{diag}\left[\mathbf{a}\right]^{-1} \mathrm{diag}\left[\mathbf{b}\right] \mathbf{Q} \mathrm{diag}\left[\boldsymbol{\lambda}\right]^{-1}) \right] && (4.96)
\end{aligned}
$$

where $\mathrm{diag}_N\left[.\right]$ is an operator that returns an $N \times N$ matrix from the diagonal of an $N^2 \times N^2$ matrix. Hence the comparative static of the equilibrium is given by:

$$
\boldsymbol{\nabla}_{\boldsymbol{\lambda}}[\boldsymbol{x}^*] = \boldsymbol{\nabla} \boldsymbol{\xi}_\lambda \mathrm{diag}\left[(\mathbf{1} \oslash \boldsymbol{\zeta}_\lambda)\right] \boldsymbol{\nabla} \boldsymbol{\zeta}_\lambda \tag{4.97}
$$

where $\oslash$ is the element by element division (numerator $\oslash$ denominator) of two arrays of the same dimension. The equation 4.97 present a sensitivity parameter that how stability varies with eigencentrality in a strongly connected network.

In the comparative static process, I applied general mathematical functions of the first-order partial derivatives into the largest eigenvalue and its associate eigenvector. However, this mathematical process cannot provide a quantitative accurate results for a further empirical analysis with real data into it. There is an essential requirement for a more specific method for the first-order partial derivatives of largest eigenvalue and its associate eigenvector. Drazin Inverse is in particular to solve this problem. Thus, in the next chapter, I will introduce and apply Drazin Inverse into the equation 4.94 from **Theorem 11** and produce a more quantitative friendly result for further practical applications.

Chapter 5

Eigencentrality and the Drazin Inverse

5.1 Background

In my final chapter, I start think about implementation and specifically the link between eigencentrality measures like Pagerank and the solution to equilibrium games.

As noted in Chapter 3, a deficiency in the network games literature is the lack of generalized comparative statics within the canon of research on network games. In this chapter I will demonstrate how weighted network games on strongly connected graphs that have equilibrium outcomes that are a function of the eigenvalues and eigenvectors can have explicit derivatives as a function of those quantities, using the group and Drazin Inverse. This approach has applications beyond simply analyzing the effect of a changing graph structure on the solution space. In empirical application, Drazin Inverse provides a high quantitative accuracy methods to first and partial derivatives of the largest eigenvalue and its associate eigenvector which value the eigencentrality of a network with strongly connected graph. Then use these facts to provide commentary on the usefulness of this approach to assess the stability of various different algorithms used extensively on graphs. The main solution is an updated equation of sensitivity of influence of various eigencentrality on stability of a weighted network with strongly connected graph in **Theorem 12** 5.29.

The remainder of this chapter is organized as follows. I will review the definition of eigencentrality from previous chapter 2.4 and then introduce a similar and well-applied centrality, PageRank in 5.1.1 which issues a variation on eigencentrality to establish node dominance as measured by the degree of centralness.

Then in 5.1.2 I will outline the various results from the Perron-Frobenius theorem and explain why the largest eigenvalue with its associate eigenvector can value the eigencentrality and the strategic change in a network.

Moreover, I will introduce the Drazin inverse and document a series of important results from the linear algebra literature that demonstrate how the 'group' version of Drazin inverse can be used to derive the exact first and second derivatives of the Perron root and the 'left' eigenvector of that root as a function of the underlying adjacency matrix.

Furthermore, I will then bring all of these pre-results together in 5.3 to illustrate a series of cases where I will compute the sensitivity of the eigencentrality to perturbations in the adjacency matrix.

In the end of the chapter, I will summerize my main achievements in the thesis in section 5.4and discuss the application of my future study on simulation in section .

5.1.1 PageRank

Generally, in a network, a centrality is a serious of mathematical methods to identify the 'degree of central importance' of vertices within networks. There are various of centralities introduced in previous chapter 2.4 such as Closeness, Betweenness, Hubs, Authorities and Eigencentrality. In contrast to the counting based approaches, eigencentrality uses eigenfunction based matrix decomposition of the adjacency matrix to deliver metrics that indicate the degree of explanatory power of specific nodes across the network. PageRank is a special case of Eigencentrality that it can only apply for directed networks. The motivation of introducing PageRank is to explore inspires of application of Eigencentrality.

Following the rapid development of World Wide Web in the lat 1990s, the then early start-up Google developed a page ranking service to search for items on the web. Previous ranking services looked at the number of times particular key search terms occurred on web pages. However, early web adopters fooled this algorithm by dictionary hacks, including hundreds of copies of common words in invisible fonts in the margins of web-pages.

To deal with the spoofing problem Brin and Page [1998], Page et al. [1999] proposed PageRank $\mathcal{C}_P$, to estimate the importance of webpages based on hyperlinks. When a query is entered the, Google search engine searches for the pages p_i that contain the same or similar (using dictionary based variations and catalogues of similar search) key words.

The pages $p_1, p_2, \ldots, p_n$ are then placed into a graph, depending on cross linkages between the ranks of the page. If page p_i links to page p_j because of the relevant information provided by p_j. The higher number of pages links to p_j, the more p_j is related to the query. However, other necessary information of links must be considered as well, such as the importance of links. The PageRank algorithm uses a modification of the eigencentrality algorithm to establish the most central collection of webpages and then reports them in a list. The key modification between PageRank and a pure eigencentrality algorithm is the addition of an arbitrary weighting matrix, which is used to adjust the linkage weights using historical searches and web traffic information.

As PageRank $\mathcal{C}_P$ is a variant of eigencentrality, they both assign weights of a vertex in networks based on itself and its linked neighbours' degrees. Algorithmically main difference between them is that PageRank takes into account link directions (out-link or in-link) as a weight.

The hyperlink web is a directly connected graph. Following definitions presented in previous chapter, let $D := (V(D), E(D))$ be a digraph $D := (V(D), E(D))$ with web HTML pages as vertices $V(D)$ and hyperlinks as edges $E(D)$. The nonnegative and irreducible adjacency matrix of this digraph D is denoted by $\boldsymbol{M} \in \mathbb{R}^{n \times n}$ in which $n = |V(D)|$ is the number of webpages in this directly connected network. For pages $i, j \in V(D)$, if page i links to page j, the value of $E(D)_{ij}$ is 1, otherwise is 0. For page i, the out-degree which is the number of i's out-link pages is denoted by $deg(i) = \sum_j E(D)_{ij}$. A transition matrix is defined as $P_{ij} = E(D)_{ij}/degi$ if $deg_i > 0$, and $P_{ij} = 0$ if $deg(i) = 0$, and i is row-stochastic.

I will now introduce the simplified version of PageRank in my notation set-up and define a random surfer model in keeping with Page et al. [1999] and Berkhin [2005]. If a surfer travels along the digraph, at k step, he locate at a page i, and at the next step $k + 1$ he uniformly and randomly reach any i's out-link neighbour j. Let $p^{(k)} = ({p_i}^{(k)})$ to be the distribution of probabilities for surfer's k step travel to page i. Then, the probability of travelling to page j at $k+1$ step can be described as following

$$\mathcal{C}_P = {p_j}^{(k+1)} = \sum_{i \to j} \frac{{p_i}^{(k)}}{deg(i)} = \sum_i P_{ij}{p_i}^{(k)} \quad or \quad {p_i}^{(k+1)} = P^T p^{(k)}. \tag{5.1}$$

This function presents the relationship of linkings and scores that the more in-link/less out-link neighbors a page has, the higher its PageRank score is. This correlation inspires the definition below Berkhin [2005].

Definition 2. *A PageRank vector is a stationary point of the transformation with nonnegative components (a steady-state vector for a Markov chain)*

$$p = \boldsymbol{M}p, \quad M = P^T \tag{5.2}$$

In this sense we have a similar case to a static weighting game as discussed in the previous chapter. Following this definition, let the sum of the $p-$component be 1 and an L_1 norm $||x|| = \sum |x_i|$ be the standard norm. However, this simple PageRank measure does not work all the time, for instance a set of pages with no outlinks, also called loops or dangling pages with $deg(i) = 0$. For solving this problem, the widely utilized method is defining a new matrix P' with added artificial links that if a surfer visits a dangling page, at next step he will uniformly and randomly move to another page in digraph Page et al. [1999], Haveliwala et al.

[2003], Kamvar et al. [2003] and Berkhin [2005].

$$
\begin{aligned}
D &= d \cdot v^T \\
P' &= P + D
\end{aligned} \tag{5.3}
$$

In this function, D is for modifying the transition probability in which v is a distribution and d is the indicator of a dangling page, $d_i = \delta(deg(i), 0)$. When we consider the web network with digraph to be strongly connected and aperiodic, the matrix P is modified as following Berkhin [2005]:

$$
P'' = cP' + (1-c)E, \quad E = (1, \ldots, 1) \cdot v^T, \quad 0 < c < 1. \tag{5.4}
$$

This function actually transforms the original probability distribution into another without change of stochastic property. If there is no dangling page, a surfer travels from page i to one of its out-link page with probability c and then move to a page j with probability $(1-c)E = (1-c)v_j$ Berkhin [2005].

Hence, our approach sits in the subset of PageRank algorithms where the collection of pages is strongly connected, that is from a surfer view point any two pages p_i and p_j can be clicked through with a finite number weblinks.

Just shortly before Brin and Page [1998], Page et al. [1999] published the PageRank measurements, Kleinberg [1999] produce a HITS algorithm for distinguishing hubs and authorities Web pages.

Following Perra and Fortunato [2008]'s definition, a Hub score and an authority score of i component is denotes by $h_i \in \mathcal{C}_H$ and $ca_i \in \mathcal{C}_A$ respectively. Vectors in the adjacency matrix $\boldsymbol{A}_G$ are denoted as $\boldsymbol{a}$, $\boldsymbol{a} \in \boldsymbol{A}_G$.

$$\boldsymbol{\lambda} ac_i = \sum_{j:j \to i} h_j = \sum_j a_{ij} h_j = (\boldsymbol{a}^T \mathcal{C}_H)_i \tag{5.5}$$

$$\boldsymbol{\mu} h_i = \sum_{j:i \to j} ac_j = \sum_j a_{ij} ac_j = (\boldsymbol{a}^T \mathcal{C}_A)_i \tag{5.6}$$

which can be written in the form of simple eigenvalue equations for both $\mathcal{C}_H$ and $\mathcal{C}_A$ by substitution

$$\boldsymbol{\lambda}\boldsymbol{\mu} h_i = (\boldsymbol{a}\boldsymbol{a}^T \mathcal{C}_H)_i \tag{5.7}$$

$$\boldsymbol{\lambda}\boldsymbol{\mu} ac_i = (\boldsymbol{a}^T \boldsymbol{a} \mathcal{C}_A)_i \tag{5.8}$$

In 5.7 and 5.8, $\boldsymbol{a}\boldsymbol{a}^T$ and $\boldsymbol{a}^T\boldsymbol{a}$ are symmetric. And also, the scores of hub and authority $\mathcal{C}_H$ and $\mathcal{C}_A$ actually correspond to the principal eigenvectors of the matrices $\boldsymbol{a}\boldsymbol{a}^T$ and $\boldsymbol{a}^T\boldsymbol{a}$.

Google's PageRank provides insight into the web structure mining, but it is not perfect. To optimize the performance of PageRank in large scale networks, there are a number of models and algorithms developed into two categories that linear algebraic methods and Monte Carlo methods.

For speeding up the computation of PageRank in linear algebraic methods, Kamvar et al. [2003] propose Quadratic Extrapolation which periodically subtracting off estimates of the nonprincipal eigenvectors and increase the computation speed of PageRank by 25 percent to 300 percent on a web network of 80 million pages. Haveliwala [2003] specify the generic PageRank vector to 16 topic-sensitive PageRank vectors following a top-level category from the Open Directory Project (ODP), and precompute scores of these vectors offline. At query time, engineers produce context-specific importance scores for pages by linearly combining these topic-sensitive PageRank scores with pages information Retrieval score. Xing and Ghorbani [2004] discuss a limitation that PageRank weight all connections in web equally, and present Weighted PageRank algorithm (WPR) which compute the importance of both forward links and back-links of webpages and distributes rank scores based on the popularity of the pages. Langville and Meyer [2004] and Bianchini et al. [2005] focus on inside PageRank, and explore the how the topological structure of the Web fundamentally affect the score distribution of pages that PageRank scores of vertices are only significantly effected when vertices are in the vicinity of the change. Zhu et al. [2005] consider the block structure of hyperlinks instead of the whole link graph, and present a distributed PageRank computation algorithm by utilizing iterative aggregation-disaggregation (IAD) method with Block Jacobi smoothing to achieve distributed computations of web pages accurately and lower time cost. Langville and Meyer [2006] efficiently update the PageRank values based on aggregation/disaggregation principles. Moreover, Charalambous et al. [2016] report a distributed coordination mechanism which can be executed to compute PageRank value even using heterogeneous update speeds.

In the second category of methods, researchers take the advantage of the probabilistic Monte Carlo (MC) methods that full personalizaion is achievable to deal with static networks. Avrachenkov et al. [2007] present a MC algorithm that accounting information from ofcourse the last visited page, moreover from all visited pages during approximations process. Bahmani et al. [2010] apply a

Monte Carlo method to achieve a more efficient incremental updates that up to a reset probability, only total work is required to maintain approximations updated of the PageRank of every pages in digraphs all the time.

Currently, researchers are interested in using PageRank to rank nodes in multiplex networks.Halu et al. [2013] define the multiplex PageRank centrality measure based on idea of biased random walks. One of the popular application is using PageRank to analyze the importance of nodes in graph neural networks International Conference on Learning Representations (ICLR) [2018].

The algorithm of Google PageRank also makes contribution to various subjects and disciplines. For example, Allesina and Pascual [2009] use it to fill the gap between qualitative and quantitative research of food webs and rank species based on their importance for coextinctions to forecast extinction risk in ecosystems. Gleich [2015] introduce diverse applications of Google's PageRank method in symbolic images and ulam networks, sports Radicchi [2011], roads and urban spaces Schlote et al. [2012].

5.1.2 Perron-Frobenius Theorem

A closely related theory of PageRank (and eigencentrality) measures is the $Perron-Frobenius$ theorem, which provides the key mathematical foundation of such centrality measures. This theorem is not only guarantee the existence and uniqueness of PageRank vector (and eigencentrality vector) for any symmetric adjacency matrices with nonnegative entries, but also provide the calculable method, convergence, for it. This theorem is defined below Frobenius [1908], Frobenius [1909] and Frobenius [1912]: If $\mathbf{M} \in R^{n \times n}$ is nonnegative and irreducible then: (a)

$\mathbf{M}$ has a positive eigenvalue, λ, equal to the spectral radius $\rho(\mathbf{M})$, which is the largest absolute value of an eigenvalue, called *Perron root*. (b) λ has algebraic multiplicity 1. (c) There is a positive eigenvector, *Perron vector*, corresponding to λ.

Following previous definitions, the web graph is a directed graph/digraph $D := (V(D), E(D))$, which is strongly connected, with web HTML pages as vertices $V(D)$ and hyperlinks as edges $E(D)$. The adjacency matrix of digraph D is nonnegative and irreducible, denoted by $\boldsymbol{M} \in \mathbb{R}^{n\times n}$. Applying the Perron-Frobenius theorem for PageRank measure, in the adjacency matrix $\boldsymbol{M}$, there exists an eigenvector $\vec{p}$ with the eigenvalue $\lambda = 1$, such that $\boldsymbol{M}\vec{p} = \mathbf{1} \cdot \vec{p}$. This vector $\vec{p}$ corresponds to the limit vector of the stationary distribution for a Markov chain, and also the sum of all its components is 1. Then this vector $\vec{p}$ actually is the PageRank vector of the transition matrix 2 for classifying web pages. Moreover, the web graph is strongly connected and aperiodic. Thus the existence and uniqueness of PageRank vector is proofed.

Perron-Frobenius theorem states that, in any nonnegative and irreducible matrix, there exists a Perron vector which is the positive eigenvector of the largest positive Perron root using for measuring eigencentrality in strongly connected digraphs.

In the following section, I will use a new method of general inverse, Drazin inverse, to analyze the sensitivity of networks by calculating the second-order partial derivatives of Perron vector and Perron root.

5.2 Drazin Inverse

Drazin Inverse is an important type of generalized inverses with spectral properties, which relates to eigenvalues and eigenvectors. It has been widely and successfully applied in many different fields, such as in exploring closed form solutions of singular differential equations with matrix coefficients Campbell and Meyer [1991], in finding difference equations Campbell et al. [1976] and Wei [1996], in numerical analysis Coll et al. [2012], and in Markov chains Meyer [1975], Meyer and Stewart [1988] and Cho and Meyer [2000]. Representations and perturbation bounds of the Drazin inverse of $n \times n$ square matrices are always important topics. They are well developed by Campbell and Meyer [1975], Rong [1982], Wei [1996], Wei and Wang [1997],Wei [1999], Wei and Wu [2000], Rakočevič and Wei [2001],Wei and Li [2003] report the representation and perturbation bounds of the Drazin inverse of $n \times n$ square matrices.

After Cline and Greville [1980] extending the Drazin inverse of a square matrix to a rectangular matrix, new characterizations of perturbation bounds of W-weighted Drazin inverse are also well studied and reported Rakočević and Wei [2002], Wei [2002], Wang and Gu [2005], Castro-González and Velez-Cerrada [2007], Chen and Xu [2008], Liao and Zhang [2013]. Recently, according to a number of useful results of nonlinear and linear recurrent neural network models have been reported Jang et al. [1988], Fa-Long and Zheng [1992], Wang [1993a],Wang [1993b], those researches provide a type of new methods to solve numerical evaluation of the inverse and generalized inverse of square and full-rank rectangular matrices. For instance, Cichocki and Unbehauen [1992] and Samardzija and Waterland [1991] explore how to use Neuron-like network architectures to compute eigenvalues and eigenvectors of real matrices. Wei [2000] investigate using Recurrent Neural Networks (RNNs) for computing weighted Moore-Penrose inverse. Stanimirović et al. [2015] apply the RNNs to compute Drazin Inverse, and Wang et al. [2017] use it to computer W-weighted Drazin inverse.

In my concave model, the network/graph W is a strongly connected digraph, its adjacency matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ is nonnegative and irreducible. As definitions used in previous chapters, eigenvalues of $\boldsymbol{M}$ is labelled as $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ in nonincreasing order: $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$. This network W is also diagnoalizable, there exist an invertible matrix $\boldsymbol{Q}$ which is the matrix of eigenvector such that $\boldsymbol{M} = \boldsymbol{Q}\boldsymbol{\lambda}\boldsymbol{Q}^{-1}$, where $\boldsymbol{\Lambda} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq |\lambda_i|$ for $2 \leq i \leq n$, $\boldsymbol{Q} = [q_1, q_2, \ldots, q_n]$, and $\boldsymbol{Q}^{-T} = [p_1, p_2, \ldots, p_n]$. The right eigenvector associated with λ_i is q_i, and the left eigenvector associated with λ_i is p_i. Following Perron-Frobenius theorem Ben-Israel and Greville [2003], $\lambda_1 > \lambda_2$ and λ_1 equals to the spectral radius $\rho(\boldsymbol{M})$ is a simple eigenvalue of $\boldsymbol{M}$, where $\lambda_1 = \boldsymbol{\lambda}^{max}$ is Perron root. In the mean time, the Perron vector corresponding to Perron root $\boldsymbol{\lambda}^{max}$ denoted $\boldsymbol{q}^{max}$ as the right Perron vector in $\boldsymbol{Q}$ and $\boldsymbol{p}^{max}$ as left Perron vector in $\boldsymbol{Q}^{-T}$. Moreover, there exist $\boldsymbol{p}^{maxT}\boldsymbol{q}^{max} = 1$, $\boldsymbol{M}\boldsymbol{q}^{max} = \boldsymbol{\lambda}^{max}\boldsymbol{q}^{max}$ and $\boldsymbol{p}^{maxT}\boldsymbol{M} = \boldsymbol{\lambda}^{max}\boldsymbol{p}^{maxT}$. Next, I will introduce generalized inverse and Drazin inverse based on my own definitions.

Moore [1920] and Moore and Barnard [1935], Penrose [1955] present a generalization of the inverse of non-singular matrix to the inverse of a singular and rectangular matrix, called *Moore-Penrose Inverse*, denoted by $\boldsymbol{A}^{\ddagger}$. For every finite matrix $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ of real or complex elements, there is a unique matrix $\boldsymbol{X} \in \mathbb{C}^{n\times m}$ satisfying the four equations (*Penrose equations*):

$$\boldsymbol{A}\boldsymbol{X}\boldsymbol{A} = \boldsymbol{A}, (1)$$

$$\boldsymbol{X}\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}, (2)$$

$$(\boldsymbol{A}\boldsymbol{X})^T = \boldsymbol{A}\boldsymbol{X}, (3)$$

$$(\boldsymbol{X}\boldsymbol{A})^T = \boldsymbol{X}\boldsymbol{A}, (4)$$

If $\boldsymbol{A} \in \mathbb{C}^{m \times m}$ is nonsingular, then $\boldsymbol{X} = \boldsymbol{A}^{-1}$. Furthermore, supplementing those four Penrose equations by the following equivalent conditions, the existence of the Drazin inverse is developed.

$$\boldsymbol{A}^k\boldsymbol{M} = \boldsymbol{A}^k, (1^k)$$

$$\boldsymbol{A}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{A}, (5)$$

$$\boldsymbol{A}^k\boldsymbol{M} = \boldsymbol{M}\boldsymbol{A}^k, (5^k)$$

$$\boldsymbol{A}\boldsymbol{M}^k = \boldsymbol{M}^k\boldsymbol{A}, (6^k)$$

In these equations k is a given positive integer. This inverse with the $1^k, 2, 5$-*inverse* of $\boldsymbol{A}$, where k is the index of $\boldsymbol{A}$, is called *Drazin inverse*, denoted by $\boldsymbol{A}^D$.

In following sections, I take second-order partial derivatives with respect to Perron root $\boldsymbol{\lambda}^{max}$ and Perron vector (right Perron vector $\boldsymbol{q}^{max}$, left Perron vector $\boldsymbol{p}^{max}$) using Drazin Inverse.

5.2.1 Second-Order Partial Derivatives of the Perron Root

In previous research, Vahrenkamp [1976] and Cohen [1978] discuss properties of the partial derivatives of the greatest eigenvalue, Perron root $\boldsymbol{\lambda}^{max}$ in nonnegative and irreducible matrices $\boldsymbol{M} \in \mathbb{R}^{n\times n}$. In order to to investigate the concavity and convexity of the Perron root as a function of the entries, Deutsch and Neumann [1984] well develop their work that extending results to second-order partial partial derivatives of Perron root $\boldsymbol{\lambda}^{max}$ at matrices $\boldsymbol{M}$.

Firstly, Deutsch and Neumann [1984] rewrite Vahrenkamp [1976]'s results of the first-order partial derivative of $\boldsymbol{\lambda}^{max}$ in terms of the Dravin Inverse $\boldsymbol{M}^D$ at nonnegative and irreducible matrices $\boldsymbol{M} \in \mathbb{R}^{n\times n}$.

$$\frac{\partial \boldsymbol{\lambda}^{max}}{\partial_{ij}} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{M}^D)^T \tag{5.9}$$

Then, Deutsch and Neumann [1984] report representations for the nonnegative and irreducible matrices of $\boldsymbol{M}$ the second-order partial derivatives.

Let $\boldsymbol{M} \in \mathbb{R}^{n\times n}$, for all $i,\ j,\ k,\ l = 1, \ldots, n$

$$\frac{\partial^2 \boldsymbol{\lambda}^{max}}{\partial_{ij}\partial_{kl}} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{M}^D)_{li}(\boldsymbol{M}^D)_{jk} + (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{M}^D{}_{jk})(\boldsymbol{M}^D)_{li} \tag{5.10}$$

Moreover, Deutsch and Neumann [1984] extend these result to matrices $\boldsymbol{M}$ with respect to the diagonal entries and obtain the representation of the mixed second partial derivatives of $\boldsymbol{\lambda}^{max}$ below:

Let $\boldsymbol{M} \in \mathbb{R}^{n\times n}$, $\boldsymbol{q}^{max} = (q_1, \ldots, q_n)^T$ and $\boldsymbol{p}^{max} = (p_1, \ldots, p_n)^T$ be right and left Perron vectors of $\boldsymbol{M}$ with $\boldsymbol{p}^{max T}\boldsymbol{q}^{max} = 1$. Set $\boldsymbol{q} = diag(\boldsymbol{q}^{max})$ and

$\boldsymbol{p} = diag(\boldsymbol{p}^{max})$, then

$$\frac{\partial^2 \boldsymbol{\lambda}^{max}}{\partial_{ij}^2} = 2(\boldsymbol{M}^D)^T \circ (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{M}^D)^T = 2\boldsymbol{p}\boldsymbol{M}^{DT}\boldsymbol{q} \tag{5.11}$$

Also, these formula of second partial derivatives of the diagonal entries can be reported as the Hessian of the Perron root Deutsch and Neumann [1984].

Let $\boldsymbol{M} \in \mathbb{R}^{n\times n}$, $\boldsymbol{q}^{max} = (q_1, \dots, q_n)^T$ and $\boldsymbol{p}^{max} = (p_1, \dots, p_n)^T$ be right and left Perron vectors of $\boldsymbol{M}$ with $\boldsymbol{p}^{maxT}\boldsymbol{q}^{max} = 1$. Set $\boldsymbol{q} = diag(\boldsymbol{q}^{max})$ and $\boldsymbol{p} = diag(\boldsymbol{p}^{max})$, then

$$H^A := \frac{\partial^2 \boldsymbol{\lambda}^{max}}{\partial_{ii}\partial_{jj}} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{M}^{DT} \circ \boldsymbol{M}^D + \boldsymbol{M}^{DT} \circ (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{M}^D) \tag{5.12}$$

$$= \boldsymbol{p}\boldsymbol{M}^D\boldsymbol{q} + \boldsymbol{q}\boldsymbol{M}^{DT}\boldsymbol{p} \tag{5.13}$$

5.2.2 Second-Order Partial Derivatives of the Perron Vector

Based on discussion of formulas of second-order partial derivatives of Perron root $\boldsymbol{\lambda}^{max}$Deutsch and Neumann [1984], Deutsch and Neumann [1985] study further up for finding representations of the second order derivatives of an appropriately normalized Perron vector with respect to the matrix entries in terms the Drazin Inverse of matrices $\boldsymbol{M} \in \mathbb{R}^{n\times n}$.

In order to investigate the convexity and concavity of the Perron vector as a function of the entries, only perturbation in the first row of the matrix $\boldsymbol{M}$ is considered and the normalization of Perron vector is to fix the value of the entry of this vector. Thus, there is an important assumption in Deutsch and Neumann

[1985]'s results that consider $\mathbb{F} \in \mathbb{R}^{n,n}$ is a set of continuous matrices, such that: at each $\boldsymbol{M} \in \mathbb{F}$

$$d^2\boldsymbol{M} = 0, \tag{5.14}$$

if $\boldsymbol{q}^{max}$ is a right Perron vector of $\boldsymbol{M} \in \mathbb{F}$, then $\boldsymbol{M}$ has already been normalized so that its first entry is a fixed positive constant σ. In other words, the first row/entry of matrix in $\mathbb{F}$ consists either linear functions in the same parameter or independent variables.

For a vector $x = (x_1 \quad \cdots \quad x_n)^T \in \mathbb{R}^n$, symbol $\bar{x}$ is denoted the $(n-1)$-vector given by $\bar{x} = (x_2 \quad \cdots \quad x_n)^T$. Deutsch and Neumann [1985]'s first result is about the second differential of the right Perron vector.

Let a matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n} = m_{ij}$ is square, nonnegative and irreducible. for all $i,\ j = 1, \dots, n$, $\boldsymbol{q}^{max} = (q_1, \dots, q_n)^T$ and $\boldsymbol{p}^{max} = (p_1, \dots, p_n)^T$ are right and left Perron vectors of $\boldsymbol{M}$ with $\boldsymbol{p}^{maxT}\boldsymbol{q}^{max} = 1$.

$$d^2\bar{\boldsymbol{q}}^{max} = 2(d\boldsymbol{\lambda}^{max})^2\boldsymbol{M}^{-2}\boldsymbol{q}^{max} - d^2\boldsymbol{\lambda}^{max}\boldsymbol{M}^{-1}\bar{\boldsymbol{q}}^{max} \tag{5.15}$$

$$= 2\left[\sum_{i=1}^{n}(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{M}^D)_{i1}dm_{1i}\right]^2\boldsymbol{M}^{-1}\bar{\boldsymbol{q}}^{max} \tag{5.16}$$

$$- 2\sum_{i,j=1}^{n}\Big[(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{M}^D)_{i1}\boldsymbol{M}^D_{j1}dm_{1i}dm_{1j}\boldsymbol{M}^{-1}\bar{\boldsymbol{q}}^{max}. \tag{5.17}$$

Then, Deutsch and Neumann [1985] show the different representation of second-order partial derivatives of Perron root if entries in the first row of $\boldsymbol{M}$ are off-diagonal (the first theorem below) or diagonal (the second theorem below). Here is an important supposition of results that the entries of $\boldsymbol{M}^D$ beneath the $(1,1)$

entry are negative. Let a matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ is square, nonnegative and irreducible. $\boldsymbol{q}^{max} = (q_1, \ldots, q_n)^T$ and $\boldsymbol{p}^{max} = (p_1, \ldots, p_n)^T$ are right and left Perron vectors of $\boldsymbol{M}$ with $\boldsymbol{p}^{maxT}\boldsymbol{q}^{max} = 1$. Set $\boldsymbol{q} = diag(\boldsymbol{q}^{max})$ and $\boldsymbol{p} = diag(\boldsymbol{p}^{max})$. Under the assumption of normalization of Perron vector that $d^2\boldsymbol{M} = 0$, for $k \neq 1$,

$$\frac{\partial^2 \bar{\boldsymbol{q}}^{max}}{\partial^2{}_{1k}} = 2\frac{\partial \boldsymbol{\lambda}^{max}}{\partial_{1k}}[\frac{\partial \boldsymbol{\lambda}^{max}}{\partial_{1k}}\boldsymbol{M}^{-1} - \boldsymbol{M}_{k1}{}^{D}\boldsymbol{I}]\boldsymbol{M}^{-1}\bar{\boldsymbol{q}}^{max} \tag{5.18}$$

$$= 2\boldsymbol{p}_1{}^{max}\boldsymbol{q}_k{}^{max}[\boldsymbol{p}_1{}^{max}\boldsymbol{q}_k{}^{max}\boldsymbol{M}^{-1} - \boldsymbol{M}_{k1}{}^{D}\boldsymbol{I}]\boldsymbol{M}^{-1}\bar{\boldsymbol{q}}^{max} \tag{5.19}$$

Thus, if $\boldsymbol{M}^D \in \mathbb{R}^{n\times n}$ or $\boldsymbol{M}_{k1}{}^D < 0$, then $\bar{\boldsymbol{q}}^{max}$ is a convex function of the $(1, k)$ entry in a neighborhood Deutsch and Neumann [1985] .

Let a matrix $\boldsymbol{M} = \boldsymbol{M}_{nn} \in \mathbb{R}^{n\times n}$ is square, nonnegative and irreducible. $\boldsymbol{q}^{max} = (q_1, \ldots, q_n)^T$ and $\boldsymbol{p}^{max} = (p_1, \ldots, p_n)^T$ are right and left Perron vectors of $\boldsymbol{M}$ with $\boldsymbol{p}^{maxT}\boldsymbol{q}^{max} = 1$. Set $\boldsymbol{q} = diag(\boldsymbol{q}^{max})$ and $\boldsymbol{p} = diag(\boldsymbol{p}^{max})$. Under the assumption of normalization of Perron vector that $d^2\boldsymbol{M} = 0$, for $k \neq 1$ Deutsch and Neumann [1985] ,

$$\frac{\partial^2 \bar{\boldsymbol{q}}^{max}}{\partial_{11}{}^2} = 2\frac{\partial \boldsymbol{\lambda}^{max}}{\partial_{11}}[\frac{\partial \boldsymbol{\lambda}^{max}}{\partial_{11}}\boldsymbol{M}^{-1} - \boldsymbol{M}_{11}{}^{D}\boldsymbol{I}]\boldsymbol{M}^{-1}\bar{\boldsymbol{q}}^{max} \tag{5.20}$$

$$= 2\boldsymbol{p}_1{}^{max}\sigma[\boldsymbol{p}_1{}^{max}\sigma\boldsymbol{M}^{-1} - \boldsymbol{M}_{11}{}^{D}\boldsymbol{I}]\boldsymbol{M}^{-1}\bar{\boldsymbol{q}}^{max} \tag{5.21}$$

$$= -2\boldsymbol{p}_1{}^{max}\sigma^2\boldsymbol{M}^{-1}\boldsymbol{M}_{n1}{}^{D} \tag{5.22}$$

Thus, if $\boldsymbol{M}^D \in \mathbb{R}^{n\times n}$ or $\boldsymbol{M}_{k1}{}^D < 0$, then $\bar{\boldsymbol{q}}^{max}$ is a convex function of the $(1, 1)$ domain.

5.3 Sensitivity and Statics

In previous chapter 4, I build a simple concave network model and achieved the first theorem 4.4.2. In the end of the chapter, the equation 4.94 from **Theorem 11** has been underpinned by comparative statics 4.97. Based on Deutsch and Neumann [1984]'s result 5.9 which provides a mathematical function of the first-order partial derivative of the greatest eigenvalue, Perron root $\boldsymbol{\lambda}^{max}$, in non-negative and irreducible matrices $\boldsymbol{M}^D \in \mathbb{R}^{n \times n}$, I extend the theorem 4.4.2 and apply Drazin Inverse to 4.4.2 to the Perron vector (dominant eigenvector) in the first derivative process.

$$
\begin{aligned}
\frac{\partial \boldsymbol{\lambda}^{max}}{\partial_{ij}} &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{M}^D)^T \\
&= (\boldsymbol{I} - \boldsymbol{x}^* \boldsymbol{x}^{*D})^T \\
&= \boldsymbol{I} - \boldsymbol{x}^{*T} \boldsymbol{x}^{*DT}
\end{aligned}
$$

For $\boldsymbol{x}^{*T}$,

$$
\begin{aligned}
\nabla \boldsymbol{T} \mathbf{1}_\lambda &= (\boldsymbol{Q} \mathrm{diag}\,[\boldsymbol{\lambda}]^{-1} \boldsymbol{Q}^{-1})^T \\
&= \boldsymbol{Q}^T \mathrm{diag}\,[\boldsymbol{\lambda}]^{-1} \boldsymbol{Q}^{-T}
\end{aligned}
\tag{5.23}
$$

$$\begin{aligned}\nabla \boldsymbol{T2}_\lambda &= (\mathrm{diag}\,[\boldsymbol{b}]\,\mathrm{diag}\,[\boldsymbol{a}]^{-1}\,(\boldsymbol{Q}\mathrm{diag}\,[\boldsymbol{\lambda}]^{-1}\,\boldsymbol{Q}^{-1})^T \\ &= \mathrm{diag}\,[\boldsymbol{b}]\,\mathrm{diag}\,[\boldsymbol{a}]^{-1}\,(\boldsymbol{Q}\mathrm{diag}\,[\boldsymbol{\lambda}]^{-1}\,\boldsymbol{Q}^{-1})^T \\ &= \mathrm{diag}\,[\boldsymbol{b}]\,\mathrm{diag}\,[\boldsymbol{a}]^{-1}\,\boldsymbol{Q}^T\mathrm{diag}\,[\boldsymbol{\lambda}]^{-1}\,\boldsymbol{Q}^{-T}\end{aligned} \tag{5.24}$$

$$\boldsymbol{x}^{*T} = \nabla \boldsymbol{T1}_\lambda \mathrm{diag}\,[(\boldsymbol{1} \oslash \boldsymbol{T}2_\lambda)]\,\nabla \boldsymbol{T2}_\lambda \tag{5.25}$$

For $\boldsymbol{x}^{*DT}$,

$$\begin{aligned}\nabla \boldsymbol{T3}_\lambda &= (\boldsymbol{Q}\mathrm{diag}\,[\boldsymbol{\lambda}]^{-1}\,\boldsymbol{Q}^{-1})^{DT} \\ &= \boldsymbol{Q}^{DT}\mathrm{diag}\,[\boldsymbol{\lambda}]^{-1}\,\boldsymbol{Q}^{-DT}\end{aligned}$$

$$\nabla \boldsymbol{T4}_\lambda = (\mathrm{diag}\,[\boldsymbol{b}]\,\mathrm{diag}\,[\boldsymbol{a}]^{-1}\,(\boldsymbol{Q}\mathrm{diag}\,[\boldsymbol{\lambda}]^{-1}\,\boldsymbol{Q}^{-1})^{DT} \tag{5.26}$$

$$\begin{aligned}&= \mathrm{diag}\,[\boldsymbol{b}]\,\mathrm{diag}\,[\boldsymbol{a}]^{-1}\,(\boldsymbol{Q}\mathrm{diag}\,[\boldsymbol{\lambda}]^{-1}\,\boldsymbol{Q}^{-1})^{DT} \\ &= \mathrm{diag}\,[\boldsymbol{b}]\,\mathrm{diag}\,[\boldsymbol{a}]^{-1}\,\boldsymbol{Q}^{DT}\mathrm{diag}\,[\boldsymbol{\lambda}]^{-1}\,\boldsymbol{Q}^{-DT}\end{aligned} \tag{5.27}$$

$$\boldsymbol{x}^{*DT} = \nabla \boldsymbol{T3}_\lambda \mathrm{diag}\,[(\boldsymbol{1} \oslash \boldsymbol{T}4_\lambda)]\,\nabla \boldsymbol{T4}_\lambda \tag{5.28}$$

where $\mathrm{diag}_N\,[.]$ is an operator that returns an $N \times N$ matrix from the diagonal of an $N^2 \times N^2$ matrix.

Hence the comparative static of the equilibrium is given by:

Theorem 12:

$$\begin{aligned}
\boldsymbol{\nabla}_{\boldsymbol{\lambda}}[\boldsymbol{x}^*] &= \boldsymbol{I} - \boldsymbol{x}^{*T}\boldsymbol{x}^{*DT} &(5.29)\\
&= \boldsymbol{I} - \nabla\boldsymbol{T1}_{\lambda}\mathrm{diag}\left[(\mathbf{1} \oslash \boldsymbol{T}2_{\lambda})\right]\nabla\boldsymbol{T2}_{\lambda}\nabla\boldsymbol{T3}_{\lambda}\mathrm{diag}\left[(\mathbf{1} \oslash \boldsymbol{T}4_{\lambda})\right]\nabla\boldsymbol{T4}_{\lambda} &(5.30)
\end{aligned}$$

where $\oslash$ is the element by element division (numerator $\oslash$ denominator) of two arrays of the same dimension.

Inspection of **Theorem 12** 5.29 presents a particular results on an interaction with Perron root $\boldsymbol{\lambda}^{max}$ (largest eigenvalue) of eigencentrality and stability (Nash equilibria in multi-player games). This result is achieved by an application of Drazin inverse which provides the partial derivative process for Perron root and Perron vector in a strongly connected network with a nonnegative adjacency matrix. Perron root of each player's or individual agent's strategic choice strongly connected to magnitude of its eigencentrality. Thus this result actually contribute a sensitivity measurement which is able to account and test the extent of interaction between changes and stability of the network system.

Theorem 12 5.29 provides a creative result that not only linking eigencentrality with stability in a network but also resulting to better understand data that could be generated by a game played on a network. One of great applications of this result will be in simulation process of quantitative financial, economic and technique problems. For instance, it can apply into comparing the various influence of bankruptcy from different levels of importance (eigencentrality) of banks on the globally stability of financial system.

5.4 Concluding Remarks

This thesis evaluates the influence of a strategic change in a strongly connected network with fixed structure on the stability of entire network. First and foremost, inspiring by the recent developed literature on Nash Equilibrium in multiplayer games by Rosen [1965], I define the stability of such strong connect network as Nash Equilibrium in these games. Chapter 4 reprove the existence and uniqueness of Nash Equilibrium in such games with diagonally concave joint payoff structures in my own definitions. Second, based on the characteristics of Eigencentrality which evaluate the 'importance' of one node in a network (or each player in a game) not only depending on the amount of links it connected but also it's connectors' (or other players') 'importance' or influence in the network, I apply Eigencentrality measurement on multiplayer games to identify that how player's strategic changes affect the stability of whole networks. Furthermore, I build my own simple concave model with a weighted network and specify a utility function of multiplayer's supermodular game. Jacobian matrix is applied into its first-order partial derivatives, and then I achieve my first theorem that a function of sensitivity of changes in a strongly connected network. I also did a comparative statics of this function.

According to the particular characteristics of Eigencentrality, Drazin Inverse Deutsch and Neumann [1984] is applied to partial derivative of Perron root and Perron vector in Chapter 5 to analysis the sensitivity in Theorem 1 of changes on a weighted network. Finally I achieve a result of new comparative statics result.

5.5 Future work

Various further works can be developed based on this thesis. An obvious and valuable direction is comparison of simulation models. 5.29 provides a sensitivity measurement which can be applied to test results of extensive simulation studies to obtain their performance in a diversity of situations.

Following Banks [2005]'s definition, simulation is 'the imitation of the operation of a real-world process or system over time'. In a simulation process, researchers design a set of assumptions to describe the relationship between real system and modelling process and then estimate the measurements of performances with simulation generated data Banks [2005]. Simulation modelling can make contributions on researches on the internal interactions of a complex system that effectively analyzing the influence of new strategies and changes for current systems and forecasting the performance of new systems. It has been widely applied to different subjects, for instance manufacturing applications Fisher and Ittner [1999], Carson and Maria [1997], Rasheed et al. [2018], semiconductor manufacturing Shanthikumar et al. [2007], construction engineering Chou [2011], Robinson [2002], logistics, transportation and distribution applications Corman and Meng [2014], business process Yan et al. [2002], organizational decision making Stasser [1988], Chung and Lee [2009] and climate changes and geographic information systems Prentice et al. [1993], Hamilton et al. [2005], He [2003].

In simulation analysis process, models and related results can be affected by a number of uncertain parameters. Sometimes, different techniques and methods may produce different results for the same problem even under same assumptions such as multi-attribute decision making problem (MADM Zanakis et al. [1998].

In various simulation scenarios, it is necessary to test or analyze the difference of sensitivity of parameters in the same model or different results from different simulation techniques and models. My theorems 5.29 provides a new and robust metrics to present and explain such interactions. Comparing with other analysis methods based on different scenarios and statistics analysis for simulation process and applied data, my comparative statics utilizes Perron root (largest eigenvalue) of eigencentrality which is insensitive to various of measurement noise such as sampling bias, missing data, aliasing and is more robust when imperfect data involved.

My model can make good contribution on macroeconomy researches. For instance, it can be applied into detecting systemic institutions in certain banking systems. Assume this banking system exists a strongly connected network, I can apply my model into this system and result the different sensitivity of different financial institutions/banks with different eigencentrality on the stability of whole system. Moreover, it can be applied into business cycle synchronization in certain countries and analyze how difference importance (eigencentrality) of different companies in the same industry or in different industries but in the same period be affected differently.

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