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Symbolic Approach to 2-Orthogonal Polynomial Solutions of a Third Order Differential Equation

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Abstract In a recent work, a generic differential operator on the vectorial space of polynomial functions was presented and applied in the study of differential relations fulfilled by polynomial sequences either orthogonal or 2-orthogonal. Considering a third order differential operator that does not increase the degree of polynomials, we search for polynomial eigenfunctions with the help of symbolic computations, assuming that those polynomials constitute a 2-orthogonal polynomial sequence. Two examples are extensively described.

Keywords d-Orthogonal polynomials · Differential operators · Symbolic computations.

Mathematics Subject Classification 42C05 · 33C45 · 68W30 · 33-04 · 34L10

1 Introduction

The families of classical orthogonal polynomials are known to fulfil a second order linear differential equation of hypergeometric type [1,2]. Since the 1990's, a significant amount of bibliography (e.g. [9]) allowed us to understand the orthogonal polynomial eigenfunctions of ordinary differential equations. With regard to the notion of d-orthogonal polynomials which generalises the standard orthogonality and it is defined by means of d functionals, for any positive integer d, much is still unknown. The d-orthogonal polynomials are characterised by a recurrence relation of order $d+1$, naturally enlarging the well known recurrence relation of order two fulfilled by the classical orthogonal polynomials. Moreover, the orthogonality conditions fulfilled by these polynomial sequences are also associated with the step-line multiple orthogonal polynomials (e.g. [3,4]). Several contributions indicate that some d-orthogonal polynomial sequences $\{P_n(x)\}_{n\geq 0}$ fulfil certain differential equations of order $d+1$ (e.g. [5,6]), though in most cases the differential operator obtained depends on n.

Focusing on the dimension $d = 2$, we define any 2-orthogonal polynomial sequence by three numerical sequences, called the recurrence coefficients. In this paper we convey a procedure that aims to compute those

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numerical sequences corresponding to polynomial eigenfunctions of ordinary differential equations with predefined coefficients, not depending on n. Some important results about a general differential operator were given in [17] and are inhere used in order to deal with the computations required for the matter in hands.

This paper has the following structure: in Sect. 2 we provide the basic concepts and the most common notation. Also in Sect. 2 the results given in [17] are listed and a new result proved. Section 3 is dedicated to a third-order differential operator (that preserves the degree of polynomials) and the above mentioned symbolic approach is depicted, step by step. In Sect. 4, we find the complete description of two 2-orthogonal polynomial sequences that are sets of polynomial eigenfunctions of two given third-order differential operators. The procedure here presented allowed us to establish, on one hand, additional differential identities fulfilled by the polynomial sequences obtained, and on the other hand, a list of some impossible cases.

2 Notation and Fundamental Results

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual space. We denote by $\langle u, p\rangle$ the action of the form or linear functional $u \in \mathcal{P}'$ on $p \in \mathcal{P}$. In particular, $\langle u, x^n\rangle := (u)_n\,, n \geq 0$, represent the moments of u. In the following, we will call polynomial sequence (PS) to any sequence $\{P_n\}_{n\geq 0}$ such that $\deg P_n = n,\ n \geq 0$, that is, for all non-negative integer. We will also call monic polynomial sequence (MPS) to a PS so that all polynomials have leading coefficient equal to one.

If $\{P_n\}_{n\geq 0}$ is a MPS, there exists a unique sequence $\{u_n\}_{n\geq 0}$, $u_n \in \mathcal{P}'$, called the dual sequence of $\{P_n\}_{n\geq 0}$, such that,

$$< u_n, P_m >= \delta_{n,m}\ ,\ n, m \geq 0. \tag{2.1}$$

On the other hand, given a MPS $\{P_n\}_{n\geq 0}$, the expansion of $x P_{n+1}(x)$, defines sequences in $\mathbb{C}$, $\{\beta_n\}_{n\geq 0}$ and $\{\chi_{n,\nu}\}_{0\leq \nu\leq n,\ n\geq 0}$, such that

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0, \tag{2.2}$$

$$x P_{n+1}(x) = P_{n+2}(x) + \beta_{n+1} P_{n+1}(x) + \sum_{\nu=0}^{n} \chi_{n,\nu} P_\nu(x)\ ,\ n \geq 0\,. \tag{2.3}$$

This relation is usually called the structure relation of $\{P_n\}_{n\geq 0}$, and $\{\beta_n\}_{n\geq 0}$ and $\{\chi_{n,\nu}\}_{0\leq \nu\leq n,\ n\geq 0}$ are called the structure coefficients (SCs) [11]. Another useful presentation is the following.

$$P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) + \sum_{\nu=0}^{n} \chi_{n,\nu} P_\nu(x)\ ,\ P_0(x) = 1, \quad P_1(x) = x - \beta_0\ ,\ n \geq 0\,.$$

When the structure coefficients fulfil $\chi_{n,\nu} = 0\ ,\ 0 \leq \nu \leq n-1\ ,\ \chi_{n,n} \neq 0$, identities (2.2)-(2.3) refer to the well known three-term recurrence associated to an orthogonal MPS. More generally, identity (2.3) may furnish a recurrence relation of a higher order corresponding to the following notion of orthogonality with respect to d given functionals.

Definition 1 [8,13,20] Given $\Gamma^1, \Gamma^2, \ldots, \Gamma^d \in \mathcal{P}'$, $d \geq 1$, the polynomial sequence $\{P_n\}_{n\geq 0}$ is called d-orthogonal polynomial sequence (d-OPS) with respect to $\Gamma = (\Gamma^1, \ldots, \Gamma^d)$ if it fulfils

$$\langle \Gamma^\alpha, P_m P_n\rangle = 0, \quad n \geq md + \alpha, \quad m \geq 0, \tag{2.4}$$

$$\langle \Gamma^\alpha, P_m P_{md+\alpha-1}\rangle \neq 0, \quad m \geq 0, \tag{2.5}$$

for each integer $\alpha = 1, \ldots, d$.

Lemma 1 [12] *For each $u \in \mathcal{P}'$ and each $m \geq 1$, the two following statements are equivalent.*

a) $\langle u, P_{m-1}\rangle \neq 0,\ \langle u, P_n\rangle = 0,\ n \geq m.$
b) $\exists \lambda_\nu \in \mathbb{C},\ 0 \leq \nu \leq m-1,\ \lambda_{m-1} \neq 0$ *such that* $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.

The conditions (2.4) are called the d-orthogonality conditions and the conditions (2.5) are called the regularity conditions. In this case, the d-dimensional functional Γ is said to be regular.

The d-dimensional functional Γ is not unique. Nevertheless, from Lemma 1, we have:

$$\Gamma^\alpha = \sum_{\nu=0}^{\alpha-1} \lambda_\nu^\alpha u_\nu, \quad \lambda_{\alpha-1}^\alpha \neq 0,\ 1 \leq \alpha \leq d.$$

Therefore, since $U = (u_0, \ldots, u_{d-1})$ is unique, we use to consider the canonical functional of dimension d, $U = (u_0, \ldots, u_{d-1})$, saying that $\{P_n\}_{n\geq 0}$ is d-orthogonal (for any positive integer d) with respect to $U = (u_0, \ldots, u_{d-1})$ if

$$\langle u_\nu, P_m P_n\rangle = 0, \quad n \geq md + \nu + 1,\ \ m \geq 0,$$
$$\langle u_\nu, P_m P_{md+\nu}\rangle \neq 0, \quad m \geq 0,$$

for each integer $\nu = 0, 1, \ldots, d-1$.

Theorem 1 [13] *Let $\{P_n\}_{n\geq 0}$ be a MPS. The following assertions are equivalent:*

a) $\{P_n\}_{n\geq 0}$ *is d-orthogonal with respect to $U = (u_0, \ldots, u_{d-1})$.*
b) $\{P_n\}_{n\geq 0}$ *satisfies a $(d+1)$-order recurrence relation ($d \geq 1$):*

$$P_{m+d+1}(x) = (x - \beta_{m+d}) P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-1-\nu}(x), \quad m \geq 0,$$

with initial conditions

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0 \ \text{ and if } d \geq 2:$$

$$P_n(x) = (x - \beta_{n-1}) P_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} P_{n-2-\nu}(x), \quad 2 \leq n \leq d,$$

and regularity conditions: $\gamma_{m+1}^0 \neq 0, \quad m \geq 0.$

In this paper, we will focus our attention on 2-orthogonal MPSs, thus fulfilling the recurrence relation

$$P_{n+3}(x) = (x - \beta_{n+2}) P_{n+2}(x) - \gamma_{n+2}^1 P_{n+1}(x) - \gamma_{n+1}^0 P_n(x),$$
$$P_0(x) = 1, \quad P_1(x) = x - \beta_0, \quad P_2(x) = (x - \beta_1) P_1(x) - \gamma_1^1, \ n \geq 0.$$

While working solely with 2-orthogonality it is usual to rename the gamma coefficients as follows (cf. [5])

$$P_{n+3}(x) = (x - \beta_{n+2}) P_{n+2}(x) - \alpha_{n+2} P_{n+1}(x) - \gamma_{n+1} P_n(x), \tag{2.6}$$
$$P_0(x) = 1, \quad P_1(x) = x - \beta_0, \quad P_2(x) = (x - \beta_1) P_1(x) - \alpha_1, \quad n \geq 0. \tag{2.7}$$

2.1 Differential Operators on $\mathcal{P}$ and Technical Identities

In this subsection, we initially list the main results indicated in [17] that will be applied along the text, and secondly, we prove new identities that are the fundamental tools for the strategy pursued. Namely, in Proposition 1 we establish an identity, fulfilled by different types of operators, which has a crucial role in the procedure outlined.

Given a sequence of polynomials $\{a_\nu(x)\}_{\nu\geq 0}$, let us consider the following linear mapping $J:\mathcal{P}\to\mathcal{P}$ (cf. [15], [19]).

$$J=\sum_{\nu\geq 0}\frac{a_\nu(x)}{\nu!}D^\nu,\quad \deg a_\nu\leq\nu,\quad \nu\geq 0. \tag{2.8}$$

Expanding $a_\nu(x)$ as follows:

$$a_\nu(x)=\sum_{i=0}^{\nu}a_i^{[\nu]}x^i,$$

and recalling that $D^\nu(\xi^n)(x)=\frac{n!}{(n-\nu)!}x^{n-\nu}$, we get the next identities about J:

$$J\left(\xi^n\right)(x)=\sum_{\nu=0}^{n}a_\nu(x)\binom{n}{\nu}x^{n-\nu}, \tag{2.9}$$

$$J\left(\xi^n\right)(x)=\sum_{\tau=0}^{n}\left(\sum_{\nu=0}^{\tau}\binom{n}{n-\nu}a_{\tau-\nu}^{[n-\nu]}\right)x^\tau,\quad n\geq 0. \tag{2.10}$$

In particular, a linear mapping J is an isomorphism if and only if

$$\deg\left(J\left(\xi^n\right)(x)\right)=n\,,\quad n\geq 0,\quad\text{and}\quad J(1)(x)\neq 0. \tag{2.11}$$

The next result establishes that any operator that does not increase the degree admits an expansion as (2.8) for certain polynomial coefficients.

Lemma 2 [17] *For any linear mapping J, not increasing the degree, there exists a unique sequence of polynomials $\{a_n\}_{n\geq 0}$, with $\deg a_n\leq n$, so that J reads as in (2.8). Further, the linear mapping J is an isomorphism of $\mathcal{P}$ if and only if*

$$\sum_{\mu=0}^{n}\binom{n}{\mu}a_\mu^{[\mu]}\neq 0,\quad n\geq 0. \tag{2.12}$$

The technique that we will implement in the next section requires the knowledge about the J-image of the product of two polynomials fg. The polynomial $J(fg)$ is then given by a Leibniz-type expansion as mentioned in the next Lemma.

Lemma 3 [17] *For any $f,g\in\mathcal{P}$, we have:*

$$J\left(f(x)g(x)\right)(x)=\sum_{n\geq 0}J^{(n)}(f)(x)\frac{g^{(n)}(x)}{n!}=\sum_{n\geq 0}J^{(n)}(g)(x)\frac{f^{(n)}(x)}{n!}, \tag{2.13}$$

where the operator $J^{(m)}$, $m\geq 0$, on $\mathcal{P}$ is defined by

$$J^{(m)}=\sum_{n\geq 0}\frac{a_{n+m}(x)}{n!}D^n\,. \tag{2.14}$$

Let us suppose that J is an operator expressed as in (2.8), and acting as the derivative of order k, for some non-negative integer k, that is, it fulfils the following conditions.

$$J\left(\xi^k\right)(x)=a_0^{[k]}\neq 0\ \text{ and }\ \deg\left(J\left(\xi^{n+k}\right)(x)\right)=n,\quad n\geq 0; \tag{2.15}$$

$$J\left(\xi^i\right)(x)=0,\quad 0\leq i\leq k-1,\ \text{ if } k\geq 1. \tag{2.16}$$

Lemma 4 [17] *An operator J fulfils (2.15)–(2.16) if and only if the next conditions hold.*

a) $a_0(x) = \cdots = a_{k-1}(x) = 0$, *if* $k \geq 1$*;*
b) $\deg(a_\nu(x)) \leq \nu - k,\ \nu \geq k$*;*
c) $\lambda_{n+k}^{[k]} := \sum_{\nu=0}^{n} \binom{n+k}{n+k-\nu} a_{n-\nu}^{[n+k-\nu]} \neq 0,\ \ n \geq 0.$

Remark 1 Note that in the definition given in item c), we find $\lambda_k^{[k]} = a_0^{[k]}$.
If $k = 0$, then it is assumed that $\lambda_n^{[0]} \neq 0,\ \ n \geq 0$, matching (2.12), so that J is an isomorphism.
If $k = 1$, then J imitates the usual derivative and is commonly called a lowering operator (e.g. [10,16]).

Applying Lemma 3 to different pairs of polynomials, we obtain immediately the next identities.

$$J(xp(x)) = xJ(p(x)) + J^{(1)}(p(x)) \tag{2.17}$$

$$J\left(x^2 p(x)\right) = x^2 J(p(x)) + 2xJ^{(1)}(p(x)) + J^{(2)}(p(x)) \tag{2.18}$$

$$J\left(x^3 p(x)\right) = x^3 J(p(x)) + 3x^2 J^{(1)}(p(x)) + 3xJ^{(2)}(p(x)) + J^{(3)}(p(x)) \tag{2.19}$$

Proposition 1 *Given an operator J defined by (2.8), and taking into account the definition of the operator $J^{(m)}$, $m \geq 0$, (see (2.14)), the following identities hold.*

$$J^{(i)}(xp(x)) = J^{(i+1)}(p(x)) + xJ^{(i)}(p(x))\ ,\ \ i = 0, 1, 2, \ldots. \tag{2.20}$$

Proof Reading $i = 0$ in (2.20) we find the identity stated in (2.17). Let us now consider (2.17) with $p(x)$ filled by the product $xp(x)$:

$$J\left(x^2 p(x)\right) = J^{(1)}(xp(x)) + xJ(xp(x))\ .$$

The last term $xJ(xp(x))$ can be rephrased taking into account (2.17), yielding

$$J\left(x^2 p(x)\right) = x^2 J(p(x)) + xJ^{(1)}(p(x)) + J^{(1)}(xp(x))\ . \tag{2.21}$$

Confronting (2.18) with (2.21), we conclude (2.20) with $i = 1$:

$$J^{(1)}(xp(x)) = J^{(2)}(p(x)) + xJ^{(1)}(p(x))\,.$$

Let us assume as induction hypotheses over $k \geq 2$ that

$$J^{(i)}(xp(x)) = J^{(i+1)}(p(x)) + xJ^{(i)}(p(x))\ ,\ \ i = 0, \ldots, k-1.$$

In view of Lemma 3, we learn that for any polynomial $p = p(x)$

$$J\left(x^{k+1} p\right) = \sum_{n \geq 0} J^{(n)}(p) \frac{\left(x^{k+1}\right)^{(n)}}{n!}\ ;$$

and thus we may write:

$$J\left(x^{k+1}p\right) = \sum_{\mu=0}^{k+1} J^{(\mu)}(p)\binom{k+1}{\mu}x^{k+1-\mu}\ ; \tag{2.22}$$

$$J\left(x^{k}p\right) = \sum_{\nu=0}^{k} J^{(\nu)}(p)\binom{k}{\nu}x^{k-\nu}\,. \tag{2.23}$$

Let us now consider (2.23) with p filled by the product xp as follows:

$$J\left(x^{k+1}p\right) = \sum_{\nu=0}^{k} J^{(\nu)}(xp)\binom{k}{\nu}x^{k-\nu}\,. \tag{2.24}$$

By means of the induction hypotheses, identity (2.24) yields the following.

$$\begin{aligned}
J\left(x^{k+1}p\right) &= \sum_{\nu=0}^{k-1} \left(J^{(\nu+1)}(p) + xJ^{(\nu)}(p)\right)\binom{k}{\nu}x^{k-\nu} + J^{(k)}(xp)\\
&= \sum_{\nu=0}^{k-1} J^{(\nu+1)}(p)\binom{k}{\nu}x^{k-\nu} + \sum_{\nu=1}^{k-1} J^{(\nu)}(p)\binom{k}{\nu}x^{k+1-\nu} + J^{(k)}(xp) + J(p)x^{k+1}\\
&= \sum_{\nu=0}^{k-2} J^{(\nu+1)}(p)\left\{\binom{k}{\nu} + \binom{k}{\nu+1}\right\}x^{k-\nu} + J^{(k)}(p)\binom{k}{k-1}x + J^{(k)}(xp) + J(p)x^{k+1}\\
&= \sum_{\nu=0}^{k-2} J^{(\nu+1)}(p)\binom{k+1}{\nu+1}x^{k-\nu} + J^{(k)}(p)kx + J^{(k)}(xp) + J(p)x^{k+1}\\
&= \sum_{\nu=0}^{k-1} J^{(\nu)}(p)\binom{k+1}{\nu}x^{k+1-\nu} + J^{(k)}(p)kx + J^{(k)}(xp)\,.
\end{aligned}$$

In brief

$$J\left(x^{k+1}p\right) = \sum_{\nu=0}^{k-1} J^{(\nu)}(p)\binom{k+1}{\nu}x^{k+1-\nu} + J^{(k)}(p)kx + J^{(k)}(xp)\,. \tag{2.25}$$

Comparing (2.25) with (2.22), we get

$$J^{(k)}(p)\binom{k+1}{k}x^{k+1-k} + J^{(k+1)}(p)\binom{k+1}{k+1} = kxJ^{(k)}(p) + J^{(k)}(xp)$$

hence $xJ^{(k)}(p) + J^{(k+1)}(p) = J^{(k)}(xp)\,,$

which ends the proof. □

3 An Isomorphism Applied to a 2-Orthogonal Sequence

In the sequel, we consider that J is an isomorphism and $a_\nu(x) = 0\,,\ \nu \geq 4$, thus

$$J = a_0(x)I + a_1(x)D + \frac{a_2(x)}{2}D^2 + \frac{a_3(x)}{3!}D^3\,, \text{ where}$$
$$a_0(x) = a_0^{[0]}\,,\ a_1(x) = a_0^{[1]} + a_1^{[1]}x\,,\ a_2(x) = a_0^{[2]} + a_1^{[2]}x + a_2^{[2]}x^2\,,$$
$$a_3(x) = a_0^{[3]} + a_1^{[3]}x + a_2^{[3]}x^2 + a_3^{[3]}x^3\,, \tag{3.1}$$

and we suppose that the MPS $\{P_n\}_{n\geq 0}$ is 2-orthogonal and fulfils

$$J\left(P_n(x)\right) = \lambda_n^{[0]}P_n(x),\ \text{with } \lambda_n^{[0]} \neq 0\,, n \geq 0\,, \tag{3.2}$$

where

$$\lambda_n^{[0]} = a_0^{[0]} + \binom{n}{1}a_1^{[1]} + \binom{n}{2}a_2^{[2]} + \binom{n}{3}a_3^{[3]}\,,\ n \geq 0\,.$$

In view of $a_\nu(x) = 0\,,\ \nu \geq 4$, the operators $J^{(1)}$, $J^{(2)}$ and $J^{(3)}$ are (see (2.14)):

$$J^{(1)}(p) = \left(a_1(x)I + a_2(x)D + \frac{1}{2!}a_3(x)D^2\right)(p) \tag{3.3}$$
$$J^{(2)}(p) = \left(a_2(x)I + a_3(x)D\right)(p) \tag{3.4}$$
$$J^{(3)}(p) = a_3(x)p$$
$$J^{(m)}(p) = 0\,,\ m \geq 4. \tag{3.5}$$

Broadly speaking, in this section we will intertwine the action of operators $J^{(k)}$, for initial values of k, with the simple multiplication by the monomial x, herein called T_x:

$$T_x\,:\ p \mapsto xp\,,$$

in order to obtain the expansions of polynomials $J^{(1)}\left(P_n(x)\right)$, $J^{(2)}\left(P_n(x)\right)$ and $J^{(3)}\left(P_n(x)\right)$ in the basis formed by the 2-orthogonal MPS $\{P_n(x)\}_{n\geq 0}$.

Most importantly, we review (2.6)-(2.7) by establishing the following definition, considering henceforth $P_{-i}(x) = 0, i = 1, 2, \ldots$.

$$T_x\left(P_n(x)\right) = P_{n+1}(x) + \beta_n P_n(x) + \alpha_n P_{n-1}(x) + \gamma_{n-1}P_{n-2}(x),\ n \geq 0\,. \tag{3.6}$$

Additionally, we can use the knowledge provided by Proposition 1, valid for all operators not decreasing the degree (2.8), that asserts

$$J^{(i)}\left(T_x\left(p\right)\right) = J^{(i+1)}\left(p\right) + T_x\left(J^{(i)}\left(p\right)\right)\,,\ \ i = 0, 1, 2, \ldots. \tag{3.7}$$

<u>`First step`</u>`: applying` J `to the four-term recurrence`

Let us apply the operator J to the recurrence relation (2.6), using both (3.7), with $i = 0$, and (3.2):

$$\lambda_{n+2}^{[0]}T_x\left(P_{n+2}(x)\right) + J^{(1)}\left(P_{n+2}(x)\right) = \lambda_{n+3}^{[0]}P_{n+3}(x)$$
$$+\,\beta_{n+2}\lambda_{n+2}^{[0]}P_{n+2}(x) + \alpha_{n+2}\lambda_{n+1}^{[0]}P_{n+1}(x) + \gamma_{n+1}\lambda_n^{[0]}P_n(x)\,. \tag{3.8}$$

Next, by (3.6) we get $J^{(1)}\left(P_{n+2}(x)\right)$ in the basis $\{P_n(x)\}_{n\geq 0}$:

$$J^{(1)}\left(P_{n+2}(x)\right)=\left(\lambda_{n+3}^{[0]}-\lambda_{n+2}^{[0]}\right)P_{n+3}(x)$$
$$+\alpha_{n+2}\left(\lambda_{n+1}^{[0]}-\lambda_{n+2}^{[0]}\right)P_{n+1}(x)+\gamma_{n+1}\left(\lambda_{n}^{[0]}-\lambda_{n+2}^{[0]}\right)P_n(x)\,,\, n\geq 0. \tag{3.9}$$

Taking into account the information retained in identities $J\left(P_0(x)\right)=\lambda_0^{[0]}P_0(x)$, $J\left(P_1(x)\right)=\lambda_1^{[0]}P_1(x)$, it is easy to verify that

$$a_1(x)=J^{(1)}\left(P_0(x)\right)=\left(\lambda_1^{[0]}-\lambda_0^{[0]}\right)P_1(x)\,,$$
$$a_1(x)P_1(x)+a_2(x)=J^{(1)}\left(P_1(x)\right)=\left(\lambda_2^{[0]}-\lambda_1^{[0]}\right)P_2(x)+\alpha_1\left(\lambda_0^{[0]}-\lambda_1^{[0]}\right)P_0(x)\,,$$

and thus we may define the image of every $P_n(x)$ through the operator $J^{(1)}$ as follows:

$$J^{(1)}\left(P_{n}(x)\right)=\left(\lambda_{n+1}^{[0]}-\lambda_{n}^{[0]}\right)P_{n+1}(x)$$
$$+\alpha_{n}\left(\lambda_{n-1}^{[0]}-\lambda_{n}^{[0]}\right)P_{n-1}(x)+\gamma_{n-1}\left(\lambda_{n-2}^{[0]}-\lambda_{n}^{[0]}\right)P_{n-2}(x)\,,\, n\geq 0. \tag{3.10}$$

Second step: applying $J^{(1)}$ to the four-term recurrence

Let us now apply operator $J^{(1)}$ to the recurrence relation (2.6) fulfilled by $\{P_n(x)\}_{n\geq 0}$:

$$J^{(1)}\left(T_x\left(P_{n+2}(x)\right)\right)=J^{(1)}\left(P_{n+3}(x)\right)$$
$$+\beta_{n+2}J^{(1)}\left(P_{n+2}(x)\right)+\alpha_{n+2}J^{(1)}\left(P_{n+1}(x)\right)+\gamma_{n+1}J^{(1)}\left(P_n(x)\right)\,. \tag{3.11}$$

We may then perform the following transformations:

$$G_1(n):\; J^{(1)}\left(T_x\left(P_{n+2}(x)\right)\right)\to J^{(2)}\left(P_{n+2}(x)\right)+T_x\left(J^{(1)}\left(P_{n+2}(x)\right)\right)\,,$$
$$I_1(n):\; J^{(1)}\left(P_n(x)\right)\to\left(\lambda_{n+1}^{[0]}-\lambda_{n}^{[0]}\right)P_{n+1}(x)$$
$$+\alpha_{n}\left(\lambda_{n-1}^{[0]}-\lambda_{n}^{[0]}\right)P_{n-1}(x)+\gamma_{n-1}\left(\lambda_{n-2}^{[0]}-\lambda_{n}^{[0]}\right)P_{n-2}(x)\,,$$
$$M(n):\; T_x\left(P_n(x)\right)\to P_{n+1}(x)+\beta_nP_n(x)+\alpha_nP_{n-1}(x)+\gamma_{n-1}P_{n-2}(x)\,.$$

These transformations are defined in a suitable computer software, allowing a symbolic implementation that executes the adequate positive increments on the variable n. In this manner, it is possible for us to obtain the expansion of the image of $P_{n+2}(x)$ by operator $J^{(2)}$, in the basis $\{P_n(x)\}_{n\geq 0}$. This procedure is expedite and provides a fourth definition also suitable for a subsequent symbolic application where the significant amount of terms involved is not a problem.

As a result of these computations, (3.11) corresponds to the next identity.

$$J^{(2)}\left(P_{n+2}(x)\right)=A_{n+4}P_{n+4}(x)+B_{n+3}P_{n+3}(x)+C_{n+2}P_{n+2}(x)$$
$$+D_{n+1}P_{n+1}(x)+F_nP_n(x)+G_{n-1}P_{n-1}(x)+H_{n-2}P_{n-2}(x)\,, \tag{3.12}$$

where

$$
\begin{aligned}
A_n &= \lambda_n^{[0]} - 2\lambda_{n-1}^{[0]} + \lambda_{n-2}^{[0]}\,;\\
B_n &= (\beta_{n-1} - \beta_n)\left(\lambda_n^{[0]} - \lambda_{n-1}^{[0]}\right)\,;\\
C_n &= 2\alpha_{n+1}\left(\lambda_n^{[0]} - \lambda_{n+1}^{[0]}\right) + 2\alpha_n\left(\lambda_n^{[0]} - \lambda_{n-1}^{[0]}\right)\,;\\
D_n &= \alpha_{n+1}\left(\beta_{n+1} - \beta_n\right)\left(\lambda_n^{[0]} - \lambda_{n+1}^{[0]}\right)\\
&\quad + \gamma_{n+1}\left(\lambda_n^{[0]} - 2\lambda_{n+2}^{[0]} + \lambda_{n+1}^{[0]}\right) + \gamma_n\left(\lambda_n^{[0]} - 2\lambda_{n-1}^{[0]} + \lambda_{n+1}^{[0]}\right)\,;\\
F_n &= \alpha_{n+2}\alpha_{n+1}\left(\lambda_n^{[0]} - 2\lambda_{n+1}^{[0]} + \lambda_{n+2}^{[0]}\right) + \gamma_{n+1}\left(\beta_{n+2} - \beta_n\right)\left(\lambda_n^{[0]} - \lambda_{n+2}^{[0]}\right)\,;\\
G_n &= \alpha_{n+3}\gamma_{n+1}\left(\lambda_n^{[0]} - 2\lambda_{n+2}^{[0]} + \lambda_{n+3}^{[0]}\right) + \alpha_{n+1}\gamma_{n+2}\left(\lambda_n^{[0]} - 2\lambda_{n+1}^{[0]} + \lambda_{n+3}^{[0]}\right)\,;\\
H_n &= \gamma_{n+3}\gamma_{n+1}\left(\lambda_n^{[0]} - 2\lambda_{n+2}^{[0]} + \lambda_{n+4}^{[0]}\right)\,. \qquad (3.13)
\end{aligned}
$$

Once more, taking into account that $J\left(P_i(x)\right) = \lambda_i^{[0]} P_i(x)$, $i = 0, 1$, and also (3.10) for $n = 0, 1, 2$, we are able to confirm that the following initial identities hold:

$$
\begin{aligned}
J^{(2)}\left(P_0(x)\right) &= A_2 P_2(x) + B_1 P_1(x) + C_0 P_0(x)\,,\\
J^{(2)}\left(P_1(x)\right) &= A_3 P_3(x) + B_2 P_2(x) + C_1 P_1(x) + D_0 P_0(x)\,,
\end{aligned}
$$

and, hence :

$$
\begin{aligned}
&J^{(2)}\left(P_n(x)\right) = A_{n+2} P_{n+2}(x) + B_{n+1} P_{n+1}(x) + C_n P_n(x)\\
&+ D_{n-1} P_{n-1}(x) + F_{n-2} P_{n-2}(x) + G_{n-3} P_{n-3}(x) + H_{n-4} P_{n-4}(x)\,,\ n \geq 0\,. \qquad (3.14)
\end{aligned}
$$

<u>`Third step`</u>: `applying` $J^{(2)}$ `to the four-term recurrence`

Let us now apply operator $J^{(2)}$ to the recurrence relation (2.6) fulfilled by $\{P_n(x)\}_{n\geq 0}$:

$$
\begin{aligned}
&J^{(2)}\left(T_x\left(P_{n+2}(x)\right)\right) = J^{(2)}\left(P_{n+3}(x)\right)\\
&+ \beta_{n+2} J^{(2)}\left(P_{n+2}(x)\right) + \alpha_{n+2} J^{(2)}\left(P_{n+1}(x)\right) + \gamma_{n+1} J^{(2)}\left(P_n(x)\right)\,.
\end{aligned}
$$

We may perform the following transformations:

$$
\begin{aligned}
G_2(n):&\ \ J^{(2)}\left(T_x\left(P_{n+2}(x)\right)\right) \to J^{(3)}\left(P_{n+2}(x)\right) + T_x\left(J^{(2)}\left(P_{n+2}(x)\right)\right)\,,\\
I_2(n):&\ \ J^{(2)}\left(P_n(x)\right) \to A_{n+2} P_{n+2}(x) + B_{n+1} P_{n+1}(x) + C_n P_n(x)\\
&\qquad + D_{n-1} P_{n-1}(x) + F_{n-2} P_{n-2}(x) + G_{n-3} P_{n-3}(x) + H_{n-4} P_{n-4}(x)\,,\\
M(n):&\ \ T_x\left(P_n(x)\right) \to P_{n+1}(x) + \beta_n P_n(x) + \alpha_n P_{n-1}(x) + \gamma_{n-1} P_{n-2}(x)\,.
\end{aligned}
$$

As before, these transformations and consequent simplifications, enhanced by the symbolic computations, permit to express $J^{(3)}\left(P_{n+2}(x)\right)$ as follows.

$$\begin{aligned}
J^{(3)}\left(P_{n+2}(x)\right) = {} & a_3^{[3]} P_{n+5}(x) \\
& + \left(A_{n+4}\beta_{n+2} - A_{n+4}\beta_{n+4} - B_{n+3} + B_{n+4}\right) P_{n+4}(x) \\
& + \left(A_{n+3}\alpha_{n+2} - A_{n+4}\alpha_{n+4} + B_{n+3}\beta_{n+2} - B_{n+3}\beta_{n+3} - C_{n+2} + C_{n+3}\right) P_{n+3}(x) \\
& + \left(A_{n+2}\gamma_{n+1} - A_{n+4}\gamma_{n+3} + B_{n+2}\alpha_{n+2} - B_{n+3}\alpha_{n+3} - D_{n+1} + D_{n+2}\right) P_{n+2}(x) \\
& + \left(B_{n+1}\gamma_{n+1} - B_{n+3}\gamma_{n+2} + C_{n+1}\alpha_{n+2} - C_{n+2}\alpha_{n+2} \right. \\
& \qquad \left. - D_{n+1}\beta_{n+1} + D_{n+1}\beta_{n+2} - F_n + F_{n+1}\right) P_{n+1}(x) \\
& + \left(C_n\gamma_{n+1} - C_{n+2}\gamma_{n+1} - D_{n+1}\alpha_{n+1} + D_n\alpha_{n+2} \right. \\
& \qquad \left. - F_n\beta_n + F_n\beta_{n+2} - G_{n-1} + G_n\right) P_n(x) \\
& + \left(-D_{n+1}\gamma_n + D_{n-1}\gamma_{n+1} - F_n\alpha_n + F_{n-1}\alpha_{n+2} \right. \\
& \qquad \left. - G_{n-1}\beta_{n-1} + G_{n-1}\beta_{n+2} - H_{n-2} + H_{n-1}\right) P_{n-1}(x) \\
& + \left(-F_n\gamma_{n-1} + F_{n-2}\gamma_{n+1} \right. \\
& \qquad \left. - G_{n-1}\alpha_{n-1} + G_{n-2}\alpha_{n+2} - H_{n-2}\beta_{n-2} + H_{n-2}\beta_{n+2}\right) P_{n-2}(x) \\
& + \left(-G_{n-1}\gamma_{n-2} + G_{n-3}\gamma_{n+1} - H_{n-2}\alpha_{n-2} + H_{n-3}\alpha_{n+2}\right) P_{n-3}(x) \\
& + \left(H_{n-4}\gamma_{n+1} - H_{n-2}\gamma_{n-3}\right) P_{n-4}(x)\,, \;\; n \geq 0\,,
\end{aligned} \tag{3.15}$$

with initial conditions:

$$\begin{aligned}
J^{(3)}\left(P_0(x)\right) = {} & a_3^{[3]} P_3(x) + \left(\left(\beta_0+\beta_1+\beta_2\right) a_3^{[3]} + a_2^{[3]}\right) P_2(x) \\
& + \left(a_3^{[3]}\left(\alpha_1+\alpha_2+\beta_0^2+\beta_1\beta_0+\beta_1^2\right) + \left(\beta_0+\beta_1\right) a_2^{[3]} + a_1^{[3]}\right) P_1(x) \\
& + \left(a_3^{[3]}\left(\alpha_1\left(2\beta_0+\beta_1\right)+\beta_0^3+\gamma_1\right) + \alpha_1 a_2^{[3]} + \beta_0\left(\beta_0 a_2^{[3]} + a_1^{[3]}\right) + a_0^{[3]}\right)\,;
\end{aligned}$$

$$\begin{aligned}
J^{(3)}\left(P_1(x)\right) = {} & a_3^{[3]} P_4(x) + \left(\left(\beta_1+\beta_2+\beta_3\right) a_3^{[3]} + a_2^{[3]}\right) P_3(x) \\
& + \left(a_3^{[3]}\left(\alpha_1+\alpha_2+\alpha_3+\beta_1^2+\beta_2\beta_1+\beta_2^2\right) + \left(\beta_1+\beta_2\right) a_2^{[3]} + a_1^{[3]}\right) P_2(x) \\
& + \left(a_3^{[3]}\left(2\left(\alpha_1+\alpha_2\right)\beta_1+\alpha_2\beta_2+\beta_1^3+\gamma_1+\gamma_2\right) + \alpha_1\beta_0 a_3^{[3]} + \left(\alpha_1+\alpha_2\right) a_2^{[3]} \right. \\
& \qquad \left. + \beta_1\left(\beta_1 a_2^{[3]} + a_1^{[3]}\right) + a_0^{[3]}\right) P_1(x) \\
& + \left(\alpha_1\left(a_3^{[3]}\left(\alpha_2+\beta_0^2+\beta_1\beta_0+\beta_1^2\right) + \left(\beta_0+\beta_1\right) a_2^{[3]} + a_1^{[3]}\right) + \alpha_1^2 a_3^{[3]} \right. \\
& \left. + \gamma_1\left(\left(\beta_0+\beta_1+\beta_2\right) a_3^{[3]} + a_2^{[3]}\right)\right)\,.
\end{aligned}$$

Recalling that $J^{(3)}(p) = a_3(x)p = \left(a_3^{[3]}x^3 + a_2^{[3]}x^2 + a_1^{[3]}x + a_0^{[3]}\right)p$, identity (3.15) enables the computation of the recurrence coefficients $(\beta_n)_{n\geq 0}$, $(\alpha_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$ of a 2-orthogonal $\{P_n\}_{n\geq 0}$ that is the solution of $J\left(P_n\right) = \lambda_n^{[0]} P_n(x)\,,\ n \geq 0\,,$ for a third-order J. We will pursuit with such computations in the next section for particular cases.

4 Finding the 2-Orthogonal Solution of Some Third-Order Differential Equations

Let us now assume that the 2-orthogonal MPS $\{P_n\}_{n\geq 0}$ fulfils $J\left(P_n\right) = \lambda_n^{[0]} P_n(x)\,,\ n \geq 0$, where J is defined by (2.8) with $a_\nu(x) = 0\,,\ \nu \geq 4$.

Initially, we consider that deg $(a_3(x)) = 0$, though $a_3(x) \neq 0$, deg $(a_2(x)) \leq 1$ and deg $(a_1(x))$=1. In other words:

$$\left(a_0(x)I + a_1(x)D + \frac{a_2(x)}{2}D^2 + \frac{a_0^{[3]}}{3!}D^3\right)(P_n(x)) = \lambda_n^{[0]}P_n(x)\,,$$

$$\text{with } a_0(x) = a_0^{[0]}\,,\ a_1(x) = a_0^{[1]} + a_1^{[1]}x\,,\ a_1^{[1]} \neq 0,$$
$$a_2(x) = a_0^{[2]} + a_1^{[2]}x\,,$$
$$a_3(x) = a_0^{[3]} \neq 0\,. \tag{4.1}$$

Consequently, $\lambda_n^{[0]} = na_1^{[1]} + a_0^{[0]}$, which we are assuming as nonzero for all non-negative integer n. Taking into account this set of hypotheses, identity (3.15) provides several difference equations due to the linear independence of $\{P_n\}_{n\geq 0}$. In particular, the coefficient of $P_{n+4}(x)$ on the right hand of (3.15) is expressed by

$$-a_1^{[1]}(\beta_{n+2} - 2\beta_{n+3} + \beta_{n+4})$$

and thus we get the equation

$$\beta_{n+4} - 2\beta_{n+3} + \beta_{n+2} = 0\,,\ n \geq 0\,. \tag{4.2}$$

Also, the coefficients of $P_{n+3}(x)$ and $P_{n+2}(x)$ on the right hand of (3.15) provide the following two identities

$$a_1^{[1]}\left(-2\alpha_{n+2} + 4\alpha_{n+3} - 2\alpha_{n+4} + (\beta_{n+2} - \beta_{n+3})^2\right) = 0\,, \tag{4.3}$$

$$-3a_1^{[1]}(\gamma_{n+1} - 2\gamma_{n+2} + \gamma_{n+3}) = a_0^{[3]}\,. \tag{4.4}$$

Before we head for the final result that describes the 2-orthogonal polynomial sequence that is formed by polynomial eigenfunctions of (4.1), let us list three identities, valid in general, that will take part in the demonstration of Propositions 2 and 4.

Lemma 5 *Given an operator J defined by (2.8), and supposing that $\{P_n(x)\}_{n\geq 0}$ is a 2-orthogonal MPS, the following identities hold.*

$$\begin{aligned} J^{(2)}(P_{n+2}(x)) &= J^{(3)}(P_{n+1}(x)) + xJ^{(2)}(P_{n+1}(x)) \\ &\quad - \beta_{n+1}J^{(2)}(P_{n+1}(x)) - \alpha_{n+1}J^{(2)}(P_n(x)) - \gamma_n J^{(2)}(P_{n-1}(x))\,; \end{aligned} \tag{4.5}$$

$$\begin{aligned} J^{(1)}(P_{n+2}(x)) &= J^{(2)}(P_{n+1}(x)) + xJ^{(1)}(P_{n+1}(x)) \\ &\quad - \beta_{n+1}J^{(1)}(P_{n+1}(x)) - \alpha_{n+1}J^{(1)}(P_n(x)) - \gamma_n J^{(1)}(P_{n-1}(x))\,; \end{aligned} \tag{4.6}$$

$$\begin{aligned} J(P_{n+3}(x)) &= J^{(1)}(P_{n+2}(x)) + xJ(P_{n+2}(x)) - \beta_{n+2}J(P_{n+2}(x)) \\ &\quad - \alpha_{n+2}J(P_{n+1}(x)) - \gamma_{n+1}J(P_n(x))\,. \end{aligned} \tag{4.7}$$

Proof Let us recall the content of Proposition (1) which says

$$J^{(i+1)}(p(x)) + xJ^{(i)}(p(x)) = J^{(i)}(xp(x))\,.$$

Let us consider $J^{(3)}(p(x)) + xJ^{(2)}(p(x)) = J^{(2)}(xp(x))$ with $p(x)$ replaced by $P_{n+1}(x)$. If the right-hand term $J^{(2)}(xP_{n+1}(x))$ is expanded by the recurrence relation

$$xP_{n+1}(x) = P_{n+2}(x) + \beta_{n+1}P_{n+1}(x) + \alpha_{n+1}P_n(x) + \gamma_n P_{n-1}(x),\ n = 0, 1, 2, \ldots,$$

the identity (4.5) is obtained. Using the same argument and identity

$$J^{(2)}\left(P_{n+1}(x)\right)+xJ^{(1)}\left(P_{n+1}(x)\right)=J^{(1)}\left(xP_{n+1}(x)\right)$$

we get (4.6). Finally, from

$$J^{(1)}\left(P_{n+2}(x)\right)+xJ\left(P_{n+2}(x)\right)=J\left(xP_{n+2}(x)\right)$$

we deduce (4.7). □

Proposition 2 *Let us consider a* 2*-orthogonal polynomial sequence* $\{P_n\}_{n\geq 0}$ *fulfilling*

$$J\left(P_n(x)\right)=\lambda_n^{[0]}P_n(x)$$

where J *is defined by (2.8) with* $a_\nu(x)=0\,,\ \nu\geq 4$, *and such that* $a_0(x)=a_0^{[0]}\,,\ a_1(x)=a_0^{[1]}+a_1^{[1]}x\,,\ a_1^{[1]}\neq 0\,,$ $a_2(x)=a_0^{[2]}+a_1^{[2]}x\,,\ a_3(x)=a_0^{[3]}\neq 0\,.$

Then the coefficient $a_1^{[2]}$ *of polynomial* $a_2(x)$ *is zero and the recurrence coefficients of the sequence* $\{P_n\}_{n\geq 0}$ *are the following.*

$$\beta_n=-\frac{a_0^{[1]}}{a_1^{[1]}}\,,\ n\geq 0\,, \tag{4.8}$$

$$\alpha_{n+1}=-\frac{a_0^{[2]}}{2a_1^{[1]}}(n+1)\,,\ n\geq 0\,, \tag{4.9}$$

$$\gamma_{n+1}=-\frac{a_0^{[3]}}{a_1^{[1]}}\left(\frac{1}{3}+\frac{1}{2}n+\frac{1}{6}n^2\right)=-\frac{a_0^{[3]}}{6a_1^{[1]}}\left(n+1\right)\left(n+2\right)\,,\ n\geq 0\,. \tag{4.10}$$

Conversely, the 2*-orthogonal polynomial sequence* $\{P_n\}_{n\geq 0}$ *defined by the recurrence coefficients (4.8)-(4.10) fulfils the third order differential equation* $J\left(P_n(x)\right)=\lambda_n^{[0]}P_n(x)\,,\ n\geq 0\,,$ *where* $a_0(x)=a_0^{[0]}\,,\ a_1(x)=a_0^{[1]}+a_1^{[1]}x\,,\ a_1^{[1]}\neq 0\,,\ a_2(x)=a_0^{[2]}\,,\ a_3(x)=a_0^{[3]}\neq 0\,,$ *and* $a_\nu(x)=0\,,\ \nu\geq 4$.

Proof Analysing the following polynomials, that are assumed to be trivial

$$\left(a_0(x)I+a_1(x)D+\frac{a_2(x)}{2}D^2+\frac{a_0^{[3]}}{3!}D^3\right)\left(P_i(x)\right)-\lambda_i^{[0]}P_i(x)\,,$$

for $i=0,1,2,3,4,5$, we deduce the next list of conditions

$$a_1^{[2]}=0\,,$$

$$\beta_i=-\frac{a_0^{[1]}}{a_1^{[1]}}\,,\ i=0,1,2,3\,,$$

$$\alpha_1=-\frac{a_0^{[2]}}{2a_1^{[1]}}\,,\ \alpha_2=-\frac{a_0^{[2]}}{a_1^{[1]}}\,,\ \alpha_3=-\frac{3a_0^{[2]}}{2a_1^{[1]}}\,,\ \alpha_4=-\frac{4a_0^{[2]}}{2a_1^{[1]}}\,,$$

$$\gamma_1=-\frac{a_0^{[3]}}{3a_1^{[1]}}\,,\ \gamma_2=-\frac{a_0^{[3]}}{a_1^{[1]}}\,,\ \gamma_3=-\frac{2a_0^{[3]}}{a_1^{[1]}}\,.$$

Equation (4.2) points to a general solution of the form $\beta_{n+2}=c_1+c_2n$ that in view of the initial data of $\beta_i\,,\ i=0,1,2,3$, yields $\beta_n=-\frac{a_0^{[1]}}{a_1^{[1]}}\,,\ n\geq 0$.

Looking at (4.3) updating the information of $\{\beta_n\}_{n\geq 0}$, and using the initial data regarding α_i , $i = 1, 2, 3, 4$, we conclude that $\alpha_{n+1} = -\frac{a_0^{[2]}}{2a_1^{[1]}}(n+1)$, $n \geq 0$. Finally, (4.4) and the initial conditions provide the global definition $\gamma_{n+1} = -\frac{a_0^{[3]}}{a_1^{[1]}}\left(\frac{1}{3} + \frac{1}{2}n + \frac{1}{6}n^2\right)$, $n \geq 0$.

It is important to stress that having defined these three sets of constants as the recurrence coefficients of the sequence $\{P_n\}_{n\geq 0}$, all the remaining terms of the right hand of relation (3.15) vanish, namely the coefficients of $P_{n+1}(x)$, $P_n(x)$, $P_{n-1}(x)$, $P_{n-2}(x)$, $P_{n-3}(x)$ and $P_{n-4}(x)$, and the entire identity (3.15) is fulfilled without the need of further restrictions.

Conversely, let us assume that the 2-orthogonal MPS is defined by the recurrence coefficients (4.8)-(4.10) and let us consider the operator

$$J = a_0(x)I + a_1(x)D + \frac{a_2(x)}{2}D^2 + \frac{a_3(x)}{3!}D^3 ,$$

with $a_0(x) = a_0^{[0]}$, $a_1(x) = a_0^{[1]} + a_1^{[1]}x$, $a_1^{[1]} \neq 0$, $a_2(x) = a_0^{[2]}$, and $a_3(x) = a_0^{[3]} \neq 0$.

It is easy to confirm that the following identities are true, for the initial values of i, like $i = 0, \ldots, nmax$ with $nmax$ equal to 4 or 5, using for that matter the definitions of the operator J, $J^{(1)}$ and $J^{(2)}$, as indicated in (3.3)-(3.4).

$$J\,(P_i(x)) = \lambda_i^{[0]} P_i(x)\ ,\ i = 0, \ldots 5.$$

$$J^{(1)}\,(P_i(x)) = \left(\lambda_{i+1}^{[0]} - \lambda_i^{[0]}\right) P_{i+1}(x)$$
$$+\, \alpha_i \left(\lambda_{i-1}^{[0]} - \lambda_i^{[0]}\right) P_{i-1}(x) + \gamma_{i-1}\left(\lambda_{i-2}^{[0]} - \lambda_i^{[0]}\right) P_{i-2}(x)\ ,\ i = 0, \ldots 4.$$

$$J^{(2)}\,(P_i(x)) = A_{i+2}P_{i+2}(x) + B_{i+1}P_{i+1}(x) + C_i P_i(x)$$
$$+\, D_{i-1}P_{i-1}(x) + F_{i-2}P_{i-2}(x) + G_{i-3}P_{i-3}(x) + H_{i-4}P_{i-4}(x)\,,\ i = 0, \ldots 4\,,$$

with A_i, B_i, C_i, D_i, F_i, G_i and H_i defined as read in (3.13).

As induction hypotheses over n, we consider to be true the following set of identities.

$$J\,(P_i(x)) = \lambda_i^{[0]} P_i(x)\ ,\ i = 0, \ldots, n+2.$$

$$J^{(1)}\,(P_i(x)) = \left(\lambda_{i+1}^{[0]} - \lambda_i^{[0]}\right) P_{i+1}(x)$$
$$+\, \alpha_i \left(\lambda_{i-1}^{[0]} - \lambda_i^{[0]}\right) P_{i-1}(x) + \gamma_{i-1}\left(\lambda_{i-2}^{[0]} - \lambda_i^{[0]}\right) P_{i-2}(x)\,,\ i = 0, \ldots, n+1.$$

$$J^{(2)}\,(P_i(x)) = A_{i+2}P_{i+2}(x) + B_{i+1}P_{i+1}(x) + C_i P_i(x)$$
$$+\, D_{i-1}P_{i-1}(x) + F_{i-2}P_{i-2}(x) + G_{i-3}P_{i-3}(x) + H_{i-4}P_{i-4}(x),\ i = 0, \ldots, n+1,$$

where A_i, B_i, C_i, D_i, F_i, G_i and H_i are the coefficients defined in (3.13).

In view of the precise definitions of the recurrence coefficients of $\{P_n\}_{n\geq 0}$ and the description of polynomials $a_3(x)$, $a_2(x)$ and $a_1(x)$, this set of identities may be rewritten as follows.

$$J\,(P_i(x)) = \lambda_i^{[0]} P_i(x)\ ,\ i = 0, \ldots, n+2\,,\ \text{with}\ \lambda_i^{[0]} = a_0^{[0]} + i a_1^{[1]} \neq 0. \tag{4.11}$$

$$J^{(1)}\,(P_i(x)) = a_1^{[1]} P_{i+1}(x) + \frac{1}{2}\, i\, a_0^{[2]} P_{i-1}(x)$$
$$+\, \frac{1}{3}\, i\,(i-1) a_0^{[3]} P_{i-2}(x)\ ,\ i = 0, \ldots, n+1. \tag{4.12}$$

$$J^{(2)}\,(P_i(x)) = a_0^{[2]} P_i(x) + i a_0^{[3]} P_{i-1}(x)\,,\ i = 0, \ldots, n+1. \tag{4.13}$$

Looking at (4.5) knowing that $J^{(3)}\left(P_{n+1}(x)\right)=a_0^{[3]}P_{n+1}(x)$, and using the four-term recurrence relation and the induction hypotheses (4.13), we conclude:

$$J^{(2)}\left(P_{n+2}(x)\right)=a_0^{[2]}P_{n+2}(x)+(n+2)a_0^{[3]}P_{n+1}(x).$$

Similarly, when we apply hypotheses (4.12) and (4.13) into (4.6), along with the four-term recurrence relation, we deduce:

$$\begin{aligned}J^{(1)}\left(P_{n+2}(x)\right)=&a_1^{[1]}P_{n+3}(x)+\frac{1}{2}\,(n+2)\,a_0^{[2]}P_{n+1}(x)\\&+\frac{1}{3}\,(n+1)(n+2)a_0^{[3]}P_n(x)\,.\end{aligned}\tag{4.14}$$

Finally, using the hypotheses (4.11), the four-term recurrence relation and (4.14) , we infer from (4.7):

$$J\left(P_{n+3}(x)\right)=\lambda_{n+3}^{[0]}P_{n+3}(x)=\left(a_0^{[0]}+(n+3)a_1^{[1]}\right)P_{n+3}(x)\,,$$

which completes the induction argument and allow us to assert that $J\left(P_n(x)\right)=\lambda_n^{[0]}P_n(x)$ for all non-negative values of n. □

Concerning the assumptions of this last Propostion, it is worth mention that, later on, it is clarified in Proposition 3 that if $\deg\left(a_3(x)\right)=0$, though $a_3(x)\neq 0$, and $\deg\left(a_2(x)\right)=0$, then $a_1^{[1]}\neq 0$.

It is also important to remark that the 2-orthogonal sequence described in Proposition 2 corresponds to a case, called E, of page 82 of [8]. We then conclude that the single 2-orthogonal polynomial sequence fulfilling the differential identity described in Proposition 2 is classical in Hahn's sense, which means that the sequence of the derivatives $Q_n(x)=\frac{1}{n+1}DP_{n+1}(x)\,,\ n\geq 0$, is also a 2-orthogonal polynomial sequence.

The classical character can be expressed by means of a vectorial functional equation fulfilled by the pair of forms (u_0,u_1) that in this particular case can be read in page 292 of [6], for the case $\beta_0=0$, or $a_0^{[1]}=0$, taking into account that it is established in [8] (p. 104) that this sequence is an Appell sequence, in other words, $Q_n(x)=P_n(x)\,,\ n\geq 0$. We review this detail while working with the intermediate relations (3.9) and (3.12) along with the proof of Proposition 2, and based on those two identities we may indicate as corollary the following two differential identities.

Corollary 1 *Let us consider the* 2*-orthogonal polynomial sequence* $\{P_n\}_{n\geq 0}$ *described in Proposition 2 that fulfils*

$$J\left(P_n(x)\right)=\lambda_n^{[0]}P_n(x)$$

where J is defined by (2.8) with $a_\nu(x)=0\,,\ \nu\geq 4$, *and such that* $a_0(x)=a_0^{[0]}\,,\ a_1(x)=a_0^{[1]}+a_1^{[1]}x\,,\ a_1^{[1]}\neq 0\,,$ $a_2(x)=a_0^{[2]}\,,\ a_3(x)=a_0^{[3]}\neq 0\,.$ *The sequence* $\{P_n\}_{n\geq 0}$ *also fulfils the following two identities*

$$\begin{aligned}&\left(a_1(x)I+a_0^{[2]}D+\frac{1}{2}a_0^{[3]}D^2\right)\left(P_n(x)\right)=a_1^{[1]}P_{n+1}(x)\\&\qquad\qquad+\frac{1}{2}n\,a_0^{[2]}P_{n-1}(x)+\frac{1}{3}(n-1)n\,a_0^{[3]}P_{n-2}(x)\,,\\&DP_n(x)=n\,P_{n-1}(x)\ ,n\geq 0\ ,\ P_{-1}(x)=0\ .\end{aligned}$$

In the next proposition, we sum up a list of further conclusions pointed out by the application of the symbolic approach detailed in Sect. 3 .

Proposition 3 *Let us consider a* 2*-orthogonal polynomial sequence* $\{P_n\}_{n\geq 0}$ *fulfilling*

$$J\left(P_n(x)\right) = \lambda_n^{[0]} P_n(x)\,,$$

where J *is defined by (2.8) with* $a_\nu(x) = 0\,,\ \nu \geq 4$.

a) *If* $a_2(x) = a_0^{[2]}$ *(constant) and* $\deg\left(a_3(x)\right) \leq 2$*, though* $a_3(x) \neq 0$*, then* $a_1^{[1]} \neq 0$.
b) *If* $a_2(x) = 0$ *and* $\deg\left(a_3(x)\right) = 1$*, then there isn't a* 2*-orthogonal polynomial sequence* $\{P_n\}_{n\geq 0}$ *such that* $J\left(P_n(x)\right) = \lambda_n^{[0]} P_n(x)\,,\, n \geq 0$.
c) *If* $a_3(x) = 0$*, then the only solution of* $J\left(P_n(x)\right) = \lambda_n^{[0]} P_n(x)$ *corresponds to* $J = a_0^{[1]} D + a_0^{[0]} I$.

Proof Inserting the hypotheses $a_2(x) = a_0^{[2]}$ (constant) and $a_3^{[3]} = 0$ into (3.15), we obtain $a_3(x) P_{n+2}(x) = a_1^{[1]} \Upsilon_n(x)$, for a given linear combination $\Upsilon_n(x)$ of elements of $\{P_n\}_{n\geq 0}$. Thus, if we consider $a_1^{[1]} = 0$, we get $a_3(x) P_{n+2}(x) = 0$ which is impossible.

With respect to item b), we proceed with the analysis of (3.15) under the assumptions $a_3(x) = a_1^{[3]} x + a_0^{[3]}$ and $a_2(x) = 0$, which allows the definition of the sequences $(\beta_n)_{n\geq 0}$, $(\alpha_n)_{n\geq 0}$ and $(\gamma_n)_{n\geq 0}$ by this order, as follows.

Firstly, identity (3.15) establishes the equation $\beta_{n+4} - 2\beta_{n+3} + \beta_{n+2} = 0$ with initial data $\beta_n = -\dfrac{a_0^{[1]}}{a_1^{[1]}}$, $n = 0, 1, 2, 3,$ yielding: $\beta_n = -\dfrac{a_0^{[1]}}{a_1^{[1]}}\ ,\ n \geq 0$.

Secondly, from the knowledge of $(\beta_n)_{n\geq 0}$, (3.15) indicates

$$\alpha_{n+4} - 2\alpha_{n+3} + \alpha_{n+2} = -\frac{a_1^{[3]}}{2a_1^{[1]}}\,,$$

and taking into account the initial data obtained, the sequence $(\alpha_n)_{n\geq 0}$ is defined by $\alpha_n = -\dfrac{a_1^{[3]}}{4a_1^{[1]}} n(n-1)\ ,\ n \geq 1\,.$

Finally, the equation about $(\gamma_n)_{n\geq 0}$ reads

$$\gamma_{n+3} - 2\gamma_{n+2} + \gamma_{n+1} = -\frac{1}{3a_1^{[1]}}\left(a_0^{[3]} - \frac{a_1^{[3]} a_0^{[1]}}{a_1^{[1]}}\right),$$

yielding, in view of γ_1 and γ_2: $\gamma_n = -\dfrac{1}{6\left(a_1^{[1]}\right)^2}\left(a_1^{[1]} a_0^{[3]} - a_0^{[1]} a_1^{[3]}\right) n(n+1)\,, n \geq 1\,.$

Nevertheless, (3.15) furnishes more than the three equations mentioned previously and the remaining equations are not fulfilled when the above three definitions are taken. In fact, both looking at (3.15) and to the initial computations

$$J\left(P_i(x)\right) - \lambda_i^{[0]} P_i(x)\,, i = 0, \cdots, 10\,,$$

we conclude that $a_1^{[3]}$ must be zero in which case we meet a particular setup of the case described in Proposition 2. Thus, we may assert that the identity $J\left(P_n(x)\right) = \lambda_n^{[0]} P_n(x)\ ,\ n \geq 0$, does not admit 2-orthogonal solutions when $\deg(a_3(x)) = 1$ and $a_2(x) = 0$.

With respect to item c), when we aim for solutions of a second degree differential equation, we may look at (3.12) because if $a_3(x) = 0$ then the operator $J^{(2)}$ defined in (3.4) is free of derivatives, since $J^{(2)}(p) = a_2(x) p$, for all $p \in \mathcal{P}$. Let us proceed by analysing identity (3.12), in particular, we apply the recurrence relation fulfilled by $\{P_n(x)\}_{n\geq 0}$:

$$x P_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \alpha_n P_{n-1}(x) + \gamma_{n-1} P_{n-2}(x)\,,$$

on the left-hand of (3.12), that is, in $a_2(x)P_{n+2}$.
On one hand, the coefficient of $P_{n-2}(x)$ of the expression

$$a_2(x)P_{n+2} - (A_{n+4}P_{n+4}(x) + B_{n+3}P_{n+3}(x) + C_{n+2}P_{n+2}(x) \\ + D_{n+1}P_{n+1}(x) + F_nP_n(x) + G_{n-1}P_{n-1}(x) + H_{n-2}P_{n-2}(x)) \tag{4.15}$$

is $-3a_2^{[2]}\gamma_{n+1}\gamma_{n-1}$ yielding $a_2^{[2]} = 0$, due to the regularity of $\{P_n(x)\}_{n\geq 0}$.
On the other hand, the computation of the first recurrence coefficients through

$$J\left(P_i(x)\right) - \lambda_i^{[0]}P_i(x) = 0\,,\, i = 0, \cdots, 3\,,$$

imply $a_1^{[1]} = 0$.

When we insert these two conditions $a_1^{[1]} = 0$ and $a_2^{[2]} = 0$ into (4.15), we gradually infer that $a_1^{[2]} = 0$ and $a_0^{[2]} = 0$ putting us on the trivial situation $J = a_0^{[1]}D + a_0^{[0]}I$ that corresponds to a known case. □

In the final Proposition 4, we find the description of the 2-orthogonal sequence that is the solution of the problem posed with respect to the third order operator J defined by the conditions $a_2(x) = 0$ and $\deg(a_3(x)) \leq 2$. Taking into consideration Proposition 3, we have assured that $\deg(a_1(x)) = 1$, or $a_1^{[1]} \neq 0$.

Proposition 4 *Let us consider a 2-orthogonal polynomial sequence $\{P_n\}_{n\geq 0}$ fulfilling*

$$J\left(P_n(x)\right) = \lambda_n^{[0]}P_n(x)\,,\; n \geq 0\,,$$

where J is defined by (2.8) with $a_\nu(x) = 0\,,\; \nu \geq 4$, and such that $a_0(x) = a_0^{[0]}\,,\; a_1(x) = a_0^{[1]} + a_1^{[1]}x\,,\; a_1^{[1]} \neq 0\,,$ $a_2(x) = 0\,,\; a_3(x) = a_0^{[3]} + a_1^{[3]}x + a_2^{[3]}x^2\,.$

Then the recurrence coefficients of the sequence $\{P_n\}_{n\geq 0}$ are the following and the coefficients of the polynomial $a_3(x) = a_2^{[3]}x^2 + a_1^{[3]}x + a_0^{[3]}$ fulfil

$$(a_1^{[3]})^2 - 4a_2^{[3]}a_0^{[3]} = 0.$$

$$\beta_n = -\frac{a_2^{[3]}}{2a_1^{[1]}}(n-1)n - \frac{a_0^{[1]}}{a_1^{[1]}}\,,\; n \geq 0\,, \tag{4.16}$$

$$\alpha_n = -\frac{a_1^{[3]}}{2a_1^{[1]}} + \frac{a_0^{[1]}a_2^{[3]}}{(a_1^{[1]})^2} + (n-2)\left(-\frac{3a_1^{[3]}}{4a_1^{[1]}} + \frac{a_2^{[3]}\left(9a_0^{[1]} + a_2^{[3]}\right)}{6(a_1^{[1]})^2}\right) \\ + (n-2)^2\left(b_0 + b_1(n-2) + b_2(n-2)^2\right)\,,\; n \geq 1\,, \tag{4.17}$$

$$\gamma_n = -\frac{1}{3a_1^{[1]}}\left(a_0^{[3]} + \frac{a_0^{[1]}\left(-a_1^{[1]}a_1^{[3]} + a_0^{[1]}a_2^{[3]}\right)}{(a_1^{[1]})^2}\right) \\ -(n-1)\left(\frac{(a_1^{[1]})^2a_0^{[3]} - a_0^{[1]}a_1^{[1]}a_1^{[3]} + (a_0^{[1]})^2\, a_2^{[3]}}{2(a_1^{[1]})^3}\right) \\ + (n-1)^2\left(f_0 + f_1(n-1) + f_2(n-1)^2 + f_3(n-1)^3 + f_4(n-1)^4\right)\,,\; n \geq 1\,; \tag{4.18}$$

where

$$f_0 = \frac{-18a_0^{[3]}(a_1^{[1]})^2 + 6a_1^{[3]}a_1^{[1]}\left(3a_0^{[1]} + a_2^{[3]}\right) + a_2^{[3]}\left(-18(a_0^{[1]})^2 - 12a_2^{[3]}a_0^{[1]} + (a_2^{[3]})^2\right)}{108(a_1^{[1]})^3},$$

$$f_1 = \frac{a_2^{[3]}\left(6a_1^{[1]}a_1^{[3]} + a_2^{[3]}\left(a_2^{[3]} - 12a_0^{[1]}\right)\right)}{72(a_1^{[1]})^3},$$

$$f_2 = -\frac{a_2^{[3]}\left(a_2^{[3]}\left(12a_0^{[1]} + a_2^{[3]}\right) - 6a_1^{[1]}a_1^{[3]}\right)}{216(a_1^{[1]})^3},$$

$$f_3 = -\frac{(a_2^{[3]})^3}{72(a_1^{[1]})^3},$$

$$f_4 = -\frac{(a_2^{[3]})^3}{216(a_1^{[1]})^3},$$

$$b_0 = \frac{1}{2}\left(-\frac{a_1^{[3]}}{2a_1^{[1]}} + \frac{a_0^{[1]}a_2^{[3]}}{(a_1^{[1]})^2} + \frac{10\,(a_2^{[3]})^2}{12\,(a_1^{[1]})^2}\right),$$

$$b_1 = \frac{(a_2^{[3]})^2}{3\,(a_1^{[1]})^2},$$

$$b_2 = \frac{(a_2^{[3]})^2}{12\,(a_1^{[1]})^2}.$$

Conversely, the 2*-orthogonal polynomial sequence* $\{P_n\}_{n\geq 0}$ *defined by the recurrence coefficients (4.16)-(4.18), under the assumption* $\gamma_n \neq 0\,,\ n \geq 1\,,$ *fulfils the differential equation* $J\left(P_n(x)\right) = \lambda_n^{[0]} P_n(x)\,,\ n \geq 0\,,$ *where* $a_0(x) = a_0^{[0]}\,,\ a_1(x) = a_0^{[1]} + a_1^{[1]}x\,,\ a_1^{[1]} \neq 0\,,\ a_2(x) = 0\,,\ a_3(x) = a_2^{[3]}x^2 + a_1^{[3]}x + a_0^{[3]}$ *with* $(a_1^{[3]})^2 - 4a_2^{[3]}a_0^{[3]} = 0$, *and* $a_\nu(x) = 0\,,\ \nu \geq 4$.

Proof The demonstration follows the same reasoning applied in the proof of Proposition 2, based on the symbolic implementation of Sect. 3.
We compute the initial data for β_i , for $i = 0, 1, 2, 3$, α_i , for $i = 1, 2, 3$ and γ_i, for $i = 1, 2$, using the equations $J\left(P_j(x)\right) - \lambda_j^{[0]} P_j(x) = 0\,,\ j = 1, \ldots, 4$.

In the equation (3.15) we have on the left-hand a linear combination of the set of polynomials $\{P_{n-2}(x), P_{n-1}(x), \ldots, P_{n+4}(x)\}$ in view of the recurrence relation fulfilled by a 2-orthogonal polynomial sequence:

$$x\left(P_n(x)\right) = P_{n+1}(x) + \beta_n P_n(x) + \alpha_n P_{n-1}(x) + \gamma_{n-1} P_{n-2}(x).$$

Comparing both members of (3.15) we identify several equations, in particular, three of those allow the definition of the sequences $(\beta_n)_{n\geq 0}$, $(\alpha_n)_{n\geq 0}$ and $(\gamma_n)_{n\geq 0}$ by this order, as follows.

Firstly, identity (3.15) establishes the equation $\beta_{n+4} - 2\beta_{n+3} + \beta_{n+2} = -\dfrac{a_2^{[3]}}{a_1^{[1]}}$ with initial data $\beta_2 = -\dfrac{a_0^{[1]} + a_2^{[3]}}{a_1^{[1]}}$, $\beta_3 = -\dfrac{a_0^{[1]} + 3a_2^{[3]}}{a_1^{[1]}}$, and $\beta_0 = \beta_1 = -\dfrac{a_0^{[1]}}{a_1^{[1]}}$, yielding (4.16).
Secondly, from the knowledge of $(\beta_n)_{n\geq 0}$, (3.15) indicates

$$\alpha_{n+4} - 2\alpha_{n+3} + \alpha_{n+2} = -\frac{a_1^{[3]}}{2a_1^{[1]}} + \frac{a_0^{[1]}a_2^{[3]}}{(a_1^{[1]})^2} + \frac{(a_2^{[3]})^2}{(a_1^{[1]})^2}(n+1)^2\,.$$

Using the values of the initial α's we conclude that the solution of this difference equation is given by (4.17). Finally, the equation for $(\gamma_n)_{n\geq 0}$ reads

$$\gamma_{n+3} - 2\gamma_{n+2} + \gamma_{n+1} = -\frac{1}{3a_1^{[1]}}\left(a_0^{[3]} + \beta_{n+2}\left(a_1^{[3]} + a_2^{[3]}\beta_{n+2}\right) + a_2^{[3]}\left(\alpha_{n+3} + \alpha_{n+2}\right)\right),$$

yielding (4.18), in view of γ_1 and γ_2.

It is important to stress that having defined these three sets of constants as the recurrence coefficients of the sequence $\{P_n\}_{n\geq 0}$, all the remaining equations defined by (3.15) are fulfilled if and only if $\left(a_1^{[3]}\right)^2 = 4a_2^{[3]}a_0^{[3]}$.

Conversely, let us assume a 2-orthogonal MPS defined by the recurrence coefficients (4.16)-(4.18) and let us consider the operator

$$J = a_0(x)I + a_1(x)D + \frac{a_2(x)}{2}D^2 + \frac{a_3(x)}{3!}D^3,$$

with $a_0(x) = a_0^{[0]}$, $a_1(x) = a_0^{[1]} + a_1^{[1]}x$, $a_1^{[1]} \neq 0$, $a_2(x) = 0$, and $a_3(x) = a_2^{[3]}x^2 + a_1^{[3]}x + a_0^{[3]}$ such that $(a_1^{[3]})^2 - 4a_2^{[3]}a_0^{[3]} = 0$.

It is easy to confirm that the following identities are true, for the initial values of i, like $i = 0, \ldots, nmax$ with $nmax$ equal to 4 or 5, using for that matter the definitions of the operator J, $J^{(1)}$ and $J^{(2)}$, as indicated in (3.3)-(3.4).

$$\begin{aligned}
&J\left(P_i(x)\right) = \lambda_i^{[0]}P_i(x),\ i = 0, \ldots 5.\\
&J^{(1)}\left(P_i(x)\right) = \left(\lambda_{i+1}^{[0]} - \lambda_i^{[0]}\right)P_{i+1}(x)\\
&\quad + \alpha_i\left(\lambda_{i-1}^{[0]} - \lambda_i^{[0]}\right)P_{i-1}(x) + \gamma_{i-1}\left(\lambda_{i-2}^{[0]} - \lambda_i^{[0]}\right)P_{i-2}(x),\ i = 0, \ldots 4.\\
&J^{(2)}\left(P_i(x)\right) = A_{i+2}P_{i+2}(x) + B_{i+1}P_{i+1}(x) + C_iP_i(x)\\
&\quad + D_{i-1}P_{i-1}(x) + F_{i-2}P_{i-2}(x) + G_{i-3}P_{i-3}(x) + H_{i-4}P_{i-4}(x),\ i = 0, \ldots 4,
\end{aligned}$$

with A_i, B_i, C_i, D_i, F_i, G_i and H_i defined as read in (3.13).

As induction hypotheses over n, we consider to be true the following set of identities.

$$J\left(P_i(x)\right) = \lambda_i^{[0]}P_i(x),\ i = 0, \ldots, n+2. \tag{4.19}$$

$$\begin{aligned}
J^{(1)}\left(P_i(x)\right) &= \left(\lambda_{i+1}^{[0]} - \lambda_i^{[0]}\right)P_{i+1}(x)\\
&\quad + \alpha_i\left(\lambda_{i-1}^{[0]} - \lambda_i^{[0]}\right)P_{i-1}(x) + \gamma_{i-1}\left(\lambda_{i-2}^{[0]} - \lambda_i^{[0]}\right)P_{i-2}(x),\ i = 0, \ldots, n+1.
\end{aligned}\tag{4.20}$$

$$\begin{aligned}
J^{(2)}\left(P_i(x)\right) &= A_{i+2}P_{i+2}(x) + B_{i+1}P_{i+1}(x) + C_iP_i(x)\\
&\quad + D_{i-1}P_{i-1}(x) + F_{i-2}P_{i-2}(x) + G_{i-3}P_{i-3}(x) + H_{i-4}P_{i-4}(x),\ i = 0, \ldots, n+1,
\end{aligned}\tag{4.21}$$

where A_i, B_i, C_i, D_i, F_i, G_i and H_i are the coefficients defined in (3.13).

Looking at (4.5) knowing that $J^{(3)}\left(P_{n+1}(x)\right) = \left(a_2^{[3]}x^2 + a_1^{[3]}x + a_0^{[3]}\right)P_{n+1}(x)$, and using the four-term recurrence relation and the induction hypotheses (4.13), we conclude:

$$\begin{aligned}
&J^{(2)}\left(P_{n+2}(x)\right) = A_{n+4}P_{n+4}(x) + B_{n+3}P_{n+3}(x) + C_{n+2}P_{n+2}(x)\\
&+ D_{n+1}P_{n+1}(x) + F_nP_n(x) + G_{n-1}P_{n-1}(x) + H_{n-2}P_{n-2}(x).
\end{aligned}$$

Similarly, when we apply hypotheses (4.20) and (4.21) into (4.6), along with the four-term recurrence relation, we deduce:

$$J^{(1)}\left(P_{n+2}(x)\right) = \left(\lambda_{n+3}^{[0]} - \lambda_{n+2}^{[0]}\right) P_{n+3}(x) \\ +\alpha_{n+2}\left(\lambda_{n+1}^{[0]} - \lambda_{n+2}^{[0]}\right) P_{n+1}(x) + \gamma_{n+1}\left(\lambda_{n}^{[0]} - \lambda_{n+2}^{[0]}\right) P_n(x)\,. \tag{4.22}$$

Finally, using the hypotheses (4.11), the four-term recurrence relation and (4.14) , we infer from (4.7):

$$J\left(P_{n+3}(x)\right) = \lambda_{n+3}^{[0]} P_{n+3}(x)\,,$$

which completes the induction argument and allow us to assert that $J\left(P_n(x)\right) = \lambda_n^{[0]} P_n(x)$ for all non-negative values of n. □

The content of Proposition 4 provides an entire solution written in terms of the polynomial coefficients of the operator J. In the next Corollary we read a specific case endowed with Hahn's property, as we may confirm computationally, for the first elements of the sequence, or prove analytically using the functionals of the dual sequence. The computational confirmation, for $n = 0, \ldots, nmax$, for a given positive integer $nmax$, was done using the recursive definition of the sequence $P_n^{[1]}(x) = (n+1)^{-1} D P_{n+1}(x)\,,\ n \geq 0$, as indicated in [18] and through the application of the routine SC_ζ that computes the structure coefficients of any given MPS $\{\zeta_n\}_{n\geq 0}$ (cf. Step 3 of the symbolic computation of [18]). The analytical proof is out of the scope of this paper, since it requires a somehow extensive work with the dual sequence along with some results of [14] and [17], regarding further knowledge on 2-orthogonality and on the transpose operator of J.

Corollary 2 *Let us consider the* 2*-orthogonal polynomial sequence* $\{P_n\}_{n\geq 0}$ *fulfilling*

$$J\left(P_n(x)\right) = \lambda_n^{[0]} P_n(x)\,,\ n \geq 0\,,$$

where J *is defined by (2.8) with* $a_\nu(x) = 0\,,\ \nu \geq 4$, *and such that* $a_0(x) = a_0^{[0]}\,,\ a_1(x) = \frac{1}{24}x\,,\ \ a_2(x) = 0\,,\ a_3(x) = (x-1)^2\,.$

Then the recurrence coefficients of the sequence $\{P_n\}_{n\geq 0}$ *are the following.*

$$\beta_n = -12(n-1)n\,,\ n \geq 0\,, \tag{4.23}$$

$$\alpha_n = 12(n-1)n(2n-3)^2\,,\ n \geq 1\,, \tag{4.24}$$

$$\gamma_n = -4n(n+1)(2n-3)^2(2n-1)^2\,,\ n \geq 1. \tag{4.25}$$

Conversely, the 2*-orthogonal polynomial sequence* $\{P_n\}_{n\geq 0}$ *defined by the recurrence coefficients (4.23)-(4.25) fulfils the differential equation*

$$\left(\frac{1}{6}(x-1)^2 D^3 + \frac{1}{24} x D + a_0^{[0]} I\right)\left(P_n(x)\right) = \lambda_n^{[0]} P_n(x)\,,\ n \geq 0\,,$$

where $\lambda_n^{[0]} = \frac{1}{24}n + a_0^{[0]}\,,\ n \geq 0\,.$

Furthermore, we remark that the polynomial sequence $\{P_n\}_{n\geq 0}$ defined by (4.23)-(4.25), fulfils the following two differential relations obtained by (3.9) and (3.12).

$$\left(\frac{1}{24}xI+\frac{1}{2}(x-1)^2D^2\right)(P_n(x))=\frac{1}{24}P_{n+1}(x)$$
$$-\frac{1}{2}(3-2n)^2(n-1)nP_{n-1}(x)+\frac{1}{3}(n-1)n(15-16n+4n^2)^2P_{n-2}(x)\,, \tag{4.26}$$
$$(x-1)^2D(P_n(x))=nP_{n+1}(x)-2n(5+4n(2n-3))P_n(x)$$
$$+(3-2n)^2n(24(n-2)n+25)P_{n-1}(x)-8(5-2n)^2(n-1)n(2n-3)^3P_{n-2}(x)$$
$$+4(3-2n)^2(5-2n)^2(7-2n)^2(n-2)(n-1)nP_{n-3}(x)\,,\ n\geq 0\,,\ P_{-i}(x)=0\,. \tag{4.27}$$

5 Conclusions

The symbolic approach here presented aims to find 2-orthogonal eigenfunctions of a third order differential operator of the form $p_3(x)D^3+p_2(x)D^2+p_1(x)D+p_0(x)I$, $\deg(p_i(x))\leq i$, and provides a research scheme that led us to the results of Sect. 4; in particular, it has brought to light the 2-orthogonal sequence defined in Corollary 2. In addition, the steps of the procedure proposed brings up other differential identities fulfilled by a 2-orthogonal solution besides the prefixed one $J\,(P_n(x))=\lambda_n^{[0]}P_n(x)\,,\ n\geq 0\,.$

An ongoing work around the functionals (u_0,u_1) demonstrates the Hahn classical feature of the sequences found so far and possibly others. Therefore, this implementation is a suitable instrument either in finding complete descriptions of such 2-orthogonal sequences or establishing the non-existence of solutions, and it is a relevant tool in the pursuit of polynomial sequences characterised as eigenfunctions of given differential operators, defined independently of n.

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