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A Lemma for a Strong Comparison Principle of Nonlinear Parabolic Equations

By

Satoshi KAWAI and Masaki OHNUMA

Satoshi KAWAI
SOLIZE Corporation, Sanban-cho UF Bldg. 3F, 6-3, Sanban-cho, Chiyoda-ku, Tokyo 102-0075, JAPAN,
e-mail address : *satokichi88@gmail.com*

and

Masaki OHNUMA
Department of Mathematical Sciences, Tokushima University, Tokushima 770-8506, JAPAN
e-mail address : *ohnuma@tokushima-u.ac.jp*

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Abstract

In this note, we prove a lemma for a strong comparison principle of nonlinear parabolic equations. We shall prove a function which is a viscosity subsolution minus a viscosity supersolution of the equation becomes a viscosity subsolution of a parabolic equation which may not coincide with the original equation. Thanks to a strong maximum principle of nonlinear parabolic equations we have a strong comparison principle.

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Introduction

We consider nonlinear parabolic equations of the form

$$u_t + F(Du, D^2u) = 0 \quad \text{in} \quad Q_T := (0, T) \times \Omega, \tag{1}$$

where $u : \overline{Q_T} \to \mathbf{R}$ is an unknown function, $F = F(q, X)$ is a given function, $T > 0$ and Ω is a domain in $\mathbf{R}^N$. Here $u_t = \partial u/\partial t$, Du and D^2u denote, respectively, the time derivative of u, the gradient of u and the Hessian of u in space variables.

Our goal is that when u and v are, respectively, a subsolution and supersolution of (1), $u - v$ becomes a subsolution of a parabolic equation which may not coincide with (1). For uniformly parabolic equations we know that a strong maximum principle holds to (1). If $u - v \leq 0$ in Q_T and there exists a point in Q_T that satisfies $u - v = 0$, then a strong maximum principle yields $u \equiv v$ in Q_T. This means a strong comparison principle holds to (1). So our goal is important to prove a strong comparison principle. For nonlinear parabolic equations we may not expect existence of classical solutions. So we deal with this problem using viscosity solutions (cf. [2], [6]).

In the study of a strong comparison principle with viscosity solutions, there are a few papers. Trudinger [9] proved a strong comparison principle for Lipschitz continuous viscosity solutions of uniformly elliptic equations. Ishii and Yoshimura [5] proved a strong comparison principle for semicontinuous viscosity solutions to uniformly elliptic equations. At the same time Giga and the second author [4] studied a strong comparison principle. Their proof [4, Proof of 3.1, p175-177] works for uniformly elliptic equations of the form $F(D^2u) = 0$ but it does not work for non-uniformly elliptic equations of the form $F(Du, D^2u) = 0$. As a special case the second author and Sakaguchi [8] proved a strong comparison principle for semicontinuous viscosity solutions to the prescribed mean curvature equation.

For parabolic problems there is a result by the second author [7]. Since the proof [7, Proof of Lemma 3.4, p159-162] is based on that of [4], it works for uniformly parabolic equations of the form $u_t + F(D^2u) = 0$ but it does not work for (1). To nonlinear parabolic equations Da Lio [3] proved a strong maximum principle for semicontinuous viscosity solutions. Once our goal is proved, thanks to the strong maximum principle we can show that a strong comparison principle holds to (1).

1 Proof of Lemma

We shall study nonlinear parabolic equations of form

(1.1) $$u_t + F(Du, D^2u) = 0 \quad \text{in} \quad Q_T.$$

We list assumptions on $F = F(p, X)$.
(F1) F is lower semicontinuous in $\mathbf{R}^N \times \mathbf{S}^N$.
(F2) F is degenerate elliptic, i.e.,

$$\text{if} \quad X \geq Y \quad \text{then} \quad F(p, X) \leq F(p, Y) \quad \text{for all} \quad p \in \mathbf{R}^N.$$

We introduce F_0 as follows

(1.2) $$F_0(p, X) := \inf\{F(p + q, X + Y) - F(q, Y); (q, Y) \in \mathbf{R}^N \times \mathbf{S}^N\}.$$

This function F_0 is introduced in [5], [6]. To consider our problem we will assume lower boundedness of F_0.

(F3) $F_0(p,X) > -\infty$ for all $p \in \mathbf{R}^N$ and $X \in \mathbf{S}^N$.

We easily see a following property about F_0.

Proposition 1.1. *If F satisfies (F1), then F_0 is lower semicontinuous.*

Proof. We fix $(\hat{p},\hat{X}) \in \mathbf{R}^N \times \mathbf{S}^N$. Since F is lower semicontinuous, we see that for all $\varepsilon > 0$ there exists $\delta > 0$ that satisfies

$$\text{if} \quad (p,X) \in B_\delta(\hat{p},\hat{X}) \quad \text{then} \quad -\varepsilon + F(\hat{p},\hat{X}) < F(p,X).$$

Here

$$B_\delta(\hat{p},\hat{X}) = \{(p,X) \in \mathbf{R}^N \times \mathbf{S}^N; \{|p-\hat{p}|^2 + ||X-\hat{X}||^2\}^{1/2} < \delta\},$$

where $||X|| := \max\{|X\xi|; \xi \in \mathbf{R}^N, |\xi| = 1\}$ for $X \in \mathbf{S}^N$. By the definition of F_0 we have that for $(r,Z) \in \mathbf{R}^N \times \mathbf{S}^N$

$$\begin{aligned} F_0(\hat{p},\hat{X}) - F_0(r,Z) \quad &= \quad \inf\{F(\hat{p}+q,\hat{X}+Y) - F(q,Y); (q,Y) \in \mathbf{R}^N \times \mathbf{S}^N\} \\ &\quad\ -\inf\{F(r+q,Z+Y) - F(q,Y); (q,Y) \in \mathbf{R}^N \times \mathbf{S}^N\}. \end{aligned}$$

Since F_0 is bounded from below, there exists $(\hat{q},\hat{Y}) \in \mathbf{R}^N \times \mathbf{S}^N$ that satisfies

$$F_0(r,Z) = F(r+\hat{q},Z+\hat{Y}) - F(\hat{q},\hat{Y}).$$

Then we observe that

$$\begin{aligned} F_0(\hat{p},\hat{X}) - F_0(r,Z) \quad &\leq \quad F(\hat{p}+\hat{q},\hat{X}+\hat{Y}) - F(\hat{q},\hat{Y}) - F(r+\hat{q},Z+\hat{Y}) + F(\hat{q},\hat{Y}) \\ &\leq \quad F(\hat{p}+\hat{q},\hat{X}+\hat{Y}) - F(r+\hat{q},Z+\hat{Y}). \end{aligned}$$

Now we have that if $(r,Z) \in B_\delta(\hat{p},\hat{X})$ then $F_0(\hat{p},\hat{X}) - F_0(r,Z) < \varepsilon$. □

Now we are in a position to state our main result.

Lemma 1.2. *Assume that (F1), (F2) and (F3) hold. Let $u \in USC([0,\infty) \times \mathbf{R}^N)$ and $v \in LSC([0,\infty) \times \mathbf{R}^N)$ be, respectively, a viscosity subsolution and a vicosity supersolution of (1.1). We set $w = u - v$. Then w is a viscosity subsoluton of*

$$u_t + F_0(Du, D^2u) = 0 \quad in \quad Q_T. \tag{1.3}$$

Proof. Let $\phi \in C^2((0,\infty) \times \mathbf{R}^N)$ and $(\hat{t},\hat{x}) \in Q_T$ satisfy $(w-\phi)(\hat{t},\hat{x}) \geq (w-\phi)(s,y)$ for all $(s,y) \in B_r(\hat{t},\hat{x})$ for some $r > 0$, where $B_r(\hat{t},\hat{x})$ denotes an open ball in $\mathbf{R}^{N+1}$ centered at $(\hat{t},\hat{x})$ with a radius r. We may assume that $w - \phi$ takes its locally strict maximum at $(\hat{t},\hat{x})$. For $\varepsilon > 0$ we set

$$\Phi(t,x,s,y) := u(t,x) - v(s,y) - \phi(t,x) - \frac{1}{2\varepsilon}(|x-y|^2 + |t-s|^2).$$

Since Φ is an upper semicontinuous function, Φ takes its maximum on $\overline{B_r(\hat{t},\hat{x})}\times\overline{B_r(\hat{t},\hat{x})}$ for some $r>0$. Let $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$ be a maximizer of Φ on $\overline{B_r(\hat{t},\hat{x})}\times\overline{B_r(\hat{t},\hat{x})}$.

Step 1. We shall show $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$ converges to $(\hat{t},\hat{x},\hat{t},\hat{x})$ as $\varepsilon\to 0$.

Since Φ takes its maximum on $\overline{B_r(\hat{t},\hat{x})}\times\overline{B_r(\hat{t},\hat{x})}$ at $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$, we see that

$$\Phi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)\geq \Phi(\hat{t},\hat{x},\hat{t},\hat{x})=(w-\phi)(\hat{t},\hat{x}).$$

As usual we may assume that $(w-\phi)(\hat{t},\hat{x})=0$. Since $u,-v\in USC([0,\infty)\times\mathbf{R}^N)$ and $\phi\in C^2((0,\infty)\times\mathbf{R}^N)$, there exists a constant C that satisfies

$$\frac{1}{2\varepsilon}\left(|x_\varepsilon-y_\varepsilon|^2+|t_\varepsilon-s_\varepsilon|^2\right)\leq u(t_\varepsilon,x_\varepsilon)-v(s_\varepsilon,y_\varepsilon)-\phi(t_\varepsilon,x_\varepsilon)\leq C \tag{1.4}$$

Then we see that

$$\lim_{\varepsilon\to 0}(x_\varepsilon-y_\varepsilon)=0,\quad \lim_{\varepsilon\to 0}(t_\varepsilon-s_\varepsilon)=0. \tag{1.5}$$

Note that $(t_\varepsilon,x_\varepsilon)\in\overline{B_r(\hat{t},\hat{x})}$, the Bolzano-Weierstrass theorem yields that there exists a sequence $\{\varepsilon_k\}$ which decreases to 0 as $k\to\infty$ and $(t_0,x_0)\in\overline{B_r(\hat{t},\hat{x})}$ satisfying

$$(t_{\varepsilon_k},x_{\varepsilon_k})\to(t_0,x_0)\quad\text{as}\quad k\to\infty.$$

By (1.5) we observe that

$$\lim_{k\to\infty}(y_{\varepsilon_k}-x_0)=\lim_{k\to\infty}(y_{\varepsilon_k}-x_{\varepsilon_k})+\lim_{k\to\infty}(x_{\varepsilon_k}-x_0)=0.$$

So we have $\lim_{k\to\infty}x_{\varepsilon_k}=\lim_{k\to\infty}y_{\varepsilon_k}=x_0$. By a similar way we have $\lim_{k\to\infty}t_{\varepsilon_k}=\lim_{k\to\infty}s_{\varepsilon_k}=t_0$. Concerning (1.4) we know that $u\in USC([0,\infty)\times\mathbf{R}^N)$ and $v\in LSC([0,\infty)\times\mathbf{R}^N)$. Then we observe that

$$\begin{aligned}
0\leq\liminf_{k\to\infty}\frac{|x_{\varepsilon_k}-y_{\varepsilon_k}|^2+|t_{\varepsilon_k}-s_{\varepsilon_k}|^2}{2\varepsilon_k} &\leq \limsup_{k\to\infty}\frac{|x_{\varepsilon_k}-y_{\varepsilon_k}|^2+|t_{\varepsilon_k}-s_{\varepsilon_k}|^2}{2\varepsilon_k}\\
&\leq u(t_0,x_0)-v(t_0,x_0)-\phi(t_0,x_0)\\
&\leq (w-\phi)(\hat{t},\hat{x})=0.
\end{aligned}$$

These inequalities yield

$$\lim_{k\to\infty}\frac{|x_{\varepsilon_k}-y_{\varepsilon_k}|^2}{\varepsilon_k}=0,\quad \lim_{k\to\infty}\frac{|t_{\varepsilon_k}-s_{\varepsilon_k}|^2}{\varepsilon_k}=0.$$

Recall that $w-\phi$ takes its locally strict maximum at $(\hat{t},\hat{x})$ we see that

$$\lim_{k\to\infty}x_{\varepsilon_k}=\lim_{k\to\infty}y_{\varepsilon_k}=\hat{x},\quad \lim_{k\to\infty}t_{\varepsilon_k}=\lim_{k\to\infty}s_{\varepsilon_k}=\hat{t}.$$

Recall again that $w - \phi$ takes its locally strict maximum at $(\hat{t}, \hat{x})$ we observe that these convergences are independent of taking a subsequence. Finally we have that

$$\lim_{\varepsilon \to 0} x_\varepsilon = \lim_{\varepsilon \to 0} y_\varepsilon = \hat{x}, \quad \lim_{\varepsilon \to 0} t_\varepsilon = \lim_{\varepsilon \to 0} s_\varepsilon = \hat{t},$$
$$\lim_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0, \quad \lim_{\varepsilon \to 0} \frac{|t_\varepsilon - s_\varepsilon|^2}{\varepsilon} = 0.$$

Step 2. We shall show $w = u - v$ is a viscosity subsolution of (1.3).

We set

$$\Psi(t,x,s,y) := \frac{1}{2\varepsilon}(|x-y|^2 + |t-s|^2).$$

Since Φ takes its maximum at $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$, $(u-\phi)(t,x) - v(s,y) - \Psi(t,x,s,y)$ takes its maximum at the same point. By step 1 we may assume that

$$(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \in B_r(\hat{t}, \hat{x}) \times B_r(\hat{t}, \hat{x}).$$

Applying Crandall-Ishii's Lemma [1] we see that for each $\alpha > 1$ there exist $X, Y \in \mathbf{S}^N$ that satisfy

$$(\Psi_t(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon), D_x\Psi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon), X) \in \overline{\mathcal{P}}^{2,+}(u-\phi)(t_\varepsilon, x_\varepsilon),$$
$$(\Psi_s(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon), D_y\Psi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon), Y) \in \overline{\mathcal{P}}^{2,+}(-v)(s_\varepsilon, y_\varepsilon),$$
$$\left(\Leftrightarrow \quad (-\Psi_s(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon), -D_y\Psi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon), -Y) \in \overline{\mathcal{P}}^{2,-} v(s_\varepsilon, y_\varepsilon)\right),$$

$$-(\alpha + ||A||)\, I_{2N} \le \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \le A + \frac{1}{\alpha}A^2. \tag{1.6}$$

Here

$$A = D^2\Psi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = \begin{pmatrix} D^2_{xx}(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) & D^2_{xy}(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \\ D^2_{yx}(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) & D^2_{yy}(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \end{pmatrix},$$

$\overline{\mathcal{P}}^{2,+}$ and $\overline{\mathcal{P}}^{2,-}$, respectively, denote closure of a set of parabolic super 2-jets $\mathcal{P}^{2,+}$ and a set of parabolic sub 2-jets $\mathcal{P}^{2,-}$ (cf. [2],[6]). By calculations we have

$$\Psi_t(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = \frac{t_\varepsilon - s_\varepsilon}{\varepsilon}, \quad \Psi_s(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = -\frac{t_\varepsilon - s_\varepsilon}{\varepsilon},$$
$$D_x\Psi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, \quad D_y\Psi(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) = -\frac{x_\varepsilon - y_\varepsilon}{\varepsilon},$$

$$A = \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{1.7}$$

By the definition of $\overline{\mathcal{P}}^{2,+}$ and $\overline{\mathcal{P}}^{2,-}$ we observe that

$$\left(\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \phi_t(t_\varepsilon, x_\varepsilon), \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + D\phi(t_\varepsilon, x_\varepsilon), X + D^2\phi(t_\varepsilon, x_\varepsilon)\right) \in \overline{\mathcal{P}}^{2,+}u(t_\varepsilon, x_\varepsilon),$$
$$\left(\frac{t_\varepsilon - s_\varepsilon}{\varepsilon}, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -Y\right) \in \overline{\mathcal{P}}^{2,-}v(s_\varepsilon, y_\varepsilon).$$

Hereafter we may suppress a point $(t_\varepsilon, x_\varepsilon)$ of ϕ_t and ϕ. Since u and v are, respectively, a viscosity subsolution and a viscosity supersolution of (1.1), we see that

$$\phi_t + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + F\left(D\phi + \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, D^2\phi + X\right) \leq 0, \tag{1.8}$$

$$\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + F\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -Y\right) \geq 0. \tag{1.9}$$

Subtracting (1.9.) from (1.8), we get

$$\phi_t + F\left(D\phi + \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, D^2\phi + X\right) - F\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -Y\right) \leq 0. \tag{1.10}$$

From (1.6) and (1.7) we have $X + Y \leq O$. By (F2) and the definiton of F_0 (1.2) we observe that

$$\begin{aligned}
&F\left(D\phi + \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, D^2\phi + X\right) - F\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -Y\right) \\
&= F\left(D\phi + \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, D^2\phi + X\right) - F\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X\right) \\
&\qquad + F\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X\right) - F\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -Y\right) \\
&\geq F\left(D\phi + \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, D^2\phi + X\right) - F\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X\right) \\
&\geq \inf\{F(D\phi + q, D^2\phi + Z) - F(q, Z); (q, Z) \in \mathbf{R}^N \times \mathbf{S}^N\} \\
&= F_0(D\phi(t_\varepsilon, x_\varepsilon), D^2\phi(t_\varepsilon, x_\varepsilon))
\end{aligned}$$

Combining (1.10) and the lower semicontinuity of F_0 we see that

$$\begin{aligned}
0 &\geq \phi_t(t_\varepsilon, x_\varepsilon) + F_0(D\phi(t_\varepsilon, x_\varepsilon), D^2\phi(t_\varepsilon, x_\varepsilon)) \\
&\geq \liminf_{\varepsilon \to 0}\{\phi_t(t_\varepsilon, x_\varepsilon) + F_0(D\phi(t_\varepsilon, x_\varepsilon), D^2\phi(t_\varepsilon, x_\varepsilon))\} \\
&\geq \phi_t(\hat{t}, \hat{x}) + F_0(D\phi(\hat{t}, \hat{x}), D^2\phi(\hat{t}, \hat{x}))
\end{aligned}$$

This means w is a viscosity subsolution of $u_t + F_0(Du, D^2u) = 0$. □

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