FORMALIZED MATHEMATICS
Vol. 30, No. 1, Pages 31–52, 2022
ISSN: 1426–2630, e-ISSN: 1898–9934
DOI: 10.2478/forma-2022-0004

sciendo
sciendo.com/journal/forma

Absolutely Integrable Functions

Noboru Endou
National Institute of Technology, Gifu College
2236-2 Kamimakuwa, Motosu, Gifu, Japan

Summary. The goal of this article is to clarify the relationship between Riemann's improper integrals and Lebesgue integrals. In previous articles [6], [7], we treated Riemann's improper integrals [1], [11] and [4] on arbitrary intervals. Therefore, in this article, we will continue to clarify the relationship between improper integrals and Lebesgue integrals [8], using the Mizar [3], [2] formalism.

MSC: 26A42 68V20

Keywords: absolutely integrable; improper integral

MML identifier: MESFUN15, version: 8.1.12 5.71.1431

1. Preliminaries

Let s be a without $-\infty$ sequence of extended reals. One can check that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is without $-\infty$.

Let s be a without $+\infty$ sequence of extended reals. One can verify that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is without $+\infty$.

Now we state the propositions:

(1) Let us consider a without $-\infty$ sequence f_1 of extended reals, and a without $+\infty$ sequence f_2 of extended reals. Then

(i) $(\sum_{\alpha=0}^{\kappa}(f_1 - f_2)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}}$, and

(ii) $(\sum_{\alpha=0}^{\kappa}(f_2 - f_1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}}$.

Proof: Set $P_1 = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}}$. Set $P_2 = (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}}$. Set $P_{12} = (\sum_{\alpha=0}^{\kappa}(f_1-f_2)(\alpha))_{\kappa\in\mathbb{N}}$. Set $P_{21} = (\sum_{\alpha=0}^{\kappa}(f_2-f_1)(\alpha))_{\kappa\in\mathbb{N}}$. Define $\mathcal{C}$[natural number] $\equiv P_{12}(\$_1) = P_1(\$_1) - P_2(\$_1)$. For every natural number k such that $\mathcal{C}[k]$ holds $\mathcal{C}[k+1]$. For every natural number k, $\mathcal{C}[k]$. For every element k of $\mathbb{N}$, $P_{12}(k) = (P_1 - P_2)(k)$. Define $\mathcal{C}$[natural number] $\equiv P_{21}(\$_1) = P_2(\$_1) - P_1(\$_1)$. For every natural number k such that $\mathcal{C}[k]$ holds $\mathcal{C}[k+1]$.

For every natural number k, $\mathcal{C}[k]$. For every element k of $\mathbb{N}$, $P_{21}(k) = (P_2 - P_1)(k)$ by [5, (7)]. □

(2) Let us consider sets X, A, and a partial function f from X to $\mathbb{R}$. If f is non-positive, then $f \restriction A$ is non-positive.

(3) Let us consider a set X, and a partial function f from X to $\mathbb{R}$. If f is non-positive, then $-f$ is non-negative.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a real number x. Now we state the propositions:

(4) If f is left convergent in a and non-decreasing, then if $x \in \operatorname{dom} f$ and $x < a$, then $f(x) \leqslant \lim_{a^-} f$.

(5) If f is left convergent in a and non-increasing, then if $x \in \operatorname{dom} f$ and $x < a$, then $f(x) \geqslant \lim_{a^-} f$.

(6) If f is right convergent in a and non-decreasing, then if $x \in \operatorname{dom} f$ and $a < x$, then $f(x) \geqslant \lim_{a^+} f$.

(7) If f is right convergent in a and non-increasing, then if $x \in \operatorname{dom} f$ and $a < x$, then $f(x) \leqslant \lim_{a^+} f$.

(8) If f is convergent in $-\infty$ and non-increasing, then if $x \in \operatorname{dom} f$, then $f(x) \leqslant \lim_{-\infty} f$.

(9) If f is convergent in $+\infty$ and non-decreasing, then if $x \in \operatorname{dom} f$, then $f(x) \leqslant \lim_{+\infty} f$.

Let us consider real numbers a, b and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(10) Suppose $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is bounded and non-negative. Then $\int\limits_{a}^{b} f(x)dx \geqslant 0$.

(11) Suppose $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is bounded and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is non-positive. Then $\int\limits_{a}^{b} f(x)dx \leqslant 0$. The theorem is a consequence of (3) and (10).

Let us consider real numbers a, b, c, d and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(12) Suppose $c \leqslant d$ and $[c, d] \subseteq [a, b] \subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is bounded and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is non-negative. Then $\int\limits_{c}^{d} f(x)dx \leqslant$

$\int_a^b f(x)dx$. The theorem is a consequence of (10).

(13) Suppose $c \leqslant d$ and $[c,d] \subseteq [a,b] \subseteq \operatorname{dom} f$ and $f{\restriction}[a,b]$ is bounded and f is integrable on $[a,b]$ and $f{\restriction}[a,b]$ is non-positive. Then $\int_c^d f(x)dx \geqslant \int_a^b f(x)dx$. The theorem is a consequence of (2) and (11).

2. Fundamental Properties of Measure and Integral

Now we state the propositions:

(14) Let us consider a non empty set X, a partial function f from X to $\mathbb{R}$, and a set E. Then $\overline{\mathbb{R}}(f){\restriction}E = \overline{\mathbb{R}}(f{\restriction}E)$.

(15) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, an element A of S, and a sequence E of subsets of S. Suppose f is A-measurable and $A = \operatorname{dom} f$ and E is disjoint valued and $A = \bigcup E$ and ($\int^+ \max_+(f)\,\mathrm{d}M < +\infty$ or $\int^+ \max_-(f)\,\mathrm{d}M < +\infty$). Then there exists a sequence I of extended reals such that

(i) for every natural number n, $I(n) = \int f{\restriction}E(n)\,\mathrm{d}M$, and

(ii) I is summable, and

(iii) $\int f\,\mathrm{d}M = \sum I$.

Proof: Consider I_1 being a non-negative sequence of extended reals such that for every natural number n, $I_1(n) = \int \max_+(f){\restriction}E(n)\,\mathrm{d}M$ and I_1 is summable and $\int \max_+(f)\,\mathrm{d}M = \sum I_1$. Consider I_2 being a non-negative sequence of extended reals such that for every natural number n, $I_2(n) = \int \max_-(f){\restriction}E(n)\,\mathrm{d}M$ and I_2 is summable and $\int \max_-(f)\,\mathrm{d}M = \sum I_2$. For every natural number n, $E(n)$ is an element of S and $E(n) \subseteq \operatorname{dom} f$. For every natural number n, $I_1(n) = \int^+ \max_+(f){\restriction}E(n)\,\mathrm{d}M$. For every natural number n, $I_2(n) = \int^+ \max_-(f){\restriction}E(n)\,\mathrm{d}M$. □

(16) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and elements A, B of S. Suppose $A \cup B \subseteq \operatorname{dom} f$ and f is $(A \cup B)$-measurable and A misses B and ($\int^+ \max_+(f{\restriction}(A \cup B))\,\mathrm{d}M < +\infty$ or $\int^+ \max_-(f{\restriction}(A \cup B))\,\mathrm{d}M < +\infty$). Then $\int f{\restriction}(A \cup B)\,\mathrm{d}M = \int f{\restriction}A\,\mathrm{d}M + \int f{\restriction}B\,\mathrm{d}M$.

(17) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, an element A of S, and

a sequence E of subsets of S. Suppose f is A-measurable and $A = \operatorname{dom} f$ and E is non descending and $\lim E \subseteq A$ and $M(A \setminus (\lim E)) = 0$ and ($\int^+ \max_+(f)\,\mathrm{d}M < +\infty$ or $\int^+ \max_-(f)\,\mathrm{d}M < +\infty$). Then there exists a sequence I of extended reals such that

(i) for every natural number n, $I(n) = \int f{\restriction}(\text{the partial unions of } E)(n)\,\mathrm{d}M$, and

(ii) I is convergent, and

(iii) $\int f\,\mathrm{d}M = \lim I$.

Proof: Reconsider $L_2 = \lim E$ as an element of S. Reconsider $F =$ the partial diff-unions of E as a sequence of subsets of S. Set $g = f{\restriction}L_2$. Consider J being a sequence of extended reals such that for every natural number n, $J(n) = \int g{\restriction}F(n)\,\mathrm{d}M$ and J is summable and $\int g\,\mathrm{d}M = \sum J$. Reconsider $I = (\sum_{\alpha=0}^{\kappa} J(\alpha))_{\kappa \in \mathbb{N}}$ as a sequence of extended reals.

For every natural number n, $g{\restriction}(\text{the partial unions of } F)(n) = f{\restriction}(\text{the partial unions of } E)(n)$. For every natural number n, (the partial unions of $E)(n) \subseteq \bigcup E$. Define $\mathcal{P}[\text{natural number}] \equiv I(\$_1) = \int g{\restriction}(\text{the partial unions of } F)(\$_1)\,\mathrm{d}M$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. For every natural number n, $I(n) = \int f{\restriction}(\text{the partial unions of } E)(n)\,\mathrm{d}M$. □

(18) Let us consider non empty sets X, Y, a set A, a sequence F of X, and a sequence G of Y. Suppose for every element n of $\mathbb{N}$, $G(n) = A \cap F(n)$. Then $\bigcup \operatorname{rng} G = A \cap \bigcup \operatorname{rng} F$.

(19) Let us consider a non empty set X, a σ-field S of subsets of X, a sequence E of S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose for every natural number n, f is $(E(n))$-measurable. Then f is $(\bigcup E)$-measurable.
Proof: For every real number r, $\bigcup E \cap \text{LE-dom}(f, r) \in S$. □

(20) Let us consider real numbers a, b, and a natural number n. If $a < b$, then $a \leqslant b - \frac{b-a}{n+1} < b$ and $a < a + \frac{b-a}{n+1} \leqslant b$.

Let us consider real numbers a, b. Now we state the propositions:

(21) Suppose $a < b$. Then there exists a sequence E of subsets of L-Field such that

(i) for every natural number n, $E(n) = [a, b - \frac{b-a}{n+1}]$ and $E(n) \subseteq [a, b[$ and $E(n)$ is a non empty, closed interval subset of $\mathbb{R}$, and

(ii) E is non descending and convergent, and

(iii) $\bigcup E = [a, b[$.

Proof: Define $\mathcal{F}(\text{element of } \mathbb{N}) = [a, b - \frac{b-a}{\$_1+1}]$. Consider E being a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $E(n) = \mathcal{F}(n)$. For every natural number n, $E(n) = [a, b - \frac{b-a}{n+1}]$. For every natural number n, $E(n) = [a, b - \frac{b-a}{n+1}]$ and $E(n) \subseteq [a, b[$ and $E(n)$ is a non empty, closed interval subset of $\mathbb{R}$. $\square$

(22) Suppose $a < b$. Then there exists a sequence E of subsets of L-Field such that

(i) for every natural number n, $E(n) = [a + \frac{b-a}{n+1}, b]$ and $E(n) \subseteq]a, b]$ and $E(n)$ is a non empty, closed interval subset of $\mathbb{R}$, and

(ii) E is non descending and convergent, and

(iii) $\bigcup E =]a, b]$.

PROOF: Define $\mathcal{F}(\text{element of } \mathbb{N}) = [a + \frac{b-a}{\$_1+1}, b]$. Consider E being a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $E(n) = \mathcal{F}(n)$. For every natural number n, $E(n) = [a + \frac{b-a}{n+1}, b]$ and $E(n) \subseteq]a, b]$ and $E(n)$ is a non empty, closed interval subset of $\mathbb{R}$. $\square$

Let us consider a real number a. Now we state the propositions:

(23) There exists a sequence E of subsets of L-Field such that

(i) for every natural number n, $E(n) = [a, a + n]$, and

(ii) E is non descending and convergent, and

(iii) $\bigcup E = [a, +\infty[$.

PROOF: Define $\mathcal{F}(\text{element of } \mathbb{N}) = [a, a + \$_1]$. Consider E being a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $E(n) = \mathcal{F}(n)$. For every natural number n, $E(n) = [a, a + n]$. $\square$

(24) There exists a sequence E of subsets of L-Field such that

(i) for every natural number n, $E(n) = [a - n, a]$, and

(ii) E is non descending and convergent, and

(iii) $\bigcup E =]-\infty, a]$.

PROOF: Define $\mathcal{F}(\text{element of } \mathbb{N}) = [a - \$_1, a]$. Consider E being a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $E(n) = \mathcal{F}(n)$. For every natural number n, $E(n) = [a - n, a]$. $\square$

(25) Let us consider a set X, a σ-field S of subsets of X, a σ-measure M on S, and a set A with measure zero w.r.t. M. Then $A \in \mathrm{COM}(S, M)$.

(26) Let us consider a real number r. Then $\{r\} \in$ L-Field. The theorem is a consequence of (25).

(27) Let us consider a non empty set X, a σ-field S of subsets of X, an element E of S, and a partial function f from X to $\overline{\mathbb{R}}$. If $E = \emptyset$, then f is E-measurable.

(28) Let us consider a non empty set X, a σ-field S of subsets of X, an element E of S, and a partial function f from X to $\mathbb{R}$. If $E = \emptyset$, then f is E-measurable. The theorem is a consequence of (27).

(29) Let us consider a real number r, an element E of L-Field, and a partial function f from $\mathbb{R}$ to $\overline{\mathbb{R}}$. If $E = \{r\}$, then f is E-measurable.
PROOF: For every real number a, $E \cap \text{LE-dom}(f, a) \in$ L-Field. $\square$

(30) Let us consider a real number r, an element E of L-Field, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. If $E = \{r\}$, then f is E-measurable. The theorem is a consequence of (29).

Let us consider real numbers a, b, a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and an element E of L-Field. Now we state the propositions:

(31) Suppose $[a, b[\subseteq \text{dom} f$ and f is right improper integrable on a and b. Then if $E \subseteq [a, b[$, then f is E-measurable. The theorem is a consequence of (21), (19), and (28).

(32) Suppose $]a, b] \subseteq \text{dom} f$ and f is left improper integrable on a and b. Then if $E \subseteq]a, b]$, then f is E-measurable. The theorem is a consequence of (22), (20), (19), and (28).

(33) Suppose $]a, b[\subseteq \text{dom} f$ and f is improper integrable on a and b. Then if $E \subseteq]a, b[$, then f is E-measurable. The theorem is a consequence of (32) and (31).

Let us consider a real number a, a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and an element E of L-Field. Now we state the propositions:

(34) Suppose $[a, +\infty[\subseteq \text{dom} f$ and f is improper integrable on $[a, +\infty[$. Then if $E \subseteq [a, +\infty[$, then f is E-measurable.
PROOF: Set $A = [a, +\infty[$. Consider K being a sequence of subsets of L-Field such that for every natural number n, $K(n) = [a, a + n]$ and K is non descending and convergent and $\bigcup K = [a, +\infty[$. Reconsider $K_1 = K$ as a sequence of L-Field. For every natural number n, $\overline{\mathbb{R}}(f)$ is $(K_1(n))$-measurable by [8, (49)]. $\overline{\mathbb{R}}(f)$ is A-measurable. $\square$

(35) Suppose $]-\infty, a] \subseteq \text{dom} f$ and f is improper integrable on $]-\infty, a]$. Then if $E \subseteq]-\infty, a]$, then f is E-measurable.
PROOF: Consider K being a sequence of subsets of L-Field such that for every natural number n, $K(n) = [a - n, a]$ and K is non descending and convergent and $\bigcup K =]-\infty, a]$. For every element n of $\mathbb{N}$, $K(n)$ is a non empty, closed interval subset of $\mathbb{R}$. Reconsider $K_1 = K$ as a sequence of L-Field. For every natural number n, $\overline{\mathbb{R}}(f)$ is $(K_1(n))$-measurable by [8, (49)]. $\overline{\mathbb{R}}(f)$ is $(\bigcup K_1)$-measurable. $\square$

(36) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\text{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$. Let us consider an element E of L-Field.

Then f is E-measurable. The theorem is a consequence of (34) and (35).

3. Relation between Improper Integral and Lebesgue Integral

Now we state the propositions:

(37) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\mathbb{R}$, and an element A of S. Suppose $A = \operatorname{dom} f$ and f is A-measurable. Then $\int -f \,\mathrm{d}M = -\int f \,\mathrm{d}M$.

(38) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\mathbb{R}$, and elements A, B, E of S. Suppose $E = \operatorname{dom} f$ and f is E-measurable and non-positive and $A \subseteq B$. Then $\int f\restriction A \,\mathrm{d}M \geqslant \int f\restriction B \,\mathrm{d}M$.

Proof: For every set x such that $x \in \operatorname{dom}(\overline{\mathbb{R}}(f))$ holds $(\overline{\mathbb{R}}(f))(x) \leqslant 0$. $\int \overline{\mathbb{R}}(f\restriction A) \,\mathrm{d}M \geqslant \int \overline{\mathbb{R}}(f)\restriction B \,\mathrm{d}M$. $\int \overline{\mathbb{R}}(f\restriction A) \,\mathrm{d}M \geqslant \int \overline{\mathbb{R}}(f\restriction B) \,\mathrm{d}M$. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and a non empty subset A of $\mathbb{R}$. Now we state the propositions:

(39) Suppose $[a, b[\subseteq \operatorname{dom} f$ and $A = [a, b[$ and f is right improper integrable on a and b and $f\restriction A$ is non-negative. Then

- (i) right-improper-integral$(f, a, b) = \int f\restriction A \,\mathrm{d}\,$L-Meas, and
- (ii) if f is right extended Riemann integrable on a, b, then $f\restriction A$ is integrable on L-Meas, and
- (iii) if f is not right extended Riemann integrable on a, b, then $\int f\restriction A \,\mathrm{d}\,$L-Meas $= +\infty$.

The theorem is a consequence of (12), (21), (31), (14), (17), (20), and (4).

(40) Suppose $[a, b[\subseteq \operatorname{dom} f$ and $A = [a, b[$ and f is right improper integrable on a and b and $f\restriction A$ is non-positive. Then

- (i) right-improper-integral$(f, a, b) = \int f\restriction A \,\mathrm{d}\,$L-Meas, and
- (ii) if f is right extended Riemann integrable on a, b, then $f\restriction A$ is integrable on L-Meas, and
- (iii) if f is not right extended Riemann integrable on a, b, then $\int f\restriction A \,\mathrm{d}\,$L-Meas $= -\infty$.

The theorem is a consequence of (3), (39), and (31).

(41) Suppose $]a, b] \subseteq \operatorname{dom} f$ and $A =]a, b]$ and f is left improper integrable on a and b and $f\restriction A$ is non-negative. Then

- (i) left-improper-integral$(f, a, b) = \int f\restriction A \,\mathrm{d}\,$L-Meas, and
- (ii) if f is left extended Riemann integrable on a, b, then $f\restriction A$ is integrable on L-Meas, and

(iii) if f is not left extended Riemann integrable on a, b, then $\int f\restriction A \, d$ L-Meas $= +\infty$.

The theorem is a consequence of (12), (22), (32), (14), (17), (20), and (7).

(42) Suppose $]a, b] \subseteq \operatorname{dom} f$ and $A =]a, b]$ and f is left improper integrable on a and b and $f\restriction A$ is non-positive. Then

(i) left-improper-integral$(f, a, b) = \int f\restriction A \, d$ L-Meas, and

(ii) if f is left extended Riemann integrable on a, b, then $f\restriction A$ is integrable on L-Meas, and

(iii) if f is not left extended Riemann integrable on a, b, then $\int f\restriction A \, d$ L-Meas $= -\infty$.

The theorem is a consequence of (3), (41), and (32).

(43) Suppose $]a, b[\subseteq \operatorname{dom} f$ and $A =]a, b[$ and f is improper integrable on a and b and $f\restriction A$ is non-negative. Then

(i) improper-integral$(f, a, b) = \int f\restriction A \, d$ L-Meas, and

(ii) if there exists a real number c such that $a < c < b$ and f is left extended Riemann integrable on a, c and right extended Riemann integrable on c, b, then $f\restriction A$ is integrable on L-Meas, and

(iii) if for every real number c such that $a < c < b$ holds f is not left extended Riemann integrable on a, c or f is not right extended Riemann integrable on c, b, then $\int f\restriction A \, d$ L-Meas $= +\infty$.

The theorem is a consequence of (31), (32), (41), (39), (26), and (33).

(44) Suppose $]a, b[\subseteq \operatorname{dom} f$ and $A =]a, b[$ and f is improper integrable on a and b and $f\restriction A$ is non-positive. Then

(i) improper-integral$(f, a, b) = \int f\restriction A \, d$ L-Meas, and

(ii) if there exists a real number c such that $a < c < b$ and f is left extended Riemann integrable on a, c and right extended Riemann integrable on c, b, then $f\restriction A$ is integrable on L-Meas, and

(iii) if for every real number c such that $a < c < b$ holds f is not left extended Riemann integrable on a, c or f is not right extended Riemann integrable on c, b, then $\int f\restriction A \, d$ L-Meas $= -\infty$.

The theorem is a consequence of (3), (43), (33), and (37).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number b, and a non empty subset A of $\mathbb{R}$. Now we state the propositions:

(45) Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and $A =]-\infty, b]$ and f is improper integrable on $]-\infty, b]$ and f is non-negative. Then

(i) $\int\limits_{-\infty}^{b} f(x)dx = \int f{\restriction}A \, \mathrm{d}\,\text{L-Meas}$, and

(ii) if f is extended Riemann integrable on $-\infty$, b, then $f{\restriction}A$ is integrable on L-Meas, and

(iii) if f is not extended Riemann integrable on $-\infty$, b, then $\int f{\restriction}A \, \mathrm{d}\,\text{L-Meas} = +\infty$.

The theorem is a consequence of (12), (24), (35), (14), (17), and (8).

(46) Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and $A =]-\infty, b]$ and f is improper integrable on $]-\infty, b]$ and f is non-positive. Then

(i) $\int\limits_{-\infty}^{b} f(x)dx = \int f{\restriction}A \, \mathrm{d}\,\text{L-Meas}$, and

(ii) if f is extended Riemann integrable on $-\infty$, b, then $f{\restriction}A$ is integrable on L-Meas, and

(iii) if f is not extended Riemann integrable on $-\infty$, b, then $\int f{\restriction}A \, \mathrm{d}\,\text{L-Meas} = -\infty$.

Proof: Reconsider $A_1 = A$ as an element of L-Field. For every object x such that $x \in \operatorname{dom}(-f)$ holds $0 \leqslant (-f)(x)$. $\int\limits_{-\infty}^{b}(-f)(x)dx = \int(-f){\restriction}A \, \mathrm{d}\,\text{L-Meas}$. $f{\restriction}A$ is A_1-measurable. $\int -f{\restriction}A \, \mathrm{d}\,\text{L-Meas} = -\int f{\restriction}A \, \mathrm{d}\,\text{L-Meas}$. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a non empty subset A of $\mathbb{R}$. Now we state the propositions:

(47) Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and $A = [a, +\infty[$ and f is improper integrable on $[a, +\infty[$ and f is non-negative. Then

(i) $\int\limits_{a}^{+\infty} f(x)dx = \int f{\restriction}A \, \mathrm{d}\,\text{L-Meas}$, and

(ii) if f is extended Riemann integrable on a, $+\infty$, then $f{\restriction}A$ is integrable on L-Meas, and

(iii) if f is not extended Riemann integrable on a, $+\infty$, then $\int f{\restriction}A \, \mathrm{d}\,\text{L-Meas} = +\infty$.

The theorem is a consequence of (12), (23), (34), (14), (17), and (9).

(48) Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and $A = [a, +\infty[$ and f is improper integrable on $[a, +\infty[$ and f is non-positive. Then

(i) $\int\limits_{a}^{+\infty} f(x)dx = \int f \restriction A \, \mathrm{d}\,\text{L-Meas}$, and

(ii) if f is extended Riemann integrable on a, $+\infty$, then $f \restriction A$ is integrable on L-Meas, and

(iii) if f is not extended Riemann integrable on a, $+\infty$, then $\int f \restriction A \, \mathrm{d}\,\text{L-Meas} = -\infty$.

PROOF: Reconsider $A_1 = A$ as an element of L-Field. For every object x such that $x \in \operatorname{dom}(-f)$ holds $0 \leqslant (-f)(x)$. $\int\limits_{a}^{+\infty} (-f)(x)dx = \int (-f) \restriction A \, \mathrm{d}\,\text{L-Meas}$. $f \restriction A$ is A_1-measurable. $\int -f \restriction A \, \mathrm{d}\,\text{L-Meas} = -\int f \restriction A \, \mathrm{d}\,\text{L-Meas}$. □

(49) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and elements A, B of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is E-measurable and f is non-negative. Then $\int^{+} f \restriction (A \cup B) \, \mathrm{d}M \leqslant \int^{+} f \restriction A \, \mathrm{d}M + \int^{+} f \restriction B \, \mathrm{d}M$.

(50) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and sets A, B. Suppose $A \subseteq \operatorname{dom} f$ and $B \subseteq \operatorname{dom} f$ and $f \restriction A$ is integrable on M and $f \restriction B$ is integrable on M. Then $f \restriction (A \cup B)$ is integrable on M. The theorem is a consequence of (49).

(51) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\mathbb{R}$, and sets A, B. Suppose $A \subseteq \operatorname{dom} f$ and $B \subseteq \operatorname{dom} f$ and $f \restriction A$ is integrable on M and $f \restriction B$ is integrable on M. Then $f \restriction (A \cup B)$ is integrable on M. The theorem is a consequence of (14) and (50).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a non empty subset A of $\mathbb{R}$. Now we state the propositions:

(52) Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$ and f is non-negative. Then

(i) $\int\limits_{-\infty}^{+\infty} f(x)dx = \int f \, \mathrm{d}\,\text{L-Meas}$, and

(ii) if f is ∞-extended Riemann integrable, then f is integrable on L-Meas, and

(iii) if f is not ∞-extended Riemann integrable, then $\int f \, \mathrm{d}\,\text{L-Meas} = +\infty$.

The theorem is a consequence of (45), (36), (26), (47), and (51).

(53) Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$ and f is non-positive. Then

(i) $\int\limits_{-\infty}^{+\infty} f(x)dx = \int f \, \mathrm{d}\, \text{L-Meas}$, and

(ii) if f is ∞-extended Riemann integrable, then f is integrable on L-Meas, and

(iii) if f is not ∞-extended Riemann integrable, then $\int f \, \mathrm{d}\, \text{L-Meas} = -\infty$.

Proof: For every object x such that $x \in \operatorname{dom}(-f)$ holds $0 \leqslant (-f)(x)$. Reconsider $E = \mathbb{R}$ as an element of L-Field. f is E-measurable. $-\int\limits_{-\infty}^{+\infty} f(x)dx = \int -f \, \mathrm{d}\, \text{L-Meas}$. $-\int\limits_{-\infty}^{+\infty} f(x)dx = -\int f \, \mathrm{d}\, \text{L-Meas}$. □

4. Absolutely Integrable Function

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(54) Suppose $[a, b[\; = \operatorname{dom} f$. Then there exists a sequence F of partial functions from $\mathbb{R}$ into $\mathbb{R}$ such that

(i) for every natural number n, $\operatorname{dom}(F(n)) = \operatorname{dom} f$ and for every real number x such that $x \in [a, b - \frac{1}{n+1}]$ holds $F(n)(x) = f(x)$ and for every real number x such that $x \notin [a, b - \frac{1}{n+1}]$ holds $F(n)(x) = 0$, and

(ii) $\lim \overline{\mathbb{R}}(F) = f$.

Proof: For every element n of $\mathbb{N}$, $[a, b - \frac{1}{n+1}] \subseteq \operatorname{dom} f$. Define $\mathcal{P}[$element of $\mathbb{N}$, object$] \equiv \$_2 = \chi_{[a, b - \frac{1}{\$_1+1}], \operatorname{dom} f}$. For every element n of $\mathbb{N}$, there exists an element $\langle$ of $\mathbb{R} \dot{\to} \mathbb{R}$ such that $P[n, \langle]$. Consider C_2 being a sequence of $\mathbb{R} \dot{\to} \mathbb{R}$ such that for every element n of $\mathbb{N}$, $P[n, C_2(n)]$. Define $\mathcal{Q}[$element of $\mathbb{N}$, object$] \equiv \$_2 = f \cdot C_2(\$_1)$. For every element n of $\mathbb{N}$, there exists an element F of $\mathbb{R} \dot{\to} \mathbb{R}$ such that $Q[n, F]$. Consider F being a sequence of $\mathbb{R} \dot{\to} \mathbb{R}$ such that for every element n of $\mathbb{N}$, $Q[n, F(n)]$. For every natural number n, $\operatorname{dom}(F(n)) = \operatorname{dom} f$ and for every real number x such that $x \in [a, b - \frac{1}{n+1}]$ holds $F(n)(x) = f(x)$ and for every real number x such that $x \notin [a, b - \frac{1}{n+1}]$ holds $F(n)(x) = 0$. For every element x of $\mathbb{R}$ such that $x \in \operatorname{dom}(\lim \overline{\mathbb{R}}(F))$ holds $(\lim \overline{\mathbb{R}}(F))(x) = (\overline{\mathbb{R}}(f))(x)$ by [9, (16)]. □

(55) Suppose $a < b$ and $[a, b[\subseteq \operatorname{dom} f$ and f is right improper integrable on a and b and $|f|$ is right extended Riemann integrable on a, b. Then

(i) f is right extended Riemann integrable on a, b, and

(ii) right-improper-integral$(f, a, b) \leqslant$ right-improper-integral$(|f|, a, b) < +\infty$.

PROOF: Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_a^x f(x)dx$ and I is left convergent in b or left divergent to $+\infty$ in b or left divergent to $-\infty$ in b. Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I = [a, b[$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int\limits_a^x |f|(x)dx$ and A_I is left convergent in b. For every real numbers r_1, r_2 such that $r_1, r_2 \in \operatorname{dom} A_I$ and $r_1 < r_2$ holds $A_I(r_1) \leqslant A_I(r_2)$. Consider r being a real number such that $0 < r < b - a$. For every real number g such that $g \in \operatorname{dom} I \cap]b - r, b[$ holds $I(g) \leqslant A_I(g)$ by [10, (8)]. □

(56) Suppose $a < b$ and $]a, b] \subseteq \operatorname{dom} f$ and f is left improper integrable on a and b and $|f|$ is left extended Riemann integrable on a, b. Then

(i) f is left extended Riemann integrable on a, b, and

(ii) left-improper-integral$(f, a, b) \leqslant$ left-improper-integral$(|f|, a, b) < +\infty$.

PROOF: Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_x^b f(x)dx$ and I is right convergent in a or right divergent to $+\infty$ in a or right divergent to $-\infty$ in a. Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I =]a, b]$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int\limits_x^b |f|(x)dx$ and A_I is right convergent in a. For every real numbers r_1, r_2 such that $r_1, r_2 \in \operatorname{dom} A_I$ and $r_1 < r_2$ holds $A_I(r_1) \geqslant A_I(r_2)$. Consider r being a real number such that $0 < r < b - a$. For every real number g such that $g \in \operatorname{dom} I \cap]a, a+r[$ holds $I(g) \leqslant A_I(g)$. □

(57) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a non empty, closed interval subset A of $\mathbb{R}$. Suppose $A \subseteq \operatorname{dom} f$. Then

(i) $\max_+(f \restriction A) = \max_+(f \restriction A)$, and

(ii) $\max_{-}(f \restriction A) = \max_{-}(f{\restriction}A)$.

(58) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and f is improper integrable on $]-\infty, b]$ and $|f|$ is extended Riemann integrable on $-\infty$, b. Then

(i) f is extended Riemann integrable on $-\infty$, b, and

(ii) $\int_{-\infty}^{b} f(x)dx \leqslant \int_{-\infty}^{b} |f|(x)dx < +\infty$.

PROOF: Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int_{x}^{b} f(x)dx$ and I is convergent in $-\infty$ or divergent in $-\infty$ to $+\infty$ or divergent in $-\infty$ to $-\infty$. Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int_{x}^{b} |f|(x)dx$ and A_I is convergent in $-\infty$. For every real numbers r_1, r_2 such that r_1, $r_2 \in \operatorname{dom} A_I$ and $r_1 < r_2$ holds $A_I(r_1) \geqslant A_I(r_2)$. For every real number g such that $g \in \operatorname{dom} I \cap]-\infty, 1[$ holds $I(g) \leqslant A_I(g)$. □

(59) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is improper integrable on $[a, +\infty[$ and $|f|$ is extended Riemann integrable on a, $+\infty$. Then

(i) f is extended Riemann integrable on a, $+\infty$, and

(ii) $\int_{a}^{+\infty} f(x)dx \leqslant \int_{a}^{+\infty} |f|(x)dx < +\infty$.

PROOF: Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int_{a}^{x} f(x)dx$ and I is convergent in $+\infty$ or divergent in $+\infty$ to $+\infty$ or divergent in $+\infty$ to $-\infty$. Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int_{a}^{x} |f|(x)dx$ and A_I is convergent in $+\infty$. For every real numbers r_1, r_2 such that r_1, $r_2 \in \operatorname{dom} A_I$ and $r_1 < r_2$ holds $A_I(r_1) \leqslant A_I(r_2)$. For every real number g such that $g \in \operatorname{dom} I \cap]1, +\infty[$ holds $I(g) \leqslant A_I(g)$. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(60) Suppose $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded. Then

(i) $\max_+(f)$ is integrable on $[a, b]$, and

(ii) $\max_-(f)$ is integrable on $[a, b]$, and

(iii) $2 \cdot (\int_a^b \max_+(f)(x)dx) = \int_a^b f(x)dx + \int_a^b |f|(x)dx$, and

(iv) $2 \cdot (\int_a^b \max_-(f)(x)dx) = -\int_a^b f(x)dx + \int_a^b |f|(x)dx$, and

(v) $\int_a^b f(x)dx = \int_a^b \max_+(f)(x)dx - \int_a^b \max_-(f)(x)dx$.

(61) Suppose $a < b$ and $]a, b] \subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b and $|f|$ is left extended Riemann integrable on a, b. Then $\max_+(f)$ is left extended Riemann integrable on a, b.

PROOF: Set $G = (R^<)\int_a^b f(x)dx$. Set $A_G = (R^<)\int_a^b |f|(x)dx$. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int_x^b f(x)dx$ and I is right convergent in a and $G = \lim_{a^+} I$.

Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I =]a, b]$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int_x^b |f|(x)dx$ and A_I is right convergent in a and $A_G = \lim_{a^+} A_I$. For every real number d such that $a < d \leqslant b$ holds $\max_+(f)$ is integrable on $[d, b]$ and $\max_+(f) \restriction [d, b]$ is bounded. There exists a partial function I_3 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_3 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_3$ holds $I_3(x) = \int_x^b \max_+(f)(x)dx$ and I_3 is right convergent in a. $\square$

(62) Suppose $a < b$ and $[a, b[\subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b and $|f|$ is right extended Riemann integrable on a, b. Then $\max_+(f)$ is right extended Riemann integrable on a, b.

Proof: Set $G = (R^{>})\int\limits_a^b f(x)dx$. Set $A_G = (R^{>})\int\limits_a^b |f|(x)dx$. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_a^x f(x)dx$ and I is left convergent in b and $G = \lim_{b^-} I$.

Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I = [a, b[$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int\limits_a^x |f|(x)dx$ and A_I is left convergent in b and $A_G = \lim_{b^-} A_I$. For every real number d such that $a \leqslant d < b$ holds $\max_+(f)$ is integrable on $[a, d]$ and $\max_+(f)\restriction[a, d]$ is bounded. There exists a partial function I_3 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_3 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_3$ holds $I_3(x) = \int\limits_a^x \max\limits_+(f)(x)dx$ and I_3 is left convergent in b. □

(63) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and f is extended Riemann integrable on $-\infty$, b and $|f|$ is extended Riemann integrable on $-\infty$, b. Then $\max_+(f)$ is extended Riemann integrable on $-\infty$, b.

Proof: Set $G = (R^{<})\int\limits_{-\infty}^b f(x)dx$. Set $A_G = (R^{<})\int\limits_{-\infty}^b |f|(x)dx$. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_x^b f(x)dx$ and I is convergent in $-\infty$ and $G = \lim_{-\infty} I$.

Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int\limits_x^b |f|(x)dx$ and A_I is convergent in $-\infty$ and $A_G = \lim_{-\infty} A_I$. For every real number d such that $d \leqslant b$ holds $\max_+(f)$ is integrable on $[d, b]$ and $\max_+(f)\restriction[d, b]$ is bounded. There exists a partial function I_3 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_3 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_3$ holds $I_3(x) = \int\limits_x^b \max\limits_+(f)(x)dx$ and I_3 is convergent in $-\infty$. □

(64) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number

a. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is extended Riemann integrable on a, $+\infty$ and $|f|$ is extended Riemann integrable on a, $+\infty$. Then $\max_+(f)$ is extended Riemann integrable on a, $+\infty$.

PROOF: Set $G = (R^>) \int\limits_a^{+\infty} f(x)dx$. Set $A_G = (R^>) \int\limits_a^{+\infty} |f|(x)dx$. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_a^x f(x)dx$ and I is convergent in $+\infty$ and $G = \lim_{+\infty} I$.

Consider A_I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} A_I = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} A_I$ holds $A_I(x) = \int\limits_a^x |f|(x)dx$ and A_I is convergent in $+\infty$ and $A_G = \lim_{+\infty} A_I$. For every real number d such that $a \leqslant d$ holds $\max_+(f)$ is integrable on $[a, d]$ and $\max_+(f) \restriction [a, d]$ is bounded. There exists a partial function I_3 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_3 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_3$ holds $I_3(x) = \int\limits_a^x \max\limits_+(f)(x)dx$ and I_3 is convergent in $+\infty$. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(65) Suppose $a < b$ and $]a, b] \subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b and $|f|$ is left extended Riemann integrable on a, b. Then $\max_-(f)$ is left extended Riemann integrable on a, b. The theorem is a consequence of (61).

(66) Suppose $a < b$ and $[a, b[\subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b and $|f|$ is right extended Riemann integrable on a, b. Then $\max_-(f)$ is right extended Riemann integrable on a, b. The theorem is a consequence of (62).

(67) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and f is extended Riemann integrable on $-\infty$, b and $|f|$ is extended Riemann integrable on $-\infty$, b. Then $\max_-(f)$ is extended Riemann integrable on $-\infty$, b. The theorem is a consequence of (63).

(68) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is extended Riemann integrable on a, $+\infty$ and $|f|$ is extended Riemann integrable on a, $+\infty$. Then $\max_-(f)$ is extended Riemann integrable on a, $+\infty$. The theorem is a consequence of

(64).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(69) Suppose $]a, b] \subseteq \operatorname{dom} f$ and $\max_+(f)$ is left extended Riemann integrable on a, b and $\max_-(f)$ is left extended Riemann integrable on a, b. Then

(i) f is left extended Riemann integrable on a, b, and

(ii) left-improper-integral$(f, a, b) =$ left-improper-integral$(\max_+(f), a, b) -$ left-improper-integral$(\max_-(f), a, b)$.

Proof: Consider I_1 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b \max\limits_+(f)(x)dx$ and I_1 is right convergent in a. Consider I_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_2 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_2$ holds $I_2(x) = \int\limits_x^b \max\limits_-(f)(x)dx$ and I_2 is right convergent in a. For every real number d such that $a < d \leqslant b$ holds f is integrable on $[d, b]$ and $f\restriction[d, b]$ is bounded. For every real number x such that $x \in \operatorname{dom}(I_1 - I_2)$ holds $(I_1 - I_2)(x) = \int\limits_x^b f(x)dx$. □

(70) Suppose $[a, b[\subseteq \operatorname{dom} f$ and $\max_+(f)$ is right extended Riemann integrable on a, b and $\max_-(f)$ is right extended Riemann integrable on a, b. Then

(i) f is right extended Riemann integrable on a, b, and

(ii) right-improper-integral$(f, a, b) =$ right-improper-integral$(\max_+(f), a, b) -$ right-improper-integral$(\max_-(f), a, b)$.

Proof: Consider I_1 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x \max\limits_+(f)(x)dx$ and I_1 is left convergent in b. Consider I_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_2 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_2$ holds $I_2(x) = \int\limits_a^x \max\limits_-(f)(x)dx$ and I_2 is left convergent in b. For every real number d such that $a \leqslant d < b$ holds f is integrable on $[a, d]$ and $f\restriction[a, d]$ is bounded. For every real number x

such that $x \in \operatorname{dom}(I_1 - I_2)$ holds $(I_1 - I_2)(x) = \int\limits_a^x f(x)dx$. □

(71) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and $\max_+(f)$ is extended Riemann integrable on $-\infty$, b and $\max_-(f)$ is extended Riemann integrable on $-\infty$, b. Then

(i) f is extended Riemann integrable on $-\infty$, b, and

(ii) $\int\limits_{-\infty}^{b} f(x)dx = \int\limits_{-\infty}^{b} \max_+(f)(x)dx - \int\limits_{-\infty}^{b} \max_-(f)(x)dx$.

Proof: Consider I_1 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b \max_+(f)(x)dx$ and I_1 is convergent in $-\infty$. Consider I_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_2 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_2$ holds $I_2(x) = \int\limits_x^b \max_-(f)(x)dx$ and I_2 is convergent in $-\infty$. For every real number d such that $d \leqslant b$ holds f is integrable on $[d, b]$ and $f{\restriction}[d, b]$ is bounded. For every real number x such that $x \in \operatorname{dom}(I_1 - I_2)$ holds $(I_1 - I_2)(x) = \int\limits_x^b f(x)dx$. □

(72) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and $\max_+(f)$ is extended Riemann integrable on a, $+\infty$ and $\max_-(f)$ is extended Riemann integrable on a, $+\infty$. Then

(i) f is extended Riemann integrable on a, $+\infty$, and

(ii) $\int\limits_{a}^{+\infty} f(x)dx = \int\limits_{a}^{+\infty} \max_+(f)(x)dx - \int\limits_{a}^{+\infty} \max_-(f)(x)dx$.

Proof: Consider I_1 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x \max_+(f)(x)dx$ and I_1 is convergent in $+\infty$. Consider I_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_2 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_2$ holds $I_2(x) = \int\limits_a^x \max_-(f)(x)dx$ and I_2 is convergent in $+\infty$. For every real number d such that $a \leqslant d$ holds

f is integrable on $[a, d]$ and $f\restriction[a, d]$ is bounded. For every real number x such that $x \in \mathrm{dom}(I_1 - I_2)$ holds $(I_1 - I_2)(x) = \int\limits_a^x f(x)dx$. □

5. Improper Integral of Absolutely Integrable Functions

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and a non empty subset A of $\mathbb{R}$. Now we state the propositions:

(73) Suppose $]a, b] \subseteq \mathrm{dom}\, f$ and $A =]a, b]$ and f is left improper integrable on a and b and $|f|$ is left extended Riemann integrable on a, b and $f\restriction A$ is non-negative. Then

(i) $f\restriction A$ is integrable on L-Meas, and

(ii) left-improper-integral$(f, a, b) = \int f\restriction A \,\mathrm{d}\,$L-Meas.

The theorem is a consequence of (56) and (41).

(74) Suppose $[a, b[\subseteq \mathrm{dom}\, f$ and $A = [a, b[$ and f is right improper integrable on a and b and $|f|$ is right extended Riemann integrable on a, b and $f\restriction A$ is non-negative. Then

(i) $f\restriction A$ is integrable on L-Meas, and

(ii) right-improper-integral$(f, a, b) = \int f\restriction A \,\mathrm{d}\,$L-Meas.

The theorem is a consequence of (55) and (39).

(75) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number b, and a non empty subset A of $\mathbb{R}$. Suppose $]-\infty, b] \subseteq \mathrm{dom}\, f$ and $A =]-\infty, b]$ and f is improper integrable on $]-\infty, b]$ and $|f|$ is extended Riemann integrable on $-\infty$, b and f is non-negative. Then

(i) $f\restriction A$ is integrable on L-Meas, and

(ii) $\int\limits_{-\infty}^{b} f(x)dx = \int f\restriction A \,\mathrm{d}\,$L-Meas.

The theorem is a consequence of (58) and (45).

(76) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a non empty subset A of $\mathbb{R}$. Suppose $[a, +\infty[\subseteq \mathrm{dom}\, f$ and $A = [a, +\infty[$ and f is improper integrable on $[a, +\infty[$ and $|f|$ is extended Riemann integrable on a, $+\infty$ and f is non-negative. Then

(i) $f\restriction A$ is integrable on L-Meas, and

(ii) $\int\limits_{a}^{+\infty} f(x)dx = \int f\restriction A \,\mathrm{d}\,$L-Meas.

The theorem is a consequence of (59) and (47).

(77) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose $a < b$ and $[a, b[\subseteq \operatorname{dom} f$ and f is right improper integrable on a and b and $|f|$ is right extended Riemann integrable on a, b. Then $\max_{+}(f)$ is right extended Riemann integrable on a, b. The theorem is a consequence of (55) and (62).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and a non empty subset A of $\mathbb{R}$. Now we state the propositions:

(78) Suppose $[a, b[\subseteq \operatorname{dom} f$ and $A = [a, b[$ and f is right improper integrable on a and b and $|f|$ is right extended Riemann integrable on a, b. Then

(i) $f \restriction A$ is integrable on L-Meas, and

(ii) right-improper-integral$(f, a, b) = \int f \restriction A \, \mathrm{d}\,$L-Meas.

The theorem is a consequence of (55), (62), (74), (66), and (70).

(79) Suppose $]a, b] \subseteq \operatorname{dom} f$ and $A =]a, b]$ and f is left improper integrable on a and b and $|f|$ is left extended Riemann integrable on a, b. Then

(i) $f \restriction A$ is integrable on L-Meas, and

(ii) left-improper-integral$(f, a, b) = \int f \restriction A \, \mathrm{d}\,$L-Meas.

The theorem is a consequence of (56), (61), (73), (65), and (69).

(80) Suppose $]a, b[\subseteq \operatorname{dom} f$ and $A =]a, b[$ and f is improper integrable on a and b and there exists a real number c such that $a < c < b$ and $|f|$ is left extended Riemann integrable on a, c and right extended Riemann integrable on c, b. Then

(i) $f \restriction A$ is integrable on L-Meas, and

(ii) improper-integral$(f, a, b) = \int f \restriction A \, \mathrm{d}\,$L-Meas.

The theorem is a consequence of (79), (78), (51), and (26).

(81) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number b, and a non empty subset A of $\mathbb{R}$. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and $A =]-\infty, b]$ and f is improper integrable on $]-\infty$, $b]$ and $|f|$ is extended Riemann integrable on $-\infty$, b. Then

(i) $f \restriction A$ is integrable on L-Meas, and

(ii) $\int\limits_{-\infty}^{b} f(x) dx = \int f \restriction A \, \mathrm{d}\,$L-Meas.

The theorem is a consequence of (58), (63), (75), (67), and (71).

(82) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a non empty subset A of $\mathbb{R}$. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and $A = [a, +\infty[$

and f is improper integrable on $[a, +\infty[$ and $|f|$ is extended Riemann integrable on $a, +\infty$. Then

(i) $f{\restriction}A$ is integrable on L-Meas, and

(ii) $\int\limits_{a}^{+\infty} f(x)dx = \int f{\restriction}A \,\mathrm{d}\,\text{L-Meas}$.

The theorem is a consequence of (59), (64), (76), (68), and (72).

(83) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$ and $|f|$ is ∞-extended Riemann integrable. Then

(i) f is integrable on L-Meas, and

(ii) $\int\limits_{-\infty}^{+\infty} f(x)dx = \int f \,\mathrm{d}\,\text{L-Meas}$.

The theorem is a consequence of (81), (82), (51), and (36).

References

[1] Tom M. Apostol. *Mathematical Analysis*. Addison-Wesley, 1969.

[2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[3] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.

[4] Vladimir Igorevich Bogachev and Maria Aparecida Soares Ruas. *Measure theory*, volume 1. Springer, 2007.

[5] Noboru Endou. Extended real-valued double sequence and its convergence. *Formalized Mathematics*, 23(**3**):253–277, 2015. doi:10.1515/forma-2015-0021.

[6] Noboru Endou. Improper integral. Part I. *Formalized Mathematics*, 29(**4**):201–220, 2021. doi:10.2478/forma-2021-0019.

[7] Noboru Endou. Improper integral. Part II. *Formalized Mathematics*, 29(**4**):279–294, 2021. doi:10.2478/forma-2021-0024.

[8] Noboru Endou. Relationship between the Riemann and Lebesgue integrals. *Formalized Mathematics*, 29(**4**):185–199, 2021. doi:10.2478/forma-2021-0018.

[9] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Basic properties of extended real numbers. *Formalized Mathematics*, 9(**3**):491–494, 2001.

[10] Noboru Endou, Yasunari Shidama, and Masahiko Yamazaki. Integrability and the integral of partial functions from $\mathbb{R}$ into $\mathbb{R}$. *Formalized Mathematics*, 14(**4**):207–212, 2006. doi:10.2478/v10037-006-0023-y.

[11] Gerald B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Wiley, 2nd edition, 1999.

Accepted April 30, 2022