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ORIGINAL PAPER

Existence of solutions of discrete fractional problem coupled to mixed fractional boundary conditions

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Abstract
In this paper, we introduce a two-point nonlinear boundary value problem for a finite fractional difference equation. An associated Green's function is constructed as a series of functions and some of its properties are obtained. Some existence results are deduced from fixed point theory and lower and upper solutions.

Keywords Fractional equations · Green's functions · Nonlinear boundary value problems

Mathematics Subject Classification 34A08 · 34B27 · 34K37

1 Introduction

In this paper, we consider the following discrete fractional problem coupled to mixed fractional boundary conditions,

$$\begin{cases} -\Delta^\alpha y(t) + a(t+\alpha-1)y(t+\alpha-1) = f(t+\alpha-1, y(t+\alpha-1)), \ t \in \mathbb{N}_0^{T+1}, \\ y(\alpha-2) = \Delta^\beta y(\alpha+T+1-\beta) = 0, \end{cases} \tag{1}$$

where Δ^α is the standard Riemman–Liouville type discrete fractional difference operator, $t \in \mathbb{N}_0^{T+1}$, $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, T is a positive integer and function $f : \mathbb{N}_{\alpha-1}^{\alpha+T} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Here we denote $\mathbb{N}_r^{K+r} = \{r, \ldots, K+r\}$ for any $r \in \mathbb{R}$ and K a positive integer. So, we will look for solutions $y : \mathbb{N}_{\alpha-2}^{T+\alpha+1} \longrightarrow \mathbb{R}$.

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The continuous fractional calculus has a long history, but the discrete fractional calculus has been investigated recently. The development of this theory starts with the paper of Díaz and Osler [5] where is defined a fractional difference as an infinite series and a generalization of the binomial formula for the n-th order difference $\Delta^n f$ operator. This study continues with the work of Miller and Ross [10] which deals with the linear υ-th order fractional differential equation as an analogue of the linear n-th order ordinary differential equation. Since them, a great progress has been made in the study of boundary value problems for fractional difference equations (see [1, 2, 6–8] and references therein).

We recall some classical definitions from discrete fractional calculus theory and preliminary results.

Definition 1 We define $t^{(\nu)} = \Gamma(t+1)/\Gamma(t+1-\nu)$, for any t and ν for which the right hand side is well defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{(\nu)} = 0$.

Definition 2 The ν-th fractional sum of a function f, for $\nu > 0$ and $t \in \mathbb{N}_{r+\nu}$, is defined as

$$\Delta^{-\nu} f(t) (\equiv \Delta^{-\nu} f(t,r)) = \frac{1}{\Gamma(\nu)} \sum_{s=r}^{t-\nu} (t-s-1)^{(\nu-1)} f(s).$$

We define the ν-th fractional difference for $\nu > 0$, by $\Delta^{\nu} f(t) := \Delta^N \Delta^{\nu-N} f(t)$, where $t \in N_{r+\nu}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 \le \nu \le N$.

Lemma 3 [9] *Let* $0 \le N-1 \le \nu \le N$. *Then* $\Delta^{-\nu}\Delta^{\nu} y(t) = y(t) + c_1 t^{(\nu-1)} + c_2 t^{(\nu-2)} + \cdots + c_N t^{(\nu-N)}$ *for some* $c_i \in \mathbb{R}$, *with* $1 \le i \le N$.

The paper is scheduled as follows: In next section we deduce some properties of the Green's function related to the linear problem

$$\begin{cases} -\Delta^{\alpha} y(t) + a(t+\alpha-1) y(t+\alpha-1) = h(t+\alpha-1), \ t \in \mathbb{N}_0^{T+1}, \\ y(\alpha-2) = \Delta^{\beta} y(\alpha+T+1-\beta) = 0, \end{cases} \tag{2}$$

for the particular case of $a(t+\alpha-1) = 0$ for all $t \in \mathbb{N}_0^{T+1}$.

The case $0 < \alpha-\beta < 1$ has been treated in [9], we will continue this study, by improving some of the obtained results in [9] and by considering the case $1 \le \alpha-\beta \le 2$. Section 3 is devoted to deduce the expression of the Green's function related to the problem (2) for a nontrivial function $a(t)$, with small enough bounded absolute value. Moreover, some a priori bounds of the Green's function are obtained. The arguments are in the line of the ones given in the paper [3]. In last section we show the applicability of the given results by obtaining some existence and uniqueness results of the nonlinear problem (1).

2 Properties of function G_0

In this section we will extend previous results related to the Green's function G_0 of problem (2) with a identically zero. Some of these results have been given in [9] for the case $0 < \alpha-\beta < 1$, we improve some of them and study also the case $1 \le \alpha-\beta \le 2$.

Lemma 4 [9] *The unique solution of the linear fractional mixed boundary value problem*

$$\begin{cases} -\Delta^{\alpha} y(t) = h(t+\alpha-1), \quad t \in \mathbb{N}_0^{T+1}, \\ y(\alpha-2) = \Delta^{\beta} y(\alpha+T+1-\beta) = 0, \end{cases}$$

has the form

$$y(t) = \sum_{s=0}^{T+1} G_0(t,s)h(s+\alpha-1),$$

where $G_0 : \mathbb{N}_{\alpha-2}^{T+\alpha+1} \times \mathbb{N}_0^{T+1} \longrightarrow \mathbb{R}$ *is given by the expression*

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{t^{(\alpha-1)}(\alpha+T-\beta-s)^{(\alpha-\beta-1)}}{(\alpha+T-\beta+1)^{(\alpha-\beta-1)}} - (t-s-1)^{(\alpha-1)}, & 0 \le s < t-\alpha+1 \le T+2, \\ \dfrac{t^{(\alpha-1)}(\alpha+T-\beta-s)^{(\alpha-\beta-1)}}{(\alpha+T-\beta+1)^{(\alpha-\beta-1)}}, & 0 \le t-\alpha+1 \le s \le T+2. \end{cases} \tag{3}$$

In the following Lemma, we will show the properties of the Green's function, which extends the one given in [9, Lemma 3.3].

Lemma 5 *The Green's function* $G_0(t,s)$ *given by* (3) *has the following properties:*

(i) *Suppose that* $0 < \alpha-\beta \le 2$*, then* $G_0(t,s) > 0$ *for* $t \in \mathbb{N}_{\alpha-1}^{\alpha+T+1}$ *and* $s \in \mathbb{N}_0^{T+1}$.
(ii) *Suppose that* $0 < \alpha-\beta \le 2$*, then*

$$\max_{t\in\mathbb{N}_{\alpha-1}^{\alpha+T+1}} G_0(t,s) = G_0(s+\alpha-1,s), \quad \textit{for each } s \in \mathbb{N}_1^{T+1}.$$

Proof The case $0 < \alpha-\beta < 1$ has been proved in [9, Lemma 3.3 (i)], so, we only need to prove the case $1 \le \alpha-\beta \le 2$.
(i) For $0 \le t-\alpha+1 \le s \le T+1$, it is clear that

$$\begin{aligned} \Gamma(\alpha)G_0(t,s) &= \frac{\Gamma(T+3)t^{(\alpha-1)}}{\Gamma(\alpha+T-\beta+2)}(\alpha+T-\beta-s)^{(\alpha-\beta-1)} \\ &= \frac{\Gamma(T+3)\Gamma(t+1)\Gamma(\alpha+T-\beta-s+1)}{\Gamma(\alpha+T-\beta+2)\Gamma(t-\alpha+2)\Gamma(T-s+2)} \\ &> 0. \end{aligned}$$

Now, for $0 \le s < t-\alpha+1 \le T+1$, we have

$$\Gamma(\alpha)G_0(t,s) = (t-s-1)^{(\alpha-1)}\left(\frac{t^{(\alpha-1)}(\alpha+T-\beta-s)^{(\alpha-\beta-1)}}{(t-s-1)^{(\alpha-1)}(\alpha+T-\beta+1)^{(\alpha-\beta-1)}} - 1\right). \tag{4}$$

Let

$$\begin{aligned} F(t,s,\alpha,\beta) &:= \frac{t^{(\alpha-1)}(\alpha+T-\beta-s)^{(\alpha-\beta-1)}}{(t-s-1)^{(\alpha-1)}(\alpha+T-\beta+1)^{(\alpha-\beta-1)}} \\ &= \frac{t^{(\alpha-1)}\Gamma(T+3)}{(t-s-1)^{(\alpha-1)}\Gamma(T-s+2)}(\alpha+T-\beta-s)^{(-s-1)}. \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial}{\partial\beta}F(t,s,\alpha,\beta) &= \frac{\Gamma(t+1)\Gamma(T+3)\Gamma(-s+t-\alpha+1)\Gamma(-s+T+\alpha-\beta+1)}{\Gamma(t-s)\Gamma(-s+T+2)\Gamma(t-\alpha+2)\Gamma(T+\alpha-\beta+2)} \\ &\quad \times(\psi^{(0)}(T+\alpha-\beta+2) - \psi^{(0)}(-s+T+\alpha-\beta+1)) > 0, \end{aligned}$$

where by $\psi^{(0)}$ it is denoted the PolyGamma special function [11]. This implies that

$$\begin{aligned} F(t,s,\alpha,\beta) &> F(t,s,\alpha,0) \\ &= \frac{t^{(\alpha-1)}(\alpha+T-s)^{(\alpha-1)}}{(t-s-1)^{(\alpha-1)}(\alpha+T+1)^{(\alpha-1)}} \\ &= \frac{t(t-1)\cdots(t-s)(T+2)(T+1)\cdots(T-s+2)}{(t-\alpha+1)(t-\alpha)\cdots(t-\alpha+1-s)(\alpha+T+1)(\alpha+T)\cdots(\alpha+T+1-s)} \\ &= \prod_{i=0}^{s} P(t,\alpha,i), \end{aligned}$$

where, for every $i \in \{0,\ldots,s\}$, we denote

$$P(t,\alpha,i) := \frac{(t-i)(T+2-i)}{(t-\alpha+1-i)(\alpha+T+1-i)}.$$

Since

$$\frac{\partial P}{\partial i}(t,\alpha,i) = \frac{(\alpha-1)(2i-t-T-2)(-\alpha+t-T-1)}{(\alpha+i-t-1)^2(\alpha-i+T+1)^2} > 0, \quad 0 \le i \le s,$$

we have that $P(t,\alpha,i) > P(t,\alpha,0) = \frac{t(T+2)}{(-\alpha+t+1)(\alpha+T+1)}$.

Now, using that $\frac{\partial P}{\partial t}(t,\alpha,0) = \frac{(1-\alpha)(T+2)}{(-\alpha+t+1)^2(\alpha+T+1)} < 0$, we deduce that $P(t,\alpha,i) > P(T+1+\alpha,\alpha,0) = 1$, for all $i \in \{1,\ldots,s\}$ and, from (4), the positiveness of the Green's function is shown.

(ii) The case $0 < \alpha-\beta < 1$ has been proved in [9, Lemma 3.3 (ii)]. Now, let $1 \le \alpha-\beta \le 2$. If $0 \le t-\alpha+1 \le s \le T+1$. Then

$$\Delta_t G_0(t,s) = \frac{(\alpha-1)\Gamma(T+3)\Gamma(t+1)\Gamma(\alpha+T-\beta-s+1)}{\Gamma(\alpha)\Gamma(\alpha+T-\beta+2)\Gamma(t-\alpha+3)\Gamma(T-s+2)} > 0.$$

Hence, we deduce that $\Delta_t G_0(t,s) > 0$ for $0 \le t-\alpha+1 \le s \le T+1$. And so, $G_0(t,s)$ is increasing with respect to t for $0 \le t-\alpha+1 \le s \le T+1$, which implies that

$$G_0(t,s) \le G_0(s+\alpha-1,s).$$

If $0 \le s < t-\alpha+1 \le T+1$. Then

$$\begin{aligned} \Gamma(\alpha)\Delta_t G_0(t,s) &= \frac{(\alpha-1)t^{(\alpha-2)}(\alpha+T-\beta-s)^{(\alpha-\beta-1)}}{(\alpha+T-\beta+1)^{(\alpha-\beta-1)}} - (\alpha-1)(t-s-1)^{(\alpha-2)} \\ &= (\alpha-1)(t-s-1)^{(\alpha-2)} \\ &\quad \times \left[\frac{t^{(\alpha-2)}(\alpha+T-\beta-s)^{(\alpha-\beta-1)}}{(\alpha+T-\beta+1)^{(\alpha-\beta-1)}(t-s-1)^{(\alpha-2)}} - 1\right]. \end{aligned}$$

Thus, to show that $\Gamma(\alpha)\Delta_t G_0(t,s) < 0$, we must prove that

$$H(t,s,\alpha,\beta) := \frac{t^{(\alpha-2)}(\alpha+T-\beta-s)^{(\alpha-\beta-1)}}{(\alpha+T-\beta+1)^{(\alpha-\beta-1)}(t-s-1)^{(\alpha-2)}} < 1. \tag{5}$$

Now, since

$$\Delta_t H(t,s,\alpha,\beta) = \frac{(2-\alpha)(s+1)\Gamma(t+1)\Gamma(T+3)\Gamma(-s+t-\alpha+2)\Gamma(-s+T+\alpha-\beta+1)}{\Gamma(-s+t+1)\Gamma(-s+T+2)\Gamma(t-\alpha+4)\Gamma(T+\alpha-\beta+2)} > 0,$$

we have that

$$H(t,s,\alpha,\beta) \le H(T+1+\alpha,s,\alpha,\beta) = \frac{(-s+T+2)\Gamma(T+\alpha+2)\Gamma(-s+T+\alpha-\beta+1)}{(T+3)\Gamma(-s+T+\alpha+1)\Gamma(T+\alpha-\beta+2)}.$$

So, by using that

$$\Delta_s H(T+1+\alpha,s,\alpha,\beta) = \frac{(-\alpha+2\beta-\beta s+s+(\beta-1)T)\Gamma(T+\alpha+2)\Gamma(-s+T+\alpha-\beta)}{(T+3)\Gamma(-s+T+\alpha+1)\Gamma(T+\alpha-\beta+2)},$$

since $-\alpha+2\beta-\beta s+s+(\beta-1)T \le 0$ for all $s \in \{0,\ldots,T\}$ if and only if $\alpha-\beta \ge \beta$, which is true whenever $\alpha-\beta \ge 1$, we have that

$$\Delta_s H(T+1+\alpha,s,\alpha,\beta) \le 0 \quad \text{for all } s \in \{0,\ldots,T\}.$$

As a direct consequence, we deduce that

$$H(T+1+\alpha,s,\alpha,\beta) \le H(T+1+\alpha,0,\alpha,\beta) = \frac{(T+2)(\alpha+T+1)}{(T+3)(\alpha-\beta+T+1)} \quad \text{for all } s \in \mathbb{N}_0^{T+1}.$$

Now, from the fact that

$$\frac{\partial}{\partial\beta}H(T+1+\alpha,0,\alpha,\beta) = \frac{(T+2)(\alpha+T+1)}{(T+3)(\alpha-\beta+T+1)^2} > 0,$$

we conclude that

$$H(t,s,\alpha,\beta) \le H(T+1+\alpha,0,\alpha,\alpha-1) = \frac{\alpha+T+1}{T+3} < 1,$$

for all $t \in \{\alpha-1,\ldots,T+1+\alpha\}$, $0 \le t-\alpha+1 \le s \le T+1$ and $0<\beta\le\alpha-1<1$, which implies, from (5), that

$$G_0(t,s) \le G_0(s+\alpha-1,s).$$

As a result, we get that $G_0(t,s)$ is increasing with respect to t for $0 \le t-\alpha+1 \le s \le T+1$ and decreasing with respect to t for $0 \le s < t-\alpha+1 \le T+1$. Hence, it follows that

$$\max_{t\in\mathbb{N}_{\alpha-1}^{\alpha+T+1}} G_0(t,s) = G_0(s+\alpha-1,s)$$

and the proof is concluded. □

In the following, we obtain a bound from above of the Green's function.

Theorem 6 *Let $G_0(t,s)$ be the Green's function defined by (3). Then, for any $(t,s) \in \mathbb{N}_{\alpha-1}^{\alpha+T+1} \times \mathbb{N}_0^{T+1}$, the following inequalities are fulfilled:*

(i) *If* $0<\alpha-\beta\leq 1$, *then*

$$G_0(t,s)\leq C_1(T,\alpha,\beta):=\frac{(T+2)\Gamma(\alpha-\beta)\Gamma(T+\alpha+1)}{\Gamma(\alpha)\Gamma(T+\alpha-\beta+2)}.$$

(ii) *If* $\alpha-\beta>1$, *then*

$$G_0(t,s) \leq C_2(T,\alpha,\beta) := \frac{\Gamma(T+3)\Gamma\left(\alpha+\frac{-\alpha T+T-\beta}{-2\alpha+\beta+2}\right)\Gamma\left(\alpha-\beta+\frac{-\alpha T+\beta T+T+\beta}{-2\alpha+\beta+2}+1\right)}{\Gamma(\alpha)\Gamma(T+\alpha-\beta+2)\Gamma\left(-\frac{(T+2)(\alpha-1)}{-2\alpha+\beta+2}\right)\Gamma\left(\frac{-\alpha T+\beta T+T+\beta}{-2\alpha+\beta+2}+2\right)}.$$

Proof From previous result we know that

$$0<G_0(t,s)\leq G_0(s+\alpha-1,s)=\frac{\Gamma(T+3)\Gamma(s+\alpha)\Gamma(-s+T+\alpha-\beta+1)}{\Gamma(\alpha)\Gamma(s+1)\Gamma(-s+T+2)\Gamma(T+\alpha-\beta+2)}.$$

So, the bounds come from the ones of the previous inequality.

Since,

$$\Delta_s G_0(s+\alpha-1,s) = \frac{\Gamma(T+3)\Gamma(s+\alpha)(\beta+s(-2\alpha+\beta+2)+(\alpha-1)T)\Gamma(-s+T+\alpha-\beta)}{\Gamma(\alpha)\Gamma(s+2)\Gamma(-s+T+2)\Gamma(T+\alpha-\beta+2)}, \quad (6)$$

we have that

$$\Delta_s G_0(s+\alpha-1,s)|_{s=0}=\frac{(T+2)(\beta+(\alpha-1)T)}{(\alpha-\beta+T)(\alpha-\beta+T+1)}>0.$$

Moreover, it is clear that (6) vanishes if and only if

$$s=s_0(\alpha,\beta):=\frac{\beta+(\alpha-1)T}{2\alpha-\beta-2}.$$

It is immediate to verify that $s_0(\alpha,\beta)\in\{0,\ldots,T+1\}$ if and only if $\alpha-\beta>1$.

(i) Consider the first case: $0<\alpha-\beta\leq 1$.
In this situation we have that $G_0(s+\alpha-1,s)$ is increasing in $s\in\mathbb{N}_0^{T+1}$. Thus

$$G_0(s+\alpha-1,s)\leq G_0(T+\alpha,T+1)=C_1(T,\alpha,\beta).$$

(ii) Assume now that $1<\alpha-\beta$.
Now the maximum is attained at $s_0(\alpha,\beta)$, in consequence:

$$G_0(s+\alpha-1,s)\leq G_0(s_0(\alpha,\beta)+\alpha-1,s_0(\alpha,\beta))=C_2(T,\alpha,\beta)$$

and the proof is concluded. □

3 Green's function of problem (2)

In this section we deduce the existence of the Green's function G related to Problem (2). The expression of the function G is given as a functional series and we prove its convergence for suitable values of the function $a(t)$. The arguments are in the line to the ones given in [3].

Let $G : \mathbb{N}_{\alpha-2}^{T+\alpha+1} \times \mathbb{N}_0^{T+1} \longrightarrow \mathbb{R}$ by

$$G(t,s) = \sum_{n=0}^{\infty} (-1)^n G_n(t,s) \tag{7}$$

where $G_0(t,s)$ is given by (3) and

$$G_n : \mathbb{N}_{\alpha-2}^{T+\alpha+1} \times \mathbb{N}_0^{T+1} \longrightarrow \mathbb{R}$$

is defined as

$$G_n(t,s) = \sum_{\tau=0}^{T+1} a(\tau+\alpha-1) G_0(t,\tau) G_{n-1}(\tau+\alpha-1,s), \quad n \geq 1. \tag{8}$$

In order to express the Green's function associated with the linear problem (2) we shall use the spectral theory in Banach space given by the following Lemma:

Lemma 7 [12, Theorem 1.B] *Let X be a Banach space and $\mathcal{A} : X \longrightarrow X$ be a linear operator with the operator norm $\|\mathcal{A}\|$. Then if $\|\mathcal{A}\| < 1$, then $(\mathcal{I} - \mathcal{A})^{-1}$ exists and $(\mathcal{I} - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} \mathcal{A}^n$, where $\mathcal{I}$ stands for the identity operator.*

Let

$$X := \{y : \mathbb{N}_{\alpha-2}^{T+\alpha+1} \longrightarrow \mathbb{R} : \quad y(\alpha-2) = \Delta^{\beta} y(\alpha+T+1-\beta) = 0\}$$

be the Banach space with norm

$$\|y\| = \max\{|y(t)|; \quad t \in \mathbb{N}_{\alpha-2}^{T+\alpha+1}\}.$$

Consider the following assumption:
(H) There exists $\overline{a} > 0$ such that

(i) If $0 < \alpha - \beta \leq 1$, then

$$|a(t+\alpha-1)| \leq \overline{a} < \frac{1}{(T+2)\, C_1(T,\alpha,\beta)} \quad \text{for all } t \in \mathbb{N}_0^{T+1}.$$

(ii) If $\alpha - \beta > 1$, then

$$|a(t+\alpha-1)| \leq \overline{a} < \frac{1}{(T+2)\, C_2(T,\alpha,\beta)} \quad \text{for all } t \in \mathbb{N}_0^{T+1},$$

where $C_1(T,\alpha,\beta)$ and $C_2(T,\alpha,\beta)$ are introduced in Theorem 6.

In next result we prove the existence and uniqueness of solution of problem (2) by means of the construction of its related Green's function.

Theorem 8 *If condition (H) is fulfilled, then the function $G(t,s)$ defined in (7) as a series of functions is convergent for all $(t,s) \in \mathbb{N}_{\alpha-2}^{T+\alpha+1} \times \mathbb{N}_0^{T+1}$. Moreover, $G(t,s)$ is the Green's function related to problem (2).*

Proof The solution y of (2) satisfies, for all $t \in \mathbb{N}_{\alpha-2}^{T+\alpha+1}$, the following equality:

$$y(t) = \sum_{s=0}^{T+1} G_0(t,s) \Big(h(s+\alpha-1) - a(s+\alpha-1) y(s+\alpha-1) \Big),$$

where G_0 is defined by (3).

This expression can be reformulated as

$$y(t) + \sum_{s=0}^{T+1} G_0(t,s)a(s+\alpha-1)y(s+\alpha-1) = \sum_{s=0}^{T+1} G_0(t,s)h(s+\alpha-1). \quad (9)$$

Now, denote operators $\mathcal{A}$ and $\mathcal{B}$ by

$$\mathcal{A}h(t) = \sum_{s=0}^{T+1} G_0(t,s)h(s+\alpha-1) \quad (10)$$

$$\mathcal{B}y(t) = \sum_{s=0}^{T+1} G_0(t,s)a(s+\alpha-1)y(s+\alpha-1). \quad (11)$$

Then, Eq. (9) becomes

$$(\mathcal{I}+\mathcal{B})y = \mathcal{A}h.$$

First, let us show that $\|\mathcal{B}\| < 1$. For any $y \in X$ with $\|y\| = 1$ and $t \in \mathbb{N}_{\alpha-2}^{T+\alpha+1}$, by (11), we have

(i) If $0 < \alpha - \beta \le 1$, then, using Theorem 6 (i) and condition (H), we have that for all $t \in \mathbb{N}_{\alpha-2}^{T+\alpha+1}$ the following inequalities are fulfilled:

$$|\mathcal{B}\, y(t)| \le \sum_{s=0}^{T+1} \overline{a} G_0(t,s) \le \sum_{s=0}^{T+1} \overline{a}\, C_1(T,\alpha,\beta) = (T+2)\,\overline{a}\, C_1(T,\alpha,\beta) < 1.$$

(ii) If $\alpha - \beta > 1$, then using Theorem 6 (ii) and condition (H), we get

$$|\mathcal{B}\, y(t)| \le \sum_{s=0}^{T+1} \overline{a} G_0(t,s) \le \sum_{s=0}^{T+1} \overline{a}\, C_2(T,\alpha,\beta) = (T+2)\,\overline{a}\, C_2(T,\alpha,\beta) < 1.$$

Therefore, by Lemma 7, we deduce that

$$y = \left(\mathcal{I}+\mathcal{B}\right)^{-1}\mathcal{A}h = \sum_{n=0}^{\infty}(-\mathcal{B})^n \mathcal{A}h. \quad (12)$$

By using analogous arguments to [3], we show that for $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ the following identity holds:

$$((-\mathcal{B})^n \mathcal{A}h)(t) = \sum_{s=0}^{T+1}(-1)^n G_n(t,s)h(s+\alpha-1). \quad (13)$$

In the sequel we obtain the following a priori bounds for function G_n for all $n \in \mathbb{N}_0$:

(i) If $0 < \alpha - \beta \le 1$, then

$$|(-1)^n G_n(t,s)| \le \overline{a}^n \left((T+2)\, C_1(T,\alpha,\beta)\right)^{n+1}. \quad (14)$$

(ii) If $\alpha - \beta > 1$, then

$$|(-1)^n G_n(t,s)| \le \overline{a}^n \left((T+2)\, C_2(T,\alpha,\beta)\right)^{n+1}, \quad (15)$$

where $\overline{a}$ is defined in (H).

In the first case, for $n = 0$, (14) holds. Assume (14) holds for $n = m \geq 0$. Then, for any $(t, s) \in \mathbb{N}_{\alpha-2}^{T+\alpha+1} \times \mathbb{N}_0^{T+1}$, we have

$$\begin{aligned} |(-1)^{m+1} G_{m+1}(t,s)| &\leq \sum_{\tau=0}^{T+1} |a(\tau+\alpha-1)| G_0(t,\tau) G_m(\tau+\alpha-1,s) \\ &\leq \sum_{\tau=0}^{T+1} \overline{a}\, C_1(T,\alpha,\beta)\, G_m(\tau+\alpha-1,s) \\ &\leq \sum_{\tau=0}^{T+1} \overline{a}^{m+1} (T+2)^{m+1} \left(C_1(T,\alpha,\beta)\right)^{m+2} \\ &= \overline{a}^{m+1} (T+2)^{m+2} \left(C_1(T,\alpha,\beta)\right)^{m+2}, \end{aligned}$$

hence (14) holds for $n = m + 1$.

By induction, (14) if fulfilled for any $n \in \mathbb{N}_0$.

Arguing as at the first case, we conclude that (15) is true for all $n \in \mathbb{N}_0$.

Finally, by (H) and (7), for all $(t, s) \in \mathbb{N}_{\alpha-2}^{T+\alpha+1} \times \mathbb{N}_0^{T+1}$, we deduce the following inequalities:

(i) If $0 < \alpha - \beta \leq 1$, then

$$|G(t,s)| = \left| \sum_{n=0}^{\infty} (-1)^n G_n(t,s) \right| \leq \sum_{n=0}^{\infty} \overline{a}^n \left((T+2)\, C_1(t,\alpha,\beta)\right)^{n+1} < \infty.$$

(ii) If $\alpha - \beta > 1$, then

$$|G(t,s)| = \left| \sum_{n=0}^{\infty} (-1)^n G_n(t,s) \right| \leq \sum_{n=0}^{\infty} \overline{a}^n \left((T+2)\, C_2(t,\alpha,\beta)\right)^{n+1} < \infty.$$

Therefore, $G(t, s)$ is uniformly convergent on $\mathbb{N}_{\alpha-2}^{T+\alpha+1} \times \mathbb{N}_0^{T+1}$.

Moreover, from (7), (12) and (13), we obtain

$$y(t) = \sum_{n=0}^{\infty} \sum_{s=0}^{T+1} (-1)^n G_n(t,s) h(s+\alpha-1) = \sum_{s=0}^{T+1} G(t,s) h(s+\alpha-1). \tag{16}$$

On the other hand, let y be defined by (16). Using (7), (10) and (11) together (8) and (13), we conclude that y satisfies (12). Then, y is the unique solution of problem (2) and G is the Green's function related to problem (2). □

As a direct consequence of Theorem 8 and Eqs. (7) and (8), we deduce the following result

Corollary 9 *If $a(t) \in [-\bar{a}, 0]$ for all $t \in \mathbb{N}_{\alpha-1}^{\alpha+T+1}$, then $G(t, s) > 0$ for all $t \in \mathbb{N}_{\alpha-1}^{\alpha+T+1}$ and $s \in \mathbb{N}_0^{T+1}$.*

In next result, we obtain an upper bound for function $|G(t, s)|$ for any s fixed.

Lemma 10 *We introduce the following function $\overline{G} : \mathbb{N}_0^{T+1} \to (0, \infty)$:*

(i) *For $0 < \alpha - \beta \leq 1$ and for all $(t, s) \in \mathbb{N}_{\alpha-2}^{\alpha+T+1} \times \mathbb{N}_0^{T+1}$*

$$\overline{G}(s) := G_0(s+\alpha-1, s) \left(\frac{T+2}{1 - \overline{a}\,(T+2)\, C_1(t,\alpha,\beta)} \right). \tag{17}$$

(ii) *For* $\alpha - \beta > 1$ *and for all* $(t, s) \in \mathbb{N}_{\alpha-2}^{\alpha+T+1} \times \mathbb{N}_0^{T+1}$

$$\overline{G}(s) := G_0(s + \alpha - 1, s) \left(\frac{T + 2}{1 - \overline{a}\,(T + 2)\, C_2(t, \alpha, \beta)} \right).$$

Then $|G(t, s)| \leq \overline{G}(s)$ *for all* $(t, s) \in \mathbb{N}_{\alpha-2}^{T+\alpha+1} \times \mathbb{N}_0^{T+1}$*, where G is given by* (7) .

Proof (i) For $0 < \alpha - \beta \leq 1$ and for all $(t, s) \in N_{\alpha-2}^{T+\alpha+1} \times \mathbb{N}_0^{T+1}$, as a direct combination of expression (8) and inequality (14), we have

$$|(-1)^n G_n(t, s)| \leq G_0(s + \alpha - 1, s)\, (T + 2)\, \left(\overline{a}\,(T + 2)\, C_1(T, \alpha, \beta)\right)^n . \tag{18}$$

Using condition (H), (17) and (18), for all $(t, s) \in \mathbb{N}_{\alpha-2}^{\alpha+T+1} \times \mathbb{N}_0^{T+1}$, we have

$$|G(t, s)| = \left| \sum_{n=0}^{\infty} (-1)^n G_n(t, s) \right| \leq G_0(s + \alpha - 1, s)\, (T + 2) \sum_{n=0}^{\infty} \left(\overline{a}\,(T + 2)\, C_1(T, \alpha, \beta)\right)^n .$$

The case $\alpha - \beta > 1$ is proved in the same way. □

4 Existence and uniqueness of solutions

In this section, we discuss the existence and uniqueness of solutions to problem (1).

Define the operator $S : X \longrightarrow X$ by

$$(Sy)(t) := \sum_{s=0}^{T+1} G(t, s) f(s + \alpha - 1, y(s + \alpha - 1)), \quad y \in X. \tag{19}$$

The proof of that S is completely continuous is analogous to the one done in [3]. It follows that the fixed points of the operator S are the solutions of the boundary value problem (1).

Arguing as in [3], one can prove the following result

Theorem 11 *Assume that condition (H) holds and f satisfies the following condition:* (H^*) *There exists a constant* $K \in (0, \left(\sum_{s=0}^{T+1} \overline{G}(s)\right)^{-1})$ ($\overline{G}$ *given in Lemma* 10) *such that*

$$|f(t, u) - f(t, v)| \leq K|u - v|, \text{ for } (t, v) \in \mathbb{N}_{\alpha}^{\alpha+T} \times \mathbb{R}.$$

Then Problem (1) *has a unique solution.*

Example 12 Let $\overline{a}$ satisfy condition (H) and $K \in (0, \left(\sum_{s=0}^{T+1} \overline{G}(s)\right)^{-1})$. Consider

$$f(t, v) := \frac{K}{2} \cot^{-1}\left(\frac{2}{v^3}\right) + g(t), \quad t \in \mathbb{N}_{\alpha-1}^{\alpha+T}, \quad v \in \mathbb{R},$$

with $g : \mathbb{N}_{\alpha-1}^{\alpha+T} \to \mathbb{R}$ an arbitrary nontrivial given function.

It is not difficult to verify that for any $t \in \mathbb{N}_{\alpha-1}^{\alpha+T}$ and $u, v \in \mathbb{R}$, $u < v$, there exist $x \in [u, v]$, such that

$$|f(t, u) - f(t, v)| = 3\,K \left| \frac{x^2}{x^6 + 4} \right| |u - v| \leq K\,|u - v|.$$

From Theorem 11 the considered problem has a unique solution.

In the sequel we deduce an existence result for the nonlinear Problem (1).

Theorem 13 *Assume that condition (H) holds and f satisfies the following condition:*

$$|f(t,v)| \leq g(t)\,h(|v|) \quad \forall (t,v) \in \mathbb{N}_\alpha^{\alpha+T} \times \mathbb{R}, \tag{20}$$

where $h : [0,+\infty) \to [0,+\infty)$ is a continuous function such that

$$\lim_{s\to+\infty} \frac{s}{h(s)} = +\infty. \tag{21}$$

Then Problem (1) has at least one solution.

Proof We know that the solutions of problem (1) are given as the fixed points of operator S defined on (19). So, to deduce the existence of a fixed point we consider, for any $\lambda \in (0,1)$, a function $y \in X$, such that $y = \lambda\, S\, y$. As a consequence, for all $t \in \mathbb{N}_\alpha^{\alpha+T}$ we have:

$$\begin{aligned}
|y(t)| &= \lambda \left| \sum_{s=0}^{T+1} G(t,s)\big(f(s+\alpha-1, y(s+\alpha-1))\big)\right| \\
&\leq \sum_{s=0}^{T+1} |G(t,s)|\,|f(s+\alpha-1, y(s+\alpha-1))| \\
&\leq \sum_{s=0}^{T+1} \overline{G}(s)\,|g(s+\alpha-1)|\,|h(y(s+\alpha-1))| \\
&\leq \left(\sum_{s=0}^{T+1} \overline{G}(s)\,g(s+\alpha-1)\right)\|h(y)\|.
\end{aligned}$$

Thus, we have that

$$\frac{\|y\|}{\|h(y)\|} \leq \sum_{s=0}^{T+1} \overline{G}(s)\,g(s+\alpha-1).$$

So, from (21) we deduce that there is a constant $K > 0$, independent of λ, such that

$$\|y\| \leq K.$$

In consequence, we deduce from Schaefer's fixed point Theorem [12], that operator S has a fixed point and it is a solution of Problem (1) □

Example 14 Let $\overline{a}$ satisfying condition (H). Let us consider Problem (1) with function

$$f(t,v) = t^{2-\alpha}(\sqrt{|v|} + \cos(v)), \quad t \in \mathbb{N}_{\alpha-1}^{\alpha+T}, \quad v \in \mathbb{R},$$

which satisfies inequality (20) and property (21) for functions $h(s) = \sqrt{s} + 1$, $s \geq 0$, and $g(t) = t^{2-\alpha}$, $t \in \mathbb{N}_{\alpha-1}^{\alpha+T}$.

So, from Theorem 13, the considered problem has at least one solution.

Remark 15 Notice that in the two previous theorems we cannot ensure that the obtained solution is not trivial. This property can be deduced when $f(t,0) \not\equiv 0$ on $\mathbb{N}_\alpha^{\alpha+T}$. In particular, the obtained solutions in the two previous examples are non trivial.

Now we will develop the monotone iterative technique for problem (1). To this end we must assume the existence of a pair of well ordered lower and upper solutions, which are defined as follows.

Definition 16 A function γ is said to be a discrete lower solution of problem (1) if

$$\begin{cases} -\Delta^{\alpha}\gamma(t) + a(t+\alpha-1)\gamma(t+\alpha-1) \leq f(t+\alpha-1, \gamma(t+\alpha-1)), \ t \in \mathbb{N}_0^{T+1}, \\ \gamma(\alpha-2) = \Delta^{\beta}\gamma(\alpha+T+1-\beta) = 0. \end{cases} \tag{22}$$

Definition 17 A function δ is said to be a discrete upper solution of problem (1) if

$$\begin{cases} -\Delta^{\alpha}\delta(t) + a(t+\alpha-1)\delta(t+\alpha-1) \geq f(t+\alpha-1, \delta(t+\alpha-1)), \ t \in \mathbb{N}_0^{T+1}, \\ \delta(\alpha-2) = \Delta^{\beta}\delta(\alpha+T+1-\beta) = 0. \end{cases} \tag{23}$$

Definition 18 We say that y is the minimal (maximal) solution of Problem (1) on the sector $[\gamma, \delta]$ if any other solution z such that $\gamma(t) \leq z(t) \leq \delta(t)$ on $\mathbb{N}_{\alpha-2}^{\alpha+T+1}$ satisfies $y(t) \leq z(t)$ $(y(t) \geq z(t))$ on $\mathbb{N}_{\alpha-2}^{\alpha+T+1}$.

The existence result is the following

Theorem 19 *Assume that γ is a lower solution of (1) and δ is an upper solution of (1) satisfying $\gamma(t) \leq \delta(t)$ on $\mathbb{N}_{\alpha-2}^{\alpha+T+1}$. Moreover function $f(t, y)$ is monotone nondecreasing with respect to $y \in \mathbb{R}$ and $a(t) \in [-\bar{a}, 0]$ on $\mathbb{N}_{\alpha}^{\alpha+T}$ ($\overline{a}$ defined in condition (H)). Then there exist two monotone sequences $\{\gamma_n\}$ and $\{\delta_n\}$, with $\gamma_0 = \gamma$ and $\delta_0 = \delta$, that converge, respectively, to the minimal and maximal solutions on $[\gamma, \delta]$ of Problem (1).*

Proof Define $\gamma_0 = \gamma$ and, for any $n \geq 1$, γ_n as the unique solution of the following linear problem:

$$\begin{cases} -\Delta^{\alpha}\gamma_n(t) + a(t+\alpha-1)\gamma_n(t+\alpha-1) = f(t+\alpha-1, \gamma_{n-1}(t+\alpha-1)), \ t \in \mathbb{N}_0^{T+1}, \\ \gamma_n(\alpha-2) = \Delta^{\beta}\gamma_n(\alpha+T+1-\beta) = 0. \end{cases}$$

From the assumptions, we have that such unique solution is given by the expression

$$\gamma_n(t) = \sum_{s=0}^{T+1} G(t,s)\, f(s+\alpha-1, \gamma_{n-1}(s+\alpha-1)), \quad t \in \mathbb{N}_{\alpha-2}^{\alpha+T+1}.$$

For $n = 1$, from the definition of lower solution given in (22) and the positiveness of the Green's function $G(t, s)$ stated at Corollary 9, we have, for all $t \in \mathbb{N}_{\alpha-2}^{\alpha+T+1}$, that

$$\begin{aligned} \gamma_1(t) - \gamma_0(t) \geq \sum_{s=0}^{T+1} G(t,s) \, \big[& f(s+\alpha-1, \gamma(s+\alpha-1)) \\ & - f(s+\alpha-1, \gamma(s+\alpha-1)) \big] = 0. \end{aligned}$$

By recurrence, by assuming that $\gamma_n \geq \gamma_{n-1}$ on $\mathbb{N}_{\alpha-2}^{\alpha+T+1}$, using that f is monotone nondecreasing with respect to the second variable, from the positiveness of the Green's function G, we deduce that

$$\begin{aligned} \gamma_{n+1}(t) - \gamma_n(t) = \sum_{s=0}^{T+1} G(t,s) \, \big[& f(s+\alpha-1, \gamma_n(s+\alpha-1)) \\ & - f(s+\alpha-1, \gamma_{n-1}(s+\alpha-1)) \big] \geq 0. \end{aligned}$$

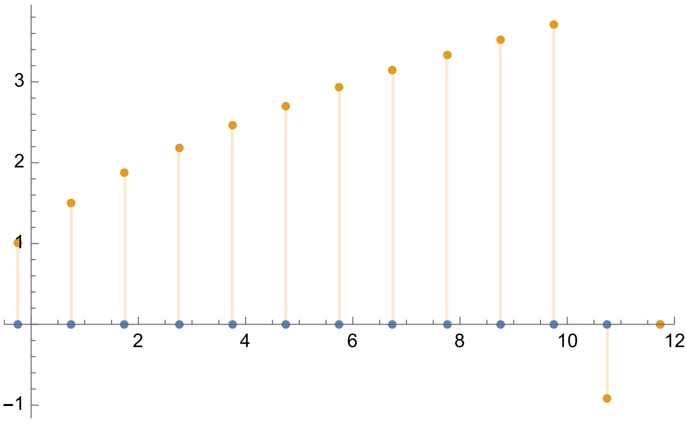

Fig. 1 Graph of $\Delta^\beta \delta_0(t)$, $t \in \mathbb{N}_{\alpha-\beta-1}^{\alpha-\beta+T+1}$

It is immediate to verify that, by defining $\delta_0 = \delta$ and, for any $n \geq 1$, δ_n as the unique solution of the following linear problem:

$$\begin{cases} -\Delta^\alpha \delta_n(t) + a(t+\alpha-1)\delta_n(t+\alpha-1) = f(t+\alpha-1, \delta_{n-1}(t+\alpha-1)), & t \in \mathbb{N}_0^{T+1}, \\ \delta_n(\alpha-2) = \Delta^\beta \delta_n(\alpha+T+1-\beta) = 0, \end{cases}$$

the sequence $\{\delta_n\}$ is monotone nonincreasing and it is satisfied that $\gamma(t) \leq \gamma_n(t) \leq \delta_n(t) \leq \delta(t)$ for all $n \geq 1$ and $t \in \mathbb{N}_{\alpha-2}^{\alpha+T+1}$.

The convergence of both sequences to the minimal and maximal solutions of Problem (1) holds immediately from the continuity of function f. □

Example 20 Let us consider Problem (1) with $\alpha = \frac{5}{4}$, $\beta = \frac{1}{2}$, $T = 10$, and $a(t) = -10^{-4}t^2$. Define the function:

$$f(t, y) = 10^{-6}(y^3 + t) - 10^5,$$

which is nondecreasing with respect to $y \in \mathbb{R}$.

Moreover

$$-\bar{a} \approx -0.0177956 < -0.0121 \leq a(t) \leq 0, \quad t \in \mathbb{N}_\alpha^{\alpha+T}.$$

Let $\gamma_0(t) = 0$ and

$$\delta_0(t) = \begin{cases} t - \alpha + 2, & \text{if } t < \alpha + 10, \\ \lambda(\alpha + 10 - \frac{t}{2}), & \text{if } t \geq \alpha + 10, \end{cases}$$

where $\lambda = \frac{12459153}{9699328}$.

It is clear that $\gamma_0 \leq \delta_0$ on $\mathbb{N}_{\alpha-2}^{\alpha+T+1}$ and that γ_0 is a lower solution of problem (1).

Let us see that δ_0 is an upper solution of problem (1).

It is obvious that $\delta_0(\alpha-2) = 0$. Moreover, a simple calculation, see Fig. 1, shows that

$$\delta_0(\alpha-2) = \Delta^\beta \delta_0(\alpha+T+1-\beta) = 0$$

and, see Fig. 2,

$$-\Delta^\alpha \delta_0(t) + a(t+\alpha-1)\delta_0(t+\alpha-1) - f(t+\alpha-1, \delta_0(t+\alpha-1)) \geq 0, \quad t \in \mathbb{N}_0^{T+1}.$$

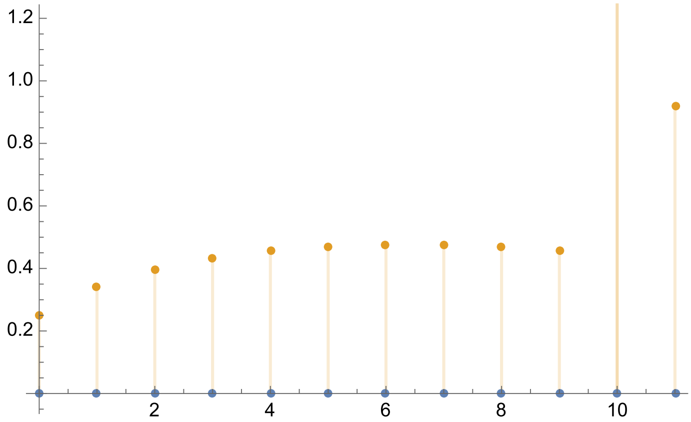

Fig. 2 $-\Delta^{\alpha}\delta_0(t) + a(t+\alpha-1)\delta_0(t+\alpha-1) - f(t+\alpha-1, \delta_0(t+\alpha-1)), t \in \mathbb{N}_0^{T+1}$

Therefore, from Theorem 19, there exist two monotone sequences $\{\gamma_n\}$ and $\{\delta_n\}$, with $\gamma_0 = \gamma$ and $\delta_0 = \delta$, that converge, respectively, to the minimal and maximal solutions on $[\gamma, \delta]$ of Problem (1).

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