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RESEARCH ARTICLE

Poisson–Cournot games

Francesco De Sinopoli[1] · **Christopher Künstler**[2] · **Claudia Meroni**[3] · **Carlos Pimienta**[4]

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Abstract

We construct a Cournot model with uncertainty in the number of firms in the industry. We model such an uncertainty as a Poisson game and characterize the set of equilibria after deriving novel properties of the Poisson distribution. When marginal costs are zero, the number of equilibria increases with the expected number of firms (n) and for $n \geq 3$ every equilibrium exhibits overproduction relative to the model with deterministic population size. For a fixed n, overproduction is robust to sufficiently small marginal costs. The set of equilibria can be Pareto ranked. If $n \geq 3$, even the expected consumer surplus induced by the lowest quantity equilibrium is larger than the consumer surplus in the model without population uncertainty.

Keywords Cournot competition · Population uncertainty · Poisson games · Poisson distribution

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✉ Claudia Meroni
claudia.meroni@unimi.it

Francesco De Sinopoli
francesco.desinopoli@univr.it

Christopher Künstler
christopher@dr-kuenstler.de

Carlos Pimienta
c.pimienta@unsw.edu.au

[1] Department of Economics, University of Verona, Verona, Italy

[2] Institute of Energy and Climate Research (IEK-8), FZ Jülich, Jülich, Germany

[3] Department of Economics, Management and Quantitative Methods, University of Milan, Milan, Italy

[4] School of Economics, The University of New South Wales, Sydney, Australia

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1 Introduction

The Cournot competition model (Cournot 1838) has been widely used to study imperfectly competitive industries.[1] The classical complete information version has been extended to account for more realistic scenarios in which competing firms face uncertainty about relevant industry characteristics such as the market demand or production costs (see, e.g., Vives 1984, 2002; Cramton and Palfrey 1990; Lagerlöf 2007; Einy et al 2010; Hurkens 2014). A different and significant source of uncertainty in many industries is the number of competitors. While in some established industries it can be reasonable to assume that there is a fixed number of firms and that their identities are well known, in emerging industries, online industries, unregulated industries, or industries facing some significant regulatory change, it may be more natural to assume that firms have uncertainty about the number of competitors they face at the time they make their strategic decisions. The same is true for industries in which the number of competitors is large or in which competition takes place at a global scale. Firms may have a good understanding about the local competition, yet they may not know the number of competitors they face globally. Some specific examples are high tech start-ups, as well as manufacturing industries such as the steel, cement, glass, and coal industries. In all these cases, firms face *population uncertainty* about their competitors.

Models with population uncertainty (Myerson 1998, 2000; Milchtaich 2004) have already been extensively used to study elections in voting and political economy models.[2] In addition to seeming more realistic than assuming common knowledge about the population size, the quite convenient properties of the Poisson distribution make Poisson games an especially useful framework. In industrial organization, Ritzberger (2009) studies the consequences of assuming population uncertainty (and of modelling it using the Poisson specification) within the Bertrand model of price competition and shows that it can resolve the Bertrand paradox.[3]

We introduce Poisson population uncertainty into the Cournot model of quantity competition. Analogously to many studies in the literature that introduce uncertainty in such a model, we also assume linear demand function and we impose a non-negativity constraint on prices.[4] In the classical Cournot model with downward sloping demand

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[1] See Daughety (1988) for a collection of relevant studies of the topic.

[2] See, e.g., Myerson (2002), Bouton and Castanheira (2012), Bouton (2013), Bouton and Gratton (2015), Hughes (2016) and De Sinopoli and Meroni (2022).

[3] That is, that two firms are enough to obtain the perfectly competitive outcome.

[4] Assuming linear demand in a Cournot model that incorporates uncertainty implies that prices can be negative with strictly positive probability. In our case, even if a firm's individual production might be small, there are large realizations of the population size under which the total quantity supplied is to the right of the production level for which a price equal to zero is needed to clear the market. Under different sources of informational asymmetries, Malueg and Tsutsui (1998); Lagerlöf (2007); Hurkens (2014) show that allowing for negative prices produces results that critically depend on that assumption, even when restricting to equilibria in which prices are positive. On the other hand, Einy et al (2010) show that an equilibrium may not even exist if prices are always restricted to be positive. As we show in Theorem 4.1, this is not an issue in our setting as Poisson-Cournot games always have an equilibrium.

function, firms' optimal choices are strategic substitutes so that a firm's best reply decreases as the total quantity produced by its opponents increases. Introducing population uncertainty induces two economic forces that operate in opposite directions. On one hand, if the expected number of firms is n, a firm that is in the industry expects its number of competitors to be larger than $n-1$, because having been recruited to compete in the industry is evidence in favor of a larger number of competing firms (cf. Myerson 1998, p. 382). Under symmetry, this translates into an increased production level of the competitors, therefore generating an incentive to underproduce relative to the equilibrium quantity without population uncertainty when the number of competitors is exactly $n-1$.

On the other hand, a firm also has the incentive to overproduce relative to the equilibrium quantity without population uncertainty to "bet" on those events in which the number of other firms is low, given that potential losses incurred in the events in which such a number is high are bounded by the fact that prices cannot be negative. Of course, the equilibrium choices that arise from these two opposite forces depend on the shape of the demand function and on how population uncertainty is introduced into the model.

Using Poisson uncertainty together with a linear and non-negative inverse demand function yields a tractable model that provides an intuitive resolution to the interaction between the two economic forces mentioned above. In particular, the *environmental equivalence* property of Poisson games (see Myerson 1998) implies that when the expected number of firms is n, the expected number of opponents for any competing firm is also n, one more than the firm's actual number of opponents in the model with deterministic population size equal to n. When n is small (namely, for $n \leq 2$) so that the probability of $n-1$ or less firms is sufficiently low, the former force is dominant and equilibrium quantities exhibit underproduction relative to the unique equilibrium in the deterministic model.[5] However, as n increases and the probability of facing a too large number of competitors also increases (i.e. the price is zero regardless of the firm's action), the second force becomes dominant and firms' equilibrium quantities are higher than in the deterministic model.[6] In particular, for n large enough every equilibrium is such that firms produce more than twice (and up to four times) as much as the equilibrium quantity when firms know the population size. Interestingly,

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[5] As it is standard, when looking at the classical model with exactly n firms we consider the symmetric equilibrium in which every firm produces the normalized quantity $\frac{1}{n+1}$, which is the only one that is robust to the introduction of an infinitesimal cost and guarantees a positive market price. With slight abuse of terminology, we refer to it as the "unique" equilibrium of that model.

[6] As we show in the Online Appendix (see Table 2 for a summary), if we focus on integer values of n to help the comparison with the Cournot model without population uncertainty, there is underproduction for n equal to 1 and 2. More precisely, for $n = 1$ there is a unique equilibrium in which each firm produces $\frac{2}{5}$, i.e. less than the monopolist's optimal quantity $\frac{1}{2}$. For $n = 2$ there are two equilibria, one in which each firm produces $\frac{5}{16}$, less than the duopolist's equilibrium quantity $\frac{1}{3}$, and one in which every firm produces more, $\frac{3}{8}$. For $n \geq 3$, every individual equilibrium quantity is greater than $\frac{1}{n+1}$.

industries that exhibit uncertainty about the number of competitors mentioned above (steel, cement, glass, and coal) often also exhibit overproduction.[7,8]

For most of the paper we consider the case in which the marginal cost is equal to zero. When firms face positive production costs, results depend on their magnitude relative to the expected number of competing firms. If we fix the expected number of firms to n, outcomes remain quantitatively and qualitatively different from those in the model without population uncertainty whenever marginal costs are sufficiently small.[9] However, not surprisingly, if we fix the marginal cost the ratio between any equilibrium quantity of the Poisson-Cournot model and the equilibrium quantity without population uncertainty converges to 1 as n goes to infinity.

There exist various models of oligopoly that analyze markets in which the number of competitors is not fixed. A vast and diversified literature has focused on entry. Initial studies have dealt with the decision problem of an established firm to delay or preclude entry of potential rivals in the market. Among them, Kamien and Schwartz (1975) extend the standard Cournot setting to allow for possible rival entry and restrict attention to its timing, which is regarded as a random variable. Entrants as rational decision makers are introduced by Milgrom and Roberts (1982a, b), where entry deterrence is analyzed in regimes of incomplete information. Other papers have focused on the properties of equilibria in Cournot oligopoly models with free entry. Novshek (1980) proves that, in competitive markets with a single homogeneous good, if firms are small relative to the market then a Cournot equilibrium with free entry exists and is approximately competitive. Amir and Lambson (2000) consider exogenous entry in the traditional Cournot oligopoly, and study how equilibrium output and profits vary as the number of firms increases under different assumptions on the primitives. More recently, Bernhard and Deschamps (2017) examine a discrete time Cournot dynamic model with probabilistic entry in which the arrival of competitors is governed by a Bernoulli process. Bernhard and Deschamps (2021) extend such a discrete time analysis and incorporate a continuous time version that models stochastic entry using a Poisson arrival process. On the other hand, entry into a Cournot market is treated endogenously by Argenziano and Schmidt-Dengler (2012, 2013), who model it as a preemption game of complete information, where each firm has to decide whether and when to enter the new market.

With the notable exception of Bernhard and Deschamps (2017, 2021), a contrasting difference in most of this literature with respect to our setting is that the number of potential entrants is taken as given and is common knowledge. Moreover, there typically is a time component. Poisson-Cournot games are not dynamic, nevertheless,

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[7] It seems reasonable to assume that, in those industries, the production choice is the most strategically relevant due to the high storage costs.

[8] For example, the *State Information Center* of China reports serious overcapacity and overproduction in many manufacturing industries in China, including those that we listed above (see http://www.sic.gov.cn/News/455/8815.htm). China has been making an effort to reduce overcapacity and overproduction in these industries, including shutting down some small, less productive firms. This entailed compensating workers who became unemployed during the process. The cost is over 100 billion RMB (around 14.3 billion USD).

[9] This is due to the fact that equilibria of the model with zero costs are typically strict, so they are robust to perturbations of any parameter of the model. In particular, they are also robust to small enough perturbations of the inverse demand function.

they could be seen as subgames in a more structured model in which, for instance, the strategic choices of an incumbent affect the mean of the Poisson distribution that determines the number of entrants.

After considering an example that highlights the main incentive implications of adding population uncertainty to Cournot competition in Sect. 2, we fully describe the model in Sect. 3. In Sect. 4 we prove existence of equilibrium and solve the model. Welfare comparisons with respect to the Cournot model without population uncertainty are in Sect. 5. In Sect. 6 we study the case in which marginal costs are strictly positive. Section 7 concludes. Appendix A provides some new results on the Poisson distribution that are needed in Sect. 4. Appendices B and C have, respectively, proofs omitted from Sects. 5 and 6.

2 An example of population uncertainty

Before introducing the Poisson distribution to model population uncertainty in the Cournot setting, we provide an illustrative example with incomplete information. This example introduces pieces of intuition and techniques that will be used to prove some results within the Poisson model. It also illustrates that the qualitative aspect of our results is not due to the specific modelling choice, that is, to the Poisson structure, but that it would be preserved in an incomplete information framework as long as an independence assumption on players' types holds.

Consider a model with a potential pool of firms each of which becomes active with some probability. We examine its equilibria as the set of firms increases. Assume that the inverse demand function is linear and constrained to be non-negative. Hence, after a normalization, it can be expressed as $p(Q) = \max\{0, 1 - Q\}$, where Q is the total produced quantity. To model uncertainty in the actual number of active firms, one can equivalently assume that a potential firm has marginal production cost either equal to zero (i.e. it will be active) or greater than 1 (i.e. it will not be active).[10]

Let each firm's marginal cost be either 0 or $\phi > 1$ with equal probability. Consider the case in which there are two potential firms, so the expected number of active firms is 1. In this case, firm i's profit when it produces q_i and the other firm produces q_j is

$$\frac{1}{2}\max\{0, 1 - q_i\}\, q_i + \frac{1}{2}\max\left\{0, 1 - q_j - q_i\right\} q_i .$$

We look for symmetric equilibria q^* such that $q^* < \frac{1}{2}$, so that both terms of the profit function are positive. Maximizing it with respect to q_i, we obtain firm i's best response function

$$\mathrm{BR}_i(q_j) = \frac{2 - q_j}{4}.$$

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[10] Janssen and Rasmusen (2002) briefly consider a Cournot model with population uncertainty which is modelled in a similar way. However, they do not impose the non-negativity constraint on prices and only consider equilibria in which prices are non-negative.

By symmetry we have $q^* = \frac{2}{5}$, which is an equilibrium since $1 - 2q^* > 0$ and there is no profitable deviation to any other quantity.[11] It can be easily seen that $\frac{1}{2}$ is not an equilibrium, so q^* is the unique symmetric equilibrium of the game. It induces an expected total quantity equal to $\frac{2}{5}$, therefore there is underproduction relative to the complete information case, where the monopolist's optimal quantity is $\frac{1}{2}$. This follows from strategic substitutability, given that the expected number of opponents for an active firm (0.5) is higher than under complete information.

We now illustrate how, as the expected number of active firms increases, the incentive to "bet" on the more profitable events in which there are few other active firms appears and rapidly becomes dominant. In particular, there might be multiple equilibria and, if the number of expected active firms is 3 or higher, every equilibrium exhibits overproduction relative to the corresponding complete information equilibrium.

Thus, let us modify the previous case so that there are four firms and, therefore, the expected number of active firms is 2. Firm i's profit when it produces $q_i \leq 1$ and every other firm produces q_j is

$$\frac{1}{8}(1 - q_i)q_i + \frac{3}{8}\max\left\{0, 1 - q_j - q_i\right\}q_i + \frac{3}{8}\max\left\{0, 1 - 2q_j - q_i\right\}q_i$$
$$+\frac{1}{8}\max\left\{0, 1 - 3q_j - q_i\right\}q_i.$$

To find symmetric equilibria we need to be aware that given an equilibrium candidate some terms in the profit function may be zero, that a deviation to a smaller quantity may render null terms strictly positive, and that a deviation to a larger quantity may render strictly positive terms null. With that in mind and after some work, it is possible to see that the game has exactly two symmetric equilibria in which, respectively, firms produce $\frac{7}{23}$ and $\frac{4}{11}$. In the complete information case with two firms, the equilibrium quantity is $\frac{1}{3}$. Thus, under incomplete information, there is one equilibrium in which firms' production is larger and one equilibrium in which firms' production is smaller than the complete information equilibrium quantity. Underproduction is again due to strategic substitutability, while overproduction comes from each firm ignoring the event in which it has two or more active opponents and makes zero profits, and focusing on maximizing profits in the event it is a monopolist or a duopolist.

If we increase the number of firms to 6 so that the expected number of active firms is 3, there exists a unique symmetric equilibrium in which firms produce $\frac{16}{57}$, which is larger than the complete information equilibrium quantity $\frac{1}{4}$. Overproduction relative to the complete information case persists when the number of firms is 8, so that the expected number of active firms is 4, and multiplicity of equilibria reappears. In this case, there are two symmetric equilibria, $\frac{32}{141}$ and $\frac{29}{107}$, both larger than $\frac{1}{5}$. As the number

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[11] Note that any profitable deviation must be greater than $\frac{3}{5}$ to make the second term in the profit function null, and the best one is $\frac{1}{2}$. However, since $\frac{1}{2} < \frac{3}{5}$, there cannot be any profitable deviation. In fact, for every q we have

$$\frac{1}{2}(1-q)q < \frac{1}{2}\left(1-\frac{1}{2}\right)\frac{1}{2} < \frac{1}{2}\left(1-\frac{1}{2}\right)\frac{1}{2} + \frac{1}{2}\left(1-q^*-\frac{1}{2}\right)\frac{1}{2} < \frac{1}{2}(1-q^*)q^* + \frac{1}{2}(1-2q^*)q^*.$$

of firms n increases, the incomplete information game can be closely approximated by a Poisson game with expected number of firms equal to $\frac{n}{2}$. In general, when p is the probability of being active and n is the cardinality of the set of players, the corresponding game with incomplete information is well approximated by a Poisson game with expected number of players equal to np.

3 The model

In a *Poisson-Cournot game* the number of *firms* in an industry is a Poisson random variable with mean n. Therefore, there are k firms with probability $P_k^n := e^{-n}\frac{n^k}{k!}$ and m or a fewer number of firms with probability $C_m^n := \sum_{k=0}^{m} P_k^n$. All firms are identical and face the same *inverse demand function* $p(Q) := \max\{0, 1 - Q\}$ where Q is the total quantity produced in the market. For the time being, we assume that the *marginal production cost* ϕ equals zero.[12] The strategy space is the set of all positive production quantities $[0, \infty)$.

A firm does not have any further information about the number of opponents. Environmental equivalence then implies that P_k^n is also the probability that a firm attaches to the event that there are k other firms in the market. Hence, if every other firm produces q', the *profit* to a firm that in turn produces q is

$$\pi(q, q' \mid n) := \sum_{k=0}^{\infty} P_k^n \max\left\{0, 1 - kq' - q\right\} q.$$

Definition 3.1 A *Nash equilibrium* of the Poisson-Cournot game is a quantity q^* such that $\pi(q^*, q^* \mid n) \geq \pi(q, q^* \mid n)$ for every other q.[13]

To make the profit maximization problem tractable, instead of working directly with the profit function, for every integer $m \geq 1$ we define the *pseudo-profit* at $m - 1$

$$\tilde{\pi}_{m-1}(q, q' \mid n) := \sum_{k=0}^{m-1} P_k^n (1 - kq' - q)q.$$

If $q, q' \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ and the realization of the number of competitors in the industry is larger than or equal to m, then the price equals zero and the realized profit equals the pseudo-profit at $m - 1$. Therefore, if $q, q' \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ then $\pi(q, q' \mid n) = \pi_{m-1}(q, q' \mid n)$. If the quantity q maximizes the pseudo-profit when every other firm produces q' then we say that q is a *pseudo-best response* against q'. Taking first order conditions to the pseudo-profit, we obtain that such a best response equals

$$\widetilde{\mathrm{BR}}_{m-1}^n(q') := \frac{1}{2} - \frac{1}{2}\frac{\sum_{k=0}^{m-1}\frac{n^k}{k!}k}{\sum_{k=0}^{m-1}\frac{n^k}{k!}}q' = \frac{1}{2} - \frac{1}{2}\frac{nC_{m-2}^n}{C_{m-1}^n}q'.$$

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[12] See Sect. 6 for the analysis with positive marginal cost.

[13] As Myerson (1998) points out, population uncertainty implies that identical firms must be treated symmetrically.

Let M^n_m denote the mean of the Poisson distribution with parameter n truncated at m, that is, conditional on its realization being smaller than or equal to m. We call M^n_m the *conditional mean at* m. Then, the pseudo-best response can be written as

$$\widetilde{\mathrm{BR}}^n_{m-1}(q') := \frac{1}{2} - \frac{1}{2}M^n_{m-1}q'.^{14}$$

Suppose $q^* \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ is an equilibrium of the Poisson-Cournot model, then it must be equal to

$$q^* = \frac{1}{M^n_{m-1}+2}. \tag{3.1}$$

However, a quantity $\tilde{q}$ may equal (3.1) and still not be an equilibrium. A necessary (but, again, still not sufficient) condition is $\frac{1}{m+1} \leq \tilde{q} < \frac{1}{m}$. Since $M^n_{m-1} \leq m-1$ is obviously true, we always have $\tilde{q} \geq \frac{1}{m+1}$. However, $\tilde{q} < \frac{1}{m}$ if and only if $M^n_{m-1} > m-2$. If $M^n_{m-1} \leq m-2$ then there exists no equilibrium in the interval $\left[\frac{1}{m+1}, \frac{1}{m}\right)$. If otherwise $M^n_{m-1} > m-2$, we say that $\tilde{q}$ is a *pseudo-equilibrium*. A pseudo-equilibrium may or may not be an equilibrium but every equilibrium is a pseudo-equilibrium.

Thus, the non-negativity constraint on prices implies that, for any production level of the competitors, there is an upper bound on how many competitors can be active in the market and prices still be positive. A profit maximizing firm ignores sufficiently high realizations of the Poisson distribution of competitors and (given the linearity of profits and risk neutrality) optimizes with respect to the expected number of competitors under such a truncation.

From Corollary A.1, we have $M^n_m - M^n_{m-1} < 1$. From Corollary A.2, we know that $M^n_m > m-1$ implies $M^n_{m-1} > m-2$. Therefore, the set of pseudo-equilibria is characterized by the unique integer $\bar{m}$ that satisfies $M^n_{\bar{m}} \leq \bar{m}-1$ and $M^n_{\bar{m}-1} > \bar{m}-2$. Moreover, there are exactly $\bar{m}$ pseudo-equilibria: the minimum pseudo-equilibrium quantity $\frac{1}{M^n_{\bar{m}-1}+2}$ belongs to the interval $\left[\frac{1}{\bar{m}+1}, \frac{1}{\bar{m}}\right)$ and, for each strictly positive integer $m < \bar{m}$, there is one pseudo-equilibrium in the interval $\left[\frac{1}{m+1}, \frac{1}{m}\right)$. Note that the largest pseudo-equilibrium quantity is given by $\frac{1}{M^n_0+2} = \frac{1}{2}$, while the smallest pseudo-equilibrium quantity is characterized by the value of $\bar{m}$ bounded in the following theorem.

Theorem 3.1 *The unique integer* $\bar{m}$ *satisfying both* $M^n_{\bar{m}-1} > \bar{m}-2$ *and* $M^n_{\bar{m}} \leq \bar{m}-1$ *obeys the double inequality* $\frac{n}{2}+1 < \bar{m} < \frac{n}{2}+3$.

Proof Follows directly from Propositions A.2 and A.3 in Appendix A. □

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[14] Recall that, in the Cournot model without population uncertainty and n firms, we have $\mathrm{BR}(q') = \frac{1}{2} - \frac{1}{2}(n-1)q'$. Note as well that the analogous expression with population uncertainty but without the non-negativity constraint on prices is $\frac{1}{2} - \frac{1}{2}nq'$. In that case, there is a unique symmetric equilibrium, $q^* = \frac{1}{n+2}$, which exhibits underproduction for every n because of the sole effect of environmental equivalence.

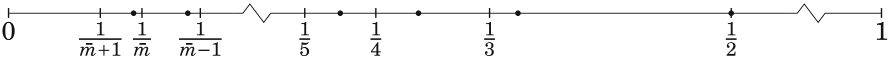

Fig. 1 Graphical representation of pseudo-equilibria

Table 1 Pseudo-equilibria for small values of n

Interval	Pseudo-equilibrium	Pseudo-equilibrium for
$\left[\frac{1}{2}, 1\right)$	$\frac{1}{2}$	$n > 0$
$\left[\frac{1}{3}, \frac{1}{2}\right)$	$\frac{1}{M_1^n+2}$	$n > 0$
$\left[\frac{1}{4}, \frac{1}{3}\right)$	$\frac{1}{M_2^n+2}$	$n > 1.41$
$\left[\frac{1}{5}, \frac{1}{4}\right)$	$\frac{1}{M_3^n+2}$	$n > 3.14$
$\left[\frac{1}{6}, \frac{1}{5}\right)$	$\frac{1}{M_4^n+2}$	$n > 4.96$
$\left[\frac{1}{7}, \frac{1}{6}\right)$	$\frac{1}{M_5^n+2}$	$n > 6.84$
$\left[\frac{1}{8}, \frac{1}{7}\right)$	$\frac{1}{M_6^n+2}$	$n > 8.75$
$\left[\frac{1}{9}, \frac{1}{8}\right)$	$\frac{1}{M_7^n+2}$	$n > 10.68$
$\left[\frac{1}{10}, \frac{1}{9}\right)$	$\frac{1}{M_8^n+2}$	$n > 12.62$

Figure 1 shows how pseudo-equilibria (represented by dots in the figure) are typically placed within their corresponding intervals. (Note that, if n is small, we can have $\bar{m} \leq 5$.) Table 1 displays, for each interval $\left[\frac{1}{m+1}, \frac{1}{m}\right)$ with $m \leq 9$, the values of n for which there is a pseudo-equilibrium in that interval so that we have $M_{m-1}^n > m-2$.[15] Furthermore, in the Online Appendix, we compute and provide analytical expressions for pseudo-equilibria and equilibria when n is small.

Every strictly positive integer smaller than $\bar{m}$ is associated with a pseudo-equilibrium. Since every equilibrium must be a pseudo-equilibrium, it follows from Theorem 3.1 that every equilibrium quantity will be strictly greater than $\frac{1}{\bar{m}+1} = \frac{2}{n+8}$. This implies that every individual equilibrium quantity will be larger than that of the model with deterministic population size, $\frac{1}{n+1}$, whenever $n \geq 6$, and will converge to at least its double as n grows to infinity. Therefore, loosely speaking, population uncertainty in the standard Cournot model induces a faster convergence to perfect competition. We formally establish such a result in Sect. 5, but first we need to show that an equilibrium always exists.

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[15] Values of n are rounded to two decimal places.

4 Equilibrium existence and characterization

We show existence of Nash equilibrium through a constructive proof that exploits the specific structure of the problem.[16] Of course, a pseudo-equilibrium is an equilibrium if there is no profitable deviation to any other quantity. To show that at least one pseudo-equilibrium is an equilibrium, we apply the following steps. First, for a given pseudo-equilibrium, we characterize the best possible deviation to a lower quantity and the best possible deviation to a higher quantity, and we show that they cannot be both profitable simultaneously. Second, we consider two pseudo-equilibria living in contiguous intervals and show that, if neither is an equilibrium, it is either because in both cases deviating to a higher quantity is profitable, or because in both cases deviating to a lower quantity is profitable. Finally, we show that from the smallest pseudo-equilibrium it is never profitable to deviate to a lower quantity. Hence, if the smallest pseudo-equilibrium quantity is not an equilibrium because deviating to a larger quantity is profitable, then either the second smallest pseudo-equilibrium quantity is an equilibrium or deviating to a higher quantity is also profitable. The same is true for any subsequent pseudo-equilibrium. Thus, an equilibrium always exists.

From the previous section, we know that there are $\bar{m}$ pseudo-equilibria. Each pseudo-equilibrium quantity $\tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ is the unique maximizer of the pseudo-profit $\tilde{\pi}_{m-1}(\cdot, \tilde{q} \mid n)$ but is not an equilibrium unless it also maximizes the profit function $\pi(\cdot, \tilde{q} \mid n)$. Indeed, when $q, \tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$, then $\tilde{\pi}_{m-1}(q, \tilde{q} \mid n)$ coincides with $\pi(q, \tilde{q} \mid n)$. However, when $q < 1 - m\tilde{q}$ or $q > 1 - (m-1)\tilde{q}$ then the true profit and the pseudo-profit differ. In the first case, a firm producing q can face up to m competitors producing $\tilde{q}$ without prices vanishing, so that $\pi(q, \tilde{q} \mid n)$ has one additional positive term (the one corresponding to $k = m$) that in the pseudo-profit is zero.[17] In the second case, one or more terms of the pseudo-profit are negative, while in the real profit they are zero. In particular, there is a largest integer i such that $q > 1 - (m - i)\tilde{q}$. For that value of i, a firm producing q can only face up to $m - i - 1$ competitors producing $\tilde{q}$ and prices still be positive, so that the last i terms of the pseudo-profit are negative, while in the real profit they are zero.

Thus, consider a pseudo-equilibrium $\tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$. If there is a profitable deviation from $\tilde{q}$ to some lower quantity $q < 1 - m\tilde{q}$, then the best of such deviations $\underline{q}$ solves

$$\max_{q} \tilde{\pi}_m(q, \tilde{q} \mid n) = \max_{q} \sum_{k=0}^{m} P_k^n (1 - k\tilde{q} - q)q,$$

and equals

$$\underline{q} = \frac{1}{2} - \frac{1}{2} M_m^n \tilde{q}.$$

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[16] The profit function is not quasi-concave, so standard topological methods cannot be used to prove existence.

[17] But if all competitors produce $\tilde{q}$, a firm cannot face $k > m$ competitors without prices falling to zero.

Note that $\underline{q}$ is not necessarily in the interval $\left[\frac{1}{m+2}, \frac{1}{m+1}\right)$ because it may also be smaller than $\frac{1}{m+2}$. On the other hand, if there is a profitable deviation from $\tilde{q}$ to some higher quantity $q > 1 - (m-1)\tilde{q}$ then the best of such possible deviations is of the form

$$\bar{q} = \frac{1}{2} - \frac{1}{2} M^n_{m-i} \tilde{q},$$

for some $i \geq 2$. We show that the best one is, in fact, $\bar{q} = \frac{1}{2} - \frac{1}{2} M^n_{m-2} \tilde{q}$.

Lemma 4.1 *Let $\tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ be a pseudo-equilibrium and let $m > 3$. Then, the best possible deviation to a higher quantity is $\bar{q} = \frac{1}{2} - \frac{1}{2} M^n_{m-2} \tilde{q}$.*

Proof Since $\tilde{q}$ is a pseudo-equilibrium, Corollary A.2 implies $M^n_{m-i} > m - i - 1$ for every $i \geq 1$. Consider quantity $\hat{q} = \frac{1}{2} - \frac{1}{2} M^n_{m-j} \tilde{q}$ with $j \geq 3$ and suppose it yields a higher expected profit than $\bar{q} = \frac{1}{2} - \frac{1}{2} M^n_{m-j+1} \tilde{q}$. Since quantity $\bar{q}$ is the maximizer of the pseudo-profit $\tilde{\pi}_{m-j+1}(\cdot, \tilde{q} \mid n)$ and $\hat{q}$ yields a higher expected profit, the latter must be maximizing a different pseudo-profit, hence, $\hat{q} > 1 - (m - j + 1)\tilde{q}$. Keeping in mind that $M^n_{m-j} > m - j - 1$, we have

$$\begin{aligned} \hat{q} = \frac{1}{2} - \frac{1}{2} M^n_{m-j} \tilde{q} &> 1 - (m - j + 1)\tilde{q} \\ \tilde{q} &> \frac{1}{2(m-j+1) - M^n_{m-j}} > \frac{1}{m-j+3} \geq \frac{1}{m}, \end{aligned}$$

but this contradicts $\tilde{q} < \frac{1}{m}$ so that $\hat{q}$ is a worse response than $\bar{q}$ against $\tilde{q}$. □

It follows that a pseudo-equilibrium $\tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ is an equilibrium if neither the higher quantity $\bar{q} = \frac{1}{2} - \frac{1}{2} M^n_{m-2} \tilde{q}$ nor the lower quantity $\underline{q} = \frac{1}{2} - \frac{1}{2} M^n_m \tilde{q}$ yield strictly higher expected profits to the deviating firm than $\tilde{q}$. These two deviations cannot be both profitable at the same time.

Lemma 4.2 *Let $\tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ be a pseudo-equilibrium. If there is a profitable deviation to the higher quantity $\bar{q}$ then there cannot be a profitable deviation to the lower quantity $\underline{q}$ and vice versa.*

Proof Recall that if $\bar{q}$ is a profitable deviation we must have $\bar{q} > 1 - (m-1)\tilde{q}$. Similarly, if $\underline{q}$ is a profitable deviation then $\underline{q} < 1 - m\tilde{q}$. Using the expressions for $\bar{q}$ and $\underline{q}$ and rearranging we obtain the inequalities

$$\tilde{q}\left(m - 1 - \frac{1}{2} M^n_{m-2}\right) > \frac{1}{2} \quad \text{and} \quad \tilde{q}\left(m - \frac{1}{2} M^n_m\right) < \frac{1}{2}.$$

Corollary A.1 implies $\frac{1}{2} M^n_m < 1 + \frac{1}{2} M^n_{m-2}$ so that, in turn, $\bar{q} > 1 - (m-1)\tilde{q}$ implies $\underline{q} > 1 - m\tilde{q}$ and $\underline{q} < 1 - m\tilde{q}$ implies $\bar{q} < 1 - (m-1)\tilde{q}$. □

We now prove that, given two pseudo-equilibria in adjacent intervals that are not equilibria, there are only two options. Either they both have a profitable deviation to a lower quantity or they both have a profitable deviation to a larger quantity.

Lemma 4.3 *Let* $\tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ *and* $\hat{q} \in \left[\frac{1}{m+2}, \frac{1}{m+1}\right)$ *be two pseudo-equilibria. If there is a profitable deviation from* $\hat{q}$ *to a higher quantity* $\bar{q}$ *then there cannot be a profitable deviation from* $\tilde{q}$ *to a lower quantity* $\underline{q}$*, and vice versa.*

Proof Substituting $\bar{q}$ and $\hat{q}$ by their corresponding values in $\bar{q} > 1 - m\hat{q}$, multiplying across by $M_m^n + 2$ and rearranging, we obtain

$$m - 1 - \frac{1}{2}M_m^n - \frac{1}{2}M_{m-1}^n > 0.$$

Similarly, substituting $\underline{q}$ and $\tilde{q}$ by their corresponding values in $\underline{q} < 1 - m\tilde{q}$, multiplying across by $M_{m-1}^n + 2$ and rearranging, we obtain

$$m - 1 - \frac{1}{2}M_m^n - \frac{1}{2}M_{m-1}^n < 0.$$

Since the two inequalities contradict each other, the result follows. □

As shown in the Online Appendix, in the largest pseudo-equilibrium quantity $\frac{1}{2}$ it is always profitable to deviate to a smaller quantity for every n. Moreover, given a pseudo-equilibrium $\tilde{q}$, the necessary condition $\bar{q} > 1 - (m-1)\tilde{q}$ for the deviation to the higher quantity $\bar{q}$ to be profitable can be rewritten as $m - 2 > \frac{1}{2}(M_{m-1}^n + M_{m-2}^n)$, which is never satisfied for $m = 1, 2$. We establish that an equilibrium always exists showing that from the smallest pseudo-equilibrium quantity deviating to a smaller quantity is never profitable.

Theorem 4.1 (Existence) *There is at least one equilibrium.*

Proof We show that if $\tilde{q} \in \left[\frac{1}{\bar{m}+1}, \frac{1}{\bar{m}}\right)$ is the smallest pseudo-equilibrium quantity then deviating to a smaller quantity is not profitable. To the contrary, suppose $\underline{q} = \frac{1}{2} - \frac{1}{2}M_{\bar{m}}^n\tilde{q}$ is a profitable deviation. Remembering that $M_{\bar{m}}^n \leq \bar{m} - 1$ then we must have

$$\begin{aligned} 1 - \bar{m}\tilde{q} &> \underline{q} = \frac{1}{2} - \frac{1}{2}M_{\bar{m}}^n\tilde{q} \geq \frac{1}{2} - \frac{1}{2}(\bar{m} - 1)\tilde{q} \\ 1 - 2\bar{m}\tilde{q} &> -(\bar{m} - 1)\tilde{q} \\ \tilde{q} &< \frac{1}{\bar{m} + 1} \end{aligned}$$

which is impossible.[18] □

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[18] An alternative proof of existence can be constructed using lattice-theoretic methods (see Amir 1996 for a complete information framework). While any selection of the best response correspondence has countably

We now turn to describing the set of equilibria. To do that, we compute the expected profit at a pseudo-equilibrium and if a firm deviates to a lower or a higher quantity. The first expected profit is

$$\begin{aligned}\pi(\tilde{q},\tilde{q}\mid n) &= \tilde{q}\left[(1-\tilde{q})\sum_{k=0}^{m-1}P_k^n - \tilde{q}\sum_{k=0}^{m-1}P_k^n k\right]\\ &= \tilde{q}\left[(1-\tilde{q})C_{m-1}^n - \tilde{q}\sum_{k=0}^{m-1}P_k^n k\right] = \tilde{q}\left[1-\tilde{q}-\tilde{q}M_{m-1}^n\right]C_{m-1}^n.\end{aligned}$$

Similarly, the expected profits from deviations $\bar{q}$ and $\underline{q}$ are

$$\begin{aligned}\pi(\bar{q},\tilde{q}\mid n) &= \bar{q}\left[1-\bar{q}-\tilde{q}M_{m-2}^n\right]C_{m-2}^n,\\ \pi(\underline{q},\tilde{q}\mid n) &= \underline{q}\left[1-\underline{q}-\tilde{q}M_{m}^n\right]C_{m}^n.\end{aligned}$$

We also have the equalities

$$\tilde{q} = \frac{1}{2}-\frac{1}{2}M_{m-1}^n\tilde{q},\qquad \bar{q} = \frac{1}{2}-\frac{1}{2}M_{m-2}^n\tilde{q},\qquad \underline{q} = \frac{1}{2}-\frac{1}{2}M_{m}^n\tilde{q},$$

which we substitute in the expressions above to obtain

$$\pi(\tilde{q},\tilde{q}\mid n) = \tilde{q}^2C_{m-1}^n,\qquad \pi(\bar{q},\tilde{q}\mid n) = \bar{q}^2C_{m-2}^n,\qquad \pi(\underline{q},\tilde{q}\mid n) = \underline{q}^2C_{m}^n.$$[19]

In equilibrium, the first one of these three profit values must be larger than the other two. We use such an observation to find a necessary condition on m such that the pseudo-equilibrium that lives in $\left[\frac{1}{m+1},\frac{1}{m}\right)$ is an equilibrium.

Proposition 4.1 *If* $q^*\in\left[\frac{1}{m+1},\frac{1}{m}\right)$ *is an equilibrium then*

$$\frac{n}{4}+\frac{1}{4} < m < \left(\frac{4}{9}n+\frac{8}{9}\right)\left(\frac{3n+12}{3n+4}\right)^2+1.$$

Proof We first show that $m\le\frac{n}{4}+\frac{1}{4}$ implies that the associated pseudo-equilibrium $\tilde{q}$ satisfies $\underline{q}^2C_m^n > \tilde{q}^2C_{m-1}^n$. On one hand, we use the upper bound $M_m^n - M_{m-1}^n < 1$

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many discontinuity points, some results obtained in this section can also be used to prove that all its jumps are upwards. Hence, any such selection is a *quasi-increasing* function so that the Tarski's intersection point theorem (see, e.g., Theorem 3 in Vives 2018) implies that it intersects the 45 degree line at least once. Features of the jumps of the best response correspondence have been exploited to prove existence of a Cournot equilibrium also by McManus (1964) for the case of identical firms and by Novshek (1985) for asymmetric ones. We chose to provide a constructive proof as it highlights some economic insights relevant to the model.

[19] Recall that in the model with deterministic population size, $q=\frac{1}{n+1}$ is the individual equilibrium quantity and equilibrium profits are given by q^2.

in Corollary A.1 to obtain

$$\left(\frac{q}{\tilde{q}}\right)^2 = \left(\frac{1}{2\tilde{q}} - \frac{1}{2}M_m^n\right)^2 = \left(1 - \frac{1}{2}(M_m^n - M_{m-1}^n)\right)^2 > \frac{1}{4}.$$

On the other hand, the upper bound $M_m^n \leq \frac{nm}{n+1}$ from Lemma A.2 combined with the the upper bound on m implies

$$\frac{C_{m-1}^n}{C_m^n} = \frac{1}{n}M_m^n \leq \frac{m}{n+1} \leq \frac{1}{4}.$$

We now show that $m \geq \left(\frac{4}{9}n + \frac{8}{9}\right)\left(\frac{3n+12}{3n+4}\right)^2 + 1$ implies $\bar{q}^2 C_{m-2}^n > \tilde{q}^2 C_{m-1}^n$. From Lemma A.3, $M_{m-1}^n - M_{m-2}^n > \frac{n-4}{n+4}$, hence

$$\left(\frac{\bar{q}}{\tilde{q}}\right)^2 = \left(\frac{1}{2\tilde{q}} - \frac{1}{2}M_{m-2}^n\right)^2 = \left(1 + \frac{1}{2}(M_{m-1}^n - M_{m-2}^n)\right)^2 > \left(\frac{3n+4}{2(n+4)}\right)^2.$$

While the lower bound on m together with the lower bound on $M_{m-1}^n > \frac{n}{n+2}(m-1)$ in Eq. (A.4) imply

$$\frac{C_{m-1}^n}{C_{m-2}^n} = \frac{n}{M_{m-1}^n} < \frac{n+2}{m-1} \leq \left(\frac{3n+4}{2(n+4)}\right)^2$$

and establish the desired result. □

Hence, equilibria of the Poisson-Cournot model are, for n sufficiently high, between two and four times as large as the equilibrium quantity without population uncertainty. We note that the upper bound for m in Proposition 4.1 is more efficient than the bound $\frac{n}{2} + 3$ found in Theorem 3.1 for pseudo-equilibria only for $n > 25.86$.

We now derive a sufficient condition which ensures that, for every value of m between two given thresholds, the interval $\left[\frac{1}{m+1}, \frac{1}{m}\right)$ contains an equilibrium. The lower threshold guarantees that there is no profitable deviation from the pseudo-equilibrium in that interval to a smaller quantity, while the higher one rules out a profitable deviation to a larger quantity.

Proposition 4.2 *If* $\tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ *is a pseudo-equilibrium such that*

$$\left(\frac{n}{4} + \frac{1}{2}\right)\left(\frac{n+12}{n+4}\right)^2 \leq m \leq \frac{4}{9}n + \frac{13}{9}$$

then $\tilde{q}$ *is an equilibrium.*

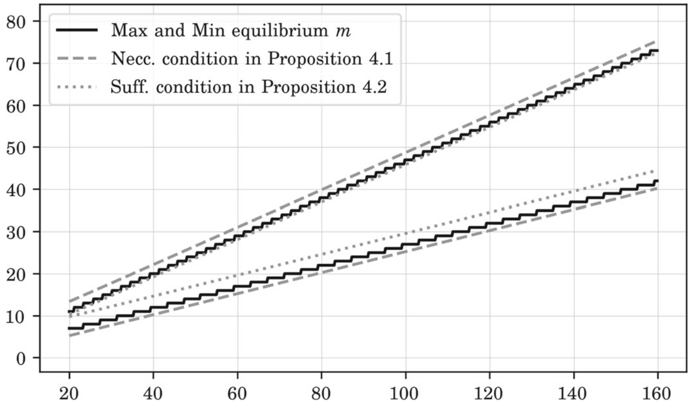

Fig. 2 Bounds for m found in Propositions 4.1 and 4.2 given $20 \leq n \leq 160$

Proof We first show that if the first inequality holds then $\tilde{q}^2 C^n_{m-1} \geq \underline{q}^2 C^n_m$. The bound $M^n_m - M^n_{m-1} > \frac{n-4}{n+4}$ found in Lemma A.3 implies

$$\left(\frac{\underline{q}}{\tilde{q}}\right)^2 = \left(1 - \frac{1}{2}(M^n_m - M^n_{m-1})\right)^2 < \left(\frac{n+12}{2(n+4)}\right)^2.$$

On the other hand, $m \geq \left(\frac{n}{4} + \frac{1}{2}\right)\left(\frac{n+12}{n+4}\right)^2$ and the lower bound on M^n_m in (A.4) imply

$$\frac{C^n_{m-1}}{C^n_m} = \frac{1}{n} M^n_m > \frac{m}{n+2} \geq \left(\frac{n+12}{2(n+4)}\right)^2.$$

It remains to show that the second inequality implies $\tilde{q}^2 C^n_{m-1} \geq \bar{q}^2 C^n_{m-2}$. Using inequality $M^n_{m-1} - M^n_{m-2} < 1$ in Corollary A.1 we have

$$\left(\frac{\bar{q}}{\tilde{q}}\right)^2 = \left(1 + \frac{1}{2}(M^n_{m-1} - M^n_{m-2})\right)^2 < \frac{9}{4}.$$

While the second inequality and the bound $M^n_m \leq \frac{n}{n+1} m$ in Lemma A.2 imply

$$\frac{C^n_{m-1}}{C^n_{m-2}} = \frac{n}{M^n_{m-1}} \geq \frac{n+1}{m-1} \geq \frac{9}{4}$$

as we wanted. □

We remark that the double inequality in the last proposition can only be satisfied for $n \geq 17.29$.

Figure 2 plots the bounds for m found in Proposition 4.1 and Proposition 4.2 for values of n between 20 and 160 together with numerical computation of the maximum

Table 2 Equilibria for small values of n

Interval	Equilibrium quantity	Equilibrium for
$\left[\frac{1}{3}, \frac{1}{2}\right)$	$\frac{1}{M_1^n+2}$	$0 < n \leq 3.61$
$\left[\frac{1}{4}, \frac{1}{3}\right)$	$\frac{1}{M_2^n+2}$	$1.69 \leq n \leq 7.46$
$\left[\frac{1}{5}, \frac{1}{4}\right)$	$\frac{1}{M_3^n+2}$	$3.69 \leq n \leq 11.39$
$\left[\frac{1}{6}, \frac{1}{5}\right)$	$\frac{1}{M_4^n+2}$	$5.79 \leq n \leq 15.33$
$\left[\frac{1}{7}, \frac{1}{6}\right)$	$\frac{1}{M_5^n+2}$	$7.93 \leq n \leq 19.3$
$\left[\frac{1}{8}, \frac{1}{7}\right)$	$\frac{1}{M_6^n+2}$	$10.11 \leq n \leq 23.27$
$\left[\frac{1}{9}, \frac{1}{8}\right)$	$\frac{1}{M_7^n+2}$	$12.29 \leq n \leq 27.26$
$\left[\frac{1}{10}, \frac{1}{9}\right)$	$\frac{1}{M_8^n+2}$	$14.5 \leq n \leq 31.24$

and minimum values of m for which there is an equilibrium in $\left[\frac{1}{m+1}, \frac{1}{m}\right)$. Table 2 provides the results of the analytical computations of equilibria for smaller values of n that are offered in the Online Appendix.[20] Focusing on integer values of n, we can see that from $n = 3$ every equilibrium of the Poisson-Cournot model exhibits overproduction relative to the model with deterministic population size. From $n = 40$, in every equilibrium, firms produce more than twice the equilibrium quantity of the deterministic case. The multiplicity of equilibria is more pervasive as n increases. It arises from the different production levels that firms can coordinate on, each production level being associated with the number of firms that can operate in the market and prices still be positive.

Proposition A.5 in Appendix A works out tighter bounds than those in Proposition 4.1 and Proposition 4.2 (cf. Fig. 3) which are, in turn, based on a tighter bound for $M_m^n - M_{m-1}^n$. Proposition A.5 implies that, as n goes to infinity, the expected total equilibrium quantity converges to at least 2.25 (and to at most 4). See Appendix A for more details.

5 Welfare

5.1 Profits

The set of equilibria can be Pareto ranked from the firms' viewpoint. Given any equilibrium, the individual profit to a firm increases under any other equilibrium associated

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[20] Formulae provided in the Online Appendix allow to compute equilibria for all values of $n \leq 14$. For instance, as mentioned in the Introduction, if $n = 2$ the equilibrium quantities are $\frac{3}{8}$ and $\frac{5}{16}$, one larger and one smaller than the equilibrium quantity $\frac{1}{3}$ in the deterministic case. For $n = 5$, the equilibrium quantities
Footnote 20 continued
are $\frac{37}{134} \approx 0.276$ and $\frac{236}{1027} \approx 0.23$, both larger than $\frac{1}{6} \approx 0.167$. For $n = 10$, the equilibrium quantities are $\frac{683}{3196} \approx 0.214$, $\frac{1933}{10696} \approx 0.181$, and $\frac{4433}{28196} \approx 0.157$, all larger than $\frac{1}{11} \approx 0.091$.

with a smaller quantity. The smaller production is compensated by the higher price for any given realization of the number of firms and, additionally, by the larger probability that prices remain strictly positive. For any m, let $\tilde{q}_{m-1} = \frac{1}{M^n_{m-1}+2}$. We have the following result.

Proposition 5.1 *Let $\tilde{q}_m$ and $\tilde{q}_{m-1}$ be two equilibria. Then $\pi(\tilde{q}_{m-1}, \tilde{q}_{m-1} \mid n) < \pi(\tilde{q}_m, \tilde{q}_m \mid n)$ for every n.*

Proof Since $\tilde{q}_m < \tilde{q}_{m-1}$ and both are equilibria, we have

$$\pi(\tilde{q}_{m-1}, \tilde{q}_{m-1} \mid n) = \sum_{k=0}^{m-1} P^n_k (1 - k\tilde{q}_{m-1} - \tilde{q}_{m-1})\tilde{q}_{m-1} <$$

$$\sum_{k=0}^{m-1} P^n_k (1 - k\tilde{q}_m - \tilde{q}_{m-1})\tilde{q}_{m-1} \le \pi(\tilde{q}_{m-1}, \tilde{q}_m \mid n) < \pi(\tilde{q}_m, \tilde{q}_m \mid n).$$

□

Nonetheless, even under the lowest equilibrium quantity, profits are still lower than in the unique equilibrium quantity if there is no population uncertainty. Thus, we claim the following result, which is proven in Appendix B.

Claim 1 Let $\tilde{q}_{m-1}$ be an equilibrium. Then $\pi(\tilde{q}_{m-1}, \tilde{q}_{m-1} \mid n) < \frac{1}{(n+1)^2}$ *for every n.*

Recall that $\bar{m}$ is the integer associated with the smallest pseudo-equilibrium quantity, so every equilibrium is larger than $\tilde{q}_{\bar{m}-1}$. In Appendix B we show that, for n large enough,

$$\tilde{q}^2_{\bar{m}-1} C^n_{\bar{m}-1} < \frac{1}{(n+1)^2}.$$

The result follows from the fact that for $m < n$ the value of C^n_m converges exponentially to zero as n increases, i.e. faster than $\frac{1}{n^2}$.

5.2 Consumer surplus

Not surprisingly, results about consumer surplus move in the opposite direction. Obviously, consumers always prefer equilibria with larger quantities as the probability distribution over prices induced by any quantity is first order stochastically dominated by the corresponding distribution induced by a smaller quantity. Moreover, even in the lowest quantity equilibrium of the Poisson-Cournot game, the consumer surplus is bigger than in the unique equilibrium of the Cournot model without population uncertainty.

When firms have common knowledge about the total number of firms n in the industry, the loss in consumer surplus relative to perfect competition equals

$$\frac{2n+1}{2(n+1)^2}.$$

In the Poisson-Cournot model, when firms produce the equilibrium quantity $\tilde{q}_{m-1} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ then, if no firm is realized, the loss in consumer surplus equals $\frac{1}{2}$; if there is 1 firm, the loss equals $\frac{1-\tilde{q}_{m-1}^2}{2}$; if there are 2 firms, the loss equals $\frac{1-4\tilde{q}_{m-1}^2}{2}$. In general, if there are $k \leq m$ firms, the loss in consumer surplus is equal to $\frac{1-k^2\tilde{q}_{m-1}^2}{2}$.[21] So, the expected loss in consumer surplus is

$$\frac{1}{2}C_m^n - \frac{1}{2}\tilde{q}_{m-1}^2 \sum_{k=0}^{m} P_k^n k^2 = \frac{1}{2}C_m^n - \frac{1}{2}\tilde{q}_{m-1}^2 (nC_{m-1}^n + n^2 C_{m-2}^n).$$

Focusing on integer values of n for a meaningful comparison with the Cournot model with deterministic population size, every equilibrium under population uncertainty exhibits overproduction when $n \geq 3$. Correspondingly, we make the following claim.

Claim 2 Let $\tilde{q}_{m-1}$ be an equilibrium. Then, for $n \geq 3$,

$$C_m^n - \tilde{q}_{m-1}^2 (nC_{m-1}^n + n^2 C_{m-2}^n) < \frac{2n+1}{(n+1)^2}.$$

Therefore, for sufficiently large n, consumer surplus under population uncertainty is always closer to the perfect competition value $\frac{1}{2}$ than consumer surplus in the deterministic model with population size equal to n. While we skip the proof to the previous claim, we note that this result can be demonstrated when $\tilde{q}_{m-1}$ is the smallest equilibrium quantity using a similar argument as in the proof of Claim 1. Furthermore, also using similar arguments, one can show that the expected total surplus of the Poisson-Cournot model is higher than the total surplus in the standard Cournot model, at least, for sufficiently high n.

6 Positive production costs

We relax the assumption of zero marginal cost and discuss the robustness of the results in the previous sections. The effect of marginal costs on outcomes depends on their magnitude relative to the expected number of firms n. First, we consider a given economy n and show that the main qualitative results remain valid if costs are sufficiently small. Then, we consider a fixed marginal cost ϕ and examine the stability of results letting n vary.

Given marginal cost $\phi > 0$, the *profit* to a firm that produces q when every other firm produces q' is given by

$$\pi(q, q' \mid n, \phi) := \sum_{k=0}^{\infty} P_k^n \max\left\{0, 1 - kq' - q\right\} q - \phi q = \pi(q, q' \mid n) - \phi q.$$

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[21] If $k > m$ the price is zero and so is the loss in consumer surplus.

The pseudo-profit, pseudo-best response, and pseudo-equilibrium can be defined analogously to Sect. 3. We provide the following essential result, whose proof follows the same arguments as the case $\phi = 0$ and can be found in Appendix C.

Theorem 6.1 *For every marginal cost $\phi > 0$ there is at least one equilibrium.*

If we fix the expected number of firms n, equilibria of the Poisson-Cournot game with positive marginal cost exhibit overproduction with respect to the Cournot model with exactly n firms, at least if such a marginal cost is sufficiently small. If an equilibrium is strict (as is typically the case), it is robust to every sufficiently small perturbation of the parameters of the model, including the marginal cost. Thus, let q^* be a non-strict equilibrium when $\phi = 0$ and let q_ϕ be the close-by pseudo-equilibrium when $\phi > 0$.[22] If a deviation from q^* to a higher quantity leads to the same profit level then, by continuity, any deviation from q_ϕ to a higher quantity leads to a strictly smaller profit level if ϕ is sufficiently small. The only event in which q_ϕ would not be an equilibrium is when a deviation from q^* to a smaller quantity leads to the same profit level. However, the same argument as the one used in the proof of Theorem 4.1 implies that this cannot be the case when q^* is the smallest pseudo-equilibrium quantity.[23] It follows that, if ϕ is sufficiently small, every equilibrium quantity is greater than the smallest pseudo-equilibrium quantity when $\phi = 0$.

When the marginal cost is substantial, firms can no longer ignore the events in which they face a large number of opponents, as in those events they now make negative profits. As a consequence, the incentive to overproduce relative to the deterministic case “betting” on the events in which opponents are few and mark-ups are high is mitigated by the possibility of incurring a substantial loss when competitors are many. Even if such a possibility may be neglected when the expected number of firms is small, it becomes more and more relevant as n increases.

We show that the ratio between any equilibrium quantity of the Poisson-Cournot model and the equilibrium quantity of the standard Cournot model converges to 1 when the expected number of firms goes to infinity. In fact, if each competitor produces a quantity relatively larger than the equilibrium quantity of the deterministic model, a firm producing that same quantity faces negative profits with probability that rapidly approaches 1 as the expected population size increases. Thus, the firm would prefer not to produce to avoid the loss. On the other hand, if each competitor produces a quantity relatively smaller than the equilibrium quantity without population uncertainty, then the total quantity that competitors produce is lower than in the deterministic case with probability that rapidly converges to 1. By strategic substitutability a firm’s best reply is then larger, and hence that quantity cannot be an equilibrium.

Recall that in the Cournot model without population uncertainty, when the total number of firms is n and the marginal cost is $0 < \phi < 1$, a firm’s equilibrium quantity is $q_n^{**}(\phi) = \frac{1-\phi}{n+1}$. Correspondingly, let $q_n^*(\phi)$ be a firm’s production under

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[22] Note that, since the profit function is continuous in ϕ, if ϕ is sufficiently small there is a pseudo-equilibrium close to every pseudo-equilibrium of the model with no costs. Indeed, recall that if $\phi = 0$ there is a quasi-equilibrium in the interval $\left[\frac{1}{m+1}, \frac{1}{m}\right)$ as long as $m - 2 < M_{m-1}^n \le m - 1$ and that the second inequality is always satisfied as a strict inequality.

[23] See also the proof of Lemma C.4.

some equilibrium of the Poisson-Cournot model with expected number of firms n and marginal cost $0 < \phi < 1$.

Proposition 6.1 *For every* $0 < \phi < 1$ *and real number* $\alpha > 1$, *there exists a value* $n_{\phi,\alpha} \in \mathbb{R}_{++}$ *such that if* $n \geq n_{\phi,\alpha}$ *then* $\frac{\alpha-1}{\alpha} q_n^{**}(\phi) \leq q_n^*(\phi) \leq \frac{\alpha+1}{\alpha} q_n^{**}(\phi)$.

Proof Let us write q_n^* and q_n^{**} instead of $q_n^*(\phi)$ and $q_n^{**}(\phi)$. We begin showing that for any $0 < \phi < 1$ and $\alpha > 1$ there exists a value $\check{n}_{\phi,\alpha}$ such that if $n \geq \check{n}_{\phi,\alpha}$ then $q_n^* \leq \frac{\alpha+1}{\alpha} q_n^{**}$.

Suppose to the contrary that $q_n^* > \frac{\alpha+1}{\alpha} q_n^{**}$ for every n, and consider a firm whose opponents all produce q_n^*. If it also produces q_n^* then, in the event in which the number of opponents is larger than $\check{m} = \left\lceil \frac{\alpha}{\alpha+1} n \right\rceil - 1$, the price is lower than

$$1 - \frac{\alpha}{\alpha+1} n q_n^* - q_n^* < 1 - \left(\frac{\alpha}{\alpha+1} n + 1\right)\left(\frac{\alpha+1}{\alpha}\frac{1-\phi}{n+1}\right) = \phi - \frac{1}{\alpha}\frac{1-\phi}{n+1}.$$

If n is sufficiently large this last estimate is positive. So, if indeed the realized number of firms is larger than $\check{m}$, profits are lower than

$$-\left(\frac{1}{\alpha}\frac{1-\phi}{n+1}\right) q_n^*.$$

Profits in the events in which the number of opponents is smaller than $\check{m}$ must be lower than the monopoly profit $(1 - q_n^* - \phi) q_n^*$. We have

$$\pi(q_n^*, q_n^* \mid n, \phi) < C_{\check{m}}^n \left(1 - \frac{\alpha+1}{\alpha}\frac{1-\phi}{n+1} - \phi\right) q_n^* - (1 - C_{\check{m}}^n)\frac{1}{\alpha}\frac{1-\phi}{n+1} q_n^*,$$

which is negative if

$$C_{\check{m}}^n < \frac{1}{\alpha n}. \tag{6.1}$$

Since $\check{m} < n$, we can use the Chernoff bound

$$C_m^n \leq \frac{e^{-n}(en)^m}{m^m}$$

to show that $C^n_{\check{m}}$ converges to zero exponentially, so faster than $\frac{1}{n}$, as n goes to infinity.[24] It follows that, for every ϕ and α, there exists a value $\check{n}_{\phi,\alpha}$ such that, for $n \geq \check{n}_{\phi,\alpha}$, we have $\pi(q^*_n, q^*_n \mid n) < 0$, so q^*_n cannot be an equilibrium.[25]

The second part of the proof consists of showing that for every $0 < \phi < 1$ and $\alpha > 1$ there exists a value $\hat{n}_{\phi,\alpha}$ such that if $n \geq \hat{n}_{\phi,\alpha}$ then $q^*_n \geq \frac{\alpha-1}{\alpha} q^{**}_n$. Suppose to the contrary that $q^*_n < \frac{\alpha-1}{\alpha} q^{**}_n$ for every n. When the number of opponents producing q^*_n is smaller than or equal to $\hat{m} = \left\lfloor \frac{\alpha}{\alpha-1}(n-1) \right\rfloor$, the total quantity they produce is less than $(n-1)q^{**}_n$ and a deviation to q^{**}_n will induce higher profits (and the lower the realized number of other firms the higher the profit). In turn, if the realized number of other firms is larger than $\hat{m}$ then the firm's losses are never greater than 1. Thus, we have

$$
\begin{aligned}
&\pi(q^{**}_n, q^*_n \mid n, \phi) - \pi(q^*_n, q^*_n \mid n, \phi) > \\
&C^n_{\hat{m}}[(1 - \hat{m}q^*_n - q^{**}_n - \phi)q^{**}_n - (1 - \hat{m}q^*_n - q^*_n - \phi)q^*_n] - (1 - C^n_{\hat{m}}) = \\
&C^n_{\hat{m}}(1 - \phi - \hat{m}q^*_n - q^{**}_n - q^*_n)(q^{**}_n - q^*_n) - (1 - C^n_{\hat{m}}) > \\
&C^n_{\hat{m}}\left[nq^{**}_n - (\hat{m}+1)\frac{\alpha-1}{\alpha}q^{**}_n\right]\left(q^{**}_n - \frac{\alpha-1}{\alpha}q^{**}_n\right) - (1 - C^n_{\hat{m}}) \geq \\
&C^n_{\hat{m}}\left[nq^{**}_n - \left(\frac{\alpha}{\alpha-1}(n-1)+1\right)\frac{\alpha-1}{\alpha}q^{**}_n\right]\frac{1}{\alpha}\frac{1-\phi}{(n+1)} - (1 - C^n_{\hat{m}}) = \\
&C^n_{\hat{m}}\left[\frac{1}{\alpha}\frac{1-\phi}{(n+1)}\right]^2 - (1 - C^n_{\hat{m}}).
\end{aligned}
$$

As in the first part of the proof, we can show that, for n sufficiently large, the last expression is greater than zero. That is, if n is large enough then $\hat{m} > n$ and the Chernoff bound $1 - C^n_m \leq \frac{e^{-n}(en)^m}{m^m}$ implies that $1 - C^n_{\hat{m}}$ converges exponentially to zero as n goes to infinity, hence faster than $\frac{1}{n^2}$. Thus, for every ϕ and α, there exists a value $\hat{n}_{\phi,\alpha}$ such that q^*_n cannot be an equilibrium if $n \geq \hat{n}_{\phi,\alpha}$.

Setting $n_{\phi,\alpha} = \max\{\check{n}_{\phi,\alpha}, \hat{n}_{\phi,\alpha}\}$, we obtain the desired result. □

---

[24] The bounds we employ for profits lead to the simple inequality (6.1) but eliminate its dependence on ϕ. Such a dependence could be restored using more efficient bounds. Note that (6.1) holds under the assumption that when the realized number of opponents is $\check{m}+1$ the price is positive, which is true if n is sufficiently large. For values of n such that $\phi - \frac{1}{\alpha}\frac{1-\phi}{n+1} \leq 0$, when the number of competitors is larger than $\check{m}+1$ the price is zero. In this case we have

$$
\pi(q^*_n, q^*_n \mid n, \phi) < C^n_{\check{m}}(1-\phi)q^*_n - (1 - C^n_{\check{m}})\phi q^*_n,
$$

which is negative if $C^n_{\check{m}} < \phi$. Since $\check{m} < n$ and $\check{m}$ depends linearly on n, this inequality is satisfied as long as n is sufficiently high. Note also that we need to consider this case for a proper comparison with the model with zero costs, in which the price is zero when the realized number of opponents is $\check{m}+1$. In particular, for any given n, $C^n_{\check{m}} < \phi$ is never satisfied if $\phi = 0$, consistently with our main results.

[25] Substituting $C^n_{\check{m}}$ with the corresponding Chernoff bound in (6.1) and using a similar approach to the proof of Claim 1, it is possible to explicitly calculate $\check{n}_{\phi,\alpha}$.

7 Conclusion

We have constructed and solved a Cournot model in which the number of firms is uncertain and distributed according to a Poisson distribution. Under symmetry (i.e. all firms have the same marginal cost) we proved that every Poisson-Cournot game has at least one equilibrium. If marginal costs are equal to zero and the expected number of firms is larger than 3, every equilibrium exhibits "severe" overproduction and, therefore, consumer surplus is larger than in the analogous situation in which the number of firms is well known. This result is robust to small enough and strictly positive marginal costs.

However, it should be noted that several simplifying assumptions, while offering tractability, also lead to some limitations. Firstly, we have already mentioned that all firms are symmetric and have the same marginal costs, so we cannot analyze how overproduction differs across productivity levels. Secondly, as in the classical Cournot model, firms are limited to choose quantities and are not able to affect market demand. Thirdly, as is typical in Poisson models, there is no possibility of entry or exit after the realization of the Poisson distribution determines the number of firms in the market. Similarly, there are no information asymmetries and there is common knowledge about the underlying distribution function (Poisson) and its parameter (the expected number of firms). Nevertheless, as the example in Sect. 2 illustrates, the main qualitative properties of our results do not fundamentally depend on the Poisson distribution.

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Appendix A: The conditional mean M_m^n

Given $n > 0$ and a positive integer m, the conditional mean M_m^n is the mean of the Poisson distribution conditional on its realization being smaller or equal than m. That is,

$$M_m^n = \frac{\sum_{k=0}^{m-1} \frac{n^k}{k!} k}{\sum_{k=0}^{m-1} \frac{n^k}{k!}} = \frac{n C_{m-1}^n}{C_m^n}.$$

Of course, M_m^n is increasing and converges to n as m grows. In this appendix we establish several other results about M_m^n that are needed to characterize the set of equilibria of the model. To simplify notation, we fix n and drop the corresponding superscript from every expression as long as it does not lead to confusion.

Recall that, whenever convenient, the Poisson distribution can be expressed in terms of the *incomplete gamma function* as follows

$$\Gamma(m+1,n) := \int_n^\infty s^m e^{-s} ds = m! C_m.$$

We can similarly use the *exponential integral*,

$$\frac{1}{n^{m+1}}\Gamma(m+1,n) = E_{-m}(n) := \int_1^\infty s^m e^{-ns} ds = e^{-n}\int_0^\infty (1+s)^m e^{-ns} ds.$$

Thus, we define the expression

$$J_m := \int_0^\infty (1+s)^m e^{-ns} ds. \tag{A.1}$$

Using the properties of the incomplete gamma function, we can write the conditional mean as $M_m = \frac{mJ_{m-1}}{J_m}$ which is, therefore, also defined for non-integer values of m. Integrating by parts, we see that J_m satisfies the recurrence

$$nJ_m = 1 + mJ_{m-1},$$

therefore, $M_m = n - \frac{1}{J_m}$.

With this in mind we can now prove the following result.

Proposition A.1 *The expression $M_m - M_{m-1}$ is decreasing in m for all $m > 0$.*[26]

Proof Since M_m is increasing in m it is enough to show that M_m is concave in m which holds if $\frac{1}{J_m}$ is convex or, taking derivatives with respect to m, if

$$\frac{2J_m'^2 - J_m'' J_m}{J_m^3} \geq 0.$$

Since J_m is always positive, we need to prove $2J_m'^2 - J_m'' J_m \geq 0$. From (A.1) we have

$$\begin{aligned}
&2J_m'^2 - J_m'' J_m \\
&\quad = 2\int_0^\infty \int_0^\infty \log(1+s_1)\log(1+s_2)(1+s_1)^m(1+s_2)^m e^{-n(s_1+s_2)} ds_1 ds_2 \\
&\qquad - \int_0^\infty \int_0^\infty \log(1+s_1)\log(1+s_1)(1+s_1)^m(1+s_2)^m e^{-n(s_1+s_2)} ds_1 ds_2.
\end{aligned}$$

---

[26] An alternative proof, only valid when m is an integer value, is available from the authors upon request.

Using the change of variables $s_2 = s - s_1$, the last expression equals

$$\int_0^\infty e^{-ns} \int_0^s (1+s_1)^m (1+s-s_1)^m \log(1+s_1) \\ \times\Big[2\log(1+s-s_1) - \log(1+s_1)\Big] ds_1 ds.$$

It is useful to define the functions

$$\begin{aligned} g_m(s_1 \mid s) &:= (1+s_1)^m (1+s-s_1)^m, \\ h(s_1 \mid s) &:= 2\log(1+s-s_1) - \log(1+s_1), \text{ and} \\ f(s_1 \mid s) &:= \log(1+s_1) h(s_1 \mid s). \end{aligned}$$

Due to the symmetry of the function $g_m(s_1 \mid s)$ around $s/2$ where it attains its unique maximum we have

$$2J_m'^2 - J_m'' J_m = \int_0^\infty e^{-ns} \int_{s/2}^s g_m(s_1 \mid s)\Big[f(s_1 \mid s) + f(s - s_1 \mid s)\Big] ds_1 ds. \quad \text{(A.2)}$$

The function $h(s_1 \mid s)$ is strictly decreasing on $0 < s_1 < s$ and is zero at $\tilde{s}_1 = s + \frac{3}{2} - \sqrt{s + \frac{9}{4}}$. It follows that $f(s_1 \mid s)$ is strictly decreasing on $\tilde{s}_1 < s_1 < s$. Furthermore, we can show that $f(s_1 \mid s)$ is increasing in $0 < s_1 < s - \tilde{s}_1$. First note that

$$\begin{aligned} f'(s_1 \mid s) &= \frac{2\log(1+s-s_1)}{1+s_1} - \frac{2(s+2)\log(1+s_1)}{(1+s-s_1)(1+s_1)} \\ &> \frac{2\log(1+\tilde{s}_1)}{1+s_1} - \frac{2(s+2)\log(1+s-\tilde{s}_1)}{(1+s-s_1)(1+s_1)}. \end{aligned}$$

Since $\tilde{s}_1$ satisfies $2\log(1+s-\tilde{s}_1) = \log(1+\tilde{s}_1)$ the right hand side of the last inequality equals

$$\frac{2\log(1+\tilde{s}_1)}{1+s_1} - \frac{(s+2)\log(1+\tilde{s}_1)}{(1+s-s_1)(1+s_1)} = \frac{\log(1+\tilde{s}_1)(s-2s_1)}{(1+s_1)(1+s-s_1)},$$

which is strictly positive because $s_1 < s - \tilde{s}_1$ and $\tilde{s}_1 > \frac{2}{3}s$ (this bound can be directly verified using the expression for $\tilde{s}_1$). Therefore, $f(s_1) + f(s - s_1)$ is strictly decreasing on $\tilde{s}_1 < s_1 < s$ and there exists a unique $\bar{s}_1$ with $\tilde{s}_1 < \bar{s}_1 < s$ at which it vanishes. Hence, Eq. A.2 is equal to

$$\int_0^\infty e^{-ns} \left(\int_{s/2}^{\bar{s}_1} g_m(s_1 \mid s)\Big[f(s_1 \mid s) + f(s - s_1 \mid s)\Big] ds_1 \right. \\ \left. + \int_{\bar{s}_1}^s g_m(s_1 \mid s)\Big[f(s_1 \mid s) + f(s - s_1 \mid s)\Big] ds_1 \right) ds$$

$$
\begin{aligned}
&> \int_0^\infty g_m(\bar{s}_1 \mid s)e^{-ns}\left(\int_{s/2}^{\bar{s}_1}\Big[f(s_1 \mid s) + f(s - s_1 \mid s)\Big]ds_1 + \right.\\
&\left.\int_{\bar{s}_1}^{s}\Big[f(s_1 \mid s) + f(s - s_1 \mid s)\Big]ds_1\right) ds\\
&= \int_0^\infty g_m(\bar{s}_1 \mid s)e^{-ns}\int_{s/2}^{s}\big[f(s_1 \mid s) + f(s - s_1 \mid s)\big]ds_1 ds\\
&= \int_0^\infty g_m(\bar{s}_1 \mid s)e^{-ns}\int_0^s f(s_1 \mid s) ds_1 ds.
\end{aligned}
$$

Consider the inner integral

$$F(s) := \int_0^s f(s_1 \mid s)\, ds_1.$$

We obviously have $F(0) = 0$. We establish the desired result by proving $F'(0) = 0$ and $F''(s) > 0$ for every $s \geq 0$. Using the Leibniz integral rule we obtain

$$F'(s) = f(s \mid s) + \int_0^s \frac{\partial f(s_1 \mid s)}{\partial s} ds_1 = -\log(1+s)^2 + \int_0^s \frac{2\ln(1+s-s_1)}{1+s_1} ds_1$$

so that $F'(0) = 0$. Furthermore,

$$F''(s) = -\frac{2\ln(1+s)}{1+s} + 0 + \int_0^s \frac{2}{(1+s_1)(1+s-s_1)} ds_1 = \frac{2s\ln(1+s)}{(1+s)(2+s)} > 0,$$

as we wanted. □

This previous result has some important implications. The first one follows directly from $M_0 = 0$ and $M_1 = \frac{n}{n+1}$.

Corollary A.1 *For any integer* $m \geq 1$ *we have* $M_m - M_{m-1} < 1$.

Corollary A.2 *For every integer* $m \geq 1$, *if* $M_m > m - 1$ *then* $M_{m-1} > m - 2$.

We are interested in finding $\bar{m}$, i.e., the greatest integer m such that $M_{m-1} > m-2$. We find bounds for $\bar{m}$ in the next two propositions.

Proposition A.2 *The greatest* m *such that* $M_{m-1} > m - 2$ *satisfies* $m > \frac{n}{2} + 1$.

Proof Using the rules of the conditional expectation, we know that the conditional mean satisfies

$$M_m = \frac{C_{m-1}}{C_m} M_{m-1} + \frac{P_m}{C_m} m. \tag{A.3}$$

If $M_{m-1} > m - 2$, then

$$M_m > \frac{C_{m-1}}{C_m}(m-2) + \frac{P_m}{C_m} m = m - \frac{2}{n} M_m.$$

Solving for M_m, we obtain that $M_{m-1} > m - 2$ implies

$$M_m > \frac{n}{n+2} m. \tag{A.4}$$

But $\bar{m}$ is the greatest integer m with $M_{m-1} > m-2$, i.e. $M_{\bar{m}} \leq \bar{m} - 1$. This inequality combined with (A.4) provides the lower bound $\frac{n}{2} + 1 < \bar{m}$. ☐

To obtain the upper bound for $\bar{m}$ we need the following basic fact about the conditional mean of a Poisson random variable.

Lemma A.1 *The conditional mean M_m^n is strictly increasing in n for every $m > 0$.*

Proof Since $P_m^n / P_{m'}^n$ is strictly increasing in n if $m > m'$, an increase in n makes any realization $m > 0$ of the Poisson random variable relatively more likely than any smaller realization m'. The result follows. ☐

Proposition A.3 *The greatest m such that $M_{m-1} > m - 2$ satisfies $m < \frac{n}{2} + 3$.*

Proof We actually show that $m \geq \frac{n}{2} + 2$ implies $M_m \leq m - 1$. Given Lemma A.1, it is enough to show that $m = \frac{n}{2} + 2$ implies $M_m \leq m - 1$. But if $m = \frac{n}{2} + 2$, the latter inequality can be written in continuous terms using the incomplete gamma function as

$$(m-3) \int_0^\infty e^{-2(m-2)s} (1+s)^m ds \leq 1.$$

With the change of variables $e^t = 1 + s$ on the left hand side we have

$$(m-3) \int_0^\infty e^{-2(m-2)(e^t - 1) + (m+1)t} dt < (m-3) \int_0^\infty e^{-2(m-2)(t + \frac{t^2}{2} + \frac{t^3}{6}) + (m+1)t} dt$$

so that, with the new change of variables $u = (1+t)^3$ and rearranging, it is enough to prove

$$(m-3)e^{-3} \int_1^\infty e^{-\frac{1}{3}(m-2)(u-1) + 3u^{1/3}} \left(\frac{1}{3} u^{-2/3} \right) du \leq 1. \tag{A.5}$$

Let I_m be the value of integral above. Integrating by parts we obtain the equality

$$(m-2) I_m = e^3 + \int_1^\infty e^{-\frac{1}{3}(m-2)(u-1) + 3u^{1/3}} \left(u^{-4/3} - \frac{2}{3} u^{-5/3} \right) du.$$

Combining the last expression with the left hand side of (A.5) we have that the latter approaches 1 as m tends to infinity. To show that (A.5) holds for every m we prove that $(m-3) I_m$ is increasing for every m.

$$\frac{d}{dm}(m-3) I_m = \frac{d}{dm}(m-2) I_m - \frac{d}{dm} I_m$$

$$= \int_1^\infty e^{-\frac{1}{3}(m-2)(u-1)+3u^{1/3}}$$

$$\left(-\frac{1}{3}u^{-1/3} + \frac{1}{9}u^{-2/3} + \frac{1}{3}u^{-4/3} - \frac{2}{9}u^{-5/3} + \frac{1}{9}u^{1/3}\right) du.$$

The derivative of $(m-3)I_m$ is positive as long as the bracketed expression is strictly positive for almost every $u \geq 1$. That is, as long as, for almost every $u \geq 1$

$$f(u) := u + 3u^{1/3} + u^2 > g(u) := 3u^{4/3} + 2.$$

Functions f and g are always positive and coincide at $u = 1$. The same properties can be verified for the pairs of functions (f', g') and (f'', g''). However, $f'''(u) > 0 > g'''(u)$ for every $u \geq 1$, thereby proving inequality (A.5) . □

When $M_{m-1} > m-2$, Eq. (A.4) provides a lower bound for M_m. The next Lemma gives a corresponding upper bound.

Lemma A.2 *For every integer $m \geq 1$ we have $M_m \leq \frac{n}{n+1}m$, with equality only if $m = 1$.*

Proof We obtain $M_1 = \frac{n}{n+1}$ by direct computation. Let $m > 1$; we obviously have $M_{m-1} < m-1$, which can be combined with (A.3) to obtain

$$M_m < \frac{C_{m-1}}{C_m}(m-1) + \frac{P_m}{C_m}m = m - \frac{C_{m-1}}{C_m} = m - \frac{1}{n}M_m.$$

The result follows after solving for M_m. □

Similarly, together with the upper bound for $M_m - M_{m-1}$ in Corollary A.1, we need a lower bound for values of m that are associated with a pseudo-equilibrium.

Lemma A.3 *If $m < \frac{n}{2} + 3$ then $M_m - M_{m-1} > \frac{n-4}{n+4}$.*

Proof From Proposition A.1 we know that $M_m - M_{m-1}$ is decreasing, so we focus on $M_{\bar{m}} - M_{\bar{m}-1}$, where $\bar{m}$ is defined as in Theorem 3.1. Subtracting $M_{\bar{m}-1}$ from both sides in (A.3) we obtain

$$M_{\bar{m}} - M_{\bar{m}-1} = \frac{P_{\bar{m}}}{C_{\bar{m}}}(\bar{m} - M_{\bar{m}-1}) \geq \frac{P_{\bar{m}}}{C_{\bar{m}}}(1 + M_{\bar{m}} - M_{\bar{m}-1}), \tag{A.6}$$

and solving for $M_{\bar{m}} - M_{\bar{m}-1}$,

$$M_{\bar{m}} - M_{\bar{m}-1} \geq \frac{P_{\bar{m}}}{C_{\bar{m}-1}} = \frac{n}{M_{\bar{m}}} - 1.$$

Since $M_{\bar{m}} \leq \bar{m} - 1$ and $\bar{m} < \frac{n}{2} + 3$ the last expression implies

$$M_{\bar{m}} - M_{\bar{m}-1} > \frac{n - \bar{m} + 1}{\bar{m} - 1} = \frac{n-4}{n+4}.$$

□

In the remainder of this appendix we find the alternative bound $M_m - M_{m-1} > \frac{n-6}{n-2}$ which is tighter whenever $n > 8$. We begin with a preliminary lemma.

Lemma A.4 *If* $n > 2$ *then*

$$\left(\frac{n}{2} - 1 - \frac{n-6}{n-2}\right) J_{\frac{n}{2}+3} \geq 1.$$

Proof Given some a such that $|a| < 1$ and some b, we begin by finding a new expression for nJ_{an+b}. We use the equality $nJ_m = 1 + mJ_{m-1}$ recursively to obtain

$$\begin{aligned} nJ_{an+b} &= 1 + (an+b)J_{an+b-1} \\ &= 1 + bJ_{an+b-1} + a\big(1 + (an+b-1)J_{an+b-2}\big) \\ &= 1 + a + bJ_{an+b-1} + a(b-1)J_{an+b-2} + a^2 nJ_{an+b-2} \\ &= \cdots \\ &= \sum_{k=0}^{N} \big(a^k + a^k(b-k)J_{an+b-1-k}\big) + a^{N+1}nJ_{an+b-1-N}. \end{aligned}$$

If $an+b$ is not an integer (so that $an+b+1-N \neq 0$ for every N) we can take the limit as N goes to infinity to obtain

$$\begin{aligned} nJ_{an+b} &= \frac{1}{1-a} + \int_0^\infty e^{-ns}(1+s)^{an+b-1} \sum_{k=0}^{\infty}(b-k)\left(\frac{a}{1+s}\right)^k ds \\ &= \frac{1}{1-a} + \int_0^\infty e^{-ns}(1+s)^{an+b} \frac{b(1+s) - a(b+1)}{(a-1-s)^2} ds. \end{aligned}$$

If $an+b$ is an integer then a continuity argument implies that the previous equality also holds. We use such an equality to obtain an expression for $n^2 J_{an+b}$.

$$n^2 J_{an+b} = \frac{n}{1-a} + n\int_0^\infty e^{-ns}(1+s)^{an+b} \frac{b(1+s) - a(b+1)}{(a-1-s)^2} ds. \tag{A.7}$$

Integrating by parts we obtain

$$\begin{aligned} &\frac{n}{1-a} + \frac{b-a-ab}{(a-1)^2} + \int_0^\infty e^{-ns}\Bigg[(an+b)(1+s)^{an+b-1}\frac{b(1+s)-a(b+1)}{(a-1-s)^2} \\ &\quad + (1+s)^{an+b}\frac{d}{ds}\frac{b(1+s)-a(b+1)}{(a-1-s)^2}\Bigg] ds \end{aligned}$$

and, rearranging,

$$\frac{n}{1-a} + \frac{b-a-ab}{(a-1)^2}$$

$$+\int_0^\infty e^{-ns}(1+s)^{an+b}\left[\frac{b}{1+s}\frac{b-a(b+1)}{(a-1-s)^2}+\frac{d}{ds}\frac{b(1+s)-a(b+1)}{(a-1-s)^2}\right]ds$$
$$+an\int_0^\infty e^{-ns}(1+s)^{an+b-1}\frac{b(1+s)-a(b+1)}{(a-1-s)^2}ds.$$

The last integral in the previous expression can be integrated by parts in the same fashion as the integral in Eq. (A.7). Doing so we obtain

$$\frac{n}{1-a}+\frac{b-a-ab}{(a-1)^2}+a\frac{b-a-ab}{(a-1)^2}$$
$$+\int_0^\infty e^{-ns}(1+s)^{an+b}\left[\frac{b}{1+s}\frac{b(1+s)-a(b+1)}{(a-1-s)^2}+\frac{d}{ds}\frac{b(1+s)-a(b+1)}{(a-1-s)^2}\right]ds$$
$$+\int_0^\infty e^{-ns}(1+s)^{an+b}\left[\frac{a(b-1)}{(1+s)^2}\frac{b(1+s)-a(b+1)}{(a-1-s)^2}\right.$$
$$\left.+\frac{a}{1+s}\frac{d}{ds}\frac{b(1+s)-a(b+1)}{(a-1-s)^2}\right]ds$$
$$+a^2n\int_0^\infty e^{-ns}(1+s)^{an+b-2}\frac{b(1+s)-a(b+1)}{(a-1-s)^2}ds.$$

Iterating the same step *ad infinitum* we have

$$\frac{n}{1-a}+\frac{b-a-ab}{(a-1)^2}\sum_{k=0}^{\infty}a^k$$
$$+\int_0^\infty e^{-ns}(1+s)^{an+b}\frac{b(1+s)-a(b+1)}{(a-1-s)^2}\sum_{k=0}^{\infty}\frac{a^k(b-k)}{(1+s)^{k+1}}ds$$
$$+\int_0^\infty e^{-ns}(1+s)^{an+b}\frac{d}{ds}\frac{b(1+s)-a(b+1)}{(a-1-s)^2}\sum_{k=0}^{\infty}\frac{a^k}{(1+s)^k}ds,$$

which we simplify by solving the infinite sums to obtain

$$n^2J_{an+b}=\frac{n}{1-a}+\frac{b-a-ab}{(1-a)^3}+\int_0^\infty e^{-ns}(1+s)^{an+b}$$
$$\frac{1}{(a-1-s)^4}\Big[b(b-1)s^2++(2b^2-2ab^2-ab+2a-2b)s$$
$$+a^2b^2+2a^2b-2ab^2+a^2-ab+b^2+2a-b\Big]ds$$

When $a=\frac{1}{2}$ and $b=3$ the expressions for n^2J_{an+b} and nJ_{an+b} become

$$nJ_{\frac{n}{2}+3}=2+4\int_0^\infty e^{-ns}(1+s)^{\frac{n}{2}+3}\frac{3s+1}{(2s+1)^2}ds,\text{ and}$$
$$n^2J_{\frac{n}{2}+3}=2n+8+8\int_0^\infty e^{-ns}(1+s)^{\frac{n}{2}+3}\frac{12s^2+5s+1}{(2s+1)^4}ds,$$

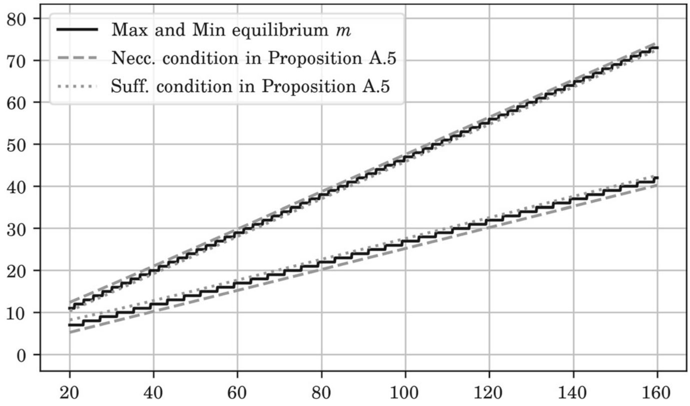

Fig. 3 New bounds for m found using $M_m - M_{m-1} > \frac{n-6}{n-2}$ given $20 \le n \le 160$

from where it readily follows

$$\left(\frac{n}{2} - 1 - \frac{n-6}{n-2}\right) J_{\frac{n}{2}+3} = \frac{n^2 - 6n + 16}{2n - 4} J_{\frac{n}{2}+3}$$
$$= 1 + \frac{16}{n-2} \int_0^\infty e^{-ns} (1+s)^{\frac{n}{2}+3} \frac{s^2(8s^2 + 7s + 3)}{(2s+1)^4} ds \ge 1.$$

□

Proposition A.4 *If $n > 2$ and $m < \frac{n}{2} + 3$ we have $M_m - M_{m-1} > \frac{n-6}{n-2}$.*

Proof Given Lemma A.4 and Propositions A.1 and A.3,

$$\begin{aligned} M_m - M_{m-1} &> M_{\frac{n}{2}+3} - M_{\frac{n}{2}+2} \\ &= \frac{1}{J_{\frac{n}{2}+2}} - \frac{1}{J_{\frac{n}{2}+3}} \\ &\ge \left(\frac{n}{2} - 1\right) - \left(\frac{n}{2} - 1 - \frac{n-6}{n-2}\right) \\ &= \frac{n-6}{n-2}. \end{aligned}$$

□

Of course, the tightness of the bounds in Propositions 4.1 and 4.2 depend on the tightness of the bounds for M_m and $M_m - M_{m-1}$. The bound found in the last proposition is tighter than the one in Lemma A.3 whenever $n > 8$. Using this new bound, we obtain the following necessary and sufficient conditions for equilibria that are represented in Fig. 3.

Proposition A.5 *Let $n > 2$. If $q^* \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ is an equilibrium then*

$$\frac{n}{4} + \frac{1}{4} < m < \left(\frac{4}{9}n + \frac{8}{9}\right)\left(\frac{3n-6}{3n-10}\right)^2 + 1.$$

Furthermore, if $\tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ is a pseudo-equilibrium such that

$$\left(\frac{n}{4} + \frac{1}{2}\right)\left(\frac{n+2}{n-2}\right)^2 \leq m \leq \frac{4}{9}n + \frac{13}{9}$$

then $\tilde{q}$ is an equilibrium.

The upper bound in Proposition A.5 is more efficient than the one in Theorem 3.1 for $n > 14$. To illustrate the efficiency of these bounds consider, e.g., $n = 100$. According to Proposition A.5 every equilibrium quantity must be larger than $\frac{1}{48}$ and smaller than $\frac{1}{26}$. Furthermore, every interval $\left[\frac{1}{m+1}, \frac{1}{m}\right)$ from $\frac{1}{46}$ to $\frac{1}{28}$ contains an equilibrium, which is given by $\frac{1}{M^{100}_{m-1}+2}$. According to numerical computations, there is an equilibrium in each interval from $\frac{1}{48}$ to $\frac{1}{27}$. (Without population uncertainty, the unique equilibrium quantity is $\frac{1}{101}$.)

Appendix B: Proof of Claim 1

Claim 1 Let $\tilde{q}_{m-1}$ be an equilibrium. Then $\pi(\tilde{q}_{m-1}, \tilde{q}_{m-1} \mid n) < \frac{1}{(n+1)^2}$ *for every n.*

Proof We begin to show that, for n large enough,

$$\tilde{q}^2_{\bar{m}-1} C^n_{\bar{m}-1} < \frac{1}{(n+1)^2}. \tag{B.1}$$

Replacing $\tilde{q}_{\bar{m}-1}$ by its value and using $M^n_{\bar{m}-1} > \bar{m} - 2$, we actually prove that for n large enough

$$C^n_{\bar{m}-1} < \left(\frac{\bar{m}}{n+1}\right)^2.$$

A tight upper bound for $C^n_{\bar{m}-1}$ can be found as follows. Since $M^n_{\bar{m}} \leq \bar{m} - 1$ and $M^n_{\bar{m}} = n\frac{C^n_{\bar{m}-1}}{C^n_{\bar{m}}} = n\left(1 - \frac{P^n_{\bar{m}}}{C^n_{\bar{m}}}\right)$ we obtain $C^n_{\bar{m}} \leq P^n_{\bar{m}}\left(\frac{n}{n+1-\bar{m}}\right)$, so that $C^n_{\bar{m}-1} = C^n_{\bar{m}} - P^n_{\bar{m}} \leq P^n_{\bar{m}}\left(\frac{\bar{m}-1}{n+1-\bar{m}}\right)$. Thus, inequality (B.1) is satisfied whenever

$$P^n_{\bar{m}}\left(\frac{\bar{m}-1}{n+1-\bar{m}}\right) < \left(\frac{\bar{m}}{n+1}\right)^2.$$

Recall that, by Theorem 3.1, $\frac{n}{2}+1<\bar{m}<\frac{n}{2}+3$. Assuming $n>6$, the previous inequality is satisfied if it holds after replacing the factorial in $P^n_{\bar{m}}$ by Stirling's approximation $\sqrt{2\pi\bar{m}}(\frac{\bar{m}}{e})^{\bar{m}}$ and $\bar{m}$ by $\frac{n}{2}+3$.[27] Making the change of variables $x=\frac{n}{2}$ we obtain the inequality

$$e^{-x+3}\frac{1}{\sqrt{2\pi(x+3)}}\left(\frac{2x}{x+3}\right)^{x+3}\left(\frac{x+2}{x-2}\right)<\left(\frac{x+3}{2x+1}\right)^{2}$$

which holds for, e.g., $x=3.174$ (i.e. $n=6.348$). To show that it also holds for every $x>3.174$ ($n>6.348$) we prove that the inequality still holds after we differentiate it with respect to x. Indeed, taking logarithms and differentiating we obtain

$$-1+\frac{x+3}{x}+\log(2x)+\frac{1}{x+2}+\frac{4}{2x+1}<\frac{2}{x+3}+\frac{1}{2(x+3)}+\log(x+3)+1 \\ +\frac{1}{x-2}.$$

Collecting the logarithms, using the bound $\log(y)\leq y-1$, and rearranging we find the simpler expression

$$\frac{x+3}{x}+\frac{4x-5}{2(x+3)}+\frac{1}{x+2}+\frac{4}{2x+1}<3+\frac{1}{x-2}.$$

This last inequality can be easily verified when $x>2$ by noticing that $\frac{4x-5}{2(x+3)}<\frac{2x-2}{x+2}$ and $\frac{4}{2x+1}<\frac{2}{x}$, and that

$$\frac{x+3}{x}+\frac{2x-2}{x+2}+\frac{1}{x+2}+\frac{2}{x}=3+\frac{10}{x(x+2)}<3+\frac{1}{x-2}.$$

Hence, (B.1) is satisfied for $n\geq 6.348$.

Furthermore, using the computation of equilibria for small values of n in the Online Appendix, Claim 1 can be directly verified *for every* $n>0$. □

Appendix C: Equilibrium existence with positive production costs

If $\phi\geq 1$ not producing is the unique equilibrium. Therefore, in this appendix we assume $0<\phi<1$. We show that there is always an equilibrium following the same strategy of the proof as when $\phi=0$. However, we previously need to show that when $0<\phi<1$ a pseudo-equilibrium always exists. To simplify notation, we fix n and drop the corresponding superscript from every expression.

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[27] To see this, note first that $\frac{m-1}{m^2(n+1-m)}$ is increasing in m when $m>\frac{n}{2}+1$. Second, we can write the density P^n_m as a continuous function of m using the gamma function instead of the factorial. If $n>6$ then, necessarily, $m<n$ and the resulting continuous function is increasing in m so that we can replace $\bar{m}$ by $\frac{n}{2}+3$. Finally, we substitute the value of the gamma function at $\frac{n}{2}+3$ with the value of Stirling's approximation.

If $q, q' \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ then the pseudo-profit equals

$$\tilde{\pi}_{m-1}(q, q' \mid n, \phi) := \sum_{k=0}^{m-1} P_k(1 - kq' - q)q - \phi q$$

which can be used to derive the pseudo-best response

$$\widetilde{\mathrm{BR}}_{m-1}(q') := \frac{1}{2} - \frac{1}{2}M_{m-1}q' - \frac{1}{2}\frac{\phi}{C_{m-1}}$$

and, if it exists, the pseudo-equilibrium

$$\tilde{q} = \frac{1 - \frac{\phi}{C_{m-1}}}{M_{m-1} + 2}.$$

Such a pseudo-equilibrium $\tilde{q}$ does exist if $\frac{1}{m+1} \leq \tilde{q} < \frac{1}{m}$. That is, if

$$\begin{aligned} L(m-1) &:= (m-2) - m\frac{\phi}{C_{m-1}} < M_{m-1} \\ &\leq H(m-1) := (m-1) - (m+1)\frac{\phi}{C_{m-1}}. \end{aligned} \tag{C.1}$$

Theorem C.1 *A pseudo-equilibrium exists.*

Proof Using the incomplete gamma function we can temporarily work with the continuous versions of C_{m-1}, $L(m-1)$, $H(m-1)$ and M_{m-1}. Since $\phi < 1$, there is an m' such that $C_{m'-1} = \phi$. For such a value, $L(m'-1) = H(m'-1) = -2 < M_{m'-1}$. Note that for every $m > m'$ we have $L(m-1) < H(m-1)$ and that, as m goes to infinity, both $L(m-1)$ and $H(m-1)$ also go to infinity while M_{m-1} converges to n. Therefore, there is some $m \in \mathbb{R}_{++}$ such that the double inequality (C.1) is satisfied. We need to show that such a double inequality is also satisfied for some integer value of m.

To the contrary assume that there is no pseudo-equilibrium. Let $\hat{m}$ be the largest integer such that $M_{\hat{m}-1} > H(\hat{m}-1)$, since there is no pseudo-equilibrium we must have $M_{\hat{m}} \leq L(\hat{m})$. Therefore,

$$M_{\hat{m}} - M_{\hat{m}-1} < L(\hat{m}) - H(\hat{m}-1) = (\hat{m}+1)\phi\frac{P_{\hat{m}}}{C_{\hat{m}}C_{\hat{m}-1}}.$$

From Eq. (A.6) in the proof of Lemma A.3 we know

$$M_{\hat{m}} - M_{\hat{m}-1} = \frac{P_{\hat{m}}}{C_{\hat{m}}}(\hat{m} - M_{\hat{m}-1}),$$

and combining the last two expressions

$$M_{\hat{m}-1} > \hat{m} - (\hat{m}+1)\frac{\phi}{C_{\hat{m}-1}} = H(\hat{m}-1) + 1.$$

Repeating this same argument but using $M_{\hat{m}-1} > H(\hat{m}-1) + 1$ we obtain

$$M_{\hat{m}-1} > \hat{m} + \frac{C_{\hat{m}}}{P_{\hat{m}}} - (\hat{m}+1)\frac{\phi}{C_{\hat{m}-1}} > H(\hat{m}-1) + 2.$$

Thus, if we iterate on the argument we conclude $M_{\hat{m}-1} > \hat{m} - 1$, which is impossible. Therefore, there is at least one pseudo-equilibrium. □

In order to show that there is always an equilibrium we follow the same strategy as in the case $\phi = 0$. That is, given a pseudo-equilibrium, we find the best possible deviation to a higher and to a lower quantity. Then we show that both deviations cannot be profitable at the same time and that, if two consecutive pseudo-equilibria are not equilibria, then either both have a profitable deviation to a smaller quantity or both have a profitable deviation to a larger quantity. The existence result follows from establishing that at the smallest pseudo-equilibrium there is no profitable deviation to a smaller quantity, and that at the largest pseudo-equilibrium there is no profitable deviation to a larger quantity. We now show each of these results in turn.

A pseudo-equilibrium $\tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ is an equilibrium if no firm can profitably deviate to either a quantity smaller than $\frac{1}{m+1}$ or to a quantity larger than $\frac{1}{m}$. The best deviation to a smaller quantity is

$$\underline{q} = \frac{1}{2} - \frac{1}{2}M_m\tilde{q} - \frac{1}{2}\frac{\phi}{C_m}.$$

And a necessary condition for it to be a profitable deviation is $\underline{q} < 1 - m\tilde{q}$. On the other hand, a necessary condition for some $\bar{q} > \frac{1}{m}$ to be a profitable deviation is $\bar{q} > 1 - (m-1)\tilde{q}$.

Lemma C.1 *The best possible deviation to a quantity higher than* $\frac{1}{m}$ *is*

$$\bar{q} = \frac{1}{2} - \frac{1}{2}M_{m-2}\tilde{q} - \frac{1}{2}\frac{\phi}{C_{m-2}}.$$

Proof We begin establishing the following fact: if $\tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ is a pseudo-equilibrium then for any $j \geq 3$ we must have

$$M_{m-j} > m - 2(j-1) - m\frac{\phi}{C_{m-j}}. \tag{C.2}$$

Indeed, using Corollary A.1 and the assumption that $\tilde{q}$ is a pseudo-equilibrium we obtain

$$M_{m-j} > M_{m-1} - (j-1) > (m-2) - (j-1) - m\frac{\phi}{C_{m-1}}$$

and it can be easily shown that if $j \geq 3$ this estimate is larger than the right-hand side of (C.2) thereby establishing such an inequality.

Suppose now that for $j \geq 3$ the deviation to

$$q_{m-j} = \frac{1}{2} - \frac{1}{2}M_{m-j}\tilde{q} - \frac{1}{2}\frac{\phi}{C_{m-j}}$$

is more profitable than the deviation to

$$q_{m-j+1} = \frac{1}{2} - \frac{1}{2}M_{m-j+1}\tilde{q} - \frac{1}{2}\frac{\phi}{C_{m-j+1}}.$$

If that is the case then $q_{m-j} > 1 - (m-j+1)\tilde{q}$. Substituting q_{m-j} by its value and solving for $\tilde{q}$ we have

$$\tilde{q} > \frac{1 + \frac{\phi}{C_{m-j}}}{2(m-j+1) - M_{m-j}} > \frac{1 + \frac{\phi}{C_{m-j}}}{m\left(1 + \frac{\phi}{C_{m-j}}\right)} = \frac{1}{m},$$

where the second inequality follows from (C.2). But this provides the desired contradiction. □

Lemma C.2 *Let $\tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ be a pseudo-equilibrium. If there is a profitable deviation to the higher quantity $\bar{q}$ then there cannot be a profitable deviation to the lower quantity $\underline{q}$ and vice versa.*

Proof If both $\bar{q}$ and $\underline{q}$ are profitable deviations, using $\bar{q} > 1-(m-1)\tilde{q}$ and $\underline{q} < 1-m\tilde{q}$, substituting $\bar{q}$ and $\underline{q}$ by their corresponding values, and applying $M_m - M_{m-2} < 2$ we obtain

$$\tilde{q}\left(m - \frac{1}{2}M_m\right) > \frac{1}{2}\left(1 + \frac{\phi}{C_{m-2}}\right) \quad \text{and} \quad \tilde{q}\left(m - \frac{1}{2}M_m\right) < \frac{1}{2}\left(1 + \frac{\phi}{C_m}\right).$$

However, they cannot both hold at the same time because $C_{m-2} < C_m$. □

Lemma C.3 *Let $\tilde{q} \in \left[\frac{1}{m+1}, \frac{1}{m}\right)$ and $\hat{q} \in \left[\frac{1}{m+2}, \frac{1}{m+1}\right)$ be two pseudo-equilibria. If there is a profitable deviation from $\hat{q}$ to a higher quantity $\bar{q}$ then there cannot be a profitable deviation from $\tilde{q}$ to a lower quantity $\underline{q}$, and vice versa.*

Proof Suppose there is a profitable deviation from $\hat{q}$ to a higher quantity and from $\tilde{q}$ to a lower quantity. From the necessary conditions for those two deviations to be profitable we obtain the inequalities

$$m-1-\frac{1}{2}M_m-\frac{1}{2}M_{m-1}>\frac{\phi}{C_{m-1}}+\frac{1}{2}\frac{\phi}{C_{m-1}}M_m+m\frac{\phi}{C_m}-\frac{1}{2}\frac{\phi}{C_m}M_{m-1}$$
$$m-1-\frac{1}{2}M_m-\frac{1}{2}M_{m-1}<\frac{\phi}{C_m}+\frac{1}{2}\frac{\phi}{C_m}M_{m-1}+m\frac{\phi}{C_{m-1}}-\frac{1}{2}\frac{\phi}{C_{m-1}}M_m.$$

We claim that the right-hand side in the second inequality is strictly smaller than the right-hand side in the first inequality. That holds if and only if

$$\frac{1}{C_{m-1}}\Big(m-1-M_m\Big)<\frac{1}{C_m}\Big(m-1-M_{m-1}\Big),$$

and this inequality holds if and only if

$$\begin{aligned}(m-1)C_m-nC_{m-1}&<(m-1)C_{m-1}-nC_{m-2}\\(m-1)P_m&<nP_{m-1}\\\frac{m-1}{n}&<\frac{P_{m-1}}{P_m}=\frac{m}{n},\end{aligned}$$

which establishes our claim and provides the desired contradiction. □

Lemma C.4 *At the highest pseudo-equilibrium quantity, deviating to a higher quantity is not profitable. Similarly, at the lowest pseudo-equilibrium quantity, deviating to a lower quantity is not profitable either.*

Proof Let $\tilde{q}\in\left[\frac{1}{m+1},\frac{1}{m}\right)$ be the smallest pseudo-equilibrium quantity so that $M_m\le L(m)$. If deviating to a lower quantity was a profitable deviation, then

$$1-\tilde{q}m>\frac{1}{2}-\frac{1}{2}\tilde{q}M_m-\frac{1}{2}\frac{\phi}{C_m}\ge\frac{1}{2}-\frac{1}{2}\tilde{q}\left(m-1-(m+1)\frac{\phi}{C_m}\right)-\frac{1}{2}\frac{\phi}{C_m}$$

and, solving from $\tilde{q}$, we have $\tilde{q}<\frac{1}{m+1}$ which is impossible.

Let $\tilde{q}\in\left[\frac{1}{m+1},\frac{1}{m}\right)$ be the highest pseudo-equilibrium quantity so that $M_{m-2}>H(m-2)$. If deviating to a higher quantity was a profitable deviation, then

$$1-(m-1)\tilde{q}<\frac{1}{2}-\frac{1}{2}\tilde{q}M_{m-2}-\frac{1}{2}\frac{\phi}{C_{m-2}}<\frac{1}{2}-\frac{1}{2}\tilde{q}\left(m-2-m\frac{\phi}{C_{m-2}}\right)$$
$$-\frac{1}{2}\frac{\phi}{C_{m-2}}.$$

Solving from $\tilde{q}$, we obtain $\tilde{q}>\frac{1}{m}$ which is also impossible. □

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