BPS Algebras in 2D String Theory

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BPS Algebras in 2D String Theory

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Abstract. We discuss a set of heterotic and type II string theory compactifications to $1 + 1$ dimensions that are characterized by factorized internal worldsheet CFTs of the form $V_1 \otimes \bar{V}_2$, where V_1, V_2 are self-dual (super) vertex operator algebras. In the cases with spacetime supersymmetry, we show that the BPS states form a module for a Borcherds–Kac–Moody (BKM) (super)algebra, and we prove that for each model the BKM (super)algebra is a symmetry of genus zero BPS string amplitudes. We compute the supersymmetric indices of these models using both Hamiltonian and path integral formalisms. The path integrals are manifestly automorphic forms closely related to the Borcherds–Weyl–Kac denominator. Along the way, we comment on various subtleties inherent to these low-dimensional string compactifications.

Contents

Ⓑ Birkhäuser

1. Introduction

Critical string theories compactified to low dimensions provide a useful formal playground for exploring various theoretical phenomena. They enjoy enhanced (nonperturbative) duality groups, uncommon supersymmetry algebras, exhibit relatively explicit but rich mathematical structures in their spacetime physics, and may still possess an interesting family of moduli enabling one to decompactify back to more physically relevant dimensions. Distinguished points in such low-dimensional moduli spaces exhibit maximal symmetry groups that can usefully organize or classify (broken) symmetries in higher-dimensional models upon decompactification. Alternatively, one can use the enhanced symmetries at special points to construct orbifolds of the original theory; such

orbifolds with respect to large symmetry groups can produce exotic models with few moduli. These models may even be completely rigid, sitting at isolated points in moduli space, and are hence distinguished in their own right. It is also interesting and non-trivial to determine the duality webs among these exceptional theories.

It is known that some of the points of maximal symmetry are given by string models whose internal worldsheet theories factorize into holomorphic and anti-holomorphic factors. Such holomorphically factorized theories must be built from consistent (self-dual)[1] vertex operator algebras (VOAs) of appropriate central charge. The latter are quite rare for low values of the central charge and have been determined at several key values of interest, including $c = 24$ for bosonic VOAs and $c = 12$ for super-VOAs (SVOAs). In this work, we will focus on compactifications of IIA string theory to two spacetime dimensions built from holomorphically factorized worldsheet theories of the form $V_1 \otimes \bar{V}_2$, where V_1, V_2 are self-dual SVOAs with $c = 12$, and $\bar{V}_2$ denotes the anti-holomorphic (right-moving) version of the SVOA V_2. Much of our analysis also carries through for heterotic string theories built from $V_1 \otimes \bar{V}_2$, where V_1 is now a self-dual VOA with $c = 24$.

One nice feature of these factorized models is that BPS states are closely related to infinite-dimensional Lie (super)algebras called Borcherds–Kac–Moody algebras [10]. For a variety of instances of BKMs in string theory, see, e.g., [15,18,31,38,39,46,54]. Such algebras can be constructed by applying a certain functor to a self-dual VOA that looks analogous to constructing a "chiral" string theory: that is, tensor in the vertex algebra corresponding to the even self-dual lattice of signature $(1, 1)$ (a chiral analogue of the light-cone directions) to obtain a theory with critical central charge, tensor in the vertex algebra corresponding to ghosts, and compute the BRST cohomology of the resulting complex. The resulting cohomology classes generate a BKM.

BKM algebras were originally defined by Borcherds in the course of his proof of the monstrous moonshine conjectures [7,9,10]. Harvey and Moore later proposed that BKM algebras should play a role as the underlying organizing structure of BPS states in string compactifications [38,39]. While a complete understanding of the elusive "algebra of BPS states" is still lacking (though see, e.g., [48] for a mathematical perspective), the fact that BKM algebras are intimately connected with BPS states is indisputable.

Let us explain this connection in some more detail. Consider heterotic string theory compactified to two dimensions with an internal CFT of the form $V^\natural \otimes \bar{V}^{f\natural}$. Here, $V^\natural$ is the monster module, constructed by Frenkel, Lepowsky and Meurman [29,30], and $V^{f\natural}$ is Duncan's super-moonshine module for the Conway group [25]. This theory was used in [55,56] to provide a physical interpretation of the genus zero property of monstrous moonshine. A key aspect is that it gives a *spacetime* description of the McKay–Thompson series, which

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[1]Throughout this paper, we use the same definition of self-dual as the authors of [21]: essentially, that an SVOA V is self-dual (or holomorphic) if V is the only irreducible, admissible V-module up to isomorphism. The name "holomorphic VOA" is often used to denote self-dual VOAs, but this name is potentially confusing in the context of this article.

appear in certain BPS indices. In particular, after compactifying the space direction on a circle S^1, one can consider the following index:

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{BPS}}}\left((-1)^F e^{2\pi i TW} e^{2\pi i UM}\right), \tag{1.1}$$

where $\mathcal{H}_{\mathrm{BPS}}$ is a second quantized space of BPS string states, F is the fermion number and (W, M) are the winding and momenta along the circle S^1. In addition, (T, U) are the associated chemical potentials which involve the spacetime radius R and inverse temperature β. It was shown in [55] that this index can be written as an infinite product

$$\left(e^{2\pi i(w_0 T + m_0 U)} \prod_{\substack{m > 0 \\ w \in \mathbb{Z}}} (1 - e^{2\pi i m U} e^{2\pi i w T})^{c(mw)}\right)^{24}, \tag{1.2}$$

where $c(n)$ are the Fourier coefficients of the modular invariant J-function. This is recognized as the (24th power of) the Borcherds–Weyl–Kac denominator formula of the monster Lie algebra $\mathfrak{m}$. The roots α of the monster Lie algebra are labeled by pairs of integers (m, w) and the Fourier coefficients $c(mw)$ encode the root multiplicities $\mathrm{mult}\,\alpha$. This implies that the denominator formula of the monster Lie algebra can be viewed as a generating function of BPS states in this model. The denominator function turns out to be a special type of automorphic form on $SO(2, 2)/(SO(2) \times SO(2))$.

One can further show that the space of BPS states is actually a *module* for the monster Lie algebra $\mathfrak{m}$. This gives $\mathfrak{m}$ a natural interpretation as a BPS algebra [55]. This analysis extends to all CHL orbifolds [14] of $V^\natural \otimes \bar{V}^{f\natural}$ in which case the relevant algebras are the $\mathfrak{m}_g$'s constructed by Carnahan in his proof of generalized monstrous moonshine [11–13].

The VOA $V^{f\natural}$ was further studied in [37], where a BKM algebra was constructed upon which the Conway group Co_0 acts faithfully. This is a candidate for an algebra of BPS states in a certain compactification of type IIA string theory. In a similar spirit, in reference [36], we constructed a family of BKM algebras associated with F_{24}, the holomorphic $c = 12$ SCFT based on 24 free fermions. More precisely, we obtained one BKM for every choice of $\mathcal{N} = 1$ superconformal structure on F_{24}. Additionally, for these BKM algebras one can identify the denominator formulas with certain higher-rank automorphic forms. A similar story unfolds if one instead starts with the self-dual SVOA based on the E_8 lattice, V^{fE_8} [61,62].

It is the purpose of the present work to provide a spacetime interpretation of the aforementioned BKM (super)algebras, along the lines of [55,56]. The models we consider are now full (non-chiral) string theory, whose spacetime BPS states moreover still enjoy a close relationship with these BKMs, in the sense that 1) they form a module for a BKM (super)algebra and 2) the BKM (super)algebras act as symmetries of amplitudes which contain insertions only of BPS vertex operators. This provides a realization of these BKM algebras as algebras of BPS states in type II string theories, such that their denominator formulas correspond to BPS indices.

Summary and outline

Now, we come to highlight the key results in the present work. Let us consider 2d type IIA and heterotic string compactifications with holomorphically factorized worldsheet theories of the form $V_1 \otimes \bar{V}_2$, as introduced above. Let $\mathfrak{g}$ be the BKM (super)algebra associated with V_1 via the standard chiral construction due to Borcherds, briefly reviewed above.[2] In this work, we show the following:

1. The spacetime BPS states in the 2d compactification associated with $V_1 \otimes \bar{V}_2$ form a representation of the BKM algebra $\mathfrak{g}$ (Sect. 4).
2. $\mathfrak{g}$ acts as a symmetry on certain genus zero BPS saturated amplitudes in the theory (Sect. 4).
3. A suitably defined spacetime supersymmetric index in the theory reproduces the (super)denominator formula of $\mathfrak{g}$ (Sect. 5).
4. This index can also be reproduced via a path integral computation on a 2d Euclidean spacetime torus. The path integral precisely reduces to a familiar theta lift in the theory of automorphic forms (Sect. 6).

The plan of the rest of the paper is as follows. In Sect. 2, we describe basic features of the 2d spacetime string theories we are interested in, including worldsheet SCFTs, massless field content, spacetime supersymmetry algebras, and gravitational anomalies. In Sect. 3, we explicitly construct the cohomology of physical states in our holomorphically factorized models, leaving some technical details to "Appendix B." In Sect. 4, we focus on the subspace of spacetime BPS states $\mathcal{H}_{\mathrm{BPS}}$ and describe the action of the BKM algebra $\mathfrak{g}$ (constructed from the (S)VOAs V_1) on $\mathcal{H}_{\mathrm{BPS}}$. We further show that $\mathfrak{g}$ is a symmetry of genus zero BPS amplitudes. In Sect. 5 we consider a second quantized version of our 2d string theories, where we allow for an arbitrary number of fundamental strings, and study the resulting Hilbert space. We compute a number of natural supersymmetric indices in the second quantized theories from the Hamiltonian point of view and explain their relation to denominators of the BKM algebras from the previous section. In Sect. 6, we revisit the computation of the indices from the path integral point of view using the formalism of theta lifts from the theory of automorphic forms and clarify some subtleties that arise in these low-dimensional models. Finally, we conclude in Sect. 7 with a discussion of several open questions that emerged during the course of this work.

This paper also contains four appendices. In "Appendix A," we give a detailed analysis of gravitational anomalies and the B-field tadpole in our models. In "Appendix B," we provide a careful treatment of the semirelative cohomology of physical states. In "Appendix C," we discuss some details related to the zero-momentum R-R spectrum in one of our models, relevant to certain computations carried out in Sect. 5. And finally, in "Appendix D" we review some standard facts about theta lifts of vector-valued modular forms.

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[2]We note that it is not a priori obvious that this construction always gives rise to a BKM algebra, although this is verified in all cases. See Sect. 4.2 for more details.

Conventions and notation

- **Gamma matrices.** We use the two-dimensional metric $\eta^{\mu\nu} = \mathrm{diag}(-,+)$. The gamma matrices $\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ obey the algebra relations $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$ and the chirality matrix is $\Gamma \equiv \Gamma^0\Gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Majorana–Weyl spinors of positive and negative chirality have the form $\begin{pmatrix} u_+ \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ u_- \end{pmatrix}$, respectively. The massless Dirac equation is

$$
0 = k_\mu \Gamma^0 \Gamma^\mu u = \begin{pmatrix} -k_0 + k_1 & 0 \\ 0 & k_0 + k_1 \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix}
$$
$$
= \begin{pmatrix} k^0 + k^1 & 0 \\ 0 & -k^0 + k^1 \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix},
$$

  such that a positive chirality fermion exists only with $k^0 = -k^1$ and a negative chirality one only for $k^0 = k^1$.

- **Supersymmetry algebra.** We say that a theory has $(\mathcal{N}_+, \mathcal{N}_-)$ supersymmetry when there are $\mathcal{N}_+$ supercharges $\mathcal{Q}_+^i$ of positive chirality and $\mathcal{N}_-$ supercharges $\mathcal{Q}_-^i$ of negative chirality. In the absence of central charges, the supersymmetry algebra in two uncompactified dimensions is

$$
\{\mathcal{Q}_\alpha^i, \mathcal{Q}_\beta^j\} = 2\delta^{ij}(\Gamma^0\Gamma^\mu)_{\alpha\beta} P_\mu = 2\delta^{ij}(P^0\delta_{\alpha\beta} + P^1\Gamma_{\alpha\beta}); \tag{1.3}
$$

i.e.,

$$
\{\mathcal{Q}_+, \mathcal{Q}_+\} = 2\delta^{ij}(P^0 + P^1), \quad \{\mathcal{Q}_-, \mathcal{Q}_-\} = 2\delta^{ij}(P^0 - P^1),
$$
$$
\{\mathcal{Q}_+, \mathcal{Q}_-\} = 0. \tag{1.4}
$$

2. Two-dimensional Models

In this section, we discuss the class of models of interest in this paper. These models are string theory compactifications to 1+1 dimensions where the worldsheet CFT takes the holomorphically factorized form $V_1 \otimes \bar{V}_2$. We will always take V_2 to be one of the three self-dual SVOAs of central charge 12 (see Sect. 2.1). In the case of type IIA compactifications, we will also take V_1 to be one of these three theories. For the heterotic models we consider, we will allow V_1 to be one of the 71 known self-dual VOAs of central charge 24 [63–66]. In all the SVOAs we consider, the representations of the Virasoro algebra obey the standard spin-statistics relation, namely that even states have integer L_0 eigenvalues, while odd states have half-integer L_0 eigenvalues.

2.1. The Three $c = 12$ SVOAs

First, we briefly review the three $c = 12$ SVOAs we will consider. In [21], using a strategy employed in [40], it was proven that there are exactly three so-called self-dual SVOAs with central charge 12. These theories are:

1. V^{fE_8}: This is the supersymmetric vertex operator algebra based on the E_8 lattice. It contains 8 chiral free bosons compactified on the E_8 root lattice, and their 8 fermionic superpartners.

2. $V^{f\natural}$: This is the unique self-dual SVOA with $c = 12$ and no fields of conformal weight $1/2$ in the NS sector. In [25] it was proved that, up to automorphisms, it admits a unique choice of $\mathcal{N} = 1$ superconformal current which is stabilized by Conway's group Co_0. (The $\mathbb{Z}_2$ center of Co_0 acts non-trivially only on the Ramond sector.)

3. F_{24}: This is a theory of 24 free chiral fermions. An $\mathcal{N} = 1$ supercurrent can be defined by taking a linear combination of cubic Fermi terms, and the 8 inequivalent choices are in one-to one correspondence with semisimple Lie algebras of dimension 24. Given a choice of $\mathcal{N} = 1$ superconformal structure, the 24 currents that are supersymmetric descendants of the free fermions generate an affine Kac–Moody algebra, of which there are eight possibilities [33]:

$$(\widehat{su}(2)_2)^{\oplus 8}, \quad (\widehat{su}(3)_3)^{\oplus 3}, \quad \widehat{su}(4)_4 \oplus (\widehat{su}(2)_2)^{\oplus 3}, \quad \widehat{su}(5)_5, \quad \widehat{so}(5)_3 \oplus \hat{g}_{2,4},$$

$$\widehat{so}(5)_3 \oplus \widehat{su}(3)_3 \oplus (\widehat{su}(2)_2)^{\oplus 2}, \quad \widehat{so}(7)_5 \oplus \widehat{su}(2)_2, \quad \widehat{sp}(6)_4 \oplus \widehat{su}(2)_2.$$

We discussed further aspects of this theory in [36].

Strictly speaking, a self-dual SVOA corresponds to the NS sector of a chiral superconformal field theory, while the Ramond sector is a (canonically) twisted module. We will be slightly imprecise in this respect and use the word self-dual SVOA to denote the sum of the NS and Ramond sector.

In the NS sector of each holomorphic SCFT, there is a canonical $\mathbb{Z}_2$ symmetry—the fermion number operator $(-1)^F$—which acts trivially on states of integral L_0 eigenvalue and as multiplication by -1 on states with half-integral eigenvalue. The Ramond sector can be seen as a twisted module with respect to this symmetry. For all the theories we consider, the definition of the fermion number can be extended to the Ramond sector in such a way that it is a symmetry of the OPE of order 2. However, there is an ambiguity by an overall sign in this definition. For F_{24} and V^{fE_8}, the zero mode of any weight $1/2$ NS field provides an isomorphism between the Ramond eigenspaces of positive and negative fermion number, so the choice of the sign of $(-1)^F$ is immaterial. For $V^{f\natural}$, there is no such weight $1/2$ NS field, and indeed all the 24 Ramond ground states (i.e., of conformal weight $1/2$) have the same $(-1)^F$ eigenvalue, with can be chosen to be $+1$ or -1. This choice will be important when we discuss the GSO projection in superstring compactifications in the next subsections. It is useful to introduce the notation $V_+^{f\natural}$ and $V_-^{f\natural}$ to denote the SVOA $V^{f\natural}$ with the choice of fermion number equal, respectively, to $+1$ and -1 on the Ramond ground states. For conformal weight strictly greater than $1/2$, the Ramond subspaces with positive and negative fermion number are again isomorphic, with the isomorphism given by the zero mode of the $\mathcal{N} = 1$ supercurrent (see below).

For each of these theories $V \in \{V^{fE_8}, V^{f\natural}, F_{24}\}$, we can define a set of four torus partition functions, corresponding to the four choices of boundary condition for the fermions,

$$\phi_{\mathrm{NS}}(\tau; V) := \mathrm{Tr}_{\mathrm{NS}}(\mathsf{q}^{L_0 - 1/2}) = \frac{\eta^{48}(\tau)}{\eta^{24}(\tau/2)\eta^{24}(2\tau)} - 24 + \chi^{\mathrm{NS}}(V) \quad (2.1)$$

$$\phi_{\widetilde{\mathrm{NS}}}(\tau; V) := \mathrm{sTr}_{\mathrm{NS}}(\mathsf{q}^{L_0 - 1/2}) = \frac{\eta^{24}(\tau/2)}{\eta^{24}(\tau)} + 24 - \chi^{\mathrm{NS}}(V) \tag{2.2}$$

$$\phi_{\mathrm{R}}(\tau; V) := \mathrm{Tr}_{\mathrm{R}}(\mathsf{q}^{L_0 - 1/2}) = 2^{12} \frac{\eta^{24}(2\tau)}{\eta^{24}(\tau)} + \left(\chi_+^{\mathrm{R}}(V) + \chi_-^{\mathrm{R}}(V)\right) \tag{2.3}$$

$$\phi_{\tilde{\mathrm{R}}}(\tau; V) := \mathrm{sTr}_{\mathrm{R}}(\mathsf{q}^{L_0 - 1/2}) = \chi_+^{\mathrm{R}}(V) - \chi_-^{\mathrm{R}}(V), \tag{2.4}$$

where by sTr we denote the supertrace, i.e., the trace with the insertion of the fermion number operator $(-1)^F$, and $\chi^{\mathrm{NS}}(V)$, $\chi_+^{\mathrm{R}}(V)$, and $\chi_-^{\mathrm{R}}(V)$ are constants. The partition functions $\{\phi_{\mathrm{NS}}, \phi_{\widetilde{\mathrm{NS}}}, \phi_{\mathrm{R}}\}$ form a three-component vector-valued representation of $SL(2, \mathbb{Z})$, whereas $\phi_{\tilde{\mathrm{R}}}$ is a singlet under the action of $SL(2, \mathbb{Z})$. We have expressed them explicitly in terms of eta quotients, where we have used the Dedekind eta function, defined as

$$\eta(\tau) := \mathsf{q}^{1/24} \prod_{n=1}^{\infty} (1 - \mathsf{q}^n), \tag{2.5}$$

and throughout we use the definition $\mathsf{q} := e^{2\pi i \tau}$. The first several terms in the q-expansion of each of these functions are

$$\phi_{\mathrm{NS}}(\tau; V) = \mathsf{q}^{-1/2} + \chi^{\mathrm{NS}}(V) + 276\mathsf{q}^{1/2} + 2048\mathsf{q} + 11202\mathsf{q}^{3/2} + \cdots \tag{2.6}$$

$$\phi_{\widetilde{\mathrm{NS}}}(\tau; V) = \mathsf{q}^{-1/2} - \chi^{\mathrm{NS}}(V) + 276\mathsf{q}^{1/2} - 2048\mathsf{q} + 11202\mathsf{q}^{3/2} + \cdots \tag{2.7}$$

$$\phi_{\mathrm{R}}(\tau; V) = \left(\chi_+^{\mathrm{R}}(V) + \chi_-^{\mathrm{R}}(V)\right) + 4096\mathsf{q} + 98304\mathsf{q}^2 + 1228800\mathsf{q}^3 + \cdots \tag{2.8}$$

$$\phi_{\tilde{\mathrm{R}}}(\tau; V) = \left(\chi_+^{\mathrm{R}}(V) - \chi_-^{\mathrm{R}}(V)\right). \tag{2.9}$$

The constant $\chi^{\mathrm{NS}}(V)$ captures the number of spin-1/2 fields of V in the NS sector. All of these fields are odd under the fermion number operator. The constants $\chi_+^{\mathrm{R}}(V)$ and $\chi_-^{\mathrm{R}}(V)$ capture the number of even and odd spin-1/2 fields in the Ramond sector, respectively. The values of these constants for each choice of $V \in \{V^{fE_8}, V^{f\natural}, F_{24}\}$ are listed in the following table.

SVOA V	$\chi^{\mathrm{NS}}(V)$	$\chi_+^{\mathrm{R}}(V)$	$\chi_-^{\mathrm{R}}(V)$
$V_{E_8}^f$	8	8	8
$V_+^{f\natural}$	0	24	0
$V_-^{f\natural}$	0	0	24
F_{24}	24	0	0.

Note that each of these theories has a total of 24 spin-1/2 fields across both of the NS and R sectors.

In order to construct a physical string model, one also needs a choice of $\mathcal{N} = 1$ supercurrent in these theories. As shown in [25], $V^{f\natural}$ admits a unique (up to isomorphism) $\mathcal{N} = 1$ supercurrent, which is stabilized by the sporadic group Co_0, the group of automorphisms of the Leech lattice. In the theory V^{fE_8}, there is a standard choice of supercurrent of the form

$$G(z) \sim \sum_{i=1}^{8} : \partial X^i \psi^i : (z),$$

which is left invariant by the group of inner automorphisms $U(1)^8 : W(E_8)$, i.e., a split extension of $W(E_8)$, the Weyl group of E_8, by the group $U(1)^8$. Finally, the theory F_{24} has 8 inequivalent choices for a supercurrent, all of the form

$$G(z) \sim \sum_{i,j,k} c_{ijk} : \lambda^i \lambda^j \lambda^k : (z),$$

where the coefficients c_{ijk} are the structure constants of a semisimple Lie algebra g of total dimension 24 [33].

As described in [21], the three theories $V^{fE_8}, V^{f\natural}, F_{24}$ can be related to each other via orbifolding by a cyclic group of symmetries, that preserve the $\mathcal{N} = 1$ superconformal structure. This is described in detail in §3 of [36] for orbifolds relating F_{24} and V^{fE_8}. When we turn to full string theory models where these SVOAs are components of the worldsheet theory, we expect these relations will be important for understanding string dualities relating these models.

2.2. Superstring Models

We can build two-dimensional compactifications of type II string theory by taking worldsheet theories of the form $V_1 \otimes \bar{V}_2$ with $V_1, V_2 \in \{V^{fE_8}, V^{f\natural}, F_{24}\}$. For each such model, we list the number of fields of spin 0 or $1/2$ in the massless spectrum (there are also fields with no propagating local degrees of freedom, namely containing only $G_{\mu\nu}$, $B_{\mu\nu}$, the dilaton ϕ, and possibly gravitinos and vector fields) and the number of spacetime supersymmetries in table 1.[3]

The theory $V^{fE_8} \otimes \bar{V}^{fE_8}$ is equivalent to the type IIA string compactified on $T^8 = R^8/\Lambda_{E_8}$ at the holomorphically factorized point in moduli space. As both $V^{f\natural}$ and F_{24} can be realized as orbifolds of V^{fE_8}, all other theories in the table can be thought of as asymmetric orbifolds of the type IIA string on T^8 at the holomorphically factorized point. Some of these models were recently considered in [28].

TABLE 1. The two-dimensional type IIA models we consider in this paper

Theory	NS-NS	R-R	NS-R	R-NS	SUSY
$V^{fE_8} \otimes \bar{V}^{fE_8}$	8×8	8×8	8×8	8×8	$(16, 16)$
$V^{fE_8} \otimes \bar{V}^{f\natural}$	0	8×24	8×24	0	$(8, 32)$
$F_{24} \otimes \bar{V}^{fE_8}$	24×8	0	24×8	0	$(8, 8)$
$V_+^{f\natural} \otimes \bar{V}_-^{f\natural}$	0	24×24	0	0	$(0, 48)$
$V_-^{f\natural} \otimes \bar{V}_-^{f\natural}$	0	24×24	0	0	$(24, 24)$
$F_{24} \otimes \bar{V}_-^{f\natural}$	0	0	24×24	0	$(0, 24)$
$F_{24} \otimes \bar{F}_{24}$	24×24	0	0	0	$(0, 0)$

We enumerate the number of massless fields from each sector in the first four columns and identify the spacetime supersymmetry algebra in the fifth column

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[3]Note that $F_{24} \otimes \bar{F}_{24}$ has massive spacetime fermions, though it has no spacetime supersymmetries.

TABLE 2. The two-dimensional heterotic models we consider in this paper

Theory	NS	R	SUSY
$V_1 \otimes \bar{V}^{fE_8}$	$N \times 8$	$N \times 8$	$(8,8)$
$V_1 \otimes \bar{V}_+^{f\natural}$	0	$N \times 24$	$(0,24)$
$V_1 \otimes \bar{F}_{24}$	$N \times 24$	0	$(0,0)$

Here, N is the number of currents in the VOA V_1

Similarly, we will also consider two-dimensional compactifications of heterotic string theory by considering worldsheet theories of the form $V_1 \otimes \bar{V}_2$, with V_1 a self-dual bosonic VOA with $c = 24$ and $V_2 \in \{V^{fE_8}, V^{f\natural}, F_{24}\}$. In Table 2, we list the number of massless fields and spacetime supersymmetries in terms of N, the number of currents in V_1.

When considering type IIA or heterotic string theory compactified to two dimensions, there is the possibility that there is a tadpole for the B-field [49,50,69] (see also [68] and [67]). Such a tadpole can be canceled by adding a certain number of spacetime-filling strings. In "Appendix A," we discuss the derivation of the tadpole for the theories we consider.

3. Physical States in Type II on $V_1 \otimes \bar{V}_2$

In this section, we will identify the physical states of type II string theory compactified on $V_1 \otimes \bar{V}_2$, where V_1 and V_2 are each one of the self-dual $\mathcal{N} = 1$ SVOAs with $c = 12$, i.e., $V^{f\natural}$, V_{E_8}, or F_{24} with a choice of $\mathcal{N} = 1$ structure. The final outcome of this section is that, roughly speaking, the space of physical states furnishes a tensor product of the physical states for the "chiral" superstring on V_1 times the physical states for the "anti-chiral" superstring on V_2.

A similar factorization holds for the physical states in the heterotic string models on $V_1 \otimes \bar{V}_2$, where V_1 is a self-dual bosonic VOA of central charge $c = 24$ and V_2 a self-dual $\mathcal{N} = 1$ SVOA with $c = 12$.

3.1. Chiral Compactification

Let us first recall the main steps in the construction of the physical states for the "chiral" version of superstrings compactified on a self-dual $c = 12$ $\mathcal{N} = 1$ SVOA V. This material is largely standard, and we closely follow the notation and treatment of [36], to which we refer for further details. One starts from a product vertex algebra with underlying Hilbert space $\mathcal{H}^m \otimes \mathcal{H}^{gh}$. The "matter sector" $\mathcal{H}^m$ is itself the product of the internal SVOA V and the super-vertex algebra (SVA) of $1 + 1$-dimensional spacetime chiral bosons X^μ and fermions ψ^μ, $\mu = 0, 1$. The "ghost" sector $\mathcal{H}^{gh}$ is the product of a bc ghost and a $\beta\gamma$ superghost system.[4] Recall that the $\beta\gamma$ system has a class of

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[4]More precisely, we restrict as always to the relevant subalgebra of the $\eta\zeta\phi$-VA, where $\beta = \partial\zeta e^{-\phi}$, $\gamma = \eta e^\phi$.

irreducible modules labeled by the "picture number" p, which takes values in $\mathbb{Z}$ (NS sector) or in $\frac{1}{2} + \mathbb{Z}$ (R sector) [59]. The module at picture number p is constructed starting from a state $|p\rangle$ satisfying $\beta_r|p\rangle = 0 = \gamma_s|p\rangle$ for all $r \geq -\frac{1}{2} - p$ and $s > \frac{1}{2} + p$, and acting in all possible ways on $|p\rangle$ with the (mutually commuting) modes β_r, γ_s with $r < -\frac{1}{2} - p$ and $s \leq \frac{1}{2} + p$. The L_0 eigenvalues of the representation at picture number p are bounded from below only for $p = -1, -1/2$ or $-3/2$; these are called the "canonical picture numbers."

The product $\mathcal{H} = \mathcal{H}^m \otimes \mathcal{H}^{gh}$ splits into four sectors $\mathcal{H}_{NS+}$, $\mathcal{H}_{NS-}$, $\mathcal{H}_{R+}$, and $\mathcal{H}_{R-}$, corresponding to the four spin structures on the torus. Here,

$$\mathcal{H}_{NS} = \mathcal{H}^m_{NS} \otimes \mathcal{H}^{gh}_{NS}$$

and

$$\mathcal{H}_R = \mathcal{H}^m_R \otimes \mathcal{H}^{gh}_R$$

denote the NS and Ramond sector, respectively, and $\mathcal{H}_{NS\pm}$ and $\mathcal{H}_{R\pm}$ are the projections onto states of positive or negative fermion number $(-1)^F$.[5] Next we perform the GSO projection, which restricts to the subspace of total fermion number $+1$, i.e.,

$$\mathcal{H}_{\text{GSO}} := \mathcal{H}_{NS_+} \oplus \mathcal{H}_{R_+}. \tag{3.1}$$

We further restrict to the $\ker b_0 \cap \ker L_0$

$$C := \mathcal{H}_{\text{GSO}} \cap \ker b_0 \cap \ker L_0. \tag{3.2}$$

The theory contains the conserved currents $j^{gh} = - : \xi\eta : + : bc :$, $j^p = \partial\phi + : \xi\eta :$ and $(P^0, P^1) = (i\partial X^0, i\partial X^1)$, whose zero modes eigenvalues correspond, respectively, to the ghost number n, the picture number p, and the winding-momenta $k := (k^0, k^1)$. The space C can be decomposed into components $C^n_p(k)$ with definite n, p, and k, as

$$C = \bigoplus_{\{k; n, p\}} C^n_p(k). \tag{3.3}$$

In [36,37], the momenta k were assumed to take values in the even unimodular lattice $\Gamma^{1,1}$; here, we leave them unspecified, for the time being.

The nilpotent BRST charge Q commutes with L_0, picture number, and winding-momenta and has ghost number 1, so that for each momentum k and picture number p it defines a complex

$$\cdots \to C^{n-1}_p(k) \to C^n_p(k) \to C^{n+1}_p(k) \to \cdots \tag{3.4}$$

graded by the ghost number n. From this complex, one builds the BRST cohomology $H^n_p(k)$. Note that the picture number takes integral values in the NS sector and half-integral values in the Ramond sector.

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[5]In the Ramond sector of the ψ^μ, X^μ theory, the fermion number acts on the ground states $|\pm\rangle$ by $\pm i$ (i.e., $(-1)^F$ coincides with $i\Gamma$, where Γ is the chirality matrix in $1+1$ dimensional spacetime), while in the β, γ theory $(-1)^F$ acts by $e^{\pi i p}$ on the ground state with picture number p. Since p is integral in the NS sector and half-integral in the Ramond sector, the fermion number $(-1)^F$ has order 2 in the full matter+ghost theory.

It is also useful to recall the picture-changing homomorphism

$$X : H_p^n(k) \to H_{p+1}^n(k), \tag{3.5}$$

which is an isomorphism for $k \neq 0$. This allows us to focus on the "canonical pictures" $p = -1$ for the NS sector and $p = -1/2$ for the R sector. Furthermore, for $k \neq 0$ the chiral cohomology groups $H_{-1}^n(k)$ and $H_{-1/2}^n(k)$ are trivial unless $n = 1$. Given this, we define the Hilbert space of (chiral) physical states as

$$\mathcal{H}^{\mathrm{phys}} := \bigoplus_k H_{-1}^1(k) \oplus H_{-\frac{1}{2}}^1(k) = \bigoplus_k H^1(k), \tag{3.6}$$

where

$$H^n(k) := H_{-1}^n(k) \oplus H_{-1/2}^n(k). \tag{3.7}$$

Note that the same construction could be implemented starting from the GSO projection $\mathcal{H}_{\mathrm{GSO}} = \mathcal{H}_{NS_+} \oplus \mathcal{H}_{R_-}$; the corresponding physical states are related by a spacetime parity transformation, which inverts the sign of X^1, ψ^1, and k^1.

Analogously, starting from anti-holomorphic rather than holomorphic vertex algebras, one can define a space of physical states for an anti-chiral superstring, in terms of the cohomology groups $\bar{H}_{\bar{p}}^{\bar{n}}(k)$ of a complex $\bar{C}_{\bar{p}}^{\bullet}(k)$ with respect to a nilpotent BRST operator Q_r.

3.2. Non-chiral Compactification

We are now ready to construct the non-chiral (left- and right-moving) compactification of type IIA on $V_1 \otimes \bar{V}_2$, where V_1 and V_2 are not necessarily the same SVOA. We define the spaces $\mathcal{H}^m, \mathcal{H}^{gh}$ and their right-moving analogues $\bar{\mathcal{H}}^m, \bar{\mathcal{H}}^{gh}$ as above. Similarly, we define NS$_\pm$ and R$_\pm$ sectors for both the left- and the right-moving spaces of states and the corresponding spaces $\mathcal{H}_{NS_\pm}$, $\mathcal{H}_{R_\pm}, \bar{\mathcal{H}}_{NS_\pm}, \bar{\mathcal{H}}_{R_\pm}$. The GSO projected space of states is

$$\mathcal{H}_{\mathrm{GSO}} := (\mathcal{H}_{NS_+} \otimes \bar{\mathcal{H}}_{NS_+}) \oplus (\mathcal{H}_{NS_+} \otimes \bar{\mathcal{H}}_{R_-}) \oplus (\mathcal{H}_{R_+} \otimes \bar{\mathcal{H}}_{NS_+})$$
$$\oplus (\mathcal{H}_{R_+} \otimes \bar{\mathcal{H}}_{R_-}). \tag{3.8}$$

Note that we are only considering type IIA GSO projection. We could also consider type IIB theories, by projecting the NS-R and R-R sectors on the subspace with positive, rather than negative right-moving fermion number. However, we would not obtain anything new in this way: As stressed in Sect. 2, there is a choice of sign for the fermion number in the Ramond sector of the internal SVOA V_2, and type IIB on $V_1 \otimes \bar{V}_2$ with one choice of sign for V_2 is equivalent to type IIA with the opposite choice. In particular, type IIA on $V_1 \otimes \bar{V}_{\pm}^{f\natural}$ is equivalent to type IIB on $V_1 \otimes \bar{V}_{\mp}^{f\natural}$; when V_2 is either V^{fE_8} or F_{24}, type IIA and type IIB GSO projections give equivalent theories.

We now restrict to the space

$$\mathsf{C} := \mathcal{H}_{\mathrm{GSO}} \cap \ker b_0 \cap \ker \bar{b}_0 \cap \ker L_0 \cap \ker \bar{L}_0, \tag{3.9}$$

which decomposes as

$$\mathsf{C} = \bigoplus_{\{k_l, k_r\}} \mathsf{C}(k_l, k_r) = \bigoplus_{\{k_l, k_r; n, \bar{n}, p, \bar{p}\}} \mathsf{C}_{p; \bar{p}}^{n, \bar{n}}(k_l, k_r), \tag{3.10}$$

where $k_l := (k_l^0, k_l^1)$ and $k_r := (k_r^0, k_r^1)$ are left- and right-moving momenta, i.e., the eigenvalues of (P_l^0, P_l^1) and (P_r^0, P_r^1), respectively. Here, P_l^μ and P_r^μ are the zero modes of $i\partial X^\mu(z)$ and $i\bar{\partial} X^\mu(\bar{z})$, respectively. We will consider two possibilities for the values of these momenta: In the "uncompactified case," where the target spacetime is $\mathbb{R}^{1,1}$, one has that $k_l^0 = k_r^0$ and $k_l^1 = k_r^1$ and both take values in $\mathbb{R}$; in the "compactified case," where a spacelike direction of spacetime is compactified on a circle of radius R, one has

$$k_l^1 = \frac{1}{\sqrt{2}} \left(\frac{m}{R} - wR \right), \qquad k_r^1 = \frac{1}{\sqrt{2}} \left(\frac{m}{R} + wR \right), \qquad m, w \in \mathbb{Z}, \tag{3.11}$$

and again $k_l^0 = k_r^0 \in \mathbb{R}$.

Throughout this section, we will construct physical states in relative cohomology for simplicity, from the space C above. We will treat the physical states of the complete, non-chiral superstring more properly in semirelative cohomology (i.e., imposing the condition $b_0 - \bar{b}_0 = 0$ rather than $b_0 = \bar{b}_0 = 0$) in "Appendix B." These two cohomologies are isomorphic at nonzero momentum, and hence in particular are both supported in ghost number $(1, 1)$. The relative and semirelative cohomologies differ, however, at zero momentum. Moreover, zero momentum states in both relative and semirelative cohomologies are supported in various ghost numbers. At ghost number 2, which are the states on which we will construct a BKM algebra action in the next section, the semirelative cohomology only differs from the relative cohomology ($\ker b_0 \cap \ker \bar{b}_0$) by a single state at zero-momentum which is largely unimportant for our subsequent analyses.

Now, if we denote by $C_p^n(k_l)$ and $\bar{C}_{\bar{p}}^{\bar{n}}(k_r)$ the respective spaces that we would obtain by the chiral and anti-chiral construction described above for V_1 and $\bar{V}_2$ (with the appropriate GSO projections), one has

$$\mathsf{C}_{p, \bar{p}}^N(k_l, k_r) = \bigoplus_{n + \bar{n} = N} C_p^n(k_l) \otimes \bar{C}_{\bar{p}}^{\bar{n}}(k_r), \tag{3.12}$$

where N is the total (holomorphic plus anti-holomorphic) ghost number. Furthermore, if Q_l and Q_r are the chiral and anti-chiral BRST charges for the corresponding left- and right-moving complexes, the operator $Q := Q_l + Q_r \equiv Q_l \otimes 1 + 1 \otimes Q_r$ is nilpotent (in particular, $\{Q_l, Q_r\} = 0$) and has total ghost number 1, so that it defines a complex

$$\cdots \to \mathsf{C}_{p, \bar{p}}^{N-1}(k_l, k_r) \to \mathsf{C}_{p, \bar{p}}^N(k_l, k_r) \to \mathsf{C}_{p, \bar{p}}^{N+1}(k_l, k_r) \to \cdots \tag{3.13}$$

This complex defines a Q-cohomology $\mathsf{H}_{p, \bar{p}}^N(k_l, k_r)$ such that

$$\mathsf{H}_{p, \bar{p}}^N(k_l, k_r) = \bigoplus_{n + \bar{n} = N} H_p^n(k_l) \otimes \bar{H}_{\bar{p}}^{\bar{n}}(k_r), \tag{3.14}$$

where $H_p^n(k_l)$ and $\bar{H}_{\bar{p}}^{\bar{n}}(k_r)$ are the corresponding left- and right-moving cohomologies of Q_l and $\bar{Q}_r$, respectively.

Finally, we restrict to the canonical pictures $p, \bar{p} \in \{-1/2, -1\}$, and total ghost number $N = 2$. If $k \neq 0$, then the cohomology group reduces to

$$\mathsf{H}^2_{p,\bar{p}}(k_l, k_r) = H^1_p(k_l) \otimes \bar{H}^1_{\bar{p}}(k_r), \tag{3.15}$$

because the other components are zero. Again, see "Appendix B" for a discussion of (unimportant for our purposes) subtleties at zero-momentum. We define the Hilbert space of physical states as

$$\mathcal{H}^{\text{phys}} := \bigoplus_{\{k_l, k_r\}} \mathsf{H}^2(k_l, k_r)$$

$$= \bigoplus_{\{k_l, k_r\}} (\mathsf{H}^2_{-1,-1}(k_l, k_r) \oplus \mathsf{H}^2_{-1,-\frac{1}{2}}(k_l, k_r) \oplus \mathsf{H}^2_{-\frac{1}{2},-1}(k_l, k_r)$$

$$\oplus \mathsf{H}^2_{-\frac{1}{2},-\frac{1}{2}}(k_l, k_r)),$$

where the four summands correspond, respectively, to the NS-NS, NS-R, R-NS and R-R sectors, and

$$\mathsf{H}^N(k_l, k_r) := \bigoplus_{p,\bar{p} \in \{-1,-1/2\}} \mathsf{H}^N_{p,\bar{p}}(k_l, k_r). \tag{3.16}$$

3.3. Heterotic Strings

Similar results hold for the physical states of heterotic strings compactified on $V_1 \otimes \bar{V}_2$, where V_1 is a bosonic self-dual VOA of central charge 24 and V_2 is one of the self-dual SVOA of central charge 12. The spacetime matter SCFT in $1 + 1$ dimensions includes only right-moving fermions $\bar{\psi}^\mu(\bar{z})$, in addition to the usual bosonic fields $X^\mu(z, \bar{z})$. Similarly, the holomorphic ghost sector includes only the bc-system and not the $\beta\gamma$-system; the anti-holomorphic ghost sector is as in type II theories.

The GSO-projected space of states is

$$\mathcal{H}_{\text{GSO}} := (\mathcal{H} \otimes \bar{\mathcal{H}}_{NS_+}) \oplus (\mathcal{H} \otimes \bar{\mathcal{H}}_{R_+}), \tag{3.17}$$

where $\mathcal{H}$ is the space of states of the holomorphic bosonic matter and ghost CFT. We restrict to the space

$$\mathsf{C} := \mathcal{H}_{\text{GSO}} \cap \ker b_0 \cap \ker \bar{b}_0 \cap \ker L_0 \cap \ker \bar{L}_0, \tag{3.18}$$

which decomposes as

$$\mathsf{C} = \bigoplus_{\{k_l, k_r\}} \mathsf{C}(k_l, k_r) = \bigoplus_{\{k_l, k_r; n, \bar{n}, \bar{p}\}} \mathsf{C}^{n,\bar{n}}_{\bar{p}}(k_l, k_r), \tag{3.19}$$

where now only the anti-holomorphic picture number $\bar{p}$ is defined. The spaces $\mathsf{C}^{n,\bar{n}}_{\bar{p}}(k_l, k_r)$ decompose as

$$\mathsf{C}^N_{\bar{p}}(k_l, k_r) = \bigoplus_{n+\bar{n}=N} C^n(k_l) \otimes \bar{C}^{\bar{n}}_{\bar{p}}(k_r), \tag{3.20}$$

where $\bar{C}^{\bar{n}}_{\bar{p}}(k_r)$ is as in the previous section, and $C^n(k_l)$ is the space one would obtain by a chiral bosonic string construction starting from the internal VOA V_1, tensoring with the chiral bosonic spacetime matter in $1 + 1$ dimensions

and bc-ghost system, and restricting to $\ker b_0 \cap \ker L_0$. One can define a nilpotent BRST charge Q of ghost number 1 that is the sum $Q = Q_l + Q_r$ of BRST charges Q_l and Q_r acting only on the left- and right-moving sectors, respectively. For each $\bar{p}$ and (k_l, k_r), one has a complex

$$\cdots \to \mathsf{C}_{\bar{p}}^{N-1}(k_l, k_r) \to \mathsf{C}_{\bar{p}}^{N}(k_l, k_r) \to \mathsf{C}_{\bar{p}}^{N+1}(k_l, k_r) \to \cdots \tag{3.21}$$

and the corresponding cohomology $\mathsf{H}_{\bar{p}}^{N}(k_l, k_r)$ is given by

$$\mathsf{H}_{\bar{p}}^{N}(k_l, k_r) = \bigoplus_{n+\bar{n}=N} H^n(k_l) \otimes \bar{H}_{\bar{p}}^{\bar{n}}(k_r). \tag{3.22}$$

Here, $\bar{H}_{\bar{p}}^{\bar{n}}(k_r)$ is as in the type II case, while $H^n(k_l)$ is the cohomology of the complex $C^\bullet(k_l)$ with respect to the nilpotent operator Q_l. For $k_l, k_r \neq 0$, the cohomology group $\mathsf{H}_{\bar{p}}^{N}(k_l, k_r)$ is non-trivial only at ghost number $N = 2$ and factorizes as

$$\mathsf{H}_{\bar{p}}^{2}(k_l, k_r) = H^1(k_l) \otimes \bar{H}_{\bar{p}}^{1}(k_r). \tag{3.23}$$

Thanks to the picture-changing isomorphism, one can focus on the canonical pictures $\bar{p} = -1$ (NS sector) and $\bar{p} = -1/2$ (Ramond sector). The space of physical states of the heterotic strings is then defined as

$$\mathcal{H}^{\mathrm{phys}} := \bigoplus_{\{k_l, k_r\}} \mathsf{H}^2(k_l, k_r) = \bigoplus_{\{k_l, k_r\}} (\mathsf{H}_{-1}^2(k_l, k_r) \oplus \mathsf{H}_{-\frac{1}{2}}^2(k_l, k_r)), \tag{3.24}$$

where the sum over momenta depends on whether the $1+1$-dimensional spacetime is compactified or not, as in the type II case.

4. BKM Algebras of BRST Exact States

Now that we have constructed the physical states of our model, we can define the corresponding action of a Borcherds–Kac–Moody (BKM) algebra on a particular subsector of the space of physical states. A BKM algebra is a type of infinite-dimensional Lie algebra introduced by Borcherds [10] expanding the notion of a Kac–Moody algebra. The construction of a BKM superalgebra on a chiral superstring is well known [61, 62], so at a point in string theory moduli space where the worldsheet theory holomorphically factorizes, one might naively expect two complementary BKM algebras acting on the left and right. However, a tensor product of two Lie algebras is not a Lie algebra, so a more complicated action of a BKM algebra must arise in the physical superstring if it is to arise at all. Moreover, the chiral superstring construction of a BKM requires all spacetime dimensions to be compactified, and it would be far more interesting if the algebra structure arose in a standard non-chiral string theory where at least the time direction remains uncompactified.

　　To set the notation, we will begin this section with a brief overview of the basic structure theory of BKMs. Then, we proceed to discuss BPS states and show that they form a representation of a BKM algebra $\mathfrak{g}$. Finally, we demonstrate that $\mathfrak{g}$ is also a symmetry of BPS saturated genus zero amplitudes. We also offer some speculations on the physical interpretation of these results.

Throughout the paper we will use the short term "BKM algebra" to collectively refer to *both* BKM Lie algebras and BKM Lie superalgebras. We hope that this will not cause confusion.

4.1. Basic Structure Theory of BKM Algebras

Let us review the definition of a Borcherds–Kac–Moody superalgebra. For more details, see [60]. Recall that, in general, a *Lie superalgebra* is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a Lie bracket satisfying the $\mathbb{Z}_2$-graded version of the usual skew-symmetry properties and Jacobi identity. A BKM Lie superalgebra $\mathfrak{g}$ is characterized by a Cartan matrix A, which is allowed to have infinite rank and is generically of indefinite signature [7,60]. Let I be a set (finite or countably infinite) indexing the simple roots of $\mathfrak{g}$ and let $S \subseteq I$ be a subset indexing the odd simple roots. Let $\mathfrak{h}_\mathbb{R}$ be a real vector space with generators $h_i, i \in I$, and equipped with a non-degenerate symmetric real-valued bilinear form $(\cdot|\cdot)$. The bilinear form satisfies the following three properties:

(1) $(h_i|h_j) \leq 0$ if $i \neq j$,

(2) If $(h_i|h_i) > 0$ then $\dfrac{2(h_i|h_j)}{(h_i|h_i)} \in \mathbb{Z}$ for all $j \in I$,

(3) If $(h_i|h_i) > 0$ and $i \in S$ then $\dfrac{(h_i|h_j)}{(h_i|h_i)} \in \mathbb{Z}$ for all $j \in I$.   (4.1)

Now set $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$. The Cartan matrix is the symmetric matrix A with entries $A_{ij} = (h_i|h_j)$. The BKM superalgebra associated with the vector space $\mathfrak{h}$ and (generalized) Cartan matrix A is the Lie superalgebra $\mathfrak{g}$ generated by the elements $h_i, e_i, f_i, i \in I$, subject to the relations:

$$[e_i, f_j] = \delta_{ij} h_i, \quad [h_i, h_j] = 0$$
$$[h, e_i] = (h|h_i)e_i, \quad [h, f_i] = -(h|h_i)f_i, \quad \forall h \in \mathfrak{h},$$
$$\deg(e_i) = \deg(f_i) = 0 \quad \text{if } i \notin S \qquad \text{and} \qquad \deg(e_i) = \deg(f_i) = 1, \quad \text{if } i \in S$$
$$\mathrm{ad}_{e_i}^{1-2A_{ij}/A_{ii}} e_j = 0, \quad \mathrm{ad}_{f_i}^{1-2A_{ij}/A_{ii}} f_j = 0 \quad \text{if } A_{ii} > 0 \quad \text{and} \quad i \neq j,$$
$$\mathrm{ad}_{e_i}^{1-A_{ij}/A_{ii}} e_j = 0, \quad \mathrm{ad}_{f_i}^{1-A_{ij}/A_{ii}} f_j = 0 \quad \text{if } i \in S, \quad A_{ii} > 0 \quad \text{and} \quad i \neq j,$$
$$[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } A_{ij} = 0. \tag{4.2}$$

Here, deg denotes the $\mathbb{Z}_2$-grading. In the special case when $A_{ii} > 0$ for all $i \in I$ the resulting algebra $\mathfrak{g}$ is a Kac–Moody Lie superalgebra.

Just like for ordinary Kac–Moody algebras, the diagonal elements h_i generate the Cartan subalgebra $\mathfrak{h}$ and the e_i's and f_j's generate nilpotent subalgebras $\mathfrak{g}^+$ and $\mathfrak{g}^-$, respectively. This implies that $\mathfrak{g}$ has a triangular decomposition

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+ \quad \text{(direct sums of vector spaces)}. \tag{4.3}$$

The bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}_\mathbb{R}$ induces a bilinear form on the dual space $\mathfrak{h}_\mathbb{R}^*$. An important difference compared to standard Kac–Moody algebras is that the diagonal entries of the Cartan matrix A are not required to be positive.

This implies that the simple roots of $\mathfrak{g}$ come in two classes: *real* simple roots satisfying $(\alpha_i|\alpha_i) > 0$, and *imaginary* simple roots satisfying $(\alpha_i|\alpha_i) \leq 0$.

We denote by Δ the set of all roots. We say that a root is positive (resp. negative) if it is a nonnegative (resp. non-positive) integer linear combination of the simple roots. We therefore have a natural splitting into positive and negative roots

$$\Delta = \Delta^+ \oplus \Delta^-. \tag{4.4}$$

Accordingly, the nilpotent subalgebras $\mathfrak{g}^-$ and $\mathfrak{g}^+$ can be written as direct sums

$$\mathfrak{g}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha, \tag{4.5}$$

where the root spaces $\mathfrak{g}_\alpha$ are finite-dimensional $\text{mult}(\alpha) := \dim(\mathfrak{g}_\alpha) < \infty$. The bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ extends uniquely to a (super-)symmetric, non-degenerate, invariant bilinear form on $\mathfrak{g}$, such that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal unless $\beta = -\alpha$. The integral span of all simple roots is the *root lattice*

$$\Phi = \sum_{i=1}^{\text{rank } \mathfrak{g}} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*. \tag{4.6}$$

The Weyl group $\mathcal{W}(\mathfrak{g})$ is the group of reflections in $\Phi \otimes \mathbb{C}$ with respect to the even real simple roots α_i, $i \in I \setminus S$. It is generated by the fundamental reflections

$$s_i : \alpha \longmapsto \alpha - 2\frac{(\alpha|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i. \tag{4.7}$$

An additional important property of a BKM algebra is the existence of a Weyl vector ρ, satisfying

$$(\rho|\alpha) \geq \frac{1}{2}(\alpha|\alpha), \tag{4.8}$$

with equality if and only if α is a simple root.

Let us focus on the case where $S = \emptyset$, so that $\mathfrak{g}$ is a (purely even) BKM Lie algebra, rather than a Lie superalgebra. For any irreducible (integrable) lowest weight representation $R(\lambda)$ of $\mathfrak{g}$ one has the Weyl–Kac–Borcherds character formula

$$\text{ch } R(\lambda) = \frac{\sum_{w \in \mathcal{W}} \epsilon(w)w(T)e^\rho}{\prod_{\alpha \in \Delta^+}(1 - e^\alpha)^{\text{mult}(\alpha)}}, \tag{4.9}$$

where $\epsilon(w) = (-1)^{\ell(w)}$ with $\ell(w)$ the length of the Weyl element w. This differs from the standard Weyl–Kac character formula by the factor $w(T)$ which contains a correction due to the imaginary simple roots

$$T = e^{\lambda-\rho}\sum_{\mu}\xi(\mu)e^\mu. \tag{4.10}$$

The sum is taken over all (unordered) sets of distinct mutually orthogonal imaginary simple roots, whose sum is denoted by μ. Here $\xi(\mu) = (-1)^m$ if μ is a sum of m distinct pairwise orthogonal imaginary simple roots which

are orthogonal to λ, and $\xi(\mu) = 0$ otherwise. For our purposes we are mainly interested in the simplest case of the trivial representation $\lambda = 0$, for which $\mathrm{ch}\, R(\lambda) = 1$, and the character formula reduces to the so-called *denominator formula*

$$\sum_{w \in \mathcal{W}} \epsilon(w) w(T) = e^{-\rho} \prod_{\alpha \in \Delta^+} (1 - e^\alpha)^{\mathrm{mult}(\alpha)}. \qquad (4.11)$$

This formula relates a sum over the Weyl group $\mathcal{W}$ to an infinite product over all positive roots of $\mathfrak{g}$.

Let us generalize the denominator formula to the case where $\mathfrak{g}$ is a BKM superalgebra. To this end, we need a little bit of additional structure. Again, we refer to [60] for details. Denote the set of roots of $\mathfrak{g}$ by Δ. This splits into even or odd roots Δ_0, Δ_1, respectively. Define the corresponding even and odd root multiplicities as follows

$$m_0(\alpha) = \dim(\mathfrak{g}_\alpha \cap \mathfrak{g}_0), \qquad m_1(\alpha) = \dim(\mathfrak{g}_\alpha \cap \mathfrak{g}_1) = \mathrm{mult}(\alpha) - m_0(\alpha). \qquad (4.12)$$

Any root $\alpha \in \Delta$ can be expanded as $\alpha = \sum_{i \in I} k_i \alpha_i$. Define the *height* $\mathrm{ht}(\alpha)$ and *even height* $\mathrm{ht}_0(\alpha)$ of α as follows

$$\mathrm{ht}(\alpha) = \sum_{i \in I} k_i, \qquad \mathrm{ht}_0(\alpha) = \sum_{i \in I \setminus S} k_i. \qquad (4.13)$$

Now introduce the following expressions where the sums are taken over all sums μ of distinct pairwise orthogonal imaginary simple roots:

$$T = e^{-\rho} \sum_\mu (-1)^{\mathrm{ht}(\mu)} e^\mu, \qquad T' = e^{-\rho} \sum_\mu (-1)^{\mathrm{ht}_0(\mu)} e^\mu, \qquad (4.14)$$

where ρ is the Weyl vector.

For any BKM superalgebra $\mathfrak{g}$, we have the *denominator formula* and *superdenominator formula*:

$$\sum_{w \in W} \epsilon(w) w(T) = \frac{e^{-\rho} \prod_{\alpha \in \Delta_0^+} (1 - e^\alpha)^{m_0(\alpha)}}{\prod_{\alpha \in \Delta_1^+} (1 + e^\alpha)^{m_1(\alpha)}},$$

$$\sum_{w \in W} \epsilon(w) w(T') = \frac{e^{-\rho} \prod_{\alpha \in \Delta_0^+} (1 - e^\alpha)^{m_0(\alpha)}}{\prod_{\alpha \in \Delta_1^+} (1 - e^\alpha)^{m_1(\alpha)}}. \qquad (4.15)$$

4.2. Lie Algebras from the Chiral Superstring

In this section, we review how infinite-dimensional Lie (super)algebras, and in particular BKM (super)algebras, can be defined on the BRST cohomology of a *chiral* (super)string. This is the first step in understanding the appearance of BKM algebras in non-chiral superstring models that will be discussed in the next subsection.

Let us first consider the case of a bosonic chiral superstring theory obtained from the compactification on a holomorphic CFT of the form $V_{\Gamma^{1,1}} \otimes V$, where $V_{\Gamma^{1,1}}$ is the lattice vertex algebra related to the even unimodular lattice $\Gamma^{1,1}$ of signature $(1,1)$, and V is one of the bosonic self-dual VOAs of

central charge $c = 24$. As mentioned in Sect. 3, upon tensoring with a bc ghost system and restricting to the $\ker b_0$ subspace, for each $k \in \Gamma^{1,1}$ one can consider a complex $C^\bullet(k)$ with respect to a nilpotent BRST operator Q, and the corresponding cohomology $H^\bullet(k)$, $k \in \Gamma^{1,1}$. For $k \neq 0$, the cohomology is concentrated at degree (ghost number) 1, so one can focus on the space $\mathcal{H}^{\text{phys}} = \bigoplus_{k \in \Gamma^{1,1}} H^1(k)$.

Following Lian and Zuckerman [51,52], one can introduce a bilinear map $\{\cdot, \cdot\} : C^n(k) \times C^{n'}(k') \to C^{n+n'-1}(k+k')$ given by

$$\{u, v\} = (b_{-1}u)_0 v, \tag{4.16}$$

for $u \in C^n(k)$ and $v \in C^{n'}(k')$, where $(b_{-1}u)_0$ denotes the zero mode of the vertex operator $b_{-1}u$. The BRST charge Q acts on the bracket as a derivation, i.e., [52]

$$Q\{u, v\} = \{Qu, v\} + \{u, Qv\}. \tag{4.17}$$

In particular, if $Qu = 0 = Qv$, then $Q\{u, v\} = 0$; furthermore, if $Qu = 0$ and $v = Qw$, then $\{u, v\} = Q\{u, w\} - \{Qu, w\} = Q\{u, w\}$ is exact. This implies that $\{\cdot, \cdot\}$ induces a well-defined bilinear map on cohomology $\{\cdot, \cdot\}$: $H^n(k) \times H^{n'}(k') \to H^{n+n'-1}(k + k')$ (we use the same symbol $\{\cdot, \cdot\}$ with abuse of notation). When acting on cohomology, the bracket is antisymmetric $\{u, v\} = -\{v, u\}$ and satisfies the Jacobi identity

$$\{u, \{v, w\}\} + \{w, \{u, v\}\} + \{v, \{w, u\}\} = 0. \tag{4.18}$$

As a consequence, one can define a Lie algebra $\mathfrak{g}$ with underlying vector space $\mathcal{H}^{\text{phys}}$ and Lie bracket $[\cdot, \cdot] : \mathcal{H}^{\text{phys}} \times \mathcal{H}^{\text{phys}} \to \mathcal{H}^{\text{phys}}$ which coincides with $\{\cdot, \cdot\}$.

For a more explicit description of $\mathfrak{g}$, first notice that all Q-closed states $u \in \ker b_0$ are necessarily in $\ker L_0$, so that $b_{-1}u$ is a current, i.e., has conformal weight 1. The Lie algebra $\mathfrak{g}$ is essentially the quotient of the algebra of zero modes of these currents by the ideal of zero modes of states of the form $b_{-1}Qw$.

This construction generalizes easily to the supersymmetric case. In this case, one has a BRST complex $C_p^\bullet(k)$ and the corresponding cohomology $H_p^n(k)$ for each momentum $k \in \Gamma^{1,1}$ and picture number $p \in \frac{1}{2}\mathbb{Z}$. For $k \neq 0$, the cohomology is concentrated at degree 1, and the picture changing homomorphism $X : H_p^n(k) \to H_{p+1}^n(k)$ is an isomorphism, so that one focuses on $\mathcal{H}^{\text{phys}} = \oplus_{k \in \Gamma^{1,1}} H_{-1}^1(k) \oplus H_{-1/2}^1(k)$.

The bilinear map of Lian and Zuckerman admits a simple generalization $\{\cdot, \cdot\} : C_p^n(k) \times C_{p'}^{n'}(k') \to C_{p+p'}^{n+n'-1}(k+k')$ given by [52]

$$\{u, v\} = (-1)^{|u|}(b_{-1}u)_0 v, \qquad u \in C_p^n(k), \quad v \in C_{p'}^{n'}(k') \tag{4.19}$$

where $|u| \in \mathbb{Z}/2\mathbb{Z}$ is defined as $|u| = n + 2p \mod 2$. A $\mathbb{Z}_2$-graded version $Q\{u, v\} = \{Qu, v\} + (-1)^{|u|-1}\{u, Qv\}$ of (4.17) holds, so that the bracket is well defined on cohomology $\{\cdot, \cdot\} : H_p^n(k) \times H_{p'}^{n'}(k') \to H_{p+p'}^{n+n'-1}(k + k')$, where it satisfies a $\mathbb{Z}_2$-graded skew-symmetry relation $\{u, v\} + (-1)^{(|u|+1)(|v|+1)}\{v, u\} =$

0 and a $\mathbb{Z}_2$-graded version of the Jacobi identity. The action of the picture-changing operator X is compatible with the bracket, in the sense that

$$X\{u,v\} = \{Xu,v\} = \{u,Xv\}. \tag{4.20}$$

One can now use this bracket to define the structure of a Lie superalgebra $\mathfrak{g}$ on $\mathcal{H}^{\mathrm{phys}}$, where $\mathfrak{g}_0 := \oplus_{k \in \Gamma^{1,1}} H^1_{-1}(k)$ is the even component and $\mathfrak{g}_1 := \oplus_{k \in \Gamma^{1,1}} H^1_{-1/2}(k)$ is the odd component. The Lie bracket $[\cdot,\cdot]$ between two odd elements is simply defined as

$$[u,v] := \{u,v\}, \qquad u,v \in \mathfrak{g}_1. \tag{4.21}$$

When one of the elements is even, for example if $u \in \oplus_{k \in \Gamma^{1,1}} H^1_{-1}(k)$ and $v \in \oplus_{k \in \Gamma^{1,1}} H^1_p(k)$, $p \in \{-1,-1/2\}$, then the Lian–Zuckerman bracket $\{u,v\} \in \oplus_{k \in \Gamma^{1,1}} H^1_{p-1}(k)$ is not contained in $\mathcal{H}^{\mathrm{phys}}$. In this case, one needs to use the picture-changing operator to define the Lie bracket

$$[u,v] := X\{u,v\}, \qquad u \in \mathfrak{g}_0, v \in \mathfrak{g} \quad \text{or} \quad u \in \mathfrak{g}, v \in \mathfrak{g}_0. \tag{4.22}$$

Both in the bosonic and in the supersymmetric case, the algebra $\mathfrak{g}$ is graded by momentum $\mathfrak{g} = \oplus_{k \in \Gamma^{1,1}} \mathfrak{g}(k)$. The zero momentum subalgebra $\mathfrak{g}(0)$ always contains two commuting (even) generators P^μ, $\mu \in \{+,-\}$ corresponding to spacetime momentum, so that the spaces $\mathfrak{g}(k)$ are the eigenspaces of P^μ (in their adjoint action on $\mathfrak{g}$) with eigenvalue k^μ. The momentum generators P^μ are usually included in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$; as a consequence, $\mathfrak{h}$ is contained in the zero-momentum subalgebra $\mathfrak{g}(0)$. In the supersymmetric case, $\mathfrak{g}(0)$ can also contain some odd elements corresponding to spacetime supersymmetry generators. In fact, the construction that we just described, when restricted to zero momentum, provides the standard spacetime supersymmetry algebras in general superstring models, see [59].

A case-by-case analysis shows that the infinite-dimensional Lie (super)algebra $\mathfrak{g}$ obtained from the BRST cohomology of a chiral (super)string on $V \otimes V_{\Gamma^{1,1}}$ is always a BKM (super)algebra. In particular, in the bosonic case, this has been first proved in [9] and [10] when V is the monster VOA $V^\natural$ or one of the lattice VOAs related to Niemeier lattices, and in [22,42,43,53] for the other bosonic VOAs of central charge 24. The proofs for the supersymmetric cases V^{fE_8}, $V^{f\natural}$, and F_{24} can be found respectively in [36,37,61]. For most of these algebras, these references also provide some descriptions of $\mathfrak{g}$ in terms of generators and relations, as in the definition of Sect. 4.1. An exception is F_{24}, where for some choices of $\mathcal{N} = 1$ superVirasoro algebra of F_{24}, the set of simple roots is not even known [36].

More generally, the Lian and Zuckerman construction can be applied to superstring compactifications where the chiral algebra is a generic VOA of central charge 26, not necessarily of the form $V \otimes V_{\Gamma^{1,1}}$ [51,52]. For these general cases, there is no reason to expect the Lie (super)algebra of the chiral BRST cohomology to be a BKM; in particular it might not admit an analog of the Borcherds–Kac–Weyl (super-)denominator formula.

4.3. BPS States in Superstring Models

In [55, 56], several of the authors conjectured the appearance of a BKM algebra in a string theory whose worldsheet matter CFT, prior to compactification on a spatial circle, was holomorphically factorized. In this section, we will expand upon and generalize the BKM action studied in [55, 56].

The basic idea of [55, 56] was as follows. We began with a heterotic string theory whose internal worldsheet CFT was given by the $c = 24$ holomorphic monster VOA, $V^\natural$, tensored with the anti-holomorphic $c = 12$ SVOA $V^{f\natural}$ (for the NS sector), or its "canonically twisted module" (for the R sector). Completing the standard string theory voodoo (adding ghost sectors, GSO projecting, etc) resulted in a $(1+1)$-d theory with $\mathcal{N} = (0, 24)$ supersymmetry. In 2d string theory, the GSO projection relates the internal fermion number of the worldsheet CFT to fermion chirality in spacetime. Compactifying the remaining spatial direction on a circle S^1 of radius R enables one to write the supersymmetry algebra as

$$\left\{ \mathcal{Q}^i, \mathcal{Q}^j \right\} = 2\delta^{ij}(P_r^0 - P_r^1), \quad i, j = 1, \ldots, 24, \tag{4.23}$$

expressed in terms of the temporal and spatial components of the *right-moving* momenta around the circle. BPS states in this model are annihilated by all supercharges and hence satisfy $k_r^0 = k_r^1$, having equal eigenvalues of the P_r^0, P_r^1 operators. BPS states furnish a subspace of physical states in the $(0+1)$-d string theory, and the latter arise as usual from BRST cohomology classes with respect to a nilpotent operator $Q = Q_l + Q_r$, which can be expressed as a sum of left- and right-moving pieces. When the right-moving momentum is nonzero, the space of physical states factorizes into a product of left- and right-moving cohomology classes, graded by momenta, picture number, and ghost number. To the left-moving factor, we can associate a BKM algebra following precisely the chiral construction; on the subspace of BPS states, the right-moving factor degenerates into 24 copies of the trivial representation of this algebra, contributing only a (24-fold) multiplicity to each BPS state. In other words, although we cannot tensor two Lie algebras to obtain a Lie algebra, a single copy of the BKM structure is preserved upon tensoring with (copies of) the trivial representation. It turns out that in this non-chiral string theory, for each choice of BRST-representative satisfying the BPS condition $k_r^0 = k_r^1$, there is a corresponding nonzero BRST-exact state obtained from it by acting with a single supercharge $\mathcal{Q}^i$. The BRST-exact state is the Q_r-image of a Q_l-closed state. One can then obtain a correspondence between generators of a BKM algebra (in this case, the monster Lie algebra) and the quotient space coming from the space of Q_r-exact Q_l-closed BPS states by the space of Q_l-exact states.

In the remainder of this section, we will expand and improve upon this algebra construction, generalize to the type II case, and clarify various subtleties and issues along the way.

Let us consider either a heterotic or type II superstring theory model compactified on $(V_1 \otimes \bar{V}_2) \otimes S^1$, where V_2 is a self-dual SVOA of central charge 12 and V_1 is either a self-dual VOA of central charge 24 (in the heterotic case)

or a self-dual SVOA of central charge 12 (in the type II case). Following the analysis of the previous section, the space of physical states has the form

$$\mathcal{H}^{\mathrm{phys}} := \left(\bigoplus_{\substack{k_l, k_r \neq 0 \\ k_l^0 = k_r^0}} H^1(k_l) \otimes \bar{H}^1(k_r) \right) \oplus \mathsf{H}^2(0,0), \qquad (4.24)$$

where $H^1(k)$ and $\bar{H}^1(k)$ are the cohomology space for the chiral and anti-chiral models with respect to the left- and right-moving BRST charges Q_l and Q_r. We have separated the nonzero-momentum states from the zero-momentum states, which do not necessarily factorize (see "Appendix B"). In the fermionic case, H^1 is a direct sum of -1 (NS) and $-1/2$ (R) picture components.

We want to focus on the subspace

$$\mathcal{H}_{\mathrm{BPS}} := \left(\bigoplus_{\substack{k_l, k_r \neq 0 \\ k_l^0 = k_r^0 = k_r^1}} H^1(k_l) \otimes \bar{H}^1(k_r) \right) \oplus \mathsf{H}^2(0,0)', \qquad (4.25)$$

of physical states satisfying

$$k_r^0 = k_r^1. \qquad (4.26)$$

The zero-momentum sector $\mathsf{H}^2(0,0)'$ is obtained from $\mathsf{H}^2(0,0)$ by taking a certain quotient of a suitable subspace; we will give the precise definition in Sect. 4.5. If the model has spacetime supersymmetry, corresponding to gravitinos of positive chirality in the NS-R sector of the theory, then $\mathcal{H}_{\mathrm{BPS}}$ is the subspace of states annihilated by those supersymmetries, hence BPS states. We use the same notation $\mathcal{H}_{\mathrm{BPS}}$ even when there are no spacetime supersymmetries. The possible momenta for states in $\mathcal{H}_{\mathrm{BPS}}$ are labeled by two integers $m, w \in \mathbb{Z}$ as follows:

$$k_l^0 = k_r^0 = k_r^1 = \frac{1}{\sqrt{2}} \left(\frac{m}{R} + wR \right), \qquad k_l^1 = \frac{1}{\sqrt{2}} \left(\frac{m}{R} - wR \right), \qquad (4.27)$$

where R is the radius of the circle.

Recall that with the self-dual VOA or SVOA V_1 is canonically associated, via the "chiral (super)string construction," a BKM algebra (if V_1 is bosonic) [10] or superalgebra (if V_1 is an SVOA) $\mathfrak{g}$ [61,62].

In the remainder of this section, we will prove the following:

1. There is a representation δ

$$\delta : \mathfrak{g} \to \mathrm{End}(\mathcal{H}_{\mathrm{BPS}}) \qquad (4.28)$$

$$x \mapsto \delta_x \qquad (4.29)$$

of the BKM (super)algebra $\mathfrak{g}$ associated with V_1 on the space $\mathcal{H}_{\mathrm{BPS}}$ of physical BPS states. The elements $x \in \mathfrak{g}$ are identified with certain Q_l-closed states v_x of ghost number $(1,0)$ in the superstring theory satisfying the "BPS" condition $k_r^0 = k_r^1$, or equivalently with their BRST variation $Q v_x$, which only has components with ghost numbers $(1,1)$. For the heterotic string, the BRST variations $Q v_x$ are exact "BPS" states in the

NS sector; in the type II case, the even and odd components $\mathfrak{g}_0$ and $\mathfrak{g}_1$ correspond to exact "BPS" states in the NS-NS and R-NS sector, respectively. Here, "BPS" is in quotes because these states satisfy the condition $k_r^0 = k_r^1$, but they are not physical states since they are cohomologically trivial. Note that the algebra action does not mix the right-moving NS sector with the right-moving R sector of $\mathcal{H}_{\mathrm{BPS}}$, so that the representation $\mathcal{H}_{\mathrm{BPS}}$ is, in general, the sum of two representations corresponding to these sectors. This is proven in Sect. 4.4.

2. The action of $\mathfrak{g}$ on $\mathcal{H}_{\mathrm{BPS}}$ is a symmetry of tree-level string theory amplitudes involving only states in $\mathcal{H}_{\mathrm{BPS}}$ (*purely BPS amplitudes*). Explicitly, for each $x \in \mathfrak{g}$ and every $\varrho_1, \ldots, \varrho_n \in \mathcal{H}_{\mathrm{BPS}}$, we have

$$\delta_x \left(\int_{\mathcal{M}_{0,n}} \left\langle \prod_{i=1}^n \mathcal{V}_{\varrho_i} \right\rangle \right) \equiv \sum_{j=1}^n \int_{\mathcal{M}_{0,n}} \left\langle \mathcal{V}_{\delta_x(\varrho_j)} \prod_{i \neq j} \mathcal{V}_{\varrho_i} \right\rangle = 0, \qquad (4.30)$$

where $\int_{\mathcal{M}_{0,n}}$ denotes the integration over the appropriate (super)moduli space of genus 0 with n punctures. Here, $\mathcal{V}_\varrho$ is the vertex operator corresponding to a state $\varrho \in \mathcal{H}_{\mathrm{BPS}}$. Roughly speaking, this result will be obtained as follows. One considers an $n + 1$-point amplitude where the additional vertex operator corresponds to the insertion of the BRST-exact state Qv_x associated with $x \in \mathfrak{g}$. By standard arguments, since this amplitude includes one BRST-exact and n BRST-closed states, it is expected to vanish. One then integrates over the modulus parametrizing the position of the exact state.[6] The amplitude is a total derivative with respect to this modulus, so it gets contributions only from the boundary of the moduli space, i.e., from the limit where the insertion of the exact vertex operator coincides with one of the other n punctures. One then just needs to prove that the sum over the boundary contributions has the form (4.30).

Claim 1 above is proven in Sect. 4.4 and claim 2 is proven in Sect. 4.6.

We further conjecture that $\mathfrak{g}$ is a symmetry of BPS amplitudes at all loops, by analogous reasoning to that outlined above. On the other hand, we do not expect the action of $\mathfrak{g}$ to extend to the whole space of physical states $\mathcal{H}^{\mathrm{phys}}$.[7]

Similarly, the superalgebra $\bar{\mathfrak{g}}$ associated with the anti-holomorphic SVOA $\bar{V}_2$ acts on the space $\bar{\mathcal{H}}_{\mathrm{BPS}}$ of states satisfying $k_l^0 = k_l^1$, and is a symmetry of the amplitudes built only from the states in such space. Note that the space $\bar{\mathcal{H}}_{\mathrm{BPS}}$ is different from $\mathcal{H}_{\mathrm{BPS}}$, so that there is apparently no space of states where both $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ act.

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[6] When x is an odd element of a superalgebra, so that v_x is in the R-NS sector, this statement needs to be refined; see Sect. 4.6.

[7] Of course, the zero-momentum subalgebra of $\mathfrak{g}$, which corresponds to the standard symmetries related to conservation of spacetime momentum and winding number, will act on all of $\mathcal{H}^{\mathrm{phys}}$.

4.4. The BPS Subspace as a Representation of the BKM Algebra

In this section, we define the action of $\mathfrak{g}$ on $\mathcal{H}_{\mathrm{BPS}}$. We begin by focusing on the states of nonzero momentum, which factorize into chiral and anti-chiral components. The algebra $\mathfrak{g}$ associated with V_1 is the direct sum of finite-dimensional components graded by "momentum" k taking values in the even unimodular lattice $\Gamma^{1,1} \cong \mathbb{Z} \oplus \mathbb{Z}$:

$$\mathfrak{g} = \bigoplus_{k \in \Gamma^{1,1}} \mathfrak{g}(k) = \bigoplus_{m,w \in \mathbb{Z}} \mathfrak{g}(m,w), \qquad (4.31)$$

where the components $\mathfrak{g}(k)$ are isomorphic, as vector spaces, to the BRST cohomology with respect to the left-moving BRST charge Q_l at momentum k,

$$\mathfrak{g}(k) \cong H^1(k), \qquad k \in \Gamma^{1,1}. \qquad (4.32)$$

There is an analogous BKM superalgebra $\bar{\mathfrak{g}}$ associated with the SVOA V_2, which admits a similar decomposition

$$\bar{\mathfrak{g}} = \bigoplus_{k \in \Gamma^{1,1}} \bar{\mathfrak{g}}(k) = \bigoplus_{m,w \in \mathbb{Z}} \bar{\mathfrak{g}}(m,w), \qquad (4.33)$$

with

$$\bar{\mathfrak{g}}(k) \cong \bar{H}^1(k), \qquad k \in \Gamma^{1,1}. \qquad (4.34)$$

We work in the relative, rather than semirelative, cohomology, so that the BPS states in $\mathcal{H}_{\mathrm{BPS}}$ admit representatives that are holomorphically factorized

$$\varrho \otimes \bar{\varsigma} \qquad (4.35)$$

as tensor products of representatives ϱ and $\bar{\varsigma}$ for classes in $H^1(k_l) \cong \mathfrak{g}(k_l)$ and $\bar{H}^1(k_r)$, respectively. (Here, the isomorphisms are to be understood as isomorphisms of vector spaces; we do not claim that an algebra structure is preserved on the tensor product.) The BPS condition $k_r^0 = k_r^1$, together with the quantization of momentum k^1 along the circle S^1 of radius R and the conditions $k_l^0 = k_r^0$ (from the uncompactified time direction), implies that

$$k_l^0 = k_r^0 = k_r^1 = \frac{1}{\sqrt{2}} \left(\frac{m}{R} + wR \right), \qquad k_l^1 = \frac{1}{\sqrt{2}} \left(\frac{m}{R} - wR \right), \qquad m, w \in \mathbb{Z} \qquad (4.36)$$

where m and w are the quantized momentum and winding number along S^1, respectively. This means that the left-moving momentum takes values in a lattice isomorphic to $\Gamma^{1,1} \cong \mathbb{Z} \oplus \mathbb{Z}$. The right-moving momentum takes values in a null subspace of $\mathbb{R}^{1,1}$, i.e., $k_r^2 = 0$, and depends on $m, w \in \mathbb{Z}$ as

$$k_r^0 = k_r^1 = \frac{1}{\sqrt{2}} \left(\frac{m}{R} + wR \right). \qquad (4.37)$$

Let us focus on the space $\bar{H}^1(k_r)$. Suppose that $\{\bar{v}^a\}_{a=1\ldots,\chi^{\mathrm{NS}}}$ and $\{\bar{u}^{i+}\}_{i=1,\ldots,\chi_+^{\mathrm{R}}}$ are bases for the spaces $\bar{V}_2^{NS}(1/2)$ and $\bar{V}_2^{R+}(1/2)$ of states of conformal weight $1/2$ in, respectively, the NS and positive fermion number Ramond sector of the anti-holomorphic SCFT $\bar{V}_2$. For $k_r^0 = k_r^1 \neq 0$, the space of Q_r-closed states

with ghost number 1, picture number -1 and weight 0 in the anti-holomorphic matter+ghost CFT is spanned by:

$$\bar{v}^a_{-1/2} e^{-\tilde{\phi}} \bar{c}_1 e^{ik_r X_r} |0\rangle, \qquad \left(\bar{\psi}^0_{-1/2} - \bar{\psi}^1_{-1/2}\right) e^{-\tilde{\phi}} \bar{c}_1 e^{ik_r X_r} |0\rangle, \qquad (4.38)$$

and the latter state is the only Q_r-exact state. It is proportional to $Q_r \bar{\beta}_{-1/2} e^{-\tilde{\phi}} \bar{c}_1 e^{ik_r X_r} |0\rangle$. In the $-1/2$ picture, the space of Q_r-closed states with $k_r^0 = k_r^1 \neq 0$ is spanned by

$$\bar{u}^{i+}_{-1/2} e^{-\tilde{\phi}/2} \bar{c}_1 e^{ik_r X_r} |0, -\rangle, \, i = 1, \ldots, \chi^R_+(V_2), \qquad (4.39)$$

where $|0, -\rangle$ is a Ramond ground state of the $(\bar{\psi}^\mu, \bar{\partial} X^\mu)$ SVOA with negative chirality. This shows that all cohomology spaces $\bar{H}^1(k_r)$, with $k_r^0 = k_r^1 \neq 0$, have dimension $\chi^{\mathrm{NS}}(V_2) + \chi^{\mathrm{R}+}(V_2)$ and are all isomorphic to each other. In fact, there is a distinguished isomorphism among them. One considers the vertex operator $e^{i(k'_r - k_r) X_r}$, which has conformal weight 0 and non-singular OPE with all the states (4.38) and (4.39). It is easy to see that the zero mode $(e^{i(k'_r - k_r) X_r})_0$ induces a well-defined map

$$\left(e^{i(k'_r - k_r) X_r}\right)_0 : \bar{H}^1(k_r) \to \bar{H}^1(k'_r), \qquad (4.40)$$

which is an isomorphism for $k_r, k'_r \neq 0$.

　　With these preparations, one can then define the action of an element $x \in \mathfrak{g}(m, w)$ on a physical state $\varrho \otimes \bar{\varsigma} \in \mathcal{H}_{\mathrm{BPS}}$ by

$$\delta_x(\varrho \otimes \bar{\varsigma}) := [x, \varrho] \otimes (e^{\frac{i}{\sqrt{2}}\left(\frac{m}{R} + wR\right) X_r})_0 \bar{\varsigma}. \qquad (4.41)$$

Here, we use the fact that the left-moving factor $\varrho \in H^1(k_l) \cong \mathfrak{g}(k_l)$ can be seen as an element of the algebra $\mathfrak{g}$ itself. The shift in the right-moving momentum is necessary to preserve the condition $k_l^0 = k_r^0 = k_r^1$.

　　If we have two elements $x \in \mathfrak{g}(m, w)$, $x' \in \mathfrak{g}(m', w')$, one has

$$(\delta_x \delta_{x'} - \delta_{x'} \delta_x)(\varrho \otimes \bar{\varsigma}) = ([x, [x', \varrho]] - [x', [x, \varrho]]) \otimes \left(e^{\frac{i}{\sqrt{2}}\left(\frac{m+m'}{R} + (w+w')R\right) X_r}\right)_0 \bar{\varsigma} \qquad (4.42)$$

$$= [[x, x'], \varrho] \otimes \left(e^{\frac{i}{\sqrt{2}}\left(\frac{m+m'}{R} + (w+w')R\right) X_r}\right)_0 \bar{\varsigma}$$

$$= \delta_{[x, x']}(\varrho \otimes \bar{\varsigma}) \qquad (4.43)$$

where in the second line we used the Jacobi identity. This shows that δ is indeed a representation of $\mathfrak{g}$—in fact, it is the tensor product of the adjoint representation of $\mathfrak{g}$ times a trivial (in general, not irreducible) representation of dimension $\chi^{NS}(V_2) + \chi^R_+(V_2)$. This proves the first claim in Sect. 4.3.

4.5. Subtleties with Zero Momentum

The procedure described in the previous subsection is problematic when either $\varrho \otimes \bar{\varsigma} \in \mathcal{H}_{\mathrm{BPS}}$ or $\delta_x(\varrho \otimes \bar{\varsigma})$ has zero right-moving momentum.[8] The problem

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[8]As explained below, we are going to exclude the ghost dilaton from the space $\mathcal{H}_{\mathrm{BPS}}$. This allows us to represent the states in $\mathcal{H}_{\mathrm{BPS}}$ as elements in the tensor product $H^1(k_l) \otimes \bar{H}^1(k_r)$ even at zero momentum $k_l = k_r = 0$. See "Appendix B" for details.

is that the operator $(e^{i(k'_r - k_r)X_r})_0$ from states of momentum k_r to states of momentum k'_r does not map, in general, Q_r-closed states to Q_r-closed states and Q_r-exact states to Q_r-exact states.

Indeed, the vertex operator $e^{ip_r X_r}(\bar{z})$, with $p_r^1 = p_r^0$, does not commute with Q

$$[Q, e^{ip_r X_r}] = p_{r\mu}(\bar{c}\bar{\partial}X^\mu + \bar{\gamma}\bar{\psi}^\mu)e^{ip_r X_r}. \tag{4.44}$$

Given this result, it is quite surprising that, at least for certain momenta, the zero mode $(e^{ip_r X_r})_0$ gives a well-defined map on the cohomology at all. Note, however, that the zero mode of $p_{r\mu}\bar{\partial}X^\mu$, when acting on states with $k_r^0 = k_r^1$, gives a term proportional to $k_r \cdot p_r = 0$, since k_r and p_r are both null and proportional to each other. As for the second term, the only cases where it is potentially nonvanishing is when the commutator $[Q, e^{ip_r X_r}]$ is applied to the ghost number 0 state $\bar{\beta}_{-1/2}e^{-\bar{\phi}}\bar{c}_1|k_r\rangle$ or the ghost number 1 states $\bar{\psi}^\mu_{-1/2}e^{-\bar{\phi}}\bar{c}_1|k_r\rangle$ and $\bar{u}^{i\mp}_{-1/2}e^{-\bar{\phi}/2}\bar{c}_1|k_r, \pm\rangle$. In these cases, we have

$$(e^{ip_r X_r})_0 Q_r \bar{\beta}_{-1/2}e^{-\bar{\phi}}\bar{c}_1|k_r\rangle = k_{r\mu}\bar{\psi}^\mu_{-1/2}e^{-\bar{\phi}}\bar{c}_1|k_r + p_r\rangle, \tag{4.45}$$

$$Q_r(e^{ip_r X_r})_0 \bar{\beta}_{-1/2}e^{-\bar{\phi}}\bar{c}_1|k_r\rangle = (k_{r\mu} + p_{r\mu})\bar{\psi}^\mu_{-1/2}e^{-\bar{\phi}}\bar{c}_1|k_r + p_r\rangle, \tag{4.46}$$

as well as

$$(e^{ip_r X_r})_0 Q_r \epsilon_\mu \bar{\psi}^\mu_{-1/2}e^{-\bar{\phi}}\bar{c}_1|k_r\rangle = \epsilon_\mu k_r^\mu \gamma_{-1/2}e^{-\bar{\phi}}\bar{c}_1|k_r + p_r\rangle, \tag{4.47}$$

$$Q_r(e^{ip_r X_r})_0 \epsilon_\mu \bar{\psi}^\mu_{-1/2}e^{-\bar{\phi}}\bar{c}_1|k_r\rangle = \epsilon_\mu(k_r^\mu + p_r^\mu)\gamma_{-1/2}e^{-\bar{\phi}}\bar{c}_1|k_r + p_r\rangle, \tag{4.48}$$

and

$$(e^{ip_r X_r})_0 Q_r \bar{u}^{i\mp}_{-1/2}e^{-\bar{\phi}/2}\bar{c}_1|k_r, \pm\rangle = k_{r\mu}\bar{\gamma}_0\bar{\psi}^\mu_0 \bar{u}^{i\mp}_{-1/2}e^{-\bar{\phi}/2}\bar{c}_1|k_r + p_r, \pm\rangle, \tag{4.49}$$

$$Q_r(e^{ip_r X_r})_0 \bar{u}^{i\mp}_{-1/2}e^{-\bar{\phi}/2}\bar{c}_1|k_r, \pm\rangle = (k_{r\mu} + p_{r\mu})\bar{\gamma}_0\bar{\psi}^\mu_0 \bar{u}^{i\mp}_{-1/2}e^{-\bar{\phi}/2}\bar{c}_1|k_r + p_r, \pm\rangle. \tag{4.50}$$

Since k_r and $k_r + p_r$ are proportional to each other, then so long as both $k_r \neq 0$ and $k_r + p_r \neq 0$, exchanging $(e^{ip_r X_r})_0$ and Q_r gives just a rescaling of the resulting state. This explains why $(e^{ip_r X_r})_0$ provides an isomorphism of cohomology in these cases. Of course, these arguments only work for states in $\mathcal{H}_{\text{BPS}}$, so there seems to be no natural way to extend the action of $\mathfrak{g}$ to the whole space of physical states $\mathcal{H}^{\text{phys}}$.

The problems arise when either k_r or $k_r + p_r$ is zero. Recall that for $k_r^0 = k_r^1 = 0$ and ghost number 1 the Q_r-closed states are

$$\nu^a := \bar{v}^a_{-1/2}e^{-\bar{\phi}}\bar{c}_1|0\rangle, \qquad a = 1, \ldots, \chi^{\text{NS}}(V_2) \tag{4.51}$$

$$\nu_- := (\bar{\psi}^0_{-1/2} - \bar{\psi}^1_{-1/2})e^{-\bar{\phi}}\bar{c}_1|0\rangle, \tag{4.52}$$

$$\nu_+ := (\bar{\psi}^0_{-1/2} + \bar{\psi}^1_{-1/2})e^{-\bar{\phi}}\bar{c}_1|0\rangle, \tag{4.53}$$

in the -1-picture (NS sector), and

$$\nu^i_+ := \bar{u}^{i-}_{-1/2}e^{-\bar{\phi}/2}\bar{c}_1|0, +\rangle, \qquad i = 1, \ldots, \chi^{\text{R}}_-(V_2) \tag{4.54}$$

$$\upsilon_-^i := \bar{u}_{-1/2}^{i+} e^{-\tilde{\phi}/2} \bar{c}_1 |0,-\rangle, \qquad\qquad i = 1,\ldots,\chi_+^{\mathrm{R}}(V_2) \qquad (4.55)$$

in the $-1/2$-picture (Ramond sector). None of these states is Q_r-exact.

Therefore, we have two kinds of problems:

1. The state $\epsilon_\mu \bar{\psi}_{-1/2}^\mu e^{-\tilde{\phi}} \bar{c}_1 |0\rangle$ at zero momentum, which is closed for every choice of polarization ϵ_μ, is mapped by $(e^{ip_r X_r})_0$ to $\epsilon_\mu \bar{\psi}_{-1/2}^\mu e^{-\tilde{\phi}} \bar{c}_1 |p_r\rangle$, which is closed only when ϵ_μ is proportional to $p_{r\mu}$. Similarly, the Q_r-closed states υ^{i+} in the Ramond sector are mapped to the states $\bar{u}_{-1/2}^{i-} e^{-\tilde{\phi}/2} \bar{c}_1 |p_r,+\rangle$ that are not closed for $p_r^0 = p_r^1$. Therefore, $(e^{ip_r X_r})_0$ maps physical states with zero momentum to non-physical states with momentum p_r.

2. The state $k_{r\mu} \bar{\psi}_{-1/2}^\mu e^{-\tilde{\phi}} \bar{c}_1 |k_r\rangle$ at momentum $k_r \neq 0$, which is exact, is mapped by $(e^{-ik_r X_r})_0$ to the state $k_{r\mu} \bar{\psi}_{-1/2}^\mu e^{-\tilde{\phi}} \bar{c}_1 |0\rangle$, which is closed but not exact. Therefore, $(e^{-ik_r X_r})_0$ maps exact states with momentum k_r to non-exact states with momentum 0.

As a consequence, $(e^{ip_r X_r})_0$ does not induce a well-defined map on cohomology in these cases.

Possible resolutions. For problem 1 above, there is actually a simple (though perhaps slightly unnatural) solution. In the NS sector, we can just exclude the states $\epsilon_\mu \bar{\psi}_{-1/2}^\mu e^{-\tilde{\phi}} \bar{c}_1 |0\rangle$ from $\mathcal{H}_{\mathrm{BPS}}$, with the exception of the combination

$$\nu_- = (\bar{\psi}_{-1/2}^0 - \bar{\psi}_{-1/2}^1) e^{-\tilde{\phi}} \bar{c}_1 |0\rangle. \qquad (4.56)$$

Similarly, in the Ramond sector, we only include the states υ_-^i and exclude the states υ_+^i. Since we are already considering a particular subspace of $\mathcal{H}^{\mathrm{phys}}$, there seems to be nothing wrong with making one further restriction. Note that by acting with the algebra $\mathfrak{g}$ on $\mathcal{H}_{\mathrm{BPS}}$ there is no way to obtain a linear combination $\epsilon_\mu \bar{\psi}_{-1/2}^\mu e^{-\tilde{\phi}} \bar{c}_1 |0\rangle$ different from (4.56), nor a state of the form υ_+^i, so the restricted space is still a representation of $\mathfrak{g}$. In the same fashion, we also exclude the ghost dilaton (see "Appendix B").

Problem 2, related to the fact that the state (4.56) is not exact at zero momentum, is more subtle. This state, when tensored with an appropriate left-moving counterpart, corresponds to a soft particle. Soft particles in amplitudes couple to other particles through conserved charges. In this case, the conserved charge is $k^0 - k^1$. (The 0 picture version of the corresponding vertex operator is proportional to the current $\bar{\partial} X^0 - \bar{\partial} X^1$.) Therefore, ν_- should decouple from any amplitude with states in $\mathcal{H}_{\mathrm{BPS}}$, i.e., amplitudes with one state (4.56) and all the other states in $\mathcal{H}_{\mathrm{BPS}}$ are always zero. In Sect. 4.6, we check this expectation by a direct calculation. This implies that, so long as we consider symmetries of purely BPS amplitudes, the state ν_- is equivalent to 0 (or to a BRST-exact state). For these reasons, it seems reasonable that the action of $\mathfrak{g}$, which is a symmetry of these purely BPS amplitudes, is only defined modulo ν_-.

The conclusion is that, at zero momentum, one should really consider a quotient of $\bar{H}^1(0)$ by the state (4.56). Altogether, the zero-momentum space $\mathsf{H}^2(0,0)'$ that should be included in $\mathcal{H}_{\mathrm{BPS}}$ is

$$\mathsf{H}^2(0,0)' = H^1(0) \otimes \bar{H}^1(0)', \tag{4.57}$$

where

$$\bar{H}^1(0)' = \left(\bigoplus_{a=1}^{\chi^{\mathrm{NS}}} \mathbb{C}\nu^a \oplus \mathbb{C}\nu_- \oplus \bigoplus_{i=1}^{\chi^{\mathrm{R}}_+} \mathbb{C}\upsilon^i_- \right) / \mathbb{C}\nu_- \cong \bigoplus_{a=1}^{\chi^{\mathrm{NS}}} \mathbb{C}\nu^a \oplus \bigoplus_{i=1}^{\chi^{\mathrm{R}}_+} \mathbb{C}\upsilon^i_-. \tag{4.58}$$

The restricted quotient space $\bar{H}^1(0)'$ has dimension $\chi^{\mathrm{NS}} + \chi^{\mathrm{R}}_+$, and it is isomorphic to the cohomology at nonzero momentum, with the isomorphism given by operators $(e^{ip_r X_r})_0$.

With this definition, the action of $\mathfrak{g}$ on $\mathcal{H}_{\mathrm{BPS}}$ is well defined, but it remains to show that it acts in the appropriate way on BPS amplitudes.

4.6. Symmetry of BPS Amplitudes

In this subsection, we will prove (4.30).

Let us consider a purely BPS genus zero string amplitude

$$A_n = \int_{\mathcal{M}_{0,n}} \left\langle \prod_{i=1}^n \mathcal{V}_{\varrho_i \otimes \bar{\varsigma}_i}(z_i, \bar{z}_i) \right\rangle \tag{4.59}$$

with $n \geq 3$ insertions of vertex operators $\mathcal{V}_{\varrho_i \otimes \bar{\varsigma}_i}$ corresponding to BRST-closed states $\varrho_i \otimes \bar{\varsigma}_i$ representing classes in the BPS subspace $\mathcal{H}_{\mathrm{BPS}}$, i.e., such that $k_r^0 = k_r^1$ for all i. The picture numbers are chosen so as to give a nonzero answer, and $n-3$ vertex operators are integrated.

For states in $\mathcal{H}_{\mathrm{BPS}}$, the integrand factorizes into a holomorphic times an anti-holomorphic factor

$$\left\langle \prod_{i=1}^n \mathcal{V}_{\varrho_i \otimes \bar{\varsigma}_i}(z_i, \bar{z}_i) \right\rangle = \left\langle \prod_{i=1}^n \mathcal{V}_\varrho(z_i) \right\rangle \left\langle \prod_{i=1}^n \mathcal{V}_{\bar{\varsigma}_i}(\bar{z}_i) \right\rangle. \tag{4.60}$$

Let us focus on the right-moving factor, which must be anti-holomorphic in the variables z_i, with possible singularities when two insertion points coincide. The BPS and physical state conditions imply that the $\mathcal{V}_{\bar{\varsigma}_i}$ are built using only the following combinations of right-moving "spacetime" operators

$$\bar{\psi}^0 - \bar{\psi}^1, \qquad \bar{\partial}X^0 - \bar{\partial}X^1, \qquad e^{ik_r X_r} \text{ (with } k_r^0 = k_r^1), \tag{4.61}$$

multiplied by (super-)ghost and internal matter operators. Using the OPE

$$\bar{\partial}X^\mu(\bar{z})\bar{\partial}X^\nu(0) = -\frac{\eta^{\mu\nu}}{\bar{z}^2} + O(1), \qquad \bar{\psi}^\mu(\bar{z})\bar{\psi}^\nu(0) = \frac{\eta^{\mu\nu}}{\bar{z}} + O(1) \tag{4.62}$$

$$\bar{\partial}X^\mu(z)e^{ik_r X_r}(0) = \frac{ik_r^\mu}{\bar{z}}e^{ik_r X_r}(0) + O(1), \tag{4.63}$$

$$e^{ik_r X_r}(\bar{z})e^{ik'_r X_r}(0) = \bar{z}^{k_r \cdot k'_r}(\pm e^{i(k_r + k'_r)X_r}(0) + O(\bar{z})), \tag{4.64}$$

it is easy to see that the spacetime operators (4.61) have non-singular OPE with each other.

Now, suppose that one of the vertex operators $\mathcal{V}_{\bar{\varsigma}}(\bar{z})$ (taken in the 0-picture, integrated form) is just a product of spacetime operators (4.61), with no further "internal" matter or ghost factor. An example is

$$\mathcal{V}_{\bar{\varsigma}}(\bar{z}) =: (\bar{\partial}X^0 - \bar{\partial}X^1)e^{ik_r X_r} : (\bar{z})d\bar{z}, \qquad (4.65)$$

which is the 0-picture vertex operator of the closed state $\bar{\varsigma} = (\bar{\psi}^0_{-1/2} - \bar{\psi}^1_{-1/2})e^{-\tilde{\phi}}\bar{c}_1 e^{ik_r X_r}|0\rangle$.

In this case, the anti-holomorphic factor $\langle \mathcal{V}_{\bar{\varsigma}}(\bar{z}) \prod_{i=1}^n \mathcal{V}_{\bar{\varsigma}_i}(\bar{z}_i) \rangle$ has no singularities in $\bar{z}$, so it must be a constant in this variable. It follows that, for these closed states, the CFT correlator $\langle \mathcal{V}_{\varrho \otimes \bar{\varsigma}}(z, \bar{z}) \prod_{i=1}^n \mathcal{V}_{\varrho_i \otimes \bar{\varsigma}_i}(z_i, \bar{z}_i) \rangle$ is meromorphic in the insertion position z.

Note that the state $\bar{\varsigma} = (\bar{\psi}^0_{-1/2} - \bar{\psi}^1_{-1/2})e^{-\tilde{\phi}}\bar{c}_1 e^{ik_r X_r}|0\rangle$ is BRST-exact if and only if $k_r \neq 0$, and in this case the full vertex operator $\mathcal{V}_{\varrho \otimes \bar{\varsigma}}(z, \bar{z})$ is a total derivative

$$\mathcal{V}_{\varrho \otimes \bar{\varsigma}}(z, \bar{z}) \propto \partial_{\bar{z}}(\mathcal{V}_{\varrho}(z)e^{ik_r X_r}(\bar{z})). \qquad (4.66)$$

Let us consider an $(n+1)$-point amplitude where the final vertex operator is of the form (4.66) for $k_r \neq 0$.

$$A_{n+1} = \int_{\mathcal{M}_{0,n}} \int_{\mathbb{P}^1} dz d\bar{z} \frac{\partial}{\partial \bar{z}} \left\langle \prod_{i=1}^n \mathcal{V}_{\varrho_i \otimes \bar{\varsigma}_i}(z_i, \bar{z}_i)\mathcal{V}_{\varrho}(z)e^{ik_r X_r}(\bar{z}) \right\rangle. \quad (4.67)$$

Since the vertex operator (4.66) corresponds to an exact state, we expect the amplitude to vanish. We will now prove that this is the case.

The integral in z needs to be regularized by cutting a small disk around each of the other insertion points $z_1, \ldots, z_n$ and then taking the limit where the radius of each disk goes to zero. Since the integrand is a total derivative of a meromorphic function in z, by Stokes' theorem we obtain

$$A_{n+1} = \int_{\mathcal{M}_{0,n}} \sum_{j=1}^n \oint_{\gamma_{z_j}} dz \left\langle \prod_{i=1}^n \mathcal{V}_{\varrho_i \otimes \bar{\varsigma}_i}(z_i, \bar{z}_i)\mathcal{V}_{\varrho}(z)e^{ik_r X_r}(\bar{z}) \right\rangle. \quad (4.68)$$

where γ_{z_j} is a small circle centered in z_j. Each contour integral picks up the residue of the correlator at the corresponding singularity. For a (single-valued) meromorphic function, the sum over all residues vanishes, so that

$$A_{n+1} = 0, \qquad (4.69)$$

as expected.

An analogous result holds even when the state $\bar{\varsigma}$ has zero momentum $k_r = 0$. In this case, $\bar{\varsigma}$ coincides with the state v^- of Eq. (4.56). The vertex operator can still be written as a total derivative, if one allows for the fields $X_r^\mu(\bar{z})$ to appear without derivatives or exponentials

$$\mathcal{V}_{\varrho \otimes \bar{\varsigma}}(z, \bar{z}) \propto \partial_{\bar{z}}(\mathcal{V}_{\varrho}(z)(X_r^0 - X_r^1)(\bar{z})). \qquad (4.70)$$

In general, there are some subtleties in trying to apply the previous argument in presence of the fields $X_r^\mu(\bar{z})$. Indeed, these fields might in principle lead to logarithmic singularities that spoil the single-valuedness of the correlator. However, this never happens in the case of the pure BPS amplitudes we are

interested in, because the combination $X_r^0 - X_r^1$ has non-singular OPE with itself and with all the operators in (4.61). We conclude that, while υ^- is *not* an exact state at zero momentum $k_r = 0$, a pure BPS amplitude with an insertion of the corresponding vertex operator always vanishes, as anticipated in the previous subsection.

Coming back to (4.68), the singularity at the point $z = z_i$ arises from the OPE of $\mathcal{V}_\varrho(z)e^{ik_r X_r}(\bar{z})$ with $\mathcal{V}_{\varrho_i \otimes \bar{\varsigma}_i}(z_i, \bar{z}_i)$. In order to prove (4.30), we just need to show that the residue of this OPE is the action of the algebra element corresponding to ϱ acting on the state $\varrho_i \otimes \bar{\varsigma}_i$, i.e.,

$$\oint_{\gamma_{z_i}} dz\, \mathcal{V}_u(z)e^{ik_r X_r}(\bar{z})\mathcal{V}_{\varrho_i \otimes \bar{\varsigma}_i}(z_i, \bar{z}_i) = \mathcal{V}_{\delta_\varrho(\varrho_i \otimes \bar{\varsigma}_i)}(z_i, \bar{z}_i). \qquad (4.71)$$

The OPE of the anti-holomorphic operators $e^{ik_r X_r}(\bar{z})$ and $\mathcal{V}_{\bar{\varsigma}_i}(\bar{z}_i)$ is non-singular and gives just a shift in the momentum of $\bar{\varsigma}_i$

$$e^{ik_r X_r}(\bar{z})\mathcal{V}_{\bar{\varsigma}_i}(\bar{z}_i) = \mathcal{V}_{(e^{ik_r X_r})_0 \bar{\varsigma}_i}(\bar{z}_i) + O(\bar{z} - \bar{z}_i). \qquad (4.72)$$

Note that the terms of order $O(\bar{z} - \bar{z}_i)$ in this OPE will not contribute to the full anti-holomorphic correlator, since, as we argued above, the correlator is constant in $\bar{z}$. Let us focus on the case where ϱ is in the NS sector, so that it is a representative of a left-moving BRST cohomology class with ghost number 1 and canonical (-1)-picture. The vertex operator $\mathcal{V}_\varrho$ is in the integrated form and in the 0-picture, so it corresponds to the state $b_{-1}X\varrho$ with picture and ghost numbers 0. The latter state has conformal weight 1 and the residue picks up the zero mode acting on the state ϱ_i:

$$\oint_{\gamma_{z_i}} dz\, \mathcal{V}_\varrho(z)\mathcal{V}_{\varrho_i}(z_i) = \mathcal{V}_{(b_{-1}X\varrho)_0 u_i}(z_i). \qquad (4.73)$$

Now, we recall that the Lie bracket of the (even part of the) BKM algebra $\mathfrak{g}$ is precisely defined to be (see Sect. 4.2)

$$[\varrho, \varrho_i] = (b_{-1}X\varrho)_0 \varrho_i. \qquad (4.74)$$

We conclude that

$$\oint_{\gamma_{z_i}} dz\, \mathcal{V}_\varrho(z)e^{ik_r X_r}(\bar{z})\mathcal{V}_{\varrho_i \otimes \bar{\varsigma}_i}(z_i, \bar{z}_i) = \mathcal{V}_{[\varrho, \varrho_i]}(z_i)\mathcal{V}_{(e^{ik_r X_r})_0 \bar{\varsigma}_i}(\bar{z}_i)$$
$$= \mathcal{V}_{\delta_\varrho(\varrho_i \otimes \bar{\varsigma}_i)}(z_i, \bar{z}_i). \qquad (4.75)$$

A similar result holds when ϱ is in the Ramond sector, and represents a left-moving BRST cohomology classes with ghost number 1 and canonical $(-1/2)$-picture. The vertex operator $\mathcal{V}_\varrho$ is in the integrated form and corresponds to the state $b_{-1}\varrho$ of conformal weight 1 and picture number $-1/2$, and the residue picks out the zero mode of this current acting on ϱ_i,

$$(b_{-1}\varrho)_0 \varrho_i. \qquad (4.76)$$

When ϱ_i is in the Ramond sector with canonical $(-1/2)$-picture, the state $(b_{-1}\varrho)_0 \varrho_i$ is in the NS sector with canonical picture number -1 and is simply the commutator $[\varrho, \varrho_i]$. When ϱ_i is in the NS sector with canonical picture

number -1, one should move the picture-changing operators so that the resulting Ramond state is $X(b_{-1}\varrho)_0\varrho_i \equiv [\varrho, \varrho_i]$ with picture number $-1/2$. This seemingly *ad hoc* prescription for the placement of picture-changing operators in fact arises naturally when one describes the superstring amplitude in terms of an integration over supermoduli.[9] This concludes the proof of the second claim in Sect. 4.3.

4.7. Physical Interpretation

Let us pause for a moment and discuss the physical interpretation of the results of this section. We have shown that the subspace $\mathcal{H}_{\mathrm{BPS}}$ of physical states is a representation of a BKM (super-)algebra $\mathfrak{g}$. Furthermore, $\mathfrak{g}$ arises as an algebra of symmetries of the string tree-level amplitudes where all insertion points correspond to states in $\mathcal{H}_{\mathrm{BPS}}$.

If the uncompactified spacetime were higher dimensional, the physical interpretation would be straightforward: $\mathfrak{g}$ is an algebra of symmetries of the S-matrix for the scattering of these BPS states, at least at tree level. In the present case, the spacetime we are considering is $0+1$-dimensional (or $1+1$ dimensional on a cylinder). Naively, it seems natural to interpret the amplitudes as transition amplitudes between an initial asymptotic state at time $t = -\infty$ and a final state at time $t = +\infty$.

However, this interpretation is problematic. Let us regard our models as theories in a $1 + 1$-dimensional cylindrical spacetime. The states in $\mathcal{H}_{\mathrm{BPS}}$, in general, represent spacetime-filling strings. The initial and final states in our transition amplitudes, therefore, represent arbitrary numbers of spacetime-filling strings. However, these states are coupled to the metric and B-field, so they cannot make sense in general: The field equations for these massless degrees of freedom require a precise number of spacetime-filling strings to cancel the B-field tadpole and cosmological constant (see "Appendix A" for a calculation of such a tadpole). In Sect. 5, we will consider a second quantized version of this theory in a zero coupling limit. In this limit, one can neglect any backreaction and regard the massless fields as a non-dynamical fixed background; these kind of inconsistencies can be safely ignored in this case. However, when considering amplitudes, as in this section, we are implicitly assuming that the string coupling is nonzero, so that this solution is not available here.

One possible attitude is to view the amplitudes as a purely formal construction. We can build a collection of functions that are symmetric under the BKM algebra. This statement is certainly true, even if the quantities do not have a sensible spacetime interpretation.

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[9] In particular, the limit where z coincides with another insertion point z_i corresponds to a degeneration of the $(n + 1)$-punctured sphere into two spheres joined through a nodal point, with the first sphere containing the two insertion points z and z_i and the second containing all the other punctures. The nodal point is an additional puncture in each of the two spheres, and the puncture is of Ramond type if ϱ and ϱ_i are one NS and one R, and of NS type otherwise. While an NS degeneration is parametrized only by one even supermodulus, a Ramond degeneration corresponds to one even and one odd supermodulus, and the integration over the odd one corresponds to the insertion of a picture-changing operator. See for example [71], sections 3 and 5, for more details.

On the other hand, the existence of this rich collection of functions is suggestive and hints that there should be a reasonable way to interpret our results. One possibility could be that the vertex operators are accompanied by a background charge that offsets their coupling to the massless fields. This would be analogous to matter coupled to Liouville gravity in two dimensions, where the Liouville and the matter part are coupled in such a way that the combined stress energy tensor is always traceless. A more suggestive possibility is that there exists a topological version of these string models, where only the states in $\mathcal{H}_{\mathrm{BPS}}$ survive, and the dangerous massless fields are either not present or decouple.

While we are not going to further investigate these proposals in the present article, let us conclude by mentioning evidence that the BKM symmetry persists in BPS amplitudes at all loops. We believe this provides an additional hint that a sensible physical interpretation of these amplitudes exists.

To start, we consider the possibility that, at least, the even subalgebra $\mathfrak{g}_0$ is $\mathfrak{g}$ is a symmetry of pure BPS amplitudes at all orders in perturbation theory. The argument of the previous subsection generalizes in a straightforward way. One considers an $(n + 1)$-point amplitude on a genus g Riemann surface Σ, where one of the vertex operators corresponds to a BRST-exact NS-NS state, and has the form (4.70), i.e., a total derivative in the insertion position. One then integrates the position of this BRST-exact state over Σ. Technically, this is an integration over the even supermodulus parametrizing this position, i.e., an integration along the fibers of the forgetful map $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$. Being a total derivative, this integral would vanish trivially if the integrand was non-singular. However, there are singularities when z collides with one of the other insertion points, so one needs to cut out from Σ small disks around each of these insertion points. For pure BPS amplitudes, the integrand still factorizes into left- and right-moving factors, and the right-moving factor is non-singular in $\bar{z}$. Therefore, the integrand is a total derivative of a meromorphic form. The integration reduces to a sum over all the residues of this meromorphic form, and this sum vanishes. The sum over residues can again be interpreted as a variation of an n-point amplitude with respect to the action of some element of the algebra $\mathfrak{g}_0$, so its vanishing implies that the n-point amplitude is invariant. This is a highly non-trivial result, that points toward the existence of a meaningful physical interpretation of these amplitudes.

The odd component of the algebra requires a more sophisticated formalism. In this case, the BRST-exact vertex operator is in the R-NS sector, and there is no even supermodulus that simply parametrizes the position of this operator. Equivalently, there is no map $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ where one "forgets" about one Ramond puncture.[10] In this case, one needs to integrate over all supermoduli at once. The integrand is still a total derivative, so one gets only

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[10]To show that such a forgetful map does not exist, it is sufficient to note that the number of Ramond punctures must always be even.

contributions from the boundary components of the Deligne–Mumford compactification of the supermoduli space. To prove that our result at tree level generalizes to higher genus, one needs to show that the only nonvanishing boundary contributions are the ones coming from degenerations of the Riemann surface Σ, of genus g and $n+1$ punctures, into a surface Σ' of the same genus g and $n-1$ punctures and a sphere with two punctures (one of them being the exact state), joined through a double point. Furthermore, the sum over all such degenerations should vanish. In the previous case of NS-NS punctures, these kind of boundary components in the Deligne–Mumford compactification correspond exactly to the limits where the exact vertex operator collides with one of the other punctures.

These conditions are similar (though not exactly the same) as the conditions for spacetime supersymmetry in general string perturbation theory. In that case, one can show (see for example [70]) that there are only a few, particular degeneration limits that can potentially spoil spacetime supersymmetry at higher loop, and that these cases are related to either tadpoles, spontaneous supersymmetry breaking, or to higher loop modifications of supersymmetry transformations. Checking that the full algebra $\mathfrak{g}$ (including the odd component) is a symmetry of purely BPS amplitudes at all orders in string perturbation theory would provide strong evidence that a consistent physical interpretation of this theory does exist.

5. Second Quantization and BPS Indices

Starting from the space of physical states in our type IIA or heterotic theory on $V_1 \otimes \bar{V}_2$, let us now define a "second quantized" Hilbert space describing an arbitrary number of strings. We will consider only the limit where the string coupling constant is zero, so that we have a theory of free strings.

We begin by constructing our second quantized Hilbert space and we define an associated supersymmetric index which only receives contributions from BPS states. We show that the second quantized space of BPS state forms a representation for the BKM algebra $\mathfrak{g}$ associated with the (super)VOA V_1, so that the index is a character for that representation. We then proceed to consider two examples in detail: a family of heterotic string compactifications on $V \otimes \bar{V}^{f\natural}$ (Sect. 5.3) and type IIA compactifications on $V \otimes \bar{V}^{f\natural}$ (Sect. 5.4). In these cases, we calculate the BPS indices and show that they correspond to known (super)denominator formulas of BKM algebras.

5.1. A Second Quantized Hilbert Space

Let us consider type IIA or heterotic string theory on $V_1 \otimes \bar{V}_2$, where $V_2 \in \{V^{fE_8}, V^{f\natural}, F_{24}\}$. Occasionally, we will compactify one further space-like direction on a circle S^1. In order to construct this second quantized space, we first introduce one operator η_a for each physical state $a \in \mathsf{H}^2(k)$ with $k \neq 0$. The operators obey a free oscillator algebra, with (anti-)commutation relations determined by the string two point functions. The $k \neq 0$ operators can be separated into creation ($k^0 > 0$) or annihilation ($k^0 < 0$) operators. The

idea is that each creation/annihilation operator η_a creates or destroys a string in the state a in the $(1 + 1)$-dimensional spacetime.

For the physical states with $k = 0$ (i.e., $k_l = k_r = 0$), the construction is slightly more complicated. First note that the $k = 0$ physical states are the same in the uncompactified and in the compactified theory; for these definitions, it is convenient to work in the uncompactified theories. We separate the space of zero-momentum physical states into two orthogonal subspaces, the "propagating" and "non-propagating" states. The "propagating" states are the ones that can be obtained starting from massless $(k_l)^2 = (k_r)^2 = 0$ states with $k \neq 0$ and then taking the $k \to 0$ limit (this limit is well defined only in the uncompactified case, which is why we prefer to give the definition in this set up). The "non-propagating" states are the states that are orthogonal to the propagating ones. The physical interpretation is clear: The propagating states are the zero modes of some local propagating degrees of freedom in the $1 + 1$-dimensional spacetime, corresponding, for example, to some massless scalars (excluding the dilaton) or spin $1/2$ fermions in spacetime. The non-propagating degrees of freedom correspond, for example, to the metric, B-field, dilaton, gravitinos and gauge vectors, that have no local propagating degrees of freedom in $1 + 1$-dimensions. The presence of the latter states in the physical spectrum of the superstring correspond to the possibility to deform the background, such as the geometry of the spacetime and the string coupling constant. In the construction of the second quantized space of states, we will introduce one operator η_a for each "propagating" state $a \in \mathsf{H}^2(0)$, and no operator for the "non-propagating" ones. For each of the latter, we can instead introduce a fixed non-dynamical background field: For example, in the compactified case, one can set the radius of the circle, a background B-field, or Wilson lines of gauge fields along the circle (besides the string coupling, which we always fix to zero).

We can now define the second quantized Hilbert space as the Fock space for the operators η_a, where a runs over a basis of $\oplus_{k \neq 0}\mathsf{H}^2(k)$ and of the subspace of states in $\mathsf{H}^2(0)$ that are "propagating," in the sense above. A ground state in this Hilbert space is a state in the kernel of all annihilation $(k^0 < 0)$ operators; physically, these states should correspond to the vacuum, i.e., no strings are present. The ground states might be degenerate if there are $k = 0$ "propagating" operators. The second quantized Hilbert space is then constructed by acting in all possible ways on the ground states with the creation operators η_a with $k^0 > 0$.

Let us consider the theory compactified on a circle S^1 of radius R. We define a supersymmetric index

$$\mathcal{Z}^{V_1 \otimes \bar{V}_2}(\beta, b, v, R, A_k) := \text{Tr}(e^{-\beta H} e^{2\pi i b W} e^{2\pi i v M} e^{2\pi i \sum_k A_k q_k} (-1)^{\mathbf{F}}), \quad (5.1)$$

where the trace is taken over the second quantized Hilbert space we have just defined, H, M, and W are the total Hamiltonian, momentum, and winding, the q_k are charges with respect to some abelian background gauge fields (corresponding to a maximal abelian subgroup of the possibly non-abelian gauge group), and $(-1)^{\mathbf{F}}$ is the spacetime fermion number. The superscript denotes

TABLE 3. The vacuum winding w_0 (i.e., the B-field tadpole) and momentum m_0 for the type IIA and heterotic compactifications we are interested in

Type IIA	w_0	m_0
$V^{fE_8} \otimes \bar{V}^{fE_8}$	0	0
$F_{24} \otimes \bar{V}^{fE_8}$	0	0
$V^{fE_8} \otimes \bar{V}^{f\natural}$	0	0
$V_+^{f\natural} \otimes \bar{V}_-^{f\natural}$	0	-24
$V_-^{f\natural} \otimes \bar{V}_-^{f\natural}$	-24	0
$F_{24} \otimes \bar{V}_-^{f\natural}$	24	24
$F_{24} \otimes \bar{F}_{24}$	0	0

Het	w_0	m_0
$V \otimes \bar{V}^{fE_8}$	0	0
$V \otimes \bar{V}_\pm^{f\natural}$	$\pm(N - 24)$	$\pm N$
$V \otimes \bar{F}_{24}$	0	0

The details of the calculations are given in "Appendix A"

the factorized worldsheet theory in which we are computing the index. The real chemical potentials β, b, v, A_k admit natural physical interpretations: β is the usual inverse temperature; b is a constant background B-field; v is a constant off-diagonal term in the $1+1$ dimensional metric; A_k are constant background for the gauge fields in a maximal abelian torus of the gauge group, giving non-trivial Wilson lines along the circle S^1. If η is an oscillator corresponding to momentum $k_l^0 = k_r^0 = E$ and $k_{l,r}^1 = \frac{1}{\sqrt{2}}(\frac{m}{R} \mp wR)$, one has

$$[H, \eta] = E\eta \qquad [M, \eta] = m\eta \qquad [W, \eta] = w\eta. \qquad (5.2)$$

The ground states of the second quantized Hilbert space are (possibly degenerate) eigenstates for H, M, W

$$H|0\rangle = E_0|0\rangle, \qquad M|0\rangle = m_0|0\rangle, \qquad W|0\rangle = w_0|0\rangle. \qquad (5.3)$$

Here, w_0 and m_0 have a straightforward physical interpretation as a B-field tadpole and a "vacuum momentum," respectively, since they couple to the background B-field b and to the off-diagonal metric component v. They arise as a 1-loop effect in superstring theory, and can be easily computed for all the theories we are considering, see "Appendix A." The "cosmological constant" E_0 is usually fixed in terms of w_0, m_0 by spacetime supersymmetry.

The theories we are considering often contain some spacetime supersymmetries $\mathcal{Q}_\alpha^i$ (see Tables 1 and 2), where $i = 1, 2, 3, \ldots$ and $\alpha \in \{\pm\}$ denote the chirality. In this case, the index gets nonvanishing contributions only from a BPS subspace of the second quantized Hilbert space, which can be constructed as a Fock space by acting on the vacuum only with the creation operators that (anti-)commute with the supercharges. In turn, these operators correspond to single string states that are annihilated by the supercharges (BPS states). Thus, we can restrict our analysis to these BPS single string states. In heterotic

models, the two-dimensional supersymmetry algebra is

$$\{\mathcal{Q}_\pm^i, \mathcal{Q}_\pm^j\} = 2\delta^{ij}(P_r^0 \pm P_r^1), \qquad \{\mathcal{Q}_+^i, \mathcal{Q}_-^j\} = 0, \tag{5.4}$$

and involves only the right-moving momentum. The same relations hold in type II theories when the supercharges arise from the NS-R sector, while for the R-NS sector we have the analogous relations with the left-moving momenta,

$$\{\mathcal{Q}_\pm^i, \mathcal{Q}_\pm^j\} = 2\delta^{ij}(P_l^0 \pm P_l^1), \qquad \{\mathcal{Q}_+^i, \mathcal{Q}_-^j\} = 0. \tag{5.5}$$

Type II supercharges coming from different sectors anticommute with each other.

When any of these supersymmetries is present, the BPS condition is equivalent to a linear relation between the energy $E = k_l^0 = k_r^0$ and the winding and momentum m, w of a single string state. By linearity, the same relation must hold between the Hamiltonian H and the momentum and winding operators M and W in the BPS subsector of the second quantized Hilbert space. For example, when negative chirality supersymmetries $\mathcal{Q}_-^i$ from the right-moving Ramond sector are present, the BPS condition implies

$$H = \frac{1}{\sqrt{2}} \left(\frac{M}{R} + WR \right). \tag{5.6}$$

Note that if we require the ground states of the second quantized Hilbert space to be supersymmetric, we have the corresponding relation $E_0 = \frac{1}{\sqrt{2}} \left(\frac{m_0}{R} + w_0 R \right)$ of vacuum eigenvalues. By restricting the trace to the BPS subsector of the second quantized Hilbert space, we can reexpress the above index as

$$\mathcal{Z}^{V_1 \otimes \bar{V}_2}(\beta, b, v, R, A_k)$$
$$= \text{Tr}_{\text{BPS}} \left(e^{-\frac{\beta}{\sqrt{2}} \left(\frac{M}{R} + WR \right)} e^{2\pi i bW} e^{2\pi i vM} \prod_i e^{2\pi i \sum_k A_k q_k} (-1)^{\mathbf{F}} \right). \tag{5.7}$$

It is convenient to reorganize the real parameters β, R, b and v into two complex variables,

$$T = b + i\frac{\beta R}{2\sqrt{2}\pi}, \qquad U = v + i\frac{\beta}{2\sqrt{2}\pi R}. \tag{5.8}$$

Both T and U take values in the upper half-plane and parametrize, respectively, the Kähler and complex structure of the Euclidean spacetime torus obtained upon Wick rotation of the time direction. In terms of these variables, the index becomes

$$\mathcal{Z}^{V_1 \otimes \bar{V}_2}(T, U, A_k) = \text{Tr}_{\text{BPS}} \left(e^{2\pi i TW} e^{2\pi i UM} \prod_i e^{2\pi i \sum_k A_k q_k} (-1)^{\mathbf{F}} \right)$$

$$= \text{Tr}_{\text{BPS}} \left(p^W q^M \prod_k y_k^{q_k} (-1)^{\mathbf{F}} \right), \tag{5.9}$$

where we introduced the variables

$$p := e^{2\pi i T}, \qquad q := e^{2\pi i U}, \qquad y_k := e^{2\pi i A_k}. \tag{5.10}$$

If there are bosonic or fermionic zero modes, this definition may need to be modified. Bosonic zero modes might lead to a divergence that needs to be

regularized. In the examples we consider below, massless scalars always come from the R-R sector. Since there is no physical string states representing the zero momentum of R-R fields, their contribution is automatically factored out. Fermionic zero modes $\lambda^1, \lambda^2 \ldots$ make the index vanish (unless they carry non-trivial charges q_k). In all the examples, we will consider, the fermionic zero modes will arise as gauginos, i.e., superpartners of the gauge fields, and thus transforming in the adjoint representation of the spacetime gauge group. The number of neutral fermionic zero modes in this case is equal to the rank of the gauge group. In order to get a nonvanishing quantity, one can absorb these zero modes by inserting them in the trace

$$\mathcal{Z}^{V_1 \otimes \bar{V}_2}(\beta, b, v, R, A_k) := \mathrm{Tr}\left(e^{-\beta H} e^{2\pi i b W} e^{2\pi i v M} e^{2\pi i \sum_k A_k q_k} (-1)^{\mathbf{F}} \prod_i \lambda^i\right).$$

(5.11)

We will use the notation $\mathcal{Z}(T, U, A_k)$ throughout this section to refer to the minimal nonvanishing index for each of the theories we study: If the theory has no fermionic zero modes, then $\mathcal{Z}$ indicates (5.9), while if the theory does have fermionic zero modes, we absorb them as in (5.11).

There are further modifications to the index which one may want to consider. First, in type II theories, the spacetime fermions may come either from the NS-R or from the R-NS sector. If there are supercharges coming from both of these sectors, then only the ground states will contribute to the index, thus giving a constant. To get a more interesting quantity, one can consider a modified index

$$\tilde{\mathcal{Z}}^{V_1 \otimes \bar{V}_2}(\beta, b, v, R, A_k) := \mathrm{Tr}\left(e^{-\beta H} e^{2\pi i b W} e^{2\pi i v M} e^{2\pi i \sum_k A_k q_k} (-1)^{\mathbf{F}_r} \prod_i \lambda^i\right),$$

(5.12)

where the fermion number $(-1)^{\mathbf{F}}$ is replaced by an operator $(-1)^{\mathbf{F}_r}$ anti-commuting only with, say, states in the right-moving R sector (NS-R and R-R) and commuting with states in the right-moving NS sector (NS-NS and R-NS). Under this definition, the states contributing to the index obey $k_r^0 = k_r^1$, but not necessarily $k_l^0 = k_l^1$. As above, we use $\tilde{\mathcal{Z}}$ to refer to the minimal nonvanishing (with any fermionic zero modes inserted) modified index for a given theory.

When the string theory has $V_2 = V_-^{f\natural}$ as its right-moving SVOA, $\mathcal{Z}, \tilde{\mathcal{Z}}$ evaluate to the 24th powers of the superdenominator and denominator (respectively) of the BKM algebra constructed from the left-moving (S)VOA. We will derive this in detail in Sects. 5.3 and 5.4, but the basic idea is that $V^{f\natural}$ has 24 states of conformal weight $1/2$ in the Ramond sector and none in the NS sector. The contribution from these 24 fermionic ground states is to produce a 24-fold multiplicity of single-string BPS states, resulting in a second quantized Fock space that has the form of a 24th tensor power. Any of the 24 factors effectively counts contributions from the left-movers, and produces the BKM (super)denominator.

There are other options to modify the index. We could impose anti-periodic boundary conditions on the fermions around the spacelike S^1 circle. This corresponds to taking a different quantization for the spacelike momenta k_l^1 and k_r^1 in the NS-R and R-NS sectors, where n takes values in $\frac{1}{2} + \mathbb{Z}$ instead of $\mathbb{Z}$. We could also impose anti-periodic boundary conditions only on (say) the NS-R fermions and periodic ones on the R-NS fermions. We will not consider these modifications in the rest of the text and focus on computing the indices in Eqs. (5.11) and (5.12).

5.2. The BPS Fock Space as an Algebra Module

As mentioned in the previous subsection, when the string theory has the right-movers fixed to be $V_2 = V^{f\natural}$, the supersymmetric indices (5.11), (5.12) reproduce the 24th power of the denominator and superdenominator of the BKM associated with the left-movers. The factor of 24 simply comes from the right-moving ground state degeneracy. In fact, the supersymmetric index can be viewed as the graded supercharacter of a certain algebra module for the left-moving BKM superalgebra $\mathfrak{g}$. Let us restrict for simplicity to the case where $\mathfrak{g}$ is bosonic, i.e., with no odd components; the generalization to the case where $\mathfrak{g}$ is a superalgebra is straightforward. Let us consider the usual triangular decomposition as $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{h} \oplus \mathfrak{g}^-$ in terms of positive and negative roots and the Cartan subalgebra. As discussed below, the module describing the second quantized states is $(\bigwedge \mathfrak{g}^-)^{\otimes 24}$, 24 copies of the graded sum of all exterior powers of the subalgebra associated with the negative roots. It is well known that for ordinary Lie algebras or Kac–Moody algebras, this is an irreducible $\mathbb{Z}_2$-graded module with highest weight the Weyl vector. It also inherits the weight space grading from $\mathfrak{g}$. In the case of the heterotic string with $V_1 = V^\natural$, the BKM algebra $\mathfrak{g}$ is the monster Lie algebra, and the module was constructed in [55]. We will briefly review its construction and the relation with the second quantized Hilbert space described in Sect. 5.1, and then comment on subtleties in the generalization to our other examples. We will focus on the construction of a single copy of $\bigwedge \mathfrak{g}^-$, since the 24 copies do not interact with one another.

The $\mathfrak{g}$-representation $\bigwedge \mathfrak{g}^-$ can be formally constructed as follows. Recall that $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, and has a non-degenerate invariant bilinear form $(\cdot|\cdot)$ with respect to which $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{h}$ and $\mathfrak{g}_\beta$, for all $\alpha, \beta \in \Delta$ with $\alpha \neq -\beta$. The basic idea is to build a module $A_\mathfrak{g}$ over the Clifford algebra whose underlying vector space is $\mathfrak{g}$ endowed with $(\cdot|\cdot)$. More precisely, let us consider an algebra of fermionic operators $\xi_a, \forall a \in \mathfrak{g}$, transforming in the adjoint representation of $\mathfrak{g}$. The fermionic modes satisfy the Clifford algebra

$$\{\xi_a, \xi_b\} = (a|b) \tag{5.13}$$

where the right-hand side vanishes whenever $a \in \mathfrak{g}_\alpha$ and $b \in \mathfrak{g}_\beta$, with $\alpha, \beta \in \Delta$, $\alpha \neq -\beta$. Let us define a module $A_\mathfrak{g}$ for this Clifford algebra as follows: We start from a space of ground states, forming an irreducible representation for the finite-dimensional Clifford algebra of operators ξ_a, $a \in \mathfrak{h}$, and annihilated by all ξ_a with $a \in \mathfrak{g}^+$; the module $A_\mathfrak{g}$ is obtained by acting in all possible ways on such ground states by the operators ξ_a, $a \in \mathfrak{g}^-$. We can view $\mathfrak{g}$ as a subalgebra

of the orthogonal algebra $so(\mathfrak{g})$ that preserves the bilinear form $(\cdot|\cdot)$, since $(\cdot|\cdot)$ is preserved by the adjoint action of the BKM on itself. This Clifford module can be understood as a spinorial representation of the orthogonal algebra $so(\mathfrak{g})$ and hence of its subalgebra $\mathfrak{g}$.

The space of ground states of $A_{\mathfrak{g}}$ forms a module for the Clifford subalgebra associated with $\mathfrak{h} \subset \mathfrak{g}$. Each of these ground states is a highest weight vector for a representation of $\mathfrak{g} \subset so(\mathfrak{g})$. This immediately shows that, in general, the Clifford module $A_{\mathfrak{g}}$ is not irreducible as a $\mathfrak{g}$-representation. In the case of the rank-2 monster Lie algebra, one has a twofold ground state degeneracy of opposite chirality. Indeed, for general $\mathfrak{g}$ of even rank, the module always splits, as a representation over $so(\mathfrak{g})$, into two spinors of definite chirality. Focusing on the rank-2 case for the moment, it is natural to consider the $\mathfrak{g}$-representation obtained by restricting to a subspace $A_{\mathfrak{g}}^+$ of definite chirality, which effectively allows us to skirt vacuum degeneracy issues relating to the Cartan operators, as follows. If we take a light-cone decomposition of the Cartan as $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$ such that $\{\xi_+, \xi_-\} = 1$ (and all other anticommutators zero), we may define a vacuum

$$\xi_a|0\rangle = 0, \quad a \in \mathfrak{g}^+ \oplus \mathfrak{h}^+. \tag{5.14}$$

The Clifford module $A_{\mathfrak{g}}$ is recovered by acting on the vacuum in all possible ways with the $\xi_a, a \in \mathfrak{g}^- \oplus \mathfrak{h}^-$. Next, we consider a chirality operator $(-1)^J$, distinct from the fermion parity operator used in the index definition, which leaves the vacuum invariant but acts on all ξ_a with a minus sign. We project to the positive chirality subspace of our module $A_{\mathfrak{g}}$, to obtain a graded vector space $A_{\mathfrak{g}}^+$ isomorphic to $\bigwedge \mathfrak{g}^-$. Namely, to obtain the even powers of $\mathfrak{g}^-$, simply act on the vacuum with an even number of $\xi_a, a \in \mathfrak{g}^-$, and to obtain the odd powers, act first with ξ_- on the vacuum, and then with an odd number of ξ_a, $a \in \mathfrak{g}^-$.

We stress that the set of modes $\{\xi_a\}$ are different from the set $\{\eta_a\}$ considered in Sect. 5.1: In that case, we exclude the modes corresponding to gravitinos, while here we include all modes. Excluding the gravitinos leads, in the rank-2 case, to a non-degenerate vacuum for the second quantized Fock space. This suggests that the second quantized space is isomorphic to the positive chirality subspace $A_{\mathfrak{g}}^+$ of the ξ_a Clifford module; as such, it is naturally a representation for the BKM algebra $\mathfrak{g}$. Indeed, the operators η_a considered in Sect. 5.1 can be identified with the operators $\xi_- \xi_a$, that are even with respect to $(-1)^J$ and carry the same grading with respect to $\Phi \subset \mathfrak{h}^*$. Implicitly, then, we may imagine that our index can be defined as a trace over the full Clifford module $A_{\mathfrak{g}}$, with an additional insertion of $(1 + (-1)^J)/2$ in the trace that has so far been suppressed. With this identification, the spacetime fermion number $(-1)^{\mathbf{F}}$ can be defined as an operator acting trivially on the vacuum, commuting with ξ_-, ξ_+ and anti-commuting with the other ξ_a. The fact that we have two $\mathbb{Z}_2$ operators to employ in defining our index, $(-1)^J$ and $(-1)^{\mathbf{F}}$, is special to supersymmetric indices in $0+1$ dimensions. We prove in [55] that the resulting module is indeed a suitable highest weight representation of $\mathfrak{g}$.

A similar construction holds for the more general, higher-rank BKMs we discuss in this paper. In the construction of the second quantized Fock space in Sect. 5.1, we consider oscillators η_a at zero momentum only when they correspond to the zero-momentum limit of "propagating" massless degrees of freedom, while we exclude the modes that exist only at zero momentum, i.e., the two gravitinos. The space spanned by the gravitino zero modes correspond to a subspace of signature $(1,1)$ in the Cartan subalgebra of $\mathfrak{g}$. In particular, gravitinos can be associated with a pair of fermionic modes ξ_+, ξ_- obeying $\{\xi_\pm, \xi_\pm\} = 0$, $\{\xi_+, \xi_-\} = 1$. The other elements of the Cartan subalgebra now correspond to gaugino zero modes, which we absorb in the definition of the index. Once again, the second quantized Fock space can be identified with the positive chirality subspace $A_{\mathfrak{g}}^+$ of the ξ_a-Clifford module $A_{\mathfrak{g}}$, where the η_a operators correspond to the generators $\xi_- \xi_a$ of the even Clifford algebra. For a BKM of even rank r, the Clifford module $A_{\mathfrak{g}}$ has ground state degeneracy $2^{r/2}$, where each chirality $A_{\mathfrak{g}}^\pm$ of the orthogonal algebra splits as $A_{\mathfrak{g}}^\pm \cong (\bigwedge \mathfrak{g}^-)^{\oplus 2^{r/2-1}}$ into $2^{r/2-1}$ representations of $\mathfrak{g}$, each isomorphic (as a graded vector space) to $\bigwedge \mathfrak{g}^-$.

The separation between the gauginos and gravitinos zero modes looks unnatural from the algebra point of view, since the adjoint action of $\mathfrak{g}$ can rotate the ξ_+, ξ_- operators into any other ξ_a, $a \in \mathfrak{h}$. In fact, the distinction between propagating and non-propagating zero modes makes sense only in the decompactification limit, where the radius R of the circle S^1 goes to infinity. When the rank of the algebra is greater than 2, there are different decompactification limits, which correspond to different choices of a null direction of the Cartan subalgebra.

We therefore have the option of defining the full BPS Fock space using the full module $A_{\mathfrak{g}}$ of the Clifford algebra, including both chiralities of the spinorial rep of $so(\mathfrak{g})$. This necessitates absorbing the gravitino modes to get a nonzero index. Alternatively, we may project onto a single chirality $A_{\mathfrak{g}}^+$ of the spinorial rep, as we did with the monster. As a graded vector space, the resulting chiral representation is isomorphic to the second quantized Fock space obtained from ignoring the gravitino zero modes. This is simpler and seems natural from the physical viewpoint, in particular if one sees the superstring model as a $(1+1)$-dimensional theory on the cylinder. The downside of privileging the gravitinos relative to the gauginos in this way is that it requires choosing a null direction in the Cartan subalgebra, viewed as a space of rank $(1, r-1)$. The resulting BKM algebra action on the Fock space module is rather unnatural.

5.3. Heterotic Strings on $V \otimes \bar{V}^{f\natural}$

Let us now apply the general formalism described in Sect. 5.1 to some specific superstring models. As a first example, let us consider the heterotic string on $V_L \otimes \bar{V}_+^{f\natural}$, where V_L is a lattice VOA for one of the 24 Niemeier lattices L and contains N currents. Each massless string state is the tensor product of a chiral and an anti-chiral physical state, which in turn are in one-to-one correspondence with, respectively, a left-moving weight 1 state and a right-moving weight 1/2 state is the internal CFT.

Let us adopt the following notation for these chiral and anti-chiral states:

- ν^a, $a = 1, \ldots, N$ denote the currents in V;
- $\bar{v}_+^i$, $i = 1, \ldots, 24$ correspond to the 24 Ramond ground states in $\bar{V}^{f\natural}$;
- $\alpha_{++}, \alpha_{--}, \bar{\psi}_{++}, \bar{\psi}_{--}$ correspond to the light-cone components of ∂X^μ and $\bar{\psi}^\mu$ from the sigma model with target $\mathbb{R}^{1,1}$.

The $+$ and $-$ subscripts keep track of the action of the $SO(1,1)$ spacetime Lorentz group: Each $+$ or $-$ is $\pm 1/2$ in the spin; spacetime parity exchanges $+$ and $-$.

By taking the tensor products of chiral and anti-chiral states, we obtain the massless content of the theory:

- Graviton $\alpha_{++}\bar{\psi}_{++}$ and $\alpha_{--}\bar{\psi}_{--}$; B-field $\alpha_{++}\bar{\psi}_{--} - \alpha_{--}\bar{\psi}_{++}$; dilaton $\alpha_{++}\bar{\psi}_{--} + \alpha_{--}\bar{\psi}_{++}$. These states are non-physical, except for $k = 0$.
- 24 chiral gravitinos $\alpha_{++}\bar{v}_+^i$ (spin $+3/2$); 24 anti-chiral dilatinos $\alpha_{--}\bar{v}_+^i$ (spin $-1/2$). These states are also only physical for $k = 0$.
- N gauge vectors $A^a = \nu^a\bar{\psi}_{++}, \nu^a\bar{\psi}_{--}$. These states are only physical for $k = 0$.
- $24N$ chiral gauginos $\lambda^{ai} = \nu^a\bar{v}_+^i$ (spin $1/2$); these states are physical for all k with $k^2 = 0$ (they are the only massless propagating degrees of freedom).

The N gauge vectors A^a correspond to a semi-simple Lie algebra g of dimension N; the propagating gauginos λ^{ai} form 24 copies of the adjoint representation for g.

The zero-momentum modes of the gauginos form a Clifford algebra, and the ground states of our second quantized Hilbert space must form a module over this Clifford algebra. The full Clifford algebra is a direct sum of 24 copies of the Clifford algebra generated by a single set (i.e., a fixed i, say $i = 1$) of N gauginos λ^{ai}, $a = 1, \ldots, N$. The Clifford module of ground states, therefore, is the tensor product of 24 copies of the Clifford module for a single set of gauginos. Thus, we can focus on one of these sets and simply take the tensor product at the end.

We choose a Cartan subalgebra $h \subset g$ and take a triangular decomposition $g = g^- \oplus h \oplus g^+$. We have an associated decomposition of the gauginos zero modes into a Cartan subspace plus the positive and negative roots subspaces. We turn on non-trivial Wilson lines $y_1, \ldots, y_r$, where $r = \operatorname{rank} g$ (for the theories we are considering we always have $r = 24$) for the gauge vectors A_a in the Cartan subalgebra, under which the gauginos in $g^- \oplus g^+$ are charged. To get a nonvanishing supersymmetric index, one needs to absorb the zero modes of the neutral λ^{ai}, i.e., the ones in the Cartan subalgebra. The ground states of the Hilbert space live in a tensor product of a Clifford module for the "Cartan" gauginos times a Clifford module for the $g^+ \oplus g^-$ gauginos. After absorbing the zero modes, the trace over the "Cartan" Clifford module just gives a nonzero constant, which, with the appropriate normalization, can be taken to be 1. Thus, we can just focus on the factor corresponding to the Clifford module for $g^+ \oplus g^-$. The latter module can be constructed by starting from a state (a highest weight vector) annihilated by all gauginos in g^- and

then acting in all possible ways with the gauginos in g^+. The ground state contribution to the supersymmetric index is a trace $\mathrm{Tr}((-1)^{\mathbf{F}} \prod y_i^{q_i})$ over this module, where q_i are the charges with respect to the background gauge fields in the Cartan torus. The final result is the Weyl denominator of the algebra g

$$y^{-\rho_g} \prod_{\alpha \in \Delta_g^+} (1 - y^\alpha), \tag{5.15}$$

where Δ_g^+ are the positive roots of the Lie algebra g and ρ_g is the charge of the lowest weight vector, which equals the Weyl vector of the Lie algebra g. (We implicitly use the notation that y^α for any vector α is shorthand for $\prod_k y_k^{\alpha_k}$). Note that there are 24 copies of this set of gauginos, and hence one needs to take the 24th power of this Weyl denominator in computing the index.[11]

The full supersymmetric index is given by multiplying this trace over the ground states by the contributions of the positive energy oscillators. For $k \neq 0$, the BRST quantization of the string is equivalent to the light-cone quantization, so that the classes in $\mathsf{H}^2(k_l, k_r)$ are in one-to-one correspondence with states in the internal CFT $V_L \otimes \bar{V}_+^{f\natural}$ satisfying the physical state conditions

$$L_0 - 1 = -(k_l)^2/2, \qquad \bar{L}_0 - \frac{1}{2} = -(k_r)^2/2. \tag{5.16}$$

The BPS condition forces $(k_r)^2 = 0$, and since $V_+^{f\natural}$ contains no NS states with weight $1/2$, all BPS states come from the Ramond sector and are therefore spacetime fermions. This means that the corresponding operators η_a obey a free fermion oscillator algebra.

In order to count the number of such oscillators, let us consider the "flavored" partition function for the lattice VOA V_L

$$Z_{V_L}(\tau, \xi) := \mathrm{Tr}_{V_L}(\mathsf{q}^{L_0 - \frac{c}{24}} e^{2\pi i \sum_k \xi_k q_k}).$$

Here, $\xi \equiv (\xi_1, \xi_2, \ldots)$ are chemical potentials, and q_k are the weights with respect to the Lie algebra generated by the zero modes of the N currents in V_L; as the notation suggests, these are also the charges of the corresponding physical string states with respect to the background gauge fields $A = (A_1, A_2, \ldots)$. For a lattice VOA V_L based on the Niemeier lattice L, this is given by

$$Z_{V_L}(\tau, \xi) = \frac{\Theta_L(\tau, \xi)}{\eta(\tau)^{24}} = \sum_{n \in \mathbb{Z}, \ell \in \mathbb{Z}^{24}} c(n, \ell) \mathsf{q}^n e^{2\pi i \sum_k \xi_k \ell_k}, \tag{5.17}$$

where Θ_L is the "flavored" lattice theta series

$$\Theta_L(\tau, \xi) = \sum_{\ell \in \mathbb{Z}^{24}} \mathsf{q}^{\frac{1}{2} \sum_{j,k} \ell_j \ell_k \mu_j \cdot \mu_k} e^{2\pi i \sum_k \xi_k \ell_k}, \tag{5.18}$$

---

[11] The ground states of the second quantized Hilbert space must form a representation of the gauge algebra g. It is clear what kind of representation this is: One considers the group G, the adjoint group of g, as a subgroup of the orthogonal group $SO(g)$ acting on the vector space of the algebra g, and preserving the Cartan Killing form. The gauginos form a vector representation for $SO(g)$. The ground states form a module for the Clifford algebra associated with the vector space underlying g, so they form a spinorial representation of $Spin(g)$, the universal cover of $SO(g)$. The group G being a subgroup of $SO(g)$ lifts to a subgroup of $Spin(g)$, and this determines the representation of G on the ground states.

where $\mu_1, \ldots, \mu_{24}$ is a basis of L.

Let us first express the ground state factor (5.15) in terms of the Fourier coefficients $c(n, \ell)$ of the partition function $Z_{V_L}(\tau, \xi)$. The number of currents in V is

$$N = \sum_{\ell \in \mathbb{Z}^{24}} c(0, \ell),$$

and the rank of the algebra is $c(0,0)$, which is always equal to 24 for L a Niemeier lattice. The set of roots $\{\lambda \in L, \ \lambda^2 = 2\}$ correspond to the roots of the algebra g, and can accordingly be written as the disjoint union of the sets of positive or negative roots. For $\ell \in \mathbb{Z}^{24}$ such that $\lambda = \sum_i \ell_i \mu_i$ is a root, we write $\ell > 0$ or $\ell < 0$ depending on whether λ is positive or negative. The ground state contribution to the index (5.15) can thus be written as

$$y^{-24\rho} \prod_{\ell > 0} \left(1 - \prod_{i=1}^{24} y_i^{\ell_i} \right)^{24c(0,\ell)}, \tag{5.19}$$

where for each $\ell \neq 0$, $c(0, \ell)$ is either 1 (if the vector $\sum_i \ell_i \mu_i$ has length 2) or 0 (otherwise).

For the contribution of positive energy string oscillators to the second quantized index, we note that, for BPS states of winding and momentum (w, m), one has $-(k_l)^2/2 = mw$. Therefore, the number of BPS single string states of winding-momentum (w, m) and charge vector $\ell \equiv (\ell_1, \ldots, \ell_{24})$ is $24c(mw, \ell)$ (the factor 24 comes from the fact that there are 24 Ramond ground states in $V^{f\natural}$). The positive energy condition $E = \frac{1}{\sqrt{2}}(\frac{m}{R} + wR) > 0$ in the large radius regime $R > 1$ is satisfied for $w > 0, m \in \mathbb{Z}$ (the only possible state with $m < 0$ is $m = -1, w = 1$) or for $w = 0, m > 0$. Therefore, the contribution from positive energy oscillators is

$$\prod_{m>0} \prod_{\ell \in \mathbb{Z}^{24}} \left(1 - q^m \prod_{i=1}^{24} y_i^{\ell_i} \right)^{24c(0,\ell)} \prod_{w>0, m \in \mathbb{Z}} \prod_{\ell \in \mathbb{Z}^{24}} \left(1 - p^w q^m \prod_{i=1}^{24} y_i^{\ell_i} \right)^{24c(mw,\ell)}. \tag{5.20}$$

Altogether, we find the index can be written as

$$\mathcal{Z}^{V_L \otimes \bar{V}_+^{f\natural}}(T, U, A_k) = p^{N-24} q^N y^{-24\rho} \prod_{\substack{m,w \in \mathbb{Z}, \ell \in \mathbb{Z}^{24} \\ (m,w,\ell) > 0}} \left(1 - p^w q^m \prod_{i=1}^{24} y_i^{\ell_i} \right)^{24c(mw,\ell)}, \tag{5.21}$$

where $(m, w, \ell) > 0$ means $w \geq 0$, with $m \geq 0$ if $w = 0$, and with $\ell > 0$ if $m = w = 0$. The winding-momentum (w_0, m_0) of the ground states is computed in "Appendix A" and is given by $w_0 = N - 24$ and $m_0 = N$. This index is equal to the 24th power of the denominator for the BKM algebra constructed from any Niemeier lattice VOA, which is known as the fake monster Lie algebra [9,10].

This algebra denominator turns out to be equal to the automorphic form Φ_{12} [34], whose appearance in string theory has been explored earlier [35]. Φ_{12} enjoys 24 expansions associated with 24 "Niemeier cusps" in moduli space corresponding to the 24 complementary constructions of the BKM. The fact that there is a unique BKM algebra associated with all the 24 VOAs V_L associated with a Niemeier lattice L is not a coincidence. Indeed, the (non-chiral) CFTs obtained as the product of any of the V_L with the CFT of a single free boson on a circle S^1 are all equivalent to each other. These CFTs admit a uniform description as theories of 25 chiral and 1 anti-chiral free boson compactified on the unique even unimodular lattice $\Gamma^{25,1}$ of signature $(25, 1)$. The Narain moduli space of this CFT is $O(25) \times O(1)\backslash O(25, 1)/O(25, 1, \mathbb{Z})$, and is spanned by the radius R and the Wilson lines $(A_1, \dots, A_{24})$. The different descriptions of this theory as a product $S^1 \times V_L$ correspond to the 24 inequivalent ways of writing the lattice $\Gamma^{25,1}$ as an orthogonal sum of a Niemeier lattice and a unimodular lattice of signature $(1, 1)$. These descriptions are related to one each other by the action of the T-duality group $O(25, 1, \mathbb{Z})$, which mixes the radius and the Wilson line moduli, as well as the winding-momentum (w, m) with the gauge charges q_k.

In general, if we turn off some of the Wilson lines (i.e., set $y_i = 1$ for some i), the index vanishes because of the zero modes of some combination of gauginos in $g^+ \oplus g^-$. The only exception is when V_L is the VOA associated with the Leech lattice $L = \Lambda$, where g is the abelian algebra $u(1)^{\oplus 24}$ and $c(0, \ell) = 0$ for all $\ell \neq 0$. In this case, we can turn off all Wilson lines, and by setting

$$c(n) = \sum_\ell c(n, \ell), \tag{5.22}$$

such that $c(n)$ are the Fourier coefficients of

$$Z_{V_\Lambda}(\tau, 0) = \frac{\Theta_\Lambda(\tau, 0)}{\eta(\tau)^{24}} = J(\tau) + 24 = \sum_{n \in \mathbb{Z}} c(n) \mathsf{q}^n = \mathsf{q}^{-1} + 24 + 196884\mathsf{q} + \cdots, \tag{5.23}$$

the index reduces to

$$\mathcal{Z}^{V_\Lambda \otimes \bar{V}_+^{f\natural}}(T, U, A_k = 0) = \left(q \prod_{m>0}(1 - q^m)^{c(0)} \prod_{w>0, m \in \mathbb{Z}}(1 - p^w q^m)^{c(mw)} \right)^{24}. \tag{5.24}$$

Recall the Koike–Norton–Zagier identity

$$p^{-1} \prod_{w>0, m \in \mathbb{Z}\backslash\{0\}}(1 - p^w q^m)^{c(mw)} = J(T) - J(U) \tag{5.25}$$

which can be interpreted as the denominator identity for the monster BKM algebra [10]. Multiplying this factor by the $w = 0$ and the $m = 0$ parts yields

$$
\begin{aligned}
\mathcal{Z}^{V_\Lambda \otimes \bar{V}_+^{f\natural}}(T, U, 0) &= \left(q \prod_{m>0} (1 - q^m)^{c(0)} \right) \left(p \prod_{w>0} (1 - p^w)^{c(0)} \right) \\
&\quad \times p^{-1} \prod_{w>0, m \in \mathbb{Z} \backslash \{0\}} (1 - p^w q^m)^{c(mw)} \\
&= \eta(U)^{24} \eta(U)^{24} (J(T) - J(U)) \\
&= \eta(U)^{24} \Theta_\Lambda(T) - \eta(T)^{24} \Theta_\Lambda(U),
\end{aligned}
\tag{5.26}
$$

which is the denominator of the fake monster Lie algebra [9].

This construction should generalize straightforwardly to all 71 self-dual VOAs V with $c = 24$. When we compactify one further direction on a circle S^1, we expect many non-trivial equivalences between these different models, as is the case for the 24 Niemeier lattice VOAs V_L. It is known [19,41] that there are only 11 isomorphism classes of holomorphic vertex algebras of central charge 26 of the form $V_{\Gamma^{1,1}} \otimes V$, where $V_{\Gamma^{1,1}}$ is the lattice vertex algebra based on the even unimodular lattice of signature $(1,1)$ and V is a self-dual VOA of central charge 24 with currents. In particular, the 24 VOAs $V_{\Gamma^{1,1}} \otimes V_L$ are all isomorphic to each other; the proof is similar to the one showing that the (non-chiral) CFTs on $S^1 \otimes V_L$ are all isomorphic. Conjecturally, there is only one additional isomorphism class $V_{\Gamma^{1,1}} \otimes V$ where V is a self-dual VOA of $c = 24$ *without* currents—it is the case where $V = V^\natural$ is the monster VOA [30]. From each of these isomorphism classes of vertex algebras, one can define a BKM algebra using Borcherds' construction based on chiral bosonic string theory compactified on $V_{\Gamma^{1,1}} \otimes V$; see [9,10,22,42,43,53] for explicit descriptions and denominator identities.[12] It is natural to conjecture that by tensoring any self-dual VOA of $c = 24$ with the bosonic (non-chiral) CFT on a circle S^1, one gets one of precisely 12 distinct CFTs of left and right central charges $(25, 1)$. If this is correct, then there are only 12 inequivalent compactifications of the heterotic string of the form $(V \otimes \bar{V}^{f\natural}) \otimes S^1$. We expect that our construction can be used to show that the corresponding second quantized indices $\mathcal{Z}$ are just the (24th powers of the) denominators of the 12 associated BKM algebras.

5.4. Type IIA Strings on $V \otimes \bar{V}^{f\natural}$

Now, we consider the type IIA string compactified on $V \otimes \bar{V}_-^{f\natural}$, where V is one of the self-dual $\mathcal{N} = 1$ SVOAs with $c = 12$.[13] We assume that V contains χ^{NS} states of weight $1/2$ in the NS sector, χ_+^{R} states of weight $1/2$ in the R_+ sector and χ_-^{R} states in the R_- sector. These VOA states correspond to

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[12]These references describe 10 out of 12 distinct BKM algebras; N. Scheithauer informed us that he calculated the denominator identities for the two remaining cases (unpublished).

[13]Note that in these type IIA examples we take $V_-^{f\natural}$ as the right-moving internal SVOA, while for the heterotic examples we took $V_+^{f\natural}$. The reason is that in heterotic string theory the GSO projection restricts to the right-moving R_+ sector, while in the type IIA string the projection is onto the right-moving R_- sector. Our choice then corresponds to having 24 gravitinos with positive chirality in all examples.

massless physical states in the chiral cohomology, denoted, respectively, by ν^a, $a = 0, \ldots, \chi^{\mathrm{NS}}$, v_+^i, $i = 1, \ldots, \chi_+^{\mathrm{R}}$, v_-^i, $i = 1, \ldots, \chi_-^{\mathrm{R}}$. As discussed in Sect. 2, we distinguish between the two cases $V = V_\pm^{f\natural}$ which have 24 Ramond ground states in the $\mathrm{R}_\pm$ sector. The anti-holomorphic $\bar{V}_-^{f\natural}$ contains 24 Ramond ground states in the R_- sector corresponding to anti-chiral physical states $\bar{v}_+^i$, $i = 1, \ldots, 24$. The subscripts $+$ or $-$ on the Ramond states $v_\pm^i$ and $\bar{v}_+^i$ refer to the spin with respect to the $SO(1,1)$ spacetime Lorentz group; note that the type IIA GSO projection identifies the sign of the spacetime spin with the internal fermion number in the holomorphic Ramond sector, and with the opposite of the fermion number in the anti-holomorphic Ramond sector. Additionally, there are the left- and right-moving states arising from the $\mathcal{N} = (1,1)$ supersymmetric nonlinear sigma model with target $\mathbb{R}^{1,1}$. The massless single string states are

- Gravitons $\psi_{++}\bar{\psi}_{++}$ and $\psi_{--}\bar{\psi}_{--}$; B-field $\psi_{++}\bar{\psi}_{--} - \psi_{--}\bar{\psi}_{++}$; dilaton $\psi_{++}\bar{\psi}_{--} + \psi_{--}\bar{\psi}_{++}$. These states are non-physical, except for $k = 0$.
- $(24 + \chi_+^{\mathrm{R}})$ chiral gravitinos $\psi_{++}\bar{v}_+^i$, $i = 1, \ldots, 24$, and $v_+^i\bar{\psi}_{++}$ $i = 1, \ldots \chi_+^{\mathrm{R}}$ (spin $+3/2$); χ_-^{R} anti-chiral gravitinos $v_-^i\bar{\psi}_{--}$ $i = 1, \ldots \chi_-^{\mathrm{R}}$ (spin $-3/2$); $(24 + \chi_+^{\mathrm{R}})$ anti-chiral dilatinos $\psi_{--}\bar{v}_+^i$, $i = 1, \ldots, 24$ and $v_+^i\bar{\psi}_{--}$ $i = 1, \ldots \chi_+^{\mathrm{R}}$ (spin $-1/2$); χ_-^{R} chiral dilatinos $v_-^i\bar{\psi}_{++}$ $i = 1, \ldots \chi_-^{\mathrm{R}}$ (spin $+1/2$). These states are only physical for $k = 0$.
- χ^{NS} gauge vectors $A^a = \nu^a\bar{\psi}_{++}$, $\nu^a\bar{\psi}_{--}$ from the NS-NS sector. These states are only physical for $k = 0$.
- From the R-R sector, $24\chi_+^{\mathrm{R}}$ states $v_+^i\bar{v}_+^j$, $i = 1, \ldots, \chi_+^{\mathrm{R}}$, $j = 1, \ldots, 24$ corresponding to derivatives $\partial_+\phi^{ij}$ of scalar chiral fields ϕ^{ij} with $\partial_-\phi^{ij} = 0$, and $24\chi_-^{\mathrm{R}}$ states $v_-^i\bar{v}_+^j$, $i = 1, \ldots, \chi_-^{\mathrm{R}}$, $j = 1, \ldots, 24$, corresponding to field strengths of $U(1)$ vectors. When the theory is uncompactified, only the scalar fields have propagating degrees of freedom. When we compactify the space direction on a circle S^1, the equations of motion force the scalars to carry no winding along S^1, while the vectors can carry winding but not momentum. There are no physical string states representing the zero modes of the scalars or of the vectors—this is standard for R-R bosons. See "Appendix C" for more detail.
- $24\chi^{\mathrm{NS}}$ chiral gauginos $\lambda^{ai} = \nu^a\bar{v}_+^i$ (spin $1/2$) from the NS-R sector; these states are physical for all k with $k^2 = 0$ (massless propagating degrees of freedom).

In total, the spacetime theory has $(\chi_-^R, 24 + \chi_+^R)$ SUSY. When we compactify the space direction on a circle S^1, the corresponding supercharges obey slightly different algebras then presented previously:

- The "standard" $(0, 24)$ supercharges $\mathcal{Q}_{r-}^i$ from the NS-R gravitinos $\psi_{++}\bar{v}_+^i$ obey the algebra

$$\{\mathcal{Q}_{r-}^i, \mathcal{Q}_{r-}^j\} = \delta^{ij}(P_r^0 - P_r^1). \tag{5.27}$$

- The $(0, \chi_+^{\mathrm{R}})$ supercharges $\mathcal{Q}_{l-}^i$ from the R-NS gravitinos $v_+^i \bar{\psi}_{++}$ obey the algebra

$$\{\mathcal{Q}_{l-}^i, \mathcal{Q}_{l-}^j\} = \delta^{ij}(P_l^0 - P_l^1). \tag{5.28}$$

  Note that here only left-moving momenta appear, while in the previous equation we had only right-moving momenta. In the uncompactified theory, one has $P_l^0 = P_r^0$ and $P_l^1 = P_r^1$ and the algebra is the same; in the compactified theory, P_l^1 and P_r^1 are independent operators.

- The $(\chi_-^{\mathrm{R}}, 0)$ supercharges $\mathcal{Q}_{l+}^i$ from the R-NS gravitinos $v_-^i \bar{\psi}_{--}$ obey the algebra

$$\{\mathcal{Q}_{l+}^i, \mathcal{Q}_{l+}^j\} = \delta^{ij}(P_l^0 + P_l^1). \tag{5.29}$$

Now, we are going to discuss the computation of the supersymmetric indices defined in Eqs. (5.11), (5.12) for these theories. Since the time direction is always uncompactified, we have $P_l^0 = P_r^0$. The spacetime index $\mathcal{Z}$ gets contributions only from states that are annihilated by all the supercharges. Since the 24 supercharges $\mathcal{Q}_{r-}^i$ are always present, this implies that $k_r^1 = k_r^0 = k_l^0$. If $\chi_+^{\mathrm{R}} \neq 0$, then the states contributing to $\mathcal{Z}$ have momenta satisfying

$$k_r^1 = k_r^0 = k_l^0 = k_l^1, \tag{5.30}$$

which means that they cannot have any winding around the circle S^1, so that the index is independent of T. If $\chi_-^{\mathrm{R}} \neq 0$, we get the condition

$$k_r^1 = k_r^0 = k_l^0 = -k_l^1 \tag{5.31}$$

which implies that states contributing to the index cannot carry any units of momentum along S^1, and $\mathcal{Z}$ is independent of U. If both χ_+^{R} and χ_-^{R} are not zero (this happens if $V = V^{fE_8}$) then the only states contributing are the ones with zero momentum

$$k_l^0 = k_r^0 = k_l^1 = k_r^1 = 0, \tag{5.32}$$

so the index is just a number.

As we will show below, the supersymmetric index $\mathcal{Z}^{V \otimes \bar{V}_-^{f\natural}}$ (5.11) and the modified index $\tilde{\mathcal{Z}}^{V \otimes \bar{V}_-^{f\natural}}$ (5.12) reproduce, respectively, the superdenominator and the denominator of the BKM superalgebra corresponding to V. Passing from $\mathcal{Z}$ to $\tilde{\mathcal{Z}}$ amounts to inserting a "left-moving" spacetime fermion number $(-1)^{\mathbf{F}_l}$ in the trace, which acts by -1 on the left-moving Ramond and by $+1$ on the left-moving NS sector. With this insertion, the states contributing to $\tilde{\mathcal{Z}}$ are the ones that are annihilated by the 24 supercharges $\mathcal{Q}_{r-}^i$ from the NS-R gravitinos $\psi_{++} \bar{v}_+^i$. Thus, the only condition is

$$k_l^0 = k_r^0 = k_r^1, \tag{5.33}$$

with no constraints on k_l^1. Indeed, all the other supercharges do not commute with the operator $(-1)^{\mathbf{F}_l}$, so there are no cancellations between states that are related by these supersymmetries. Then, $\tilde{\mathcal{Z}}$ will in general be a function of both T and U for all algebras. This is indeed what happens for the algebra denominators.

The only propagating massless fermions are the $24\chi^{\mathrm{NS}}$ chiral gauginos λ^{ai}, that are the superpartners of the gauge vectors A^a. The gauge algebra $g^{\oplus 24}$ consists of 24 copies of a χ^{NS}-dimensional Lie algebra g. Therefore, when $\chi^{\mathrm{NS}} \neq 0$, we can consider Wilson lines corresponding to the Cartan generators of g. Furthermore, in order to get a nonvanishing result for the index, one needs to absorb the zero modes of the gauginos corresponding to Cartan generators, which are neutral under the Wilson lines. This situation occurs when $V = V^{fE_8}$ ($\chi^{\mathrm{NS}} = 8$) and for $V = F_{24}$ ($\chi^{\mathrm{NS}} = 24$).

Let us now consider a case-by-case analysis of the three possibilities for V.

1. For $V = V_+^{f\natural}$ (or $V = V_-^{f\natural}$) one has $\chi_+^{\mathrm{R}} \neq 0$ and $\chi^{\mathrm{NS}} = \chi_-^{\mathrm{R}} = 0$ (resp., $\chi_-^{\mathrm{R}} \neq 0$ and $\chi^{\mathrm{NS}} = \chi_+^{\mathrm{R}} = 0$), so the index $\mathcal{Z}$ receives contributions only from states with $w = 0$ (resp., $m = 0$) and therefore is a function of U only (resp., of T only). This fits with the fact that the superdenominator of the Conway superalgebra associated with $V^{f\natural}$ depends on one variable only (see equation 4.30 and the surrounding discussion in [37]). The two cases $V = V_+^{f\natural}$ and $V = V_-^{f\natural}$ are related by exchanging winding and momentum along S^1, i.e., by T-duality along this circle; this fits with the idea that T-duality is a "left-moving parity" exchanging the left-moving R_+ and R_- sectors while keeping the right-moving sectors fixed. The conditions (5.30) (or (5.31)) on momenta imply that only propagating massless states contribute to the index. The only such states are the 24^2 massless bosons in the R-R sector; they are either scalars carrying only momentum for $V = V_+^{f\natural}$, or vectors carrying only winding number for $V = V_-^{f\natural}$. The index is therefore

$$\mathcal{Z}^{V_+^{f\natural} \otimes \bar{V}_-^{f\natural}}(U) = \left(q^{-1} \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{24}}\right)^{24} = \left(\frac{1}{\eta^{24}(U)}\right)^{24}, \quad (5.34)$$

$$\mathcal{Z}^{V_-^{f\natural} \otimes \bar{V}_-^{f\natural}}(T) = \left(p^{-1} \prod_{w=1}^{\infty} \frac{1}{(1-p^w)^{24}}\right)^{24} = \left(\frac{1}{\eta^{24}(T)}\right)^{24}, \quad (5.35)$$

that are indeed the 24th power of the superdenominator of Conway BKM superalgebra [37].

The modified index $\tilde{\mathcal{Z}}$ receives contributions from both massive and massless states with $k_r^0 = k_r^1$. Let $c_{NS}(n)$ and $c_R(n)$ be the multiplicities of the GSO-projected NS and R states in the internal SVOA V at level $L_0 - \frac{1}{2} = n$—these multiplicities are the same for both $V_-^{f\natural}$ and $V_+^{f\natural}$. For winding and momenta m, w, we have $24c_{NS}(mw)$ fermions from the NS-R sector and $24c_R(mw)$ bosons from the R-R sector. The multiplicities are nonzero only for $mw \geq 0$, and the positive energy condition $E = \frac{1}{\sqrt{2}}(\frac{m}{R} + wR) > 0$ implies that $m, w \geq 0$. As discussed above, the $24c_R(0) = 24^2$ massless bosons must have $w = 0$ for $V = V_+^{f\natural}$ or $m = 0$ for $V = V_-^{f\natural}$. The product $(-1)^{\mathbf{F}}(-1)^{\mathbf{F}_l} = (-1)^{\mathbf{F}_r}$ of spacetime fermion number $(-1)^{\mathbf{F}}$ and the "left-moving" fermion number $(-1)^{\mathbf{F}_l}$ acts by a

minus sign on all of these states. The final result for the modified index is therefore

$$\tilde{\mathcal{Z}}^{V_+^{f\natural} \otimes \bar{V}_-^{f\natural}}(T, U) = \left(q^{-1} \prod_{m=1}^{\infty} \frac{1}{(1+q^m)^{24}} \prod_{w=1}^{\infty} \frac{(1-p^w q^m)^{c_{NS}(mw)}}{(1+p^w q^m)^{c_R(mw)}} \right)^{24},$$

(5.36)

$$\tilde{\mathcal{Z}}^{V_-^{f\natural} \otimes \bar{V}_-^{f\natural}}(T, U) = \left(p^{-1} \prod_{w=1}^{\infty} \frac{1}{(1+p^w)^{24}} \prod_{m=1}^{\infty} \frac{(1-p^w q^m)^{c_{NS}(mw)}}{(1+p^w q^m)^{c_R(mw)}} \right)^{24}.$$

(5.37)

These are indeed the 24th power of the superalgebra denominator (as computed in [37]).

2. When $V = F_{24}$, we have $\chi^{NS} = 24$ and $\chi_+^{R} = \chi_-^{R} = 0$. Thus, for both the index and the modified index, the states giving nonzero contribution to the trace may carry any value of winding and momentum. To be precise, there are 8 different string models, depending on the choice of a $\mathcal{N} = 1$ supercurrent for F_{24} (as described in Sect. 2.) Each choice is associated with a semi-simple Lie algebra g of dimension 24; g is also the algebra of gauge symmetries in spacetime for the corresponding model. One can turn on r Wilson lines for these gauge fields where r is the rank of the algebra. We are left with r neutral massless gauginos that need to be absorbed in order to get a nonzero result. The treatment of the massless modes is completely analogous to the case of heterotic strings with N currents that we saw in the previous section. The only difference is that in this case we have massive BPS bosons, while in the heterotic case all BPS states were fermions. Let us denote by $c_{NS}(n, \lambda)$ and $c_R(n, \lambda)$ the GSO-projected NS and Ramond multiplicities of states in the internal CFT F_{24} with level $L_0 - \frac{1}{2} = n$ and charge λ with respect to the algebra g. The result for the index is then

$$\mathcal{Z}^{F_{24} \otimes \bar{V}_-^{f\natural}}$$
$$= \left(pqy^{-\rho} \prod_{\lambda \in \Delta_g^+} (1-y^\lambda)^{c_{NS}(0,\lambda)} \prod_{\substack{m,w=0 \\ (m,w) \neq (0,0)}}^{\infty} \prod_{\lambda \in \Delta^g} \frac{(1-p^w q^m y^\lambda)^{c_{NS}(mw,\lambda)}}{(1-p^w q^m y^\lambda)^{c_R(mw,\lambda)}} \right)^{24},$$

(5.38)

and for the modified index

$$\tilde{\mathcal{Z}}^{F_{24} \otimes \bar{V}_-^{f\natural}}$$
$$= \left(pqy^{-\rho} \prod_{\lambda \in \Delta_g^+} (1-y^\lambda)^{c_{NS}(0,\lambda)} \prod_{\substack{m,w=0 \\ (m,w) \neq (0,0)}}^{\infty} \prod_{\lambda \in \Delta^g} \frac{(1-p^w q^m y^\lambda)^{c_{NS}(mw,\lambda)}}{(1+p^w q^m y^\lambda)^{c_R(mw,\lambda)}} \right)^{24}.$$

(5.39)

where, for any vector α, y^α is shorthand for $\prod_k y_k^{\alpha_k} = e^{2\pi i \sum_k A_k \alpha_k}$. Again, up to the power 24, these are, respectively, the superdenominator and denominator of the associated BKM superalgebra (see [36]).

3. Finally, for $V = V^{fE_8}$, we have $\chi^{NS} = \chi_+^R = \chi_-^R = 8$. The $8 \cdot 24$ vectors from the NS-NS sectors are the gauge bosons of an abelian $U(1)^{8 \cdot 24}$ algebra. The $8 \cdot 24$ massless gauginos from the NS-R sector are all neutral, so their zero modes force the indices to vanish independently of the Wilson lines, unless they are absorbed. For simplicity, we consider the case where the Wilson lines are set to zero. The theory contains supersymmetries of the form $\mathcal{Q}_{l+}^i$ and $\mathcal{Q}_{l-}^i$ from the R-NS sector and $\mathcal{Q}_{r-}^i$ from the NS-R sector; as discussed above, this implies that the contributions to the index $\mathcal{Z}$ from states of nonzero momentum always cancel. So, the index $\mathcal{Z}^{V^{fE_8} \otimes \bar{V}^{f\natural}}$ is simply a nonzero constant that can be normalized to 1. The modified index is more interesting, and gives

$$\tilde{\mathcal{Z}}^{V^{fE_8} \otimes \bar{V}^{f\natural}}(T,U) = \left(\prod_{\substack{m,w=0 \\ (m,w) \neq (0,0)}}^{\infty} \frac{(1 - p^w q^m)^{c_{NS}(mw)}}{(1 + p^w q^m)^{c_R(mw)}} \right)^{24}, \qquad (5.40)$$

where $c_{NS}(n) = c_R(n)$ are the Fourier coefficients of the GSO projected NS and Ramond characters of V^{fE_8} (that are actually equal to each other). This is the denominator of the fake monster superalgebra (to the power 24) [10,61,62].

6. Path Integrals and Theta Lifts

The index described in the previous section, and its variations, can be calculated using a path integral. In this section, we discuss this point of view, first reviewing the structure and modular properties of second quantized partition functions (Sect. 6.1), and then discussing the relationship between the path integral and operatorial formalism in more detail (Sect. 6.2). In Sect. 6.3, we show that the path integral reduces to a so-called theta lift in the theory of automorphic forms. We then demonstrate how to recover the (super)denominator identities using this approach (Sects. 6.4 and 6.5).

6.1. First and Second Quantized Partition Functions

Let us first consider the relation between first and second quantized partition functions in particle physics/ quantum field theory. For a single point particle of mass m in a d-dimensional spacetime, one defines a (first quantized) partition function by summing over worldlines all possible ways to embed a single (Euclidean) loop in spacetime

$$Z_{particle}(m^2) = V_d \int \frac{d^d k}{(2\pi)^d} \int_0^{+\infty} \frac{dl}{2l} e^{-\frac{1}{2}(k^2 + m^2)l}. \qquad (6.1)$$

Here, l is the length of the loop, $\frac{1}{2}(k^2 + m^2)$ is the worldline Hamiltonian, and the factor $2l$ takes into account the translation and reflection invariance

of the worldline parametrization. To obtain the (second quantized) partition function in a quantum field theory, one needs to sum over an arbitrary number n of disconnected worldlines describing n particles going around a closed loop in spacetime, where on each worldline one sums over all possible particles in the theory. One should also divide by a factor $n!$ to take into account the permutations of the n worldlines. The final result is

$$Z_{QFT} = \exp\left(\sum_i Z_{particle}(m_i^2)(-1)^{\mathbf{F}_i}\right), \qquad (6.2)$$

where the factor $(-1)^{\mathbf{F}_i}$ takes into account that fermions running in the loop contribute with negative sign with respect to the bosons (see [58], §7.3.).

Now, let us turn to the analogue of this in string theory. In order to obtain the second quantized partition function in string theory, one needs to exponentiate the (first quantized) torus partition function for a single string, with opposite signs for spacetime fermions and bosons, namely

$$\mathscr{Z} = \exp\left(\frac{1}{2}\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} \frac{\mathrm{d}^2\tau}{\tau_2} \mathrm{Tr}\left(\mathsf{q}^{L_0}\bar{\mathsf{q}}^{\bar{L}_0}e^{2\pi i\sum_k A_k q_k}(-1)^{\mathbf{F}}\right)\right), \qquad (6.3)$$

where $\tau = \tau_1 + i\tau_2$ and $\mathsf{q} = e^{2\pi i\tau}$. The trace is over the GSO-projected single string states and $(-1)^{\mathbf{F}}$ is a spacetime fermion number. The factors of $1/2$ and $1/\tau_2$ in the exponential are needed in the torus amplitude to take into account, respectively, of the reflection and translation automorphisms of the worldsheet torus [58]. The A_k are background gauge fields (if present), and q_k are the corresponding charges. The Virasoro operators L_0 and $\bar{L}_0$ depend implicitly on the geometry of the spacetime and on the background B-field. The target space should be a Euclidean two-dimensional spacetime on a torus $S^1 \times S^1$, where the first S^1 corresponds, in Lorentzian signature, to the spacelike circle of radius R, while the second S^1 is a thermal circle, i.e., a Euclidean time circle with radius the inverse temperature β. This means that both k^0 and k^1 are quantized in the path integral. The standard quantization of momenta in the path integral will lead, on the operatorial side, to the index with fermions that are periodic along the spacelike circle S^1 and with $(-1)^{\mathbf{F}}$ inserted. Non-standard quantizations of momenta in the path integral (i.e., with half-integral rather than integral momentum, either in the space or in the Euclidean time directions) for *some* of the sectors will lead, in the operatorial side, to either anti-periodic fermions around the space circle, or to modifications of the $(-1)^{\mathbf{F}}$ factor in the trace.

The path integral representation of the index should make invariance under a suitable subgroup of $O(2,2,\mathbb{Z}) \sim SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ manifest. The precise invariance subgroup depends on the periodicity conditions imposed on the fields around the spatial and Euclidean time circles. The two $SL(2,\mathbb{Z})$ factors in $O(2,2,\mathbb{Z})$ act by fractional linear transformations on the complex variables T, U parametrizing, respectively, the complexified Kähler and the complex structure moduli of the spacetime torus. Thus, duality invariance implies that the index is a modular function in both variables T and U. This

invariance may be modified in the presence of non-trivial Wilson lines, which give rise to (possibly multivariate) Jacobi forms rather than modular functions.

The path integral might have infrared divergences in the presence of massless or tachyonic modes. These can be regularized in a way that preserves spacetime diffeomorphism invariance and T-duality invariance, by cutting a suitable region in the integration over the fundamental domain $SL(2,\mathbb{Z})\backslash\mathbb{H}$. Therefore, one expects the path integral to be $O(2,2,\mathbb{Z})$-invariant even in this case.

This is in contrast with the expected modular properties of the index $\mathcal{Z}$, as obtained from the operatorial formalism of Sect. 5. For example, suppose that our theory contains some fermionic zero modes that need to be absorbed as in Eqs.(5.11) and (5.12). In this case, one expects a nonzero modular weight depending on the number of fermions one needs to absorb. For example, consider a modular transformation $U \to \frac{aU+b}{cU+d}$ of the complex structure of the Euclidean torus. Spacetime fermions are sections of a spinorial bundle on the torus; in particular, a left-moving fermion is a section of $K^{1/2}$, a square root of the canonical bundle on the torus, and thus transforms as $\mathrm{d}z^{1/2}$, where z is a complex coordinate on the torus. Similarly, right-moving fermions transform as $d\bar{z}^{1/2}$. This means that if one needs to absorb n_l left-moving and n_r right-moving fermions, the corresponding index is a modular form of weights (n_l, n_r), rather than a modular invariant function.

In order to explain this mismatch, we will now discuss in more detail the relationship between the path integral $\mathscr{Z}$ and the corresponding index $\mathcal{Z}$.

6.2. Path Integral Versus Operatorial Formalism

Some of the theories we are considering contain chiral massless fermions or bosons, whose path integral description is notoriously problematic. For a single chiral free boson or fermion, a standard method to overcome this issue consists in introducing additional massless degrees of freedom with the opposite chirality, so as to make the whole theory non-chiral. One can then easily compute the path integral $\mathscr{Z}$ for this non-chiral theory. Finally, the chiral partition function $\mathcal{Z}$ is obtained by extracting a suitable "holomorphic square root" of the non-chiral one $\mathscr{Z}$—this last step might be quite non-trivial, as discussed below.

Including spurious degrees of freedom. Since we are interested in computing indices in free theories, a similar method can be generalized to our case. Actually, it is simpler to modify the indices computed in Sect. 5 directly in the operatorial formalism, by introducing suitable spurious degrees of freedom that make the theory non-chiral. The "non-chiral" indices we obtain this way can be compared with the path integral results we discuss in the following subsections.

Let us take a closer look at the chiral content of the superstring compactifications. Chiral massless fields can appear when the SVOA $V^{f\natural}$ is the holomorphic or anti-holomorphic factor in the internal CFT. All 24 Ramond ground states of this SVOA have the same worldsheet fermion number, which

can be taken to have either positive (for $V_+^{f\natural}$) or negative ($V_-^{f\natural}$). The GSO projection relates the fermion number of these ground states to the spacetime spin of the corresponding massless fields. Finally, the mass-shell condition, arising from the requirement that the state is BRST closed, relates the spin to the spacetime momentum. For example, for chiral fermions of spin $+1/2$ or $-1/2$, the massless Dirac equation admits a nonzero solution only for $k^0 = -k^1$ or $k^0 = k^1$, respectively. As a result, we obtain massless fields that are either purely left-moving or purely right-moving in spacetime. This phenomenon does not occur for V^{fE_8} or F_{24}, for which the Ramond sector contains the same number of states with positive or negative worldsheet fermion number.

In a path integral formulation, the mass-shell condition is not imposed on fields, so that the relationship between spacetime momentum and spin (and, in turn, with the internal fermion number) is lost. One simply sums over all possible spacetime momenta and, independently, over all possible internal states. Nevertheless, this procedure gives the correct result if the number of internal states with positive and negative worldsheet fermion numbers is the same. Therefore, it is clear how to modify the operatorial description so as to obtain a "non-chiral" index that can be compared with the path integral: For each (holomorphic or anti-holomorphic) internal SVOA $V^{f\natural}$, containing 24 Ramond ground states having fermion number $+1$ or -1, one needs to add 24 "spurious" ground states with the opposite fermion number.

Het/Type IIA on $V \otimes \bar{V}_-^{f\natural}$. Let us analyze how the index is modified by these spurious degrees of freedom. Let us first suppose that the theory is heterotic string theory on $V \otimes \bar{V}_+^{f\natural}$ or type IIA on $V \otimes \bar{V}_-^{f\natural}$, where V is a VOA or SVOA different from $V^{f\natural}$. As described in Sect. 5, in this case, the only single string physical states relevant to the supersymmetric index are the ones with $k_r^0 = k_r^1$, and correspond to products $v \otimes \bar{v}_+^i$, where v is a state in V and $\bar{v}_+^1, \ldots, \bar{v}_+^{24}$ are the anti-holomorphic Ramond ground states in $\bar{V}_+^{f\natural}$ (heterotic) or $\bar{V}_-^{f\natural}$ (type IIA) (the subscript $+$ in $\bar{v}_+^i$ denotes positive spacetime chirality, and not the worldsheet fermion number; notice that, according to our conventions, the GSO projection on the anti-holomorphic internal factor for type IIA is the opposite than for heterotic). For each of these states, there is a creation or annihilation (depending on the sign of the energy) operator acting on the second quantized Hilbert space. Adding 24 further anti-holomorphic Ramond ground states $\bar{v}_-^1, \ldots, \bar{v}_-^{24}$ with opposite worldsheet fermion number to $\bar{V}^{f\natural}$ corresponds to introducing a second copy of each creation/annihilation operator, but now with right-moving momenta satisfying $k_r^0 = -k_r^1$ rather than $k_r^0 = k_r^1$. This means that, for these new operators, the relation between energy and winding-momentum changes sign and becomes $E = -\frac{1}{\sqrt{2}}(\frac{m}{R} + wR)$, so that the second copy of a creation (i.e., positive energy) operator has the same energy and the opposite winding and momentum with respect to the first copy. Changing the sign of m and w while keeping the energy fixed is equivalent to the transformation $(T, U) \mapsto (-\bar{T}, -\bar{U})$. In turn, this is equivalent to applying

complex conjugation $\overline{\mathcal{Z}(T,U)}$ to the index, because the latter is a power series in $p = e^{2\pi i T}$ and $q = e^{2\pi i U}$ with integral coefficients.

This construction generalizes to the case where Wilson lines are present. In this case, one has

$$\overline{\mathcal{Z}(T,U,A_k)} = \mathcal{Z}(-\bar{T},-\bar{U},-A_k) = \mathcal{Z}(-\bar{T},-\bar{U},A_k). \tag{6.4}$$

The last equality follows because the spectrum of a self-dual VOA V is invariant under charge conjugation—in general, applying charge conjugation to a VOA V leads to a different V-module, but since V is self-dual the charge conjugate V-module must be isomorphic to V itself.[14]

Finally, the "non-chiral" index obtained by including both the original and the spurious degrees of freedom, therefore, is just the modulus square,

$$|\mathcal{Z}(T,U,A_k)|^2 = \mathcal{Z}(T,U,A_k)\overline{\mathcal{Z}(T,U,A_k)}, \tag{6.5}$$

of the original index. A similar result was already discussed in [55] in the case of a heterotic string theory on $V^\natural \otimes V^{f\natural}$.

Type IIA on $V_\pm^{f\natural} \otimes \bar{V}_-^{f\natural}$. The treatment is slightly more complicated for type IIA compactified on $V_+^{f\natural} \otimes \bar{V}_-^{f\natural}$ or $V_-^{f\natural} \otimes \bar{V}_-^{f\natural}$, because one needs to add spurious degrees of freedom to both the holomorphic and the anti-holomorphic factor. Let us start with $V_+^{f\natural} \otimes \bar{V}_-^{f\natural}$. For the index $\mathcal{Z}$ (corresponding to the superdenominator of the associated BKM superalgebra), the only relevant single string physical states are the ones satisfying $k_l^1 = k_l^0 = k_r^0 = k_r^1$, i.e., with zero winding $w = 0$ along the circle, so that the index equals $\mathcal{Z}(T,U) = (1/\eta(U)^{24})^{24}$. Let us first add 24 spurious ground states with fermion number -1 to the holomorphic factor $V_+^{f\natural}$. This amounts to introducing creation/annihilation operators with momentum $-k_l^1 = k_l^0 = k_r^0 = k_r^1$, i.e., with $m = 0$, which multiply the index by a factor $(1/\eta(T)^{24})^{24}$. Similarly, the modified index $\tilde{\mathcal{Z}}$, corresponding to the denominator of the associated BKM superalgebra, gets multiplied by $\left(p^{-1}\prod_{w=1}^\infty \frac{1}{(1+p^w)^{24}}\right)^{24}$. Therefore, once these holomorphic spurious states are introduced, the index $\mathcal{Z}$ becomes

$$\mathcal{Z}_{sp}^{V_+^{f\natural}\otimes\bar{V}_-^{f\natural}}(T,U) = \left(\frac{1}{\eta(T)^{24}\eta(U)^{24}}\right)^{24}, \tag{6.6}$$

and the modified index $\tilde{\mathcal{Z}}$ becomes

$$\tilde{\mathcal{Z}}_{sp}^{V_+^{f\natural}\otimes\bar{V}_-^{f\natural}}(T,U)$$
$$= \left(q^{-1}\prod_{m=1}^\infty \frac{1}{(1+q^m)^{24}}\right)^{24} \left(p^{-1}\prod_{w=1}^\infty \frac{1}{(1+p^w)^{24}}\right)^{24}$$
$$\left(\prod_{m,w=1}^\infty \frac{(1-p^w q^m)^{c_{NS}(mw)}}{(1+p^w q^m)^{c_R(mw)}}\right)^{24}. \tag{6.7}$$

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[14]In VOA language, charge conjugation associates a VOA module with its contragredient module; here, we are simply observing that a self-dual VOA V is necessarily self-contragredient.

Exactly the same result can be obtained by adding spurious states to the holomorphic factor of $V_-^{f\natural} \otimes \bar{V}_-^{f\natural}$. This is obvious, since the only difference between these two theories is the fermion number of the Ramond ground states.

But we are not done yet: One needs to add also the spurious states for the anti-holomorphic factor $\bar{V}^{f\natural}$. The treatment is similar to the other $V \otimes \bar{V}^{f\natural}$ compactifications: The new spurious degrees of freedom just introduce a second copy of each creation and annihilation operator with the same energy but with opposite winding and momentum. Therefore, the complete non-chiral index (respectively, non-chiral modified index) is just the modulus square of $\mathcal{Z}_{sp}$ in (6.6) (respectively, of $\tilde{\mathcal{Z}}_{sp}$ in (6.7)).

Scalar and fermionic zero modes. One final subtlety occurs when there are scalar or fermionic zero modes. It is well known that the path integral representation for the partition function of a free massless scalar field on a two dimensional torus is not holomorphically factorized, i.e., it is not simply the modulus square of a function depending holomorphically on the complex structure modulus U. Something similar can happen for a free fermion on the torus with doubly periodic boundary conditions, due to the zero modes. In Euclidean signature, the fermion is necessarily complex, and its partition function is a regularized determinant $\mathrm{det}' \Delta_{1/2}$, where $\Delta_{1/2}$ is the two-dimensional (spacetime) Laplace operator acting on $1/2$-differentials, and the symbol det' means that the zero eigenvalues are excluded, i.e., the zero modes have been absorbed. At first sight, there seems to be no problem with the holomorphic factorization of $\mathrm{det}' \Delta_{1/2}$. Indeed, if one chooses a conformal gauge $ds^2 = \rho(z, \bar{z})\mathrm{d}z\,\mathrm{d}\bar{z}$ for the metric of the spacetime torus, the functional determinant can be regularized in such a way that it is the modulus square of a chiral determinant $\mathrm{det}' \bar{\partial}_{1/2}$ depending holomorphically on U (in fact, it is the modulus square of an eta function). This is the standard result one obtains in two-dimensional conformal field theory for the torus 1-point function of the free fermion with periodic boundary conditions. This factorized expression for $\mathrm{det}' \Delta_{1/2}$, however, is not invariant under diffeomorphisms of the spacetime torus: Indeed, being a 1-point function on a torus, it transforms as a section of $K^{1/2}$, a square root of the cotangent bundle. On the other hand, the path integral we are considering in this section is manifestly diff-invariant—it depends on the spacetime metric only through diff-invariant quantities, such as the volume, the lengths of geodesics or the angle between them. Therefore, it must correspond to a different, diff-invariant regularization of the functional determinant $\mathrm{det}' \Delta_{1/2}$.

A classical result by Belavin and Knizhnik [3] shows that, in general, a diff-invariant regularized determinant $\mathrm{det}' \Delta_n$ of the Laplacian acting on n-differentials on a Riemann surface Σ fails to be holomorphically factorized. More precisely, $\mathrm{det}' \Delta_n$ is related to a chiral determinant $\mathrm{det}' \bar{\partial}_n$ by

$$\frac{\mathrm{det}' \Delta_n}{\mathrm{det}\langle \phi_a^n | \phi_b^n \rangle \, \mathrm{det}\langle \phi_a^{1-n} | \phi_b^{1-n} \rangle} = e^{-c_n S_L(\rho)} |\mathrm{det}' \bar{\partial}_n|^2. \qquad (6.8)$$

Here, $\{\phi_1^n, \phi_2^n, \ldots\}$ and $\{\phi_1^{1-n}, \phi_2^{1-n}, \ldots\}$ are bases of, respectively, the kernel and cokernel of $\bar{\partial}_n$, i.e., holomorphic n- and $1-n$-differentials, $S_L(\rho)$ is the

Liouville action, depending on the metric $\rho(z,\bar{z})$ used to define the Laplacian, c_n is the conformal anomaly, and $(\det'\bar{\partial}_n)\,\phi_1^n \wedge \phi_2^n \wedge \cdots \wedge \phi_1^{1-n} \wedge \phi_2^{1-n} \wedge \cdots$ is a section of a holomorphic line bundle on the moduli space of Riemann surfaces. Furthermore, the scalar product $\langle \phi_a^n | \phi_b^n \rangle$ is defined by

$$\langle \phi_a^n | \phi_b^n \rangle = \int_\Sigma (\rho(z,\bar{z})\mathrm{d}z\,\mathrm{d}\bar{z})^{1-n} \phi_a^n(z)\overline{\phi_b^n(z)}, \qquad (6.9)$$

and an analogous definition holds for the scalar product between $1-n$ differentials.

In our case, Σ is the Euclidean spacetime torus T^2, which can be parametrized by the complex coordinate z taking values in $\mathbb{C}/(\mathbb{Z}+U\mathbb{Z})$, with respect to which the metric is of the form $ds^2 = \rho(z,\bar{z})\mathrm{d}z\,\mathrm{d}\bar{z}$. The holomorphic 1-differential $\mathrm{d}z$ is canonically normalized $\oint_A \mathrm{d}z = 1$, $\oint_B \mathrm{d}z = U$, where the A- and B-cycles are identified with the spatial and the Euclidean time circle, respectively. The metric is constant $\rho(z,\bar{z}) \equiv \rho$ and normalized so that the torus volume is $\mathrm{Im}\,T$. These conditions imply that

$$\int_{T^2} \mathrm{d}z\,\mathrm{d}\bar{z} = \mathrm{Im}\,U, \qquad \int_{T^2} \rho\,\mathrm{d}z\,\mathrm{d}\bar{z} = \mathrm{Im}\,T. \qquad (6.10)$$

There is a one-dimensional space of holomorphic 1/2-differentials with doubly periodic boundary conditions, and we can take a generator to be $\phi^{1/2}(z) = (\mathrm{d}z)^{1/2}$. Note that free chiral fermions with standard normalization of the OPE, such as the ones used to absorb the zero modes in the indices $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ in eqs. (5.11) and (5.12), correspond, up to a numerical $(T,U$-independent) factor, to the half-differential $(\mathrm{d}z)^{1/2}$. The square norm of $\phi^{1/2}$ is

$$\langle \phi^{1/2} | \phi^{1/2} \rangle = \int_{T^2} (\rho\,\mathrm{d}z\,\mathrm{d}\bar{z})^{1/2}(\mathrm{d}z)^{1/2}\,(\mathrm{d}\bar{z})^{1/2} = (\mathrm{Im}\,U)^{1/2}(\mathrm{Im}\,T)^{1/2}. \qquad (6.11)$$

For a flat torus, the Liouville factor e^{-S_L} admits holomorphic factorization so that it can be reabsorbed into the definition of the chiral determinants

$$\det{}'\Delta_{1/2} = (\mathrm{Im}\,U)(\mathrm{Im}\,T)e^{c_{1/2}S_L}|\det{}'\bar{\partial}_{1/2}|^2 = (\mathrm{Im}\,U)(\mathrm{Im}\,T)|\widetilde{\det}'\bar{\partial}_{1/2}|^2. \qquad (6.12)$$

An analogous result holds for scalar fields. The partition function of a free *complex* scalar field is the inverse $(\det'\Delta_0)^{-1}$ of the regularized functional determinant for the scalar Laplacian. We can choose $\phi^1(z) = \mathrm{d}z$ and $\phi^0(z) = 1$ as generators of the spaces of holomorphic 1- and 0-differentials on the torus, respectively, so that

$$\langle \phi^1 | \phi^1 \rangle = \int_{T^2} \mathrm{d}z\,\mathrm{d}\bar{z} = \mathrm{Im}\,U, \qquad \langle \phi^0 | \phi^0 \rangle = \int_{T^2} \rho\,\mathrm{d}z\,\mathrm{d}\bar{z} = \mathrm{Im}\,T. \qquad (6.13)$$

We conclude that

$$\det{}'\Delta_0 = (\mathrm{Im}\,U)(\mathrm{Im}\,T)|\widetilde{\det}'\bar{\partial}_0|^2, \qquad (6.14)$$

where, once again, the Liouville factor is absorbed in a redefinition of the chiral determinant.

This discussion finally suggests the correct relation between the indices $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ of Sect. 5 and their path integral representation $\mathscr{Z}$ considered in

this section. In particular, if the string theory we are considering contains n_b massless scalars and n_f massless fermions, then the path integral equals

$$\mathscr{Z} = [(\text{Im } U)(\text{Im } T)]^{n_f - n_b} |\mathcal{Z}_{sp}|^2, \qquad (6.15)$$

where $\mathcal{Z}_{sp}$ possibly contains additional "spurious" degrees of freedom, as in Eq. (6.6).

Note that if the massless degrees of freedom are charged with respect to some non-trivial background gauge fields, then the corresponding Laplacian is modified in such a way that the zero modes are lifted. This means that one can take the full regularized determinant $\det \Delta$, without projecting onto nonzero modes, and such a determinant automatically factorizes. As a consequence, in the presence of non-trivial Wilson lines for the gauge fields, the correct power $[(\text{Im } U)(\text{Im } T)]^{n_f - n_b}$ in Eq. (6.15) is given by the number n_f of *neutral* massless fermions minus the number n_b of *neutral* massless scalars.

An analogous formula to (6.15) holds for the path integral $\tilde{\mathscr{Z}}$ corresponding to $\tilde{\mathcal{Z}}$. The insertion of the factor $(-1)^{\mathbf{F}_l}$ in the trace corresponds, in the path integral formulation, to requiring the fermions and bosons to obey periodic or anti-periodic boundary conditions along the Euclidean time circle, depending on their $(-1)^{\mathbf{F}_l}$ eigenvalue. Fermions and bosons with anti-periodic boundary conditions on the torus have no zero modes; therefore, the analog (6.15) holds if n_f and n_b are the number of zero modes of *periodic* fermions and bosons.

Examples. Let us consider some examples:

- For heterotic strings on $V_L \otimes V_+^{f\natural}$, where V_L is the lattice VOA of the Niemeier lattice L, the number of massless fermions is $24N$, where N is the number of currents in the VOA V_L. Only 24^2 of them are neutral with respect to the background gauge fields in a maximal abelian torus of the gauge group. Therefore, $n_f = 24^2$. There are no massless propagating scalars, so $n_b = 0$. Thus, we expect the path integral to reproduce

$$\mathscr{Z}^{V_L \otimes V_+^{f\natural}}(T, U, A_k) = [(\text{Im } U)(\text{Im } T)]^{24^2} |\mathcal{Z}^{V_L \otimes V_+^{f\natural}}(T, U, A_k)|^2, \quad (6.16)$$

  where $\mathcal{Z}^{V_L \otimes V_+^{f\natural}}(T, U, A_k)$ is given in (5.21).

- Consider type IIA on $V_\pm^{f\natural} \otimes V_-^{f\natural}$. Once the spurious states are included, the theory contains 24^2 massless scalars from the R-R sector, and no propagating massless fermions. When we consider the index $\mathcal{Z}$, we have $n_b = 24^2$ and $n_f = 0$, so that the path integral will reproduce

$$\mathscr{Z}(T, U) = [(\text{Im } U)(\text{Im } T)]^{-24^2} |\mathcal{Z}_{sp}(T, U)|^2$$
$$= \frac{1}{[(\text{Im } U)(\text{Im } T)]^{24^2}} \left| \frac{1}{\eta(T)^{24^2} \eta(U)^{24^2}} \right|^2, \qquad (6.17)$$

  where $\mathcal{Z}_{sp}$ was calculated in (6.6). As for the modified index $\tilde{\mathcal{Z}}$, recall that $(-1)^{\mathbf{F}_l}$ acts by -1 on the R-NS and R-R sector and trivially on the NS-R and NS-NS sector. Therefore, all 24^2 massless R-R scalars obey anti-periodic boundary conditions along the Euclidean time circle, and

there are no zero modes. It follows that $n_b = n_f = 0$ and the path integral $\tilde{\mathscr{Z}}$ factorizes as the modulus square

$$\tilde{\mathscr{Z}}^{V_\pm^{f\natural} \otimes V_-^{f\natural}} = |\tilde{\mathcal{Z}}_{sp}^{V_\pm^{f\natural} \otimes V_-^{f\natural}}|^2 \tag{6.18}$$

of the index in Eq. (6.7).

- In type IIA on $F_{24} \otimes V_-^{f\natural}$, there are 24^2 massless fermions from the NS-R sector and no massless bosons. Only $24r$ of these fermions are neutral, where r is the rank of the gauge group, which depends on the choice of the $\mathcal{N} = 1$ supercurrent in F_{24}. Thus, both for the index $\mathcal{Z}$ and the modified index $\tilde{\mathcal{Z}}$, one has $n_f = 24r$ and $n_b = 0$, so that the path integral is expected to give

$$\mathscr{Z}^{F_{24} \otimes V_-^{f\natural}}(T, U, A_k) = [(\mathrm{Im}\, U)(\mathrm{Im}\, T)]^{24r} |\mathcal{Z}^{F_{24} \otimes V_-^{f\natural}}(T, U, A_k)|^2 \tag{6.19}$$

and

$$\tilde{\mathscr{Z}}^{F_{24} \otimes V_-^{f\natural}}(T, U, A_k) = [(\mathrm{Im}\, U)(\mathrm{Im}\, T)]^{24r} |\tilde{\mathcal{Z}}^{F_{24} \otimes V_-^{f\natural}}(T, U, A_k)|^2 \tag{6.20}$$

  where $\mathcal{Z}^{F_{24} \otimes V_-^{f\natural}}$ and $\tilde{\mathcal{Z}}^{F_{24} \otimes V_-^{f\natural}}$ are given in (5.38) and (5.39), respectively.

- Finally, in type IIA on $V^{fE_8} \otimes V_-^{f\natural}$, there are $8 \cdot 24$ bosons from the R-R sector and $8 \cdot 24$ fermions from the NS-R sector. The gauge group is abelian and all of these states are neutral. For the index $\mathcal{Z}$, one has $n_b = n_f = 8 \cdot 24$, so that the path integral $\mathscr{Z}$ is holomorphically factorized (in fact, it is just a constant). For the modified index, the scalars obey anti-periodic boundary conditions, so that $n_f = 8 \cdot 24$ and $n_b = 0$. Thus, we expect the path integral to reproduce

$$\tilde{\mathscr{Z}}^{V^{fE_8} \otimes V_-^{f\natural}} = [(\mathrm{Im}\, U)(\mathrm{Im}\, T)]^{8 \cdot 24} |\tilde{\mathcal{Z}}^{V^{fE_8} \otimes V_-^{f\natural}}|^2, \tag{6.21}$$

  where $\tilde{\mathcal{Z}}^{V^{fE_8} \otimes V_-^{f\natural}}$ is given in Eq. (5.40).

In Sects. 6.4 and 6.5, we will verify some of these expectations by explicit path integral calculations. In particular, we compute explicitly the path integrals corresponding to the superdenominators of type IIA on $F_{24} \otimes V_-^{f\natural}$, $V^{fE_8} \otimes V_-^{f\natural}$ and $V_\pm^{f\natural} \otimes V_-^{f\natural}$, as well as the denominators of type IIA on $V_\pm^{f\natural} \otimes V_-^{f\natural}$.

6.3. Path Integrals and Theta Lifts

Let us consider heterotic or type II superstring theory on a generic internal CFT $V_1 \otimes \bar{V}_2$. Let us first consider the case where there are no Wilson lines. As mentioned above, the path integral representation of the second quantized index is given by

$$\mathscr{Z} = \exp\left(\frac{1}{2} \int_{SL(2,\mathbb{Z}) \backslash \mathbb{H}} \frac{\mathrm{d}^2\tau}{\tau_2} \mathrm{Tr}\left(\mathsf{q}^{L_0} \bar{\mathsf{q}}^{\bar{L}_0} (-1)^{\mathbf{F}}\right)\right). \tag{6.22}$$

The trace in the integrand is taken over the full GSO-projected ghost and matter CFT, restricted to $\ker b_0 \cap \ker \bar{b}_0$. The latter condition corresponds to the insertion of suitable ghost vertex operators in the path integral. The operator $(-1)^{\mathbf{F}}$ is a spacetime fermion number; for example, in type II theories it acts trivially on the NS-NS and R-R sectors and by a minus sign on the

NS-R and R-NS sectors. As usual, the contributions from nonzero modes of the ghosts and superghosts exactly cancel the bosonic and fermionic matter oscillators in the $1 + 1$ spacetime directions. Thus, the trace is the product of a term coming from a sum over winding-momenta along the two-dimensional Euclidean spacetime torus, times the partition function of the internal CFT $V_1 \otimes \bar{V}_2$.

The winding-momenta take values in an even unimodular lattice $\Gamma^{2,2}$ of signature $(2, 2)$, so that their contribution gives a Narain–Siegel theta function

$$\Theta_{\Gamma^{2,2}}(\tau; T, U) = \tau_2 \sum_{m_1, w_1, m_0, w_0 \in \mathbb{Z}} \mathsf{q}^{\frac{k_l^2}{2}} \bar{\mathsf{q}}^{\frac{k_r^2}{2}}, \tag{6.23}$$

(we included a factor τ_2 to make it modular invariant and match with the mathematical literature) where

$$k_l^2 = \frac{\left| (T\ 1) \begin{pmatrix} w_0 & w_1 \\ -m_1 & m_0 \end{pmatrix} \begin{pmatrix} -U \\ 1 \end{pmatrix} \right|^2}{2 T_2 U_2} \tag{6.24}$$

$$k_r^2 = k_l^2 - 2 m_1 w_1 - 2 m_0 w_0. \tag{6.25}$$

The partition function of the CFT $V_1 \otimes \bar{V}_2$ factorizes as a product $f(\tau, V_1) \overline{f(\tau, V_2)}$ of a holomorphic and an anti-holomorphic factor.

Let us focus on type II models; in this case, the spacetime fermion number factorizes as $(-1)^{\mathbf{F}} = (-1)^{\mathbf{F}_l} (-1)^{\mathbf{F}_r}$, where $(-1)^{\mathbf{F}_{l,r}}$ acts by a minus sign on the (left- or right-moving) Ramond sector. In terms of the traces of Eq. (2.1), for each given SVOA V, one has

$$f(\tau, V) = \frac{\phi_{NS}(\tau, V) - \phi_{\widetilde{NS}}(\tau, V)}{2} - \frac{\phi_R(\tau, V) \pm \phi_{\tilde{R}}(\tau, V)}{2}$$

$$= \begin{cases} 0 & \text{for } V = V^{fE_8}, \\ -24 & \text{for } V = V_{\pm}^{f\natural}, \\ 24 & \text{for } V = F_{24}. \end{cases} \tag{6.26}$$

In this formula, the first two terms yield the GSO-projected NS sector. The relative sign between the NS and the Ramond sector is due to the spacetime fermion number. The last two terms can be explained as follows. In the Ramond sector, the GSO projection should be implemented by taking just one of the two spacetime chiralities for each Ramond state of the internal SVOA V, depending on the eigenvalue of the internal worldsheet fermion number $(-1)^F$. However, in our Euclidean path integral, the description of chiral fermions is problematic. A way out is as follows. For $V = F_{24}$ or $V = V^{fE_8}$, one observes that $\phi_{\tilde{R}}(\tau, V) = 0$, i.e., the number of states with positive and negative fermion number is the same for each L_0 eigenvalue. This means that the states in the Ramond sector come in pairs with opposite fermion number, and we can count one complex (non-chiral) spacetime fermion for each such pair, thus giving $\frac{\phi_R(\tau, V)}{2}$ as a counting function. The same reasoning applies to $V_{\pm}^{f\natural}$ with $L_0 > 1/2$: The number of states with positive and negative fermion number is the same for $L_0 > 1/2$, so we can count one complex fermion for each pair

of states. On the other hand, the 24 Ramond ground states ($L_0 = 1/2$) in $V_\pm^{f\natural}$, only appear for either positive (for $V_+^{f\natural}$) or negative (for $V_-^{f\natural}$) fermion number, i.e., they would correspond to a chiral spacetime fermion. Since we are not able to describe these chiral fermions in our Euclidean path integral formulation, we simply count one complex non-chiral fermion for each of these 24 states, irrespective of their fermion number; this gives $\frac{\phi_R(\tau, V_\pm^{f\natural}) \pm \phi_{\tilde{R}}(\tau, V_\pm^{f\natural})}{2}$. In this way, we have a slight overcounting of Ramond ground states: This is the origin of the "spurious states" discussed in Sect. 6.2.

By (6.26), in type IIA theories, the partition function $f(\tau, V_1)\overline{f(\tau, V_2)}$ is just a constant. Note that when either V_1 or V_2 is V^{fE_8}, the partition function vanishes.

For heterotic models on $V_1 \otimes \bar{V}_2$, we have a similar result for $f(\tau, V_1)\overline{f(\tau, V_2)}$, where the anti-holomorphic factor $\overline{f(\tau, V_2)}$ is a constant given in (6.26), while the holomorphic factor

$$f(\tau, V_1) = J(\tau) + N = \mathsf{q}^{-1} + N + 196884\mathsf{q} + \cdots \tag{6.27}$$

is the partition function of the bosonic VOA V_1, which equals the J-function up to the constant term.

Thus, in all cases we are interested in, the integrand $\tau_2 \operatorname{Tr}(\mathsf{q}^{L_0} \bar{\mathsf{q}}^{\bar{L}_0}(-1)^{\mathbf{F}})$ is just the product $F(\tau)\Theta_{\Gamma^{2,2}}(\tau; T, U)$ of a theta series and a holomorphic function $F(\tau) = f(\tau, V_1)\overline{f(\tau, V_2)}$.

We conclude that the path integral will amount to performing an integral of the form

$$\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} F(\tau)\Theta_{\Gamma^{2,2}}(\tau; T, U)\frac{\mathrm{d}^2\tau}{\tau_2^2}. \tag{6.28}$$

This is recognized as a *theta lift* in the mathematics literature. More generally one can lift a modular form F on $\mathbb{H}$ to an automorphic form on $SO(m, n)$ via [6]

$$F \quad \longmapsto \quad \Theta_F(g) = \int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} F(\tau)\Theta_{\Gamma^{m,n}}(\tau; g)\frac{\mathrm{d}^2\tau}{\tau_2^2}. \tag{6.29}$$

Here, $\Theta_{\Gamma^{m,n}}$ is a Siegel–Narain theta series for the lattice $\Gamma^{m,n}$, depending on $\tau \in \mathbb{H}$ and $g \in SO(m, n; \mathbb{R})$. The image $\Theta_F(g)$ of the theta lift is then an automorphic form on

$$\Gamma\backslash SO(m, n; \mathbb{R})/(SO(m) \times SO(n)), \tag{6.30}$$

where $\Gamma \subset SO(m, n; \mathbb{R})$ is an arithmetic subgroup (T-duality group).

Consider the case of the lattice $\Gamma^{2,d}$, which is the level of generality that we are interested in. The case when $d > 2$ corresponds to including Wilson lines. Then $\Theta_{\Gamma^{2,d}}$ transforms like a modular form of weight $d/2 - 1$ with respect to $SL(2, \mathbb{Z})$. Assume further that $F(\tau)$ is a weakly holomorphic modular form of weight $1 - d/2$ with Fourier coefficients $c(m, n)$. Since F may have singularities at the cusps the integral diverges. However, it can be regularized by introducing a cutoff $\Im(\tau) \leq t$ and computing the integral in the limit $t \to \infty$

[24,38]. Physically, the divergence is an IR effect, due to the presence of mass-less or tachyonic modes (the latter only arises in off-shell heterotic strings). The result is of the form

$$\Theta_F(g) = \log ||\Phi_F(g)||_{\text{Pet}}^2 + \text{const}, \tag{6.31}$$

where $|| \cdot ||_{\text{Pet}}$ denotes the Petersson metric on the line bundle $\mathcal{L}$ of modular forms of weight $c(0,0)/2$ over $SO(2,d;\mathbb{R})/(SO(2) \times SO(d))$. Moreover, in the neighborhood of a cusp, $\Phi_F(g)$ has an infinite product expansion. The theta lift generalizes to the case where Θ_Λ is the theta series of a lattice Λ that is not unimodular, and $F(\tau)$ is a vector-valued modular form for a suitable $SL(2,\mathbb{Z})$-representation, see "Appendix D" for details and references.

To illustrate this, let us restrict again to the case of having no Wilson lines, i.e., $d = 2$, as in the beginning of this section. In this case, $F(\tau)$ is a weight 0 modular form. The simplest possibility is to let F be a constant, say c. Then we obtain

$$\Theta_c(T,U) = \int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} c \cdot \Theta_{\Gamma^{d,d}}(\tau;T,U) \frac{\mathrm{d}^2\tau}{\tau_2^2}$$
$$= -c \log ||\eta(T)^2 \eta(U)^2||_{\text{Pet}}^2 + \text{const}, \tag{6.32}$$

where the Petersson metric is given explicitly by

$$||\eta(T)^2 \eta(U)^2||_{\text{Pet}}^2 = \Im(T)\Im(U)\big|\eta^2(T)\eta^2(U)\big|^2. \tag{6.33}$$

The constant in (6.32) is explicit and involves the Euler–Mascheroni constant (see, e.g., [38]). We recognize the general structure of the path integral (6.15), where we recall that the prefactors $\Im(T)\Im(U)$ arise from the bosonic and fermionic zero modes.

As mentioned in Sect. 6.2, the path integral we started from is manifestly invariant under spacetime diffeomorphisms. The dependence on the metric is only through the complex moduli T and U, which in turn are defined in terms of diff-invariant quantities, such as the volume of the spacetime torus, the length of geodesics or the angle between them. The infrared regularization is also manifestly diff-invariant—we just cut a region of integration over the moduli of the *worldsheet* torus. Thus, the final result is expected to be free of gravitational anomalies. In particular, the arithmetic T-duality group $\Gamma \subset SO(2,d;\mathbb{R})$ contains a group of large spacetime diffeomorphisms, so that the absence of gravitational anomalies can be seen as the physical counterpart of the automorphy properties of the theta lift.

6.4. The Path Integral for the Superdenominator

Let us consider a type IIA model on $V \otimes \bar{V}_-^{f\natural}$, where V is a self-dual $\mathcal{N} = 1$ SVOA with $c = 12$. The second quantized supersymmetric index $\mathcal{Z}$ corresponding to this model is (the 24th power of) the superdenominator of the BKM superalgebra associated with V.

In the following subsections, we will describe how the superdenominator arises explicitly from the path integral for the type II string on $V \otimes V^{f\natural}$, with $V = F_{24}$, $V^{f\natural}$, and V^{fE_8}.

The case $F_{24} \otimes \bar{V}_-^{f\natural}$. For $V = F_{24}$ we have $\chi_+^{\mathrm{R}} = \chi_-^{\mathrm{R}} = 0$. Let us consider the case when we turn off all Wilson lines, i.e., $y_i = 1$ for all i, so that the lattice is $\Gamma^{2,2}$. We expect the index to depend on both T and U, the Kähler and complex structure moduli of the torus. By (6.26), we have $f(\tau, F_{24}) = 24$ and $f(\tau, V_-^{f\natural}) = -24$. The result of the theta lift is thus

$$
\Theta_{F_{24} \otimes \bar{V}^{f\natural}}(T, U) = \int_{SL(2,\mathbb{Z}) \backslash \mathbb{H}} 24 \cdot (-24) \cdot \Theta_{\Gamma^{2,2}}(\tau; T, U) \mathrm{d}\mu
$$
$$
= 24 \cdot 24 \log \|\eta^2(T)\eta^2(U)\|_{\mathrm{Pet}}^2. \tag{6.34}
$$

The full non-holomorphic partition function in this case is given by

$$
\mathscr{Z}^{F_{24} \otimes \bar{V}^{f\natural}}\Big|_{y=1} = e^{\frac{1}{2}\Theta_{F_{24} \otimes \bar{V}^{f\natural}}(T,U)} = \text{const} \cdot \|(\eta^{24}(T))^{24}(\eta^{24}(U))^{24}\|_{\mathrm{Pet}}^2. \tag{6.35}
$$

Now, let us give a rough idea of the generalization to the case where the Wilson lines are turned on. Recall that $y_i = e^{2\pi i A_i}$, $i = 1, \ldots, r$, where A_i are Wilson line moduli and r is the rank of the gauge group G. The integral now involves the Siegel–Narain theta function $\Theta_\Lambda(\tau; T, U, A)$ for a lattice $\Lambda = \Gamma^{2,2} \oplus L$ of signature $(2, 2+r)$, where L is the r-dimensional lattice of charges with respect to gauge group G. The partition function F becomes a Jacobi form, with theta decomposition of the form $F = \sum F_\gamma \theta_\gamma$, where γ label the cosets in $L^\vee/L$. Using the vector-valued theta lift of F discussed in "Appendix D," we arrive at

$$
\mathscr{Z}^{F_{24} \otimes \bar{V}_-^{f\natural}} = e^{\frac{1}{2}\Theta_{F_{24} \otimes \bar{V}^{f\natural}}(T,U,A)} = \text{const} \cdot \|\Phi(T, U, A)^{24}\|_{\mathrm{Pet}}^2, \tag{6.36}
$$

where $\Phi(T, U, A)$ has the following infinite product representation

$$
\Phi = e^{-2\pi i(T+U+(\rho,A))} \prod_{\lambda \in \Delta_g^+} (1 - e^{2\pi i(\lambda,A)})^{c(0,\lambda)}
$$
$$
\prod_{\substack{m,w=0 \\ (m,w)\neq(0,0)}}^{\infty} \prod_{\lambda \in \Delta^g} (1 - e^{2\pi i(mT+wU+(\lambda,A))})^{c(mw,\lambda)} \tag{6.37}
$$

This matches perfectly with the index $\mathscr{Z}^{F_{24} \otimes \bar{V}_-^{f\natural}}$ obtained in Sect. 5.4.

The case $V^{fE_8} \otimes \bar{V}^{f\natural}$. In this case, we have both $\chi_+^{\mathrm{R}} \neq 0$ and $\chi_-^{\mathrm{R}} \neq 0$, and only states with all momenta vanishing will contribute. Thus, we expect the theta lift to be independent of T and U. Indeed, the contribution to $F(\tau)$ from the $f(\tau, V^{fE_8})$ vanishes (see Eq. (6.26)). Hence, the partition function is just

$$
\mathscr{Z}^{V^{fE_8} \otimes \bar{V}^{f\natural}} = e^{\frac{1}{2}\Theta_0} = 1, \tag{6.38}
$$

which agrees with the result in Sect. 5.4.

The cases $V_{\pm}^{f\natural} \otimes \bar{V}_{-}^{f\natural}$. For $V_{\pm}^{f\natural}$, we have $\chi_{\pm}^R \neq 0$ (but $\chi_{\mp}^R = 0$). Recall from Sect. 5.4 the case $\chi_{+}^R \neq 0$ implies that, on-shell, there is a constraint on the winding and momenta such that there is no winding around the spacelike S^1. Similarly, the case $\chi_{-}^R \neq 0$ implies that there is no momenta around the S^1. Therefore, in the operatorial formalism, one has that $\mathcal{Z}^{V_{+}^{f\natural} \otimes \bar{V}_{-}^{f\natural}}$ depends only on U, while $\mathcal{Z}^{V_{+}^{f\natural} \otimes \bar{V}_{-}^{f\natural}}$ depends only on T. In the path integral (6.22), these conditions are not imposed, so one expects $\mathcal{Z}$ to depend on both T and U and be the same function for both $V_{+}^{f\natural} \otimes \bar{V}_{-}^{f\natural}$ and $V_{-}^{f\natural} \otimes \bar{V}_{-}^{f\natural}$.

The calculation is very similar to the case $F_{24} \otimes \bar{V}_{-}^{f\natural}$, but now

$$c = f(\tau, V_{\pm}^{f\natural})\overline{f(\tau, V_{-}^{f\natural})} = (-24)^2. \tag{6.39}$$

The result is

$$\mathscr{Z}^{V_{\pm}^{f\natural} \otimes \bar{V}_{-}^{f\natural}}(T, U) = \text{const} \cdot ||\eta^{-24}(T)\eta^{-24}(U)||_{\text{Pet}}^{2 \cdot 24}. \tag{6.40}$$

The superdenominator formula should correspond to the 24th power of the holomorphic piece of this partition function, namely $\eta^{-24}(T)$ (or $\eta^{-24}(U)$). This indeed agrees with Eq. (6.17).

6.5. Path Integral for the Denominator of Type II on $V_{\pm}^{f\natural} \otimes \bar{V}_{-}^{f\natural}$

We shall now move on to compute the path integral for the modified index $\tilde{\mathcal{Z}}^{V_1 \otimes \bar{V}_{-}^{f\natural}}$ for type IIA on $V_1 \otimes \bar{V}_{-}^{f\natural}$, corresponding to the denominator function of the BKM superalgebra associated with V_1. The modified index $\tilde{\mathcal{Z}}$ is obtained by inserting an operator $(-1)^{\mathbf{F}_l}$ in the trace over the second quantized Hilbert space. The left-moving fermion number $(-1)^{\mathbf{F}_l}$ acts by $+1$ on the physical states in the left-moving NS sector (which give the even elements of the BKM superalgebra) and by -1 on physical string states in the left-moving R sector (corresponding to odd elements in the BKM superalgebra). In the path integral formulation of the index, this insertion corresponds to taking either integral or half-integral quantization for the momenta in the time direction, depending on the $(-1)^{\mathbf{F}_l}$ eigenvalue of the string running in the loop. Formally, this is equivalent to taking the standard path integral for an orbifold theory, analogous to the CHL string models. One starts from a theory whose Euclidean time circle has twice the radius, and then takes an orbifold by a symmetry $\mathfrak{z}$ acting as a half-period shift along the Euclidean time circle, together with the action of $(-1)^{\mathbf{F}_l}$. As a result, we get a theory whose fields obey $(-1)^{\mathbf{F}_l}$-twisted boundary conditions along the Euclidean time circle. From a mathematical viewpoint, this procedure amounts to taking the theta lift of vector-valued (rather than scalar) modular forms in the integrand. The general structure of such integrals is recalled in "Appendix D."

Now, let us apply this machinery to our setting. Let us consider the example of type IIA string theory on $V_{-}^{f\natural} \otimes \bar{V}_{-}^{f\natural}$. Recall that the path integral for the superdenominator, in this example, is given by (6.22), where the integrand is $\Theta_{\Gamma^{2,2}}(T, U, \tau)f(\tau, V_{-}^{f\natural})\overline{f(\tau, V_{-}^{f\natural})}$. In particular, in the CFT trace inside the integral, the spacetime oscillators cancel against the ghost and superghost

contributions, and we are left with the theta series for the winding-momentum lattice on the Euclidean torus times the trace over the internal CFT.

Now, in order to compute the path integral for the denominator, we have to take an orbifold by an order 2 symmetry $\mathfrak{z}$ acting by a half-period shift along the Euclidean time circle and simultaneously by $(-1)^{\mathbf{F}_l}$. This symmetry acts trivially on the spacetime oscillators and (super)ghosts, so that the same cancellations as for the superdenominator case occur. Therefore, the CFT trace in the integral of (6.22) is still $\overline{f(\tau, V_-^{f\natural})} = -24$ times the contributions from the holomorphic internal SVOA and from the winding-momentum. We claim that the integrand has the form

$$
\sum_{r\in\mathbb{Z}/2\mathbb{Z}} \mathrm{Tr}_{\mathfrak{z}^r}\left(\mathsf{q}^{L_0}\bar{\mathsf{q}}^{\bar{L}_0}(-1)^{\mathbf{F}}\frac{1}{2}\sum_{s\in\mathbb{Z}/2\mathbb{Z}}\mathfrak{z}^s\right)
$$
$$
= \frac{1}{2}\sum_{r,s\in\mathbb{Z}/2\mathbb{Z}} f(r,s;\tau)\Theta^{r,s}(T,U;\tau)\overline{f(\tau,V_-^{f\natural})}. \tag{6.41}
$$

Here, $\mathrm{Tr}_{\mathfrak{z}^r}$ denotes the trace over the $\mathfrak{z}^r$-twisted sector, and $\frac{1}{2}\sum_s\mathfrak{z}^s$ is the projection onto the $\mathfrak{z}$-invariant states. The anti-holomorphic SVOA $\bar{V}_-^{f\natural}$ contributes with the usual factor $\overline{f(\tau, V_-^{f\natural})} = -24$, since $\mathfrak{z}$ acts trivially on this theory. For each $r,s\in\mathbb{Z}/2\mathbb{Z}$, $f(r,s;\tau)$ is the contribution from the holomorphic internal SVOA $V_-^{f\natural}$, while $\Theta^{r,s}(T,U;\tau)$ comes from the sum over winding and momentum along the spacetime torus. Explicitly,

$$
f(0,0;\tau) = \frac{\phi_{\mathrm{NS}}(\tau) - \phi_{\tilde{\mathrm{NS}}}(\tau)}{2} - \frac{\phi_{\mathrm{R}}(\tau) - \phi_{\tilde{\mathrm{R}}}(\tau)}{2} = -24
$$
$$
f(0,1;\tau) = \frac{\phi_{\mathrm{NS}}(\tau) - \phi_{\tilde{\mathrm{NS}}}(\tau)}{2} + \frac{\phi_{\mathrm{R}}(\tau) - \phi_{\tilde{\mathrm{R}}}(\tau)}{2}
$$
$$
= 24 + 4096\mathsf{q} + 98304\mathsf{q}^2 + 1228800\mathsf{q}^3 + \cdots
$$
$$
f(1,0;\tau) = \frac{\phi_{\mathrm{NS}}(\tau) + \phi_{\tilde{\mathrm{NS}}}(\tau)}{2} - \frac{\phi_{\mathrm{R}}(\tau) + \phi_{\tilde{\mathrm{R}}}(\tau)}{2}
$$
$$
= \mathsf{q}^{1/2} + 276\mathsf{q}^{1/2} - 2048\mathsf{q} + 11202\mathsf{q}^{3/2} + \cdots
$$
$$
f(1,1;\tau) = -\frac{\phi_{\mathrm{NS}}(\tau) + \phi_{\tilde{\mathrm{NS}}}(\tau)}{2} - \frac{\phi_{\mathrm{R}}(\tau) + \phi_{\tilde{\mathrm{R}}}(\tau)}{2}
$$
$$
= -\mathsf{q}^{-1/2} - 276\mathsf{q}^{1/2} - 2048\mathsf{q} - 11202\mathsf{q}^{3/2} + \cdots
$$

where in the second line we use the fact that $\mathfrak{z} = (-1)^{\mathbf{F}_l}$ on $V_-^{f\natural}$, and the third and fourth line follow from modular transformations of the second (recall that f_{NS}, $f_{\tilde{\mathrm{NS}}}$, and f_{R} span a three-dimensional representation of $SL(2,\mathbb{Z})$, while $f_{\tilde{\mathrm{R}}}$ is $SL(2,\mathbb{Z})$-invariant). As for the theta series, we have

$$
\Theta^{r,s}(T,U;\tau) = \tau_2 \sum_{m_1,w_1,m_0\in\mathbb{Z}}\sum_{w_0\in\mathbb{Z}+\frac{r}{2}} \mathsf{q}^{\frac{k_l^2}{2}}\bar{\mathsf{q}}^{\frac{k_r^2}{2}}(-1)^{sm_0}, \tag{6.42}
$$

where k_l, k_r are given in (6.24) and (6.25). This follows from the fact that the winding number w_0 is half-integral in the $\mathfrak{z}$-twisted sector, and that a shift by half a period corresponds to multiplication by $(-1)^{m_0}$.

The connection with vector-valued theta lifts can be easily made manifest (see "Appendix D" for notation). Let us consider the lattice $\Lambda = \Gamma^{1,1} \oplus \Gamma^{1,1}(2)$, where $\Gamma^{1,1}$ is the unique even unimodular lattice of signature $(1,1)$, and $\Gamma^{1,1}(2)$ is the same lattice with the quadratic form rescaled by 2. This implies

$$\Lambda^\vee = \Gamma^{1,1} \oplus \tfrac{1}{2}\Gamma^{1,1}(2), \qquad \Lambda^\vee/\Lambda \cong \tfrac{1}{2}\Gamma^{1,1}(2)/\Gamma^{1,1}(2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2. \quad (6.43)$$

The elements $\lambda \in \tfrac{1}{2}\Gamma^{1,1}(2)$ can be parametrized by

$$\lambda = (m/2, n/2), \qquad \lambda^2 = mn. \quad (6.44)$$

The cosets $\gamma \in \Lambda^\vee/\Lambda$ are represented by pairs $(r/2, s/2)$ with $r, s \in \mathbb{Z}/2\mathbb{Z}$. With the lattice Λ is associated a vector-valued representation ρ_Λ (Weil representation) of $SL(2,\mathbb{Z})$, as in (D.3). We define a vector-valued modular form F for ρ_Λ via the discrete Fourier transform as follows:

$$F_{r,s}(\tau) := \frac{1}{2} \sum_{t \in \mathbb{Z}/2\mathbb{Z}} e^{-\pi i t s} f(r, t; \tau)\overline{f(\tau, V_-^{f\natural})}. \quad (6.45)$$

This yields

$$\begin{aligned}
F_{00}(\tau) &= -24(2048q + \cdots) \\
F_{01}(\tau) &= -24(-24 - 2048q - \cdots) \\
F_{10}(\tau) &= -24(-2048q - \cdots) \\
F_{11}(\tau) &= -24(q^{-1/2} + 276q^{1/2} + \cdots).
\end{aligned} \quad (6.46)$$

For later reference, let us denote the Fourier expansions of these functions by

$$F_{r,s}(\tau) = -24 \sum_{n \in \frac{1}{2}\mathbb{Z}} C_{r,s}(n)q^n. \quad (6.47)$$

These functions represent the components $F_\gamma(\tau)$ of the vector-valued modular form $F(\tau)$.

Now, let us turn to the theta function. The shifted theta series $\theta_{\Lambda+\gamma}$ can be written as follows in terms of the shifted Narain theta series $\Theta^{r,s}$:

$$\theta_{\Lambda+\gamma} = \frac{1}{2} \sum_{t \in \mathbb{Z}/2\mathbb{Z}} e^{\pi i s t}\Theta^{r,t}(T, U; \tau), \quad (6.48)$$

where $\gamma \equiv (r/2, s/2) \in \Lambda^\vee/\Lambda$.

The inner product in the integrand of the theta lift can now be written out explicitly as follows:

$$\begin{aligned}
\left(\overline{\Theta}_\Lambda, F\right) &= \sum_{\gamma, \gamma' \in \Lambda^\vee/\Lambda} \theta_{\Lambda+\gamma} F_{\gamma'} \delta_{\gamma'-\gamma,0} \\
&= \frac{1}{2} \sum_{r,s \in \mathbb{Z}/2\mathbb{Z}} f(r, s; \tau)\Theta^{r,s}(T, U; \tau)\overline{f(\tau, V_-^{f\natural})}.
\end{aligned} \quad (6.49)$$

Thus, the theta integral that we wish to perform is explicitly given by

$$\frac{1}{2} \int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} \left(\overline{\Theta}_\Lambda, F\right) d\mu. \quad (6.50)$$

We can now invoke the general theorems of Borcherds [6] and Carnahan [12] to deduce that the result of this integral is[15]

$$\log ||\Phi_F(T,U)||_{\text{Pet}} + \text{const}, \tag{6.51}$$

where $\Phi_F(T,U)$ is an automorphic form on $O(2,2;\mathbb{R})/(O(2)\times O(2))$, with the following product representation:

$$\Phi_F(T,U) = p^{-24}q^{-24} \prod_{\substack{m,w\in\mathbb{Z}\\(\lambda,W)>0}} \prod_{t\in\mathbb{Z}/2\mathbb{Z}} (1-(-1)^t p^w q^m)^{24C_{0,t}(mw)}. \tag{6.52}$$

The condition $(\lambda,W)>0$ includes all states $\lambda=(m,w)$ such that $m,w\geq 0$, and $(m,w)\neq(0,0)$.[16] We may separate this into states with $w=0, m>0$ and states with $w\geq 1, m\geq 0$. Since $C_{0,0}(0)=0$ and $C_{0,1}(0)=-24$ the first set of states (with the q^{-24} factor) give

$$\left(q^{-1}\prod_{m=1}^{\infty}(1+q^m)^{-24}\right)^{24} \tag{6.53}$$

which we can rewrite in terms of η-functions according to

$$\left(\frac{\eta(U)^{24}}{\eta(2U)^{24}}\right)^{24}. \tag{6.54}$$

The complete function Φ_F may now be written as

$$\Phi_F(T,U) = \left(q^{-1}\prod_{m=1}^{\infty}\frac{1}{(1+q^m)^{24}}\right)^{24}\Psi(T,U)^{24}, \tag{6.55}$$

where we defined

$$\Psi(T,U) = p^{-1}\prod_{w=1}^{\infty}\prod_{m\geq 0}(1-p^w q^m)^{C_{0,0}(mw)}(1+p^w q^m)^{C_{0,1}(mw)}. \tag{6.56}$$

Here, $C_{0,0}$ and $C_{0,1}$ are the Fourier coefficients defined in (6.47). The function $\Psi(T,U)^{24}$ matches exactly with the modified index $\tilde{\mathcal{Z}}^{V^{f\natural}\otimes\bar{V}^{f\natural}}_{-}$ in Eq. (5.37), and its 24th root $\Psi(T,U)$ is the denominator formula in Eq. (4.18) of [37]. The factor $\left(q^{-1}\prod_{m=1}^{\infty}\frac{1}{(1+q^m)^{24}}\right)^{24}$ is the contribution of the spurious states, so that Φ_F matches with $\tilde{\mathcal{Z}}^{V^{f\natural}\otimes\bar{V}^{f\natural}}_{sp,-}(T,U)$ in Eq. (6.7).

Using similar techniques, one may compute the path integrals for the other cases and reproduce the corresponding denominator functions obtained in Sect. 5.4.

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[15]The function Φ_F considered here is the inverse of the automorphic function Φ_M defined in theorem 13.3 of [6]. With our conventions, Φ_F can be directly identified with the path integral $\tilde{\mathcal{Z}}_{sp}$.

[16]We thank Scott Carnahan for clarifications about this point.

7. Discussion

In this paper, we construct a family of distinguished type IIA and heterotic string compactifications to two spacetime dimensions. These compactifications are distinguished by the fact that there exists an action of a BKM algebra on the BPS subspace of the Hilbert space of physical states in the models, which we illustrate in Sect. 4, following the original construction for the monster BKM algebra in [55,56]. Furthermore, we demonstrate how a spacetime supersymmetric index in our theories—computed from both a trace in the Hilbert space (Sect. 5) and via a path integral (Sect. 6)—furnishes an automorphic form which is (closely related to) a denominator function for the corresponding BKM algebra. In this family we find a spacetime string-theoretic setting for the BKM algebras constructed in [36,37,61], among many others.

The compactifications we consider in this paper are also distinguished because they are highly symmetric, and the corresponding BKM algebras which arise have large finite symmetry groups, in some cases sporadic groups. For example, $V^{f\natural}$ has an action of the Conway group [25], which carries over to an action on the corresponding BKM arising from the chiral string compactification [37]. The lattice VOAs V_L based on the Niemeier lattices L have symmetry group corresponding to (extensions of) the automorphism group of the lattice.[17] (For L the Leech lattice, this is the group Co_0, while other cases have symmetry groups M_{24} and $2.M_{12}$ arising from the Mathieu groups.) In light of this fact, a natural question is to explore so-called CHL models [14] based on these compactifications, which arise from orbifolds of string theory on $\mathcal{C} \times T^d$. These orbifolds arise by quotienting by a symmetry of the internal CFT $\mathcal{C}$ combined with a translation along the torus T^d. For a general discussion, see [57]. In the monster case of [55,56], where $\mathcal{C} = V^\natural \otimes \bar{V}^{f\natural}$, considering these CHL models leads to a family of "twisted" denominator formulas for the spacetime BKM algebra. It would be interesting to generalize this to the case of models considered in this paper.

Furthermore, these CHL models also inspired a proposed physical explanation for the genus zero property of monstrous moonshine [20]. As the Conway moonshine arising from the SVOA $V^{f\natural}$ also satisfies a genus zero property [26],[18] it would be very interesting to offer a physical explanation along the lines of that in [55,56] based on the constructions in this paper and their corresponding CHL models.

Besides finding explicit physical models where one can explore the physics of BPS algebras, another one of our original motivations for this work was to uncover new 2d spacetime string-theoretic dualities, where one can explore connections to BKM algebras and automorphic forms. It is the case that a single automorphic form on the moduli space $\mathcal{M}$ of some family of string

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[17]These are the groups appearing in the umbral moonshine phenomenon [17]. It would be very interesting if there was a connection between these physical models and umbral moonshine.

[18]To be precise, Conway moonshine is usually expressed in terms of $V^{s\natural}$, which is an SVOA isomorphic to $V^{f\natural}$, but with a different action of the Conway group. In our setup, it would be more natural to re-formulate this moonshine phenomenon in terms of $V^{f\natural}$.

compactifications can have multiple distinct Fourier expansions at different "cusps" of $\mathcal{M}$, and that the expansion at each cusp corresponds to the (super)denominator formula a unique BKM (super)algebra, as in [34]. In a string theory context, these BKMs arise as symmetries of different perturbative duality frames of a given compactification, which can be related by discrete duality transformations. We imagine that this furnishes a physical explanation for the following observations:

1. The denominator of the Conway BKM superalgebra constructed in [37] arises as the Fourier expansion of automorphic form on $(\mathbb{H} \times \mathbb{H})/O(2, 2, \mathbb{Z})$ at a given cusp. The authors of this paper gave evidence for the existence of a distinct BKM superalgebra, which also has an action of the Conway group, and whose denominator and superdenominator should correspond to the expansion of the same automorphic form at a different cusp of $(\mathbb{H} \times \mathbb{H})/O(2, 2, \mathbb{Z})$. The construction in the present article provides a natural description of this new mysterious superalgebra. In Sect. 6, we described the denominator of the Conway superalgebra as a path integral with "twisted" boundary conditions along the Euclidean time circle, i.e., with the fields being periodic or anti-periodic depending on their eigenvalue with respect to the left-moving spacetime fermion number $(-1)^{\mathbf{F}_l}$. The expansions of this automorphic function at different cusps are related to one another by $O(2, 2, \mathbb{Z})$ T-duality transformation. A general T-duality does not necessarily preserve these boundary conditions, but can map them to $(-1)^{\mathbf{F}_l}$-twisted boundary conditions along the spacelike circle or around both circles. In particular, the path integrals obtained from these two boundary conditions are related, respectively, to the denominator and superdenominator of a different BKM superalgebra, the one obtained by taking the CHL model by $(-1)^{\mathbf{F}_l}$ along the spacelike circle.

2. Along similar lines, one of the BKM superalgebras discussed in [36] (and originally constructed by Borcherds in [8–10]) arises from the expansion of an automorphic form for a group of the form $\Gamma \backslash SO(2, 10)/SO(2) \times SO(10)$ (where Γ is a discrete group of automorphisms of a particular lattice.) Expanding this automorphic form at a different cusp gives rise to another BKM superalgebra, based of the SVOA V^{fE_8} and originally constructed by Scheithauer in [61]. See example 13.7 in [6] for a description of this relation, and sections 6 and 7 of [36] for further elaboration of this point. We expect this connection arises physically from dualities of certain models we consider in this paper.

These are just two examples where we believe string dualities acting on models of the type we study in this paper lead to non-trivial relations among different BKM algebras, though we expect there are many more.

Finally, we close with a number of additional natural questions which are raised by our analysis:

- While our analysis in this paper has been limited to two-dimensional string compactifications, we hope that BKM algebras act as symmetries

of special string solutions in higher dimensions in a similar way. For example, it would be interesting to study decompactification limits of our models to higher dimensions (perhaps along similar lines as [44]), and understand what, if any, symmetries of the BKM algebras are preserved in these limits. One could also try to construct such algebras directly from worldsheet theories of certain higher-dimensional string compactifications.

- It would be very interesting to understand the D-brane states in the family of type II string compactifications considered here, which we imagine may furnish representations of the BKM algebras we have constructed.

- The denominator of the fake monster superalgebra, arising as the modified index for type IIA strings on $V^{fE_8} \otimes \bar{V}_-^{f\natural}$, also appears as the genus one topological amplitude F_1 in type II string theory on the Enriques Calabi–Yau threefold X (the "FHSV-model") [47]. From this point of view the modified index can be interpreted as a generating function for Gromov–Witten invariants on X. It would be interesting to understand more generally whether such geometric interpretations exist for other denominator formulas obtained in this paper. In view of heterotic-type II duality, the perturbative BPS states contributing to our indices should correspond to non-perturbative states in another duality frame. Mathematically, this would correspond to counting Donaldson–Thomas invariants on $K3, T^4$, or the Enriques Calabi–Yau threefold X.

- Many of the (S)VOAs which appear as part of the worldsheet theory in our models have connections to large sporadic symmetry groups, arising as automorphisms that preserve some suitable subVOA (see, e.g., [2,16, 27].) It would be interesting to understand if these structures are present in the BKM algebras which appear in the spacetime of our models.

- In [45], it was observed that the denominator formula for one of the BKM superalgebras in [36] can be obtained starting from the $SL(2, \mathbb{Z})$-invariant J-function. Can this connection be explained in terms of heterotic-type II string duality?

- It would be interesting to determine if any of our models arise as special points in the moduli space of any of the 2d string models considered in [67], where certain dual pairs of 2d superstring compactifications were identified.

- As discussed in more detail in Sect. 4, it would be satisfying to study a twisted version of our models, so that our space of BPS states becomes a space of physical states in a topological string theory, perhaps with a natural action of the BKM algebras.

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A Anomalies and Tadpoles

In this section, we discuss the tadpole for the B-field which may arise when compactifying string theory to two dimensions. This was first computed in [49,50,69]; see also [67] and [68]. We will adapt their methods to the theories of interest to us in this paper.

As a consistency check, we also verify that gravitational anomalies cancel in all the models we are interested in. In the second quantized theory considered in Sect. 5, where one further spatial direction is compactified on a circle S^1, the B-field tadpole can be interpreted as a vacuum winding number w_0 along S^1. In this case, we argue that the vacuum might also carry non-trivial momentum m_0 along S^1, and we compute m_0 for the models we are interested in.

A.1 Type II Models

Let us compute the B-field tadpole w_0 in the type IIA models we are interested in. As argued in [69], the relevant contribution to the tadpole is a 1-loop string amplitude with the insertion of one B-field vertex operator at zero-momentum, as well as the suitable ghost and superghost insertions to make the amplitude nonvanishing. The only nonzero contribution to this 1-point function comes from path integral with either (even,odd) or (odd,even) spin structure.[19]

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[19]A reminder about spin structures: The odd spin structure means fully periodic boundary conditions for fermions on the torus. In the Hamiltonian formalism, this is the Ramond sector with the insertion of worldsheet fermion number in the trace, which we denote by $\tilde{R}$;

According to [69], the (even,odd) and the (odd,even) spin structures give the same contribution, so they just compute the (even,odd) case and put a factor 2 in front of the final result. This step assumes that the internal CFTs are left-right symmetric, which is not necessarily true in our case; therefore, with respect the formulas in [69], we drop the factor 2 and consider the explicit sum of the (even,odd) and the (odd,even) spin structures. The computation of the (even,odd) case is completely analogous to [69]; after the usual cancellations between ghost, superghosts and light-cone oscillators, plus some further non-trivial steps, the final result is that one has to take the coefficient of the q^0-term of the combination

$$-\frac{1}{48}(Z_{NS,\tilde{R}}(\tau) - Z_{\widetilde{NS},\tilde{R}}(\tau) - Z_{R,\tilde{R}}(\tau))_{\mathsf{q}^0}. \tag{A.1}$$

where $Z_{X,Y}$ denotes the partition functions of the internal CFT in the X, Y sector. The relative signs arise because the central charge of the internal CFT is not a multiple of 24, so that $Z_{NS,\tilde{R}}(\tau+1) = -Z_{\widetilde{NS},\tilde{R}}(\tau)$. Note that $Z_{NS,\tilde{R}}(\tau)$ and $-Z_{\widetilde{NS},\tilde{R}}(\tau)$ only differ for the sign of the half-integral powers of q, while the integral powers of q (in particular, the q^0 term) are the same. Thus, we can set $n_{NS,\tilde{R}} = (Z_{NS,\tilde{R}}(\tau))_{\mathsf{q}^0} = (-Z_{\widetilde{NS},\tilde{R}}(\tau))_{\mathsf{q}^0}$ and $n_{R,\tilde{R}} = (Z_{R,\tilde{R}})_{\mathsf{q}^0}$, and get the formula

$$-\frac{1}{48}(2n_{NS,\tilde{R}} - n_{R,\tilde{R}}), \tag{A.2}$$

which is, up to a factor 2, Eq. (2.6) in [67]. The sum $Z_{NS,\tilde{R}}(\tau) - Z_{\widetilde{NS},\tilde{R}}(\tau)$ effectively gives twice the GSO-projected partition function in the NS sector, so $n_{NS,\tilde{R}}$ is also the q^0 term in the GSO-projected partition function.[20]

The (odd,even) contribution is completely analogous (it is given by the $\bar{\mathsf{q}}^0$ term in the analogous combination of partition functions), so that the total contribution to the tadpole is

$$w_0 = -\frac{1}{48}(2n_{NS,\tilde{R}} - n_{R,\tilde{R}} + 2n_{\tilde{R},NS} - n_{\tilde{R},R}). \tag{A.3}$$

For left-right symmetric internal CFTs, one has $n_{NS,\tilde{R}} = n_{\tilde{R},NS}$ and $n_{R,\tilde{R}} = n_{\tilde{R},R}$, so that one reobtains formula (2.6) in [67]. When the internal CFT is holomorphically factorized and of the form $V_1 \otimes \bar{V}_2$, then one has $n_{X,Y} = n_X \bar{n}_Y$, where n_X refers to the left-moving SVOA V_1 and n_Y to the right-moving SVOA $\bar{V}_2$, so that the tadpole is

$$w_0 = -\frac{1}{48}[(2n_{NS} - n_R)\bar{n}_{\tilde{R}} + n_{\tilde{R}}(2\bar{n}_{NS} + \bar{n}_R)]. \tag{A.4}$$

---

it is $SL(2,\mathbb{Z})$-invariant. The even spin structures have some anti-periodic boundary condition for fermions. In the Hamiltonian formalism, these are NS sector with or without fermion number insertion, or the Ramond sector without fermion number. We denote these sectors by $\widetilde{NS}$, NS and R, respectively; the three sectors are permuted by $SL(2,\mathbb{Z})$ transformations.

[20]This explains Sen's comments in [67] that the $n_{NS,\tilde{R}}$ count states that are GSO-projected on the left- but not on the right-moving side, while $n_{R,\tilde{R}}$ counts states that are not GSO-projected either on the left or on the right.

In the following table, we report $n_{NS} = -n_{\widetilde{NS}} = \chi^{\mathrm{NS}}$, $n_R = \chi_+^{\mathrm{R}} + \chi_-^{\mathrm{R}}$, $n_{\tilde{R}} = \chi_+^{\mathrm{R}} - \chi_-^{\mathrm{R}}$ and $2n_{NS} - n_R = 2\chi^{\mathrm{NS}} - \chi_+^{\mathrm{R}} - \chi_-^{\mathrm{R}}$ for the SVOAs we are interested in. The quantities $\chi^{\mathrm{NS}}, \chi_\pm^{\mathrm{R}}$ are the same as presented in Sect. 2.1.

SVOA V	$\chi^{\mathrm{NS}}(V)$	$\chi_+^{\mathrm{R}}(V)$	$\chi_-^{\mathrm{R}}(V)$	n_R	$n_{\tilde{R}}$	$2n_{NS} - n_R$
$V_{E_8}^f$	8	8	8	16	0	0
$V_-^{f\natural}$	0	0	24	24	-24	-24
$V_+^{f\natural}$	0	24	0	24	$+24$	-24
F_{24}	24	0	0	0	0	48.

Note that for V^{fE_8} one has $n_{\tilde{R}} = 2n_{NS} - n_R = 0$, so that when either V_1 or V_2 is V^{fE_8} there is no tadpole. More generally, the tadpole is always an integer, so it can be canceled by a suitable number of spacetime-filling strings.

In terms of χ^{NS}, χ_+^{R}, χ_-^{R} and their right-moving counterparts $\bar{\chi}^{\mathrm{NS}}$, $\overline{\chi}_+^{\mathrm{R}}$, $\overline{\chi}_-^{\mathrm{R}}$ the tadpole is

$$w_0 = \frac{1}{48}\left[(2\chi^{\mathrm{NS}} - \chi_+^{\mathrm{R}} - \chi_-^{\mathrm{R}})(\overline{\chi}_-^{\mathrm{R}} - \overline{\chi}_+^{\mathrm{R}}) + (\chi_-^{\mathrm{R}} - \chi_+^{\mathrm{R}})(2\overline{\chi}^{\mathrm{NS}} - \overline{\chi}_+^{\mathrm{R}} - \overline{\chi}_-^{\mathrm{R}})\right].$$

(A.5)

The values of w_0 for the theories we are interested in are given in Table 3 in the main text.

Let us now check that the gravitational anomaly always cancels. The chiral massless content of these theories, as a function of χ^{NS}, χ_+^{R}, χ_-^{R}, $\bar{\chi}^{\mathrm{NS}}$, $\overline{\chi}_+^{\mathrm{R}}$, $\overline{\chi}_-^{\mathrm{R}}$, is (the NS-NS sector does not contain chiral bosons):

Spin	NS-R	R-NS	R-R	Total
$+3/2$	$\overline{\chi}_-^{\mathrm{R}}$	χ_+^{R}	0	$\chi_+^{\mathrm{R}} + \overline{\chi}_-^{\mathrm{R}}$
$-3/2$	$\overline{\chi}_+^{\mathrm{R}}$	χ_-^{R}	0	$\chi_-^{\mathrm{R}} + \overline{\chi}_+^{\mathrm{R}}$
$+1/2$ (dilatinos)	$\overline{\chi}_+^{\mathrm{R}}$	χ_-^{R}	0	$\chi_-^{\mathrm{R}} + \overline{\chi}_+^{\mathrm{R}}$
$-1/2$ (dilatinos)	$\overline{\chi}_-^{\mathrm{R}}$	χ_+^{R}	0	$\chi_+^{\mathrm{R}} + \overline{\chi}_-^{\mathrm{R}}$
$+1/2$ (gauginos)	$\chi^{\mathrm{NS}}\overline{\chi}_-^{\mathrm{R}}$	$\chi_+^{\mathrm{R}}\overline{\chi}^{\mathrm{NS}}$	0	$\chi^{\mathrm{NS}}\overline{\chi}_-^{\mathrm{R}} + \chi_+^{\mathrm{R}}\overline{\chi}^{\mathrm{NS}}$
$-1/2$ (gauginos)	$\chi^{\mathrm{NS}}\overline{\chi}_+^{\mathrm{R}}$	$\chi_-^{\mathrm{R}}\overline{\chi}^{\mathrm{NS}}$	0	$\chi^{\mathrm{NS}}\overline{\chi}_+^{\mathrm{R}} + \chi_-^{\mathrm{R}}\overline{\chi}^{\mathrm{NS}}$
chiral $+1$	0	0	$\chi_+^{\mathrm{R}}\overline{\chi}_-^{\mathrm{R}}$	$\chi_+^{\mathrm{R}}\overline{\chi}_-^{\mathrm{R}}$
anti-chiral -1	0	0	$\chi_-^{\mathrm{R}}\overline{\chi}_+^{\mathrm{R}}$	$\chi_-^{\mathrm{R}}\overline{\chi}_+^{\mathrm{R}}$

The contribution to the gravitational anomaly polynomial of each of these massless chiral fields is [1][21]

- Gravitinos (spin $\pm\frac{3}{2}$): $\pm\frac{23}{48}p_1$;
- Chiral fermions (spin $\pm\frac{1}{2}$): $\mp\frac{1}{48}p_1$;
- Chiral bosons (spin ±1): $\mp\frac{1}{24}p_1$.

---

[21] Here, we consider Majorana–Weyl spin $1/2$ and $3/2$ fields and real chiral and anti-chiral bosons. The contributions of the spin $1/2$ and $3/2$ fields are half of the ones calculated in [1], because in that case complex Weyl fermions were considered.

Here,

$$p_1 = \frac{1}{16\pi^2} \operatorname{Tr}(R \wedge R) \tag{A.6}$$

is the first Pontryagin class and R denotes Riemann tensor. Therefore, the total gravitational anomaly is (note that each gravitino of spin $\pm\frac{3}{2}$ comes with a dilatino of spin $\mp\frac{1}{2}$, so it is easier to consider these fields together, contributing $\pm\frac{24}{48}p_1$ each)

$$
\begin{aligned}
&\left(\chi_+^{\mathrm{R}} + \overline{\chi}_-^{\mathrm{R}} - \chi_-^{\mathrm{R}} - \overline{\chi}_+^{\mathrm{R}}\right) \frac{24}{48} p_1 \\
&\quad + \left(\chi^{\mathrm{NS}}\overline{\chi}_+^{\mathrm{R}} + \chi_-^{\mathrm{R}}\overline{\chi}^{\mathrm{NS}} - \chi^{\mathrm{NS}}\overline{\chi}_-^{\mathrm{R}} - \chi_+^{\mathrm{R}}\overline{\chi}^{\mathrm{NS}}\right) \frac{p_1}{48} + \left(\chi_-^{\mathrm{R}}\overline{\chi}_+^{\mathrm{R}} - \chi_+^{\mathrm{R}}\overline{\chi}_-^{\mathrm{R}}\right) \frac{p_1}{24} \\
&= \frac{p_1}{48}\left[\left(\chi^{\mathrm{NS}} + \chi_-^{\mathrm{R}} + \chi_+^{\mathrm{R}} - 24\right)\left(\overline{\chi}_+^{\mathrm{R}} - \overline{\chi}_-^{\mathrm{R}}\right)\right. \\
&\quad \left. + \left(\chi_-^{\mathrm{R}} - \chi_+^{\mathrm{R}}\right)\left(\overline{\chi}^{\mathrm{NS}} + \overline{\chi}_+^{\mathrm{R}} + \overline{\chi}_-^{\mathrm{R}} - 24\right)\right] = 0 \tag{A.7}
\end{aligned}
$$

where the last equality follows because all internal SVOAs satisfy $\chi^{\mathrm{NS}} + \chi_-^{\mathrm{R}} + \chi_+^{\mathrm{R}} = 24$. Therefore, there is no gravitational anomaly.

Finally, when the theory is further compactified on a circle of radius R, there might be a nonzero vacuum momentum $\frac{2\pi m_0}{R}$ [23, 32].[22] This momentum is related to the B-field tadpole w_0 of the T-dual theory. A similar phenomenon arises in two-dimensional conformal field theory on a cylinder, where $L_0 - \bar{L}_0$ along the circle might have a nonzero vacuum level. More precisely, $L_0 - \bar{L}_0$ is the generator of translations when the radius is rescaled so that $R = 2\pi$, so that m_0 is precisely the vacuum value of $L_0 - \bar{L}_0$. This analysis shows that a massless chiral boson of spin ± 1 contributes $\mp\frac{1}{24}$ to m_0, while a chiral fermion of spin $\pm\frac{1}{2}$ with periodic boundary conditions contributes $\pm\frac{1}{24}$ (with anti-periodic boundary conditions, the contributions is $\mp\frac{1}{48}$). For this calculation, we take into account only propagating degrees of freedom, so from the previous description of massless chiral fields we exclude all gravitinos of spin $\pm\frac{3}{2}$ and the same number of dilatinos of spin $\mp\frac{1}{2}$. Therefore, for periodic fermions, one has

$$m_0 = \frac{1}{24}\left[\left(\chi^{\mathrm{NS}}\overline{\chi}_-^{\mathrm{R}} + \chi_+^{\mathrm{R}}\overline{\chi}^{\mathrm{NS}} + \chi_-^{\mathrm{R}}\overline{\chi}_+^{\mathrm{R}}\right) - \left(\chi^{\mathrm{NS}}\overline{\chi}_+^{\mathrm{R}} + \chi_-^{\mathrm{R}}\overline{\chi}^{\mathrm{NS}} + \chi_+^{\mathrm{R}}\overline{\chi}_-^{\mathrm{R}}\right)\right]. \tag{A.8}$$

This expression can be obtained from the B-field tadpole (A.5) by exchanging χ_-^R and χ_+^R, as expected from T-duality

$$m_0 = \frac{1}{48}\left[\left(2\chi^{\mathrm{NS}} - \chi_+^{\mathrm{R}} - \chi_-^{\mathrm{R}}\right)\left(\overline{\chi}_-^{\mathrm{R}} - \overline{\chi}_+^{\mathrm{R}}\right) + \left(\chi_+^{\mathrm{R}} - \chi_-^{\mathrm{R}}\right)\left(2\overline{\chi}^{\mathrm{NS}} - \overline{\chi}_+^{\mathrm{R}} - \overline{\chi}_-^{\mathrm{R}}\right)\right]. \tag{A.9}$$

---

[22]The vacuum momentum can arise in type IIB theories, while it is usually forbidden in type IIA theories, because it violates spacetime parity. In our models, while we impose a type IIA GSO projection, the final theory in general is not invariant under spacetime parity, due to the fact that the internal CFT is not necessarily left-right symmetric; therefore, there is no reason to expect m_0 to be 0.

The values of m_0 for the theories that are relevant to our construction are listed in Table 3.

A.2 Heterotic models

In the heterotic string case, the B-field tadpole is given by [49,50]

$$w_0 = \frac{1}{24}(Z_{\tilde{R}}(\tau)E_2(\tau))_{\mathsf{q}^0}, \tag{A.10}$$

where $Z_{\tilde{R}}(\tau)$ is the partition function of the internal CFT in the $\tilde{R}$ sector, and $E_2(\tau) = 1 - 24\mathsf{q} + O(\mathsf{q}^2)$ is the Eisenstein series of weight 2.

For a holomorphically factorized theory $V_1 \otimes \bar{V}_2$, one has $Z_{\tilde{R}}(\tau) = \bar{n}_{\tilde{R}} Z_{V_1}(\tau)$, where $\bar{n}_{\tilde{R}} = \bar{\chi}_+^{\mathrm{R}} - \bar{\chi}_-^{\mathrm{R}}$, $Z_{V_1}(\tau) = N + J(\tau)$ is the partition function of the internal bosonic VOA V_1, N is the number currents in V_1, and

$$J(\tau) = \mathsf{q}^{-1} + 0 + 196884\mathsf{q} + \cdots, \tag{A.11}$$

is the modular invariant J-function with vanishing constant term. Thus, in this case, the B-field tadpole is equal to (see [55] for the case of $V_1 = V^\natural$)

$$\begin{aligned}
w_0 &= \frac{1}{24}(\bar{\chi}_+^{\mathrm{R}} - \bar{\chi}_-^{\mathrm{R}})(Z_{V_1}(\tau)E_2(\tau))_{\mathsf{q}^0} \\
&= \frac{1}{24}(\bar{\chi}_+^{\mathrm{R}} - \bar{\chi}_-^{\mathrm{R}})[(\mathsf{q}^{-1} + N + \cdots)(1 - 24\mathsf{q} + \cdots)]_{\mathsf{q}^0} \\
&= \frac{1}{24}(\bar{\chi}_+^{\mathrm{R}} - \bar{\chi}_-^{\mathrm{R}})(N - 24).
\end{aligned} \tag{A.12}$$

We summarize the result for the heterotic theories of interest to us in Table 3 in the main text. Again, the tadpole is always an integer, so that it can be canceled by a number of spacetime-filling string with suitable orientation.

Let us verify that the gravitational anomaly cancels in this case as well. Let $\bar{\chi}_+^{\mathrm{R}}$ and $\bar{\chi}_-^{\mathrm{R}}$ be the number of weight $1/2$ states in the R_+ and R_- sector of the SVOA $\bar{V}_2$, so that $\bar{n}_{\tilde{R}} = \bar{\chi}_+^{\mathrm{R}} - \bar{\chi}_-^{\mathrm{R}}$. Then, the chiral massless fermions are:

- $\bar{\chi}_+^{\mathrm{R}}$ gravitinos with spin $+3/2$ and $\bar{\chi}_-^{\mathrm{R}}$ with spin $-3/2$;
- $\bar{\chi}_+^{\mathrm{R}}$ dilatinos and $N\bar{\chi}_-^{\mathrm{R}}$ gauginos of spin $-1/2$, and $\chi_-^{\mathrm{R}} + N\bar{\chi}_+^{\mathrm{R}}$ of spin $+1/2$.

Furthermore, in heterotic string, every unit of B-field tadpole contributes $p_1/2$ to the gravitational anomaly. The total anomaly, therefore, is

$$(\bar{\chi}_+^{\mathrm{R}} - \bar{\chi}_-^{\mathrm{R}})\frac{24}{48}p_1 + N(\bar{\chi}_-^{\mathrm{R}} - \bar{\chi}_+^{\mathrm{R}})\frac{p_1}{48} + \frac{(N - 24)(\bar{\chi}_+^{\mathrm{R}} - \bar{\chi}_-^{\mathrm{R}})}{24}\frac{p_1}{2} = 0. \tag{A.13}$$

When one inserts the spacetime-filling strings to cancel the B-field tadpole, the additional degrees of freedom must be such that the gravitational anomaly still vanishes. Each spacetime-filling string contributes $\frac{c_R - c_L}{24}p_1$ to the gravitational anomaly.[23] One has to take the static gauge for the string, so that c_L and c_R

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[23] The normalization and sign can be deduced by comparison with the case of a single chiral Majorana–Weyl fermion or a chiral real scalar.

are the left- and right-moving central charges of the internal CFT. Depending on the orientation of the string, we might have either $(c_L, c_R) = (24, 12)$ if the supersymmetric sector is right-moving, or $(c_L, c_R) = (12, 24)$ in the other case. Matching the anomaly contribution of the B-field and of the spacetime-filling strings, one gets that the number of spacetime-filling strings must be $|w_0| = \frac{|(N-24)(\overline{\chi}_+^R - \overline{\chi}_-^R)|}{24}$, with the supersymmetric sector being left-moving for $w_0 > 0$ and right-moving if $w_0 < 0$.

As in the type II case, when the theory is compactified on a circle of radius R, there might be a vacuum momentum $\frac{2\pi m_0}{R}$. In this case, only the chiral gauginos contribute, and for periodic boundary conditions we obtain

$$m_0 = \frac{N(\overline{\chi}_+^R - \overline{\chi}_-^R)}{24}. \tag{A.14}$$

See Table 3 for the relevant values of the tadpole w_0 and momentum m_0 in the heterotic models we are interested in.

B Semirelative Cohomology

When combining left- and right-moving sectors of a closed string theory, the condition that the corresponding states are annihilated by b_0 and $\bar{b}_0$ is in general too strong. One need only require that the state is annihilated by the combination $b_0 - \bar{b}_0$. We can follow the treatment of [72] to discuss the relevant distinctions. We denote the total left- and right-moving BRST operator by $Q = Q_l + Q_r$. The three varieties of cohomology, all graded by ghost number N,[24] relevant to this discussion are

1. Absolute cohomology $\mathsf{H} = \oplus_N \mathsf{H}^N$ of Q-closed states $|\Psi\rangle$ modulo states of the form $Q|\Lambda\rangle$.

2. Relative cohomology $\mathsf{H}_R = \oplus_N \mathsf{H}_R^N$ of Q-closed states $|\Psi\rangle$ that satisfy $b_0|\Psi\rangle = \bar{b}_0|\Psi\rangle = 0$, modulo states of the form $Q|\Lambda\rangle$ which satisfy $b_0|\Lambda\rangle = \bar{b}_0|\Lambda\rangle = 0$.

3. Semirelative cohomology $\mathsf{H}_S = \oplus_N \mathsf{H}_S^N$ of Q-closed states $|\Psi\rangle$ that satisfy $(b_0 - \bar{b}_0)|\Psi\rangle = 0$, modulo states of the form $Q|\Lambda\rangle$ which satisfy $(b_0 - \bar{b}_0)|\Lambda\rangle = 0$.

At each total ghost number N, the relative and absolute cohomologies factorize between left- and right-movers: $\mathsf{H}^N = \oplus_{n+\bar{n}=N} H_l^n \otimes \bar{H}_r^{\bar{n}}$, but the semirelative cohomology does not. The cohomologies fit into the exact sequences

$$\cdots \mathsf{H}_R^N \xrightarrow{i'} \mathsf{H}_S^N \xrightarrow{b_0^+} \mathsf{H}_R^{N-1} \xrightarrow{\{Q, c_0^+\}} \mathsf{H}_R^{N+1} \xrightarrow{i'} \mathsf{H}_S^{N+1} \cdots \tag{B.1}$$

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[24] We suppress the additional grading by picture number and momenta.

$$\cdots \mathsf{H}_S^N \xrightarrow{i''} \mathsf{H}^N \xrightarrow{b_0^-} \mathsf{H}_S^{N-1} \xrightarrow{\{Q,c_0^-\}} \mathsf{H}_S^{N+1} \xrightarrow{i''} \mathsf{H}^{N+1} \cdots \qquad (\text{B.2})$$

where $b_0^{\pm} = b_0 \pm \bar{b}_0$, and similarly $c_0^{\pm} = \frac{1}{2}(c_0 \pm \bar{c}_0)$. The maps i', i'' are the maps that forget the relevant conditions involving the b-ghosts, and the other maps are given by multiplication by the indicated operators.

For the $k \neq 0$ states, [52] proved that for a chiral critical superstring at picture number -1 in the NS sector (to which we may always specialize canonically at nonzero momentum), the cohomology vanishes for all ghost numbers not equal to 1, so for a full (chiral and anti-chiral) superstring the only nonvanishing relative cohomology group is at ghost number 2. Then, the exact sequence collapses to isomorphisms between the following relative and semirelative cohomologies:

$$0 \to \mathsf{H}_R^2 \to \mathsf{H}_S^2 \to 0$$
$$0 \to \mathsf{H}_S^3 \to \mathsf{H}_R^2 \to 0.$$

The first map is the forgetful inclusion and the second map produces a state in H_S^3 from one in H_R^2 by acting with c_0^+, which effectively inverts b_0^+ at nonzero momentum. Similar arguments apply for the theory in the Ramond sector at the canonical picture numbers $-1/2, -3/2$. Hence, for the states of nonzero momentum, we therefore lose nothing by considering the relative cohomology, which is reassuring since the cohomology at nonzero momentum correctly reproduces the physical states one expect from a light-cone gauge calculation.

B.1 Zero-momentum States

To complete our understanding of the physical string states, we would like to understand what kind of zero-momentum states we can obtain (at any ghost number), and whether this has any interesting impact on the semirelative cohomology. For this analysis and all subsequent analyses, we will only consider the canonical pictures where all positive modes of worldsheet fields annihilate the vacuum: -1 in NS and $-1/2, -3/2$ in R, though a priori one could consider more general pictures.

We can start more simply in a chiral superstring model and identify the $k = 0$ states there. We focus on the SVOA $V_+^{f\natural}$; the generalizations to the other self-dual $c = 12$ SVOAs is trivial. These states would contribute to the relative cohomology factors. $V_+^{f\natural}$ has 24 weight $1/2$ states in the R_+ sector which we denote by u^{i+}, $i = 1, \ldots, 24$, and 0 such states in the NS, R_- sectors. Consequently, there are 2 states of ghost number 1

$$\psi_{-1/2}^{\mu} e^{-\phi} c_1 |0\rangle, \quad \mu = \pm, \qquad (\text{B.3})$$

1 state of ghost number 2,

$$\gamma_{-1/2} e^{-\phi} c_1 |0\rangle, \qquad (\text{B.4})$$

and 1 state of ghost number 0,

$$\beta_{-1/2} e^{-\phi} c_1 |0\rangle, \qquad (\text{B.5})$$

which can each be combined with their anti-holomorphic counterparts to get states in relative cohomology.

There are also zero-momentum states in the Ramond sector (chirally, from the R_+ sector), in this case at all ghost numbers. Again, we start with the states that appear in relative cohomology by writing down the chiral states of interest (which may then be combined with anti-chiral states). At picture number $-1/2$, there are 24 states of ghost number 1:

$$u^{i+}_{-1/2} e^{-\phi/2} c_1 |0, +\rangle, \tag{B.6}$$

acting on the ground state of zero momentum and positive spacetime chirality (which in this 2d model is tied to the fermion number). Additionally, we get 24 states at each positive ghost number ≥ 2:

$$(\gamma_0)^n u^{i+}_{-1/2} e^{-\phi/2} c_1 |0, +\rangle, \quad n \geq 1. \tag{B.7}$$

Correspondingly, there are states at picture number $-3/2$ for all ghost numbers $1 - m$ with $m \geq 0$:

$$(\beta_0)^m u^{i+}_{-1/2} e^{-3\phi/2} c_1 |0, -\rangle, \quad m \geq 1. \tag{B.8}$$

In the relative cohomology then, we get new NS-NS states of total ghost numbers 0, 1, 2, 3, 4 and new R-R states at picture number $-1/2$ at all ghost numbers 2 and higher, etc. Note that in the literature, the extra R sector conditions $\ker \beta_0 = 0 = \ker \bar{\beta}_0$ (the analogue of the relative cohomology conditions in the NS sector) and $G_0 = \bar{G}_0 = 0$ are usually imposed although the former conditions, like their NS sector counterpart, may be too strong in general.

B.2 Semirelative Cohomology at Zero Momentum

With the chiral zero-momentum states in hand, let us ask if we can understand the semirelative cohomology using the exact sequences. We start with the NS-NS sector at the canonical picture number, since it turns out that the picture-changing operator no longer has an inverse at zero-momentum and therefore we must treat that sector with more care.

The zeroth semirelative cohomology is isomorphic to the zeroth relative cohomology since the long exact sequence truncates due to the vanishing of negatively moded cohomology groups: $0 \to \mathsf{H}^0_R \to \mathsf{H}^0_S \to 0$. These are just one-dimensional groups generated by the vacuum (which is constructed by acting on the $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$-invariant vacuum $|0\rangle$[25] with various ghosts): $\beta_{-1/2} \bar{\beta}_{-1/2} e^{-\phi} e^{-\bar{\phi}} c_1 \bar{c}_1 |0\rangle$.

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[25]In the non-chiral setting, we use the shorthand $|0\rangle$ for $|0, 0\rangle$ when we think no confusion can arise.

Given the nonvanishing relative cohomology representatives we can construct at zero-momentum, the long exact sequence splits into the following exact sequences (plus the zeroth cohomology sequence above):

$$0 \longrightarrow \mathsf{H}^5_S \xrightarrow{\;b^+_0\;} \mathsf{H}^4_R \longrightarrow 0 \tag{B.9}$$

$$0 \longrightarrow \mathsf{H}^1_R \xrightarrow{\;i\;} \mathsf{H}^1_S \xrightarrow{\;b^+_0\;} \mathsf{H}^0_R \xrightarrow{\{Q,c^+_0\}} \mathsf{H}^2_R \xrightarrow{\;i\;} \mathsf{H}^2_S \xrightarrow{\;b^+_0\;} \mathsf{H}^1_R \xrightarrow{\{Q,c^+_0\}} \mathsf{H}^3_R \xrightarrow{\;i\;}$$

$$\mathsf{H}^3_S \xrightarrow{\;b^+_0\;} \mathsf{H}^2_R \xrightarrow{\{Q,c^+_0\}} \mathsf{H}^4_R \xrightarrow{\;i\;} \mathsf{H}^4_S \xrightarrow{\;b^+_0\;} \mathsf{H}^3_R \longrightarrow 0.$$

From this, we can calculate the rest of the semirelative cohomology groups. For convenience, let us define the shorthand $|\Omega, \bar{\Omega}\rangle := e^{-\phi} e^{-\bar{\phi}} c_1 \bar{c}_1 |0\rangle$ such that $\beta_{-1/2} \bar{\beta}_{-1/2} |\Omega, \bar{\Omega}\rangle$ generates the zeroth (semi)relative cohomology groups at picture number -1, and denote $|\Omega\rangle := e^{-\phi} c_1 |0\rangle$ (similarly for $|\bar{\Omega}\rangle$) the chiral counterparts at ghost numbers (1,0) (respectively, (0,1)).

Then, at ghost number 1 there are 4 states: $\psi^\mu_{-1/2} \bar{\beta}_{-1/2} |\Omega, \bar{\Omega}\rangle$, $\bar{\psi}^\mu_{-1/2} \beta_{-1/2} |\Omega, \bar{\Omega}\rangle$. These are identical in both relative and semirelative cohomology.

At ghost number 2, we have the representatives

$$\left\{ \psi^\mu_{-1/2} \bar{\psi}^\nu_{-1/2} |\Omega, \bar{\Omega}\rangle, \gamma_{-1/2} \bar{\beta}_{-1/2} |\Omega, \bar{\Omega}\rangle, \bar{\gamma}_{-1/2} \beta_{-1/2} |\Omega, \bar{\Omega}\rangle \right\} \in \mathsf{H}^2_R$$

$$\left\{ \psi^\mu_{-1/2} \bar{\psi}^\nu_{-1/2} |\Omega, \bar{\Omega}\rangle, (\gamma_{-1/2} \bar{\beta}_{-1/2} - \bar{\gamma}_{-1/2} \beta_{-1/2}) |\Omega, \bar{\Omega}\rangle \right\} \in \mathsf{H}^2_S$$

The second state in H^2_S is called the ghost dilaton [4] and is non-trivial only in the semirelative, but not the relative, cohomology. As mentioned in the main text, this state is excluded to construct the BKM algebra on BPS states.

It is also simple to write down representatives for the remaining relative cohomology classes at ghost numbers 3 and 4:

$$\left\{ \bar{\gamma}_{-1/2} \psi^\mu_{-1/2} |\Omega, \bar{\Omega}\rangle, \gamma_{-1/2} \bar{\psi}^\mu_{-1/2} |\Omega, \bar{\Omega}\rangle \right\} \in \mathsf{H}^3_R$$

$$\left\{ \gamma_{-1/2} \bar{\gamma}_{-1/2} |\Omega, \bar{\Omega}\rangle \right\} \in \mathsf{H}^4_R.$$

From these, we use the exact sequence to obtain the remaining semirelative cohomology classes at ghost numbers 3, 4, and 5:

$$\left\{ c^+_0 \psi^\mu_{-1/2} \bar{\psi}^\nu_{-1/2} |\Omega, \bar{\Omega}\rangle, c^+_0 (\gamma_{-1/2} \bar{\beta}_{-1/2} - \bar{\gamma}_{-1/2} \beta_{-1/2}) |\Omega, \bar{\Omega}\rangle \right\} \in \mathsf{H}^3_S$$

$$\left\{ c^+_0 \bar{\gamma}_{-1/2} \psi^\mu_{-1/2} |\Omega, \bar{\Omega}\rangle, c^+_0 \gamma_{-1/2} \bar{\psi}^\mu_{-1/2} |\Omega, \bar{\Omega}\rangle \right\} \in \mathsf{H}^4_S$$

$$\left\{ c^+_0 \gamma_{-1/2} \bar{\gamma}_{-1/2} |\Omega, \bar{\Omega}\rangle \right\} \in \mathsf{H}^5_S.$$

We focus on the ghost number 2 cohomology (or ghost number 1 for chiral superstrings) in the main text, but record these additional zero-momentum states for completeness.

To deal properly with the Ramond sector, we can largely follow the analysis of [5]. Now, for states of nonzero momentum, chiral relative cohomologies and closed string semirelative cohomologies at all picture numbers are isomorphic to one another (with a canonical isomorphism given by an invertible picture-changing operator) but this is not strictly true for the closed string semirelative cohomology at zero momentum. This is essentially because b_0^- does not commute with the picture-changing operators; it does turn out that zero-momentum RR states are inequivalent at different picture numbers (though nonzero-momentum physical states are equivalent in all pictures for all sectors). For the NS-R sectors, we have the options of being at picture numbers $(-1, -1/2), (-1, -3/2)$ which do turn out to be equivalent to one another. The computations presented in [5] essentially apply to this case as well.

For example, if we take the R_+ sector on the holomorphic side in the $-1/2$ picture and the NS sector on the anti-holomorphic side in the -1 picture the cohomology (given by listing representatives) is[26]

$$\mathsf{H}_S^0 = \left\{ \bar{\beta}_{-1/2} |-\tfrac{1}{2}, \bar{\Omega}\rangle^i \right\}$$

$$\mathsf{H}_S^1 = \left\{ \bar{\psi}^\mu_{-1/2} |-\tfrac{1}{2}, \bar{\Omega}\rangle^i, \ \bar{\beta}_{-1/2}\gamma_0 |-\tfrac{1}{2}, \bar{\Omega}\rangle^i \right\}$$

$$\mathsf{H}_S^2 = \left\{ (\bar{\gamma}_{-1/2} - \gamma_0^2 \bar{\beta}_{-1/2}) |-\tfrac{1}{2}, \bar{\Omega}\rangle^i, \ \bar{\psi}^\mu_{-1/2}\gamma_0 |-\tfrac{1}{2}, \bar{\Omega}\rangle^i \right\}$$

$$\mathsf{H}_S^3 = \left\{ \gamma_0 \left(\bar{\gamma}_{-1/2} - \gamma_0^2 \bar{\beta}_{-1/2} \right) |-\tfrac{1}{2}, \bar{\Omega}\rangle^i \right\},$$

where we have defined the state $|-\tfrac{1}{2}\rangle^i := u^{i+}_{-1/2} e^{-\phi/2} c_1 |0, +\rangle$. Isomorphic (but more complicated-looking) expressions can be obtained for the $(-3/2, -1)$ picture.

In the R-R sector, there are subtle differences among the various picture numbers, which in turn is related to distinctions between finite and infinite cohomologies arising from the possibility of allowing arbitrary number of superghost zero modes. In particular, in the mixed picture numbers, $(\gamma_0 \bar{\beta}_0)^n$, respectively $(\bar{\gamma}_0 \beta_0)^n$, do not annihilate the vacuum for arbitrary nonnegative n. In the diagonal $(-1/2, -1/2)$ and $(-3/2, -3/2)$ pictures, we do not have the possibility of such combinations. In [5], it is proven that the diagonal $-3/2$ picture cohomology is in fact isomorphic to the finite $(-3/2, -1/2)$ cohomology, while the diagonal $-1/2$ picture cohomology is isomorphic to the infinite $(-3/2, -1/2)$ cohomology. For simplicity, in the main text, we focus on the analogue of the relative cohomology condition in the R sector, namely $\ker\beta_0 = \ker\bar{\beta}_0 = 0$, as we do not believe these subtleties impact the construction of the BKM algebra.

A more refined analysis can be undertaken using the results of [5]. When $V_2 = V^{f\natural}$, for example, one finds a nonvanishing contribution in the $(-3/2, -3/2)$

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[26] Our ghost number convention differs slightly from [5].

picture, where there is a non-trivial H_S^2 generated by the state given by c_0^+ acting on the vacuum. This can be excluded from considerations of the BKM algebra action in the manner of the ghost dilaton.

C Zero-momentum R-R Sector

In this section, we describe in detail the massless R-R spectrum of type IIA superstring theory compactified on $V^{f\natural} \otimes \bar{V}^{f\natural}$. These results are needed in the calculation of the spacetime indices $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ in Sect. 5.

Both the holomorphic and anti-holomorphic SVOAs have 24 Ramond ground states, whose corresponding chiral and anti-chiral physical states we denote by v_α^i and $\bar{v}_\beta^j$ where $i, j = 1, \ldots, 24$ and $\alpha, \beta \in \{+, -\}$ are Dirac spinor indices in two-dimensional spacetime. There are two inequivalent choices for the internal fermion number in each of these SVOAs, leading to two inequivalent GSO projections (up to parity). In particular, for $V_+^{f\natural} \otimes \bar{V}_-^{f\natural}$, the spinors v_α^i and $\bar{v}_\beta^j$ have all the same spacetime spin $\alpha = \beta = +$. For $V_-^{f\natural} \otimes \bar{V}_-^{f\natural}$, the 24 holomorphic v_α^i have spin $\alpha = -$ and the 24 anti-homolorphic $\bar{v}_\beta^j$ have spin $\beta = +$.

In the R-R sector, one has $24^2 = 576$ ground states $v_\alpha^i \bar{v}_\beta^j$ where $i, j = 1, \ldots, 24$. Since these are actually Majorana–Weyl spinors, only one of these Dirac components is nonzero, for example $\bar{v}_+^j$ and either v_+^i or v_-^i. Now, we impose the physical state condition. For $k \neq 0$, BRST closedness implies that, for all $i, j = 1, \ldots, 24$, v_α^i and $\bar{v}_\beta^j$ should satisfy the massless Dirac equations

$$\begin{pmatrix} k_l^0 + k_l^1 & 0 \\ 0 & -k_l^0 + k_l^1 \end{pmatrix} \begin{pmatrix} v_+^i \\ v_-^i \end{pmatrix} = 0, \tag{C.1}$$

$$\begin{pmatrix} k_r^0 + k_r^1 & 0 \\ 0 & -k_r^0 + k_r^1 \end{pmatrix} \begin{pmatrix} \bar{v}_+^j \\ \bar{v}_-^j \end{pmatrix} = 0. \tag{C.2}$$

Since only one of the two components v_+^i and v_-^i is nonzero, the first equation either eliminates that component or it is trivially satisfied, depending on whether $k_l^0 = k_l^1$ or $k_l^0 = -k_l^1$. An analogous argument holds for the anti-holomorphic $\bar{v}_\beta^i$.

In the uncompactified theory, one has $k_l^\mu = k_r^\mu$, so that physical states exist only if the v_α^i and v_β^j have the same nonzero component, say $\alpha = \beta = +$; this corresponds to type IIA on $V_+^{f\natural} \otimes \bar{V}_-^{f\natural}$. In this case, the Dirac equations are satisfied if and only if $k^0 = -k^1$. From the spacetime point of view, this means that there are chiral massless propagating degrees of freedom of spin 1. We recall that, since the states corresponding to R-R fields obey physical state conditions that are of first order in momenta, they are naturally interpreted as quanta of field strengths rather than potentials. In the case of type IIA on $V_+^{f\natural} \otimes \bar{V}_-^{f\natural}$, this means that the fields corresponding to $v_+^i \bar{v}_+^j$ are the derivative $\partial_+ \phi^{ij}$ of 576 massless chiral scalar fields ϕ^{ij}, $i, j = 1, \ldots, 24$.

On the other hand, for type IIA on $V_-^{f\natural} \otimes \bar{V}_-^{f\natural}$, where v_α^i and v_β^j have nonzero components of opposite spin ($\alpha = -$, $\beta = +$), there cannot be any

physical propagating degrees of freedom from the massless R-R fields. In this case, the corresponding spacetime fields would be the field strengths of massless vector fields C_μ^{ij}, which must be the gauge bosons of an abelian gauge symmetry. The fact that there are no physical states is consistent with the fact that massless gauge bosons in two dimensions carry no propagating degrees of freedom.

At zero momentum, all 576 states $v_\alpha^i \bar{v}_\beta^j$ are physical. For $V_+^{f\natural} \otimes \bar{V}_-^{f\natural}$, these are just the zero modes of the scalars ϕ^{ij}. For $V_-^{f\natural} \otimes \bar{V}_-^{f\natural}$, these zero modes are the fluxes of the field strengths in the two-dimensional spacetime.

If the spatial direction is compactified on a circle S^1, for $V_+^{f\natural} \otimes \bar{V}_-^{f\natural}$ the physical states carry zero winding ($k_l^1 = k_r^1$) but possibly nonzero momentum along S^1; for the $V_-^{f\natural} \otimes \bar{V}_-^{f\natural}$, the physical states carry zero momentum ($k_l^1 = -k_r^1$) but possibly nonzero winding along the circle.

D Theta Lifts of vector-valued Modular Forms

The theta lift of a vector-valued modular form $F(\tau)$ is of the form [6]

$$\Theta_F := \int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} \left(\overline{\Theta}_\Lambda(\tau), F(\tau)\right) \mathrm{d}\mu, \tag{D.1}$$

where $\tau = \tau_1 + i\tau_2 \in \mathbb{H}$, Λ is an even lattice of signature $(2, d)$, F is a weight $k = 1 - d/2$ vector-valued modular form and $\overline{\Theta}$ is a weight $-k$ Siegel–Narain theta series for Λ. In particular, if $\Lambda^\vee$ denotes the dual lattice of Λ, then we can write F as follows:

$$F(\tau) = \sum_{\gamma \in \Lambda^\vee/\Lambda} F_\gamma(\tau) e_\gamma, \tag{D.2}$$

where the components F_γ are modular forms for a congruence subgroup $\Gamma \subseteq SL(2,\mathbb{Z})$, and F transforms in the metaplectic representation ρ_Λ of $\widetilde{SL(2,\mathbb{Z})}$ on the group ring $\mathbb{C}[\Lambda^\vee/\Lambda]$. In practice, this means that the components F_γ are in one-to-one correspondence with the cosets $\Lambda^\vee/\Lambda$ and under the action of the modular group we have

$$F_\gamma(\tau + 1) = e^{\pi i \gamma^2} F_\gamma(\tau)$$
$$F_\gamma(-1/\tau) = \frac{i^{(2-d)/2}}{\sqrt{|\Lambda/\Lambda^\vee|}} \sum_{\delta \in \Lambda/\Lambda^\vee} e^{-2\pi i (\gamma,\delta)} F_\delta(\tau). \tag{D.3}$$

The Siegel theta series is similarly written as

$$\Theta_\Lambda = \sum_{\gamma \in \Lambda^\vee/\Lambda} \theta_{\Lambda+\gamma} e_\gamma, \tag{D.4}$$

where $\theta_{\Lambda+\gamma}$ is a shifted theta series. In addition, the integrand involves the scalar product $(\, ,\,)$ defined by

$$(e_\gamma, e_{\gamma'}) = \delta_{\gamma+\gamma',0}, \qquad \gamma \in \Lambda^\vee/\Lambda. \tag{D.5}$$

Furthermore, we define complex conjugation to act by $\overline{e_\gamma} = e_{-\gamma}$ on the elements of $\mathbb{C}[\Lambda^\vee/\Lambda]$. With these definitions, the integrand $\big(\overline{\Theta}_\Lambda(\tau), F(\tau)\big)$ is $SL(2,\mathbb{Z})$-invariant, and it makes sense to integrate it on the upper half-plane $\mathbb{H}$ modulo $SL(2,\mathbb{Z})$-transformations. The theta lift produces an automorphic form on the Grassmannian $G(2,d)$ of positive definite 2-planes in $\mathbb{R}^{2,d}$, for a discrete group $O(\Lambda, F)$ (a subgroup of the group $O(\Lambda)$ of automorphisms of the lattice Λ).

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