Robust estimation of high-order phase dynamics using Variational Bayes inference

Fabio Fabozzi*, Stéphanie Bidon* and Sébastien Roche†
*ISAE-SUPAERO, Université de Toulouse, France
Email: {fabio.fabozzi,stephanie.bidon}@isae-supaero.fr
†Airbus Defence and Space SAS, Toulouse, France
Email: sebastien.roche@airbus.com

***Abstract*—Cycle slips strongly impact the performance of phase tracking algorithm leading, in the worst case, to a permanent loss of lock of the signal. In this paper, we propose a new nonlinear phase estimator to obtain more robust tracks. The latter stems from a Variational Bayes (VB) approximation used within the optimal Bayesian filtering formulation in case of high-order phase dynamics. A comparison with a more conventional technique, namely a Kalman filter based PLL (Phase Lock Loop), is performed in terms of mean square error of the phase estimate and mean time to first slip. Results show that the proposed method outperforms the conventional linear filter with respect to both metrics, especially at low signal-to-noise ratio.**

***Index Terms*—Nonlinear Bayesian filtering, Variational Bayes approximation, Phase tracking, Cycle slips**

I. Introduction

Carrier phase estimation has become a fundamental task in many various engineering applications from sonar/radar [1] to guidance/navigation [2]. Concerning the former, phase measurements are directly related to parameters such as range, bearing, and velocity which are crucial to the successful detection, tracking, and imaging of targets; about the latter, phase measurements provide high accuracy of a user/system's position (*e.g.*, in case of GNSS (Global Navigation Satellite System), carrier phase measurements offer a centimeter-level position estimation against the metric-level accuracy provided by the conventional code measurements [2]). However, phase measurements obtained by traditional phase tracking techniques may be strongly weakened by the presence of ambiguous phase jumps, known as cycle slips. The latter are unpredictable, nonlinear phenomena which makes their mathematical analysis extremely difficult. Cycle slipping especially occurs in harsh environment leading, in the worst-case, to a complete drop-lock of the signal. A reacquisition process is then necessary which afflicts the tracking performance. In the literature, the phenomenon is well studied for the conventional PLL [3]–[8]. To avoid cycle slipping, various solutions have been presented over the years. Starting from the optimized PLL architecture [9], a plethora of robust filtering techniques have been adapted to the phase tracking problem such as Kalman filter (KF) based techniques, Particle Filter (PF), Gaussian sum filter [10], [11].

The work of Fabio Fabozzi is supported by the DGA (Direction Générale de l'Armemement) under grant 2018 60 0014 and by Airbus Defence and Space.

In this paper, we propose a nonlinear filtering method based on Variational Bayes inference [12], [13]. We actually continue the work of [14], where only a first-order phase dynamics was considered. Herein, we assume any order phase dynamics so that not only the phase but its derivatives can be tracked too. Applying then the Restricted Variational Bayesian (RVB) approximation [12], we obtain a closed-form nonlinear update equation which can be practically implemented with moderate complexity and drastically decreases slipping occurrence.

The paper is organized as follows. In Section II, the state-space model is presented. Principle of the VB filtering is then introduced and applied in Section III. Numerical simulations are described in Section IV. Finally, conclusion is given in Section V.

II. State-space model

A. Measurement equation

In this study, we consider the observation

$$z_k = \alpha e^{j[\boldsymbol{x}_k]_1} + n_k \tag{1}$$

where α is a real amplitude. The measurement (1) only involves the first element of the n_x-length state vector $\boldsymbol{x}_k$ (5), namely $[\boldsymbol{x}_k]_1$ (or ϕ_k), which is the phase to be tracked by our estimator. The noise component n_k is supposed to be complex white and Gaussian with known power σ_n^2, namely at instant k $n_k \sim \mathcal{CN}(0, \sigma_n^2)$. The Signal-to-Noise-Ratio (SNR) is defined by

$$\text{SNR} = \frac{\mathcal{E}\big\{|\alpha e^{j[\boldsymbol{x}_k]_1}|^2\big\}}{\mathcal{E}\big\{|n_k|^2\big\}} = \frac{\alpha^2}{\sigma_n^2} \tag{2}$$

where $\mathcal{E}(\cdot)$ indicates the expected value operator. Given the noise term n_k and (1), the likelihood function is

$$\begin{aligned} f(z_k|\boldsymbol{x}_k) &= f(z_k|[\boldsymbol{x}_k]_1) \\ &= \frac{1}{\pi\sigma_n^2}\exp\Big\{-\frac{1}{\sigma_n^2}\Big[|z_k|^2 + \alpha^2 - 2\alpha|z_k|\cos([\boldsymbol{x}_k]_1 - \psi_k)\Big]\Big\} \end{aligned} \tag{3}$$

with $\angle z_k \overset{\text{def}}{=} \psi_k = \text{atan2}(\Im\{z_k\}, \Re\{z_k\})$ the angle that lies between $[-\pi, \pi]$. In this work, we assume known the amplitude α and the noise power σ_n^2. For conciseness reason, they are omitted in the conditional terms. Accordingly, the sensor factor associated with (3) is

$$s([\boldsymbol{x}_k]_1) \propto f(z_k|\boldsymbol{x}_k) \propto \exp\Big\{\beta_k \cos([\boldsymbol{x}_k]_1 - \psi_k)\Big\} \tag{4}$$

where $\beta_k = {}^{2\alpha|z_k|}/_{\sigma_n^2}$. In (4), we recognize a Tikhonov distribution (or Von Mises distribution) with mean direction ψ_k and concentration parameter β_k. Its probability density function (pdf) is denoted $T(\phi_k|\psi_k, \beta_k)$ [15].

B. Dynamics equation

The time-varying evolution of the assumed high-order phase dynamics is described through the so-called transition equation

$$\boldsymbol{x}_k = \begin{bmatrix} \phi_k \\ \dot{\phi}_k \\ \vdots \\ \overset{n}{\dot{\phi}}_k \end{bmatrix} = \boldsymbol{A}\boldsymbol{x}_{k-1} + \boldsymbol{w}_{k-1} \tag{5}$$

where $\overset{n}{\dot{\phi}}_k$ is the nth derivative of the phase and $\boldsymbol{w_k}$ represents a centered Gaussian noise with a covariance matrix $\boldsymbol{Q}$, *viz.* $\boldsymbol{w}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q})$. The so-called state-transition matrix $\boldsymbol{A}$ is with appropriate dimension. The transitional density is thus

$$f(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}) = (2\pi)^{(-n_x/2)}|\boldsymbol{C}|^{1/2} \exp(-1/2((\boldsymbol{x}_k - \boldsymbol{A}\boldsymbol{x}_{k-1})^\intercal \boldsymbol{C}(\boldsymbol{x}_k - \boldsymbol{A}\boldsymbol{x}_{k-1})) \tag{6}$$

where $\boldsymbol{C}$ is the precision matrix of $\boldsymbol{x}_k|\boldsymbol{x}_{k-1}$ given by

$$\boldsymbol{C} = \boldsymbol{Q}^{-1}. \tag{7}$$

III. Variational Bayes tracking algorithm

Herein, we introduce the principle of the RVB approximation and apply it to high-order phase dynamics. The methodology was originally described in [12] and applied for first-order phase dynamics only in [14], [16].

A. Optimal filtering problem

The optimal Bayes filtering that iteratively evaluates the filtering distribution $f(\boldsymbol{x}_k|\boldsymbol{Z}_k)$ is obtained by alternating between (8) and (9) as follows [17]

$$f(\boldsymbol{x}_k|\boldsymbol{Z}_{k-1}) = f(\boldsymbol{x}_1), \qquad k = 1 \tag{8a}$$

$$f(\boldsymbol{x}_k|\boldsymbol{Z}_{k-1}) = \int f(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}) f(\boldsymbol{x}_{k-1}|\boldsymbol{Z}_{k-1}) d\boldsymbol{x}_{k-1}, k > 1 \tag{8b}$$

and

$$f(\boldsymbol{x}_k|\boldsymbol{Z}_k) \propto f(\boldsymbol{z}_k|\boldsymbol{x}_k) f(\boldsymbol{x}_k|\boldsymbol{Z}_{k-1}), \qquad k > 1 \tag{9}$$

where $f(\boldsymbol{x}_1)$ is the prior distribution at k=1 and $\boldsymbol{Z}_k = [z_1, \ldots, z_k]$ denotes the aggregated set of observations till the instant k. However, given the likehood function (3) and the transitional density (6), the recursive propagation (9) cannot be analytically determined. Though, using the RVB approximation [12], we can consider a fixed functional form for the filtering distribution, *viz* $f(\boldsymbol{x}_k|\boldsymbol{Z}_k) \sim \tilde{f}(\boldsymbol{x}_k|\boldsymbol{Z}_k)$. In our case, we can show as presented after that it allows us to obtain a tractable Bayes filter.

B. Principle of the RVB approximation

The RVB-based method relies on a twofold approximation of the filtering problem. Following [12], in the first stage a local Variational Bayes (VB) approximation is made, in particular a *conditional independence* between $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k-1}$ is assumed

$$\tilde{f}(\boldsymbol{x}_k, \boldsymbol{x}_{k-1}|\boldsymbol{Z}_k) = \tilde{f}(\boldsymbol{x}_k|\boldsymbol{Z}_k)\tilde{f}(\boldsymbol{x}_{k-1}|\boldsymbol{Z}_k) \tag{10}$$

where $\tilde{f}(\cdot)$ refers to the approximated posterior distribution. The latter is then chosen to minimize the Kullback-Leibler (KL) divergence [12]. This approximation leads to recursively evaluate the posterior distribution $f(\boldsymbol{x}_k|\boldsymbol{Z}_k)$ as follows

- Prediction and data update for $k = 1$

$$\tilde{f}(\boldsymbol{x}_k|\boldsymbol{Z}_{k-1}) \propto \exp\Big(\mathrm{E}_{\tilde{f}(\boldsymbol{x}_{k-1}|\boldsymbol{Z}_k)}\Big[\ln(f(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}))\Big]\Big) \tag{11}$$

with

$$\tilde{f}(\boldsymbol{x}_{k-1}|\boldsymbol{Z}_k) \propto \exp\Big(\mathrm{E}_{\tilde{f}(\boldsymbol{x}_k|\boldsymbol{Z}_k)}\Big[\ln(f(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}))\Big]\Big) \times \tilde{f}(\boldsymbol{x}_{k-1}|\boldsymbol{Z}_{k-1}).$$

- Prediction and data update for $k > 1$

$$\tilde{f}(\boldsymbol{x}_k|\boldsymbol{Z}_k) \propto f(\boldsymbol{z}_k|\boldsymbol{x}_k)\tilde{f}(\boldsymbol{x}_k|\boldsymbol{Z}_{k-1}). \tag{12}$$

The operator $\mathrm{E}_{f(\boldsymbol{x})}\Big[g(\boldsymbol{x})\Big]$ denotes the expected value of the function $g(\boldsymbol{x})$ with respect to the density function $f(\boldsymbol{x})$.

To obtain a closed-form filter, a second approximation is made. In particular, through *RVB approximation* the distribution $\tilde{f}(\boldsymbol{x}_{k-1}|\boldsymbol{Z}_k)$ in (11) is replaced by the fixed posterior distribution $\tilde{f}(\boldsymbol{x}_{k-1}|\boldsymbol{Z}_{k-1})$. The prediction distribution becomes then

$$\tilde{f}(\boldsymbol{x}_k|\boldsymbol{Z}_{k-1}) \propto \exp\Big(\mathrm{E}_{\tilde{f}(\boldsymbol{x}_{k-1}|\boldsymbol{Z}_{k-1})}\Big[\ln(f(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}))\Big]\Big). \tag{13}$$

Using the sensor factor (4) and the transitional density (6), we can show that RVB filtering has the following closed-form expressions

- Prediction and data update for $k = 1$

$$\tilde{f}(\boldsymbol{x}_1|\boldsymbol{Z_0}) \overset{\text{def}}{=} f(\boldsymbol{x}_1) \tag{14a}$$

$$\tilde{f}(\boldsymbol{x}_1|\boldsymbol{Z_1}) \propto T([\boldsymbol{x}_1]_1|\psi_1, \beta_1) \times f(\boldsymbol{x}_1). \tag{14b}$$

- Prediction and data update for $k > 1$

$$\tilde{f}(\boldsymbol{x}_k|\boldsymbol{Z}_{k-1}) = \mathrm{N}\Big(\boldsymbol{x}_k|\boldsymbol{A}\mathrm{E}_{\tilde{f}(\boldsymbol{x}_{k-1}|\boldsymbol{Z}_{k-1})}[\boldsymbol{x}_{k-1}], \boldsymbol{C}^{-1}\Big) \tag{15a}$$

$$\tilde{f}(\boldsymbol{x}_k|\boldsymbol{Z}_k) \propto T([\boldsymbol{x}_1]_1|\psi_k, \beta_k) \times \tilde{f}(\boldsymbol{x}_k|\boldsymbol{Z}_{k-1}) \tag{15b}$$

where $\mathrm{N}(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Omega})$ is the multivariate normal pdf with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Omega}$.

(Demonstrations are not given here but will be detailed in a future work.)

The functional form of the filtering distribution is stable through the iteration (14)-(15) and is tractable if one can derive the MMSE (Minimum Mean Square Error) estimator with respect to the approximated RVB distribution (15b) that we denote as

$$\hat{\boldsymbol{x}}_k^{\text{rvb}} \overset{\text{def}}{=} \mathrm{E}_{\tilde{f}(\boldsymbol{x}_k|\boldsymbol{Z}_k)}[\boldsymbol{x}_k] = \int \boldsymbol{x}_k \tilde{f}(\boldsymbol{x}_k|\boldsymbol{Z}_k) d\boldsymbol{x}_k. \tag{16}$$

C. RVB estimator

1) *Initialization*: We propose a factorized form to describe the initial condition: $f(\boldsymbol{x}_1) = f(x_{1,1})f(\boldsymbol{x}_{1_{-1}})$ where the notation $\boldsymbol{x}_{-1}$ designates the (n_x-1)-length vector $\boldsymbol{x}$ to which the first element has been removed. In that case, using (16) with (14b) the RVB estimator at $k = 1$ is

$$\hat{\boldsymbol{x}}_1^{\text{rvb}} \stackrel{\text{def}}{=} \begin{bmatrix} \hat{x}_{1,1}^{\text{rvb}} \\ \hat{\boldsymbol{x}}_{1_{-1}}^{\text{rvb}} \end{bmatrix} \tag{17}$$

where

$$\hat{x}_{1,1}^{\text{rvb}} = \frac{\int x_{1,1} T(x_{1,1}|\psi_1, \beta_1) f(x_{1,1}) dx_{1,1}}{\int T(x_{1,1}|\psi_1, \beta_1) f(x_{1,1}) dx_{1,1}} \tag{18a}$$

$$\hat{\boldsymbol{x}}_{1_{-1}}^{\text{rvb}} = \int \boldsymbol{x}_{1_{-1}} f(\boldsymbol{x}_{1_{-1}}) d\boldsymbol{x}_{1_{-1}}. \tag{18b}$$

Without any knowledge of the initial phase and its derivatives, we can choose as an example a uniform distribution for the prior of $\boldsymbol{x}_1$, *i.e.*

$$f(x_{1,1}) \stackrel{\text{def}}{=} f(\phi_1) \propto \mathbb{I}_{[-\pi,\pi]}(\phi_1) \tag{19a}$$

$$f(\boldsymbol{x}_{-1}) \propto \mathbb{I}_{\mathcal{I}}(\boldsymbol{x}_{-1}) \tag{19b}$$

where $\mathcal{I}$ denotes a set of symmetrical intervals. Using (19a) and (18a), the expression of $\hat{x}_{1,1}^{\text{rvb}}$ (or ϕ_1) turns out to be the same as that in [14]. Using (19b) and (18b), we obtain then $\hat{\boldsymbol{x}}_{1_{-1}}^{\text{rvb}} = \boldsymbol{0}$.

2) *Recursion*: The recursive equation of the estimator for $k > 1$ is derived substituting the posterior distribution (15b) into (16). Its expression is shown in (20) with

$$P_1 \stackrel{\text{def}}{=} [C_{1,1} - \boldsymbol{c}_{-1}^{\mathsf{T}} \boldsymbol{C}_{-1}^{-1} \boldsymbol{c}_{-1}]^{-1} \tag{21}$$

where

- $[\boldsymbol{A}\hat{\boldsymbol{x}}_{k-1}^{\text{rvb}}]_1$ is the first element of $\boldsymbol{A}\hat{\boldsymbol{x}}_{k-1}^{\text{rvb}}$;
- $I_q(\cdot)$'s are the modified Bessel functions of the first kind at qth order;
- $C_{1,1}$ is the first diagonal element of $\boldsymbol{C}$;
- $\boldsymbol{C}_{-1}$ is the matrix $\boldsymbol{C}$ to which the first column and row have been removed;
- $\boldsymbol{c}_{-1}$ is the first column of $\boldsymbol{C}$ where the first element has been removed.

As can be seen, the expression (20) maintains the nonlinear nature of the measurement equation (1). Moreover, we can see a similarity between (20) and the traditional KF update state expression. As a matter of fact, (20) is the sum of the predicted state estimate $\boldsymbol{A}\hat{\boldsymbol{x}}_{k-1}^{\text{rvb}}$ plus a function of the innovation term $\psi_k - [\boldsymbol{A}\hat{\boldsymbol{x}}_{k-1}^{\text{rvb}}]_1$ that is nonlinear (unlike the Kalman gain function). Note that the vector $\left[1; -\boldsymbol{C}_{-1}^{-1}\boldsymbol{c}_{-1}\right]^{\mathsf{T}}$ remains constant over the time. Moreover, since the Bessel function decreases rapidly with respect to (wrt) the summation index q, a truncation of the infinite sum in (20) to $q_{\max}$ is enforced. Finally, it is worth noticing that at first-order, namely $n_x = 1$, then the filtering of [14] is recovered.

IV. Numerical simulations

In what follows, performance of the proposed RVB estimator (20) is assessed numerically on synthetic data and compared to a KF-based PLL [18].

A. Scenario

In this study, the received signal is generated as in (1). The phase dynamics follows a polynomial evolution as

$$\phi_k = \phi_0 + \dot{\phi}_0 T k + \frac{\ddot{\phi}_0 T^2 k^2}{2} \tag{22}$$

where ϕ_0, $\dot{\phi}_0$, $\ddot{\phi}_0$ are respectively the initial phase (step), phase rate (ramp), acceleration (parabola) and T is the tracking update time. Accordingly, we choose a third-order phase model known as the PVA (Position, Velocity and Acceleration) model [19]–[21]. The state matrix $\boldsymbol{A}$ is thus equal to

$$\boldsymbol{A} = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

and the process noise covariance $\boldsymbol{Q}$ is [19], [22]

$$\begin{aligned} \boldsymbol{Q} = \Sigma_{\text{pva}} &\begin{bmatrix} T^5/20 & T^4/8 & T^3/6 \\ T^4/8 & T^3/3 & T^2/2 \\ T^3/6 & T^2/2 & T \end{bmatrix} \\ + \Sigma_{\text{pv}} &\begin{bmatrix} T^3/3 & T^2/2 & 0 \\ T^2/2 & T & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Sigma_{\text{p}} \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{24}$$

where the notation Σ represents the power spectral density (PSD) of the continuous-time white noise [20].

B. Processing parameters

The input parameters of the received signal and of both algorithms are presented in Tab. I. Note that tuning the PSDs of the process noise covariance matrices $\boldsymbol{Q}$ (24) is not simple. Particularly for the RVB estimator, its update equation (20) is not only nonlinear but also depends only partially on the covariance matrix $\boldsymbol{Q}$, via the terms P_1 and $\left[1; -\boldsymbol{C}_{-1}^{-1}\boldsymbol{c}_{-1}\right]^{\mathsf{T}}$. So far, we have thus experimentally set its PSD values so that both KF-PLL and RVB filters result in a non-informative estimator below approximately the same SNR value (see later Fig. 1 when so-called RMSE-mod is approximately equal to $\pi/\sqrt{3}$).

1) RVB initialization: A value of $q_{\max} = 50$ is chosen which leads to, in a first attempt, a good balance between computational cost and accuracy of estimation.

2) KF-based PLL initialization: To have a fair comparison, an ATAN2 discriminator is used for the KF-based PLL. Accordingly, the phase noise power is approximated as in [22, A-13].

C. Phase tracking

1) Performance metrics: We focus on our main parameter of interest, that is the phase estimate $[\boldsymbol{x}_k]_1$. Given the nonlinear nature of phase tracking, two metrics are chosen to characterize on the one hand the precision of estimation letting alone the slip phenomenon and on the other hand the occurrence of slips. More precisely, the first metrics is the Root Mean Square Error of the phase error modulo-2π (denoted as RMSE-mod) which is defined as $\tilde{e}_k = (\phi_k - \hat{\phi}_k)_{[-\pi,\pi]}$. In our simulations, a steady-state is always reached after a

$$\hat{\boldsymbol{x}}_k^{\text{rvb}} = \boldsymbol{A}\hat{\boldsymbol{x}}_{k-1}^{\text{rvb}} + 2P_1 \frac{\sum_{q=1}^{+\infty} q I_q(\beta_k) \sin[q(\psi_k - [\boldsymbol{A}\hat{\boldsymbol{x}}_{k-1}^{\text{rvb}}]_1)] e^{-\frac{q^2 P_1}{2}}}{I_0(\beta_k) + 2\sum_{q=1}^{+\infty} I_q(\beta_k) \cos[q(\psi_k - [\boldsymbol{A}\hat{\boldsymbol{x}}_{k-1}^{\text{rvb}}]_1)] e^{-\frac{q^2 P_1}{2}}} \begin{bmatrix} 1 \\ -\boldsymbol{C}_{-1}^{-1}\boldsymbol{c}_{-1} \end{bmatrix} \tag{20}$$

TABLE I: Input parameters

Parameter	Variable	Value
Phase	ϕ_0	0 rad
Phase rate	$\dot{\phi}_0$	0 rad/s
Phase acceleration	$\ddot{\phi}_0$	$\pi/0.16$ rad/s^2
Monte Carlo realizations	M_c	1000
Tracking update time	T	0.02 s
RVB PSDs	$\sqrt{\Sigma_{\text{p}} T}$	0.8π rad
	$\sqrt{\Sigma_{\text{pv}} T}$	20π rad/s
	$\sqrt{\Sigma_{\text{pva}} T}$	100π rad/s^2
KF-PLL PSDs	$\sqrt{\Sigma_{\text{p}} T}$	0.2π rad
	$\sqrt{\Sigma_{\text{pv}} T}$	0.8π rad/s
	$\sqrt{\Sigma_{\text{pva}} T}$	0.2π rad/s^2

certain transient time. We depict only the value of RMSE at steady-state. The second metrics of interest is the Mean Time to First cycle Slip (MTFS). We assess it as described in [14]. Both metrics are evaluated via Monte Carlo simulations and presented wrt the SNR (2).

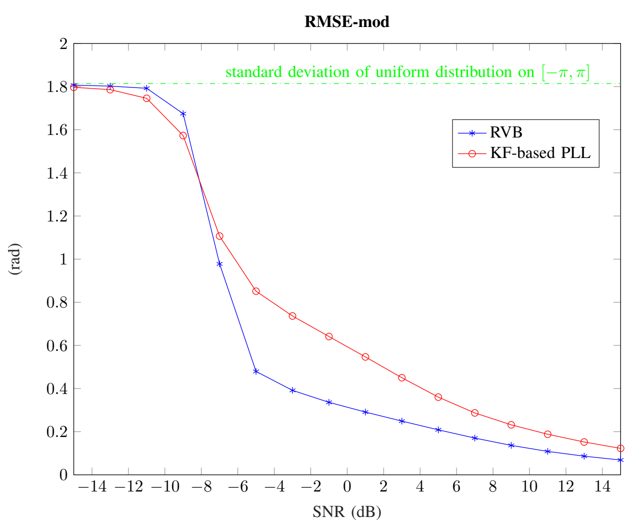

Fig. 1: RMSE-mod

2) Results: As can be seen in Figs. 1 and 2, the RVB estimator outperforms the KF estimator wrt both metrics. In particular, in Fig. 1 the RMSE-mod of RVB appears to be lower for most of the SNR values. Below a SNR of ≈ -10 dB, both phase estimators can be considered as noninformative since their RMSE-mod is equal to that of a uniform distribution on [-π,π]. Above, the precision of estimation of both algorithms increases wrt the SNR. Though, the RVB outperforms here clearly the KF-PLL particularly at low to medium SNR (i.e., approximately from -6 to 8 dB). The MTFS is shown in Fig. 2. In practice, a time limit is set to observe the occurrence of a slip. It is fixed to half-a-day. As can be seen, the RVB has tremendously higher MTFS at low to medium SNRs indicating a low probability of slip occurrence compared to the KF-PLL. Note that at high SNR, RMSE-mods and MTFS of both algorithms are comparable. As a matter of fact, they both act as a linear filter in absence of cycle slips.

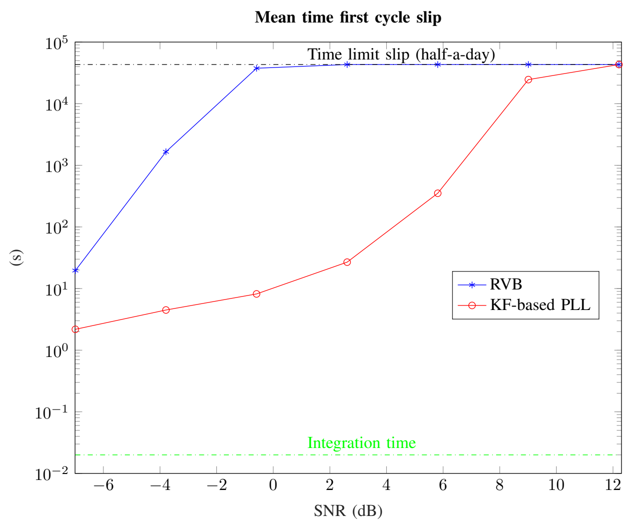

Fig. 2: MTFS

V. Conclusion

In this paper we proposed a robustified phase tracking technique to avoid cycle slipping at any order dynamics. The method is based on a RVB approximation which leads to a nonlinear and tractable filter with closed-form expression and moderate computational complexity. Performance of the associated MMSE estimator is evaluated in terms of RMSE-mod and MTFS and compared with a conventional KF-based PLL. The new nonlinear filter shows significant performance improvement with low cycle slip occurrence.

In future work, we intend to implement our algorithm in a more practical scenario, namely in a software-based receiver using real GNSS data. Compared to [14], a wider range of scenarios will be addressed since our new algorithm is suited for high-order dynamics.

REFERENCES

[1] M. Richards, W. Melvin, J. Scheer, J. Scheer, and W. Holm, *Principles of Modern Radar: Radar Applications, Volume 3*, ser. Electromagnetics and Radar. Institution of Engineering and Technology, 2013.

[2] E. Kaplan and C. Hegarty, *Understanding GPS: principles and applications*. Artech house, 2005.

[3] G. Ascheid and H. Meyr, "Cycle slips in phase-locked loops: A tutorial survey," *IEEE Transactions on Communications*, vol. 30, no. 10, pp. 2228–2241, October 1982.

[4] B. Hofmann-Wellenhof, H. Lichtenegger, and J. Collins, *Global positioning system: theory and practice*. Springer Science & Business Media, 2012.

[5] F. J. Charles and W. C. Lindsey, "Some analytical and experimental phase-locked loop results for low signal-to-noise ratios," *Proceedings of the IEEE*, vol. 54, no. 9, pp. 1152–1166, Sep. 1966.

[6] A. J. Viterbi, "Phase-locked loop dynamics in the presence of noise by Fokker-Planck techniques," in *The Foundations Of The Digital Wireless World: Selected Works of AJ Viterbi*. World Scientific, 2010, pp. 13–29.

[7] W. C. Lindsey and C. M. Chie, "A survey of digital phase-locked loops," *Proceedings of the IEEE*, vol. 69, no. 4, pp. 410–431, April 1981.

[8] S. Brunt, M. Darnell, and M. Grayson, "Cycle-slipping probability of first-order phase-locked loop using transient analysis," *IEE Proceedings-Communications*, vol. 144, no. 5, pp. 357–360, 1997.

[9] R. Jaffe and E. Rechtin, "Design and performance of phase-lock circuits capable of near-optimum performance over a wide range of input signal and noise levels," *IRE Transactions on Information Theory*, vol. 1, no. 1, pp. 66–76, 1955.

[10] J. A. Lopez-Salcedo, J. A. D. Peral-Rosado, and G. Seco-Granados, "Survey on robust carrier tracking techniques," *IEEE Communications Surveys Tutorials*, vol. 16, no. 2, pp. 670–688, Second 2014.

[11] P. Tam and J. Moore, "A gaussian sum approach to phase and frequency estimation," *IEEE Transactions on Communications*, vol. 25, no. 9, pp. 935–942, 1977.

[12] V. Šmídl and A. Quinn, "Variational Bayesian filtering," *IEEE Transactions on Signal Processing*, vol. 56, no. 10, pp. 5020–5030, 2008.

[13] V. Šmídl and A. Quinn, *The variational Bayes method in signal processing*. Springer Science & Business Media, 2006.

[14] F. Fabozzi, S. Bidon, S. Roche, and B. Priot, "Robust GNSS phase tracking in case of slow dynamics using Variational Bayes inference," in *2020 IEEE/ION Position, Location and Navigation Symposium (PLANS)*, 2020, pp. 1189–1195.

[15] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. Courier Corporation, 1965, vol. 55.

[16] S. Bidon and S. Roche, "Variational Bayes phase tracking for correlated dual-frequency measurements with slow dynamics," *Signal Processing*, vol. 113, pp. 182–194, 2015.

[17] V. Peterka, "Bayesian approach to system identification," in *Trends and Progress in System identification*. Elsevier, 1981, pp. 239–304.

[18] J. A. Del Peral-Rosado, J. A. López-Salcedo, G. Seco-Granados, J. M. López-Almansa, and J. Cosmen, "Kalman filter-based architecture for robust and high-sensitivity tracking in GNSS receivers," in *2010 5th ESA Workshop on Satellite Navigation Technologies and European Workshop on GNSS Signals and Signal Processing (NAVITEC)*, Dec 2010, pp. 1–8.

[19] R. G. Brown, P. Y. Hwang *et al.*, *Introduction to random signals and applied Kalman filtering*. Wiley New York, 1992, vol. 3.

[20] X. R. Li and V. P. Jilkov, "Survey of maneuvering target tracking: dynamic models," in *Signal and Data Processing of Small Targets 2000*, vol. 4048. International Society for Optics and Photonics, 2000, pp. 212–235.

[21] R. A. Singer, "Estimating optimal tracking filter performance for manned maneuvering targets," *IEEE Transactions on Aerospace and Electronic Systems*, vol. AES-6, no. 4, pp. 473–483, July 1970.

[22] C. Gernot, *Development of combined GPS L1/L2C acquisition and tracking methods for weak signals environments*. University of Calgary, Department of Geomatics Engineering, 2009.