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INSIDE-OUTSIDE DUALITY FOR MODIFIED TRANSMISSION EIGENVALUES

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Abstract. We introduce a new modified spectrum associated with the scattering from penetrable objects using the modified background technique. We prove that the inside-outside duality method allows to reconstruct this spectrum from full aperture far field measurements at a fixed frequency. The method is numerically tested and validated on some synthetic examples.

1. **Introduction.**

Our objective in this work is to analyze the inside-outside duality method for reconstructing some carefully designed spectra from full aperture far field measurements at a fixed frequency. The inside-outside duality method has been introduced in [10] for identifying Dirichlet eigenvalues from far field measurements. It has been extended to inhomogeneous media in [16] to reconstruct so called transmission eigenvalues [5] but under the assumption that the refractive index is a sufficiently large constant (or small perturbation of a large constant). In order to remove this restriction, the authors in [3] proposed to modify the background in a region containing the inhomogeneity so that it has a refractive index that scales proportionally to $1/k^2$ where k is the wave number. We propose in this work to follow this approach but keep the wave number fixed and define as spectral parameter a constant ρ such that the index of refraction in ρ/k^2 inside the inhomogeneous part of the modified background. We refer to these spectral values as modified transmission eigenvalues. Indeed, in practice, this option is more interesting as it requires only measurements of the far field operator at a fixed wave number. Other modifications of the background have been proposed in the literature [2, 4, 7, 6, 8, 9]. However, these modifications would lead to the same difficulty as for homogeneous backgrounds for the inside-outside duality method. In those cases, the use of approaches based on the generalized sampling method would be a possibility [2]. The latter is numerically more expensive.

Following a similar theoretical approach as in [3] we first provide a necessary and sufficient condition on the eigenvalues of a modified far field operator so that ρ is a modified transmission eigenvalue. This result is a consequence of the asymptotic expansion of the solution of the scattering problem at a modified transmission

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eigenvalue when the incident field coincides with a an eigenfunction. We also prove that an approximation of this eigenfunction can be computed.

We provide some numerical validating results in a two dimensional setting of the problem. We first test the approach for circular domains where the modified transmission eigenvalues can be computed as the root of some analytic determinant. For non circular domains we compute reference values of the modified transmission eigenvalues by solving the spectral problem using a finite element solver. We then validate the inside-outside duality method against these numerical values. We show that the method is quite effective and has relative robustness with respect to small noise in the data if the domain is convex. The accuracy may be deteriorated in the case of non convex domains. Exploiting these values for solving some inverse problems is a prospect of the current work. We refer to [1] for a possible use of modified transmission eigenvalues in some non destructive testing applications.

The paper is organized as follows. In Section 2 we introduce the notion of modified transmission eigenvalues and review some known results on the modified far field operator and associated factorization. Section 3 contains the statements and the proofs of the main theoretical results of this paper, namely necessary and sufficient conditions for ρ to be an eigenvalue in terms of the modified far field operator. Section 4 is dedicated to the numerical validation of the method.

2. Definition of Modified Transmission Eigenvalues and Some Classical Related Results.

2.1. Definition of Modified Transmission Eigenvalues.

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected complement and let the refractive index $n \in \mathrm{L}^\infty(\mathbb{R})$ be a real valued function such that $n = 1$ outside Ω. Consider an incident wave u^i that satisfies the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^3$ at the frequency $k > 0$. The scattering problem can be formulated as: the total field $u = u^i + u^s$ and is such that

$$\left| \begin{array}{l} \Delta u + k^2 n u = 0 \text{ in } \mathbb{R}^3, \\ \lim\limits_{r\to+\infty} \displaystyle\int_{|x|=r} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 ds(x) = 0. \end{array} \right. \tag{1}$$

The last condition is referred to later as the Sommerfeld radiation condition. This problem has a unique solution $u \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$ and the scattered field u_s has the expansion

$$u^s(r\hat{x}) = \frac{e^{ikr}}{r}\Big(u^\infty(\hat{x}) + O(1/r) \Big), \tag{2}$$

as $r = |x| \to +\infty$, uniformly in $\hat{x} \in \mathbb{S}^2 := \{\theta \in \mathbb{R}^3;\ |\theta| = 1\}$ (see for instance [5]). The function $u^\infty : \mathbb{S}^2 \to \mathbb{C}$ is the so-called far field pattern of u^s. When u^i coincides with the incident plane wave $u^i(\cdot, d) := e^{ikd\cdot x}$, with $d \in \mathbb{S}^2$, we denote respectively by $u^s(\cdot, d)$ and $u^\infty(\cdot, d)$ the corresponding scattered field and far field pattern. We define the far field operator $F : \mathrm{L}^2(\mathbb{S}^2) \to \mathrm{L}^2(\mathbb{S}^2)$ such that

$$(Fg)(\hat{x}) = \int_{\mathbb{S}^2} g(d)\, u^\infty(\hat{x}, d)\, ds(d). \tag{3}$$

Remark 2.1. *The analysis of the inside outside duality heavily relies of the normality of the far field operator. This is mainly why we imposed that n is real valued. Our analysis/procedure can be extended to the anisotropic cases as long as the coefficients are also assumed to be real valued.*

If we define for $v \in \mathrm{L}^2(\Omega)$ a scattered field $w \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$ that satisfies $\Delta w + k^2 n w = k^2(1-n)v$ in $\mathbb{R}^3$ and the Sommerfeld radiation condition, then by linearity of the scatting problem with respect to the incident wave, we obtain $Fg = w^\infty$, the far field pattern associated with w the solution for $v = v_g$ in Ω where

$$v_g(x) := \int_{\mathbb{S}^2} g(d) e^{ikd\cdot x}\, ds(d). \tag{4}$$

It is known, that the set $\{v_g|_\Omega,\, g \in \mathrm{L}^2(\mathbb{S}^2)\}$ is dense in $\{\varphi \in \mathrm{L}^2(\Omega);\, \Delta\varphi + k^2\varphi = 0 \text{ in } \Omega\}$. Moreover, the scattering operator $S : \mathrm{L}^2(\mathbb{S}^2) \to \mathrm{L}^2(\mathbb{S}^2)$ defined by

$$S := \mathrm{Id} + \frac{ik}{2\pi} F \tag{5}$$

is unitary and the operator F is normal (see for instance [5, 15, 9]).
We define the classical Transmission Eigenvalues (TEs) as the values of $k > 0$ for which there exists a non trivial incident field $v \in \mathrm{L}^2(\Omega)$ for which $w = 0$ outside Ω. This leads to the equivalent definition of TEs as the values of $k \in \mathbb{R}^*_+ := (0; +\infty)$ for which the problem

$$\left|\begin{array}{rcll} \Delta w + k^2 n w &=& k^2(1-n)v & \text{in } \Omega \\ \Delta v + k^2 v &=& 0 & \text{in } \Omega \end{array}\right. \tag{6}$$

admits a non trivial solutions $(v, w) \in \mathrm{L}^2(\Omega) \times \mathrm{H}^2_0(\Omega)$.

This definition of TEs uses as a reference media (or background) the vacuum. Similarly one can define other transmission eigenvalues by changing this reference media. Let $n_b \in \mathrm{L}^\infty(\mathbb{R}^3)$ a given real valued function such that $n_b = 1$ in $\mathbb{R}^3 \setminus \overline{\Omega_b}$ where $\Omega_b \supset \Omega$ is a bounded Lipschitz domain with connected complement. We denote by $u_b \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$ the total field associated with scattering of the incident plane wave u^i by the inhomogeneous medium defined by the refractive index n_b. This field satisfies, $u_b = u^i + u^s_b$ and

$$\Delta u_b + k^2 n_b u_b = 0 \text{ in } \mathbb{R}^3 \tag{7}$$

and u^s_b satisfies the Sommerfeld radiation condition. We denote by $u_b(\cdot, d)$ and $u^s_b(\cdot, d)$ the total and scattered fields associated with $u^i = u^i(\cdot, d)$. We define the background far field operator $F_b : \mathrm{L}^2(\mathbb{S}^2) \to \mathrm{L}^2(\mathbb{S}^2)$ by

$$(F_b g)(\hat{x}) := \int_{\mathbb{S}^2} g(d)\, u^\infty_b(\hat{x}, d)\, ds(d),$$

where $u^\infty_b(\cdot, d)$ is the far field pattern associated with $u^s_b(\cdot, d)$. Similarly to (5), we define the background scattering operator $S_b : \mathrm{L}^2(\mathbb{S}^2) \to \mathrm{L}^2(\mathbb{S}^2)$ by

$$S_b := \mathrm{Id} + \frac{ik}{2\pi} F_b. \tag{8}$$

We then define the modified far field operator $F_m : \mathrm{L}^2(\mathbb{S}^2) \to \mathrm{L}^2(\mathbb{S}^2)$ as

$$F_m := F - F_b. \tag{9}$$

Inspired by [3], we make the following specific choice of the values of n_b inside Ω_b, where $\rho \in \mathbb{R}$ will play the role of the spectral parameter

$$n_b := \rho / k^2 \text{ in } \Omega_b. \tag{10}$$

From the definition of F and F_b, we observe that $F_m g = w^\infty$ where w^∞ is the far field pattern associated with $w \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$ satisfying

$$\left| \begin{array}{l} \Delta w + k^2 n w = k^2(n_b - n)v \text{ in } \mathbb{R}^3, \\ \displaystyle\lim_{r\to+\infty} \int_{|x|=r} \left| \frac{\partial w}{\partial r} - ikw \right|^2 ds(x) = 0, \end{array} \right. \tag{11}$$

with $v = u_b$ solution of problem (7) with u^i replaced by $v = v_g$. We notice for later use that we have, by the well posedness of the scattering problem associated with n,

$$\|w\|_{\mathrm{H}^2(\Omega_b)} \leq C \|v\|_{\mathrm{L}^2(\Omega_b)} \tag{12}$$

where $C > 0$ is a constant independent of $v \in \mathrm{L}^2(\Omega_b)$ and independent of ρ in a compact set of $\mathbb{R}$. We also recall that the far field w^∞ associated with this w can be expressed as

$$w^\infty(\hat{x}) = \frac{1}{4\pi} \int_{\Omega_b} (k^2(n_b - n)v + k^2(1-n)w)(y) e^{-ik\hat{x}\cdot y} ds(y) \quad \forall \hat{x} \in \mathbb{S}^2 \tag{13}$$

We are now in position to define the modified transmission eigenvalues (MTEs)

Definition 2.2. *Let $k > 0$ be fixed. We define the MTEs as the values of $\rho \in \mathbb{R}$ for which there exists $u^i \in \mathrm{L}^2(\Omega_b) \setminus \{0\}$ satisfying $\Delta u^i + k^2 u^i = 0$ in Ω_b and such that the corresponding u^s and u^s_b respectively defined via (1) and (7) are equal outside Ω_b.*

Clearly, if ρ is a MTE then we have $w = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega_b}$ where w is the solution of (11) with $v = u_b$. Therefore MTEs can be equivalently defined as the values of $\rho \in \mathbb{R}$ for which there exists a non trivial solution $(v, w) \in \mathrm{L}^2(\Omega_b) \times \mathrm{H}^2_0(\Omega_b)$ to the modified interior transmission problem

$$\left| \begin{array}{rcll} \Delta w + k^2 n w &=& (\rho - k^2 n) v & \text{in } \Omega_b, \\ \Delta v + \rho v &=& 0 & \text{in } \Omega_b. \end{array} \right. \tag{14}$$

The goal of this paper is to show how the inside-outside duality method can be used to characterize MTEs from the knowledge of the operator F^m. The latter can be in practice constructed from the measurement operator F and the numerically evaluated operator F^b.

2.2. **Collection of some needed technical results.** Since ρ is the spectral parameter that will vary, we shall make the dependence of the operators explicit on ρ in the notation. For instance the operators F^b and F^m will be respectively denoted by $F^b(\rho)$ and $F^m(\rho)$ in the sequel.

We define the Herglotz operator for the background media as $H(\rho) : \mathrm{L}^2(\mathbb{S}^2) \to \mathrm{L}^2(\Omega_b)$ such that

$$H(\rho)g := \int_{\mathbb{S}^2} g(d) u_b(\cdot, d) ds(d)|_{\Omega_b}. \tag{15}$$

We define the operator $T(\rho) : \mathrm{L}^2(\Omega_b) \to \mathrm{L}^2(\Omega_b)$ such that

$$T(\rho)v := \frac{1}{4\pi}(k^2 n - \rho)(v + w(\rho)), \tag{16}$$

where $w(\rho) \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$ is the unique solution of (11). Following [17, 3, 12], if we consider the modified far field operator $\mathscr{F}(\rho) : \mathrm{L}^2(\mathbb{S}^2) \to \mathrm{L}^2(\mathbb{S}^2)$ defined as $\mathscr{F}(\rho) := (S_b(\rho))^* F_m(\rho)$, one has the factorization

$$\mathscr{F}(\rho) = H^*(\rho) T(\rho) H(\rho), \tag{17}$$

and $\mathscr{F}(\rho)$ is normal. Moreover, the modified scattering operator $\mathscr{S}(\rho) : \mathrm{L}^2(\mathbb{S}^2) \to \mathrm{L}^2(\mathbb{S}^2)$ defined as

$$\mathscr{S}(\rho) := \mathrm{Id} + \frac{ik}{2\pi}\mathscr{F}(\rho) \tag{18}$$

is unitary.

We now collect some known properties of the operator $T(\rho)$ that can be found in [3]. First, the continuity of $\rho \mapsto w(\rho)$ from $\mathbb{R} \to \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$ implies the continuity of $\rho \mapsto T(\rho)$ in the operator norm. The second property is related to the imaginary part of $T(\rho)$,

$$v \in \mathrm{L}^2(\Omega_b), \qquad 4\pi\, \Im m\, (T(\rho)v, v)_{\mathrm{L}^2(\Omega_b)} = k \int_{\mathbb{S}^2} |w^\infty|^2\, ds, \tag{19}$$

where $w^\infty(\rho)$ is the far field pattern associated with $w(\rho)$. Additionally, if ρ is not a MTE then $T(\rho)$ is coercive on $\mathrm{H}_{inc}(\rho) = \overline{\mathcal{R}(H(\rho))}^{\mathrm{L}^2(\Omega_b)} = \{\varphi \in \mathrm{L}^2(\Omega_b);\ \Delta\varphi + \rho\varphi = 0 \text{ in } \Omega_b\}$ namely

$$|(T(\rho)v, v)_{\mathrm{L}^2(\Omega_b)}| \geq \gamma \|v\|^2_{\mathrm{L}^2(\Omega_b)} \quad \forall v \in \mathrm{H}_{\mathrm{inc}}(\rho) \tag{20}$$

for some positive constant γ. We finally indicate some obvious connections between MTEs and the operator $T(\rho)$. From (19), Rellich's Lemma and the unique continuation principle, one has that

$$\rho \text{ is a MTE of (14)} \quad \Leftrightarrow \quad \exists v \in \mathrm{H}_{\mathrm{inc}}(\rho) \setminus \{0\} \text{ such that } \Im m\, (T(\rho)v, v)_{\mathrm{L}^2(\Omega_b)} = 0. \tag{21}$$

Now let $(v, w) \in \mathrm{L}^2(\Omega_b) \times \mathrm{H}^2_0(\Omega_b)$ be an eigenpair associated with a MTE ρ. Then indeed $v \in \mathrm{H}_{\mathrm{inc}}(\rho)$ and using the second Green's formula,

$$\int_{\Omega_b} (T(\rho)v) v\, dx = \frac{1}{4\pi} \int_{\Omega_b} (k^2 n - \rho)(v + w) v\, dx = -\frac{1}{4\pi} \int_{\Omega_b} (\Delta w + \rho w) v\, dx = 0. \tag{22}$$

The operator $T(\rho)$ verifies also the symmetry property

Lemma 2.3. *Let $\rho \in \mathbb{R}$ and and let $v, u \in \mathrm{L}^2(\Omega_\mathrm{b})$. Then*

$$\int_{\Omega_b} (T(\rho)v)\, u dx = \int_{\Omega_b} (T(\rho)u)\, v dx$$

Proof. According to the definition of $T(\rho)$ in (16), we have $T(\rho)v = \frac{1}{4\pi}(k^2 n - \rho)(v + w_v)$ (respectively $T(\rho)u = \dfrac{1}{4\pi}(k^2 n - \rho)(u + w_u)$) where $w_v \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$ (respectively $w_u \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$) and verify (11) (respectively verify (11) with $v = u$). Let $R > 0$ large enough such that $\Omega_b \subset B_R = B(O, R)$, using Green's formula twice we obtain

$$\begin{aligned}
\int_{B_R} (k^2 n - \rho)(u + w_u) w_v dx &= \int_{B_R} (\Delta w_u + \rho w_u) w_v dx \\
= \int_{\partial B_R} (\frac{\partial w_u}{\partial \nu} w_v &- \frac{\partial w_v}{\partial \nu} w_u) ds + \int_{B_R} w_u(\Delta w_v + \rho w_v) dx \\
= \int_{B_R} w_u(\Delta w_v + \rho w_v) dx &= \int_{B_R} (k^2 n - \rho)(v + w_v) w_u dx
\end{aligned}$$

due to the Sommerfeld radiation condition and the fact that w_v satisfies (11). The desired identity can be straightforwardly deduced using the definition of $T(\rho)$. $\square$

The analysis below requires also the introduction of the orthogonal projection $P(\rho) : \mathrm{L}^2(\Omega_b) \to \mathrm{H}_{\mathrm{inc}}(\rho)$. We here clarify some of the arguments used in [16]. Let us denote by $V_0(\Omega_b)$ the completion of $\mathscr{C}_0^\infty(\Omega_b)$ with respect to the norm $\|u\|^2_{V_0(\Omega_b)} := \|u\|^2_{L^2(\Omega_b)} + \|\Delta u\|^2_{L^2(\Omega_b)}$. From the Green formula, one deduces that $V_0(\Omega_b)$ is a subspace of $H_0^1(\Omega_b)$ and the Poincaré inequality shows that $\|\Delta u\|_{L^2(\Omega_b)}$ defines an equivalent norm in this space. In order to shorten the notation, below and except other mention, the scalar product in $\mathrm{L}^2(\Omega_b)$ will be denoted by $(\cdot,\cdot)$.

Lemma 2.4. *$P(\rho)$ is explicitly given by*

$$P(\rho)g = g - (\varDelta\hat{w}(\rho) + \rho\hat{w}(\rho)), \tag{23}$$

where $\hat{w}(\rho) \in V_0(\Omega_b)$ is the unique solution of the variational problem

$$\int_{\Omega_b} (\varDelta\hat{w}(\rho) + \rho\hat{w}(\rho))(\varDelta\psi + \rho\psi)dx = \int_{\Omega_b} g(\varDelta\psi + \rho\psi)dx \qquad \textit{for all } \psi \in V_0(\Omega_b). \tag{24}$$

Proof. Since $P(\rho)$ is the orthogonal projection on $\mathrm{H}_{\mathrm{inc}}(\rho)$, for $g \in \mathrm{L}^2(\Omega_b)$, we have

$$(P(\rho)g - g, v) = 0 \qquad \forall\, v \in \mathrm{H}_{\mathrm{inc}}(\rho). \tag{25}$$

Since v verifies $\Delta v + \rho v = 0$ in $\mathrm{L}^2(\Omega_b)$, which is equivalent to $\int_{\Omega_b} (\Delta\psi + \rho\psi)\overline{v}dx = 0$ for all $\psi \in \mathscr{C}_0^\infty(\Omega_b)$, by density,

$$\int_{\Omega_b} (\Delta\psi + \rho\psi)\overline{v}dx = 0 \qquad \forall\psi \in V_0(\Omega_b). \tag{26}$$

Now using (25) and (26), we obtain $P(\rho)g - g \in V(\rho)$ where $V(\rho)$ is the closure of $\{\Delta\psi + \rho\psi, \psi \in V_0(\Omega_b)\}$ in $\mathrm{L}^2(\Omega_b)$. In order to conclude the proof we show that $V(\rho) = \{\Delta\psi + \rho\psi, \psi \in V_0(\Omega_b)\}$. To proceed, let $v \in \mathrm{L}^2(\Omega_b)$ such there exists a sequence $(\psi_n)_n \in V_0(\Omega_b)$ and $v_n = \Delta\psi_n + \rho\psi_n \longrightarrow v$ strongly in $\mathrm{L}^2(\Omega_b)$. In order to prove that $(\psi_n)_n$ converges strongly in $V_0(\Omega_b)$, we prove that the norm $\|\Delta\psi + \rho\psi\|_{\mathrm{L}^2(\Omega_b)}$ defines an equivalent norm in $V_0(\Omega_b)$. Using the fact that $\|\Delta\psi\|_{\mathrm{L}^2(\Omega_\mathrm{b})}$ is an equivalent norm in $V_0(\Omega_b)$, it is sufficient to prove that $\|\psi\|_{\mathrm{L}^2(\Omega_\mathrm{b})} \leq C\|\Delta\psi + \rho\psi\|_{\mathrm{L}^2(\Omega_b)}$ for all $\psi \in V_0(\Omega_b)$.
We employ a contradiction argument. Suppose there exists a sequence $(\psi_n)_n \in V_0(\Omega_b)$ such that $\|\psi_n\|_{\mathrm{L}^2(\Omega_b)} = 1\, \forall n \in \mathbb{N}$ and $\|\Delta\psi_n + \rho\psi_n\|_{\mathrm{L}^2(\Omega_b)} \to_{n\to\infty} 0$. This implies $\|\Delta\psi_n\|_{\mathrm{L}^2(\Omega_b)}$ is bounded, and therefore the sequence $(\psi_n)_n$ is bounded in $V_0(\Omega_b)$. Up to a subsequence, we conclude that ψ_n converges to some $\psi \in V_0(\Omega_b)$ weakly in $V_0(\Omega_b)$ and strongly in $\mathrm{L}^2(\Omega_\mathrm{b})$. The limit $\psi \in V_0(\Omega_b)$ verifies $\Delta\psi + \rho\psi = 0$ and therefore $\psi = 0$ by a unique continuation argument (the extension by 0 of ψ verifies $\Delta\psi + \rho\psi = 0$ in $\mathbb{R}^3$). It also verifies $\|\psi\|_{\mathrm{L}^2(\Omega_\mathrm{b})} = 1$ which is a contradiction.
We now conclude from $P(\rho)g - g \in V_0(\Omega_b)$ that there exists $\hat{w} \in V_0(\Omega_b)$ such that $P(\rho)g - g = -(\varDelta\hat{w}(\rho) + \rho\hat{w}(\rho))$. The variational equation (24) comes from (25). This equation uniquely determines $\hat{w}(\rho)$ by the Lax-Milgram theorem thanks to the norm equivalence indicated above. □

Observe that $\hat{w}(\rho)$ in Lemma 2.4 satisfies in the distributional sense

$$(\varDelta + \rho)(\varDelta + \rho)\hat{w}(\rho) = (\varDelta + \rho)g \qquad \text{in } \Omega_b. \tag{27}$$

One can also show that $\rho \to P(\rho)$ is continuous and differentiable in the operator norm as indicated in the following lemma.

Lemma 2.5. *The map $\rho \to P(\rho)$ is continuous and differentiable and for all $\rho \in \mathbb{R}$. Moreover for all $\rho_0 \in \mathbb{R}$ there is $\varepsilon > 0$ such that*

$$P(\rho)g = P(\rho_0)g - (\rho - \rho_0)\left[\Delta\hat{w}'(\rho_0) + \rho_0\hat{w}'(\rho_0) + \hat{w}(\rho_0)\right] + (\rho - \rho_0)^2\mu(\rho) \quad (28)$$

where $\hat{w}'(\rho_0) \in V_0(\Omega_b)$ satisfies $(\Delta + \rho_0)(\Delta + \rho_0)\hat{w}'(\rho_0) = g - 2(\Delta\hat{w}(\rho_0) + \rho_0\hat{w}(\rho_0))$ in Ω_b and $\mu(\rho)$ is such that $\|\mu(\rho)\|_{\mathrm{L}^2(\Omega_b)} \leq C\|g\|_{\mathrm{L}^2(\Omega_b)}$ with $C > 0$ independent from $\rho \in]\rho_0 - \varepsilon, \rho_0 + \varepsilon[$.

Proof. It is sufficient to only prove (28). In the following the constant C refers to a constant which value may differ but remains independent from $\rho \in]\rho_0 - \varepsilon, \rho_0 + \varepsilon[$ for some $\varepsilon > 0$ fixed later. Set $\eta(\rho) = \hat{w}(\rho) - \hat{w}(\rho_0)$ where $\hat{w}(\rho) \in V_0(\Omega_b)$ is defined in Lemma 2.4. From (23),

$$P(\rho)g - P(\rho_0)g = -\Delta\eta(\rho) - \rho_0\eta(\rho) - (\rho - \rho_0)\hat{w}(\rho_0) - (\rho - \rho_0)\eta(\rho). \quad (29)$$

The function $\eta(\rho) \in V_0(\Omega_b)$ verifies

$$(\Delta + \rho_0)(\Delta + \rho_0)\eta(\rho) = (\rho - \rho_0)g - 2(\rho - \rho_0)\Delta\hat{w}(\rho) - (\rho^2 - \rho_0^2)\hat{w}(\rho) \qquad \text{in } \Omega_b. \quad (30)$$

Denote by $Q(\rho) : V_0(\Omega_b) \to V_0(\Omega_b)$ the Riesz representation of the operator defined by left hand side of (24). We have that $Q(\rho) = Q(\rho_0) + (\rho - \rho_0)Q_0(\rho)$ where $Q_0(\rho) : V_0(\Omega_b) \to V_0(\Omega_b)$ is uniformly bounded with respect to ρ in neighbourhood of ρ_0. Using the Laurent series we deduces that $(Q(\rho))^{-1}$ is uniformly bounded with respect to $\rho \in]\rho_0 - \varepsilon, \rho_0 + \varepsilon[$ for some sufficiently small $\varepsilon > 0$. In this sequel, we always assume that $\rho \in]\rho_0 - \varepsilon, \rho_0 + \varepsilon[$. Consequently $\|\hat{w}(\rho)\|_{V_0(\Omega_b)} \leq C\|g\|_{\mathrm{L}^2(\Omega_b)}$ and from (30) $\|\eta(\rho)\|_{V_0(\Omega_b)} \leq C|\rho - \rho_0|\|g\|_{\mathrm{L}^2(\Omega_b)}$.
The function $\gamma(\rho) := \eta(\rho) - (\rho - \rho_0)\hat{w}'(\rho_0)$ verifies

$$(\Delta + \rho_0)(\Delta + \rho_0)\gamma(\rho) = -2(\rho - \rho_0)\Delta\eta(\rho) - (\rho - \rho_0)^2\hat{w}(\rho) \qquad \text{in } \Omega_b$$

which implies $\|\gamma(\rho)\|_{V_0(\Omega_b)} \leq C|\rho - \rho_0|^2\|g\|_{\mathrm{L}^2(\Omega_b)}$. The result of the lemma follows by observing that from (29), identity (28) holds with

$$\mu(\rho) = -\Delta\gamma(\rho) - \rho_0\gamma(\rho) - (\rho - \rho_0)\gamma(\rho).$$

□

3. **Analysis of the inside outside duality.** We first consider the case when ρ is not a MTE. Since $\mathscr{F}(\rho)$ is a compact normal operator, there is an orthonormal complete basis $(g_j)_{j\in\mathbb{N}}$ of $\mathrm{L}^2(\mathbb{S}^2)$ such that $\mathscr{F}(\rho)g_j = \lambda_j g_j$ where $\{\lambda_j, j = 1, \cdots, +\infty\}$ are the eigenvalues of $\mathscr{F}(\rho)$ that accumulate at 0. If ρ is not a MTE then $\mathscr{F}(\rho)$ is injective and therefore $\lambda_j \neq 0 \; \forall j \in \{1, \cdots, +\infty\}$. Since $\mathscr{S}(\rho)$ is unitary, the eigenvalues of $\mathscr{F}(\rho)$ lie on the circle of radius $2\pi/k$ and centre $2i\pi/k$. We set $\lambda_j := 2\pi/ik(e^{i\delta_j} - 1)$ with $\delta_j \in (0, 2\pi)$ so that $e^{i\delta_j}$ are the eigenvalues of $\mathscr{S}(\rho)$.
The following proposition indicates the region of the complex plane where the λ_j accumulate at zero according to the sign of $k^2n - \rho$. We refer to [3] for the proof of this result.

Proposition 3.1. *Assume that ρ is not a MTE.*

- *If $k^2n - \rho \geq \alpha > 0$ in Ω_b, then the sequence (δ_j) accumulates only at 0.*
- *If $\rho - k^2n \geq \alpha > 0$ in Ω_b, then the sequence (δ_j) accumulates only at 2π.*

When ρ is not a MTE we set

$$\delta_\star(\rho) := \min_{j\geq 1}\delta_j \quad \text{if } \rho - k^2n \geq \alpha > 0 \text{ and } \delta_\star(\rho) := \max_{j\geq 1}\delta_j \quad \text{if } k^2n - \rho \geq \alpha > 0 \quad (31)$$

and denote

$$\lambda_\star(\rho) := \frac{2\pi}{ik}(e^{i\delta_\star(\rho)} - 1).$$

The inside-outside duality method relies on the behaviour of $\delta_\star(\rho)$ as ρ approaches a MTE. The analysis of this behaviour is topic of the following subsections.

3.1. **A sufficient condition for the detection of a MTE.** In this paragraph, we provide a sufficient condition allowing one to detect a MTE of (14).

Theorem 3.2. *Let $\rho_0 \in \mathbb{R}$ and $I = (\rho_0 - \varepsilon; \rho_0 - \varepsilon) \setminus \{\rho_0\}$ such that no $\rho \in I$ is a MTE of (14). Assume that there is a sequence (ρ_j) of elements of I such that*

$$\lim_{j\to+\infty} \rho_j = \rho_0 \quad \textit{and} \quad \lim_{j\to+\infty} \delta_\star(\rho_j) = \left| \begin{array}{ll} 2\pi & \textit{when } k^2 n - \rho_0 \geq \alpha > 0 \\ 0 & \textit{when } \rho_0 - k^2 n \geq \alpha > 0. \end{array} \right.$$

Then ρ_0 is a MTE of (14). Moreover, the sequence (v_j), with

$$v_j := \frac{H(\rho_j)g_j}{\|H(\rho_j)g_j\|_{\mathrm{L}^2(\Omega_b)}},$$

admits a subsequence which converges strongly to $v \in \mathrm{L}^2(\Omega_b)$, where (v, w) is an eigenpair of (14) associated with ρ_0. Here g_j is a normalised eigenfunction of $\mathscr{F}(\rho)$ associated with $\lambda_\star(\rho_j)$ and w satisfies (11) with $\rho = \rho_0$.

Proof. The proof follows the lines of a similar result in [3] but we provide the details here for the sake of completeness. We consider only the case $k^2n - \rho_0 \geq \alpha > 0$ since the other one follows replacing $\mathscr{F}(\rho)$ with $-\mathscr{F}(\rho)$. Considering j sufficiently large we can assume that $k^2 n - \rho_j \geq \alpha/2 > 0$. Set

$$\psi_j := \frac{H(\rho_j)g_j}{\sqrt{|\lambda_\star(\rho_j)|}} \in \mathrm{L}^2(\Omega_b).$$

The sequence (ψ_j) satisfies, according to the assumptions and the factorisation (17),

$$(T(\rho_j)\psi_j, \psi_j) = \frac{\lambda_\star(\rho_j)}{|\lambda_\star(\rho_j)|}(g_j, g_j)_{\mathrm{L}^2(\mathbb{S}^2)} = \frac{\lambda_\star(\rho_j)}{|\lambda_\star(\rho_j)|} \underset{j\to+\infty}{\to} -1. \tag{32}$$

Suppose that ρ_0 is not a MTE. Then the operator $T(\rho_0)$ is coercive on $\mathrm{H}_{\mathrm{inc}}(\rho_0)$. We deduce from (20), the continuity of $\rho \mapsto T(\rho)$ and the continuity of $\rho \mapsto P(\rho)$ where $P(\rho)$ is the orthogonal projection defined earlier, that one can choose the coercivity constant γ in (20) to be independent from $\rho \in I$. The identity (32) then shows that the sequence (ψ_j) is bounded in $\mathrm{L}^2(\Omega_b)$ and consequently, up to a subsequence, one can assume that (ψ_j) weakly converges to some ψ_0 in $\mathrm{L}^2(\Omega_b)$. Since $\psi_j \in \mathrm{H}_{\mathrm{inc}}(\rho_j)$ for all $j \in \mathbb{N}$, the weak limit satisfies $\Delta\psi_0 + \rho_0\psi_0 = 0$ in Ω_b, therefore $\psi_0 \in \mathrm{H}_{\mathrm{inc}}(\rho_0)$. Let us denote by $w_j \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$ (resp. $w_0 \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$) the solution of (11) with $v = \psi_j$, $\rho = \rho_j$ (resp. $v = \psi_0$, $\rho = \rho_0$). From (19) that

$$4\pi\, \Im m(T(\rho_j)\psi_j, \psi_j) = k \int_{\mathbb{S}^2} |w^\infty(\rho_j)|^2 ds. \tag{33}$$

From expression (13) we have $\rho \mapsto w^\infty(\rho)$ from $\mathbb{R} \to \mathrm{L}^2(\Omega_\mathrm{b})$ is compact and consequently, using (32)

$$4\pi\, \Im m\, (T(\rho_j)\psi_j, \psi_j) \underset{j\to+\infty}{\to} 4\pi\, \Im m\, (T(\rho_0)\psi_0, \psi_0) = 0.$$

Therefore (ψ_0, w_0) is a solution of the interior transmission problem (14) for $\rho = \rho_0$. The hypothesis on ρ_0 implies $\psi_0 = 0$. Using the definition of $T(\rho)$ we have

$$\frac{1}{4\pi}((k^2n-\rho_j)\psi_j,\psi_j) = (T(\rho_j)\psi_j,\psi_j) - \frac{1}{4\pi}((k^2n-\rho_j))\psi_j, w_j) \tag{34}$$

where $((k^2n-\rho_j)\psi_j, w_j) \to ((k^2n-\rho_0)\psi_0, w_0)$ when $j \to +\infty$. The latter property is a consequence of (12) and the fact that $\mathrm{H}^2(\Omega_b)$ is compactly embedded in $\mathrm{L}^2(\Omega_b)$. Consequently

$$0 \leq ((k^2n-\rho_j)\psi_j,\psi_j) \underset{j\to+\infty}{\to} -1$$

which is a contradiction.
We now consider the second part of the theorem. Since $\|v_j\|_{\mathrm{L}^2(\Omega_b)} = 1$, the sequence (v_j) admits a subsequence that we still denote the same, that weakly converges to some $v \in \mathrm{L}^2(\Omega_b)$. We have

$$(T(\rho_j)v_j, v_j) = \theta_j\,\frac{\lambda_\star(\rho_j)}{|\lambda_\star(\rho_j)|} \tag{35}$$

with $\theta_j := |\lambda_\star(\rho_j)|/\|H(\rho_j)g_j\|^2_{\mathrm{L}^2(\Omega_b)}$. Using the definition $\lambda_\star(\rho_j)$ we have $\theta_j \leq \|T(\rho_j)\|$ and therefore by continuity of $\rho \mapsto T(\rho)$ one conclude that θ_j is bounded and can assume up to changing the subsequence v_j that $\theta_j \to \theta_0 \geq 0$. We then conclude from (35) that $\Im m\,(T(\rho_j)v_j, v_j) \to 0$ as $j \to +\infty$. The same arguments as above prove that the pair (v, w) solves problem (14) for $\rho = \rho_0$, w being the solution of (11) with $\rho = \rho_0$. Identity (34) with ψ_j is replaced by v_j and the fact that $(T(\rho_j)v_j, v_j) \to -\theta_0 \leq 0$ imply

$$\limsup_{j\to+\infty}\,(k^2n-\rho_j)v_j, v_j) \leq -((k^2n-\rho_0)v, w) \tag{36}$$

where we used the strong convergence of w_j to w in $\mathrm{L}^2(\Omega_b)$. Since the pair (v, w) solves (14), we have $((k^2n-\rho_0)(v+w), v) = 0$. Consequently,

$$\limsup_{j\to+\infty}\,((k^2n-\rho_j)v_j, v_j) \leq ((k^2n-\rho_0)v, v).$$

Since $((\rho_j-\rho_0)v, v) \to 0$ as $j \to +\infty$, we finally get

$$\limsup_{j\to+\infty}\alpha\|v_j - v\|^2_{\mathrm{L}^2(\Omega_\mathrm{b})} \leq \limsup_{j\to+\infty}\,((k^2n-\rho_j)(v_j-v), v_j - v) \leq 0,$$

which proves that (v_j) strongly converges to v in $\mathrm{L}^2(\Omega_b)$ and finishes the proof. □

3.2. **Necessary conditions for MTE.** In order to derive necessary conditions for the characterization of MTE, we need to study the behaviour of $(T(\rho)H(\rho)g, H(\rho)g)$ for ρ approaching some MTE ρ_0. For that purpose we the approach in [16]. If ρ is not a MTE then $\mathrm{L}^2(\mathbb{S}^2) = \overline{\mathcal{R}(\mathscr{F}(\rho))}^{\mathrm{L}^2(\mathbb{S}^2)}$ and 1 is not an eigenvalue of $\mathscr{S}(\rho)$. Therefore we can define the Cayley transform $\mathfrak{S}(\rho) : \mathrm{L}^2(\mathbb{S}^2) \to \mathrm{L}^2(\mathbb{S}^2)$ as $\mathfrak{S}(\rho) = i(\mathrm{Id} + \mathscr{S}(\rho))(\mathrm{Id} - \mathscr{S}(\rho))^{-1}$. This operator is self adjoint, its spectrum is discrete and we have the equivalence: $e^{i\delta_\star(\rho)}$ is an eigenvalue of $\mathscr{S}(\rho)$ if and only if $-\cot(\delta_\star(\rho)/2) \in \mathbb{R}$ is an eigenvalue of $\mathfrak{S}(\rho)$. Since $\mathfrak{S}(\rho)$ is selfadjoint, we can apply Courant-Fischer min-max principle and get in the case $k^2n - \rho \geq \alpha > 0$ in Ω_b (using Proposition 3.1)

$$-\cot\frac{\delta_\star(\rho)}{2} = \sup_{f\in\mathcal{R}(\mathscr{F})}\frac{(\mathfrak{S}(\rho)f, f)}{\|f\|_{\mathrm{L}^2(\mathbb{S}^2)}} = \sup_{g\in\mathrm{L}^2(\mathbb{S}^2)}\frac{\Im m\,(\mathscr{S}(\rho)g, g)}{\Re e\,(\mathscr{S}(\rho)g, g) - \|g\|^2_{\mathrm{L}^2(\mathbb{S}^2)}}.$$

Using expression (18) and the factorization (17), we obtain

$$-\cot\frac{\delta_\star(\rho)}{2} = \sup_{g\in \mathrm{L}^2(\mathbb{S}^2)} \frac{\Re e\,(\mathscr{F}(\rho)g,g)}{-\Im m\,(\mathscr{F}(\rho)g,g)}$$

$$= \sup_{g\in \mathrm{L}^2(\mathbb{S}^2)} \frac{\Re e\,(T(\rho)H(\rho)g,H(\rho)g)}{-\Im m\,(T(\rho)H(\rho)g,H(\rho)g)} = \sup_{\varphi\in \mathrm{H}_{\mathrm{inc}}(\rho)} \frac{\Re e\,(T(\rho)\varphi,\varphi)}{-\Im m\,(T(\rho)\varphi,\varphi)}.$$

Property 21 ensures that the denominator of the previous expression does not vanishes. We summarize these results in the following proposition.

Proposition 3.3. *Assume that $\rho > 0$ is not a MTE of (14).*

1. *Assume that $k^2 n - \rho \geq \alpha > 0$ in Ω_b. Then*

$$\cot\frac{\delta_\star(\rho)}{2} = \inf_{\varphi\in \mathrm{H}_{\mathrm{inc}}(\rho)} \frac{\Re e\,(T(\rho)\varphi,\varphi)}{\Im m\,(T(\rho)\varphi,\varphi)}. \tag{37}$$

2. *Assume that $\rho - k^2 n \geq \alpha > 0$ in Ω_b. Then*

$$\cot\frac{\delta_\star(\rho)}{2} = \sup_{\varphi\in \mathrm{H}_{\mathrm{inc}}(\rho)} \frac{\Re e\,(T(\rho)\varphi,\varphi)}{\Im m\,(T(\rho)\varphi,\varphi)}. \tag{38}$$

We now state and prove the key technical result of this section.

Proposition 3.4. *Let ρ_0 be a MTE of (14) and $(v_0, w_0) \in \mathrm{L}^2(\Omega_b)\times V_0(\Omega_b)$ an associated eigenpair. Then there is $\varepsilon > 0$ and $\eta(\rho)$ such that*

$$4\pi(T(\rho)v_0,v_0) = -(\rho-\rho_0)\Big(\|\, v_0\,\|^2_{\mathrm{L}^2(\Omega_b)} + 2\Re e(v_0,w_0)\Big) + (\rho-\rho_0)^2\eta(\rho) \tag{39}$$

for all ρ such that $|\rho-\rho_0|\leq\varepsilon$, where $|\eta(\rho)| \leq C \,\|\, v_0\,\|^2_{\mathrm{L}^2(\Omega_b)}$ with $C > 0$ independent from ρ.

Proof. According to the definition of $T(\rho)$ in (16), we have

$$4\pi(T(\rho)v_0,v_0) = ((k^2n-\rho)(v_0+w(\rho)),v_0) \tag{40}$$

where $w(\rho)$ is the solution of (11) with $v = v_0$. We remark that, according to the definition of MTEs, the solution $w(\rho_0)$ of (11) with $v = v_0$ and $\rho = \rho_0$ is such that $w(\rho_0) = w_0$ in Ω_b and $w(\rho_0) = 0$ outside Ω_b. Let w' the derivative of w at $\rho = \rho_0$ then we have an expansion as $\rho \to \rho_0$ of the form

$$w(\rho) - w(\rho_0) = (\rho-\rho_0)w' + (\rho-\rho_0)^2\tilde{w}(\rho), \tag{41}$$

where w' is independent from ρ and where $\tilde{w}(\rho)$ have bounded norm as $\rho \to \rho_0$. Let B_R a ball such that $\overline{\Omega_b} \subset B_R$ and consider the Dirichlet-to-Neumann operator $\Lambda(k) : \mathrm{H}^{1/2}(\partial B_R) \to \mathrm{H}^{-1/2}(\partial B_R)$ such that $\Lambda(k)\varphi = \partial_\nu\psi$ (ν is oriented to the exterior of B_R) where $\psi \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus B_R)$ is the outgoing function solving

$$\left|\begin{array}{rcll} \Delta\psi + k^2\psi &=& 0 & \text{in } \mathbb{R}^3\setminus B_R \\ \psi &=& \varphi & \text{on } \partial B_R. \end{array}\right. \tag{42}$$

Then we have, for all $\varphi \in \mathrm{H}^1(B_R)$,

$$\begin{aligned} (\nabla w(\rho),\nabla\varphi) - k^2(nw(\rho),\varphi) - \langle\Lambda(k)w(\rho),\varphi\rangle_{\partial B_R} &= ((k^2n-\rho)v_0,\varphi), \\ (\nabla w(\rho_0),\nabla\varphi) - k^2(nw(\rho_0),\varphi) - \langle\Lambda(k)w(\rho_0),\varphi\rangle_{\partial B_R} &= ((k^2n-\rho_0)v_0,\varphi) \end{aligned} \tag{43}$$

where $\langle \cdot,\cdot\rangle_{\partial B_R}$ denotes the $\mathrm{H}^{-1/2}(\partial B_R) - \mathrm{H}^{1/2}(\partial B_R)$ duality product. As $\rho \to \rho_0$, the difference of the two lines of (43) leads us to prove that $w' \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$ satisfies, for all $\varphi \in \mathrm{H}^1(B_R)$,

$$(\nabla w', \nabla\varphi) - k^2(nw', \varphi) - < \Lambda(k)w', \varphi >_{\partial B_R} = -(v_0, \varphi).$$

Using, $(T(\rho_0)v_0(\rho_0), v_0(\rho_0)) = 0$, we obtain

$$\begin{aligned}4\pi(T(\rho)v_0, v_0) &= ((k^2n-\rho)(v_0+w(\rho)), v_0) - ((k^2n-\rho_0)(v_0+w(\rho_0)), v_0)\\ &= -(\rho-\rho_0) \parallel v_0 \parallel^2_{\mathrm{L}^2(\Omega_b)} + k^2(n(w(\rho)-w(\rho_0)), v_0) - (\rho-\rho_0)(w(\rho_0), v_0)\\ &\quad -(\rho-\rho_0)(w(\rho)-w(\rho_0), v_0) - \rho_0(w(\rho)-w(\rho_0), v_0)\end{aligned}$$

Thus, using (41) we have

$$\begin{aligned}4\pi(T(\rho)v_0, v_0) = (\rho-\rho_0)\Big(&- \parallel v_0 \parallel^2_{\mathrm{L}^2(\Omega_b)} + ((k^2n-\rho_0)w', v_0)\\ &- (w_0, v_0) + (\rho-\rho_0)\eta(\rho)\Big)\end{aligned}$$

where $\eta(\rho) = (k^2n\tilde{w}(\rho) - w' - \rho_0\tilde{w}(\rho) - (\rho-\rho_0)\tilde{w}(\rho), v_0)_{\mathrm{L}^2(\Omega_b)}$ and $|\eta(\rho)| \le C\|v_0\|^2_{\mathrm{L}^2(\Omega_b)}$ with $C>0$ independent of ρ sufficiently close to ρ_0 .
We recall that $w_0 \in \mathrm{H}^2_0(\Omega_b)$ and satisfies $\Delta w_0 + k^2nw_0 = -(k^2n-\rho_0)v_0$. This allows us to write, since n and ρ are real

$$\begin{aligned}((k^2n-\rho_0)w', v_0) &= (w', (k^2n-\rho_0)v_0) = -(w', \Delta w_0 + k^2nw_0)\\ &= -(\Delta w' + k^2nw', w_0) = -(v_0, w_0)\end{aligned}$$

Thus, we obtain,

$$\begin{aligned}4\pi(T(\rho)v_0, v_0)_{\mathrm{L}^2(\Omega_b)} = (\rho-\rho_0)\Big(&- \parallel v_0 \parallel^2_{\mathrm{L}^2(\Omega_b)} - (v_0, w_0) - (w_0, v_0)\Big)\\ &+ (\rho-\rho_0)^2\eta(\rho)\end{aligned}$$

which is exactly the identity (39) of the proposition. □

Proposition 3.5. *Let ρ_0 be a MTE of (14) and $(v_0, w_0) \in \mathrm{L}^2(\Omega_b)\times V_0(\Omega_b)$ an associated eigenpair. Then*

$$(T(\rho)v(\rho), v(\rho)) = \frac{-1}{4\pi}(\rho-\rho_0) \parallel v_0 \parallel^2_{\mathrm{L}^2(\Omega_b)} + (\rho-\rho_0)^2\xi(\rho), \tag{44}$$

where $v(\rho) = P(\rho)v_0$ and $|\xi(\rho)| \le C$ with $C>0$ independent from ρ sufficiently close to ρ_0.

Proof. We hereafter denote by $O((\rho-\rho_0)^m)$ any quantity of the form $(\rho-\rho_0)^m\xi(\rho)$ where $\xi(\rho)$ is as in the proposition. We consider the following obvious splitting,

$$\begin{aligned}(T(\rho)v(\rho), v(\rho)) = (T(\rho)v_0, v_0) + (T(\rho)v_0, (v(\rho)-v_0)) + (T(\rho)(v(\rho)-v_0), v_0)\\ + (T(\rho)(v(\rho)-v_0), (v(\rho)-v_0))\end{aligned} \tag{45}$$

and separately analyze each of the terms appearing on the right hand side. From Lemma 2.5 applied to $g = v_0$ and using $P(\rho_0)v_0 = v_0$ and $\hat{w}(\rho_0) = 0$, we have

$$v(\rho) - v_0 = -(\rho-\rho_0)(\Delta+\rho_0)\hat{w}'(\rho_0) + (\rho-\rho_0)^2\mu(\rho) \tag{46}$$

where $\hat{w}'(\rho_0) \in V_0(\Omega_b)$ satisfies

$$(\Delta+\rho_0)(\Delta+\rho_0)\hat{w}'(\rho_0) = v_0 \qquad \text{in } \Omega_b \tag{47}$$

and $\mu(\rho)$ is such that $\|\mu(\rho)\|_{\mathrm{L}^2(\Omega_b)} \le C\|v_0\|_{\mathrm{L}^2(\Omega_b)}$ with $C>0$ independent from $\rho \in]\rho_0-\varepsilon, \rho_0+\varepsilon[$ for some sufficiently small $\varepsilon>0$. Let w' be the derivative of w

at $\rho = \rho_0$ defined in (41). Using definition (16) of the operator $T(\rho)$ and expansion (46), we obtain

$$4\pi(T(\rho)v_0, (v(\rho)-v_0)) = ((k^2 n-\rho)(v_0+w(\rho)), -(\rho-\rho_0)(\Delta+\rho_0)\hat{w}'(\rho_0)+O((\rho-\rho_0)^2).$$

$$= (\rho-\rho_0)((\Delta+\rho)w(\rho), (\Delta+\rho_0)\hat{w}'(\rho_0)) + O((\rho-\rho_0)^2)$$

where we used the equation satisfied by $w(\rho)$ in (11). We observe that

$$(\Delta+\rho)w(\rho) = (\Delta+\rho_0)w_0 + \Theta(\rho)$$

where $\Theta(\rho) := (\Delta+\rho)(w(\rho)-w(\rho_0)) + (\rho-\rho_0)w_0$. Using (41) we obtain that $\|\Theta(\rho)\|_{\mathrm{L}^2(\Omega_\mathrm{b})} = O((\rho-\rho_0))$ and therefore

$$4\pi(T(\rho)v_0, (v(\rho)-v_0)) = (\rho-\rho_0)((\Delta+\rho_0)w_0, (\Delta+\rho_0)\hat{w}'(\rho_0)) + O((\rho-\rho_0)^2)$$

Using (47) and the fact that $w_0 \in V_0(\Omega_b)$ we end up with

$$4\pi(T(\rho)v_0, (v(\rho)-v_0)) = (\rho-\rho_0)(w_0, v_0) + O((\rho-\rho_0)^2). \tag{48}$$

Since ρ_0 and n are real, if (v_0, w_0) solves (14) then $(\overline{v_0}, \overline{w_0})$ solves (14) and therefore we can suppose without loss of generality that v_0 and$w_0)$ are real valued. Using Lemma 2.3, we then deduce that

$$(T(\rho)(v(\rho)-v_0), v_0) = (T(\rho)v_0, (v(\rho)-v_0)). \tag{49}$$

The desired result follows from expression (45) by substituting (49) and (48) and using Proposition 3.2. □

We are now ready to state and prove the main result that provides a necessary and sufficient condition for an isolated MTE (i.e. MTE for which there exists a neighborhood in $\mathbb{R}$ where it is the only MTE).

Theorem 3.6. *Let $\rho_0 \in \mathbb{R}$ and $\delta_\star(\rho)$ defined by (31) for ρ not being a MTE. Then we have the following equivalences.*

1. *Assume $k^2 n - \rho_0 \geq \alpha > 0$ in Ω_b. Then ρ_0 is an isolated MTE if and only if* $\lim_{\rho \searrow \rho_0} \delta_\star(\rho) = 2\pi$.
2. *Assume that we have $\rho_0 - k^2 n \geq \alpha > 0$ in Ω_b.Then ρ_0 is an isolated MTE if and only if* $\lim_{\rho \nearrow \rho_0} \delta_\star(\rho) = 0$.

Proof. Let us consider the case $k^2 n - \rho_0 \geq \alpha > 0$ in Ω_b. Let ρ_0 be a MTE of (14) and $(v_0, w_0) \in \mathrm{L}^2(\Omega_b) \times V_0(\Omega_b)$ a corresponding eigenpair. According to Proposition 3.5, we have

$$\cot \frac{\delta_\star(\rho)}{2} = \inf_{\varphi \in \mathrm{H}_{\mathrm{inc}}(\rho)} \frac{\Re e\,(T(\rho)\varphi, \varphi)}{\Im m\,(T(\rho)\varphi, \varphi)} \leq \frac{\Re e\,(T(\rho)v_0, v_0)}{\Im m\,(T(\rho)v_0, v_0)}.$$

As we have $\Im m(T(\rho)v_0, v_0) > 0$ if ρ is not a MTE, then we obtain

$$\frac{\Re e(T(\rho)v_0(\rho), v_0(\rho))}{\Im m(T(\rho)v_0(\rho), v_0(\rho))} = \frac{\frac{-1}{4\pi} \parallel v_0 \parallel^2_{\mathrm{L}^2(\Omega_\mathrm{b})} + (\rho-\rho_0)\Re e\,\eta(\rho)}{(\rho-\rho_0)\Im m\,\eta(\rho)} \rightarrow_{\rho \searrow \rho_0} -\infty$$

and we conclude that $\lim_{\rho \searrow \rho_0} \delta_\star(\rho) = 2\pi$ The case $\rho_0 - k^2 n \geq \alpha > 0$ in Ω_b can be proved similarly. These provide the necessary conditions that complement Theorem 3.2 and end the proof of this theorem. □

4. **Numerical experiments and validation .** The goal of this section is to illustrate the theoretical results of Theorem 3.6 with synthetic 2D numerical examples. It also aims at testing the precision and the stability of the underlying numerical scheme to retrieve modified transmission eigenvalues. When the geometry is circular, one can derive an analytical expression of the modified scattering operator $\mathscr{S}$ and characterize the MTEs as the zeros of some determinant that can be computed using standard routines (for instance the ones implemented in NumPy). For the other geometries we numerically approximate the far field operators F and F_b and the operators S and S_b using `FreeFem++` finite elements library [13].

4.1. **Validation in the case where the domain Ω is a disc.** In this first example we consider a case where analytical expressions can be derived. We take n equal to a real constant in $\Omega = B_R$ and $n = 1$ in $\mathbb{R}^2 \setminus \overline{\Omega}$ where B_R is the ball of radius R centred at the origin. Moreover, for the modified background in (7), we take $\Omega_b = \Omega$. Let us recall that in 2D, in order to have unitary operators, S and S_b are respectively defined from F and F_b by

$$S := \mathrm{Id} + 2ik\frac{e^{-i\pi/4}}{\sqrt{8\pi k}}F \qquad \text{and} \qquad S_b := \mathrm{Id} + 2ik\frac{e^{-i\pi/4}}{\sqrt{8\pi k}}F_b.$$

Following [3], the non zero eigenvalues of $\mathscr{F}(\rho) = (S_b(\rho))^* F_m(\rho)$ coincide with the set $\{\gamma_\ell;\, \ell \in \mathbb{Z}\}$, with

$$\gamma_\ell := (1 + 2\overline{D_{b,\ell}})(1 + 2D_\ell).$$

where

$$D_{b,\ell} := \frac{-\sqrt{\rho}J_\ell'(\sqrt{\rho}R)J_\ell(kR) + kJ_\ell'(kR)J_\ell(\sqrt{\rho}R)}{-kJ_\ell(\sqrt{\rho}R){H_\ell^{(1)}}'(kR) + \sqrt{\rho}J_\ell'(\sqrt{\rho}R)H_\ell^{(1)}(kR)}$$

$$D_\ell := \frac{-\sqrt{n}J_\ell'(k\sqrt{n}R)J_\ell(kR) + J_\ell'(kR)J_\ell(k\sqrt{n}R)}{-J_\ell(k\sqrt{n}R){H_\ell^{(1)}}'(kR) + \sqrt{n}J_\ell'(k\sqrt{n}R)H_\ell^{(1)}(kR)}$$

and J_ℓ and $H_\ell^{(1)}$ stand for the Bessel and Hankel functions of the first kind of order ℓ. We denote by $\hat{\delta}_\ell \in [0; 2\pi)$ the phase of the γ_ℓ.

Figure 1 displays the curves $\rho \mapsto \hat{\delta}_\ell(\rho) \in [0, 2\pi)$ for $\rho \in (1, 40)$ for $n = 4$, $k = 5$ and $R = 2\pi/k$. Each colour corresponds to a different value of $\ell \in \{0, \dots, 7\}$. The vertical dashed lines indicates the MTEs computed as the zeros of the determinant

$$\det\begin{pmatrix} J_\ell(\sqrt{\rho}R) & J_\ell(k\sqrt{n}R) \\ \sqrt{\rho}J_\ell'(\sqrt{\rho}R) & k\sqrt{n}J_\ell'(k\sqrt{n}R) \end{pmatrix} \tag{50}$$

that characterizes the existence of non trivial solutions to (14) using separation of variables. For $\rho \in [1, 40[$, $k^2 n - \rho > 0$ and we indeed observe in Figure 1 that there exists ℓ_0 such that the phase $\rho \mapsto \hat{\delta}_{\ell_0}(\rho)$ goes to 2π as $\rho \searrow \rho_0$ where ρ_0 is an MTE. For ρ sufficiently close to ρ_0 the $\hat{\delta}_{\ell_0}(\rho)$ coincides with the $\delta_\star(\rho)$ of Theorem 3.6.

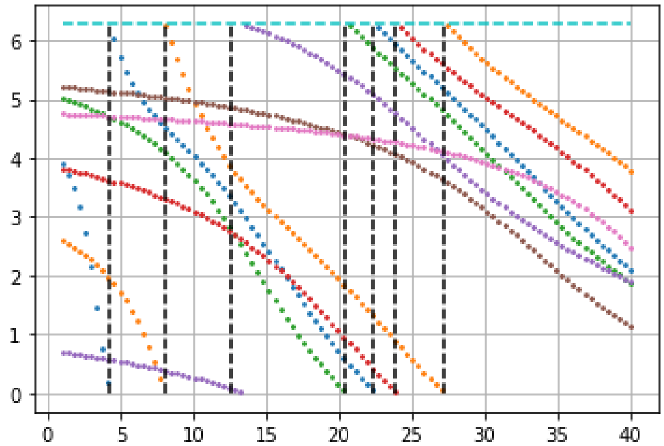


FIGURE 1. The curves $\rho \mapsto \hat{\delta}_\ell(\rho) \in [0, 2\pi)$ for $\rho \in (1, 40)$ in the case where $\Omega = \Omega_b = B_R$, $n = 4$, $k = 5$ and $R = 2\pi/k$. Each colour corresponds to a different value of $\ell \in \{0, \ldots, 7\}$. The vertical dashed lines indicate the MTEs computed as the zeros of (50).

4.2. **Numerical validation for other geometries.** We here rely on the finite elements library `FreeFem++` [13] to numerically solve the scattering problem and synthetically generate F_b, F, S and S_b for a given geometry and set of physical parameters. We use M uniformly distributed incident directions $d_j = (\cos(\theta_j), \sin(\theta_j))$, $\theta_j = j2\pi/M$, $j = 0, \ldots, M-1$ and numerically evaluate the far fields at $\hat{x}_i = (\cos(\theta_i), \sin(\theta_i))$, $i = 0, \ldots, M-1$. This gives for instance an $M \times M$ matrix F whose entries are $F_{i,j} = u_\infty(\hat{x}_i, d_j)$. Similar procedure holds for F_b. In all numerical examples we take $M = 35$.

After generating the matrices F and $F_b(\rho)$ for a given ρ in an interval of values (we uniformly discretize the interval of values for ρ) we use Python to evaluate the eigenvalues of $\mathscr{F}(\rho)$ and and display the curves $\rho \mapsto \delta_\ell(\rho)$. Note that the index ℓ here has a different meaning than in the case of the disc above. It corresponds to the index of the eigenvalues generated by the eigenvalue solver, which has nothing to do with the index of the Fourier mode in the case of the disc.

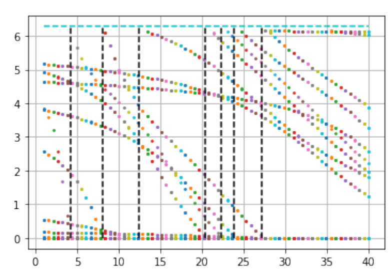
According to the Inside-Outside Duality (IOD) theorem above, the MTEs should correspond with the values of ρ for which the curves $\rho \mapsto \delta_\ell(\rho)$ reach 2π (respectively 0) if these values are smaller (respectively larger) than $k^2 n$. It is easy to identify visually these curves as shown in the examples below. However, it appears delicate to design an automatic procedure that determines these values. We hereafter explain how we obtain the numerical values indicated in the tables below. We give the details for the values of $\rho < k^2 n$ (similar procedure applies in the other case). Let us first remark that due to numerical errors, we always have spurious phases $\rho \mapsto \delta_\ell(\rho)$ (for some indexes ℓ) that accumulate at 2π. These are discarded in the Figures below by displaying only the values of $\delta_\ell(\rho)$ that are in the interval $[0, 2\pi - \eta]$ (in the numerical examples $\eta \sim 10^{-2}$). This allows to better visualize the curves that converge to 2π (The value of η can be easily adapted to any example). In order to find the value of MTEs, we choose an interval of the form $[\gamma, 2\pi - \eta]$ where γ is chosen (visually) so that only these curves have values in that interval (this choice is always possible if we subdivide carefully the intervals for ρ). We then identify the values of MTEs as the first values of ρ for which the number of $\delta_\ell(\rho) \in [\gamma, 2\pi - \eta]$ is incremented.

In order to validate the identified MTEs by the inside-outside algorithm, we numerically evaluate the MTEs by solving the eigenvalue problem (14). For that purpose we use a H^1-variational formulation of the problem (as in [11]). Multiplying the second equation by $w' \in \mathrm{H}^1_0(\Omega_b)$ and the first equation by $v' \in \mathrm{H}^1(\Omega_b)$, we obtain after integration by parts and using that the normal derivative of w vanishes on $\partial\Omega_b$

$$\begin{cases} \displaystyle\int_{\Omega_b} (\nabla w \nabla v' + k^2 n w v')dx = \int_{\Omega_b} (\rho - k^2 n) v v' dx, \\ \displaystyle\int_{\Omega_b} (\nabla v \nabla w' + \rho v w')dx = 0, \end{cases} \tag{51}$$

for all $(w', v') \in \mathrm{H}^1_0(\Omega_b) \times \mathrm{H}^1(\Omega_b)$. This eigenvalue problem is discretized for $(w, v) \in \mathrm{H}^1_0(\Omega_b) \times \mathrm{H}^1(\Omega_b)$ using P_1 Lagrange finite elements with the help of `FreeFem++` [13]. The discrete eigenvalue problem is then solved using the package `PETSc` embedded in `FreeFem++`. Since we relaxed the condition on the normal trace of w, we observed that some spurious numerical eigenvalues may show up. They correspond to eigenfunctions that do not have vanishing normal derivative for w on $\partial\Omega_b$. Theses spurious modes are discarded by checking whether the latter property is verified or not.

We first validate the procedure in the case of the disc for which the numerical results have already been obtained above using analytical expression of the solutions. Figure 4-left shows the obtained results in the case of the experiment illustrated by Figure 1. We observe that we obtain the same results. In order to verify the stability of the inside outside duality method with respect to noise, we repeat the experiment by adding a random noise to the far field operator F. More specifically, we replace each entry $F_{i,j}$ of the far field with $\tilde{F}_{i,j} := (1 + \eta_{i,j})F_{i,j}$ where $\eta_{i,j} \in \mathbb{C}$ has a real and imaginary values randomly drawn between $-\epsilon$ and $\epsilon > 0$. Figure 4-right displays the obtained results for $\epsilon = 2\%$.



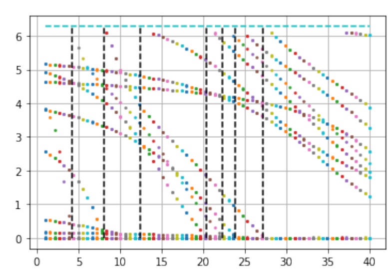


FIGURE 2. The two figures display the curves $\rho \mapsto \delta_\ell(\rho) \in [0, 2\pi)$ for $\rho \in (1, 40)$ in the case where $\Omega = \Omega_b = B_R$, $n = 4$, $k = 5$ and $R = 2\pi/k$. This corresponds to the same experiment as in Figure 1 but using finite elements method to numerically evaluate the far fields and MTEs. The vertical dashed lines correspond to the numerical MTEs obtained by solving (51). Figure on the left is obtained with far field without added random noise. The figure on the right is obtained with noisy far field with noise level $\epsilon = 2\%$.

The following examples are for other simple geometries. The first two are convex domains shown in Figure 3. The associate spectrum is relatively simple and the inside outside duality is able to correctly identify the MTEs as attested by Figure 4.

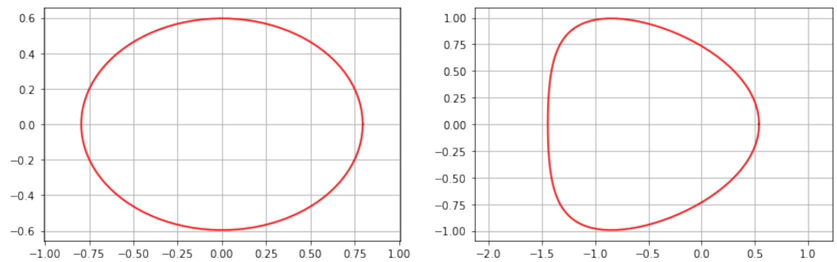


FIGURE 3. Geometry of the convex domains for our experiments

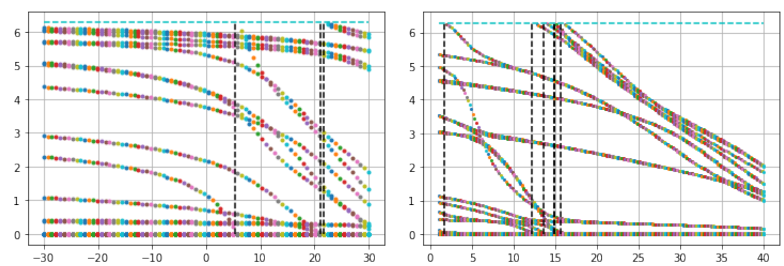


FIGURE 4. The left and right figures display the curves $\rho \mapsto \delta_\ell(\rho) \in [0, 2\pi)$ for $\rho \in (1, 40)$ in the case where $n = 2$, $k = 5$ and the domains $\Omega = \Omega_b$ are respectively shown in Figure 3 left and right. The vertical dashed lines correspond to the numerical MTEs obtained by solving (51). The far fields are corrupted with a noise level $\epsilon = 2\%$.

Reference numerical values	5.4	21.13	21.68
IOD with noise level $\epsilon = 2\%$	6.16	21.76	22.47

TABLE 1. MTEs extracted from Figure 4 left for the ellipse in Figure 3 left. The reference numerical values are obtained by solving (51).

Reference numerical values	1.61	12.18	13.62	14.864	14.867	15.62
IOD with noise level $\epsilon = 2\%$	1.97	12.89	14.36	15.18	15.18	16.19

TABLE 2. MTEs extracted from Figure 4 right for the kite in Figure 3 right. The reference numerical values are obtained by solving (51).

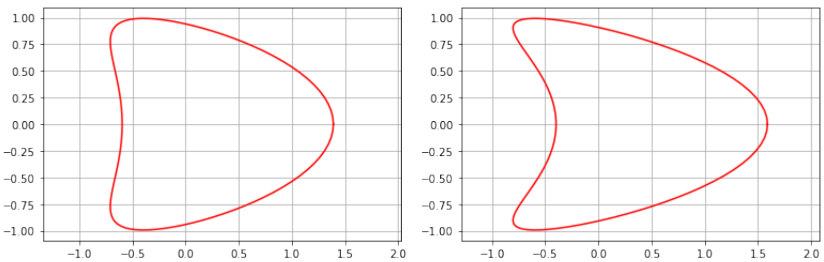


FIGURE 5. The examples of non convex domains with different concavity.

In the last two examples we consider non convex domains having the shape of a kite as shown in Figure 5. For these domains, we observe that the number of MTEs is higher in the same interval as for the previous geometries as shown in Figure 6 that corresponds to noise free far field operators. The MTEs are correctly identified but the higher is the concavity the lesser is the precision, especially for closely separated eigenvalues. Figure 7 corresponds with noisy far field operators with noise level $\epsilon = 2\%$. For these examples, it seems difficult to identify the exact values of MTEs. One should then be cautious when exploiting the MTEs in solutions to inverse problems: precise values maybe out of reach using the inside outside duality for non convex geometries. Comparing Figures 6 and 7 we observe that the overall behaviour of the spectrum of the far field operator is relatively stably reproduced in the case of noisy operators. Consequently, if only qualitative behaviour of these quantities is of interest (as for instance for the inversion method developed in [1]), then the use of the inside outside duality can be useful. For further discussions on the numerical implementation/accuracy of the inside outside duality, we refer to [14].

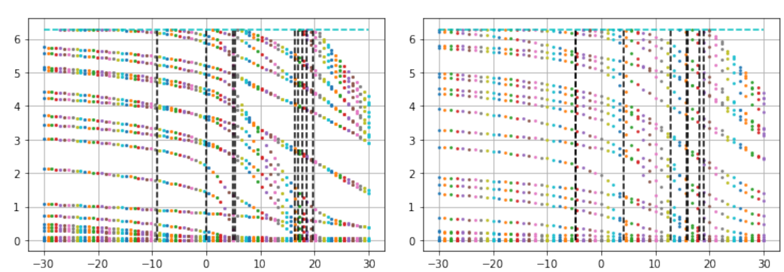


FIGURE 6. The left and right figures display the curves $\rho \mapsto \delta_\ell(\rho) \in [0, 2\pi)$ for $\rho \in (1, 40)$ in the case where $n = 2$, $k = 5$ and the domains $\Omega = \Omega_b$ are respectively shown in Figure 5 left and right. The vertical dashed lines correspond to the numerical MTEs obtained by solving (51).

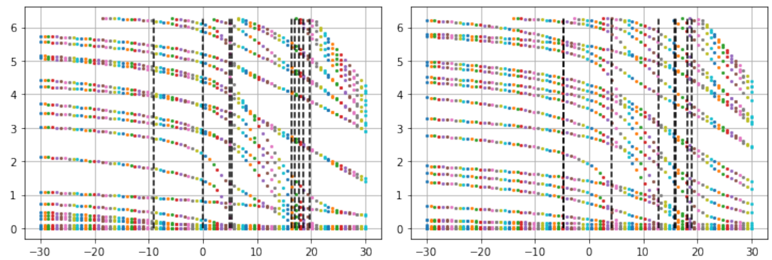


FIGURE 7. The left and right figures display the curves $\rho \mapsto \delta_\ell(\rho) \in [0, 2\pi)$ for $\rho \in (1, 40)$ in the case where $n = 2$, $k = 5$ and the domains $\Omega = \Omega_b$ are respectively shown in Figure 5 left and right. The vertical dashed lines correspond to the numerical MTEs obtained by solving (51). The far fields are corrupted with a noise level $\epsilon = 2\%$.

Ref. values	-9	0.08	4.9	5.3	16.31	16.93	17.72	18.6	19.7
IOD without noise	-7.5	0.75	5.26	6.01	16.52	17.25	18.02	19.51	21.02
IOD, noise $\epsilon = 2\%$	-5.98	1.5	6.01	6.01	16.50	17.27	18.01	19.72	21.01

TABLE 3. MTEs extracted from Figure 6 left and Figure 7 left, which correspond with kite in Figure 5 left. The reference numerical values are obtained by solving (51).

Ref. values	-4.81	-4.61	4.17	12.75	15.82	16.02	18.05	18.89
IOD without noise	-3.71	-3.71	4.51	14.26	16.72	18.00	18.78	19.51
IOD, noise $\epsilon = 2\%$	-2.98	-2.98	5.25	14.26	16.51	18.01	19.50	21.30

TABLE 4. MTEs extracted from Figure 6 right and Figure 7 right, which correspond with kite in Figure 5 right. The reference numerical values are obtained by solving (51).

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