Erratum: Constraint algorithm for singular field theories in the k-cosymplectic framework

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A simple geometric description of singular autonomous field theories is provided by k-presymplectic geometry. Consistency of field equations can be analyzed by means of a constraint algorithm. In our recent paper [1] we extended this analysis to the non-autonomous case. In this case the geometric setting is provided by the notion of k-precosymplectic structure. However, to ensure the existence of Reeb vector fields and Darboux coordinates, we restricted our attention to k-precosymplectic manifolds of the form $\mathbb{R}^k \times P$, with P a k-presymplectic manifold.

As a typical example, we analized the case of affine Lagrangians of the type

$$L(x^\alpha, q^i, v^i_\alpha) = f^\mu_j(x^\alpha, q^i) v^j_\mu + g(x^\alpha, q^i) \tag{1}$$

on the manifold $\mathbb{R}^k \times T^1_k Q$, and a particular academic example (sections 6.1 and 6.2). Nevertheless, such Lagrangians do not result in k-precosymplectic structures of the above mentioned type, and their analysis as presented in the paper is not correct (for instance, Reeb vector fields may not be well defined).

In this note we correct this mistake by restricting our study to the family of affine Lagrangians of the type $L(x^\alpha, q^i, v^i_\alpha) = f(q^i) v^j_\mu + g(x^\alpha, q^i)$, which lead to k-precosymplectic structures as previously indicated. We also analyze a particular example in this class that replaces the one in section 6.2.

Affine Lagrangians

Let Q be the configuration manifold of a field theory. The bundle $\bar\tau_1\colon \mathbb{R}^k \times T^1_k Q \to \mathbb{R}^k$ represents its non-autonomous phase space of k-velocities, and has coordinates $(x^\alpha, q^i, v^i_\alpha)$. We consider an affine Lagrangian $L\colon \mathbb{R}^k \times T^1_k Q \to \mathbb{R}$ of the form

$$L(x^\alpha, q^i, v^i_\alpha) = f^\mu_j(q^i) v^j_\mu + g(x^\alpha, q^i) \,. \tag{2}$$

Such a function is the sum of the pullbacks to $\mathbb{R}^k \times T^1_k Q$ of two functions:

- a linear function $T^1_k Q \to \mathbb{R}$ on the fibers of the bundle $T^1_k Q \to Q$;

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- an arbitrary function $\mathbb{R}^k \times Q \to \mathbb{R}$.

In this way, the difference in the treatment of the Lagrangian (2) with respect to the Lagrangian (1) is that $\dfrac{\partial f^\mu_j}{\partial x^\alpha} = 0$, both in the Lagrangian and the Hamiltonian formalisms. With these changes, most of the equations of section 6.1 of [1] are correct, as well as its conclusions. Notice in particular that, for Lagrangians of the form (2), Reeb vector fields always exist and can be taken to be $\mathcal{R}_\alpha = \dfrac{\partial}{\partial x^\alpha}$.

A simple affine Lagrangian model

Here we analyze a simple Lagrangian of type (2), which should replace example 6.2 in [1].

Lagrangian formalism: The configuration manifold is $\mathbb{R}^2 \times Q = \mathbb{R}^2 \times \mathbb{R}^2$, with coordinates $(x^1, x^2; q^1, q^2)$. The Lagrangian formalism takes place in $\mathbb{R}^2 \times \oplus^2 TQ$, with coordinates $(x^1, x^2, q^1, q^2, v^1_1, v^1_2, v^2_1, v^2_2)$, and we consider the Lagrangian function

$$L = q^2 v^1_1 - q^1 v^2_2 + q^1 q^2 x^1 \,.$$

We have the forms

$$\eta^1 = \mathrm{d}x^1 \,, \quad \eta^2 = \mathrm{d}x^2 \,; \quad \omega^1_L = \mathrm{d}q^1 \wedge \mathrm{d}q^2 \,, \quad \omega^2_L = \mathrm{d}q^1 \wedge \mathrm{d}q^2 \,,$$

and the Reeb vector fields $\mathcal{R}^L_1 = \dfrac{\partial}{\partial x^1}$, $\mathcal{R}^L_2 = \dfrac{\partial}{\partial x^2}$. The energy is simply $E_L = -q^1 q^2 x^1$, and, if $\mathcal{X} = (X_1, X_2) \in \mathfrak{X}^2(\mathbb{R}^2 \times \oplus^2 TQ)$ a generic 2-vector field with

$$X_\alpha = \frac{\partial}{\partial x^\alpha} + F^1_\alpha \frac{\partial}{\partial q^1} + F^2_\alpha \frac{\partial}{\partial q^2} + G^1_{\alpha 1} \frac{\partial}{\partial v^1_1} + G^1_{\alpha 2} \frac{\partial}{\partial v^1_2} + G^2_{\alpha 1} \frac{\partial}{\partial v^2_1} + G^2_{\alpha 2} \frac{\partial}{\partial v^2_2} \,,$$

then the Lagrangian equation $i_{X_\alpha} \omega^\alpha_L = \mathrm{d}E_L - \mathcal{R}^L_\alpha(E_L)\,\mathrm{d}x^\alpha$ is

$$F^1_1 \mathrm{d}q^2 - F^2_1 \mathrm{d}q^1 + F^1_2 \mathrm{d}q^2 - F^2_2 \mathrm{d}q^1 = -q^2 x^1 \mathrm{d}q^1 - q^1 x^1 \mathrm{d}q^2 \,,$$

which leads to

$$F^2_1 + F^2_2 = q^2 x^1 \,, \quad F^1_1 + F^1_2 = -q^1 x^1 \,.$$

Imposing the second order condition, $F^l_\mu = v^l_\mu$, we have that the 2-vector field $\mathcal{X} = (X_1, X_2)$ is

$$X_\alpha = \frac{\partial}{\partial x^\alpha} + v^1_\alpha \frac{\partial}{\partial q^1} + v^2_\alpha \frac{\partial}{\partial q^2} + G^l_{\alpha\nu} \frac{\partial}{\partial v^l_\nu}$$

and we get the two constraints

$$\zeta_1 = v^2_1 + v^2_2 - q^2 x^1 = 0 \quad , \quad \zeta_2 = v^1_1 + v^1_2 + q^1 x^1 = 0$$

The constraints ζ_1 and ζ_2 define the submanifold $\mathcal{S}_1 \hookrightarrow \mathbb{R}^2 \times \oplus^2 TQ$. Next, the tangency conditions on this submanifold lead to

$$\begin{cases} X_1(\zeta_1) = -q^2 + G^2_{11} + G^2_{12} - x^1 v^2_1 = 0 \\ X_2(\zeta_1) = -x^1 v^2_2 + G^2_{21} + G^2_{22} = 0 \end{cases} \quad , \quad \begin{cases} X_1(\zeta_2) = q^1 + x^1 v^1_1 + G^1_{11} + G^1_{12} = 0 \\ X_2(\zeta_2) = x^1 v^1_2 + G^1_{21} + G^1_{22} = 0 \end{cases}$$

which allow us to partially determine the coefficients $G^l_{\alpha\nu}$. Notice that no new constraints appear. Thus, the final constraint submanifold is $\mathcal{S}_1$.

Hamiltonian formalism: The Hamiltonian formalism takes place in the bundle $\mathbb{R}^2 \times \oplus^2 T^*Q$, which has coordinates $(x^1, x^2, y^1, y^2, p_1^1, p_1^2, p_2^1, p_2^2)$. The Legendre map $\mathcal{F}L\colon \mathbb{R}^2 \times \oplus^2 TQ \to \mathbb{R}^2 \times \oplus^2 T^*Q$ is given by

$$(x^1, x^2, y^1, y^2, p_1^1, p_1^2, p_2^1, p_2^2) = \mathcal{F}L(x^1, x^2, q^1, q^2; v_1^1, v_2^1, v_1^2, v_2^2) = (x^1, x^2, q^1, q^2; q^2, 0, 0, -q^1)\,.$$

Its image is the submanifold $\mathcal{P}$ of $\mathbb{R}^2 \times \oplus^2 T^*Q$ given by the primary constraints

$$p_1^1 = q^2\,, \quad p_1^2 = 0\,, \quad p_2^1 = 0\,, \quad p_2^2 = -q^1\,;$$

so, we can describe $\mathcal{P}$ with coordinates (x^1, x^2, q^1, q^2). In $\mathcal{P}$ we have the forms

$$\eta^1 = \mathrm{d}x^1\,, \quad \eta^2 = \mathrm{d}x^2\,; \quad \omega^1 = \mathrm{d}q^1 \wedge \mathrm{d}q^2\,, \quad \omega^2 = \mathrm{d}q^1 \wedge \mathrm{d}q^2\,,$$

and the Reeb vector fields $\mathcal{R}_1 = \dfrac{\partial}{\partial x^1}$, $\mathcal{R}_2 = \dfrac{\partial}{\partial x^2}$. The Hamiltonian function is $h = -q^1 q^2 x^1$. Let $\mathcal{X} = (X_1, X_2) \in \mathfrak{X}^2(\mathcal{P})$ be a generic 2-vector field with

$$X_\alpha = \frac{\partial}{\partial x^\alpha} + B_\alpha^1 \frac{\partial}{\partial q^1} + B_\alpha^2 \frac{\partial}{\partial q^2}\,;$$

then the Hamiltonian equation $i_{X_\alpha}\omega^\alpha = \mathrm{d}h - \mathcal{R}_\alpha(h)\,\mathrm{d}x^\mu$ gives

$$B_1^1 \mathrm{d}q^2 - B_1^2 \mathrm{d}q^1 + B_2^1 \mathrm{d}q^2 - B_2^2 \mathrm{d}q^1 = -x^1 q^2 \mathrm{d}q^1 - x^1 q^1 \mathrm{d}q^2\,,$$

which leads to

$$B_1^2 + B_2^2 = x^1 q^2\,, \quad B_1^2 + B_2^1 = -x^1 q^1\,.$$

This allows us to partially determine the coefficients B_α^j. Notice that, in this case, no new constraints appear.

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References

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