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Exponentiated Weibull Burr Type X Distribution Properties and Its Applications

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This study proposes a new distribution called exponentiated Weibull Burr type X distribution which provides greater flexibility in fitting the survival data. We derive several statistical properties of the proposed distribution, namely the quantile function, moment, order statistics, and Renyi entropy. We use the maximum likelihood approach to estimate the proposed distribution's parameters. Simulation study is then conducted with varying sample sizes and parameter values to examine the proposed distribution's performance. Lastly, real data are used to illustrate the flexibility and performance of the proposed distribution, its sub-models, some extension of Burr type X distribution and some nonnested models. The results show that our exponentiated Weibull Burr type X distribution is very competitive and hence can be used as an alternative approach of the competing models. In conclusion, the proposed distribution can model a wide range of survival data, including decreasing, increasing, bathtub, and unimodal hazard functions. It perform better than its sub-models in fitting the survival data and is a strong competitor of some five-parameter and four-parameter distributions.

keywords: survival analysis, Burr type X, exponentiated, Weibull generalized.

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1 Introduction

Statistical distributions have great importance for modelling survival data. However, in some cases, the survival data could not be fitted by the existing distribution, especially for survival data with bathtub and unimodal shaped hazard functions, which are common in survival analysis. Hence, there has been a revival of interest in forming distributions with greater flexibility by adding parameters to the baseline distribution, which fit most of the survival data with various shapes of hazard function. In view of that, several families of distribution have been introduced to obtain new form of distributions that are more flexible in term of fitting real data, such as beta-G (Eugene et al., 2002), exponentiated generalized family of distributions (Cordeiro et al., 2013), gamma-G (Ristić and Balakrishnan, 2012), Kum-G (Cordeiro and de Castro, 2009), and Weibull generalized family of distribution (Bourguignon et al., 2014).

Burr type X distribution is one of the popular distributions in lifetime data modelling as its density and cumulative functions are closed form and thus can apply for data with or without censored data. It is one of the twelve new cumulative distribution functions proposed by Burr (Burr, 1942) using the differential equation method. The initial form of Burr type X distribution is a single-parameter distribution (BX1). Then, in the early 20th century, Surles and Padgett (2001) developed a scaled Burr type X distribution, namely two-parameter Burr type X (BX). Over the past few years, BX distribution has been widely studied, and several new distributions have been generated where the Burr type X distribution is used as the baseline distribution. These include gamma Burr type X (Khaleel et al., 2016), beta-Burr type X (Merovci et al., 2016), exponentiated generalized Burr type X (Khaleel et al., 2018), beta Kumaraswamy Burr type X (Madaki et al., 2018), and Weibull Burr type X (Ibrahim et al., 2017). Weibull Burr type X distribution (WBX) is generated using Weibull-G (Bourguignon et al., 2014) and Burr type X distribution as the baseline distribution. The WBX probability density function (pdf) is defined as

$$\begin{aligned} f(x,\alpha,\beta,\lambda,\theta) =& \frac{2\alpha\beta\lambda^2\theta x e^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\theta\beta-1}}{\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}\right]^{\beta+1}} \\ &\times \exp\left(-\alpha\frac{\left(1-e^{-(\lambda x)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}\right]^{\beta}}\right) \end{aligned} \tag{1}$$

and the corresponding cdf is derived as

$$F(x,\alpha,\beta,\lambda,\theta) = 1-\exp\left(-\alpha\left[\frac{(1-e^{-(\lambda x)^2})^{\theta}}{1-(1-e^{-(\lambda x)^2})^{\theta}}\right]^{\beta}\right), \tag{2}$$

where $\alpha,\beta,\lambda,\theta > 0$. By adding two additional parameters α and β, WBX forms a four-parameter distribution that can model the decreasing, increasing, and bathtub hazard functions. However, it is not able to fit data with unimodal hazard function, which is very common in survival analysis. Hence, this study aims to generate a new distribution, namely exponentiated Weibull Burr type X, that will increase the flexibility of WBX distribution by covering hazard functions in all the shapes mentioned above, including unimodal. The proposed distribution is expected to be superior to overcome the deficiency of WBX.

This paper is outlined and constructed as follows. Section 2 presents the proposed distribution exponentiated Weibull Burr Type X. Section 3 obtains the linear representation of the probability density function and cumulative distribution function. The statistical properties and likelihood function are presented in Sections 4 and 5, respectively. Section 6 shows and examines the performance of EWBX by using the chi-square goodness of fit test and several goodness of fit statistics. Real data sets are utilized to illustrate the performance of EWBX in Section 7. Finally, the findings of this paper are reported in Section 8.

2 Exponentiated Weibull Burr Type X

This study proposed a five-parameter distribution, namely exponentiated Weibull Burr type X distribution (EWBX), which is derived through the exponentiated type of distribution suggested by Gupta et al. (1998) and the WBX distribution in equation (1) as baseline distribution. Hence, the pdf of EWBX is given as

$$\begin{aligned} f\left(x,\alpha,\beta,\gamma,\lambda,\theta\right) =& \frac{2\alpha\beta\lambda^2\theta x e^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{\theta\beta-1}}{\left[1-\left(1-e^{-(\lambda x)^2}\right)^{\theta}\right]^{\beta+1}} \\ &\times \exp\left(-\alpha\left[\frac{(1-e^{-(\lambda x)^2})^{\theta}}{1-(1-e^{-(\lambda x)^2})^{\theta}}\right]^{\beta}\right) \\ &\times\left[1-\exp\left(-\alpha\left[\frac{(1-e^{-(\lambda x)^2})^{\theta}}{1-(1-e^{-(\lambda x)^2})^{\theta}}\right]^{\beta}\right)\right]^{\gamma-1}, \end{aligned} \tag{3}$$

while the corresponding cdf is

$$F\left(x,\alpha,\beta,\gamma,\lambda,\theta\right) = \left[1-\exp\left(-\alpha\left[\frac{(1-e^{-(\lambda x)^2})^{\theta}}{1-(1-e^{-(\lambda x)^2})^{\theta}}\right]^{\beta}\right)\right]^{\gamma}, \tag{4}$$

where $\alpha,\ \beta,\ \gamma,\ \lambda,\ \theta > 0$. Figures 1 and 2 present the EWBX pdf and hazard function for different parameter values. The flexibility of EWBX is evident with the formation of its hazard function in various shapes, including increasing, decreasing, bathtub, and unimodal. This shows that EWBX is very flexible.

EWBX distribution can be reduced to several well-known distributions by changing its parameter values. These reduced models are also known as sub-models of EWBX distributions. The EWBX sub-models are listed in Table 1. For example, EWBX is reduced to WBX when $\gamma = 1$.

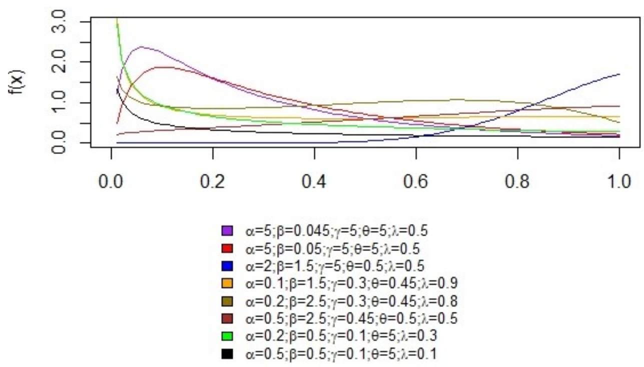

Figure 1: EWBX Distribution Probability Density Functions

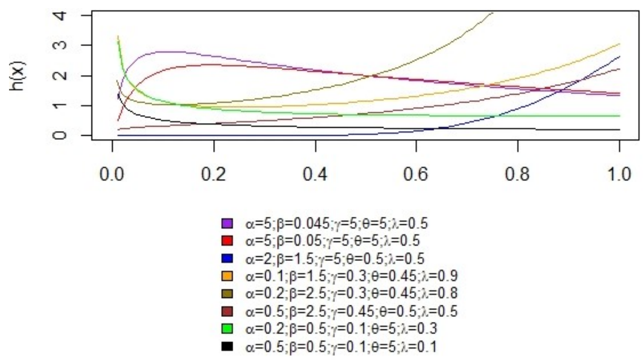

Figure 2: EWBX Distribution Hazard Functions

Table 1: Exponentiated Weibull Burr Type X Distribution Sub-models

	Parameters Values				
Distribution	α	β	γ	λ	θ
WBX			1		
Weibull one-parameter Burr type X			1	1	
Weibull-Rayleigh			1		1

3 Linear Representation

In this section, we derive the linear form of EWBX distribution pdf and cdf that are useful in exploring the statistical properties of the distribution. Recall the EWBX distribution pdf equation (3) and we now let

$$A = e^{\left(-\alpha\left[\frac{(1-e^{-(\lambda x)^2})^\theta}{1-(1-e^{-(\lambda x)^2})^\theta}\right]^\beta\right)}.$$

By rewriting the exponential function as power series, we obtain

$$A = \sum_{i=0}^{\infty} \frac{(-1)^i \alpha^i (1-e^{-(\lambda x)^2})^{\theta\beta i}}{i!\left[1-(1-e^{-(\lambda x)^2})^\theta\right]^{\beta i}}.$$

Now let

$$B = \left[1 - e^{-\alpha\left(\frac{(1-e^{-(\lambda x)^2})^{\theta\beta}}{\left[1-(1-e^{-(\lambda x)^2})^\theta\right]^\beta}\right)}\right]^{\gamma-1}.$$

Expanding the binomial term, B becomes

$$B = \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \alpha^j \frac{(1-e^{-(\lambda x)^2})^{\theta\beta j}}{\left[1-(1-e^{-(\lambda x)^2})^\theta\right]^{\beta j}}.$$

Then equation (3) is reduced to

$$\begin{aligned} f(x,\alpha,\beta,\gamma,\lambda,\theta) &= 2\alpha\beta\gamma\lambda^2\theta x e^{-(\lambda x)^2} \\ &\times \sum_{i=0}^{\infty}\sum_{j=0}^{r-1}\left[\binom{r-1}{j}(-1)^{i+j}\alpha^{i+j}\frac{(1-e^{-(\lambda x)^2})^{\theta\beta(i+j+1)-1}}{\left[1-(1-e^{-(\lambda x)^2})^\theta\right]^{\beta(i+j+1)+1}}\right]. \end{aligned} \tag{5}$$

Using binomial expansion and letting

$$w_{i,j,k} = \binom{r-1}{j}\binom{\beta(i+j+1)+k}{k}(-1)^{i+j+k}\alpha^{i+j},$$

we can rewrite equation (5) as

$$\begin{aligned} f(x,\alpha,\beta,\gamma,\lambda,\theta) =& 2\alpha\beta\gamma\lambda^2\theta x e^{-(\lambda x)^2} \\ &\times \sum_{i=0}^{\infty}\sum_{j=0}^{r-1}\sum_{k=0}^{\infty}\left[w_{i,j,k}(1-e^{-(\lambda x)^2})^{\theta[\beta(i+j+1)+k]-1}\right]. \end{aligned}$$

Hence, the EWBX pdf is given by

$$f(x,\alpha,\beta,\gamma,\lambda,\theta) = \alpha\beta\sum_{i=0}^{\infty}\sum_{j=0}^{r-1}\sum_{k=0}^{\infty}\left[\frac{w_{i,j,k}}{\beta(i+j+1)+k} g(x,\lambda,\theta[\beta(i+j+1)+k])\right], \tag{6}$$

where $g(x,\lambda,\theta[\beta(i+j+1)+k])$ is the BX distribution pdf with parameters λ and $\theta[\beta(i+j+1)+k]$.

By integrating equation (6), we can write the EWBX distribution cdf in the linear form

$$F(x,\alpha,\beta,\gamma,\lambda,\theta)=\alpha\beta\sum_{i=0}^{\infty}\sum_{j=0}^{r-1}\sum_{k=0}^{\infty}\frac{w_{i,j,k}G(x,\lambda,\theta[\beta(i+j+1)+k])}{\beta(i+j+1)+k}, \tag{7}$$

where $G(x,\lambda,\theta[\beta(i+j+1)+k])$ is the cdf of BX distribution.

In summary, the linear form of the EWBX distribution pdf and cdf consists of the BX distribution pdf.

4 Statistical Properties

This section discusses several statistical properties of EWBX distribution. These include quantile function (qf), moments, moment generating function (mgf), order statistics, and φ order Renyi entropy.

4.1 Quantile Function

The EWBX qf can be derived by inverting its cdf equation (4) to give,

$$Q(y)=x=\frac{1}{\lambda}\left(-\ln\left[1-\left(\frac{\left[-\ln\left(1-y^{\frac{1}{\gamma}}\right)\right]^{\frac{1}{\beta}}}{\left(\alpha^{\frac{1}{\beta}}+\left[-\ln\left(1-y^{\frac{1}{\gamma}}\right)\right]^{\frac{1}{\beta}}\right)}\right)^{\frac{1}{\theta}}\right]\right)^{\frac{1}{2}}. \tag{8}$$

The EWBX random variable $x=Q(y)$ can be simulated with $\boldsymbol{Y}\sim U(0,1)$.

4.2 Moment

Moments of distributions are essential to determine the shape of the distribution. It can be applied to obtain the measures of central tendency, kurtosis, and skewness. The r^{th} moment of EWBX is given by

$$E(x^r)=\int_{-\infty}^{\infty}x^r f(x,\alpha,\beta,\gamma,\lambda,\theta)\,dx.$$

Then using equation (6), we obtain

$$E(x^r)=\alpha\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left[\frac{w_{i,j,k}}{\beta(j+1)+k}\int_{-\infty}^{\infty}x^r g(x,\lambda,\theta[\beta(j+1)+k])\,dx\right],$$

where the integral term is the r^{th} moment of BX distribution that is given by

$$\frac{\theta}{\lambda^r}\Gamma\left(\frac{r}{2}+1\right)\sum_{i=0}^{\theta-1}\binom{\theta-1}{i}\frac{(-1)^i}{(i+1)^{\frac{r}{2}+1}}.$$

Thus the r^{th} moment of EWBX is written as

$$\begin{aligned} E\left(x^{r}\right) =& \frac{\alpha\beta\gamma\theta}{\lambda^{r}}\Gamma\left(\frac{r}{2}+1\right) \\ &\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\theta[\beta(j+1)+k]-1}\left[\binom{\theta\left[\beta\left(j+1\right)+k\right]-1}{l}\frac{\left(-1\right)^{l}w_{i,j,k}}{\left(l+1\right)^{\frac{r}{2}+1}}\right]. \end{aligned} \tag{9}$$

4.3 Moment Generating Function

As its name implies, mgf can be used to generate the r^{th} moments of a distribution. Each distribution has a unique mgf and it is useful in studying the properties of a distribution. The EWBX distribution mgf is expressed as

$$M_X\left(t\right)=E\left(e^{tX}\right)=\int_{-\infty}^{\infty}e^{tx}f(x,\alpha,\beta,\gamma,\lambda,\theta)dx.$$

Using the linear representation in equation (6), we have

$$M_X\left(t\right)=\alpha\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left[\frac{w_{i,j,k}}{\beta\left(j+1\right)+k}\int_{-\infty}^{\infty}e^{tx}g\left(x,\lambda,\theta\left[\beta\left(j+1\right)+k\right]\right)dx\right].$$

Then by using the power series, we can write

$$M_X\left(t\right)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\frac{\alpha\beta\gamma w_{i,j,k}t^{l}}{\left[\beta\left(j+1\right)+k\right]l!}\int_{-\infty}^{\infty}x^{l}g\left(x,\lambda,\theta\left[\beta\left(j+1\right)+k\right]\right)dx\right], \tag{10}$$

where the integral term is the l^{th} moment of BX. Then equation (10) can be defined in another form as follows

$$\begin{aligned} M_X\left(t\right)=&\alpha\beta\gamma\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\theta[\beta(j+1)+k]-1}\left[\binom{\theta\left[\beta\left(j+1\right)+k\right]-1}{m}\right. \\ &\left.\times\frac{\left(-1\right)^{m}t^{l}w_{i,j,k}}{\left(m+1\right)^{\frac{l}{2}+1}\lambda^{l}l!}\Gamma\left(\frac{l}{2}+1\right)\right]. \end{aligned} \tag{11}$$

4.4 Order Statistics

Let $X_1 < X_2 < \ldots < X_n$ be EWBX random variables. The i^{th} order statistic of EWBX distribution is written as

$$f_{x(i)}\left(x\right)=\frac{n!}{\left(i-1\right)!\left(n-i\right)!}f\left(x\right)\left[F\left(x\right)\right]^{i-1}\left[1-F\left(x\right)\right]^{n-i}, \tag{12}$$

where $f(x)$ and $F(x)$ are the EWBX distribution pdf and cdf, respectively. Making use of beta function and binomial expansion, we can rewrite equation (12) as

$$f_{x(i)}\left(x\right)=\frac{f(x)}{B(i,n-i+1)}\sum_{j=0}^{n-i}\left(-1\right)^{j}\binom{n-i}{j}\left[F\left(x\right)\right]^{i+j-1}. \tag{13}$$

Let

$$A = 1 - e^{-(\lambda x)^2},$$

$$B = e^{-\alpha\left(\frac{A^{\theta\beta}}{(1-A^{\theta})^{\beta}}\right)},$$

and

$$C = \frac{2\alpha\beta\gamma\lambda^2\theta x e^{-(\lambda x)^2} A^{\theta\beta-1}}{[1-A^{\theta}]^{\beta+1}},$$

then equation (3) and equation (4) can be expressed as

$$f(x) = CB\left(1-B\right)^{\gamma-1} \tag{14}$$

and

$$F\left(x\right) = \left(1-B\right)^{\gamma}, \tag{15}$$

respectively. Then we have

$$f_{x(i)}\left(x\right) = \frac{C}{B\left(i, n-i+1\right)} B \sum_{j=0}^{n-i} \left[(-1)^j \binom{n-i}{j} [1-B]^{\gamma(i+j)-1}\right]. \tag{16}$$

By expanding $[1-B]^{\gamma(i+j)-1}$ in equation (16), we obtain

$$\begin{aligned} f_{x(i)}\left(x\right) =& \frac{C}{B\left(i, n-i+1\right)} \\ & \times \sum_{j=0}^{n-i} \sum_{k=0}^{\gamma(i+j)-1} \left[\binom{n-i}{j} \binom{\gamma\left(i+j\right)-1}{k} (-1)^{j+k} e^{-\alpha\left(\frac{A^{\theta\beta}}{(1-A^{\theta})^{\beta}}\right)(k+1)}\right]. \end{aligned} \tag{17}$$

Expanding the exponential term in equation (17), we get

$$\begin{aligned} f_{x(i)}\left(x\right) =& \frac{2\alpha\beta\gamma\lambda^2\theta x e^{-(\lambda x)^2}}{B\left(i, n-i+1\right)} \sum_{j=0}^{n-i} \sum_{k=0}^{\gamma(i+j)-1} \sum_{l=0}^{\infty} \left[\binom{n-i}{j} \binom{\gamma\left(i+j\right)-1}{k} \right. \\ & \left. \times (-1)^{j+k+l} \frac{\alpha^l \left(k+1\right)^l A^{\theta\beta(l+1)-1}}{l! \left(1-A^{\theta}\right)^{\beta(l+1)+1}}\right]. \end{aligned}$$

Using the binomial expansion, we now have

$$\begin{aligned} f_{x(i)}\left(x\right) =& \frac{2\alpha\beta\gamma\lambda^2\theta x e^{-(\lambda x)^2}}{B\left(i, n-i+1\right)} \sum_{j=0}^{n-i} \sum_{k=0}^{\gamma(i+j)-1} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \left[\binom{n-i}{j} \binom{\gamma\left(i+j\right)-1}{k} \right. \\ & \left. \times \binom{\beta\left(l+1\right)+m}{m} (-1)^{j+k+l+m} \frac{\alpha^l \left(k+1\right)^l A^{\theta[\beta(l+1)+m]-1}}{l!}\right]. \end{aligned} \tag{18}$$

Let

$$w_{i,j,k,l,m} = \binom{n-i}{j}\binom{\gamma(i+j)-1}{k}\binom{\beta(l+1)+m}{m}\frac{\alpha^l(-1)^{j+k+l+m}(k+1)^l}{l!B(i,n-i+1)},$$

then equation (18) can be rewritten as

$$f_{x(i)}(x) = \alpha\beta\gamma\sum_{j=0}^{n-i}\sum_{k=0}^{\gamma(i+j)-1}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{w_{i,j,k,l,m}\; g(x,\lambda,\theta[\beta(l+1)+m])}{[\beta(l+1)+m]}, \tag{19}$$

where $g(x,\lambda,\theta[\beta(l+1)+m])$ is the BX distribution pdf.

4.5 Renyi Entropy

Renyi entropy is essential in ecology and statistics. It measures the diversity, variation, and uncertainty of a distribution. The φ order Renyi entropy of EWBX distribution is given by

$$I_{\varphi}(x) = \frac{1}{1-\psi}\log\left(\int_{-\infty}^{\infty}[f(x)]^{\varphi}\; dx\right), \tag{20}$$

where $\varphi \geq 0$ and $\varphi \neq 1$. Making use of equation (14) and binomial expansion in equation (20), we have

$$[f(x)]^{\varphi} = C^{\varphi}\sum_{i=0}^{\infty}\binom{\varphi(\gamma-1)}{i}(-1)^i B^{i+\varphi}.$$

By means of power series and binomial expansion, we get

$$\begin{aligned}[f(x)]^{\varphi} = \left(2\alpha\beta\gamma\lambda^2\theta x e^{-(\lambda x)^2}\right)^{\varphi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}&\left[\binom{\beta(\varphi+j)+\varphi-1}{k}\binom{\varphi(\gamma-1)}{i}\right.\\ &\left.\times(-1)^{i+j+k}\frac{\alpha^j(i+\varphi)^j A^{\theta[\beta(\varphi+j)+k]-\varphi}}{j!}\right].\end{aligned} \tag{21}$$

Using $A = 1-e^{-(\lambda x)^2}$ and binomial expansion, then $A^{\theta[\beta(\varphi+j)+k]-\varphi}$ in equation (21) can be written as

$$A^{\theta[\beta(\varphi+j)+k]-\varphi} = \sum_{l=0}^{\infty}\left[\binom{\theta[\beta(\varphi+j)+k]-\varphi}{l}e^{-l(\lambda x)^2}(-1)^l\right]. \tag{22}$$

Combining equation (21) and (22), give us

$$\begin{aligned}[f(x)]^{\varphi} = \left(2\alpha\beta\gamma\lambda^2\theta x\right)^{\varphi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}&\left[\binom{\varphi(\gamma-1)}{i}\binom{\beta(\varphi+j)+\varphi-1}{k}\right.\\ &\left.\times\binom{\theta[\beta(\varphi+j)+k]-\varphi}{l}\frac{(-1)^{i+j+k+l}\alpha^j(i+\varphi)^j}{j!}e^{-(l+\varphi)(\lambda x)^2}\right].\end{aligned} \tag{23}$$

Let

$$V_{i,j,k,l}=\binom{\varphi(\gamma-1)}{i}\binom{\beta(\varphi+j)+\varphi-1}{k}\binom{\theta\left[\beta(\varphi+j)+k\right]-\varphi}{l}\frac{(-1)^{i+j+k+l}\,\alpha^{j}\,(i+\varphi)^{j}}{j!},$$

then equation (23) can be rewritten as

$$[f(x)]^{\varphi}=\left(2\alpha\beta\gamma\lambda^{2}\theta x\right)^{\varphi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}V_{i,j,k,l}e^{-(l+\varphi)(\lambda x)^{2}}\ .$$

Thus,

$$\int_{0}^{\infty}[f(x)]^{\varphi}\,dx=\left(2\alpha\beta\gamma\lambda^{2}\theta\right)^{\varphi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[V_{i,j,k,l}\int_{0}^{\infty}x^{\varphi}e^{-(l+\varphi)(\lambda x)^{2}}dx\right]$$

and the integral term can be written in the form of gamma function as follows

$$\int_{0}^{\infty}x^{\varphi}e^{-(l+\varphi)(\lambda x)^{2}}dx=\int_{0}^{\infty}\left(x^{2}\right)^{\frac{\varphi}{2}}e^{-(l+\varphi)\lambda^{2}x^{2}}dx=\frac{\Gamma\left(\frac{\varphi}{2}+1\right)}{\lambda^{\varphi+2}\left(l+\varphi\right)^{\frac{\varphi}{2}+1}}.$$

Subsequently, we obtain the EWBX distribution Renyi entropy as

$$I_{\varphi}(x)=\frac{1}{1-\psi}\log\left(\left(2\alpha\beta\gamma\lambda^{2}\theta\right)^{\varphi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[V_{i,j,k,l}\frac{\Gamma\left(\frac{\varphi}{2}+1\right)}{\lambda^{\varphi+2}\left(l+\varphi\right)^{\frac{\varphi}{2}+1}}\right]\right). \tag{24}$$

5 Estimation

In this study, the EWBX parameters estimates are obtained by adopting maximum likelihood estimation (MLE) approach. Suppose $x_1,\ x_2,\ \ldots,\ x_n$ is a random sample of EWBX distribution. Then the log-likelihood function is expressed as

$$\begin{aligned}l(\boldsymbol{\phi})=&\sum_{i=1}^{n}\left[\ln\left(2\alpha\beta\gamma\lambda^{2}\theta x_{i}\right)-(\lambda x_{i})^{2}+(\theta\beta-1)\ln(1-e^{-(\lambda x_{i})^{2}})\right]\\&-(\beta+1)\sum_{i=1}^{n}\ln\left[1-\left(1-e^{-(\lambda x_{i})^{2}}\right)^{\theta}\right]-\alpha\sum_{i=1}^{n}\frac{\left(1-e^{-(\lambda x_{i})^{2}}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda x_{i})^{2}}\right)^{\theta}\right]^{\beta}}\\&+(\gamma-1)\sum_{i=1}^{n}\ln\left[1-e^{\left(-\alpha\left[\frac{\left(1-e^{-(\lambda x_{i})^{2}}\right)^{\theta}}{1-\left(1-e^{-(\lambda x_{i})^{2}}\right)^{\theta}}\right]^{\beta}\right)}\right].\end{aligned} \tag{25}$$

where $\boldsymbol{\phi}=[\alpha\ \beta\ \gamma\ \lambda\ \theta]^{T}$.

To maximize the log-likelihood function, we obtain its first-order partial derivatives:

$$\begin{aligned}\frac{\partial l}{\partial \alpha} =& \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{\left(1-e^{-(\lambda x_i)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}\right]^{\beta}} \\ &+ (\gamma-1)\sum_{i=1}^{n} \frac{\left(1-e^{-(\lambda x_i)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}\right]^{\beta}} \left[e^{\left(\alpha\left[\frac{\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}}{1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}}\right]^{\beta}\right)} - 1\right]^{-1},\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\partial l}{\partial \beta} =& \frac{n}{\beta} + \sum_{i=1}^{n} \left(\theta \ln\left[1-e^{-(\lambda x_i)^2}\right] - \ln\left[1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}\right]\right) \\ &\times \left(1 - \frac{\alpha\left(1-e^{-(\lambda x_i)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}\right]^{\beta}}\right) \\ &+ \alpha(\gamma-1)\sum_{i=1}^{n} \frac{\left(1-e^{-(\lambda x_i)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}\right]^{\beta}} \left[1 - e^{\left(-\alpha\left[\frac{\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}}{1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}}\right]^{\beta}\right)}\right]^{-1} \\ &\times \left[\left(\theta \ln\left[1-e^{-(\lambda x_i)^2}\right] - \ln\left[1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}\right]\right) e^{\frac{-\alpha\left(1-e^{-(\lambda x_i)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}\right]^{\beta}}}\right],\end{aligned} \tag{27}$$

$$\frac{\partial l}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n} \ln\left[1 - e^{\left(-\alpha\left[\frac{\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}}{1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}}\right]^{\beta}\right)}\right], \tag{28}$$

$$\begin{aligned}
\frac{\partial l}{\partial \lambda} =& \frac{2n}{\lambda} - 2\lambda \sum_{i=1}^{n} x_i^2 + 2\lambda\left(\theta\beta - 1\right)\sum_{i=1}^{n} \frac{x_i^2}{e^{(\lambda x_i)^2} - 1} \\
&+ 2\lambda\theta\left(\beta+1\right)\sum_{i=1}^{n} \frac{x_i^2 e^{-(\lambda x_i)^2}\left(1-e^{-(\lambda x_i)^2}\right)^{\theta-1}}{1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}} \\
&- 2\alpha\beta\lambda\theta \sum_{i=1}^{n} \frac{x_i^2 e^{-(\lambda x_i)^2}\left(1-e^{-(\lambda x_i)^2}\right)^{\theta\beta-1}}{\left[1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}\right]^{\beta+1}} \\
&+ 2\alpha\beta\lambda\theta\left(\gamma-1\right)\sum_{i=1}^{n} \frac{x_i^2 e^{-(\lambda x_i)^2}\left(1-e^{-(\lambda x_i)^2}\right)^{\theta\beta-1}}{\left[1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}\right]^{\beta+1}} \\
&\times \left[e^{\left(\alpha\left[\frac{\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}}{1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}}\right]^{\beta}\right)} - 1\right]^{-1},
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
\frac{\partial l}{\partial \theta} =& \frac{n}{\theta} + \left(\beta+1\right)\sum_{i=1}^{n} \frac{\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}\ln\left(1-e^{-(\lambda x_i)^2}\right)}{1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}} \\
&- \alpha\sum_{i=1}^{n} \frac{\beta\left(1-e^{-(\lambda x_i)^2}\right)^{\beta\theta}\ln\left(1-e^{-(\lambda x_i)^2}\right)}{\left[1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}\right]^{\beta+1}} + \beta\sum_{i=1}^{n}\ln\left(1-e^{-(\lambda x_i)^2}\right) \\
&+ \alpha\beta\left(\gamma-1\right)\sum_{i=1}^{n}\left[\frac{\left(1-e^{-(\lambda x_i)^2}\right)^{\beta\theta}\ln\left(1-e^{-(\lambda x_i)^2}\right)}{\left[1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}\right]^{\beta+1}}\right. \\
&\left.\times\left[e^{\left(\alpha\left[\frac{\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}}{1-\left(1-e^{-(\lambda x_i)^2}\right)^{\theta}}\right]^{\beta}\right)} - 1\right]^{-1}\right].
\end{aligned} \tag{30}$$

To estimate the EWBX parameters, we set all the first-order partial derivatives equal to zero. The complicated system of equations calls for quasi-Newton (BFGS) approach in the R software to obtain the parameter estimates.

6 Simulations

This section assesses the performance of the EWBX parameters maximum likelihood estimates (MLEs) via Monte Carlo simulations. We used EWBX distribution quantile function equation (8) to obtain the EWBX random variable. We consider different sets of parameter values and sample sizes $n = 50$, $n = 150$, and $n = 300$. The simulation is repeated 2000 times for each case. The average (AVE), root mean square errors (RMSE) and bias of the estimation are displayed in Table 2. The small values of RMSE and bias show that MLE works well in estimating the EWBX parameters. From Table 2, it can be observed that the AVEs are near to the true values, and RMSEs and bias are decreased toward zero as the sample size increases. On the whole, MLE approach is appropriate for estimating EWBX distribution parameters.

7 Application

As an illustration, the EWBX distribution is fitted to two real data sets. The first data set includes the failure time of 85 aircraft windshields which are reported in Tahir et al. (2015). Meanwhile, the second data set records the remission time of 128 bladder cancer patients (Lee and Wang, 2003). The flexibility and performance of EWBX are compared with its sub-models, some extended Burr type X distributions and nonnested model. The sub-models and extended Burr type X distribution include WBX, beta Burr type X (BBX), exponentiated Generalized Burr type X (EGBX), gamma Burr type X (GBX) and BX distributions. Meanwhile, the nonnested model includes three five-parameter and one four-parameter distributions: exponentiated Burr type XII Poisson (EBXIIP) (da Silva et al., 2015), exponentiated Weibull Burr type XII (EWBXII) (Abouelmagd et al., 2017), beta Burr type XII (BBXII) (Paranaíba et al., 2011), and generalised Marshall-Olkin extended Burr-XII (GMOBXII) (Handique and Chakraborty, 2018)distributions. The performances of these models are assessed using chi-square goodness of fit test and the goodness of fit statistics, including Akaike (AIC), Bayesian (BIC), corrected Akaike (CAIC), and Hannan-Quinn (HQ) information criterion. The chi-square goodness of fit test determines if the data came from a particular distribution. While the information criterion are used to evaluate how well a model fits the data and select which model gives the best fit for the data, a smaller value of these criteria denotes a superior fit. The goodness of fit statistics has different advantages and disadvantages. For instance, AIC emphasis more on model performance, BIC penalises more for the model complexity, and CAIC and HQ are suitable for cases with large sample size. Besides, we employed MLE to estimate the parameters of the distributions and all the analyses were done by using R-software. The chi-square goodness of fit test results is presented in Table 3. Meanwhile, MLEs, values of negative log-likelihood $-l$, AIC, BIC, CAIC, and HQ of all competing models are displayed in Tables 4 and 5. The plot of the survival functions for the two data sets are presented in Figure 3 and Figure 4, respectively.

All the competing models can fit the first data set well as the p-values in Table 3 are all greater than 0.05. Table 4 shows that EWBX has the lowest value of AIC, CAIC, and HQ and the second lowest value of BIC compared to its submodels and extended Burr type X distribution. Apart from that, it has the second lowest values for all criteria when compared with the nonnested models. These indicate that EWBX gives the finest fit to the first data set compared to its submodels and extended Burr type X distribution and is a strong competitor to the nonnested models.

For the second data set, all the models fit the data well except BBX, EGBX, and BX as their p-values in Table 3 are less than 0.05. From Table 5, we can observe that EWBX has the second smallest AIC, CAIC, and HQ values and the third smallest BIC values compared to its submodels and extended Burr type X distributions. In addition, the differences between its AIC, BIC, CAIC, and HQ values and the nonnested models are small. This implies that EWBX can be used as an alternative approach for all the competing models. In Figures 3 and 4, we can observe that EWBX gives a good fit for the two data sets. In conclusion, EWBX is a strong competitor to all competing models for both data sets.

Table 2: Ave, RMSE, and Bias for Different Set of Parameter Values

Set 1									
n	$\alpha = 1.5$			$\beta = 1.5$			$\gamma = 1.5$		
	AVE	RMSE	Bias	AVE	RMSE	Bias	AVE	RMSE	Bias
50	1.4791	0.1558	-0.0209	1.5021	0.0985	0.0021	1.4568	0.2991	-0.0432
150	1.491	0.0993	-0.009	1.5013	0.0607	0.0013	1.4802	0.1977	-0.0198
300	1.4924	0.0746	-0.0076	1.4959	0.0492	-0.0041	1.481	0.1451	-0.019
n	$\lambda = 0.25$			$\theta = 0.5$					
	AVE	RMSE	Bias	AVE	RMSE	Bias			
50	0.2607	0.0664	0.0107	0.5565	0.1392	0.0565			
150	0.2525	0.0493	0.0025	0.5216	0.0798	0.0216			
300	0.2529	0.0359	0.0029	0.5161	0.061	0.0161			
Set 2									
n	$\alpha = 1.5$			$\beta = 1.5$			$\gamma = 1.5$		
	AVE	RMSE	Bias	AVE	RMSE	Bias	AVE	RMSE	Bias
50	1.4985	0.0221	-0.0015	1.5127	0.0888	0.0127	1.5065	0.0656	0.0065
150	1.4994	0.0291	-0.0006	1.5074	0.0602	0.0074	1.5031	0.0743	0.0031
300	1.5001	0.0226	0.0001	1.5049	0.0694	0.0049	1.5023	0.0786	0.0023
n	$\lambda = 0.15$			$\theta = 1.5$					
	AVE	RMSE	Bias	AVE	RMSE	Bias			
50	0.1402	0.0564	-0.0098	1.5081	0.048	0.0081			
150	0.1429	0.0471	-0.0071	1.5038	0.0294	0.0038			
300	0.146	0.0376	-0.004	1.5018	0.0405	0.0018			
Set 3									
n	$\alpha = 0.5$			$\beta = 0.5$			$\gamma = 3.5$		
	AVE	RMSE	Bias	AVE	RMSE	Bias	AVE	RMSE	Bias
50	0.4962	0.0514	-0.0038	0.5027	0.0512	0.0027	3.5006	0.009	0.0006
150	0.4975	0.0352	-0.0025	0.5015	0.0426	0.0015	3.5004	0.0062	0.0004
300	0.4988	0.0226	-0.0012	0.5014	0.0213	0.0014	3.5002	0.0039	0.0002
n	$\lambda = 0.05$			$\theta = 0.5$					
	AVE	RMSE	Bias	AVE	RMSE	Bias			
50	0.043	0.0279	-0.007	0.5018	0.0307	0.0018			
150	0.0467	0.0192	-0.0033	0.5009	0.0206	0.0009			
300	0.0483	0.0135	-0.0017	0.5004	0.0139	0.0004			

Table 3: Chi-Square Goodness of Fit Test

Data Set 1		
Model	**Test Statistics**	**p-value**
EWBX	7.20475	0.1255
WBX	3.242282	0.662689
BBX	3.783407	0.581002
EGBX	4.221055	0.518048
GBX	3.789692	0.705112
BX	3.464346	0.838987
EBXIIP	2.979797	0.561212
EWBXII	5.606855	0.230495
BBXII	7.195479	0.125912
GMOBXII	3.148541	0.677097
Data Set 2		
EWBX	1.5589	0.2118
WBX	1.8892	0.3888
BBX	6.3571	0.0416
EGBX	16.388	0.0003
GBX	3.0558	0.3831
BX	23.28	0.0001
EBXIIP	1.0487	0.3058
EWBXII	0.8348	0.3609
BBXII	1.0246	0.3114
GMOBXII	0.8028	0.6694

Table 4: MLEs, $-l$, AIC, BIC, CAIC and HQ for Failure Time of Aircraft Windshields

Model	MLEs	-l	AIC	BIC	CAIC	HQ
EWBX	$\hat{\alpha} = 0.02447$ $\hat{\beta} = 0.85639$ $\hat{\gamma} = 4.88100$ $\hat{\lambda} = 0.22679$ $\hat{\theta} = 0.00465$	125.5207	261.0413	273.1954	261.8105	265.9271
WBX	$\hat{\alpha} = 109.6290$ $\hat{\beta} = 0.10869$ $\hat{\lambda} = 0.05065$ $\hat{\theta} = 11.14409$	129.349	266.697	276.4203	267.2033	270.6057
BBX	$\hat{\alpha} = 11.76432$ $\hat{\beta} = 0.42007$ $\hat{\lambda} = 0.57382$ $\hat{\theta} = 0.08239$	130.067	268.1342	277.8574	268.6405	271.0429
EGBX	$\hat{\alpha} = 11.19570$ $\hat{\beta} = 0.34823$ $\hat{\lambda} = 0.19342$ $\hat{\theta} = 2.76908$	127.23	262.4608	272.1841	262.9671	266.3695
GBX	$\hat{\gamma} = 0.41840$ $\hat{\lambda} = 0.57497$ $\hat{\theta} = 0.94562$	130.06	266.1199	273.4123	266.4199	269.0514
BX	$\hat{\lambda} = 0.38012$ $\hat{\theta} = 1.9883$	130.47	264.9391	269.8008	265.0873	266.8935
EBXIIP	$\hat{\alpha} = 9.15920$ $\hat{\beta} = 2.74395$ $\hat{\gamma} = 0.55735$ $\hat{\lambda} = 9.71469$ $\hat{\theta} = 4.35239$	129.718	269.4367	281.5908	270.2059	274.3225
EWBXII	$\hat{\alpha} = 6.77928$ $\hat{\beta} = 3.12810$ $\hat{\gamma} = 0.75182$ $\hat{\lambda} = 0.02549$ $\hat{\theta} = 0.14972$	124.348	258.6951	270.8492	259.4644	263.581
BBXII	$\hat{\alpha} = 6.64048$ $\hat{\beta} = 3.48602$ $\hat{\gamma} = 4.00343$ $\hat{\lambda} = 0.35704$ $\hat{\theta} = 4.93921$	126.785	263.5693	275.7234	264.3386	268.4552
GMOBXII	$\hat{\alpha} = 130.35920$ $\hat{\beta} = 1.90189$ $\hat{\lambda} = 1.66131$ $\hat{\theta} = 4.11652$	128.054	264.1073	273.8306	264.6137	268.016

Table 5: MLEs, $-l$, AIC, BIC, CAIC and HQ for Survival Time of Bladder Cancer

Model	MLEs	-l	AIC	BIC	CAIC	HQ
EWBX	$\hat{\alpha} = 4.53335$ $\hat{\beta} = 0.07585$ $\hat{\gamma} = 1.55666$ $\hat{\lambda} = 0.03299$ $\hat{\theta} = 6.25106$	409.759	829.5181261.0413	843.7783	830.0100	835.3121
WBX	$\hat{\alpha} = 3.72278$ $\hat{\beta} = 0.06373$ $\hat{\lambda} = 0.04141$ $\hat{\theta} = 9.90887$	410.668	831.3369	845.5971	831.8287	837.1309
BBX	$\hat{\alpha} = 0.00767$ $\hat{\beta} = 0.12837$ $\hat{\lambda} = 0.07625$ $\hat{\theta} = 2.72744$	411.403	830.8070	842.2151	831.2988	838.6009
EGBX	$\hat{\alpha} = 0.24876$ $\hat{\beta} = 0.15018$ $\hat{\lambda} = 0.07625$ $\hat{\theta} = 2.72744$	410.016	828.0321	839.4402	828.3573	832.6672
GBX	$\hat{\gamma} = 18.38230$ $\hat{\lambda} = 0.00183$ $\hat{\theta} = 2.01989$	412.569	833.1378	844.5459	833.4630	833.7730
BX	$\hat{\lambda} = 0.04764$ $\hat{\theta} = 0.36427$	420.352	848.7047	860.1129	849.0299	853.3399
EBXIIP	$\hat{\alpha} = 10.34460$ $\hat{\beta} = 1.42855$ $\hat{\gamma} = 0.74335$ $\hat{\lambda} = 0.00505$ $\hat{\theta} = 1.77907$	409.651	829.3029	843.5631	829.7947	835.0969
EWBXII	$\hat{\alpha} = 2.27377$ $\hat{\beta} = 2.47111$ $\hat{\gamma} = 8.71612$ $\hat{\lambda} = 1.10386$ $\hat{\theta} = 0.08206$	409.209	828.4183	842.6784	828.9101	834.2122
BBXII	$\hat{\alpha} = 10.60420$ $\hat{\beta} = 0.18051$ $\hat{\gamma} = 7.56962$ $\hat{\lambda} = 0.74947$ $\hat{\theta} = 1.76993$	409.663	829.3253	843.5855	829.8171	835.1193
GMOBXII	$\hat{\alpha} = 45.72230$ $\hat{\beta} = 1.56492$ $\hat{\lambda} = 1.11245$ $\hat{\theta} = 1.41616$	409.500	826.9993	838.4074	827.3245	831.6344

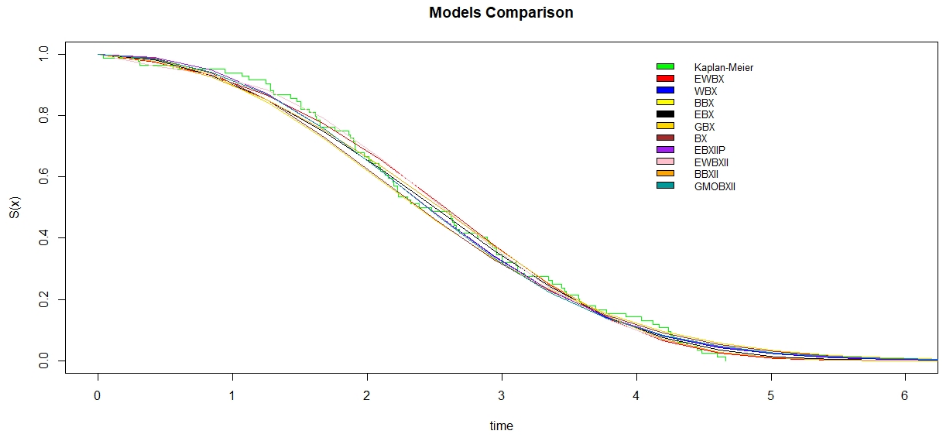

Figure 3: Estimated Survival Function for Failure Time of Aircraft Windshield

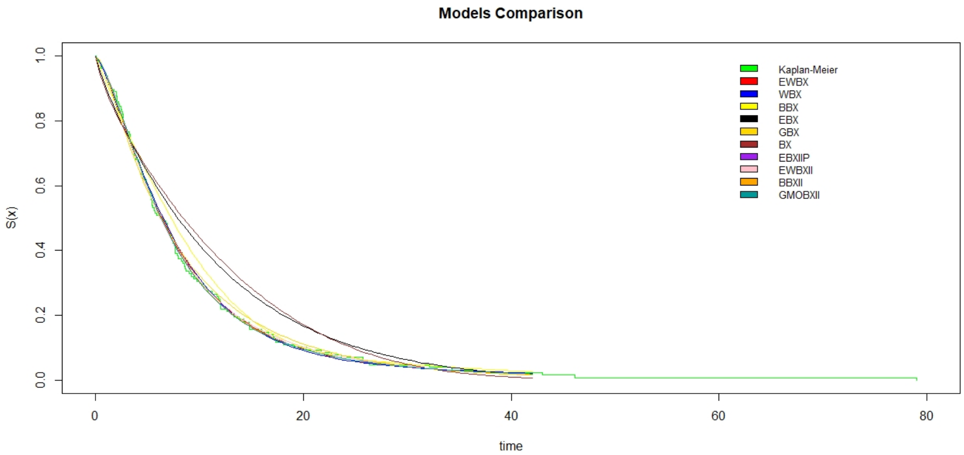

Figure 4: Estimated Survival Function for Survival Time of Bladder Cancer

8 Conclusion

This study generated a new five parameters distribution, namely the exponentiated Weibull Burr type X distribution (EWBX), to overcome the deficiency of the Weibull Burr type X distribution. The proposed distribution is very flexible; it can fit various hazard functions, including increasing, decreasing, bathtub, and unimodal. We derived its pdf and cdf, and presented several statistical properties, including quantile function, moment generating function, pdf of order statistics, and φ order Renyi entropy. The proposed distribution parameters are then estimated by using the maximum likelihood estimation approach. The simulation study is conducted with varying sample sizes and parameter values to examine the performance of the proposed distribution. The real data illustrates that the proposed distribution fits superior to its submodels and extended Burr type X distribution. In addition, the proposed distribution is a strong competitor to some five-parameter and four-parameter distributions. Besides, the proposed distribution can be applied to model survival data in medical and engineering fields. This study does not include the presence of censoring observations and covariates. Thus, we would recommend that the presence of censoring observations and covariates be considered for future research.

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