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# A force acting on an oblate spheroid with discontinuous surface temperature in a slightly rarefied gas

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An oblate spheroid, whose respective hemispheroids are kept at different uniform temperatures, placed in a rarefied gas at rest is considered. The explicit formula of the force acting on the spheroid (radiometric force) is obtained for small Knudsen numbers. This is a model of a vane of the Crookes radiometer. The analysis is performed for the general axisymmetric distribution of the surface temperature of the spheroid allowing abrupt changes. Although the generalized slip flow theory, established by Sone [SONE, Y., 1969. In *Rarefied Gas Dynamics* (ed. L. Trilling & H. Y. Wachman), vol. 1, pp. 243–253. New York: Academic Press], is available for general rarefied gas flows at small Knudsen numbers, it cannot be applied to the present problem because of the abrupt temperature changes. However, if it is combined with the symmetry relations for the linearized Boltzmann equation developed recently by Takata [TAKATA, S., 2009. *J. Stat. Phys.* **136**, 751–784], one can bypath the difficulty. To be more specific, the force acting on the spheroid in the present problem can be generated from the solution of the adjoint problem to which the generalized slip flow theory can be applied, i.e., the problem in which the same spheroid with a uniform surface temperature is placed in a uniform flow of a rarefied gas. The analysis of the present paper follows this strategy.

**Key words:** Kinetic theory, Molecular dynamics, Microfluidics, Non-continuum effects

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## 1. Introduction

When a body with a non-uniform surface temperature is placed in a rarefied gas at rest, a steady flow is induced around the body, and a force acts on it. Photophoresis [see, e.g., Preining (1966); Sone & Aoki (1977); Chernyak & Beresnev (1993)] and thermophoresis [see, e.g., Hidy & Brock (1970); Sone & Aoki (1977); Takata & Sone (1995)] of aerosol particles and the vanes of the Crookes radiometer [see Ketsdever *et al.* (2012) and references therein] are typical examples. When the gas is slightly rarefied, that is, when the Knudsen number, the ratio of the mean free path of gas molecules to the characteristic length of the body, is small, the induced flow and the resulting force can be explained in the framework of the generalized slip flow theory (Sone 1969, 1971, 1991, 2002, 2007).

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This theory is a consequence of a systematic asymptotic analysis of the boundary-value problem of the Boltzmann equation for small Knudsen numbers and provides a general recipe to obtain a correct asymptotic solution with the help of appropriate fluid-dynamic-type equations and their slip or jump boundary conditions, as well as kinetic corrections in the vicinity of the boundary (the Knudsen layer). For instance, when the temperature variation along the surface of the body is small, the dominant flow is the thermal creep flow (Kennard 1938; Sone 1966; Ohwada, Sone & Aoki 1989), which is induced by the tangential velocity slip on the boundary proportional to the tangential temperature gradient there.

The generalized slip flow theory is very powerful and has been applied successfully to various problems containing flows induced by temperature fields [e.g., Sone (1972); Sone & Aoki (1977); Sone & Tanaka (1980)]. However, the basic assumption of the theory is that the shape of the boundary as well as the boundary data is smooth enough. More specifically, the local radius of curvature of the boundary and the local length scale of variation of the temperature and velocity of the boundary should be much longer than the molecular mean free path. Therefore, if the body has a sharp edge or a surface-temperature distribution with an abrupt change, as in the case of the vanes of a radiometer, the generalized slip flow theory cannot be applied. However, if we restrict ourselves to abrupt but small changes of the surface temperature leaving the body shape smooth, there is a way to bypath the difficulty and to obtain the force acting on the body by the use of the generalized slip flow theory. The key is to use the symmetry relations associated with the linearized Boltzmann equation (Takata 2009*a,b*).

In the present study, we consider an oblate spheroid, with an axisymmetric distribution of the surface temperature, placed in an infinite expanse of a gas at an equilibrium state at rest. The temperature distribution is arbitrary and can be abrupt. Indeed, our aim is to derive an explicit formula for the force acting on the oblate spheroid when a hemispheroid is kept at a lower temperature while the other hemispheroid is kept at a higher temperature. This may be a model of a vane of the Crookes radiometer [cf. Ketsdever *et al.* (2012) and references therein; Taguchi & Aoki (2011, 2012); Chen *et al.* (2012)]. For this purpose, following Takata (2009*a*), we consider the following adjoint problem: The same oblate spheroid with a uniform surface temperature is placed in an infinite expanse of a gas; there is a slow uniform flow, parallel to the axis of revolution of the spheroid, at infinity; the surface temperature of the spheroid is the same as that of the gas at infinity. The symmetry relation tells that the force acting on the spheroid in the original problem is obtained from the information on the local heat flow on the spheroid in the adjoint problem. Since the shape of the boundary as well as the boundary data is perfectly smooth in the adjoint problem, the generalized slip flow theory can be applied to it safely to obtain the information on the heat flow in the adjoint problem. In this way, we can obtain the force acting on the oblate spheroid with an abrupt temperature distribution by the use of the generalized slip flow theory. This is the strategy in the present paper.

The paper is organized as follows. In §2, we describe the original problem and its adjoint problem with a mention of the necessary formula originating from the symmetry relations of the linearized Boltzmann equation. §3 gives a summary of the generalized slip flow theory. We perform analysis of the adjoint problem on the basis of the generalized slip flow theory in §4 and obtain the force acting on the spheroid in the original problem in §5. §6 contains some discussions, and §7 is devoted to concluding remarks.

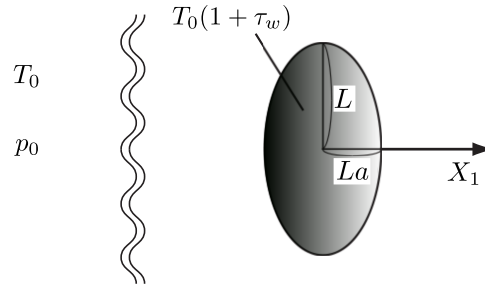


FIGURE 1. Problem

## 2. Problem and adjoint problem

### 2.1. Problem

[Problem I] Consider an oblate spheroid with long radius  $L$  and short radius  $aL$  ( $0 < a \leq 1$ ) placed in an infinite expanse of a rarefied gas in an equilibrium state at rest at pressure  $p_0$  and temperature  $T_0$  (figure 1). The Cartesian coordinate system  $X_i$  is taken as in figure 1, that is, the origin is at the center of the spheroid and the  $X_1$  axis along its axis. Then, the temperature of the surface of the spheroid, which is assumed to be axisymmetric but arbitrary, is denoted by  $T_0(1 + \tau_w)$ , where  $\tau_w$  varies with  $X_1/L$ . In particular, we are interested in a discontinuous temperature distribution:

$$\tau_w = \begin{cases} 0 & (0 < X_1/L \leq a), \\ \tau_{wc} & (-a \leq X_1/L < 0), \end{cases} \quad (2.1)$$

where  $\tau_{wc}$  is a constant.

We investigate the force acting on the oblate spheroid (radiometric force) under the following assumptions:

- (i) The behavior of the gas is described by the steady Boltzmann equation with its kinetic boundary condition.
- (ii) The temperature variation  $|\tau_w|$  over the surface of the spheroid is so small that the Boltzmann equation and the boundary condition can be linearized around the uniform equilibrium state at rest at pressure  $p_0$  and temperature  $T_0$ .
- (iii) The Knudsen number, the ratio of the molecular mean free path at the reference equilibrium state to the characteristic length, say  $L$ , is small.

Neither the model for intermolecular collisions (such as the hard-sphere molecules) nor that for the boundary condition (such as the diffuse reflection) are specified here, but are left arbitrary until we need quantitative results. However, they should be common to the present problem and the adjoint problem in the next subsection.

One of the reasons why we are interested in the discontinuous surface temperature (2.1) is that an oblate spheroid with such a surface temperature may be a three-dimensional model of a vane of the Crookes radiometer [see Ketsdever *et al.* (2012); Taguchi & Aoki (2011, 2012); Chen *et al.* (2012)].

### 2.2. Adjoint problem

We introduce the following problem adjoint to the above problem (adjoint problem).

[Problem II] Consider the same oblate spheroid but with uniform surface temperature  $T_0$  placed in a uniform flow of a rarefied gas in the direction of the axis of the spheroid (figure 2). Using the same Cartesian coordinate system as in figure 1, we denote the pressure,

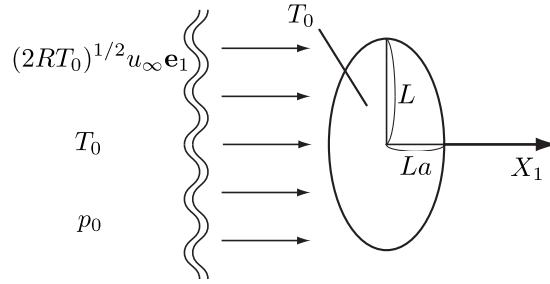


FIGURE 2. Adjoint problem

temperature and flow velocity of the gas at infinity by  $p_0$ ,  $T_0$  and  $(2RT_0)^{1/2} u_\infty \mathbf{e}_1$ , where  $R$  is the gas constant per unit mass ( $R = k_B/m$  with  $k_B$  the Boltzmann constant and  $m$  the mass of a gas molecule), and  $\mathbf{e}_1$  is the unit vector in the  $X_1$  direction.

We investigate the steady behavior of the gas under the same assumptions as (i) and (iii) in § 2.1 and the following assumption:

(ii)' The flow speed at infinity  $u_\infty$  is so small that the Boltzmann equation and the boundary condition can be linearized.

### 2.3. Formula for radiometric force

Let us denote by  $F^I \mathbf{e}_1$  the force (radiometric force) acting on the spheroid in the original problem (Problem I) and by  $p_0(2RT_0)^{1/2} Q_n^{\text{II}}$  the normal component (in the direction pointing into the gas) of the heat-flow vector on the spheroid in the adjoint problem (Problem II). According to Takata (2009a) (see Example 2 in § 5.1 in this reference), the force acting on the spheroid in Problem I is expressed in terms of the surface temperature of Problem I and the normal component of the local heat flow on the spheroid in Problem II, that is,

$$F^I = -(p_0 L^2 / u_\infty) \int_S \tau_w(x_1) Q_n^{\text{II}}(x_1) dS, \quad (2.2)$$

where  $x_i = X_i/L$  is the dimensionless Cartesian coordinate system corresponding to  $X_i$ , and  $S$  and  $dS$  indicate, respectively, the surface of the oblate spheroid and the surface element on it in the dimensionless  $x_i$  space. It is shown explicitly in (2.2) the fact that  $\tau_w$  and  $Q_n^{\text{II}}$  depends only on the  $x_1$  coordinate of a point on the surface because of axial symmetry. It should be mentioned that this formula holds for the whole range of the Knudsen number, that is, without assumption (iii) in § 2.1.

Problem II does not contain any abrupt change in the boundary temperature, so that it can be analyzed by the use of the generalized slip flow theory when the Knudsen number is small. Once the heat-flow vector in this problem is obtained, the force acting on the spheroid in Problem I follows immediately from (2.2). In the next section, we give a brief summary of the generalized slip flow theory.

We remark that, in Takata (2009a), the formula (2.2) is derived for a body of arbitrary shape as an example of the general theory, so that some delicate mathematical problems relevant to the present Problems I and II are not discussed explicitly. In order to see the mathematical problems more clearly, we derive (2.2) again directly from the basic equations of Problems I and II in Appendix A. Incidentally, the reader is referred to Takata (2009b) for the relationship between the symmetry relations for the linearized Boltzmann equation and the Onsager reciprocity relations.

### 3. Generalized slip flow theory: Summary

#### 3.1. Stokes equations and slip boundary conditions

The linearized version of the generalized slip flow theory has been established in Sone (1969, 1971, 1991, 2002). Here, we basically follow the notations in Sone (2007).

Let us consider a gas around a solid boundary of arbitrary but smooth shape with smooth variation of the surface temperature. The boundary is the ordinary one across which there is no net mass flow of the gas (simple solid boundary). We repeat assumptions (i) and (iii) in §2.1 for the present more general setting and put the following assumption:

(ii)'' The deviation of the system from a reference equilibrium state at rest is so small that the Boltzmann equation and its boundary condition can be linearized.

The model for intermolecular collisions as well as that for the boundary condition are unspecified here. When quantitative results are necessary, we will assume hard-sphere molecules or the Bhatnagar–Gross–Krook (BGK) model (Bhatnagar, Gross & Krook 1954; Welander 1954) for the former and diffuse reflection for the latter (Sone 2007).

In this section, we redefine  $X_i$ ,  $L$ ,  $p_0$  and  $T_0$  as the Cartesian coordinate system, the reference length, the reference pressure and the reference temperature, respectively, appropriate to the general problem described in the preceding paragraph. Then, let  $x_i = X_i/L$  be the dimensionless Cartesian coordinates,  $\rho_0(1 + \omega)$  the density,  $(2RT_0)^{1/2}u_i$  the flow velocity,  $T_0(1 + \tau)$  the temperature,  $p_0(1 + P)$  the pressure,  $p_0(\delta_{ij} + P_{ij})$  the stress tensor, and  $p_0(2RT_0)^{1/2}Q_i$  the heat-flow vector, where  $\rho_0 = p_0/RT_0$  is the reference density, and  $\delta_{ij}$  the Kronecker delta. In addition,  $\ell_0$  denotes the mean free path of the gas molecules at the reference equilibrium state at rest at pressure  $p_0$  and temperature  $T_0$ ; for instance,

$$\ell_0 = 1/\sqrt{2}\pi d_m^2(\rho_0/m) \quad (\text{for hard-sphere molecules}), \quad (3.1a)$$

$$\ell_0 = (8RT_0/\pi)^{1/2}/A_c\rho_0 \quad (\text{for the BGK model}), \quad (3.1b)$$

where  $d_m$  is the diameter of a molecule, and  $A_c$  is a constant such that  $A_c\rho_0$  is the collision frequency of a molecule in the reference equilibrium state at rest.

The deviations  $h$ , where  $h$  stands for  $\omega$ ,  $u$ ,  $\tau$ , etc., are split into two parts,  $h_G$  and  $h_K$ , and each part is expanded in power series of a small parameter  $k$  corresponding to the Knudsen number  $\text{Kn} = \ell_0/L$ , i.e.,

$$h = h_G + h_K \quad (3.2a)$$

$$h_G = h_{G0} + h_{G1}k + h_{G2}k^2 + \cdots, \quad (3.2b)$$

$$h_K = h_{K1}k + h_{K2}k^2 + \cdots, \quad (3.2c)$$

$$k = (\sqrt{\pi}/2)\text{Kn} = (\sqrt{\pi}/2)(\ell_0/L). \quad (3.2d)$$

The  $h_G$ , which we call the Grad-Hilbert solution, describes the overall behavior of the gas, whereas the  $h_K$ , which we call the Knudsen-layer correction, is the correction to  $h_G$  in the thin layer with thickness of the order of the mean free path adjacent to the boundary. The expansion of  $h_K$  starts from the first order.

The equations for the Grad-Hilbert solution are the Stokes system of equations to *any*

|            | hard sphere   | BGK        | boundary condition |
|------------|---------------|------------|--------------------|
| $\gamma_1$ | 1.270 042 427 | 1          | -                  |
| $\gamma_2$ | 1.922 284 066 | 1          | -                  |
| $\gamma_3$ | 1.947 906 335 | 1          | -                  |
| $k_0$      | -1.254 0      | -1.016 191 | diffuse reflection |
| $K_1$      | -0.646 5*     | -0.383 16  | diffuse reflection |
| $d_1$      | 2.400 1       | 1.302 72   | diffuse reflection |
| $\gamma_A$ | 0.153 0       | 0.116 84   | diffuse reflection |
| $\gamma_B$ | 1.276         | 0.415 56   | diffuse reflection |

TABLE 1. Coefficients contained in (3.3c), (3.6), and (3.7). The asterisk \* indicates the value taken from Takata & Hattori (2012a), which is more accurate than that shown in Sone (2007) (-0.6463).

order of expansion, that is,

$$\frac{\partial P_{G0}}{\partial x_i} = 0 \quad (3.3a)$$

$$\frac{\partial u_{jGm}}{\partial x_j} = 0, \quad (3.3b)$$

$$\frac{\partial P_{Gm+1}}{\partial x_i} - \gamma_1 \frac{\partial^2 u_{iGm}}{\partial x_j^2} = 0, \quad (3.3c)$$

$$\frac{\partial^2 \tau_{Gm}}{\partial x_j^2} = 0, \quad (3.3d)$$

$$P_{Gm} = \omega_{Gm} + \tau_{Gm}, \quad (3.3e)$$

$$(m = 0, 1, 2, \dots),$$

where  $\gamma_1$  is a constant depending on the model for intermolecular collisions and is related to the viscosity  $\mu$  at temperature  $T_0$  and pressure  $p_0$  as

$$\mu = (\sqrt{\pi}/2)\gamma_1 p_0 (2RT_0)^{-1/2} \ell_0. \quad (3.4)$$

The values of  $\gamma_1$  for hard-sphere molecules and for the BGK model, taken from Sone (2007), are shown in Table 1. (3.3a) is the degenerated Euler equation, (3.3b) is the continuity equation, (3.3c) is the Stokes equation, (3.3d) is the steady heat-conduction equation, and (3.3e) is the equation of state.

The boundary conditions for (3.3b)–(3.3d) are of the form of slip or jump conditions and have been obtained up to the order of  $k^2$  (Sone 1969, 1971, 1991, 2002, 2007). Here, we show them only up to the order of  $k$ . Let  $n_i$  and  $t_i$  denote the unit normal vector (pointing into the gas) and unit tangential vector to the boundary, respectively, and let  $T_0(1 + \tau_w)$  and  $(2RT_0)^{1/2} u_{wi}$  denote the temperature and velocity of the boundary, respectively (note that  $u_{wi} n_i = 0$  because the problem is steady). Then, the boundary conditions read as follows:

$$u_{iG0} = u_{wi}, \quad (3.5a)$$

$$\tau_{G0} = \tau_w, \quad (3.5b)$$

$$u_{iG1}t_i = -k_0 \left( \frac{\partial u_{iG0}}{\partial x_j} + \frac{\partial u_{jG0}}{\partial x_i} \right) n_i t_j - K_1 \frac{\partial \tau_{G0}}{\partial x_i} t_i, \quad u_{iG1}n_i = 0, \quad (3.6a)$$

$$\tau_{G1} = d_1 \frac{\partial \tau_{G0}}{\partial x_i} n_i, \quad (3.6b)$$

where the derivatives on the right-hand sides of (3.6) are evaluated on the boundary, and the constants  $k_0$  ( $< 0$ ),  $K_1$  ( $< 0$ ) and  $d_1$  ( $> 0$ ) are the so-called slip coefficients, which depend on the model of interaction between gas molecules as well as that of gas-surface interaction. The values of  $k_0$ ,  $K_1$  and  $d_1$  for hard-sphere molecules and for the BGK model under the diffuse-reflection condition, taken from Sone (2007), are shown in Table 1. (3.5) is the no-slip condition for viscous fluids, and (3.6a) and (3.6b) are the so-called slip (jump) conditions.

It should be mentioned that boundary conditions (3.5) and (3.6) (and the higher-order conditions) are obtained only by the analysis of the Knudsen-layer corrections and are determined together with the latter. The reader is referred to Sone (2002, 2007) for the Knudsen-layer corrections as well as the second-order slip boundary conditions. We note that, although the complete list of the slip coefficients in the second-order slip boundary condition is available in Sone (2002, 2007) for the BGK model with diffuse reflection, only three of them are given there for hard-sphere molecules. The list for hard-sphere molecules (with diffuse reflection) has been completed recently by Takata & Hattori (2012a,b).

The solution procedure is as follows. We first determine the constant  $P_{G0}$  from an appropriate condition, say the pressure at infinity. Then, we solve (3.3b)–(3.3d) with  $m = 0$  under the no-slip condition (3.5). The flow velocity  $u_{iG0}$  and pressure  $P_{G1}$  are nothing but the classical Stokes flow solution. With the solution for  $m = 0$ , we evaluate the right-hand sides of (3.6). The next step is to solve (3.3b)–(3.3d) with  $m = 1$  under the boundary conditions (3.6). We repeat the same procedure for (3.3b)–(3.3d) with  $m = 2$  under the second-order boundary conditions.

### 3.2. Heat flow on the boundary

In the framework of the generalized slip flow theory, the Grad-Hilbert solution and the Knudsen-layer correction for the (dimensionless) stress tensor  $P_{ij}$  and heat-flow vector  $Q_i$  have been obtained [cf. (3.21), (3.22), and (3.49)–(3.54) in Sone (2007)]. As seen from (2.2), we need information about the normal component  $Q_i n_i$  of the heat-flow vector on the boundary. Therefore, we only give its expression, in the form of power series in  $k$ . That is,  $Q_i n_i$  on the boundary is given by

$$\begin{aligned} Q_i n_i = & -\frac{5}{4} \gamma_2 \frac{\partial \tau_{G0}}{\partial x_i} n_i k + \left\{ -\frac{5}{4} \gamma_2 \frac{\partial \tau_{G1}}{\partial x_i} n_i + \frac{\gamma_3}{2} \frac{\partial^2 u_{iG0}}{\partial x_j^2} n_i \right. \\ & - \frac{1}{2} \gamma_A \left[ \frac{\partial}{\partial x_k} \left( \frac{\partial u_{iG0}}{\partial x_j} + \frac{\partial u_{jG0}}{\partial x_i} \right) \right] n_i n_j n_k \\ & \left. - \gamma_B \left( \frac{\partial^2 \tau_{G0}}{\partial x_i \partial x_j} n_i n_j + 2\bar{\kappa} \frac{\partial \tau_{G0}}{\partial x_i} n_i \right) \right\} k^2 + O(k^3), \end{aligned} \quad (3.7)$$

where  $\gamma_2$  and  $\gamma_3$  are constants depending on the model for intermolecular collisions, and  $\gamma_2$  is related to the thermal conductivity  $\lambda$  at temperature  $T_0$  and pressure  $p_0$  as

$$\lambda = (5\sqrt{\pi}/4) \gamma_2 R p_0 (2RT_0)^{-1/2} \ell_0. \quad (3.8)$$



In addition,  $\gamma_A$  and  $\gamma_B$  are constants depending on the model for intermolecular collisions as well as that for gas-surface interaction, and  $\bar{\kappa}$  is the mean curvature of the boundary in the dimensionless  $x_i$  space. The terms containing  $\gamma_2$  correspond to the Fourier law, the term  $\gamma_3$  indicates the heat flow due to the pressure gradient [cf. (3.3c)], which is a typical effect of gas rarefaction, and the terms containing  $\gamma_A$  and  $\gamma_B$  are the contribution from the Knudsen-layer correction. The constants  $\gamma_A$  and  $\gamma_B$  are, respectively, expressed as integrals of functions  $H_A(\eta)$  and  $H_B(\eta)$  occurring in the profile of  $Q_{iK1}t_i$  inside the Knudsen layer [cf. (3.53) and (3.54) in Sone (2007)], i.e.,

$$\gamma_A = \int_0^\infty H_A(\eta)d\eta, \quad \gamma_B = \int_0^\infty H_B(\eta)d\eta. \quad (3.9)$$

The values of  $\gamma_2$ ,  $\gamma_3$ ,  $\gamma_A$  and  $\gamma_B$  for hard-sphere molecules as well as the BGK model (and for the diffuse-reflection condition for the latter two) are shown in Table 1. The values of  $\gamma_2$  and  $\gamma_3$  are taken from Sone (2007), whereas those of  $\gamma_A$  and  $\gamma_B$  were obtained by using the data of  $H_A$  and  $H_B$  recomputed in Takata & Hattori (2012a).

#### 4. Solution to adjoint problem

In this section, we try to obtain the solution to the adjoint problem, Problem II in § 2.2, with the help of the generalized slip flow theory. It is seen from (2.2) and (3.7) that we only need  $u_{iG0}$ ,  $\tau_{G0}$  and  $\tau_{G1}$  to obtain the force  $F^I$  acting on the spheroid in the original problem (Problem I in § 2.2) up to the order of  $k^2$ .

In Problem II, the condition at infinity for the Stokes system (3.3) is

$$u_{iG0} \rightarrow (u_\infty, 0, 0), \quad u_{iGm+1} \rightarrow 0, \quad P_{Gm} \rightarrow 0, \quad \tau_{Gm} \rightarrow 0, \quad (m = 0, 1, 2, \dots), \quad (4.1)$$

while

$$u_{wi} = \tau_w = 0, \quad (4.2)$$

in boundary condition (3.5). Therefore, it is obvious from (3.3a), (3.3d) ( $m = 0, 1$ ), (3.5b), and (3.6b) that

$$P_{G0} = 0, \quad \tau_{G0} = 0, \quad \tau_{G1} = 0. \quad (4.3)$$

Thus, the remaining task is to obtain the solution,  $u_{iG0}$  and  $P_{G1}$ , to the boundary-value problem, (3.3b) and (3.3c) ( $m = 0$ ) with boundary conditions (4.1) [ $u_{iG0} \rightarrow (u_\infty, 0, 0)$ ] and (3.5a). This solution is nothing but the classical solution for the uniform flow of a Stokes fluid past an oblate spheroid, which is available in the literature, e.g., Payne & Pell (1960). Therefore, we just need to rewrite the solution in Payne & Pell (1960) in terms of the present notations.

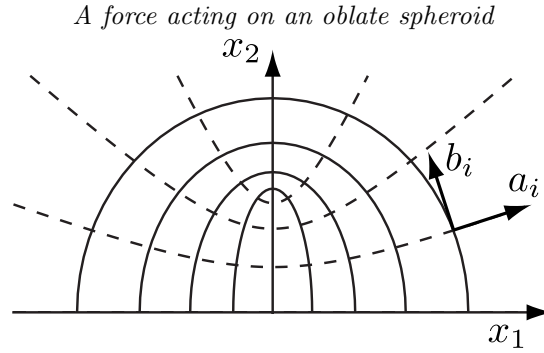
We first introduce the oblate spheroidal coordinate system  $(\alpha, \theta, \varphi)$  ( $0 \leq \alpha < \infty$ ,  $0 \leq \theta \leq \pi$ ,  $0 \leq \varphi < 2\pi$ ) (figure 3), i.e.,

$$x_1 = c_0 \sinh \alpha \cos \theta, \quad x_2 = c_0 \cosh \alpha \sin \theta \cos \varphi, \quad x_3 = c_0 \cosh \alpha \sin \theta \sin \varphi, \quad (4.4)$$

with

$$c_0 = 1/\cosh \alpha_0 = \sqrt{1 - a^2}, \quad \tanh \alpha_0 = a, \quad (4.5)$$

where  $a$  ( $0 < a \leq 1$ ) is the aspect ratio of the oblate spheroid (cf. § 2.1). The body (oblate spheroid) occupies the domain  $\alpha \leq \alpha_0$ , and the gas occupies  $\alpha > \alpha_0$ . The coordinate system approaches the spherical coordinate system as  $\alpha$  becomes large. Let  $a_i$ ,  $b_i$  and  $c_i$  denote the unit vectors in the directions of increasing  $\alpha$ ,  $\theta$  and  $\varphi$ , respectively, and  $f_\alpha$ ,



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FIGURE 3. Oblate spheroidal coordinate system

$f_\theta$  and  $f_\varphi$  denote the  $\alpha$ ,  $\theta$  and  $\varphi$  components of a vector  $f_i$ , i.e.,  $f_\alpha = f_i a_i$ ,  $f_\theta = f_i b_i$  and  $f_\varphi = f_i c_i$ .

According to Payne & Pell (1960), the solution,  $u_{iG0}$  and  $P_{G1}$ , is given as follows:

$$u_{\alpha G0} = u_\infty \left\{ \cosh \alpha - \frac{1}{A} [(1 + t_0^2) \tanh \alpha + (1 - t_0^2) \cosh \alpha \cot^{-1}(\sinh \alpha)] \right\} \frac{1}{\rho} \cos \theta, \quad (4.6a)$$

$$u_{\theta G0} = -u_\infty \left\{ \sinh \alpha - \frac{1}{A} [t_0^2 + (1 - t_0^2) \sinh \alpha \cot^{-1}(\sinh \alpha)] \right\} \frac{1}{\rho} \sin \theta, \quad (4.6b)$$

$$u_{\varphi G0} = 0, \quad (4.6c)$$

$$P_{G1} = -2\gamma_1 u_\infty \frac{t_0}{aA} \frac{1}{\rho^2} \cos \theta, \quad (4.6d)$$

where

$$\rho = \sqrt{\sinh^2 \alpha + \cos^2 \theta}, \quad t_0 = \sinh \alpha_0 = \frac{a}{\sqrt{1 - a^2}}, \quad A = t_0 + (1 - t_0^2) \cot^{-1} t_0. \quad (4.7)$$

Substituting (4.3) and (4.6) into (3.7) and noting that  $n_i = a_i$  ( $\alpha = \alpha_0$ ), we obtain the following expression of the heat flow  $Q_i n_i$  on the surface of the oblate spheroid:

$$\frac{Q_i n_i}{u_\infty} = 2(\gamma_3 - 4\gamma_A) \frac{t_0^4}{a^3 A} \frac{\cos \theta}{(t_0^2 + \cos^2 \theta)^{5/2}} k^2 + O(k^3). \quad (4.8)$$

In this way, we can obtain the  $k^2$ -order term, which is the leading-order term, of the normal component of the heat-flow vector on the oblate spheroid in Problem II only from the classical solution (4.6). Note that  $t_0$  and  $A$  are functions of the aspect ratio  $a$  [see (4.7)]. Figure 4 shows  $Q_i n_i$  [with  $O(k^3)$  term neglected] versus  $\theta$  for various values of  $a$ .

## 5. Force on oblate spheroid with discontinuous surface temperature

Now we go back to the original problem, Problem I (cf. §2.1). Let us express the surface temperature  $\tau_w(x_1)$  of the oblate spheroid as the function of the variable  $\theta$  and denote it by  $\tau_w(\theta)$ . Then, noting that  $Q_n^{\text{II}}$  in (2.2) is given by  $Q_i n_i$  in (4.8) and that the surface element  $dS$  is expressed as  $dS = c_0 \sqrt{t_0^2 + \cos^2 \theta} \sin \theta d\theta d\varphi$  in terms of  $\theta$  and  $\varphi$ ,

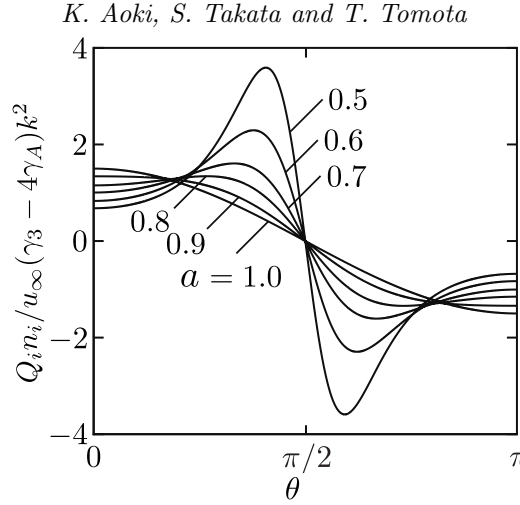


FIGURE 4. Heat flow on the surface of the oblate spheroid versus  $\theta$  for various values of the aspect ratio  $a$ .

we have the following expression of the force acting on the spheroid:

$$F^I = -4\pi(\gamma_3 - 4\gamma_A) p_0 L^2 k^2 \frac{t_0^3}{a^2 A} \int_0^\pi \tau_w(\theta) \frac{\sin \theta \cos \theta}{(t_0^2 + \cos^2 \theta)^2} d\theta, \quad (5.1)$$

where  $O(k^3)$  terms have been neglected.

When the surface temperature is discontinuous as given by (2.1), i.e.,

$$\tau_w(\theta) = \begin{cases} 0 & (0 \leq \theta < \pi/2), \\ \tau_{wc} & (\pi/2 < \theta \leq \pi), \end{cases} \quad (5.2)$$

(5.1) becomes

$$F^I = 2\pi(\gamma_3 - 4\gamma_A) p_0 L^2 \tau_{wc} \frac{1}{At_0} k^2. \quad (5.3)$$

The force  $F^I$  versus the aspect ratio of the oblate spheroid  $a \leq 1$  is shown in figure 5. For  $a = 1$  (a spherical body), (5.3) reduces to

$$F^I = (3/2)\pi(\gamma_3 - 4\gamma_A) p_0 L^2 \tau_{wc} k^2. \quad (5.4)$$

In this way, we obtain the analytic formula of the leading-order term (i.e.,  $k^2$ -order term) of the radiometric force acting on the oblate spheroid with discontinuous surface temperature with the help of the adjoint problem (Problem II). It should be emphasized that in this case, the direct application of the generalized slip flow theory to the original problem (Problem I) is beyond its applicability.

## 6. Some discussions

### 6.1. Direct application of generalized slip flow theory

As mentioned repeatedly, the original problem (Problem I) cannot be handled by the generalized slip flow theory when the boundary temperature is discontinuous. But, it would be interesting to see what happens if we apply the generalized slip flow theory to the problem formally ignoring its applicability. To simplify the situation, we restrict ourselves to the case of a spherical body ( $a = 1$ ). Here, we use the spherical coordinate

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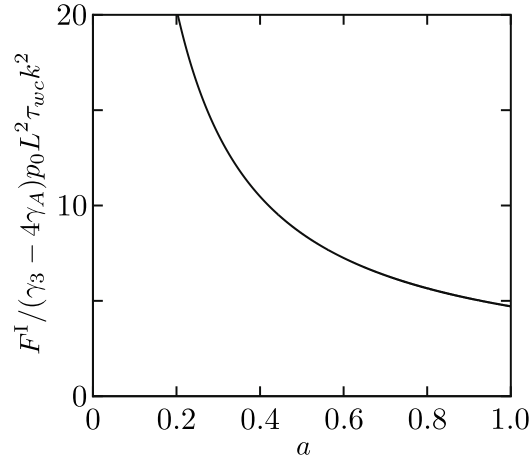


FIGURE 5. Radiometric force  $F^I$  versus the aspect ratio  $a$  of the oblate spheroid

system  $(r, \theta, \varphi)$  ( $0 \leq r < \infty$ ,  $0 \leq \theta \leq \pi$ ,  $0 \leq \varphi < 2\pi$ ):

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta \cos \varphi, \quad x_3 = r \sin \theta \sin \varphi, \quad (6.1)$$

and denote by  $u_{rGm}$ ,  $u_{\theta Gm}$  and  $u_{\varphi Gm} (\equiv 0)$  the  $r$ ,  $\theta$  and  $\varphi$  components of  $u_{iGm}$ .

In this case, the condition at infinity for the Stokes equations (3.3) becomes

$$u_{iGm} \rightarrow 0, \quad P_{Gm} \rightarrow 0, \quad \tau_{Gm} \rightarrow 0, \quad (m = 0, 1, 2, \dots), \quad (6.2)$$

while

$$u_{wi} = 0, \quad \tau_w(\theta) = \begin{cases} 0 & (0 \leq \theta \leq \pi/2), \\ \tau_{wc} & (\pi/2 < \theta \leq \pi), \end{cases} \quad (6.3)$$

in boundary condition (3.5). The formal solution of (3.3) satisfying conditions (6.2), (3.5) [with (6.3)] and (3.6) is obtained as

$$P_{G0} = 0, \quad (6.4a)$$

$$u_{rG0} = u_{\theta G0} = P_{G1} = 0, \quad \tau_{G0} = \tau_{wc} \sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} \mathcal{P}_n, \quad (6.4b)$$

$$v_{rG1} = -\frac{1}{2} K_1 \tau_{wc} \sum_{n=1}^{\infty} n(n+1) \left( \frac{1}{r^n} - \frac{1}{r^{n+2}} \right) a_n \mathcal{P}_n, \quad (6.4c)$$

$$v_{\theta G1} = \frac{1}{2} K_1 \tau_{wc} \sum_{n=1}^{\infty} \left( \frac{n-2}{r^n} - \frac{n}{r^{n+2}} \right) a_n \frac{d\mathcal{P}_n}{d\theta}, \quad (6.4d)$$

$$P_{G2} = -\gamma_1 K_1 \tau_{wc} \sum_{n=1}^{\infty} \frac{n(2n-1)}{r^{n+1}} a_n \mathcal{P}_n, \quad (6.4e)$$

where

$$a_n = \begin{cases} 1/2 & (n = 0), \\ (-1)^{m+1} \frac{(4m+3)}{2} \cdot \frac{(2m-1)!!}{(2m+2)!!} & (n = 2m+1; m = 0, 1, \dots), \\ 0 & (n = 2m+2; m = 0, 1, \dots), \end{cases} \quad (6.5)$$

with  $(2m)!! = 2m \cdot (2m-2) \cdots 4 \cdot 2$ ,  $(2m+1)!! = (2m+1) \cdot (2m-1) \cdots 3 \cdot 1$ ,  $0!! = (-1)!! = 1$ , and  $\mathcal{P}_n = \mathcal{P}_n(\cos \theta)$  is the Legendre polynomial defined by

$$\mathcal{P}_n = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} \cos^{n-2k} \theta, \quad (6.6)$$

with  $\lfloor n/2 \rfloor$  being the largest integer less than or equal to  $n/2$ . Here, we have formally performed term-by-term differentiation, which is not applicable at  $(r, \theta) = (1, \pi/2)$ . In fact, the resulting solution (6.4) is singular there and should be interpreted as a generalized function (distribution) or the fundamental solution for the problem with a smooth  $\tau_w(\theta)$  [cf. Sone & Aoki (1977)]. However, we can formally obtain the force acting on the sphere  $F^I \mathbf{e}_1$  from this solution. In fact, only the  $r^{-1}$  terms in  $u_{rG1}$  and  $u_{\theta G1}$  and the  $r^{-2}$  term in  $P_{G2}$  contribute to the force  $F^I$  and give

$$F^I = -3\pi\gamma_1 K_1 p_0 L^2 \tau_{wc} k^2. \quad (6.7)$$

This coincides with (5.4) because  $\gamma_3 - 4\gamma_A = -2\gamma_1 K_1$  holds (Takata 2009a). Thus, the formal solution, which violates the basic assumption for the generalized slip flow theory, happens to give the correct result for the radiometric force acting on the sphere. This is just an accidental coincidence without theoretical validation.

### 6.2. Free molecular limit

In the present study, we are concerned with the force acting on an oblate spheroid with nonuniform (axisymmetric) surface temperature at small Knudsen numbers. But if we consider the other extreme limit, the free-molecular flow, where the gas is so rarefied that the collision between gas molecules can be neglected, we can readily obtain the solution and thus the force acting on the spheroid. To be more specific, we consider Problem I (cf. §2.1) with assumption (iii) replaced with the assumption that the Knudsen number is infinitely large. Then, the force acting on the spheroid is obtained as

$$F^I = -\frac{\pi}{2} p_0 L^2 \int_0^\pi \tau_w(\theta) \sin \theta \cos \theta d\theta. \quad (6.8)$$

It should be noted that the force does not depend on the aspect ratio of the spheroid  $a$ . For the discontinuous surface temperature (2.1) or (5.2), (6.8) reduces to

$$F^I = (\pi/4) p_0 L^2 \tau_{wc}. \quad (6.9)$$

### 6.3. Final velocity of the oblate spheroid set free in the gas

If the oblate spheroid in Problem I is set free in the gas, it starts moving in the direction of the force (5.1) or (5.3). Let us denote by  $V_f \mathbf{e}_1$  its final velocity (in dimensional form) after steady motion has been reached. This  $V_f \mathbf{e}_1$  can be obtained by the superposition of Problem I and Problem II because of the linearity of the problems as follows. In the superposed problem, where the oblate spheroid with a non-uniform surface temperature  $T_0[1 + \tau_w(x_1)]$  is placed in a uniform flow of gas  $(2RT_0)^{1/2} u_\infty \mathbf{e}_1$ , the total force acting on the spheroid is  $(F^I + F^{II}) \mathbf{e}_1$ . Here,  $F^{II} \mathbf{e}_1$  is the force acting on the oblate spheroid in Problem II, which is given in Appendix B in two special cases, i.e., the Stokes flow and the free-molecular flow. If we let  $u_\infty^*$  be the value of  $u_\infty$  for which the total force vanishes, then the final velocity mentioned above is given as  $V_f \mathbf{e}_1 = -(2RT_0)^{1/2} u_\infty^* \mathbf{e}_1$ . In the case of the discontinuous temperature distribution (2.1) [or (5.2)],  $F^I$  of (5.3) [or (6.8)] should be combined with  $F^{II}$  of (B1) (with  $u_\infty$  of the order of  $k$ ) [or (B3)]. In

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summary, we obtain the following results: For small Knudsen numbers,

$$V_f = \frac{\gamma_3 - 4\gamma_A}{4\gamma_1 a} (2RT_0)^{1/2} \tau_{wc} k, \quad (6.10)$$

with the terms of  $O(k^2)$  neglected, and for the free-molecular gas,

$$V_f = \frac{\sqrt{\pi}}{2} \frac{1}{D(a)} (2RT_0)^{1/2} \tau_{wc}, \quad (6.11)$$

where  $D(a)$  is given by (B 4).

## 7. Concluding remarks

In the present study, we considered an oblate spheroid with a non-uniform and axisymmetric surface temperature placed in a rarefied gas at an equilibrium state at rest when the temperature variation is small (Problem I). In this case, according to the formula [cf. (2.2)] derived by Takata (2009a) on the basis of the symmetry relations for the linearized Boltzmann equation, we can obtain the force acting on the spheroid (radiometric force) by solving the adjoint problem (Problem II), where the same oblate spheroid with a uniform temperature is placed in a slow uniform flow of a gas.

This approach is particularly useful for small Knudsen numbers. When the Knudsen number is small, the generalized slip flow theory developed by Sone is available for any smooth geometry and smooth boundary data (Sone 1969, 1971, 1991, 2002, 2007). According to the theory, the zeroth-order solution in the Knudsen number of Problem II, which is necessary to obtain the leading-order term in the Knudsen number of the radiometric force in Problem I, is nothing but the classical Stokes solution (Payne & Pell 1960). Therefore, we can readily obtain the leading-order term of the force acting on the spheroid in Problem I by the formula (2.2) for any temperature distribution. In particular, when the surface temperature is not smooth, such as the discontinuous temperature distribution given by (2.1), this approach is very powerful because Problem I itself is beyond the applicability of the generalized slip flow theory that is based on the smoothness of the boundary data. In other words, if one tries to solve Problem I in this case, there is no way other than a direct numerical approach even for small Knudsen numbers. Nevertheless, we were able to obtain an analytical formula for the radiometric force (5.1) by the present approach.

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## Appendix A. Direct derivation of (2.2)

The formula (2.2) has been derived in Takata (2009a) in a much more general setting. Here, we derive it again directly from the basic equations for Problem I and Problem II in order to check some mathematical problems that have not been mentioned explicitly in Takata (2009a). For this purpose, we introduce some additional quantities. Let  $(2RT_0)^{1/2} \zeta_i$  be the velocity of gas molecules and  $\rho_0 (2RT_0)^{-3/2} E[1 + \phi(x_i, \zeta_i)]$  be the velocity distribution function of gas molecules, where  $\rho_0 = p_0/RT_0$  and  $E = \pi^{-3/2} \exp(-\zeta_i^2)$ . Then, the deviations of the macroscopic quantities  $\omega$ ,  $u_i$ ,  $\tau$ ,  $P$ ,  $P_{ij}$ , and

$Q_i$  are expressed in terms of  $\phi$  as follows:

$$\omega = \langle \phi \rangle, \quad u_i = \langle \zeta_i \phi \rangle, \quad \tau = \frac{2}{3} \left\langle \left( \zeta_i^2 - \frac{3}{2} \right) \phi \right\rangle, \quad P = \omega + \tau = \frac{2}{3} \langle \zeta_i^2 \phi \rangle, \quad (\text{A } 1a)$$

$$P_{ij} = 2 \langle \zeta_i \zeta_j \phi \rangle, \quad Q_i = \langle \zeta_i \zeta_j^2 \phi \rangle - \frac{5}{2} u_i. \quad (\text{A } 1b)$$

Here,  $\langle \rangle$  indicates

$$\langle g(\zeta_i) \rangle = \int g(\zeta_i) E d\zeta, \quad (\text{A } 2)$$

where  $g(\zeta_i)$  is an arbitrary function of  $\zeta_i$ ,  $d\zeta = d\zeta_1 d\zeta_2 d\zeta_3$ , and the domain of integration is the whole space of  $\zeta_i$ . If we introduce the notation  $\Phi(\zeta_i)^- = \Phi(-\zeta_i)$  for any function  $\Phi$  of  $\zeta_i$ , it follows that

$$(\Phi^-)^- = \Phi, \quad \langle \Phi^- \rangle = \langle \Phi \rangle. \quad (\text{A } 3)$$

Let us consider Problem I and Problem II for arbitrary Knudsen numbers under assumptions (i) and (ii) [or (ii)'] with the diffuse reflection condition as the boundary condition for the Boltzmann equation. Then, each problem is formulated as follows:

[Problem I] The basic equation is

$$\zeta_i \frac{\partial \phi^I}{\partial x_i} = \frac{1}{k} \mathcal{L}(\phi^I), \quad (\text{A } 4)$$

and the boundary conditions are

$$\phi^I = \sigma_w^I + \left( \zeta_i^2 - \frac{3}{2} \right) \tau_w, \quad (\text{for } \zeta_i n_i > 0 \text{ on the spheroid}), \quad (\text{A } 5a)$$

$$\text{with } \sigma_w^I = -\frac{1}{2} \tau_w - 2\sqrt{\pi} \int_{\zeta_i n_i < 0} \zeta_j n_j \phi^I E d\zeta, \quad (\text{A } 5b)$$

$$\phi^I \rightarrow 0, \quad (\text{for } |x_i| \rightarrow \infty). \quad (\text{A } 5c)$$

[Problem II] The basic equation is

$$\zeta_i \frac{\partial \phi^{\text{II}}}{\partial x_i} = \frac{1}{k} \mathcal{L}(\phi^{\text{II}}), \quad (\text{A } 6)$$

and the boundary conditions are

$$\phi^{\text{II}} = \sigma_w^{\text{II}} \quad (\text{for } \zeta_i n_i > 0 \text{ on the spheroid}), \quad (\text{A } 7a)$$

$$\text{with } \sigma_w^{\text{II}} = -2\sqrt{\pi} \int_{\zeta_i n_i < 0} \zeta_j n_j \phi^{\text{II}} E d\zeta, \quad (\text{A } 7b)$$

$$\phi^{\text{II}} \rightarrow 2\zeta_1 u_\infty, \quad (\text{for } |x_i| \rightarrow \infty). \quad (\text{A } 7c)$$

Here and in what follows, the superscript I (or II) is put for the solution and corresponding quantities of Problem I (or Problem II). In (A 4) and (A 6),  $\mathcal{L}(\phi)$  is the linearized collision operator of the Boltzmann equation, whose explicit form is omitted here [see Sone (2002, 2007)]. We just note that it has the following basic properties: For any functions  $\Phi$  and  $\Psi$  of  $\zeta_i$ ,

$$\mathcal{L}(\Phi^-) = \mathcal{L}(\Phi)^-, \quad (\text{A } 8)$$

$$\langle \Phi \mathcal{L}(\Psi) \rangle = \langle \Psi \mathcal{L}(\Phi) \rangle, \quad (\text{self-adjointness}), \quad (\text{A } 9)$$

$$\langle \Phi \mathcal{L}(\Phi) \rangle \leq 0, \quad (\text{non-positivity}). \quad (\text{A } 10)$$

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In (A 10), the equality sign holds if and only if  $\Phi$  is a linear combination of 1,  $\zeta_i$  and  $\zeta_i^2$ .

Now let us consider  $\langle \phi^{\text{II-}} \times (\text{A 4}) \rangle - \langle \phi^{\text{I-}} \times (\text{A 6}) \rangle$ , that is,

$$\left\langle \phi^{\text{II-}} \zeta_j \frac{\partial \phi^{\text{I}}}{\partial x_j} \right\rangle - \left\langle \phi^{\text{I-}} \zeta_j \frac{\partial \phi^{\text{II}}}{\partial x_j} \right\rangle = \frac{1}{k} [\langle \phi^{\text{II-}} \mathcal{L}(\phi^{\text{I}}) \rangle - \langle \phi^{\text{I-}} \mathcal{L}(\phi^{\text{II}}) \rangle]. \quad (\text{A 11})$$

The right-hand side (RHS) of (A 11) vanishes because of the properties (A 3), (A 8) and (A 9), i.e.,

$$\text{RHS} = \langle \phi^{\text{II-}} \mathcal{L}(\phi^{\text{I}}) \rangle - \langle \phi^{\text{I-}} \mathcal{L}(\phi^{\text{II-}}) \rangle = \langle \phi^{\text{II-}} \mathcal{L}(\phi^{\text{I}}) \rangle - \langle \phi^{\text{II-}} \mathcal{L}(\phi^{\text{I}}) \rangle = 0. \quad (\text{A 12})$$

On the other hand, because of the same properties, the left-hand side (LHS) of (A 11) is transformed as

$$\text{LHS} = \left\langle \phi^{\text{II-}} \zeta_j \frac{\partial \phi^{\text{I}}}{\partial x_j} \right\rangle + \left\langle \phi^{\text{I-}} \zeta_j \frac{\partial \phi^{\text{II-}}}{\partial x_j} \right\rangle = \left\langle \zeta_j \frac{\partial \phi^{\text{I}} \phi^{\text{II-}}}{\partial x_j} \right\rangle = \frac{\partial}{\partial x_j} \langle \zeta_j \phi^{\text{I}} \phi^{\text{II-}} \rangle. \quad (\text{A 13})$$

However, in order that the last equality holds, we implicitly assume that  $\phi^{\text{I}}$  and  $\phi^{\text{II}}$  are smooth enough. In reality, in Problems I and II, where a gas around a convex body (oblate spheroid) is considered,  $\phi^{\text{I}}$  and  $\phi^{\text{II}}$  have discontinuities in the  $\zeta_i$  space (Sone & Takata 1992; Sone 2002, 2007). More specifically, at a fixed point  $x_i$  in the gas, they are discontinuous for the velocities in the direction of the lines (in the  $x_i$  space) that are tangent to the body. In addition, in Problem I, discontinuities are also produced by a discontinuous boundary temperature, such as (2.1) [or (5.2)]. It should be noted here that, at any point  $x_i$ , all the discontinuities in the  $\zeta_i$  space are located along radial lines parallel to  $\zeta_i$  [i.e., the characteristic line of (A 4) or (A 6)]. As pointed out by Sone in the derivation of the conservation equations from the Boltzmann equation [see §1.2 of the Supplementary Notes in Sone (2007)], if we take this property into account, we can show that the derivative  $\partial/\partial x_i$  and the integral  $\langle \cdot \rangle$  with respect to  $\zeta_i$  are interchangeable, that is, the last equality of (A 13) holds. Then, from (A 12) and (A 13), we have

$$\frac{\partial}{\partial x_j} \langle \zeta_j \phi^{\text{I}} \phi^{\text{II-}} \rangle = 0. \quad (\text{A 14})$$

Let us consider the domain  $\mathcal{V}$  between the surface of the spheroid  $S$  and the surface  $S_r$  of a sphere of radius  $r$  centered at the origin and containing the spheroid. If we integrate (A 14) over  $\mathcal{V}$  and apply the Gauss theorem formally, we obtain

$$\int_{\mathcal{V}} \frac{\partial}{\partial x_j} \langle \zeta_j \phi^{\text{I}} \phi^{\text{II-}} \rangle d\mathbf{x} = \int_{S_r} \langle \zeta_j \phi^{\text{I}} \phi^{\text{II-}} \rangle n_j dS - \int_S \langle \zeta_j \phi^{\text{I}} \phi^{\text{II-}} \rangle n_j dS = 0, \quad (\text{A 15})$$

where  $d\mathbf{x} = dx_1 dx_2 dx_3$ ,  $n_i$  is the outward unit normal of the surface of the sphere  $S_r$  or the unit normal of the surface of the spheroid  $S$  pointing into the gas, and  $dS$  is the surface element on  $S_r$  or  $S$  in the dimensionless  $x_i$  space. Here, the integrand  $\langle \zeta_j \phi^{\text{I}} \phi^{\text{II-}} \rangle n_j$  in the integral over  $S$  is interpreted as the limit from the gas side. As mentioned in the preceding paragraph,  $\phi^{\text{I}}$  and  $\phi^{\text{II}}$  exhibit discontinuities. However,

Property 1:  $\langle \zeta_j \phi^{\text{I}} \phi^{\text{II-}} \rangle$ , which is the integral over the whole space of  $\zeta_i$ , is a continuous (and differentiable) function of  $x_i$  in the gas, as other macroscopic quantities in Problems I and II.

Therefore, the only problem in (A 15) is that  $\langle \zeta_j \phi^{\text{I}} \phi^{\text{II-}} \rangle$  may have singularities on  $S$ . For instance, when the temperature of the spheroid in Problem I is discontinuous,

Property 2:  $\langle \zeta_j \phi^{\text{I}} \phi^{\text{II-}} \rangle$  is discontinuous on  $S$ , as other macroscopic quantities in Problem I; this means that  $\langle \zeta_j \phi^{\text{I}} \phi^{\text{II-}} \rangle n_j$  is integrable on  $S$ .



Properties 1 and 2 are the consequences based on the assumption that  $\phi^I$  and  $\phi^{II}$  are piecewise smooth functions with discontinuities in  $x_i$  and  $\zeta_i$  as in Sone & Takata (1992). We must say that it is very hard to prove these properties as well as the properties of  $\phi^I$  and  $\phi^{II}$  in a rigorous mathematical way from the basic systems, Eqs. (A 4) and (A 5) and Eqs. (A 6) and (A 7). However, these consequences are supported indirectly by the previous numerical analysis (Aoki, Takata, Aikawa & Golse 2001) of a problem with discontinuous boundary temperature using the (nonlinear) BGK model and by the rigorous mathematical study (Aoki, Bardos, Dogbe & Golse 2001) of a boundary-value problem with discontinuous boundary data for a simple radiative transfer equation, which has a structure similar to the linearized BGK model. Thus, if we take the limit  $r \rightarrow \infty$  in (A 15), we have

$$\lim_{r \rightarrow \infty} \int_{S_r} \langle \zeta_j \phi^I \phi^{II-} \rangle n_j dS = \int_S \langle \zeta_j \phi^I \phi^{II-} \rangle n_j dS. \quad (\text{A } 16)$$

According to Takata (2009a),  $\phi^I$  and  $\phi^{II}$  behave, for large  $|x_i|$ , as

$$\phi^I = 2\zeta_j c_j^I + \left( \zeta_j^2 - \frac{5}{2} \right) c^I + O(|x_i|^{-2}), \quad (\text{A } 17a)$$

$$\phi^{II} = 2\zeta_j (u_{\infty j} + c_j^I) + \left( \zeta_j^2 - \frac{5}{2} \right) c^{II} + O(|x_i|^{-2}), \quad (\text{A } 17b)$$

where  $u_{\infty i}$  indicates the vector  $u_{\infty} \mathbf{e}_1$ , and  $c_i^I$ ,  $c^I$ ,  $c_i^{II}$  and  $c^{II}$  are quantities of  $O(|x_i|^{-1})$ . Using (A 17a) and (A 17b) and noting that  $-2 \int_{S_r} \langle \zeta_1 \zeta_j \phi^I \rangle n_j dS$  is nothing else than  $F^I/p_0 L^2$ , we can transform the LHS of (A 16) as

$$\begin{aligned} \lim_{r \rightarrow \infty} \int_{S_r} \langle \zeta_j \phi^I \phi^{II-} \rangle n_j &= \lim_{r \rightarrow \infty} \int_{S_r} [-2u_{\infty} \langle \zeta_1 \zeta_j \phi^I \rangle n_j + O(|x_i|^{-3})] dS. \\ &= (u_{\infty}/p_0 L^2) F^I. \end{aligned} \quad (\text{A } 18)$$

On the other hand, since  $\phi^I$  for  $\zeta_i n_i > 0$  on the spheroid is given by (A 5a) and  $\phi^{II-}$  for  $\zeta_i n_i < 0$  there, which is equivalent to  $\phi^{II}$  for  $\zeta_i n_i > 0$ , is given by (A 7a), the integrand of the RHS, which is evaluated on the spheroid, is transformed as follows:

$$\begin{aligned} &\langle \zeta_j \phi^I \phi^{II-} \rangle n_j \\ &= \int_{\zeta_j n_j < 0} \zeta_j n_j \phi^I \phi^{II-} E d\zeta + \int_{\zeta_j n_j > 0} \zeta_j n_j \phi^I \phi^{II-} E d\zeta \\ &= \sigma_w^{II} \int_{\zeta_j n_j < 0} \zeta_j n_j \phi^I E d\zeta + \int_{\zeta_j n_j > 0} \zeta_j n_j \left[ \sigma_w^I + \left( \zeta_i^2 - \frac{3}{2} \right) \tau_w \right] \phi^{II-} E d\zeta \\ &= \sigma_w^{II} \int_{\zeta_j n_j < 0} \zeta_j n_j \phi^I E d\zeta - \int_{\zeta_j n_j < 0} \zeta_j n_j \left[ \sigma_w^I + \left( \zeta_i^2 - \frac{3}{2} \right) \tau_w \right] \phi^{II} E d\zeta \\ &= -\tau_w \left( \frac{1}{\sqrt{\pi}} \sigma_w^{II} + \int_{\zeta_j n_j < 0} \zeta_j n_j \zeta_i^2 \phi^{II} E d\zeta \right), \end{aligned} \quad (\text{A } 19)$$

where (A 5b) and (A 7b) have been used to obtain the last line. However, if we calculate the normal component of the heat-flow vector  $Q_i^{II} n_i$  corresponding to  $\phi^{II}$  on the spheroid using (A 1b) and (A 7a) and note that  $u_i n_i = 0$  there, we find that the last line of (A 19)

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is equal to  $-\tau_w Q_i^{\text{II}} n_i$ . Therefore, the RHS of (A 16) becomes

$$\int_S \langle \zeta_j \phi^{\text{I}} \phi^{\text{II-}} \rangle n_j dS = - \int_S \tau_w Q_i^{\text{II}} n_i dS. \quad (\text{A } 20)$$

In summary, (A 16), (A 18) and (A 20) lead to the relation

$$(u_\infty/p_0 L^2) F^{\text{I}} = - \int_S \tau_w Q_i^{\text{II}} n_i dS, \quad (\text{A } 21)$$

i.e., (2.2).

## Appendix B. Drag acting on an oblate spheroid in Problem II

From the solution (4.6) for the Problem II, we can obtain the leading-order term in  $k$  ( $\ll 1$ ) of the drag force acting on the oblate spheroid placed in a slow uniform flow of the gas, which is nothing but the classical Stokes drag (Payne & Pell 1960). That is, the drag force  $F^{\text{II}} \mathbf{e}_1$  is obtained as

$$F^{\text{II}} = 8\pi\gamma_1 p_0 L^2 u_\infty \frac{a}{At_0} k, \quad (\text{B } 1)$$

with  $k^2$ -order term neglected, where  $A$  and  $t_0$  are the functions of the aspect ratio  $a$  [see (4.7)]. In the case of a sphere ( $a = 1$ ) and a circular disk ( $a = 0$ ), the drag (B 1) reduces, respectively, to

$$F^{\text{II}} = \begin{cases} 6\pi\gamma_1 p_0 L^2 u_\infty k & (a = 1), \\ 16\pi\gamma_1 p_0 L^2 u_\infty k & (a = 0). \end{cases} \quad (\text{B } 2)$$

On the other hand, in the free molecular flow ( $k = \infty$ ), the drag force is given by

$$F^{\text{II}} = (\sqrt{\pi}/2) p_0 L^2 u_\infty D(a), \quad (\text{B } 3)$$

where

$$D(a) = \frac{4 + \pi - 2a^2}{c_0^2} \left[ 2 - a^2 c_0^{-1} \ln \left( \frac{1 + c_0}{1 - c_0} \right) \right] + 4a^2 c_0^{-1} \ln \left( \frac{1 + c_0}{1 - c_0} \right), \quad (\text{B } 4)$$

and  $c_0 = \sqrt{1 - a^2}$ , see (4.5). In the case of a sphere ( $a = 1$ ) and a circular disk ( $a = 0$ ), (B 3) reduces, respectively, to

$$F^{\text{II}} = \begin{cases} (2\sqrt{\pi}/3)(8 + \pi) p_0 L^2 u_\infty & (a = 1), \\ \sqrt{\pi}(4 + \pi) p_0 L^2 u_\infty & (a = 0). \end{cases} \quad (\text{B } 5)$$

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