



A family of non-modular covariant AQFTs

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Abstract

Based on the construction provided in our paper “Covariant homogeneous nets of standard subspaces”, *Comm Math Phys* 386:305–358, (2021), we construct non-modular covariant one-particle nets on the two-dimensional de Sitter spacetime and on the three-dimensional Minkowski space.

Contents

1	Introduction	1
2	Preliminaries	4
2.1	One-particle subspaces	4
2.2	Euler wedges	5
2.3	Nets of standard subspaces	9
2.4	Symmetric and non-symmetric Euler elements	10
3	One-particle nets which are not modular covariant	12
3.1	A general construction principle	12
3.2	Non-modular covariant nets	15
3.2.1	On de Sitter space dS^2	15
3.2.2	An explicit example on de Sitter space dS^2	16
3.2.3	Counterexamples on Minkowski spacetime	18
4	Outlook: Disintegration, locality and higher dimensions	20
	References	24

1 Introduction

The modular theory of von Neumann algebras, based on the Tomita–Takesaki Theorem has applications in many different areas of mathematics and physics because it

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generates rich structures from very basic assumptions, such as the existence of cyclic separating vectors. In Algebraic Quantum Field Theory (AQFT), the verification of the Bisognano–Wichmann property was a fundamental breakthrough [1]. This property deeply relates the geometry of the models to the modular structure of von Neumann algebras of observables.

The description of the QFT models through operator algebras is based on an assignment of von Neumann algebras $A(\mathcal{O})$ to spacetime regions \mathcal{O} satisfying isotony. Basic features of QFT can be expressed as natural properties of such a Haag–Kastler net $\mathcal{O} \mapsto A(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ ([13]): Algebras associated with spacelike separated regions are required to commute (locality), and covariance is formulated in terms of a unitary positive energy representation of the symmetry group (the Poincaré group on the Minkowski spacetime or the Lorentz group on de Sitter spacetime) that acts covariantly on the observable algebras. Together with an invariant vacuum vector state, this defines the vacuum representation of a net of observables as introduced by Haag and Kastler in 1964 ([14]).

Given a von Neumann algebra \mathcal{M} with a cyclic and separating vector, the Tomita–Takesaki Theorem provides a unitary one-parameter group, called modular group, whose adjoint action on the algebra defines a one-parameter group of automorphism. The modular groups of some algebras with particular localization properties actually correspond to global geometric symmetries. This is called the Bisognano–Wichmann (BW) property. It marked one of the formidable successes of modular theory: The modular structure of some observable algebras in the vacuum sector has geometric meaning.

The BW property has been verified for large number of models (see e.g. [5, 11, 26]) and applied in various ways with feedback both for mathematics and for physics. For recent developments concerning entropy of QFT's we refer to [9, 18, 21, 22, 32, 33] and for some new constructions exploiting geometric symmetries and modular theory to [19, 25]. The property of so-called “modular covariance” of a net is weaker than the Bisognano–Wichmann property [4, 8, 34]. On one hand, the Bisognano–Wichmann property assumes covariance with respect to a Poincaré (on Minkowski spacetime) or Lorentz (on de Sitter spacetime) representation and states that the unitary representation of the Lorentz boosts coincides with the modular group of the algebra associated to a wedge domain, properly parametrized. On the other hand, modular covariance ensures that the modular groups of wedge region algebras act on the net geometrically as the Lorentz boosts, but not that they belong to the given representation of the symmetry group. On Minkowski spacetime the modular group of the wedge regions generate a covariant representation of the Poincaré group satisfying the positive energy condition, under the Reeh-Schlieder property assumption for space-like cones [12]. This result does not directly extend to de Sitter spacetime because a specific domain for the modular operator of orthogonal wedges has not been proved yet to be dense in the case of complementary series representations of $\mathrm{PSL}_2(\mathbb{R})$ as a subgroup of the Lorentz group (cf. [24, §4.4]).

A long standing question is how to characterize models where just modular covariance fails or, more generally, that are not modular covariant. Most of the models violating the Bisognano–Wichmann property are still modular covariant, see [10, 20, 23]. A first family of models without modular covariance in Minkowski space with

dimension $1 + d > 2$ has been provided by Yngvason in [34]. For a certain two-point function, he constructs translation covariant nets of von Neumann algebras with positive energy, computes the modular operators of wedge algebras and concludes that they do not act covariantly on the net. However, these models are not expected to be Lorentz covariant.

A new approach to geometric features in AQFT models has been provided in [24], where wedge regions are replaced by abstract data associated to a graded Lie group, which in representations correspond to modular operators and conjugations as produced by the Tomita–Takesaki Theorem. This Lie theoretic perspective on the wedge–boost correspondence creates a means to construct AQFT models. In [24] we also determine a class of Lie groups which are compatible with a notion of wedge localization, including the known cases from physics (see also [29–31] for related recent work in this direction).

The construction of the models is based on the Brunetti–Guido–Longo (BGL) construction of the free fields: Nets of von Neumann algebras are constructed through the second quantization canonical procedure starting from one-particle nets of standard subspaces defined by (anti-)unitary representations of graded Lie groups ([7]). Thus, in this context, in order to obtain a net of von Neumann algebras, one has to construct nets of standard subspaces on the one-particle Hilbert space.

In this paper we present a structural condition that can be used to construct non-modular covariant nets in the abstract setting of [24]. Then we construct a family of Lorentz and Poincaré covariant models based on one-particle nets that are non-modular covariant, on two-dimensional de Sitter spacetime and on three-dimensional Minkowski spacetime. The idea is to construct a BGL one-particle net of real subspaces from a representation of a “large” group G that has a restriction to the de Sitter or to the Minkowski symmetry group which is not modular covariant. It is central that the Lie algebra \mathfrak{g} of G contains non-symmetric Euler elements. A non-symmetric Euler element is associated to a wedge orbit under the G -action that does not contain complement wedges. Note that this is not the case of the two (or higher) dimensional de Sitter or three (or higher) dimensional Minkowski spacetime with respect to their symmetry groups. Several groups of this kind are described in [24]. With the proper identifications, we first obtain non-local nets on two-dimensional de Sitter spacetime which are Lorentz covariant. Then we generalize the construction to three-dimensional Minkowski space.

The paper is organized as follows. In Sect. 2 we present the construction of the generalized AQFT models provided in [24]. In Sect. 3 we present a general construction of non-modular covariant nets and give sufficient criteria for its applicability. In Sect. 3.2.2 we present an explicit example of this method. In Sect. 3.2.3 we show how the construction can be applied to Minkowski space. An outlook on disintegration of covariant representations, locality and higher dimensional examples is given in Sect. 4.

2 Preliminaries

In this section we collect background on several concepts and their properties: standard subspaces, abstract Euler wedges and the BGL construction. In Sect. 2.4 we prepare the group theoretic background for our construction of non-modular covariant nets.

2.1 One-particle subspaces

We call a closed real subspace H of the complex Hilbert space \mathcal{H} *cyclic* if $H+iH$ is dense in \mathcal{H} , *separating* if $H \cap iH = \{0\}$, and *standard* if it is cyclic and separating. The symplectic “complement” of a real subspace H is defined by the symplectic form $\text{Im}\langle \cdot, \cdot \rangle$ via

$$H' = \{ \xi \in \mathcal{H} : (\forall \eta \in H) \text{Im}\langle \xi, \eta \rangle = 0 \}.$$

Then H is separating if and only if H' is cyclic, hence H is standard if and only if H' is standard. For a standard subspace H , we define the *Tomita operator* as the closed antilinear involution

$$S_H : H + iH \rightarrow H + iH, \quad \xi + i\eta \mapsto \xi - i\eta.$$

The polar decomposition $S_H = J_H \Delta_H^{\frac{1}{2}}$ defines an antiunitary involution J_H (a conjugation) and the modular operator Δ_H . For the modular group $(\Delta_H^{it})_{t \in \mathbb{R}}$, we then have

$$J_H H = H', \quad \Delta_H^{it} H = H \quad \text{and} \quad J_H \Delta_H^{it} J_H = \Delta_H^{-it} \quad \text{for every } t \in \mathbb{R}.$$

One also has $H = \ker(S_H - \mathbf{1})$ ([17, Thm. 3.4]). This construction leads to a one-to-one correspondence between Tomita operators and standard subspaces:

Proposition 2.1 ([17, Prop. 3.2]) *The map $H \mapsto S_H$ is a bijection between the set of standard subspaces of \mathcal{H} and the set of closed, densely defined, antilinear involutions on \mathcal{H} . Moreover, polar decomposition $S = J \Delta^{1/2}$ defines a one-to-one correspondence between such involutions and pairs (Δ, J) , where J is a conjugation and $\Delta > 0$ selfadjoint with $J \Delta J = \Delta^{-1}$.*

The modular operators of symplectic complements satisfy the following relations

$$S_{H'} = S_H^*, \quad \Delta_{H'} = \Delta_H^{-1}, \quad J_{H'} = J_H.$$

From Proposition 2.1 we easily deduce:

Lemma 2.2 ([23, Lemma 2.2]) *Let $H \subset \mathcal{H}$ be a standard subspace and $U \in \text{AU}(\mathcal{H})$ be a unitary or anti-unitary operator. Then UH is also standard and $U \Delta_H U^* = \Delta_{UH}^{\varepsilon(U)}$ and $U J_H U^* = J_{UH}$, where $\varepsilon(U) = 1$ if U is unitary and $\varepsilon(U) = -1$ if it is antiunitary.*

2.2 Euler wedges

Let G be a finite dimensional \mathbb{Z}_2 -graded Lie group (G, ε_G) , i.e., G is a Lie group and $\varepsilon_G : G \rightarrow \{\pm 1\}$ a continuous homomorphism. We write

$$G^\uparrow = \varepsilon_G^{-1}(1) \quad \text{and} \quad G^\downarrow = \varepsilon_G^{-1}(-1),$$

so that $G^\uparrow \trianglelefteq G$ is a normal subgroup of index 2 and $G^\downarrow = G \setminus G^\uparrow$. As the subgroup G^\uparrow is open and closed, it contains the connected component G_e of the neutral element e in G .

Definition 2.3 (a) We call an element x of the finite dimensional real Lie algebra \mathfrak{g} an *Euler element* if $\text{ad } x$ is non-zero and diagonalizable with $\text{Spec}(\text{ad } x) \subseteq \{-1, 0, 1\}$. In particular the eigenspace decomposition with respect to $\text{ad } x$ defines a 3-grading of \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{g}_1(x) \oplus \mathfrak{g}_0(x) \oplus \mathfrak{g}_{-1}(x), \quad \text{where} \quad \mathfrak{g}_\nu(x) = \ker(\text{ad } x - \nu \text{id}_\mathfrak{g})$$

Then $\sigma_x(y_j) = (-1)^j y_j$ for $y_j \in \mathfrak{g}_j(x)$ defines an involutive automorphism of \mathfrak{g} .

We write $\mathcal{E}(\mathfrak{g})$ for the set of Euler elements in \mathfrak{g} . The orbit of an Euler element x under the group $\text{Inn}(\mathfrak{g}) = \langle e^{\text{ad } \mathfrak{g}} \rangle$ of *inner automorphisms* is denoted with $\mathcal{O}_x = \text{Inn}(\mathfrak{g})x \subseteq \mathfrak{g}$.¹

We say that x is *symmetric* if $-x \in \mathcal{O}_x$.

(b) The set

$$\mathcal{G} := \mathcal{G}(G) := \{(x, \sigma) \in \mathfrak{g} \times G^\downarrow : \sigma^2 = e, \text{Ad}(\sigma)x = x\}$$

is called the *abstract wedge space of G* . We assign to $W = (x, \sigma) \in \mathcal{G}$ the one-parameter group

$$\lambda_W : \mathbb{R} \rightarrow G^\uparrow \quad \text{by} \quad \lambda_W(t) := \exp(tx). \tag{2.1}$$

An element $(x, \sigma) \in \mathcal{G}$ is called an *Euler couple* or *Euler wedge* if

$$\text{Ad}(\sigma) = \sigma_x := e^{\pi i \text{ad } x}. \tag{2.2}$$

Then σ is called an *Euler involution*. We write $\mathcal{G}_E \subseteq \mathcal{G}$ for the subset of Euler couples.

(c) For the graded group (G, ε) , we consider on \mathfrak{g} the *twisted adjoint action* which changes the sign on odd group elements:

$$\text{Ad}^\varepsilon : G \rightarrow \text{Aut}(\mathfrak{g}), \quad \text{Ad}^\varepsilon(g) := \varepsilon_G(g) \text{Ad}(g). \tag{2.3}$$

It extends to an action of G on \mathcal{G} by

$$g.(x, \sigma) := (\text{Ad}^\varepsilon(g)x, g\sigma g^{-1}). \tag{2.4}$$

¹ For a Lie subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$, we write $\text{Inn}_\mathfrak{g}(\mathfrak{s}) = \langle e^{\text{ad } \mathfrak{s}} \rangle \subseteq \text{Aut}(\mathfrak{g})$ for the subgroup generated by $e^{\text{ad } \mathfrak{s}}$.

(d) (Duality operation) We define the notion of a “causal complement” on the abstract wedge space: For $W = (x, \sigma) \in \mathcal{G}$, we define the *dual wedge* by $W' := (-x, \sigma) = \sigma.W$. Note that $(W')' = W$ and $(gW)' = gW'$ for $g \in G$ by (2.4).

The relation $\sigma.W = W'$ is our main motivation to work with the twisted adjoint action. This relation fits the geometric interpretation in the context of wedge domains in spacetime manifolds.

(e) (Order structure on \mathcal{G}) For a given invariant closed convex cone $C \subseteq \mathfrak{g}$, we obtain an order structure on \mathcal{G} as follows. We associate to $W = (x, \sigma) \in \mathcal{G}$ a semigroup \mathcal{S}_W whose unit group is $\mathcal{S}_W \cap \mathcal{S}_W^{-1} = G_W^\uparrow$, the stabilizer of W ([27, Thm. III.4]). It is specified by

$$\mathcal{S}_W := \exp(C_+)G_W^\uparrow \exp(C_-) = G_W^\uparrow \exp(C_+ + C_-),$$

where the convex cones C_\pm are the following intersections

$$C_\pm := \pm C \cap \mathfrak{g}^{-\sigma} \cap \ker(\text{ad } x \mp \mathbf{1}) \quad \text{and} \quad \mathfrak{g}^{\pm\sigma} := \{y \in \mathfrak{g} : \text{Ad}(\sigma)(y) = \pm y\}.$$

Then \mathcal{S}_W defines a G^\uparrow -invariant partial order on the orbit $G^\uparrow.W \subseteq \mathcal{G}$ by

$$g_1.W \leq g_2.W \quad :\iff \quad g_2^{-1}g_1 \in \mathcal{S}_W. \tag{2.5}$$

In particular, $g.W \leq W$ is equivalent to $g \in \mathcal{S}_W$.

Lemma 2.4 ([24, Lemma 2.6]) *For every $W = (x_W, \sigma_W) \in \mathcal{G}$, $g \in G$, and $t \in \mathbb{R}$, the following assertions hold:*

- (i) $\lambda_W(t).W = W$, $\lambda_W(t).W' = W'$ and $\sigma_W.W = W'$.
- (ii) $\sigma_{W'} = \sigma_W$ and $\lambda_{W'}(t) = \lambda_W(-t)$.
- (iii) σ_W commutes with $\lambda_W(\mathbb{R})$.

Remark 2.5 Let $W = (x, \sigma) \in \mathcal{G}$ and consider $y \in \mathfrak{g}$. Then $\exp(\mathbb{R}y)$ fixes W if and only if

$$[y, x] = 0 \quad \text{and} \quad y = \text{Ad}(\sigma)y.$$

If (x, σ) is an Euler couple, then $\text{Ad}(\sigma)y = e^{\pi i \text{ad } x}y = y$ follows from $[y, x] = 0$, so that

$$\mathfrak{g}_W := \{y \in \mathfrak{g} : \exp(\mathbb{R}y) \subseteq G_W^\uparrow\} = \ker(\text{ad } x). \tag{2.6}$$

Definition 2.6 (The abstract wedge space) From here on, we always assume that $\mathcal{G} \neq \emptyset$, i.e., that G^\downarrow contains an involution σ . Then

$$G \cong G^\uparrow \rtimes \{\text{id}, \sigma\}$$

For a fixed couple $W_0 = (h, \sigma) \in \mathcal{G}$, the orbits

$$\mathcal{W}_+(W_0) := G^\uparrow.W_0 \subseteq \mathcal{G} \quad \text{and} \quad \mathcal{W}(W_0) := G.W_0 \subseteq \mathcal{G}$$

are called the *positive* and the *full wedge space containing* W_0 .

Remark 2.7 (Lorentz wedges on de Sitter spacetime) The de Sitter spacetime is the manifold $dS^d = \{(t, \mathbf{x}) \in \mathbb{R}^{1+d} : \mathbf{x}^2 - t^2 = 1\}$, endowed with the metric obtained by restriction of the Minkowski metric

$$ds^2 = dt^2 - dx_1^2 - \dots - dx_d^2$$

to dS^d .

The generator $k_1 \in \mathfrak{so}_{1,d}(\mathbb{R})$ of the Lorentz boost on the (x_0, x_1) -plane

$$k_1(x_0, x_1, x_2, \dots, x_d) = (x_1, x_0, 0, \dots, 0)$$

is an Euler element. It combines with the spacetime reflection

$$j_1(x) = (-x_0, -x_1, x_2, \dots, x_d)$$

to the Euler couple (k_1, j_1) for the graded Lie group $SO_{1,d}(\mathbb{R})$. The spacetime region

$$W_{x_1} = \{x \in \mathbb{R}^{1+d} : |x_0| < x_1\}$$

is called the *standard right wedge* and we put

$$W_{x_1}^{dS} := W_{x_1} \cap dS^d.$$

Note that W_{x_1} and therefore $W_{x_1}^{dS}$ are invariant under $\exp(\mathbb{R}k_1)$. Lorentz transforms $W_{x_1}^{dS} = g \cdot W_{x_1}^{dS}$ of $W_{x_1}^{dS}$ with $g \in SO_{1,d}(\mathbb{R})$ are called *wedge regions in de Sitter space*. They are in 1-1 correspondence with Euler couples in $\mathcal{G}(SO_{1,d}(\mathbb{R}))$ and one can associate to W_{x_1} the couple $(k_1, j_1) \in \mathcal{G}_E(SO_{1,d}(\mathbb{R}))$ (cf. [28, Lemma 4.13] and [7, Sect. 5.2]). For $\ell = 2, \dots, d$, one likewise obtains couples (k_ℓ, j_ℓ) corresponding to the wedges $W_{x_\ell} = \{x \in \mathbb{R}^{1+d} : |x_0| < x_\ell\}$.

In this paper we will focus on 2-dimensional de Sitter spacetime. The orbit of the wedge $W_{x_1}^{dS}$ under the Lorentz group is indicated with $\mathcal{W}_+^{dS} = \mathcal{L}_+^\uparrow W_{x_1}$ where $\mathcal{L}_+^\uparrow := SO_{1,2}(\mathbb{R})^\uparrow$. In this case the orthochronous symmetry group $SO_{1,2}(\mathbb{R})^\uparrow$ is isomorphic to $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm 1\}$ and the action on dS^2 can be visualized by the adjoint action of $PSL_2(\mathbb{R})$ on the orbit generated by the matrix $\frac{1}{2} \text{diag}(1, -1)$. We shall use the following matrix picture of dS^2 , which is implemented by the bijection

$$dS^2 \rightarrow dS_{\text{mat}}^2 := \left\{ X \in M_2(\mathbb{R}) : \text{tr } X = 0, \det X = -\frac{1}{4} \right\} \subseteq \mathfrak{sl}_2(\mathbb{R})$$

$$x = (x_0, x_1, x_2) \mapsto \tilde{x} := \frac{1}{2} \begin{pmatrix} x_1 & -x_0 - x_2 \\ x_0 - x_2 & -x_1 \end{pmatrix},$$

and

$$\sigma_0 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that, for $X \in \text{dS}_{\text{mat}}^2$, the vector $y = (-2 \text{Tr}(X\sigma_0), 2 \text{Tr}(X\sigma_2), -2 \text{Tr}(X\sigma_1)) \in \text{dS}^2$ satisfies $\tilde{y} = X$. The Minkowski quadratic form $x^2 = x_0^2 - x_1^2 - x_2^2$ corresponds to the determinant by $x^2 = 4 \det \tilde{x}$, so that $x \in \text{dS}^2$ if and only if $\det \tilde{x} = -\frac{1}{4}$.

We write $\Lambda : \text{SL}_2(\mathbb{R}) \rightarrow \mathcal{L}_+^\uparrow$ for the quotient map defined by the relation:

$$(\Lambda(g)x)^\sim = g\tilde{x}g^{-1} \quad \text{for } x \in \text{dS}^2, g \in \text{SL}_2(\mathbb{R}). \tag{2.7}$$

Then it is easy to see that $\text{dS}_{\text{mat}}^2 = \{g\sigma_1g^{-1} : g \in \text{SL}_2(\mathbb{R})\}$. The one-parameter groups

$$\lambda_{x_i}(t) = \exp(\sigma_i t) \in \text{SL}_2(\mathbb{R}), i = 1, 2, \tag{2.8}$$

are the lifts of the boosts $\Lambda_{W_{x_i}}(t) \in \mathcal{L}_+^\uparrow$, and $r(\theta) = \exp(-\sigma_0\theta)$ is the one-parameter group lifting the one-parameter group

$$\Lambda(r(\theta)) = R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \tag{2.9}$$

of space rotations.

Theorem 2.8 ([24, Thm. 3.10]) *Suppose that \mathfrak{g} is a non-compact simple real Lie algebra and that $\mathfrak{a} \subseteq \mathfrak{g}$ is maximal ad-diagonalizable with restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \subseteq \mathfrak{a}^*$ of type X_n . We follow the conventions of the tables in [2] for the classification of irreducible root systems and the enumeration of the simple roots $\alpha_1, \dots, \alpha_n$. For each $j \in \{1, \dots, n\}$, we consider the uniquely determined element $h_j \in \mathfrak{a}$ satisfying $\alpha_k(h_j) = \delta_{jk}$. Then every Euler element in \mathfrak{g} is conjugate under inner automorphism to exactly one h_j . For every irreducible root system, the Euler elements among the h_j are the following:*

$$\begin{aligned} A_n &: h_1, \dots, h_n, & B_n &: h_1, & C_n &: h_n, & D_n &: h_1, h_{n-1}, h_n, \\ E_6 &: h_1, h_6, & E_7 &: h_7. \end{aligned} \tag{2.10}$$

For the root systems BC_n, E_8, F_4 and G_2 no Euler element exists (they have no 3-grading). The symmetric Euler elements are

$$\begin{aligned} A_{2n-1} &: h_n, & B_n &: h_1, & C_n &: h_n, & D_n &: h_1, & D_{2n} &: h_{2n-1}, h_{2n}, \\ E_7 &: h_7. \end{aligned} \tag{2.11}$$

Remark 2.9 The preceding theorem shows that non-symmetric Euler elements exist for the root systems of type $A_n, n \geq 2, D_n, n \geq 4$, and E_6 .

Example 2.10 For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$, the subspace

$$\mathfrak{a} = \left\{ \text{diag}(x_1, \dots, x_n) : \sum_j x_j = 0 \right\}$$

of diagonal matrices is maximal abelian. In terms of the linear functionals $\varepsilon_j(\text{diag}(x)) = x_j$ on \mathfrak{a} , the root system is

$$A_{n-1} = \{\varepsilon_i - \varepsilon_j : i \neq j \in \{1, \dots, n\}\}.$$

The matrices

$$h_j = \frac{1}{n} \begin{pmatrix} (n-j)\mathbf{1}_j & 0 \\ 0 & -j\mathbf{1}_{n-j} \end{pmatrix}, \quad j = 1, \dots, n-1,$$

are Euler elements. They are symmetric if and only if $n = 2j$. A corresponding graded Lie group is $G = \text{PGL}_n(\mathbb{R})$ with $G^\uparrow = \text{PSL}_n(\mathbb{R})$.

2.3 Nets of standard subspaces

Hereafter we will consider orbits of Euler elements $\mathcal{W}_+ \subset \mathcal{G}_E(G)$ (Definition 2.6).

Definition 2.11 Let $G = G^\uparrow \rtimes \{e, \sigma\}$ be as above, $C \subseteq \mathfrak{g}$ be a closed convex $\text{Ad}^\varepsilon(G)$ -invariant cone in \mathfrak{g} , and fix a G^\uparrow -orbit $\mathcal{W}_+ \subseteq \mathcal{G}_E(G)$. Let (U, \mathcal{H}) be a unitary representation of G^\uparrow and

$$N: \mathcal{W}_+ \rightarrow \text{Stand}(\mathcal{H}) \tag{2.12}$$

be a map, also called a *net of standard subspaces*. In the following we denote this data as (\mathcal{W}_+, U, N) . We consider the following properties:

- (HK1) **Isotony:** $N(W_1) \subseteq N(W_2)$ for $W_1 \leq W_2$.²
- (HK2) **Covariance:** $N(gW) = U(g)N(W)$ for $g \in G^\uparrow, W \in \mathcal{W}_+$.
- (HK3) **Spectral condition:** $C \subseteq C_U := \{x \in \mathfrak{g} : -i\partial U(x) \geq 0\}$, where $U(\exp tx) = e^{t\partial U(x)}$ for $t \in \mathbb{R}$. We then say that U is *C-positive*.
- (HK4) **Locality:** If $W \in \mathcal{W}_+$ is such that $W' \in \mathcal{W}_+$, then $N(W') \subset N(W)'$.
- (HK5) **Bisognano–Wichmann (BW) property:** $U(\lambda_W(t)) = \Delta_{N(W)}^{-it/2\pi}$ for all $W \in \mathcal{W}_+, t \in \mathbb{R}$.
- (HK6) **Haag Duality:** $N(W') = N(W)'$ for all $W \in \mathcal{W}_+$ with $W' \in \mathcal{W}_+$.
- (HK7) **G-covariance:** There exists an (anti-)unitary extension of U from G^\uparrow to G such that

$$N(g.W) = U(g)N(W) \quad \text{for } g \in G, W \in \mathcal{W}_+. \tag{2.13}$$

- (HK8) **PCT property:** Suppose that (HK7) is satisfied and that U is the corresponding representation. Then $U(\sigma_W) = J_{N(W)}$ for $W \in \mathcal{W}_+$ with $W' \in \mathcal{W}_+$.

Theorem 2.12 (Brunetti–Guido–Longo (BGL) net generalization, [24]) *If (U, G) is an (anti-)unitary representation, then we obtain a G -equivariant map*

² Here we refer to the order structure on \mathcal{W}_+ introduced in Definition 2.3(e).

$N_U : \mathcal{G} \rightarrow \text{Stand}(\mathcal{H})$ determined for $W = (k_W, \sigma_W)$ by

$$J_{N_U(W)} = U(\sigma_W) \quad \text{and} \quad \Delta_{N_U(W)}^{-it/2\pi} = U(\exp tk_W) \quad \text{for } t \in \mathbb{R}. \quad (2.14)$$

The BGL net associates to every wedge $W \in \mathcal{G}$ a standard subspace $N_U(W)$. We shall denote with (\mathcal{W}_+, N_U, U) the restriction of the BGL net to the G^\uparrow -orbit $\mathcal{W}_+ \subseteq \mathcal{G}_E(G)$.

Theorem 2.13 ([24, Thm. 4.12, Prop. 4.16]) *The restriction of the BGL net N_U associated to an (anti-)unitary C -positive representation U of $G = G^\uparrow \rtimes \{e, \sigma\}$ to a G^\uparrow -orbit $\mathcal{W}_+ \subseteq \mathcal{G}_E$ satisfies all the axioms (HK1)–(HK8).*

We are interested in models where the following property fails.

(MC) **Modular covariance:** $\Delta_{N(W_a)}^{-it} N(W_b) = N(\lambda_{W_a}(2\pi t).W_b)$ for $W_a, W_b \in \mathcal{W}_+$, $t \in \mathbb{R}$.

Modular covariance is an immediate consequence of the Bisognano–Wichmann property. Indeed, (HK2) and the BW property imply

$$\Delta_{N(W_a)}^{-it} N(W_b) = U(\lambda_{W_a}(2\pi t))N(W_b) = N(\lambda_{W_a}(2\pi t).W_b).$$

2.4 Symmetric and non-symmetric Euler elements

For the graded group $G := \text{PGL}_2(\mathbb{R})$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, the abstract wedge space \mathcal{W}_+ can be identified with the set $\mathcal{E}(\mathfrak{g})$ of Euler elements in $\mathfrak{sl}_2(\mathbb{R})$. Since $\ker(\text{Ad})$ is trivial, for any Euler element $h \in \mathfrak{g}$, we have

$$G_h = \{g \in G : \text{Ad}(g)h = h\} = G_{(h, \sigma_h)},$$

and

$$\mathcal{E}(\mathfrak{g}) \cong \text{Ad}(G^\uparrow)h \cong G^\uparrow / G_h^\uparrow. \quad (2.15)$$

Depending on the choice of the positive cone in \mathfrak{g} we have wedge spaces with different order structures:

Remark 2.14 The (ordered) symmetric space $\mathcal{E}(\mathfrak{sl}_2(\mathbb{R}))$ can be identified with the following spaces:

- For a non-trivial order (corresponding to $C \neq \{0\}$): The set of non-dense open intervals in \mathbb{S}^1 (wedge space of the conformal structure on \mathbb{S}^1) ([24, Rem. 2.9(c)]) and the set of wedge domains in two-dimensional Anti-de Sitter space AdS^2 ([31, §11]).
- For a trivial order (corresponding to $C = \{0\}$): The set of wedge domains in two-dimensional de Sitter space dS^2 (see Remark 2.7) ([28, Lemma 4.13] and [30]).

Let (U, \mathcal{H}) be an (anti-)unitary representation of G and $\mathbb{V} \subseteq \mathcal{H}$ be a standard subspace with modular objects $(\Delta_{\mathbb{V}}, J_{\mathbb{V}})$. Then there exists a well-defined G -equivariant map

$$\mathcal{E}(\mathfrak{g}) \rightarrow \text{Stand}(\mathcal{H}), \quad \text{Ad}(g)h \mapsto U(g)\mathbb{V}$$

if and only if G_h^\uparrow is contained in the stabilizer group $G_{\mathbb{V}}^\uparrow$ of \mathbb{V} :

$$\begin{aligned} G_{\mathbb{V}}^\uparrow &= \{g \in G^\uparrow : U(g)\mathbb{V} = \mathbb{V}\} \\ &= \{g \in G^\uparrow : U(g)J_{\mathbb{V}}U(g)^{-1} = J_{\mathbb{V}}, U(g)\Delta_{\mathbb{V}}U(g)^{-1} = \Delta_{\mathbb{V}}\}. \end{aligned} \quad (2.16)$$

For $h := \frac{1}{2} \text{diag}(1, -1) \in \mathcal{E}(\mathfrak{g})$, the stabilizer group in $\text{PSL}_2(\mathbb{R}) = \text{Ad}(G^\uparrow)$ is the adjoint image of

$$\text{SL}_2(\mathbb{R})_h = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^\times \right\} \cong \mathbb{R}^\times,$$

hence connected because $-1 \in Z(\text{SL}_2(\mathbb{R}))$. Therefore

$$G_h^\uparrow = \exp(\mathbb{R}h),$$

and thus $G_h^\uparrow \subseteq G_{\mathbb{V}}^\uparrow$ is equivalent to $U(\exp \mathbb{R}h)$ commuting with $J_{\mathbb{V}}$ and $\Delta_{\mathbb{V}}$.

Lemma 2.15 *Let \mathfrak{g} be a finite-dimensional Lie algebra and $h \in \mathcal{E}(\mathfrak{g})$ an Euler element. If the image of h in the semisimple quotient $\mathfrak{g}/\text{rad}(\mathfrak{g})$ is non-zero, then there exists a Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ containing h such that*

- (a) $\mathfrak{b} \cong \mathfrak{sl}_2(\mathbb{R})$ if and only if h is symmetric, and
- (b) $\mathfrak{b} \cong \mathfrak{gl}_2(\mathbb{R})$ if h is not symmetric.
- (c) If h is symmetric, then $\text{Inn}_{\mathfrak{g}}(\mathfrak{b}) \cong \text{PSL}_2(\mathbb{R})$.
- (d) If h is not symmetric and \mathfrak{g} is simple, then $\text{Inn}_{\mathfrak{g}}([\mathfrak{b}, \mathfrak{b}]) \cong \text{SL}_2(\mathbb{R})$.

Proof (a) As all Euler elements in $\mathfrak{sl}_2(\mathbb{R})$ are symmetric ($\text{Inn}(\mathfrak{sl}_2(\mathbb{R}))$ acts transitively on $\mathcal{E}(\mathfrak{sl}_2(\mathbb{R}))$), this follows from [24, Thm. 3.13].

- (b) Suppose that h is not symmetric and pick a maximal abelian hyperbolic subspace $\mathfrak{a} \subseteq \mathfrak{g}$ containing h . With [16, Prop. I.2] we find an \mathfrak{a} -invariant Levi complement $\mathfrak{s} \subseteq \mathfrak{g}$. Then $\mathfrak{a}_{\mathfrak{s}} := \mathfrak{a} \cap \mathfrak{s}$ is maximal hyperbolic in \mathfrak{s} and $\mathfrak{a} = \mathfrak{a}_{\mathfrak{s}} + \mathfrak{z}_{\mathfrak{a}}(\mathfrak{s})$. We pick a root $\alpha \in \Delta(\mathfrak{s}, \mathfrak{a})$ with $\alpha(h) = 1$ and root vectors $x_\alpha \in \mathfrak{s}_\alpha$ and $y_\alpha \in \mathfrak{s}_{-\alpha}$ with $h_\alpha := [x_\alpha, y_\alpha] \neq 0$. We stress that $x_\alpha \in \mathfrak{s}_1(h)$. We use that

$$[x_\alpha, y_\alpha] = \kappa(x_\alpha, y_\alpha)a_\alpha,$$

where $a_\alpha \in \mathfrak{a}$ is the unique element with $\alpha(a) = \kappa(a_\alpha, a)$ for all $a \in \mathfrak{a}$, and that the Cartan–Killing form κ induces a dual pairing $\mathfrak{s}_\alpha \times \mathfrak{s}_{-\alpha} \rightarrow \mathbb{R}$. Then

$$\mathfrak{b}_\alpha := \mathbb{R}x_\alpha + \mathbb{R}y_\alpha + \mathbb{R}h_\alpha \cong \mathfrak{sl}_2(\mathbb{R})$$

and $[h, \mathfrak{b}_\alpha] \subseteq \mathfrak{b}_\alpha$. Hence $\mathfrak{b} := \mathbb{R}h + \mathfrak{b}_\alpha$ is a Lie subalgebra of \mathfrak{g} . As h is not symmetric, $h \notin \mathfrak{b}_\alpha$, and therefore $\mathfrak{b} \cong \mathfrak{gl}_2(\mathbb{R})$.

- (c) If h is symmetric, then $\mathfrak{b} = [\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{sl}_2(\mathbb{R})$ by (a) and the fact that \mathfrak{b} contains an Euler element of \mathfrak{g} implies that all simple \mathfrak{b} -submodules of \mathfrak{g} are either trivial or isomorphic to the adjoint representation of $\mathfrak{sl}_2(\mathbb{R})$ (consider eigenspaces of $\text{ad } h$). This implies that $\text{Inn}_{\mathfrak{g}}(\mathfrak{b}) \cong \text{PSL}_2(\mathbb{R})$.
- (d) Suppose that \mathfrak{g} is simple. If h is not symmetric, then the Weyl group reflection s_α corresponding to the root α from above satisfies

$$s_\alpha(h) = h - \alpha(h)\alpha^\vee = h - \alpha^\vee.$$

As h is not contained in $\mathbb{R}\alpha^\vee \subseteq \mathfrak{b}_\alpha$, we have $s_\alpha(h) \notin \mathbb{R}h$.

The simplicity of \mathfrak{g} ensures that the root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ is irreducible and 3-graded by $h \in \mathfrak{a}$. Therefore

$$\Delta_0 := \{\alpha \in \Delta : \alpha(h) = 0\}$$

spans a hyperplane in \mathfrak{a}^* , which coincides with h^\perp , and thus $\mathbb{R}h = \Delta_0^\perp$ by duality. Since $s_\alpha(h)$ is not contained in $\mathbb{R}h$, there exists a $\beta \in \Delta_0$ with $\beta(s_\alpha(h)) \neq 0$. Now $\beta(h) = 0$ implies

$$0 \neq \beta(s_\alpha(h)) = -\beta(\alpha^\vee).$$

As $s_\alpha(h)$ is an Euler element, we obtain $|\beta(\alpha^\vee)| = 1$. Therefore the central element $e^{\pi i \text{ad } \alpha^\vee}$ of $\text{Inn}_{\mathfrak{g}}(\mathfrak{b}_\alpha)$ acts non-trivially, and this implies that $\text{Inn}_{\mathfrak{g}}(\mathfrak{b}_\alpha) \cong \text{SL}_2(\mathbb{R})$ because it is a linear group with non-trivial center ([15, Ex. 9.5.18]).

Remark 2.16 Note that the $\mathfrak{sl}_2(\mathbb{R})$ -subalgebra \mathfrak{b}_α generated by $x_\alpha, y_\alpha, h_\alpha$ does not centralize h . We actually have

$$h = h_c + \frac{1}{2}\alpha^\vee \quad \text{with} \quad h_c \neq 0 \quad \text{and} \quad [h_c, \mathfrak{b}_\alpha] = \{0\}.$$

3 One-particle nets which are not modular covariant

Based on the preceding discussion, we describe in Sect. 3.1 a general principle that can be used to construct one-particle nets that are not modular covariant. This is then applied to obtain such nets on two-dimensional de Sitter space dS^2 and three-dimensional Minkowski space.

3.1 A general construction principle

We describe a construction principle for non-modular covariant nets of standard subspaces. Let $G = G^\uparrow \rtimes \{\mathbf{1}, \tau\}$ be a graded Lie group and (U, \mathcal{H}) an (anti-)unitary representation of G . We consider the following situation:

- a graded subgroup $H \subseteq G$,
- $W_1 = (h_1, \tau_1) \in \mathcal{G}_E(H)$ and an Euler couple $W_2 = (h_2, \tau_2) \in \mathcal{G}_E(G)$, so that $\text{Ad}(\tau_2) = e^{\pi i \text{ad} h_2}$.
- the stabilizer $H_{W_1}^\uparrow$ of W_1 in H^\uparrow fixes W_2 . As $\exp(\mathbb{R}h_1) \subseteq H_{W_1}^\uparrow$, this implies $[h_1, h_2] = 0$ (Remark 2.5).

Then the BGL construction provides a standard subspace $N_2 = N(h_2, \tau_2, U)$ with

$$J_{N_2} = U(\tau_2) \quad \text{and} \quad \Delta_{N_2} = e^{2\pi i \cdot \partial U(h_2)}.$$

As the BGL net $\mathcal{G}(G) \rightarrow \text{Stand}(\mathcal{H})$ is G -equivariant, maps W_2 to N_2 , and H^\uparrow fixes W_2 , we obtain an H^\uparrow -equivariant map

$$N: \mathcal{G}(H) \supseteq \mathcal{W}_+ := \mathcal{W}_+(H, h_1, \tau_1) := H^\uparrow \cdot W_1 \rightarrow \text{Stand}(\mathcal{H}), \quad g \cdot W_1 \mapsto U(g)N_2$$

which is uniquely determined by

$$N(W_1) = N_2. \tag{3.1}$$

Lemma 3.1 *The net N on $\mathcal{W}_+(H, h_1, \tau_1)$ satisfies modular covariance if and only if, for all $g \in H^\uparrow, t \in \mathbb{R}$, the operator*

$$U(g)U(\exp t(h_1 - h_2))U(g)^{-1}$$

fixes the standard subspace N_2 , i.e.,

$$g \exp(t(h_1 - h_2))g^{-1} \in G_{N_2} \quad \text{for} \quad g \in H^\uparrow, t \in \mathbb{R}. \tag{3.2}$$

Proof The net N satisfies modular covariance if and only if

$$N(\lambda_{g_1 W_1}(t)g_2 W_1) \stackrel{!}{=} \Delta_{N(g_1 W_1)}^{-it/2\pi} N(g_2 W_1) \quad \text{for} \quad g_1, g_2 \in H^\uparrow, t \in \mathbb{R}. \tag{3.3}$$

By covariance of N , the left hand side equals

$$\begin{aligned} U(\lambda_{g_1 W_1}(t))N(g_2 W_1) &= U(g_1 \exp(th_1)g_1^{-1})U(g_2)N(W_1) \\ &= U(g_1)U(\exp(th_1))U(g_1^{-1}g_2)N_2, \end{aligned}$$

and the right hand side is

$$\begin{aligned} \Delta_{N(g_1 W_1)}^{-it/2\pi} U(g_2)N_2 &= \Delta_{U(g_1)N_2}^{-it/2\pi} U(g_2)N_2 = U(g_1)\Delta_{N_2}^{-it/2\pi} U(g_1)^{-1}U(g_2)N_2 \\ &= U(g_1) \exp(th_2)U(g_1^{-1}g_2)N_2. \end{aligned}$$

Note that $[h_1, h_2] = 0$ implies that $U(\exp th_1)U(\exp -th_2) = U(\exp t(h_1 - h_2))$. So (3.3) means that

$$\begin{aligned} U(g_2^{-1}g_1)U(\exp -th_2)U(\exp th_1)U(g_1^{-1}g_2) \\ = U(g_2^{-1}g_1)U(\exp t(h_1 - h_2))U(g_1^{-1}g_2) \end{aligned}$$

fixes N_2 for $g_1, g_2 \in H^\uparrow, t \in \mathbb{R}$. This is (3.2).

Remark 3.2 (a) Condition (3.2) is not easy to evaluate, but one can easily formulate sufficient conditions for it to be satisfied. As $gG_{\mathbb{N}_2}g^{-1} = G_{U(g)\mathbb{N}_2}$ by covariance, it is equivalent to

$$\exp(t(h_1 - h_2)) \in G_{g.\mathbb{N}_2} \quad \text{for all } g \in H^\uparrow, t \in \mathbb{R} \tag{3.4}$$

or

$$U(\exp(\mathbb{R}(h_1 - h_2))) \subseteq \{U \in U(\mathcal{H}) : (\forall H \in \mathbb{N}(\mathcal{W}_+)) UH = H\}. \tag{3.5}$$

(b) If $\ker(U)$ is discrete, then the representation of the Lie algebra \mathfrak{g} is faithful, so that any element in the Lie algebra $\mathfrak{g}_{\mathbb{N}_2}$ of the stabilizer group $G_{\mathbb{N}_2}$ commutes with h_2 , hence is contained in $\mathfrak{g}_{W_2} = \ker(\text{ad } h_2)$ (cf. Remark 2.5). We thus have

$$\mathfrak{g}_{\mathbb{N}_2} = \mathfrak{g}_{W_2} = \ker(\text{ad } h_2).$$

Therefore (3.2) is equivalent to

$$\text{Ad}(H^\uparrow)(h_1 - h_2) \subseteq \mathfrak{g}_{\mathbb{N}_2} = \ker(\text{ad } h_2), \tag{3.6}$$

which implies in particular, by derivation and by the closedness of the Lie subalgebra, that

$$[\mathfrak{h}, h_1 - h_2] \subseteq \ker(\text{ad } h_2). \tag{3.7}$$

We conclude that (3.6) is violated if

$$[\mathfrak{h}, h_1 - h_2] \not\subseteq \ker(\text{ad } h_2). \tag{3.8}$$

(c) If, in addition, $\text{ad}(h_2)|_{\mathfrak{h}} = -\text{ad}(h_1)$, then $[\mathfrak{h}, h_1 - h_2] = [\mathfrak{h}, h_1] \subseteq \mathfrak{h}$, so that (3.8) is equivalent to $[h_1, \mathfrak{h}] \subseteq \ker(\text{ad } h_1)$, which, by semisimplicity of $\text{ad } h_1$ on \mathfrak{h} , is equivalent to $h_1 \in \mathfrak{z}(\mathfrak{h})$. Therefore $h_1 \notin \mathfrak{z}(\mathfrak{h})$ implies (3.8).

(d) The condition under (c) is always satisfied if

$$\mathfrak{h} = \mathfrak{g} \quad \text{and} \quad h_1 = -h_2 \in \mathcal{E}(\mathfrak{g}).$$

Example 3.3 An easy example satisfying the construction is the following. Let U be an (anti-)unitary representation of the proper Lorentz group $\mathcal{L}_+ = \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\downarrow$ on a Hilbert space \mathcal{H}_U and let \mathbb{N}_U be the corresponding BGL net of standard subspaces. Fix a wedge $W \in \mathcal{G}_E(\mathcal{L}_+)$ and consider the net of standard subspaces obtained by $\mathbb{N}'_U(W) := \mathbb{N}_U(W)'$. Then \mathbb{N}'_U is a local net in the sense that it satisfies (HK4). But it does not satisfy modular covariance, since $\Delta_{\mathbb{N}'_U(W)}^{it} = U(\Lambda_W(2\pi t))$ acts with the inverted boost flow with respect to the Lorentz action on Minkowski spacetime. In particular it is easy to see that, with the previous notation $h_1 = -h_2$.

3.2 Non-modular covariant nets

3.2.1 On de Sitter space dS^2

Here we apply the previous prescription in order to construct nets of standard subspaces **without** the modular covariance property. We can specify the assumption (HK2) for the (double covering) of the Lorentz group as follows. We say that a net of standard subspaces on the two-dimensional de Sitter spacetime $W^{dS} \mapsto N(W^{dS})$ is **Lorentz covariant** with respect to a unitary representation U of $SL_2(\mathbb{R}) \simeq \tilde{\mathcal{L}}_+^\uparrow$ if

$$U(g)N(W^{dS}) = N(\Lambda(g)W^{dS}) \quad \text{for } g \in SL_2(\mathbb{R}), \tag{3.9}$$

where $\Lambda : SL_2(\mathbb{R}) \rightarrow SO_{1,2}(\mathbb{R})^\uparrow$ is the covering homomorphism, see (2.7). We shall say that a net of standard subspaces on the three-dimensional Minkowski spacetime $\mathbb{R}^{1+2} \supset W^{\mathbb{R}^{1+2}} \mapsto N(W^{\mathbb{R}^{1+2}})$ is **Poincaré covariant** with respect to a unitary representation U of the (double covering of the) Poincaré group $\tilde{\mathcal{P}}_+^\uparrow \simeq \mathbb{R}^{1+2} \rtimes_\Lambda SL_2(\mathbb{R})$ if

$$U(g)N(W^{dS}) = N(\Lambda(s)W^{dS} + a) \quad \text{for } g = (a, s) \in \mathbb{R}^{1+2} \rtimes SL_2(\mathbb{R}). \tag{3.10}$$

Let $G^\uparrow = \text{Inn}(\mathfrak{g})$ with \mathfrak{g} a simple non-compact Lie algebra as in Theorem 2.8 and let G be the graded extension $G^\uparrow \rtimes \{1, \sigma\}$, where σ is an Euler involution. We consider an (anti-)unitary representation U of G on a Hilbert space \mathcal{H} . We pick a **non-symmetric** Euler element $h \in \mathfrak{g}$ and the associated wedge $W_h = (h, \sigma_h) \in \mathcal{G}_E(G)$. By Lemma 2.15 there exists a \mathfrak{gl}_2 -subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ containing h such that $\mathfrak{h} := [\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{sl}_2(\mathbb{R})$ satisfies $[\mathfrak{h}, h] \neq 0$. We consider $H^\uparrow := \text{Inn}_{\mathfrak{g}}(\mathfrak{h}) \cong SL_2(\mathbb{R})$ and $H = H^\uparrow \rtimes \{1, \sigma_h\}$ (Lemma 2.15(d)). The Euler element $h \in \mathfrak{b}$ has a central component h_c , so that

$$h = h_c - h_1 \quad \text{with } h_c \in \mathfrak{z}(\mathfrak{b}) \quad \text{and } h_1 \in \mathcal{E}(\mathfrak{h}). \tag{3.11}$$

Choosing the isomorphisms $\mathfrak{h} \rightarrow \mathfrak{sl}_2(\mathbb{R})$ suitably, we may henceforth assume that

$$h_1 = \sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.12}$$

Let V be the restriction $U|_H$. The group $H^\uparrow \cong SL_2(\mathbb{R})$ is the double covering of $PSL_2(\mathbb{R}) \cong \mathcal{L}_+^\uparrow$. So it acts on de Sitter space dS^2 through the covering map $\Lambda : SL_2(\mathbb{R}) \ni g \mapsto \Lambda(g) \in PSL_2(\mathbb{R})$ from Remark 2.7.

We thus obtain a net H^{dS} on de Sitter spacetime.

Theorem 3.4 *There exists a net of standard subspaces on de Sitter spacetime dS^2*

$$\mathcal{W}^{dS} \ni W^{dS} \mapsto H^{dS}(W^{dS}) \subset \mathcal{H}_U$$

such that

$$H^{dS}(\Lambda(g)W_{x_1}^{dS}) := V(g)N_U(W_h) \quad \text{for } g \in H^\uparrow \cong \text{SL}_2(\mathbb{R}), \quad (3.13)$$

where N_U is the BGL net defined by U , satisfying the Lorentz covariance property (3.9) w.r.t. the representation V of H^\uparrow and the action defined by (2.7). The net H^{dS} is Lorentz covariant and does not satisfy modular covariance.

We remark that since all the wedge inclusions are trivial and since the positive cone C is $\{0\}$ for the Lorentz group, (HK1) and (HK3) are also satisfied by H^{dS} for trivial reasons.

Proof For H^{dS} to be well-defined, we have to argue that the stabilizer of $W_{x_1}^{dS}$ in $\text{SL}_2(\mathbb{R})$, which is the centralizer $H_{h_1}^\uparrow = \{\pm \mathbf{1}\} \exp(\mathbb{R}h_1)$, fixes the standard subspace $N_U(W_h)$. This follows from the fact that $h = h_c - h_1$, where h_c is fixed by H^\uparrow . Hence h is fixed by $H_{h_1}^\uparrow$, and therefore $H_{h_1}^\uparrow$ leaves $N_U(W_h)$ invariant by covariance of the BGL net N_U under $V = U|_H$.

The covariance of the net H^{dS} follows immediately from its definition in (3.13). To see that it is not modular covariant, recall that the Euler element h is not symmetric, so that it is not contained in any \mathfrak{sl}_2 -subalgebra ([24, Thm. 3.13]), and the same holds for all elements in $\text{Ad}(H^\uparrow)h$.

We put $h_2 := h$ and, as above, we write $h = h_c - h_1$, where $h_c \in \mathfrak{z}(\mathfrak{b})$ and $h_1 \in \mathfrak{h} = [\mathfrak{b}, \mathfrak{b}]$ is an Euler element in \mathfrak{h} . Then $h_2 - h_1 = h_c - 2h_1$ satisfies

$$[h_2 - h_1, \mathfrak{h}] = [h_1, \mathfrak{h}] \not\subseteq \ker(\text{ad } h_1),$$

so that Remark 3.2(b),(c) imply that the net is not modular covariant. More concretely,

$$\text{Ad}(H^\uparrow)(h_2 - h_1) = h_c - 2 \text{Ad}(H^\uparrow)h_1 \in h_c - 2\mathcal{E}(\mathfrak{h})$$

is not contained in the centralizer of $h = h_2$ (cf. (3.6)). So the action of the modular group $U(\exp(-2\pi th)) = \Delta_{H^{dS}(W_{x_1}^{dS})}^{it}$ on the net differs from that of the one-parameter group $U(\exp(-2\pi th_1))$ of the corresponding boost. \square

3.2.2 An explicit example on de Sitter space dS^2

In this section we will present an explicit construction in $\text{PSL}_n(\mathbb{R})$ for the previous construction.

With the previous notation we specify the case $\text{PGL}_n(\mathbb{R})$ as an example. Let $G^\uparrow := \text{PSL}_n(\mathbb{R})$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$, so that $G^\uparrow \cong \text{Inn}(\mathfrak{g})$. The conjugacy classes of Euler elements in \mathfrak{g} are represented by the diagonal matrices $h_j, j = 1, \dots, n - 1$, from Example 2.10. As the dimensions of the eigenspaces show, no two of these elements are conjugate. This follows from the classification for the root system A_{n-1} (Theorem 2.8). The Euler element h_j is symmetric if n is even and $j = \frac{n}{2}$.

More generally, there exist non-symmetric Euler elements which are diagonal matrices of the form

$$h = \text{diag}(a_1, \dots, a_n) \quad \text{with} \quad \sum_{i=1}^n a_i = 0 \quad \text{and} \quad a_i - a_j \in \{-1, 0, 1\} \quad \text{for} \quad i \neq j. \tag{3.14}$$

We fix such a non-symmetric Euler element h . After a permutation of the entries, we may assume that $a_2 - a_1 = 1$, so that h does not commute with the subalgebra $\mathfrak{h} \cong \mathfrak{sl}_2(\mathbb{R})$, generated by the Euler elements

$$k_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mathbf{0}_{n-2} \end{pmatrix} \quad \text{and} \quad k_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mathbf{0}_{n-2} \end{pmatrix}.$$

Then

$$h = h_c - k_1 \quad \text{with} \quad 0 \neq h_c = \text{diag}(a_1 + \frac{1}{2}, a_1 + \frac{1}{2}, a_3, \dots, a_n). \tag{3.15}$$

Let $\sigma_h \in \text{Aut}(\mathfrak{g})$ be the involution defined by h and $G := G^\uparrow\{\mathbf{1}, \sigma_h\} \subseteq \text{Aut}(\mathfrak{g})$. Let $H \subset G$ be the graded subgroup generated by the two one-parameter groups $\exp(\mathbb{R}k_{1,2})$ and σ_h . The Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ generated by h, k_1 and k_2 is easily seen to be isomorphic to $\mathfrak{gl}_2(\mathbb{R})$ with commutator algebra $\mathfrak{h} = [\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{sl}_2(\mathbb{R})$. By Lemma 2.15(d), we have $H^\uparrow \cong \text{SL}_2(\mathbb{R})$, realized via

$$\text{SL}_2(\mathbb{R}) \ni g \mapsto g_{\text{PSL}_n} := \begin{bmatrix} g & 0 \\ 0 & \mathbf{1}_{n-2} \end{bmatrix} \in \text{PSL}_n(\mathbb{R}).$$

Observe that $\sigma_h|_{\mathfrak{h}} = \sigma_{k_1}$. Let $W_h = (h, \sigma_h) \in \mathcal{G}_E(G)$ be the Euler wedge associated to h and observe that its stabilizer in H^\uparrow coincides with the stabilizer of $(k_1, \sigma_h) \in \mathcal{G}(H)$.

Let (U, \mathcal{H}_U) be an (anti-)unitary representation of $G = \text{PGL}_n(\mathbb{R})$, so that U is unitary on $G^\uparrow = \text{PSL}_n(\mathbb{R})$ (see [28, Lemma 2.10] for existence). In order to fit with Remark 2.7 (note that k_1 corresponds to σ_2), we construct the net with base point $W_{x_2}^{\text{dS}}$. We consider the net \mathcal{N}^{dS} of standard subspaces on de Sitter spacetime defined as in (3.13) by

$$\mathcal{W}_+ \ni W^{\text{dS}} \mapsto \mathcal{N}^{\text{dS}}(W^{\text{dS}}) \subset \mathcal{H}_U \quad \text{defined by} \quad \mathcal{N}^{\text{dS}}(\Lambda(g)W_{x_2}^{\text{dS}}) := V(g)N_U(W_h). \tag{3.16}$$

By Theorem 3.4, this net is Lorentz covariant but not modular covariant. To match notation, take $h_1 := k_1$ and $h_2 := h$.

Remark 3.5 Consider the previous identification of $\text{SL}_2(\mathbb{R}) \cong H^\uparrow \subset \text{PSL}_n(\mathbb{R})$, where $h = h_c - k_1$, and k_1 and k_2 are the Euler elements in $\mathfrak{h} \cong \mathfrak{sl}_2(\mathbb{R})$ associated to the wedge domains $W_{x_2}^{\text{dS}}$ and $W_{x_1}^{\text{dS}}$ in dS^2 , respectively. Then

$$k_1 = r(\pi/2)k_2r(\pi/2)^{-1}, \quad \text{where} \quad r(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

By the BGL construction and (2.9),

$$U(e^{-2\pi it \cdot h}) = U(e^{-2\pi it \cdot (h_c - k_1)}) = \Delta_{N^{\text{dS}}(W_{x_2}^{\text{dS}})}^{it} \quad \text{and}$$

$$U(e^{-2\pi it \cdot (h_c - k_2)}) = \Delta_{N^{\text{dS}}(W_{x_1}^{\text{dS}})}^{it}.$$

As the two Euler elements $h = h_c - k_1$ and $h_c - k_2$ generate $\mathfrak{b} \cong \mathfrak{gl}_2(\mathbb{R})$, we also observe that modular covariance manifestly fails because these two one-parameter groups do not generate a representation of $\widetilde{\text{SL}}_2(\mathbb{R})$ (cf. [6, Thm. 1.7]).

3.2.3 Counterexamples on Minkowski spacetime

Let N be a net of standard subspaces:

$$\mathcal{W}_+^{\text{dS}} \ni W^{\text{dS}} \mapsto N(W^{\text{dS}}) \subset \mathcal{H}$$

which is covariant for a unitary representation V of the double covering $\text{SL}_2(\mathbb{R}) \simeq \widetilde{\mathcal{L}}_+^\uparrow$ of the Lorentz group on the Hilbert space \mathcal{H} . We assume that the couple (V, N) satisfies (3.9). We further assume that the net N does not satisfy modular covariance, i.e., that there exist $g \in \text{SL}_2(\mathbb{R})$ and $t \in \mathbb{R}$ such that

$$\Delta_{N(W_{x_1}^{\text{dS}})}^{-it} N(g W_{x_1}^{\text{dS}}) \neq N(\Lambda_{W_{x_1}^{\text{dS}}}(2\pi t) g W_{x_1}^{\text{dS}}) = N(\Lambda(\exp(th_1)) g W_{x_1}^{\text{dS}}). \quad (3.17)$$

(Note that the two wedges $W_{x_1}^{\text{dS}}$ and $W_{x_2}^{\text{dS}}$ are Lorentz conjugate, so that the base point in the wedge space does not matter.) We constructed such nets in Sects. 3.2.1 and 3.2.2.

The proper Poincaré group

$$\mathcal{P}_+ = \mathcal{P}_+^\uparrow \cup \mathcal{P}_+^\downarrow \quad \text{with} \quad \mathcal{P}_+^{\uparrow\downarrow} = \mathbb{R}^{1+2} \rtimes \mathcal{L}_+^{\uparrow\downarrow}$$

is the inhomogeneous Lorentz group acting on the Minkowski spacetime: the real space \mathbb{R}^{1+2} endowed with the Lorentzian metric with signature $(1, -1, -1)$. We write

$$\mathcal{W}_+^{\mathbb{R}^{1+2}} = \mathcal{P}_+^\uparrow \cdot W_{x_1}$$

for the set of wedges of $(1 + 2)$ -dimensional Minkowski spacetime. Note that $\text{dS}^2 \subset \mathbb{R}^{1+2}$ and that, for every $W^{\text{dS}} \subseteq \text{dS}^2$, there exists a unique $W^{\mathbb{R}^{1+2}} \in \mathcal{L}_+^\uparrow \cdot W_{x_1}^{\mathbb{R}^{1+2}} \subseteq \mathcal{W}_+^{\mathbb{R}^{1+2}}$ such that $W^{\text{dS}} = W^{\mathbb{R}^{1+2}} \cap \text{dS}^2$ (cf. Remark 2.7). We may thus identify $\mathcal{W}_+^{\text{dS}}$ with a subset of $\mathcal{W}_+^{\mathbb{R}^{1+2}}$. The Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^{1+2} \rtimes \mathcal{L}_+^\uparrow$ act on $\mathcal{W}_+^{\text{dS}}$ through the projection onto \mathcal{L}_+^\uparrow , so that the translation group acts trivially.

With this identification, N extends to a net \widehat{N} on Minkowski wedges by

$$\mathcal{W}_+^{\mathbb{R}^{1+2}} \ni W^{\mathbb{R}^{1+2}} \mapsto \widehat{N}(W^{\mathbb{R}^{1+2}}) \subset \mathcal{H}$$

with

$$\widehat{\mathbf{N}}(\Lambda^{\mathbb{R}^{1+2}}(a, g)W_{x_1}^{\mathbb{R}^{1+2}}) := V(g)\mathbf{N}(W_{x_1}^{\text{dS}}), \quad (a, g) \in \widetilde{\mathcal{P}}_+^\uparrow$$

and $\Lambda^{\mathbb{R}^{1+2}} : \widetilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^{1+2} \rtimes \widetilde{\mathcal{L}}_+^\uparrow \ni (a, g) \mapsto (a, \Lambda(g)) \in \mathcal{P}_+^\uparrow$ is the quotient map. As above, this construction requires that the stabilizer subgroup of the standard right wedge $W_R := W_{x_1}^{\mathbb{R}^{1+2}}$ in \mathcal{P}_+^\uparrow preserves $W_{x_1}^{\text{dS}}$ ([28, Lemma 4.13]).

Now let U_0 be an (anti-)unitary positive energy³ representation of \mathcal{P}_+ on the Hilbert space \mathcal{K} . We write \mathbf{N}_{U_0} for the BGL-net of standard subspaces associated to U_0 (Theorem 2.13). We define a unitary positive energy representation of $\widetilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^{1+2} \rtimes \text{SL}_2(\mathbb{R})$, the double covering of \mathcal{P}_+^\uparrow , on the tensor product Hilbert space $\mathcal{H} \otimes \mathcal{K}$ by

$$U : \widetilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^{1+2} \rtimes \text{SL}_2(\mathbb{R}) \ni (a, g) \mapsto V(g) \otimes U_0(a, g) \in \text{U}(\mathcal{H} \otimes \mathcal{K}). \quad (3.18)$$

The tensor product subspaces $\widehat{\mathbf{N}}(W^{\mathbb{R}^{1+2}}) \otimes \mathbf{N}_{U_0}(W^{\mathbb{R}^{1+2}}) \subset \mathcal{H} \otimes \mathcal{K}$ are standard by [20, Prop. 2.6]. We obtain a (non-local) net of standard subspaces on Minkowski spacetime

$$\mathbf{H} : \mathcal{W}_+^{\mathbb{R}^{1+2}} \ni W^{\mathbb{R}^{1+2}} \mapsto \mathbf{H}(W^{\mathbb{R}^{1+2}}) \subset \mathcal{H} \otimes \mathcal{K},$$

where $\mathbf{H}(W_R) = \widehat{\mathbf{N}}(W_R) \otimes \mathbf{N}_{U_0}(W_R)$ and, for the standard right wedge $W_R := W_{x_1}^{\mathbb{R}^{1+2}}$:

$$\begin{aligned} \mathbf{H}(W^{\mathbb{R}^{1+2}}) &= \mathbf{H}(\Lambda^{\mathbb{R}^{1+2}}((a, g)) \cdot W_R) = U((a, g))\mathbf{H}(W_R) \\ &= (V(g) \otimes U_0(a, g))(\widehat{\mathbf{N}}(W_R) \otimes \mathbf{N}_{U_0}(W_R)) \\ &\text{for } (a, g) \in \widetilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^{1+2} \rtimes \text{SL}_2(\mathbb{R}). \end{aligned}$$

Poincaré covariance is satisfied by construction, but modular covariance is broken. Indeed, by [20, Sect. 2],

$$\Delta_{\widehat{\mathbf{N}}(W) \otimes \mathbf{N}_{U_0}(W)}^{it} = \Delta_{\widehat{\mathbf{N}}(W)}^{it} \otimes \Delta_{\mathbf{N}_{U_0}(W)}^{it}$$

and since $\widehat{\mathbf{N}}$ does not satisfy modular covariance, there exists a $g \in \widetilde{\mathcal{L}}_+^\uparrow \subseteq \widetilde{\mathcal{P}}_+^\uparrow$ such that we obtain:

$$\begin{aligned} \Delta_{\mathbf{H}(W_R)}^{-it} \mathbf{H}(gW_R) &= (\Delta_{\widehat{\mathbf{N}}(W_R)}^{-it} \otimes \Delta_{\mathbf{N}_{U_0}(W_R)}^{-it})(\widehat{\mathbf{N}}(gW_R) \otimes \mathbf{N}_{U_0}(gW_R)) \\ &= (\Delta_{\widehat{\mathbf{N}}(W_R)}^{-it} \widehat{\mathbf{N}}(gW_R) \otimes \Delta_{\mathbf{N}_{U_0}(W_R)}^{-it} \mathbf{N}_{U_0}(gW_R)) \\ &= (\Delta_{\widehat{\mathbf{N}}(W_R)}^{-it} \widehat{\mathbf{N}}(gW_R)) \otimes U_0(\Lambda_{W_R}(2\pi t))\mathbf{N}_{U_0}(gW_R) \\ &= (\Delta_{\widehat{\mathbf{N}}(W_R)}^{-it} \widehat{\mathbf{N}}(gW_R)) \otimes \mathbf{N}_{U_0}(\Lambda_{W_R}(2\pi t)gW_R) \end{aligned}$$

³ Positive energy means that the joint spectrum of the translation generator lies inside the forward light cone $C = \{x \in \mathbb{R}^{1+2} : x_0^2 - x_1^2 - x_2^2 \geq 0, x_0 \geq 0\}$

$$\neq \widehat{N}(\Lambda_{W_R}(2\pi t)gW_R) \otimes N_{U_0}(\Lambda_{W_R}(2\pi t)gW_R)$$

where the last inequality follows from (3.17).

4 Outlook: Disintegration, locality and higher dimensions

Comments on the representation theory. Let N_U be the BGL-net associated to an anti-unitary (positive energy) representation U of a \mathbb{Z}_2 -graded Lie group G whose Lie algebra \mathfrak{g} contains Euler elements, cf. [24]. The counterexamples to the BW property described in [20, 23] (see also [3, Thm. 3.1]) can be interpreted in our general setting in the following sense:

Proposition 4.1 *Let (U, \mathcal{H}) be an (anti-)unitary representation of the graded group G and*

$$\zeta : G \rightarrow U(G)' \cap U(\mathcal{H})$$

be a group homomorphism. Then

$$\widetilde{U}(g) := U(g)\zeta(g) = \zeta(g)U(g)$$

defines an (anti-)unitary representation of G on \mathcal{H} , and if $N := N_U : \mathcal{G}(G) \rightarrow \text{Stand}(\mathcal{H})$ is the BGL net corresponding to U , then N is \widetilde{U} -covariant and satisfies the modular covariance condition.

Proof The net N is modular covariant by construction and since each $\zeta(g)$ fixes all subspaces in the net by Lemma 2.2, the N is \widetilde{U} -covariant.

In the setting of Proposition 4.1, if U is an infinite multiple of an irreducible Poincaré representation, then \widetilde{U} disintegrates with infinitely many disjoint Poincaré representations, see e.g. [23, Sect. 5], [20, Sect. 7].

In the current paper, our family of counterexamples to the modular covariance property on de Sitter spacetime relies on the inclusion

$$H^\uparrow = \text{SL}_2(\mathbb{R}) \subset G := \text{GL}_2(\mathbb{R}) \quad \text{with} \quad \mathfrak{h} := \mathfrak{sl}_2(\mathbb{R}) \subset \mathfrak{g} = \mathfrak{gl}_2(\mathbb{R}),$$

where we consider a non-symmetric Euler element $h \in \mathfrak{gl}_2(\mathbb{R})$ and observe that $\mathfrak{g}^h \not\supseteq [\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}_2(\mathbb{R})$. Note that $\mathfrak{gl}_2(\mathbb{R}) = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{z}(\mathfrak{g})$ where $\mathfrak{z}(\mathfrak{g}) = \mathbb{R} \cdot \mathbf{1}$.

Let (V, \mathbb{H}) be a net of standard subspaces satisfying (3.9) on the two-dimensional de Sitter spacetime, where V is a representation of the (covering of the) Lorentz group acting covariantly on \mathbb{H} . Assume that \mathbb{H} has been constructed as in Sect. 3.2.1, with respect the group $G = \text{GL}_2(\mathbb{R})$. Then h is a non-symmetric Euler element in $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$ as above and $H^\uparrow = \langle \exp \mathfrak{h} \rangle = \text{SL}_2(\mathbb{R}) \subset \text{GL}_2(\mathbb{R})$.

An irreducible unitary representation of $G^\uparrow = \mathbb{R}_+^\times \text{SL}_2(\mathbb{R})$ is of the form $U_p(g, a) = V(g)e^{ipa}$ where V is an irreducible representation of $\text{SL}_2(\mathbb{R})$ and

$\mathbb{R} \ni p \neq 0$. The representation V extends to an (anti-)unitary irreducible representation of $H = H^\uparrow \rtimes \mathbb{Z}_2$ on the same space (see e.g. [24, Thm. 2.24]). An (anti-)unitary representation of $G = G^\uparrow \rtimes \mathbb{Z}_2$ which is non-trivial on the center $\mathbf{1} \times \mathbb{R}_+^\times \subset G^\uparrow$ decomposes on G^\uparrow as $U := U_p \oplus U_{-p}$ because of the antilinearity of the action of the involution.

Let $V = U|_H$ and $H(W_{x_1}^{\text{dS}}) = N_U(W_h)$. So the net $H(gW_{x_1}^{\text{dS}}) = V(g)H(W_h)$ is defined by covariance but it does not satisfy modular covariance. We conclude that, on de Sitter spacetime, even if V is a two-fold direct sum of irreducible representations of $\text{SL}_2(\mathbb{R})$, it is possible that a Lorentz covariant net of standard subspaces (V, H) is not modular covariant. On the other hand, on Minkowski spacetime, due to the tensor product representation (3.18), the Poincaré representation U contains infinitely many inequivalent representations in the direct integral decomposition (cf. [23, Sect. 5], [20, Sect. 7]).

Counterexamples with bosonic representations. Lemma 2.15 claims that if \mathfrak{g} is simple then $\text{Inn}_{\mathfrak{g}}([\mathfrak{b}, \mathfrak{b}]) \simeq \text{SL}_2(\mathbb{R})$. Then, going back to the construction presented in Sect. 3.2.1, the covariant representation of the symmetry group acting on the two-dimensional de Sitter spacetime is actually a representation of the double covering of the Lorentz group that does not factor through the Lorentz group. In order to consider representations of the Lorentz group, one can get rid of the assumption on \mathfrak{g} to be simple. Note that Lemma 2.15(a)-(c) do not require \mathfrak{g} to be simple.

With the notation of Sect. 3.2.1, consider a (non simple) Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}\xi$, where $\mathfrak{z}(\mathfrak{g}) = \mathbb{R} \cdot \xi$ is the one-dimensional center of \mathfrak{g} . Then $h_2 = \xi - h_1 \in \mathfrak{g}$ is a non-symmetric Euler element of \mathfrak{g} and h_1 is an Euler element in $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{R})$. It is clear that $[\mathfrak{h}, h_1 - h_2] \not\subseteq \ker(h_2)$. Let $G^\uparrow = \text{PSL}_2(\mathbb{R}) \times \mathbb{R}$ and $H^\uparrow = \text{PSL}_2(\mathbb{R}) \times \{0\}$, then $\text{Lie}(G^\uparrow) = \mathfrak{g}$ and $\text{Lie}(H^\uparrow) = \mathfrak{h}$. Consider the \mathbb{Z}_2 -graded extension H of H^\uparrow obtained adding an Euler involution. Then $G = H \times \mathbb{R}$ is a \mathbb{Z}_2 -graded extension of G^\uparrow . Let U be an (anti-)unitary representation of G on a Hilbert space \mathcal{H}_U and N_U be the BGL-net with respect to the wedge set $\mathcal{W}_+(W_{h_2})$. One can replicate the construction of the non-modular covariant net presented in Sect. 3.2.1 since $H_{h_1} = \exp(\mathbb{R}h_1)$, and obtain a net of standard subspaces indexed by wedge regions on two-dimensional de Sitter spacetime

$$\mathcal{W}^{\text{dS}} \supset W^{\text{dS}} \longmapsto N(W^{\text{dS}}) \subset \mathcal{H}_U.$$

The net N is Lorentz covariant with respect to $V = U|_H$ as in (3.9), where V is already defined on \mathcal{L}_+^\uparrow and not only on the 2-fold covering.

Twisted-local non modular Lorentz covariant nets on de Sitter spacetime. The counterexamples to the modular covariance property we provided in this paper are not compatible with locality since the Euler element we start the construction with is not symmetric. Indeed, the subspaces corresponding to causal complementary wedges (cf. Definition 2.3(d)) are not local in the sense that, if $W_1 \subset W_2'$, then $H(W_1) \subset H(W_2)'$. Clearly, on de Sitter spacetime, since there are no wedge inclusions, the locality condition becomes $H(W^{\text{dS}'}) \subset H(W^{\text{dS}'})'$. On the other hand we can construct twisted-local nets that are not modular covariant as follows.

We refer to the notation contained in Sect. 3.2.1. Let (N, U) be a non-modular covariant net on de Sitter spacetime as constructed in Sect. 3.2.1, with $N(W_{x_1}) = N_U(W_h)$ where h is a non-symmetric Euler element in \mathfrak{g} . We can define a second Lorentz covariant net of standard subspaces on de Sitter spacetime by covariance under U , starting from

$$K(W'_{x_1}{}^{\text{dS}}) := N_U(W_{-h}) = N_U(W_h)' \subset \mathcal{H},$$

and putting

$$K(\Lambda(g)W_{x_1}{}^{\text{dS}'}) := U(g)N_U(W_h)'.$$

Note that $W_{x_1}{}^{\text{dS}'} = R(\pi)W_{x_1}{}^{\text{dS}}$, so $K(W_{x_1}{}^{\text{dS}}) = U(r(\pi))N(W_{x_1}{}^{\text{dS}})'$, and by covariance

$$N(W^{\text{dS}'}) = K(W^{\text{dS}})'. \tag{4.1}$$

The net is Lorentz covariant (3.9) since, for $g \in H^\uparrow = \text{SL}_2(\mathbb{R})$, $\Lambda(g)W^{\text{dS}} = W^{\text{dS}}$ is equivalent to $\Lambda(g)W^{\text{dS}'} = W^{\text{dS}'}$. With the notation of Sect. 3.1, we put $h_2 := h$ and $h_1 := h_{W_{x_1}}$, so that (3.8) holds:

$$[\mathfrak{h}, h_1 - h_2] \not\subseteq \ker(h_2).$$

In particular, for $h_{W'_h} = -h_2$ and $h_{W'_{x_1}} = -h_1$, we get again $[\mathfrak{h}, h_2 - h_1] \not\subseteq \ker(h_2)$, hence K is not modular covariant. The net

$$\mathcal{W}_+^{\text{dS}} \ni W^{\text{dS}} \mapsto \tilde{H}(W^{\text{dS}}) = N(W^{\text{dS}}) \oplus K(W^{\text{dS}}) \subset \mathcal{H} \oplus \mathcal{H}$$

is Lorentz covariant with respect to the representation

$$\text{SL}_2(\mathbb{R}) \ni g \mapsto \tilde{U}(g) = U(g) \oplus U(g).$$

Consider the operator

$$Z = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \in \tilde{U}(\text{SL}_2(\mathbb{R}))'.$$

The net \tilde{H} is **twisted Haag dual** (cf. [GL95, MN21]) with respect to the twist operator Z , in the sense that

$$\tilde{H}(W') = Z\tilde{H}(W)'. \tag{4.2}$$

Indeed, (4.2) follows from

$$\begin{aligned}
 \tilde{H}(W_{x_1}^{\text{dS}'}) &= \tilde{U}(r(\pi))\tilde{H}(W_{x_1}^{\text{dS}}) = U(r(\pi))N(W_{x_1}^{\text{dS}}) \oplus U(r(\pi))K(W_{x_1}^{\text{dS}}) \\
 &= U(r(\pi))N(W_{x_1}^{\text{dS}}) \oplus U(r(\pi))U(r(\pi))N(W_{x_1}^{\text{dS}})' \\
 &= N(W_{x_1}^{\text{dS}'}) \oplus U(r(2\pi))N(W_{x_1}^{\text{dS}})' \\
 &\stackrel{(4.1)}{=} K(W_{x_1}^{\text{dS}})' \oplus N(W_{x_1}^{\text{dS}})' \\
 &= Z(N(W_{x_1}^{\text{dS}})' \oplus K(W_{x_1}^{\text{dS}})') \\
 &= Z\tilde{H}(W_{x_1}^{\text{dS}})'.
 \end{aligned}$$

By covariance we obtain $\tilde{H}(W^{\text{dS}'}) = Z\tilde{H}(W^{\text{dS}})'$ for every wedge on de Sitter spacetime. An analogous example can be constructed on Minkowski spacetime.

Higher dimensional spacetimes. In this paper we provide counterexamples to modular covariance on two-dimensional de Sitter spacetime and three-dimensional Minkowski spacetime. The general construction principle (Sect. 3.1) depends neither on the spacetime dimension nor on the manifold we consider. Here we sketch how to extend the concrete construction presented in Sect. 3.2.2 to higher dimensional de Sitter or Minkowski spacetimes. Let $\mathfrak{h} := \mathfrak{so}_{1,n}(\mathbb{R}) \subset \mathfrak{g} := \mathfrak{sl}_m(\mathbb{R})$ and h be a non-symmetric Euler element of $\mathfrak{sl}_m(\mathbb{R})$ such that $\{0\} \neq [\mathfrak{so}_{1,n}(\mathbb{R}), h] \subseteq \mathfrak{so}_{1,n}(\mathbb{R})$ and let U be an (anti-)unitary representation of the \mathbb{Z}_2 -graded extension $\text{PGL}_m(\mathbb{R})$ of $\text{PSL}_m(\mathbb{R})$. Let $W_{x_1}^{\text{dS}, \mathbb{R}^{1+n}}$ be the wedge in the x_1 -direction on de Sitter or on Minkowski spacetime, with the identification $H(W_{x_1}) = N_U(W_h)$. One can define by covariance a Lorentz (3.9) or a Poincaré covariant (3.10) net of standard subspaces if and only if

$$\text{Stab}_{\tilde{\mathcal{L}}_+^\uparrow} W_{x_1} = \{g \in \tilde{\mathcal{L}}_+^\uparrow : \Lambda(g)W_{x_1} = W_{x_1}\} \subset (\tilde{\mathcal{L}}_+^\uparrow)^h$$

or

$$\text{Stab}_{\tilde{\mathcal{P}}_+^\uparrow} W_{x_1} = \{g \in \tilde{\mathcal{P}}_+^\uparrow : \Lambda(g)W_{x_1} = W_{x_1}\} \subset (\tilde{\mathcal{P}}_+^\uparrow)^h,$$

respectively. In this case, the covariance conditions (3.9) and (3.10) define standard subspaces

$$H(\Lambda(g)W_{x_1}^{\text{dS}, \mathbb{R}^{1+s}}) := N_U(gW_h) \quad \text{for } g \in \tilde{\mathcal{L}}_+^\uparrow \quad \text{or } g \in \tilde{\mathcal{P}}_+^\uparrow,$$

where $\tilde{\mathcal{L}}_+^\uparrow$ and $\tilde{\mathcal{P}}_+^\uparrow$ are the double (and universal) covering of the Lorentz and the Poincaré group respectively, and Λ is the covering homomorphism.

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