

Why is the Weyl double copy local in position space?

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ABSTRACT: The double copy relates momentum-space scattering amplitudes in gauge and gravity theories. It has also been extended to classical solutions, where in some cases an exact double copy can be formulated directly in terms of products of fields in position space. This is seemingly at odds with the momentum-space origins of the double copy, and the question of why exact double copies are possible in position space — and when this form will break — has remained largely unanswered. In this paper, we provide an answer to this question, using a recently developed twistorial formulation of the double copy. We show that for certain vacuum type-D solutions, the momentum-space, twistor-space and position-space double copies amount to the same thing, and are directly related by integral transforms. Locality in position space is ultimately a consequence of the very special form of momentum-space three-point amplitudes, and we thus confirm suspicions that local position-space double copies are possible only for highly algebraically-special spacetimes.

KEYWORDS: Black Holes, Classical Theories of Gravity, Scattering Amplitudes

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1 Introduction

Recent years have seen intense study of the relations between different field theories. One such relation is the *double copy*, whose original incarnation relates scattering amplitudes in gauge and gravity theories [1, 2], and was itself inspired by earlier work in string theory [3]. Since then, similar correspondences have been found for amplitudes in a variety of field theories (see e.g. refs. [4–6] for recent reviews). Relevant for the present study is *biadjoint scalar theory*, consisting of a single scalar field carrying two different types of colour charge. Its various copy relationships with other relevant theories are shown in figure 1, and subsequent work has attempted to establish how generally we are allowed to interpret this scheme. That it extends beyond scattering amplitudes was first argued in ref. [7], which showed that certain types of exact classical solution could be copied between theories (see also refs. [8, 9] for earlier work in a different context), namely those that are of *Kerr-Schild* form in gravity. Whilst algebraically special, this family of solutions includes cases of astrophysical relevance, such as certain black holes, and cosmologies (e.g. de Sitter space). As explored in this and many follow-up works [10–20], the Kerr-Schild double copy involves products of certain scalar and vector fields directly in position space. A second exact classical double copy was formulated in ref. [21], and further explored in refs. [22–28]. It uses the spinorial formalism of field theory, and is known as the *Weyl double copy*. Although it looks rather different to the Kerr-Schild approach, it is equivalent where overlap exists, and also involves products of spacetime fields directly in position space. Again, however,

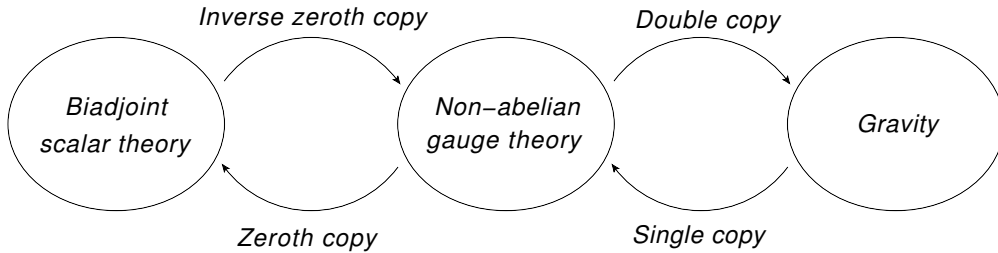


Figure 1. Different field theories, and the relationships between them.

the set of solutions that are amenable to being double-copied is restricted to those that are algebraically special. In terms of the well-known Petrov classification for gravitational solutions, the original Weyl double copy was argued to hold for all vacuum solutions that are of Petrov type D. Particular type-N solutions have also been explored in ref. [24].

It is possible to double copy more-complicated classical gauge theory solutions to gravity, at the expense of having to work order-by-order in perturbation theory, using suitable gauge (or other) choices on both sides [29–40]. Typically, however, one must formulate such double copies in momentum space, analogous to how the original double copy for scattering amplitudes was formulated in the latter. This creates a clear puzzle: even if direct position-space double copies are restricted to certain classes of solution only, why should they exist in the first place? The “natural” home of the double copy is apparently momentum space, and one then expects that gravity fields in position space should be obtainable as *convolutions* of spacetime biadjoint and gauge fields. Indeed, there is an approach that does just this [41–48], which can work in any gauge in principle. Why then, for certain solutions, can one obtain a *product* in position space? This issue has been addressed recently in ref. [49], which looked in detail at the convolution integrals relating spacetime gravity solutions to gauge / scalar counterparts, and showed that these factorise in certain cases into a local product. The Kerr-Taub-NUT solution was found to be part of this class, linking with the earlier observations of refs. [7, 10]. However, it was noted that a local product in position space was not possible if one generalises to solutions that include additional scalar degrees of freedom in the double copy of pure Yang-Mills theory, such as the dilaton. It will also be the case that many solutions even in pure gravity do not have a “simple” double copy in position space, and thus ref. [49] is certainly not the last word on this matter.

In this paper, we take a different approach to examining locality of the Weyl double copy, using various ideas from twistor theory [50–52]. The latter is a branch of mathematical physics that combines various elements of algebraic geometry and complex analysis (see e.g. refs. [53–57] for pedagogical reviews), and allows us to visualise certain physical questions in geometric and / or topological terms. Points in spacetime are mapped non-locally to objects in an abstract *twistor space*, and vice versa. This already tells us that issues relating to locality in spacetime may benefit from viewing them through a twistorial lens, and indeed a procedure for “deriving” the (position-space) Weyl double copy using data in twistor space has been given in refs. [25, 26]. At its heart is a relationship known as

the *Penrose transform*, that relates certain contour integrals in twistor space to fields in spacetime. The integrands of these formulae involve certain twistor “functions”, albeit defined only up to equivalence transformations that leave the integrals invariant. More formally, these quantities are representatives of cohomology classes, and the twistor double copy proposed in ref. [25] is in terms of so-called *Čech cohomology*.

Representative functions exist for all of the spacetime fields (scalar, gauge and gravity) entering the Weyl double copy, and refs. [25, 26] demonstrated that a certain non-linear product of functions in twistor space corresponds to the Weyl double copy in position space. This is already intriguing, given that the map between twistor space and spacetime is non-local. Furthermore, the non-linear relationship required in twistor space is obviously at odds with the ability to first perform equivalence transformations of the various functions that appear. It seems, then, that particular representatives must be selected for the twistor double copy to work, but it is not known *a priori* what procedure must be used to systematically fix them. This issue was explored further in ref. [58], which showed that spacetime data at null infinity could be used to fix representatives in twistor space, at least for radiative solutions. Reference [28] considered a different approach, by first translating from the Čech cohomology language to the alternative framework of *Dolbeault cohomology*, in which the Penrose transform integral is interpreted in terms of differential forms. In Euclidean signature, one may uniquely choose *harmonic* representatives of each required form, upon which the spacetime Weyl double copy can indeed be shown to correspond to a product structure in twistor space. However, none of the methods discussed in refs. [28, 58] obviously matches the original Čech framework of refs. [25, 26], which is arguably simpler to work with (see e.g. ref. [27] for a physical application). Until recently, a simple way of identifying the Čech representatives used in refs. [25, 26] has been lacking. Furthermore, any such procedure should ideally relate to previously known aspects of the double copy.

We can in fact address both the choice of representatives in the Čech twistor double copy, and the question of why the Weyl double copy is local in position space, using the ideas of refs. [59]. This showed, building on the previous work of e.g. refs. [49, 60, 61], how certain classical spacetimes can be obtained from momentum-space scattering amplitudes. Naïvely one might think that amplitudes have nothing to say about classical spacetimes in general: the former have all external legs on-shell, corresponding to particles that are radiated to / from past or future null infinity, whereas the latter are non-radiative in general, and have an off-shell external line (corresponding to where the spacetime field is being evaluated). However, as argued in refs. [59–61], non-radiative modes of spacetime fields can indeed probe null infinity provided one works in $(2, 2)$ signature, rather than the usual $(1, 3)$ Lorentzian signature of relativistic quantum field theory. One may then indeed establish a link between momentum-space scattering amplitudes in $(2, 2)$ signature, and classical solutions in position space, where one must perform an inverse Fourier transform as expected.

Cleverly, ref. [59] takes the equation expressing classical solutions as inverse Fourier transforms of $(2, 2)$ amplitudes, and splits it into two steps. The first, which we will refer to as the *half-transform*, converts the amplitudes into objects in twistor space, such that the second step is precisely the Penrose transform from twistor to position space, which happens

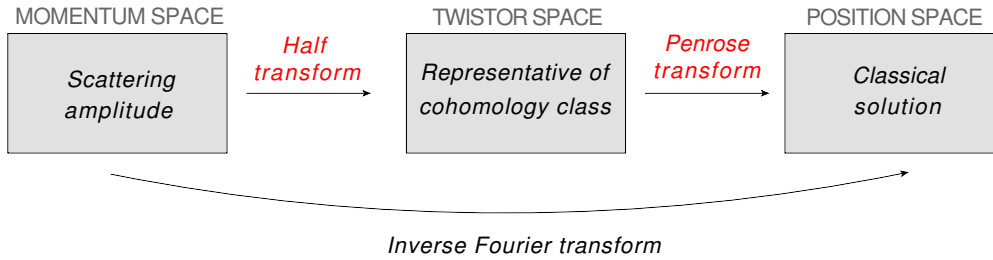


Figure 2. Scheme proposed in ref. [59] for obtaining position-space classical solutions from momentum-space scattering amplitudes in $(2, 2)$ signature. The form of the “half transform” is explained in the main text.

to be in the Čech language. This scheme is shown in figure 2, and it is straightforward to apply it to the classical solutions entering the twistor double copy considered in refs. [25, 26]. As mentioned above and explained in ref. [49], it is known that many of the type-D solutions entering the Weyl double copy of ref. [21] can be obtained from scattering amplitudes in momentum space. The relevant gravity amplitudes can be obtained from corresponding results in gauge theory, using the double copy as it was originally formulated. The latter can then be translated into a relationship between the twistor “functions” (representatives of cohomology classes) living in the middle of figure 2, which we will show is the twistor double copy of refs. [25, 26]. Finally, this translates into the known Weyl double copy in position space, which is equivalent to the Kerr-Schild double copy where appropriate. We will see the form of the half transform appearing in figure 2, but it uniquely fixes the representatives in twistor space that are obtained from given momentum-space amplitudes. Crucially, these representatives are *precisely* those Čech representatives that appear in the original twistor double copy. Thus, the latter is a true consequence of the double copy for scattering amplitudes, and this even suggests how the twistor approach may be extended (e.g. by translating higher-order amplitudes into the twistor language).

In summary, by fleshing out the details of figure 2, we firmly establish the complete equivalence of the BCJ double copy for three-point amplitudes [1, 2], the twistor double copy of refs. [25, 26], and the type D Weyl double copy of ref. [21], at least for those type-D solutions where corresponding amplitudes are known. This is itself puzzling: the maps between all three spaces are non-local, and yet the double copy takes a manifestly local form in all three! We will be able to ascertain why this is the case, and it will only turn out to be true due to the highly-special form of the relevant three-point amplitudes in momentum space. Not only does this settle the question of why local position-space double copies are possible, but it also confirms that such situations are not generic, but rely on very special circumstances.

The structure of our paper is as follows. In section 2, we review ideas relating to the twistor double copy of refs. [25, 26]. In section 3, we apply the methods of ref. [59] to demonstrate that scattering amplitudes in momentum space can be used to pick out the Čech cohomology representatives entering the twistor double copy. In section 4, we explain why locality of the double copy is simultaneously manifest in momentum, twistor and position space, for type-D solutions. Finally, we discuss our results and conclude in section 5.

2 The twistor double copy

In this section, we review various aspects of the twistor double copy introduced in refs. [25, 26], both to make the paper reasonably self-contained, and also to set up notation needed for what follows. As mentioned above, the twistor double copy reproduces the Weyl double copy in position space, so we must first recap the definition of the latter.

2.1 Spinors and the Weyl double copy

The Weyl double copy relies on the spinorial formalism of field theory, in which all equations of motion are written in terms of two-component Weyl spinors π_A , or conjugate spinors $\pi_{A'}$, and their multi-index generalisations. Here the indices $A, A' \in \{0, 1\}$, where indices may be raised and lowered using the two-dimensional Levi-Civita symbols:

$$\pi_A = \epsilon_{AB}\pi^B, \quad \pi^B = \pi_A\epsilon^{AB}, \quad (2.1)$$

where

$$\epsilon_{AB}\epsilon^{CB} = \delta_A^C, \quad \epsilon_{01} = 1. \quad (2.2)$$

Similar equations hold for raising and lowering indices of conjugate spinors, but using $\epsilon_{A'B'}$ etc. such that

$$\epsilon_{A'B'}\epsilon^{C'B'} = \delta_{A'}^{C'}, \quad \epsilon_{0'1'} = 1. \quad (2.3)$$

To convert between spacetime indices¹ and spinor indices, one may use the *Infeld-van-der-Waerden symbols* $\{\sigma_{AA'}^a, \sigma_a^{AA'}\}$. Given that we wish to make contact with refs. [59, 60], we will work in a (2, 2) spacetime signature throughout, for which a suitable choice for the Infeld-van-der-Waerden symbols is [60]:

$$\sigma^a = (1, i\sigma_y, \sigma_z, \sigma_x), \quad (2.4)$$

where 1 denotes the 2×2 identity matrix, and we have used the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.5)$$

According to these conventions, a spacetime 4-vector has the following spinorial translation:

$$V_{AA'} \equiv V_a \sigma_{AA'}^a = \begin{pmatrix} V_0 + V_2 & V_1 + V_3 \\ V_3 - V_1 & V_0 - V_2 \end{pmatrix}, \quad (2.6)$$

from which one obtains the handy formula

$$V \cdot W = \frac{1}{2} V_{AA'} W^{AA'}. \quad (2.7)$$

The determinant of the matrix in eq. (2.6) is

$$|V_{AA'}| = (V_0^2 + V_1^2 - V_2^2 - V_3^2) = V^2, \quad (2.8)$$

¹Throughout the paper, we use lower-case Latin letters, upper-case Latin letters and Greek letters for tensor, spinor and twistor indices respectively.

which therefore vanishes for null vectors ($V^2 = 0$). This in turn implies that one may factorise the matrix into an outer product of two spinors:

$$V_{AA'} = \pi_A \tilde{\pi}_{A'}, \quad V^2 = 0. \quad (2.9)$$

A consequence of the limited range of spinorial indices is that all multi-index spinors can be decomposed into products of fully-symmetrised spinors, and Levi-Civita symbols. As an example, the spinorial translation of the field strength tensor in electromagnetism can be written as follows:

$$F_{ab} \rightarrow F_{AA'BB'} = \phi_{AB} \epsilon_{A'B'} + \bar{\phi}_{A'B'} \epsilon_{AB}. \quad (2.10)$$

Here the symmetric spinors ϕ_{AB} and $\bar{\phi}_{A'B'}$ respectively represent the anti-self-dual and self-dual degrees of freedom in the electromagnetic field. Another important case is that of vacuum gravitational solutions, for which the Riemann curvature tensor R_{abcd} reduces to the *Weyl tensor*, with spinorial translation

$$C_{abcd} \rightarrow \phi_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'} + \bar{\phi}_{A'B'C'D'} \epsilon_{AB} \epsilon_{CD}. \quad (2.11)$$

Again, (un-)barred quantities correspond to the (anti-)self-dual parts of the field. The various quantities appearing in eqs. (2.10), (2.11) obey special cases of the general *massless free field equations*

$$\nabla^{AA'} \phi_{AB\dots C} = 0, \quad \nabla^{AA'} \bar{\phi}_{A'B'\dots C'} = 0, \quad (2.12)$$

with $\nabla^{AA'}$ the spinorial translation of the covariant derivative. A spin- n spacetime field leads to a multi-spinor field with $2n$ indices. Following convention, we will refer to the $n = 1$ and $n = 2$ cases as electromagnetic and Weyl spinors respectively.

Again due to the two-valued nature of spinor indices, it turns out that all symmetric multi-index spinors can be factorised into a symmetrised product of 1-index *principal spinors*. For electromagnetic and Weyl spinors, this takes the explicit form

$$\phi_{AB} = \alpha_{(A} \beta_{B)}, \quad \phi_{ABCD} = \alpha_{(A} \beta_B \gamma_C \delta_{D)}. \quad (2.13)$$

We may then classify solutions of electromagnetism and gravity into qualitatively different types, according to the degeneracy of their principal spinors. Electromagnetic fields are referred to as (non-)null, if their principal spinors are (not) proportional. There are many more possibilities for gravity solutions, which we list in table 1. This is known as the *Petrov classification*, and different patterns of principal spinors constitute different *Petrov types*. Given a principal spinor ξ_A , we may take its complex conjugate $\tilde{\xi}^{A'}$ and form a spacetime vector according to

$$x^a = \sigma_{AA'}^a \xi^A \tilde{\xi}^{A'}, \quad (2.14)$$

which will be null in accordance with eq. (2.9). Thus, principal spinors translate to so-called *principal null directions* in the tensorial language.

Given certain electromagnetic spinors $\{\phi_{AB}^{(1)}, \phi_{AB}^{(2)}\}$ and a scalar field ϕ , the *Weyl double copy* states that

$$\phi_{ABCD} = \frac{\phi_{(AB}^{(1)} \phi_{CD)}^{(2)}}{\phi} \quad (2.15)$$

Weyl type	Petrov label
{1, 1, 1, 1}	I
{2, 1, 1}	II
{3, 1}	III
{4}	N
{2, 2}	D
{-}	O

Table 1. Different types of Weyl spinor classified by: (i) the pattern of degenerate principal null directions; (ii) the equivalent Petrov type.

is a Weyl spinor, corresponding to a particular gravitational solution [21]. The original incarnation of this formula applied to only those cases in which $\phi_{AB}^{(1)} = \phi_{AB}^{(2)}$, and was argued to hold for arbitrary Petrov type-D solutions. Further work has established the existence of *mixed* Weyl double copies with $\phi_{AB}^{(1)} \neq \phi_{AB}^{(2)}$, with applications to certain type N solutions [24], as well as other Petrov types at linearised level only [25, 26]. Other implications have been explored in refs. [23, 62–65], and a novel three-dimensional counterpart of the Weyl double copy (the *Cotton double copy*) has recently been proposed in refs. [66, 67]. Note that eq. (2.15) involves products of fields in position space, which is mysterious given that the original double copy for scattering amplitudes is naturally expressed in momentum space. This implies that one should expect *convolutions* of fields in position space, and indeed refs. [41–47] imply that this will be true in general for classical fields. For specific solutions, ref. [49] has pointed out that the mathematical properties of the relevant convolution integrals are such that products of fields can indeed be made manifest in both position and momentum space. Here, we shed more light on this issue by using the twistor methods outlined below.

2.2 The twistor double copy

We may define a twistor to be a composite object containing two spinors of opposite chirality

$$Z^\alpha = (\lambda_A, \mu^{A'}), \tag{2.16}$$

whose components satisfy the *incidence relation*²

$$\mu^{A'} = x^{AA'} \lambda_A. \tag{2.17}$$

Twistor space \mathbb{T} consists of all objects of the form of eq. (2.16). However, eq. (2.17) is invariant under rescalings of both sides (and thus the twistor of eq. (2.16)) by a common factor λ . Thus, twistors satisfying the incidence relation are points in *projective twistor space* \mathbb{PT} . Unless otherwise stated, we will consider complexified flat spacetime in (2, 2) signature, with Cartesian line element

$$ds^2 = dt^2 + dx^2 - dy^2 - dz^2, \quad t, x, y, z \in \mathbb{C}. \tag{2.18}$$

²One way to interpret eqs. (2.16), (2.17) is that the spinors in Z^α characterise independent solutions of the *twistor equation* $\nabla_A^{(A'} \Lambda^{B')} = 0$, for some spinor field $\Lambda^{B'}$. The incidence relation then arises by defining the location of a twistor in spacetime by $\Lambda^{A'} = 0$. See e.g. ref. [54].

All twistor components are then real, and it is straightforward to ascertain that the incidence relation comprises a *non-local* map between \mathbb{PT} and spacetime. For example, a given point in \mathbb{PT} is associated with all spacetime points satisfying eq. (2.17), which are of the form

$$x^{AA'} = x_0^{AA'} + \lambda^A \alpha^{A'}. \quad (2.19)$$

Here $x_0^{AA'}$ is a fixed point in spacetime, and λ_A is also fixed for a given point in \mathbb{PT} . Equation (2.9) then reveals that the second term on the right-hand side generates a null direction in spacetime for a given $\alpha^{A'}$. Thus, varying $\alpha^{A'}$ generates a set of null directions, and thus a *null plane* in (complex) spacetime: a plane such that all tangent vectors are null. These are called α -planes, and we may also note that were we to restrict to a real spacetime in Lorentzian signature, we would obtain a null geodesic (line) rather than a null plane in spacetime. To see this, note that in Lorentzian signature, eq. (2.6) would be replaced with

$$V_{AA'} \equiv V_a \sigma_{AA'}^a = \begin{pmatrix} V_0 + V_2 & V_3 + iV_1 \\ V_3 - iV_1 & V_0 - V_2 \end{pmatrix}, \quad (2.20)$$

where all coordinates V_a are real. This in turn implies that the spinor $\alpha^{A'}$ appearing in eq. (2.19) must be related to the complex conjugate $\tilde{\lambda}^{A'}$ of λ_A up to a constant factor:

$$\alpha^{A'} \propto \tilde{\lambda}^{A'}.$$

This picks out the unique null direction specified by λ_A (which is fixed for a given twistor), as required.

So much for the map from \mathbb{PT} to complex spacetime. To go the other way round, we may note that a point in twistor space has (from eq. (2.16)) 4 complex degrees of freedom, reducing to 3 if we consider \mathbb{PT} . The incidence relation of eq. (2.17) then provides a further 2 constraints, so that a fixed point in spacetime constitutes a single degree of freedom, or (complex) line, in \mathbb{PT} . We can take points on this line to be specified by the twistor components $\pi_{A'}$ which, given the projective nature of the space, we may parametrise according to either

$$\pi_{A'} = (1, \xi) \quad \text{or} \quad \pi_{A'} = (\eta, 1), \quad \xi, \eta \in \mathbb{C}. \quad (2.21)$$

These define two coordinate patches covering a Riemann sphere, which has a nice geometric interpretation in the real Lorentzian case. Given $\pi_{A'}$ corresponds to a null direction emanating from the fixed point $x_0^{AA'}$, the Riemann sphere corresponding to a fixed spacetime point constitutes the *celestial sphere* of all possible null directions from $x_0^{AA'}$ (up to reparametrisations). We will refer to the Riemann sphere corresponding to a specific spacetime point $x^{AA'}$ as X in what follows.

Given the twistor of eq. (2.16), one may also define a *dual twistor*

$$W_\alpha = (\tilde{\mu}^A, \tilde{\lambda}_{A'}). \quad (2.22)$$

This allows one to define an inner product between (dual) twistors:

$$Z^\alpha W_\alpha = \tilde{\mu}^A \lambda_A + \mu^{A'} \tilde{\lambda}_{A'}. \quad (2.23)$$

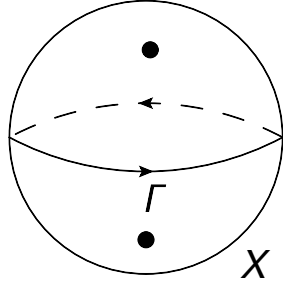


Figure 3. The Penrose transform involves the integral of a twistor “function” $f(Z^\alpha)$ around a contour Γ on the Riemann sphere X corresponding to a given spacetime point x . For a non-zero result, there must be at least one pole on either side of the contour.

As discussed in the introduction, a key result of twistor theory is the fact that solutions of the massless free field equations of eq. (2.12) can be represented using certain integral formulae in \mathbb{P}^T . More specifically, the *Penrose transform* expresses the self-dual part of a spin- n field as

$$\phi_{AB\dots C}(x) = \frac{1}{2\pi i} \oint_{\Gamma} \lambda_E d\lambda^E \lambda_A \lambda_B \dots \lambda_C [\rho_x f(Z^\alpha)]. \quad (2.24)$$

The right-hand contains a holomorphic “function” of a single twistor variable $f(Z^\alpha)$, where the symbol ρ_x denotes restriction to the Riemann sphere X corresponding to spacetime point $x^{AA'}$. The remaining integrand contains factors of the spinor λ_A that enters the twistor Z^α (as in eq. (2.16)), and the contour Γ is such as to separate any poles of $f(Z^\alpha)$ on X . An example is shown in figure 3, and for the contour integral to give a non-zero answer, one must clearly have at least one pole on either side of the contour Γ . For a given spacetime field, the function $f(Z^\alpha)$ is not uniquely defined: it may be subjected to equivalence transformations of the form

$$f(Z^\alpha) \rightarrow f(Z^\alpha) + f_N(Z^\alpha) + f_S(Z^\alpha), \quad (2.25)$$

where $f_N(Z^\alpha)$ ($f_S(Z^\alpha)$) has poles only in the northern (southern) hemisphere of X , without changing the result of the contour integral. Mathematically speaking, $f(Z^\alpha)$ is a representative of a *cohomology class*, and one may formalise this discussion in terms of *Čech cohomology groups*, which are themselves approximations to *sheaf cohomology groups*, as discussed in refs. [54, 55, 68]. An alternative formulation exists using the language of differential forms and *Dolbeault cohomology*, as reviewed e.g. in ref. [57], and a recent discussion of a comparison between the two approaches can be found in ref. [28]. We will use the Čech approach throughout, and the next point we need to note is that the requirement that eq. (2.24) make sense as an integral in projective twistor space imposes a restriction on $f(Z^\alpha)$. That is, the integral must give the same answer under rescalings $Z^\alpha \rightarrow \lambda Z^\alpha$, which can only be true if one has

$$f(Z^\alpha) \rightarrow \lambda^{-2n-2} f(Z^\alpha), \quad (2.26)$$

if there are $2n$ indices on the left-hand side of the Penrose transform. That is, a spin- n field corresponds to a cohomology representative $f(Z^\alpha)$ with homogeneity $(-2n - 2)$. For

scalar, electromagnetic and gravitational fields respectively, this implies homogeneities -2 , -4 and -6 . We have here addressed the case of the self-dual part of a massless free field. The anti-self-dual part can be obtained using an alternative Penrose transform in terms of dual twistors, as discussed in e.g. refs. [54, 55].

We are now able to state the twistor expression of the Weyl double copy that first appeared in refs. [25, 26]. Given certain cohomology representatives $f_{-2}(Z^\alpha)$, $f_{-4}^{(1)}(Z^\alpha)$ and $f_{-4}^{(2)}(Z^\alpha)$, where the subscript denotes the homogeneity, one may construct a homogeneity -6 representative via the product

$$f_{-6}(Z^\alpha) = \frac{f_{-4}^{(1)}(Z^\alpha)f_{-4}^{(2)}(Z^\alpha)}{f_{-2}(Z^\alpha)}. \tag{2.27}$$

By the above remarks, this will correspond to a gravitational field. However, the constituent “functions” on the right-hand side correspond to a pair of electromagnetic fields, and a scalar. There must then be a relationship between the corresponding spacetime fields, and refs. [25, 26] showed that one may choose representatives such that this spacetime relationship is precisely the type-D Weyl double copy. To do this, one may rely on the observation made in ref. [54], that — for representatives $f(Z^\alpha)$ that involve only two poles — a pole of order m in twistor space leads to a $(n - m + 1)$ -fold degenerate principal spinor in spacetime, for n the spin. A type-D solution has two 2-fold degenerate principal spinors, so that it may be generated using a cohomology representative of form

$$f_{-6}(Z^\alpha) = \left[Q^{\alpha\beta} Z^\alpha Z^\beta \right]^{-3}, \tag{2.28}$$

where $Q^{\alpha\beta}$ is a constant twistor. Likewise, one may generate scalar and electromagnetic fields via the choices

$$f_{-2}(Z^\alpha) = \left[Q^{\alpha\beta} Z^\alpha Z^\beta \right]^{-1}, \quad f_{-4}^{(1)}(Z^\alpha) = f_{-4}^{(2)}(Z^\alpha) = \left[Q^{\alpha\beta} Z^\alpha Z^\beta \right]^{-2}. \tag{2.29}$$

It is easily checked that these representatives obey eq. (2.27). Furthermore, choosing different forms for $Q^{\alpha\beta}$ is sufficient to map out the complete space of vacuum type-D solutions [69].

As remarked above, the quantities $f(Z^\alpha)$ entering the Penrose transform integral are representatives of cohomology classes which, in more pedestrian terms, amount to functions defined only up to the equivalence transformations of eq. (2.25). The product of eq. (2.27), needed to reproduce the Weyl double copy in position space, is clearly incompatible with the ability to first perform equivalence transformations of the gauge and / or scalar functions. Furthermore, this is unavoidable, given that the combination of twistor “functions” required by the double copy is necessarily non-linear. It seems, then, that the product-like nature of the twistor space double copy is only possible if special representatives of each cohomology class are chosen, and it is not clear a priori what these representatives should be.

Reference [58] was the first work to provide a potential solution to this issue, at least for radiative spacetimes that can be fully defined by specifying data at future null infinity. A certain procedure exists [70] for using this data to fix twistor representatives of spacetime fields, in the Dolbeault cohomology framework alluded to above. Reference [58] then argued

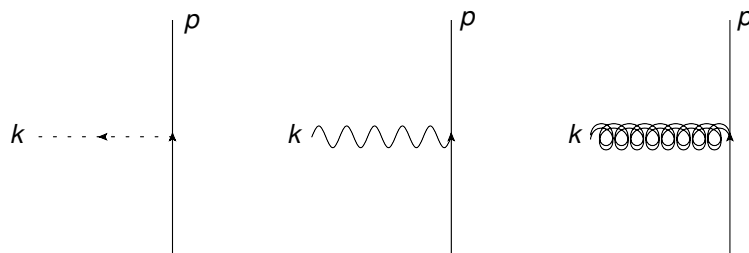


Figure 4. Three-point amplitudes for the emission of a scalar, photon or graviton from static sources.

that a twistorial double copy naturally emerges for these representatives. Reference [28] considered both the Čech and Dolbeault languages, first showing that one may translate the original Čech double copy of refs. [25, 26] into the Dolbeault approach, albeit subject to the same conceptual puzzle regarding how to pick cohomology representatives. It then showed that, for solutions in Euclidean signature, established techniques imply that there are unique choices of Dolbeault representative — namely those that are *harmonic* differential forms [56] — such that the Weyl double copy in position space yields a product structure in twistor space. Whilst this is encouraging, it is not known how to directly relate these representatives to those in the Čech language, nor is it known what the harmonic condition implies for the latter. It also not known whether this procedure can be directly related to that of ref. [58]. From a mathematical point of view, it is not clear whether the different double copy procedures in twistor space amount to the same double copy in position space, or a set of physically distinct double copy procedures. If the latter turns out the case, one can then ask which twistor double copy, if any, corresponds to the original double copy for scattering amplitudes. We provide an answer to this question in the following section.

3 Cohomology representatives from scattering amplitudes

Above, we have posed the question of which twistor double copy procedure, if any, can be related to the double copy for scattering amplitudes. In fact, the recent developments of refs. [59, 60], allow us to precisely answer this question. We begin by showing how the scheme of figure 2 can be made precise.

3.1 From scattering amplitudes to twistor space

Let us consider three-point amplitudes for emission of a scalar, photon or graviton from a static source particle, as shown in figure 4. Following ref. [59], we may write the spinorial translation of the radiation momentum k^μ as

$$k_{AA'} = \omega \lambda_A \tilde{\lambda}_{A'} + \xi q_{AA'}, \tag{3.1}$$

where $\omega = k^0$ is the energy, and $\lambda_A, \tilde{\lambda}_{A'}$ are dimensionless spinors. In what follows, we will parametrise these by

$$\lambda_A = (1, z), \quad \tilde{\lambda}_{A'} = (1, \tilde{z}) \tag{3.2}$$

for $z, \tilde{z} \in \mathbb{C}$, such that we may think of each spinor as defining a point on a Riemann sphere, whose meaning will be clarified shortly.³ We have also introduced a null reference vector $q_{AA'}$ in eq. (3.1), such that ξ parametrises the off-shellness of k^μ .

Denoting the amplitude for spin- n radiation by $\mathcal{A}_\pm^{(n)}$, where \pm denotes the helicity of the emitted boson as appropriate, one may obtain the classical unprimed spinor field for the emitted radiation via the integral formula:

$$\phi_{A_1 A_2 \dots A_{2n}} = N_n \operatorname{Re} \int d\Phi(k) 2\pi\delta(2p \cdot k) \omega^n \lambda_{A_1} \lambda_{A_2} \dots \lambda_{A_{2n}} e^{-ik \cdot x} \mathcal{A}_+^{(n)}, \quad (3.3)$$

where

$$d\Phi(k) \equiv \frac{d^4 k}{(2\pi)^4} 2\pi\delta(k^2) \Theta(\omega), \quad (3.4)$$

and we have introduced the constants $\{N_n\}$, which collect numerical factors and coupling constants. In words, eq. (3.3) represents the spacetime field as an on-shell inverse Fourier transform of the momentum-space amplitude. It was derived in ref. [60] using the so-called KMOC formalism for obtaining classical observables from quantum field theory.⁴ A similar conclusion was presented in ref. [59], using different but related arguments. However, whereas ref. [60] examined the position-space implications of eq. (3.3) directly by carrying out the inverse Fourier transform in one go, ref. [59] split this into two stages, according to the scheme of figure 2. To see how this works in the present context (i.e. for eq. (3.3) taken from ref. [60]), we may recast eq. (3.4) so as to involve the variables appearing in eq. (3.1). To do this, we may equate $k_{AA'}$ (obtained from eq. (2.6)) with the right-hand side of eq. (3.1), and solve for the components k_a . In doing so, one may use eq. (3.2) and also the fact that nullity of the reference vector q_a implies

$$q_{AA'} = q_A \tilde{q}_{A'},$$

for some spinors $q_A, \tilde{q}_{A'}$. One finds

$$\begin{aligned} k_0 &= \frac{1}{2} [\xi (q_0 \tilde{q}_{0'} + q_1 \tilde{q}_{1'}) + \omega(1 + z\tilde{z})]; \\ k_1 &= \frac{1}{2} [\xi (q_0 \tilde{q}_{1'} - q_1 \tilde{q}_{0'}) + \omega(z - \tilde{z})]; \\ k_2 &= \frac{1}{2} [\xi (q_0 \tilde{q}_{0'} - q_1 \tilde{q}_{1'}) + \omega(1 - z\tilde{z})]; \\ k_3 &= \frac{1}{2} [\xi (q_0 \tilde{q}_{1'} + q_1 \tilde{q}_{0'} + \omega(z + \tilde{z}))]. \end{aligned} \quad (3.5)$$

The Jacobian is given by [71]

$$J = \frac{i\omega^2}{4\nu} \lambda^A q_A \tilde{\lambda}^{A'} \tilde{q}_{A'}, \quad \nu = \begin{cases} 1 & \text{in (1, 3) signature} \\ i & \text{in (2, 2) signature,} \end{cases} \quad (3.6)$$

³To cover the Riemann sphere, we would need to consider a second coordinate patch in which $\lambda_1 \neq 0, \tilde{\lambda}_{1'} \neq 0$. We will not need to consider this explicitly for our purposes.

⁴To convert to the notation of ref. [60], one must write $\omega^n \lambda_{A_1} \dots \lambda_{A_{2n}} \equiv |k\rangle|k\rangle \dots |k\rangle$ in eq. (3.3), such that there are $2n$ factors of the spinor $|k\rangle$. Furthermore, retarded boundary conditions for the radiated field are implicit in eq. (3.3), which is equivalent to slightly deforming the radiated energy according to $k^0 \rightarrow k^0 + i\varepsilon, \varepsilon > 0$.

from which one finds

$$\begin{aligned} d\Phi(k) &= \frac{dz d\bar{z} d\omega d\xi}{4\nu(2\pi)^3} \omega^2(q_1 - q_0\bar{z})(\tilde{q}_{1'} - \tilde{q}_{0'}z) \delta[\xi\omega(q_1 - q_0\bar{z})(\tilde{q}_1 - \tilde{q}_0z)] \\ &= \frac{dz d\bar{z} d\omega d\xi}{4\nu(2\pi)^3} \omega\delta(\xi). \end{aligned} \quad (3.7)$$

Note that the on-shell delta function simply becomes the requirement that $\xi = 0$, as expected from the parametrisation of eq. (3.1). Substituting eq. (3.7) into eq. (3.3), one obtains⁵

$$\begin{aligned} \phi_{A_1\dots A_{2n}} &= \frac{N_n}{4(2\pi)^2} \int dz d\bar{z} d\omega d\xi \delta(\xi) \delta(2p \cdot k) \omega^{n+1} \lambda_{A_1} \dots \lambda_{A_{2n}} e^{-\frac{i\omega}{2} \lambda_A \tilde{\lambda}_{A'} x^{AA'}} e^{-i\xi q \cdot x} \mathcal{A}_+(k) \\ &= \frac{N_n}{4(2\pi)^2} \int dz d\bar{z} d\omega \delta(2p \cdot k) \omega^{n+1} \lambda_{A_1} \dots \lambda_{A_{2n}} e^{-\frac{i\omega}{2} \lambda_A \tilde{\lambda}_{A'} x^{AA'}} \mathcal{A}_+(k), \end{aligned} \quad (3.8)$$

where we have used eq. (2.7), and eliminated the on-shell delta function in the second line. For brevity, we have also left implicit the overall real part from eq. (3.3), and will continue to do so in what follows. Regarding the remaining integral measure, we may write this in a spinor-invariant form as follows. First, from eq. (3.2), we may rewrite

$$\int dz \rightarrow \oint \lambda_E d\lambda^E,$$

where the latter is the conventional measure on the Riemann sphere associated with λ_E . Note that, from the parametrisation of eq. (3.2), the integral over z is in the complex plane one obtains by stereographic projection. Rewritten in terms of λ_E , this will become a closed contour integral on the Riemann sphere itself. Next, we may define

$$\tilde{\xi}_{A'} = \omega \tilde{\lambda}_{A'} = \omega(1, \tilde{z}), \quad (3.9)$$

such that one has

$$d\omega d\bar{z} = \frac{1}{\omega} d\tilde{\xi}_{0'} d\tilde{\xi}_{1'} \equiv \frac{1}{\omega} d^2\tilde{\xi}. \quad (3.10)$$

Equation (3.8) then becomes

$$\phi_{A_1\dots A_{2n}} = \frac{1}{2\pi i} \oint \lambda_E d\lambda^E \lambda_{A_1} \dots \lambda_{A_{2n}} \rho_x [\mathfrak{M}_+(Z^\alpha)], \quad (3.11)$$

where we have defined

$$\mathfrak{M}_+ = \frac{iN_n}{4(2\pi)^2} \int d^2\tilde{\xi} e^{-\frac{i}{2} \lambda_A \tilde{\xi}_{A'} x^{AA'}} \delta(2p \cdot k) \omega^n \mathcal{A}_+(k). \quad (3.12)$$

We may recognise eq. (3.11) as precisely the Penrose transform of eq. (2.24), where the spinor λ_A entering the spinorial decomposition of the radiation momentum of eq. (3.2) forms half of the twistor components defined in eq. (2.22). Using the incidence relation of

⁵Carrying through the $i\varepsilon$ prescription for the retarded boundary conditions in eq. (3.7) amounts to the deformation $\xi \rightarrow \xi - i\omega\varepsilon$ in the first line of eq. (3.8), which guarantees convergence of the energy integral in the second line.

eq. (2.17), we may then recognise the combination $\lambda_A x^{AA'}$ appearing in the exponent in eq. (3.12) as the remaining half of the twistor $\mu^{A'}$. Hence, one has

$$\mathfrak{M}_+(Z^\alpha) = \int d^2\tilde{\xi} e^{-\frac{i}{2}\tilde{\xi}_{A'}\mu^{A'}} A(k), \quad A(k) = \frac{iN_n}{4(2\pi)^2} \delta(2p \cdot k) \omega^n \mathcal{A}_+(k). \quad (3.13)$$

The object on the left-hand side depends on the spinors λ_A and $\mu^{A'}$, and hence the single twistor argument Z^α . We note that the little group properties of the amplitude, i.e. that it transforms as $\mathcal{A} \rightarrow t^{-2n}\mathcal{A}$ under a little group transformation of the massless leg, ensures that \mathfrak{M} transforms with the required homogeneity of $-2n-2$, since the measure transforms as $d^2\tilde{\xi} \rightarrow t^{-2}d^2\tilde{\xi}$. The integral transform appearing in eq. (3.13) takes a certain dressed momentum-space amplitude, and maps it into a quantity in twistor space. We will call this the “half transform” as in figure 2 given that, as pointed out in ref. [59], eq. (3.13) is related to the well-known “half Fourier transform” of refs. [72–74] that takes momentum-space amplitudes into twistor space. Here and in ref. [59], however, one integrates only over manifestly positive energies $\omega > 0$ such that the half transform, considered as an integral in ω , is equivalent to a Laplace transform, as we will see in explicit examples.

The above results indeed realise the scheme of figure 2: eq. (3.13) is the half transform mapping momentum-space quantities into twistor space. Subsequently, eq. (3.11) is the Penrose transform that takes the quantity $\mathfrak{M}(Z^\alpha)$ in twistor space, and associates it with a classical spacetime field. Thus, in a well-defined sense, twistor space sits “in between” momentum and position space, allowing us to address conceptual questions regarding the double copy.

3.2 Cohomology representatives from half-transformed amplitudes

As discussed in section 2, an open problem in the twistor double copy is to make sense of the product of cohomology representatives occurring in eq. (2.27). Whilst various ideas for choosing representatives have occurred in recent literature [26, 28, 58], it is not clear that any of these correspond to the original Čech double copy presented in refs. [25, 26]. Furthermore, it would be reassuring to know that any incarnation of the twistor double copy can be shown to be equivalent to the original BCJ double copy for scattering amplitudes [1, 2], which would immediately put the twistor approach on a much firmer footing. In fact, the scheme of figure 2 allows us to do just this. First, note that the half transform of eq. (3.13) relates a given momentum-space amplitudes to a specific (unambiguous) cohomology representative in twistor space. From a double copy point of view, there is then a natural choice of representative for a given classical solution in position space, namely that which is picked out by a momentum-space amplitude. For certain amplitudes relating to known static solutions in position space, we show that the cohomology representatives in twistor space are *precisely* those entering the Čech double copy of eq. (2.27). This in turn implies that for these solutions, the BCJ double copy for amplitudes [1, 2], the twistor double copy of refs. [25, 26], and the Weyl double copy of ref. [21], amount to the same thing. Whilst the connection between the Weyl double copy and three-point amplitudes in momentum space was already noted in ref. [60], the linking of both of these to an intermediate twistor space is both new and useful, as we will see later on.

Let us now find the cohomology representatives in twistor space corresponding to given amplitudes. If we take a spinless static source with

$$p^\mu = Mu^\mu, \tag{3.14}$$

where u^μ is the 4-velocity, then the delta function appearing in eq. (3.13) becomes, when translated using eq. (3.2),

$$\delta(2p \cdot k) = \delta(Mu^{AA'}\lambda_A\tilde{\xi}_{A'}), \tag{3.15}$$

which implies

$$\tilde{\xi}_{A'} \propto \omega u^A_{A'}\lambda_A. \tag{3.16}$$

In the rest frame where $u_a = (1, \mathbf{0})$, we find that

$$\omega u^A_{A'}\lambda_A = z\tilde{\xi}_{A'}. \tag{3.17}$$

We recall the fact that we have the freedom to perform a scaling of the form $\lambda \rightarrow t\lambda$ and $\tilde{\lambda} \rightarrow t^{-1}\tilde{\lambda}$, and that the curvature spinor is invariant under such scaling. We choose to scale

$$\lambda \rightarrow \frac{1}{\sqrt{z}}\lambda, \quad \tilde{\lambda} \rightarrow -\frac{1}{\sqrt{-\bar{z}}}\tilde{\lambda}, \tag{3.18}$$

using the fact that eq. (3.15) fixes $z\bar{z} = -1$. By fixing this symmetry, eq. (3.16) becomes an equality

$$\tilde{\xi}_{A'} = \omega u^A_{A'}\lambda_A. \tag{3.19}$$

Equation (3.13) can be expressed as⁶

$$\mathfrak{M}(Z^\alpha) = \int_0^\infty d\omega \exp\left[-\frac{\omega}{2}u^A_{A'}\mu^{A'}\lambda_A\right] \mathcal{A}(k) \tag{3.20}$$

where, following ref. [59], we have rotated $x^\mu \rightarrow ix^\mu$. The latter replacement merely corresponds to reversing the (2, 2) signature of our spacetime metric, and avoids proliferation of factors of i in what follows. The amplitude for a scalar is simply a coupling, and so one may straightforwardly carry out the ω integral to give

$$\mathfrak{M} \propto \frac{1}{u^A_{A'}\mu^{A'}\lambda_A}. \tag{3.21}$$

For a static particle we must have $u_a = (1, \mathbf{0})$, and it then follows from eq. (2.6) that

$$u^A_{A'} = \epsilon^{BA}u_{BA'} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.22}$$

Then eq. (3.21) may be rewritten as

$$\mathfrak{M} \propto \frac{1}{Q_{\alpha\beta}Z^\alpha Z^\beta}, \quad Q_{\alpha\beta} = \begin{pmatrix} 0 & u^A_{B'} \\ u^B_{A'} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.23}$$

⁶We have removed the helicity subscript in eq. (3.20), given that this is not relevant for the scalar case.

We have thus obtained a cohomology representative in twistor space for a static scalar field, and we can immediately note that this has precisely the form required by the Čech form of the twistor double copy i.e. eq. (2.29). Furthermore, the explicit form of $Q_{\alpha\beta}$ is indeed that used to obtain the zeroth copy of the Schwarzschild black hole in refs. [25, 26] (see also refs. [75, 76] for the original context in which this quadratic form was presented).

A generic amplitude of two scalars with mass M and a positive-helicity spin- n radiated field is given by

$$\mathcal{A}_+^{(n)}(k) = g_n M^n X^n, \quad X = \sqrt{2}u \cdot \epsilon_+(k). \quad (3.24)$$

We will focus on the electromagnetic and gravitational cases, where the relevant amplitudes are given by taking $g_1 = -\sqrt{2}Q$ and $g_2 = -\frac{\kappa}{2}$ (see e.g. ref. [60]) and

$$\mathcal{A}_+^{\text{EM}} = -\sqrt{2}MQX, \quad \mathcal{A}_+^{\text{grav.}} = -\frac{\kappa}{2}M^2X^2. \quad (3.25)$$

The fact that the gravity amplitude in eq. (3.25) is related to the square of the electromagnetic case is a manifestation of the BCJ double copy for amplitudes. At this stage it is useful to introduce the bispinor

$$T_{AA'} = V_a \sigma_{AA'}^a, \quad (3.26)$$

where the vector V_a is a fixed constant vector with $V^2 = \pm 1$ that points in only one direction. This bispinor has an important property, namely that

$$T_{AA'} T^{A'B} = \pm \delta_A^B. \quad (3.27)$$

To see this, we use the Clifford algebra

$$T_{AA'} T^{A'B} = V_a \sigma_{AA'}^a V^b \bar{\sigma}_b^{A'B} = \frac{1}{2} V_a V_b \left[\sigma_{AA'}^a \bar{\sigma}^{b,A'B} + \sigma_{AA'}^b \bar{\sigma}^{a,A'B} \right] = V^2 \delta_A^B = \pm \delta_A^B, \quad (3.28)$$

where the bar denotes Infeld-van-der-Waerden symbols acting on conjugate spinors. For static solutions, $u_a = (1, \mathbf{0})$ and the bispinor $u^{AA'}$ is exactly of the form of eq. (3.26). This means we can recast the delta function constraint, in the static case, to be

$$u^{BA'} \tilde{\xi}_{A'} = \omega u^A_{A'} u^{A'B} \lambda_A = \omega \lambda_B. \quad (3.29)$$

Using this, the delta function constrains the three-particle amplitudes to be simple constant factors. One way to see this is to use the explicit spinor form of the polarisation vector⁷

$$\epsilon_+^a = \frac{1}{\sqrt{2}} \frac{(\sigma^a)^{AA'} q_A \tilde{\lambda}_{A'}}{\lambda^A q_A}, \quad (3.30)$$

where q_A is a so-called *reference spinor*, corresponding to a null reference vector in the tensorial language. The form of $u_a = (1, \mathbf{0})$ for a static solution implies

$$X = \sqrt{2}u \cdot \epsilon_+ = \frac{u^{AA'} q_A \tilde{\lambda}_{A'}}{\lambda^A q_A} = \frac{1}{\omega} \frac{u^{AA'} q_A \tilde{\xi}_{A'}}{\lambda^A q_A} = 1, \quad (3.31)$$

⁷In the conventional spinor helicity notation of ref. [60], eq. (3.30) reads $\epsilon_+^a = \frac{[k|\tilde{\sigma}^a|q\rangle}{\sqrt{2}\langle kq\rangle}$, where $\tilde{\sigma}^a$ denotes an Infeld-van-der-Waerden symbol with upstairs spinor indices.

where we have again used (3.29). We see then that the X -factor is set to unity and the spin- n amplitude, under the integral of the Laplace transform, is simply proportional to some coupling times a mass. Using this fact, and carrying out similar steps to the scalar case, we find that the spin- n version of eq. (3.13) satisfies

$$\begin{aligned} \mathfrak{M}_{n,+}(Z^\alpha) &\propto \int_0^\infty d\omega \omega^n \exp\left[-\frac{\omega}{2} u^A{}_{A'} \mu^{A'} \lambda_A\right] \\ &\propto \frac{1}{(Q_{\alpha\beta} Z^\alpha Z^\beta)^{n+1}}, \end{aligned} \tag{3.32}$$

where $Q_{\alpha\beta}$ is given by eq. (3.23) as before. For spinless static solutions — corresponding physically to the Schwarzschild black hole and its single / zeroth copies — we have thus reproduced the cohomology representatives of eq. (2.28), (2.29).

It is straightforward to generalise the above arguments to other classical solutions, that are related to three-point amplitudes according to eq. (3.3). As argued in refs. [49, 60, 77], for example, one may modify the 3-point amplitudes for a spinless static particle to include both the effects of rotation, and also a dual charge e.g. a NUT charge in gravity, corresponding to a magnetic monopole in gauge theory [10]. Furthermore, this modification is remarkably simple: one simply replaces the three-point amplitude for helicity η according to⁸

$$\mathcal{A}_\eta \rightarrow e^{\eta(i k \cdot a + \theta)} \mathcal{A}_\eta, \tag{3.33}$$

where θ is related to the NUT charge, and a^μ is the classical spin vector. In twistor space, this has the effect of simply multiplying each representative by e^θ , whilst simultaneously replacing $x \rightarrow x - a$, so that we get

$$\begin{aligned} \mathfrak{M}_{n,+} &\propto \frac{e^\theta}{(u^A{}_{A'} \mu^{A'} \lambda_A - u^{(A}{}_{A'} a^{B)A'} \lambda_A \lambda_B)^{n+1}} \\ &\propto \frac{1}{(Q_{\alpha\beta} Z^\alpha Z^\beta)^{n+1}}, \end{aligned} \tag{3.34}$$

where we now have

$$Q_{\alpha\beta} = e^{-\theta} \begin{pmatrix} 0 & u^A{}_{B'} \\ u^B{}_{A'} & 2\mu^{AB} \end{pmatrix}, \quad \mu^{AB} = -u^{(A}{}_{A'} a^{B)A'}. \tag{3.35}$$

Up to our overall normalisation, the result for $Q_{\alpha\beta}$ (in the case $\theta = 0$) is precisely the so-called *kinematic twistor* that encodes the (angular) momentum of a spinning particle [51, 54].⁹ Again, we find that the cohomology representatives picked out by momentum-space three-point amplitudes are precisely the quadratic forms required by eqs. (2.28), (2.29).

Above, we remarked that the half transform from momentum to twistor space assumes the form of a Laplace transform, and we can see this directly in our explicit examples.

⁸For the scalar field, one chooses the sign of η according to whether one is taking the zeroth copy of the self-dual or anti-self-dual electromagnetic field strength spinor.

⁹Note that the lower-right components in eq. (3.35) have an accompanying factor of i in refs. [51, 54], owing to the choice of Lorentzian rather than (2, 2) signature.

First, note that the right-hand side of eq. (3.32) may be written as

$$\mathcal{L}[\omega^n](U) \equiv \int_0^\infty d\omega \omega^n e^{-\omega U}, \quad U = \frac{1}{2} u^A{}_{A'} \mu^{A'} \lambda_A, \quad (3.36)$$

which is a manifest Laplace transform in the energy ω , with U playing the role of the conjugate variable. Likewise, the shift $x \rightarrow x - a$ above eq. (3.34) amounts to the replacement

$$U \rightarrow U - V, \quad V = \mu^{AB} \lambda_A \lambda_B. \quad (3.37)$$

This is commonly referred to as the *frequency-shifting* property of Laplace transforms:

$$\mathcal{L}[e^{\omega V} f(\omega)](U) = \mathcal{L}[f(\omega)](U - V), \quad (3.38)$$

which we collect for later use.

Let us take stock of what has happened. The twistor double copy of refs. [25, 26] gave a way to “derive” the Weyl double copy for type-D vacuum solutions in position space, but suffered from the conceptual puzzle of how to multiply together the relevant quantities in twistor space or, in other words, how to choose appropriate cohomology representatives so that a simple product structure is obtained. Procedures for achieving the latter were discussed in refs. [28, 58], but it remains unclear whether or not these are equivalent. Furthermore, none of them reproduces the original choice of representatives in the original twistor double copy of refs. [25, 26]. In this section, we have used the methods of refs. [59, 60] to show that, for those stationary fields which can be obtained from three-point amplitudes in momentum space, then the Weyl double copy [21], the twistor double copy [25, 26], and the BCJ double copy for scattering amplitudes [1, 2] *are completely equivalent*. They are related by integral transforms according to the scheme of figure 2, and our findings are significant in that they immediately put the twistor double copy on a much firmer footing. Furthermore, they suggest how it may be extended (e.g. by half-transforming considering more complicated amplitudes in momentum space). For the remainder of this paper, we discuss some of the conceptual implications of figure 2, specifically regarding locality of the Weyl double copy.

4 Why is the type-D Weyl double copy local in position space?

In the previous section, we have shown that the twistor version of the type-D Weyl double copy can be straightforwardly obtained by half-transforming three-point amplitudes from momentum space. As well as justifying the use of twistor methods in studying the double copy, this allows us also to examine certain conceptual questions regarding exact classical double copies. As remarked above, the fact that the exact position-space double copies are possible is puzzling, given that the traditional BCJ double copy for scattering amplitudes is set up in momentum space. In fact, the construction of figure 2 extends this interesting phenomenon, given that the momentum-, twistor- and position-space double copies are all related by integral transforms. Each one of these transforms is non-local, and yet the type-D Weyl double copy is manifestly local in all three spaces, involving products of quantities evaluated at the same point. How can this possibly be true?

4.1 Locality in twistor space

The first step in answering these questions is to consider the half transform of momentum-space amplitudes into twistor space. The BCJ double copy in momentum space is multiplicative, in that the amplitudes entering eq. (3.13) for different theories are related by

$$\mathcal{A}_+^{\text{grav.}} = \frac{\mathcal{A}_+^{\text{EM}} \mathcal{A}_+^{\text{EM}}}{\mathcal{A}_+^{\text{scal.}}} \tag{4.1}$$

This multiplicative structure survives upon including the energy dependence from eq. (3.13) i.e. the overall power of ω . To go to twistor space in each theory, we must use a Laplace transform in the energy, as discussed in eqs. (3.36), (3.38). But this then creates the puzzle of why the twistor-space representatives are related by the simple product of eq. (2.27), rather than the convolution one expects upon taking the Laplace transform of a product.

The resolution of this puzzle lies in the very particular form of three-point amplitudes in momentum space. The product in twistor space will emerge provided that the amplitudes (and related energy factors) in momentum space are such that their convolution is equivalent to a product of similar functions. Clearly this is not true for most functions, but it does happen to be true for pure power-like functions of energy. Considering

$$f(\omega) = \omega^\alpha, \quad g(\omega) = \omega^\beta, \tag{4.2}$$

one has

$$f(\omega) \star g(\omega) \equiv \int_0^\omega du f(u)g(\omega - u) = \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} \omega^{\alpha+\beta+1}. \tag{4.3}$$

That is, the convolution of two power-like functions is also a power-like function, up to an overall numerical factor. Denoting the three-point amplitude for a spin- n emission together with its accompanying energy factor by

$$\tilde{\mathcal{A}}^{(n)} = \omega^n \tilde{\mathcal{A}}_+^{(n)}, \tag{4.4}$$

we may then write the gravity case of eq. (3.32) as

$$\begin{aligned} \mathfrak{M}_{2,+}(Z^\alpha) &\propto \int_0^\infty e^{-\omega U} \tilde{\mathcal{A}}^{(1)} \tilde{\mathcal{A}}^{(1)} [\tilde{\mathcal{A}}^{(0)}]^{-1} \\ &\propto \int_0^\infty e^{-\omega U} \tilde{\mathcal{A}}^{(1)} \star \tilde{\mathcal{A}}^{(1)} \star [\tilde{\mathcal{A}}^{(0)}]^{-1}, \end{aligned} \tag{4.5}$$

where U has been defined in eq. (3.36), and the second line follows from eq. (4.3) and associativity of the convolution. The convolution theorem then implies that the gravitational twistor space representative is given by

$$\mathfrak{M}_{2,+}(Z^\alpha) \propto \frac{\mathcal{L}[\tilde{\mathcal{A}}^{(1)}] \mathcal{L}[\tilde{\mathcal{A}}^{(1)}]}{\mathcal{L}[\tilde{\mathcal{A}}^{(0)}]} \propto \frac{\mathfrak{M}_{1,+}(Z^\alpha) \mathfrak{M}_{1,+}(Z^\alpha)}{\mathfrak{M}_{0,+}(Z^\alpha)}, \tag{4.6}$$

which is precisely the twistor-space product of eq. (2.27). Note that the power-like functions in eqs. (4.2), (4.3) are not the only possibilities that lead to a product in twistor space. One may also perform a *frequency shift* in the Laplace transform, according to eq. (3.38).

Combining the latter with the convolution theorem, it is easy to prove that the frequency shift operation commutes with a convolution:

$$\mathcal{L}[e^{\omega V} f(\omega) \star g(\omega)](U) = \mathcal{L} \left[\left(e^{\omega V} f(\omega) \right) \star \left(e^{\omega V} g(\omega) \right) \right] (U). \quad (4.7)$$

The Kerr-Taub-NUT twistor representative of eq. (3.34) is obtained by frequency shifting eq. (4.5) (as well as multiplying by a constant factor):

$$\mathfrak{M}_{2,+} = e^{\theta} \mathcal{L}[e^{\omega V} \tilde{\mathcal{A}}^{(1)} \star \tilde{\mathcal{A}}^{(1)} \star (\tilde{\mathcal{A}}^{(0)})^{-1}]. \quad (4.8)$$

Using eq. (4.7), this is equivalent to shifting each amplitude combination *before* taking the convolution:

$$\mathfrak{M}_{2,+} = \mathcal{L}[(e^{\omega V + \theta} \tilde{\mathcal{A}}^{(1)}) \star (e^{\omega V + \theta} \tilde{\mathcal{A}}^{(1)}) \star (e^{\omega V + \theta} \tilde{\mathcal{A}}^{(0)})^{-1}], \quad (4.9)$$

In twistor space, this means that the product form of eq. (4.6) remains the same, even for the full Kerr-Taub-NUT solution. The procedure of eq. (3.33), that relates the amplitudes for the Kerr-Taub-NUT solution to those generating pure Schwarzschild, is a momentum-space counterpart of the well-known *Newman-Janis shift* for the corresponding classical fields [78]. Here we see a novel interpretation of this shift, namely that it acts as a frequency shift in the energy Laplace transform of the momentum-space amplitude, whose consequence is to ensure locality of the double copy in twistor space!

In summary, the twistor double copy for type-D solutions involves a local product of cohomology representatives because: (i) these cohomology representatives can be obtained as a half transform (equivalent to a Laplace transform in energy) of momentum-space amplitudes; (ii) the form of the amplitudes is precisely such that their product in momentum space is equivalent to a convolution. Next, let us consider why locality in twistor space implies locality in position space.

4.2 Locality in position space

As we have seen above, locality of the double copy in momentum space implies locality in twistor space given the specific form of three-point amplitudes, and also the mathematical properties of the half transform, which is equivalent to a Laplace transform in energy. The map between twistor space and position space also involves some non-locality, although the nature of the integral transform is different, as is clear from figure 2. Thus, a different argument is needed to explain why the type-D Weyl double copy is local in position space, given locality in twistor space.

First, let us remind ourselves that all of the Čech cohomology representatives for type-D solutions involve inverse powers of a quadratic form in the twistor variable Z^α . This implies the presence of two poles in twistor space, that will appear on the Riemann sphere X corresponding to each spacetime point $x^{AA'}$ once the appropriate incidence relation of eq. (2.17) is imposed. The Penrose transform is a contour integral that will pick out the residue of one of these poles, which amounts to the vanishing of a twistor function. This has a nice interpretation in position space, due to the following result known as the *Kerr*

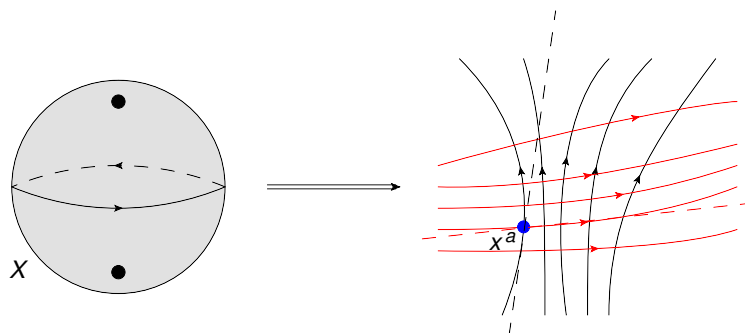


Figure 5. The poles of a twistor “function” $f(Z^\alpha)$ define shear-free null geodesic congruences in spacetime. For a given spacetime point, carrying out the Penrose transform on the Riemann sphere X in \mathbb{PT} picks out the null directions at a single spacetime point (shown in blue). One thus obtains a local product of principal spinors in spacetime, even though the twistor product implies a non-local statement in position space.

Theorem (see e.g. ref. [54] for an extended discussion, and refs. [79, 80] for a complementary application of the Kerr theorem to understanding the classical double copy):

given a holomorphic, homogeneous twistor function $\chi(Z^\alpha)$, the requirement $\chi(Z^\alpha) = 0$ defines a null shear-free geodesic vector field in Minkowski space.

The two poles in the inverse quadratic form for type-D solutions thus imply the presence of two null shear-free geodesic vector fields in spacetime, which is indeed a characteristic feature of type D solutions. Note that the Kerr theorem applies for all spacetime points simultaneously, given that it applies to the full twistor function entering the Penrose transform, *before* restriction to a given spacetime point $x^{AA'}$. This situation is depicted in figure 5, where on the right-hand side we draw two null shear-free vector fields in spacetime. In twistor space, restriction to a given spacetime point leads to a particular Riemann sphere X , corresponding to the blue point x^a on the right-hand side of the figure. The poles of the general twistor function, upon restriction to this Riemann sphere X , correspond to fixed points in projective twistor space \mathbb{PT} . As discussed in section 2, points in \mathbb{PT} correspond to null geodesics in Minkowski space. The latter will be tangent to the null shear-free vector fields generated by the general twistor function corresponding to a given pole, as shown on the right-hand side of the figure. These null directions are in one-to-one correspondence with the principal spinors α_A, β_A of the spacetime field at the point x^a .

The twistor double copy for type-D solutions states that one may combine the representatives of scalar and gauge fields of eq. (2.29) in order to obtain the gravity representative of eq. (2.28). This does not change the location of the poles in twistor space, such that all of the spacetime fields entering the correspondence have the *same* pair of null shear-free geodesic vector fields associated with them. In the spinor language, this means that the spacetime fields have the same pair of principal spinors $\alpha_A(x)$ and $\beta_A(x)$, such that only their multiplicity differs between theories. The type-D Weyl double copy of eq. (2.15) then simply amounts to the statement that the multiplicity of the principal spinors of a gravity solution can be simply obtained by appropriately combining the principal spinors of gauge

and scalar fields. This is both a non-local and a local statement. It is non-local in that it is a statement about *directions*, and thus entire null geodesics associated with a given spacetime point. This is precisely the non-locality one expects upon transforming a local product in twistor space into a space-time statement. However, the Weyl double copy is local in that it refers to the principal spinors at a given spacetime point, which are associated with the null tangent directions on the right-hand side of figure 5. The null directions are potentially different at all points in spacetime, and this is reflected in the Weyl double copy by the fact that it applies point-by-point in spacetime.

4.3 Beyond type-D solutions

In the previous two sections, we have used the construction of figure 2 to argue why the type-D Weyl double copy is local in position space. The intermediate twistor step is useful in this regard, as it provides another layer of information that allows us to visualise properties of spacetime fields geometrically. It also allows to address when the simple properties embodied by the type-D Weyl double copy might fail. We will exclude the case of non-vacuum solutions, for which the techniques of this paper — which heavily rely on the Penrose transform for massless free fields — do not apply. Indeed, ref. [49] provided an example of a non-vacuum solution which indeed does not have a local position-space double copy. It was also not a pure gravity solution, due to a non-zero dilaton field.

Assuming that the “true” double copy is in momentum-space [1, 2], the results of section 4.1 tell us that locality in twistor space is expected to fail for those amplitudes which are not proportional to pure exponentials in the energy. In that case, the Laplace transform to twistor space will involve non-power-like functions, such that a more complex structure in twistor space is obtained, rather than the simple product that is needed to reproduce the Weyl double copy. Thus, local twistor-space double copies will not be obtained in general beyond linearised level, given that the appropriate amplitudes in momentum space will have non-trivial momentum dependence, including poles in Mandelstam invariants.

Even if restricting to linearised level, one can ask if the simple form of the Weyl double copy of eq. (2.15) is generic for arbitrary (approximate) Petrov types. In section 4.2, we used the Kerr theorem to argue that the Penrose transform will automatically lead to a local position-space double copy, for scalar, gauge and gravity fields that share the same poles in twistor space (or principal spinors in spacetime). Scrutiny of this argument reveals that it only depends on there being a pair of common poles in the twistor representatives for the fields. Allowing the multiplicity of these poles to be different to that in the type-D case, one may obtain type III or N solutions, which (from table 1) all have at most a pair of distinct spinors associated with them. Indeed, examples of such linearised double copies have been given in refs. [25, 26]. Where more than two poles are present in twistor space, the situation is more complicated. Performing the Penrose transform integral means that one must take the residue of more than one pole in twistor space. The resulting spacetime gravity field is given by a sum of Weyl double-copy-like terms, such that the total principal spinors of the gravity field are not necessarily easily related to those of the constituent gauge and scalar fields [26]. Thus, the simple form of the Weyl double copy is indeed

highly special, and is not expected to be true in general for either non-linear fields, or linear fields that have more than two principal null directions.

5 Discussion

In this paper, we have explored the issue of why the well-known *Weyl double copy* relating fields in scalar, gauge and gravity theories, is local in position space. This question arises given that the original BCJ double copy for scattering amplitudes [1, 2] is local in momentum space. Although recent works have shown that mathematical properties of Fourier integrals from momentum to position space indeed imply locality for some solutions [49], we have here sought a more underlying explanation. To this end, we have used the ideas of ref. [59], that say that one may split the Fourier transform from momentum to position space into two steps. The first takes three-point amplitudes into twistor space, which we have shown leads to the twistor double copy of refs. [25, 26]. The second step is a Penrose transform, which produces the Weyl double copy for type D solutions. By using known three-point amplitudes in momentum space, we have shown that, for type-D solutions where relevant amplitudes are known, the BCJ, twistor and Weyl double copies amount to the same thing.

As a byproduct, our analysis resolves a lingering puzzle in the twistor double copy, which involves products of “functions” in twistor space. These functions should actually be interpreted as representatives of cohomology classes, and one must then provide a prescription for picking out special representatives. Various such procedures have been given in the literature [28, 58], but none of them obviously corresponds to the twistor double copy of refs. [25, 26]. In this paper, we have shown that the required representatives in twistor space are precisely those picked out by three-point amplitudes! This observation may prove very useful in extending the use of twistor methods in the double copy.

The mechanism by which the type-D classical double copy inherits locality in position space is interesting. First, the known amplitudes corresponding to type-D fields in spacetime are such that their convolution is equivalent to a product of similar functions. Thus, the “half transform” that takes amplitudes into twistor space implies the presence of a local product in twistor space. This criterion will fail beyond linearised level, confirming that local position space double copies are highly special.

Secondly, the Penrose transform from twistor to position space has both a local and a non-local character. The Weyl double copy is a statement about *directions* (principal spinors), which are non-local objects, in keeping with the fact that points in twistor space are associated with null geodesics in position space. However, the principal spinors of a field are different at different spacetime points in general, so that the local information in the Weyl double copy is simply that the principal spinors of gravity fields are obtained from their gauge and scalar counterparts point-by-point in spacetime. The simple nature of the Weyl double copy is restricted to those solutions that share the same pair of poles in twistor space, and hence have only two distinct principal spinors. For type II or type I fields, this will no longer be true.

We hope that our results clarify the nature of exact position space double copies, and confirm their rigour where applicable. We believe our results also suggest that further use of twistor ideas will prove fruitful in clarifying other aspects of the double copy correspondence, which continues to fascinate and intrigue in equal measure.

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