# Essays in Economic Theory 

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Abstract<br>Essays in Economic Theory<br>Suneil Parimoo

This dissertation looks at models in which agents make decisions under various economic frictions, and examines the role of their preferences. The first two chapters analytically characterize an infinite-horizon open economy under the friction of a stock collateral constraint, whereby borrowing is limited by the value of capital assets available. The model that is considered allows for general subjective discounting of households and fully variable productivity. The third chapter looks at a model of an ambiguity-averse benevolent mediator tasked with choosing a price contract at which a risk neutral buyer and seller transact an indivisible good under the friction of unquantifiable uncertainty of their reservation values.

The first chapter establishes that it is possible for households to enjoy the allocation they would obtain absent a stock collateral constraint under a condition that relates to their patience; this condition requires a long-run depression when agents are impatient relative to the market, and allows for an economic expansion when agents are more patient relative to the market. When this condition is not met, households are tightly constrained at least once and experience debt deleveraging in all periods and deflation of asset prices in periods preceding the constrained period relative to their unconstrained allocation. Households also ration their consumption more when they expect to be more tightly constrained in the future.

The second chapter is a sequel to the first chapter and shows that under constant output, agents who are impatient relative to the market can face two and three-period cycles in
consumption, debt, and asset prices. Further, large initial debt can lead to multiple equilibria. The third chapter considers a mediator who plays a Stackelberg game against Nature to maximize the distributionally worst-case expected weighted Nash product subject to known mean and boundary constraints on buyer and seller reservation values. We study the role of bargaining power and show that relative to what the buyer and seller themselves would choose when equipped with the mediator's information, the mediator's price contract has a shallow dependency on bargaining power, which is only exacerbated by the possibility of dependent buyer and seller values. Comparative statics results are obtained.

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## Dedication

To my guru, who is a source of inspiration and wisdom.
sukhārthī vā tyajedvidyāṃ vidyārthī vā sukham tyajet।I
"The seeker of ease must give up knowledge, and the seeker of knowledge must give up ease." (Mahabharata 5.40.5)

# Chapter 1: Open Economies with Stock Collateral Constraints Under General Impatience 

### 1.1 Introduction

This paper analytically studies a model of an open economy with a stock collateral constraint, whereby external debt is restricted by the collateralizable value of capital assets. Stock collateral pledging plays an important role in financing activities of firms, and more broadly, in business cycles. Its cyclical role has been referred to as the "collateral channel" (Gan 2007, Chaney, Sraer, and Thesmar 2012), a vicious circle whereby a business downturn deteriorates asset values, thus reducing borrowing capacity, leading to depressed investment, further exacerbating the downturn (Figure 1.1). ${ }^{1}$ The collateral channel has been regarded as a suspect for the severity of the Great Depression (Bernanke 1983). Conversely, its virtuous circle counterpart, whereby an economic boom leads to increased investment, has been cited as contributing to the expansion of the Japanese economy at the end of the 1980s (Cutts 1990). In addition to its role in amplifying business cycles, can asset collateral pledging in a debtor open economy be an engine in generating business cycles in the first place? Can asset collateral pledging engender self-fulfilling crises, inducing households to pessimistically choose a "bad" cycle over a "good" one? Further, what role does household patience play in such an economy?

The model considered here is a generalization of one that is presented in Schmitt-Grohé and Uribe 2017a (henceforth SGU), which in part approaches the question of self-fulfilling crises and considers the amplificatory role of stock collateral constraints. SGU focus on the effects of onetime deterministic initial shocks to an otherwise constant output on equilibria in a perfect-foresight

[^0]

Figure 1.1: Collateral channel
model. In their setup, SGU show that under relatively small negative shocks to productivity-what they refer to as "regular shocks"-the economy can be sustained in a steady-state equilibrium without being affected by the collateral constraint. Small shocks induce households to borrow internationally to smooth consumption, the equilibrium price of capital unaffected. In contrast, they show that shocks that are sufficiently large-surpassing the threshold defined by "regular shocks"-put households up against their borrowing constraint in at least one period. When up against the collateral constraint at least once, agents are shown to initially deleverage (reduce debt) and contract their initial consumption relative to the steady-state, while experiencing Fisherian deflation (a reduced price of capital) relative to the steady state in all periods prior and up to the first period in which they are up against the constraint. The other key finding of SGU is that open economies may be vulnerable to nonfundamental (or sunspot) shocks in the sense that poor fundamentals (including sufficiently high initial debt) can support both the steady-state equilibrium that is free from the collateral constraint as well as a welfare-inferior equilibrium where households are up against their borrowing constraint at the outset and thereafter enjoy a constraint-free steadystate allocation.

The approach of this paper is to explore equilibrium dynamics of the SGU model under two primary extensions: 1) a more general impatience assumption, and 2) fully variable output (sub-
ject to having finite net present value). In approaching the role of patience, we define the relative patience ratio $\delta:=\beta(1+r)$, which is the ratio of the subjective discount factor of households, $\beta \in(0,1)$, to the market discount factor under a constant interest rate $r>0, \frac{1}{1+r}$. While the SGU model makes a standard simplifying assumption that $\delta=1$, we allow for more general $\delta$, with the focus of this paper being the case when $\delta<1$. That is, agents are permitted to subjectively discount the future relatively more than the market, in which case they are said to be impatient relative to the market. In fact, many results of this paper apply to $\delta \geq 1$ as well, although to ease exposition, the default assumption for much of this paper (sections 1.2-2.2) is $\delta<1$, with the case $\delta \geq 1$ discussed in section 2.3. This extension that permits general impatience relative to the market is certainly of theoretical interest, but it is also of empirical interest in light of calibrations that feature $\delta<1$ (c.f. Bianchi 2011 and Ottonello 2021). Further, we shall see $\delta<1$ is key to explaining the emergence of business cycles. Absent any collateral constraint (which we refer to as the unconstrained model), when $\delta=1$, agents enjoy constant consumption, thus perfectly smoothing their consumption profile. In contrast, when $\delta<1$, absent any collateral constraint, agents frontload their consumption through borrowing internationally via the current account, allowing their future consumption to diminish at the rate of the relative patience ratio, approaching zero in the long run (a property we refer to as eventual starvation). Further, absent the collateral constraint, in the long run, such impatient households approach their limiting natural debt limit, an exogenous upper bound on borrowing equivalent to the net present value of output that is independent of the collateral constraint. Given this dynamic nature of debt insofar as its tendency to approach the natural debt limit under relative impatience, the second key extension we allow for, namely fully general-instead of constant-output, enters as a reasonable complement to the model since output controls the natural debt limit.

In allowing for these extensions, the methods we use in studying model equilibria are different from those pursued in SGU. While SGU generally use the recursive-form equilibrium conditions directly, which are often aptly visualized in terms of fixed-point iteration in a state space, we aim to obtain closed-form solutions throughout where possible. After recapping the SGU model in
section 1.2.1, we first show in section 1.2.2 how the model equilibria can be explicitly characterized in terms of the path of a single endogenous variable, namely the shadow value associated with the collateral constraint. This novel characterization supplies a powerful framework that applies under general $\delta$ and facilitates a straightforward approach to obtain stronger results than those in SGU. In section 1.3, we solve for the steady-state equilibrium. SGU show that the steady-state equilibrium when $\delta=1$ has households always free from the collateral constraint, which is always satisfied under the aforementioned "regular shocks" condition. In contrast, we show that when $\delta<1$, the steady-state equilibrium has households always up against the constraint and exists under knife-edge initial debt (admitting other parameters as given). In section 1.4, we solve for the unconstrained equilibrium, the equilibrium of the unconstrained model (the model without the collateral constraint). A key contribution of this paper is that under $\delta<1$, households can enjoy the allocation of the unconstrained equilibrium (which we refer to as the eternally slack equilibrium when it is supported by the full model with the collateral constraint) under a generalization of the "regular shocks" condition. This more generalized condition requires that the economy should endure a depressionary scenario in the long run, with productivity tending to zero over time. In this respect, allowing for fully variable output is clearly useful for appreciating the equilibrium dynamics. The intuition of this generalized condition is that when households expect to face a long-run depression, they are incentivized to deleverage over time and possibly switch to an asset (lending) position to fund their frontloaded consumption. This deleveraging behavior protects them against the collateral constraint. In the event households assume lending, as consumption diminishes over time, households require less of the asset position over time, and thus diminish their lending to nil in the long run, all the while free from the collateral constraint. Thus, under $\delta<1$, we have two benchmark equilibria: the steady-state equilibrium in which households are ever up against their collateral constraint, and the eternally slack equilibrium in which they are ever free from their constraint.

With these benchmark equilibria in hand, we show in section 1.5 how the key properties obtained in SGU of initial consumption rationing, initial deleveraging, and Fisherian deflation-all
relative to the unconstrained equilibrium (instead of the steady-state equilibrium) - apply under general $\delta$. These properties are obtained using the equilibria characterization from section 1.2.2. This analysis provides more insight than do recursive-form equilibrium conditions, and hence allows for a strengthening of the SGU results. In section 1.5.1, we show how Fisherian deflation occurs relative to the unconstrained equilibrium in all periods during or preceding those in which households are up against the collateral constraint, while no such deflation occurs in a period if households expect to never be up against their constraint. In section 1.5.2, we discuss the dynamics of consumption. Agents are shown to feature a rationing property, whereby the less tightly constrained they were in the past or the more tightly constrained they expect to be from today onward, the more they plan to ration today. This analysis allows us to obtain an upper bound on consumption: it can never exceed the future value of the initial consumption of the unconstrained equilibrium. A special case of this result is initial consumption rationing relative to the unconstrained equilibrium, as obtained in SGU. In section 1.5.3, we show how deleveraging occurs in all periods —not just the initial period as predicted in SGU—relative to the unconstrained equilibrium when households are up against the collateral constraint at least once.

Section 1.5.4 considers long-run equilibrium behavior and shows how under reasonable output paths, it is not possible for the shadow value of the collateral constraint to be persistently too large or persistently too small in the long run as large shadow values in the long run would be accompanied by agents taking a long-term asset position to fund growing consumption (thus slackening the constraint), while small shadow values would be accompanied by eventual starvation and are only supported under long-run depressionary output. Indeed, the eternally slack equilibrium is one special case of this latter behavior where the shadow value is always zero. In section 2.1 , we study when equilibria can take the form of periodic cycles. Section 2.1 . 1 characterizes cycles of arbitrary periodicity. The prospect of cycles emerges due to both the friction of the collateral constraint as well as impatience relative to the market. In particular, the constraint introduces a shadow cost of borrowing needed to fund consumption and thus induces households to push forward consumption, while impatience relative to the market induces households to frontload consumption; the compe-
tition between these two forces thus allows for the possibility of cycles. Section 2.1.2 examines the special case of 2-cycles, and we show that under plausible parametrizations, any kind of 2-cyclic equilibrium can exist under a constant output trend. Section 2.1.3 obtains a particular 3-cyclic equilibrium, and considers the prospect of cycles of higher periodicity and chaos. In section 2.2, we look at when multiple equilibria can coexist. Section 2.2.1 considers variable output regimes, demonstrating an extension of a result in SGU, namely coexistence of the eternally slack equilibrium and a welfare-inferior one in which households are initially borrowing-constrained. Section 2.2.2 considers constant output regimes, and shows how different 2 -cyclic equilibria can coexist with each other and with the steady-state equilibrium. In this sense, our model shows that a nonfundamental shock can trigger the collateral channel, precipitating a choice of a "bad equilibrium" associated with the effects in Figure 1.1. Households may choose a welfare-inferior equilibrium due to the fact they do not internalize the effect of their demand for capital on the market price for capital, thus allowing room for macroprudential policy intervention. Section 2.3 discusses how the results from preceding sections extend to $\delta \geq 1$. Section 2.4 concludes.

## Related literature

This work contributes to a growing literature on open economies with collateral constraints. Often, a fundamental problem in open economy macroeconomic models is aptly constructing a debtor economy as a means of explaining emerging countries with substantial external debt that we observe in the data. Such models that incorporate a financial friction (such as a collateral constraint) and impatience relative to the market $(\delta<1)$ have the desirable feature of inducing nontrivial borrowing behavior along a balanced growth path. In contrast, closed economy models generally do not seek to explain external borrowing behavior (the model economy having a nil current account), and as such, many closed economy models adopt the assumption $\delta=1$ as featured in real business-cycle models in the tradition of Kydland and Prescott 1982. Consequently, these differing models posit quite distinct explanations for the emergence of business cycles: While Kydland and Prescott 1982 explain how cycles arise from productivity shocks, this paper shows how
cycles arise from the forces of a stock collateral constraint and impatience relative to the market when productivity is constant.

The friction of stock collateral constraints has been analytically treated in Kiyotaki and Moore 1997, which departs from the Kydland and Prescott 1982 tradition by introducing a closed economy model (with endogenous interest rate) with risk neutral lenders and borrowers (the latter assumed to be impatient relative to the market) and shows how the collateral constraint amplifies shocks, in that temporary shocks to income can result in persistent fluctuations in asset prices. ${ }^{2}$ Subsequent theory work on stock collateral constraints has built on this model, ${ }^{3}$ or has obtained this amplification result in a simplified small open economy setting that assumes $\delta=1$, as in SGU (c.f. also Kocherlakota 2000). Often, the interest in such work is studying the effect of transitory shocks to an economy in steady state, and the analytical approach that is typically adoptedparticularly in looking at model simulations- involves looking at linearized approximations about the steady state. Jeanne and Korinek 2019 consider a stochastic model that is more similar to the one presented in this paper, and they focus on the case where $\delta<1$ under a stochastic binomial trend in output in their calibration. Methodologically, they offer a heuristic analysis to determine conditions for equilibrium uniqueness, a counterpoint to this paper's aim of searching for multiple equilibria. ${ }^{4}$ In comparison to many of these papers, our work builds on a fairly tractable model while more completely exploring the role of household patience and providing a precise analytical characterization of equilibria in terms of the path of the constraint's shadow value. This characterization may be of methodological service to related theory work on collateral constraints.

A cousin to the class of models with stock collateral constraints in this literature is one with flow collateral constraints, where debt is restricted by income or output. ${ }^{5}$ Relatively recent progress

[^1]on flow constraints includes Schmitt-Grohé and Uribe 2021 and Schmitt-Grohé and Uribe 2020 among others, both of which analytically explore a model under flow collateral constraints that is quite similar to the SGU model. The former work considers $\delta<1$ and fully characterizes a debt policy function, establishing the existence of debt cycles in which periods of debt growth are followed by periods of debt deleveraging (c.f. also Aghion, Bacchetta, and Banerjee 2004), akin to our result on deterministic cycles; ${ }^{6}$ however, while they only show cycles in which debt oscillates above and below its steady-state level, we allow for the full debt policy correspondence and show cycles in which debt never exceeds its steady-state level. The latter work considers $\delta=1$ and shows the existence of multiple equilibria (c.f. also Mendoza, Bergoeing, and Roubini 2005, which considers conditions for equilibrium uniqueness), similar to the result discussed in this paper; however, they focus on coexistence between an equilbirum with an ever slack constraint and one where the constraint initially binds, while we consider this coexistence as well as coexisting cycles. While some insights thus apply to both models with stock and flow collateral constraints, conceptually, the former class of models is analytically more challenging in a certain sense. The challenge arises from the fact that such flow collateral constraint models restrict debt according to exogenous income or output, while stock collateral constraint models restrict debt according to an endogenous level of assets and effectively incorporate an additional Euler equation. Nonetheless, by abstracting away from uncertainty and other complications, our model allows for a fruitful exploration of equilibrium properties while uniting results from the "stock" and "flow" strands of the collateral constraint theory literature.

While this work is of theoretical interest on its own, it is a serviceable companion to the emerging quantitative literature on collateral constraints. For instance, Mendoza $2010^{7}$ considers a

[^2]stochastic model calibrated to Mexico data and finds a low sensitivity of the standard deviation of output to the model both with and without a stock collateral constraint, suggesting such constraints do not amplify regular business cycles. However, under large negative output shocks, Mendoza finds that a binding constraint exacerbates aggregate dynamics. Devereux, Young, and Yu 2016 work with a stock constraint model calibrated under $\delta<1$ to ensure net borrowing behavior in the steady state, and focus more on the normative question of the optimal policy to internalize the pecuniary externality ${ }^{8}$ of the model, while quantifying welfare effects. While flow constraint models may not be particularly suited for describing the aforementioned collateral channel effects on real business cycles, both stock and flow models share this pecuniary externality feature, and there has been extensive theoretic and quantitative work on this topic along with the role of macroprudential policy in addressing this friction. ${ }^{9}$ Such quantitative work on policy questions premised on collateral constraint models would proceed in largely a conjectural manner absent a full analytical characterization of model equilibria. In this context, this paper's exact approach in studying equilibrium properties provides a theoretic foundation to facilitate further quantitative work.

### 1.2 Model

### 1.2.1 Setup and definition of equilibrium

In this section, we summarize the SGU model and our definition of an equilibrium under general impatience and variable output. The model is characterized by a perfect-foresight small open economy with a large number of infinitely-lived homogeneous households with intertemporally separable log-utility preferences, with lifetime utility at time 0 given by

$$
\sum_{t=0}^{\infty} \beta^{t} \log c_{t},
$$

[^3]where $c_{t}$ denotes consumption and $\beta \in(0,1)$ is the subjective discount factor of households. Households face a sequential budget constraint given by
$$
\left(\mathrm{SBC}_{t}\right) \quad c_{t}+d_{t}+q_{t}\left(k_{t+1}-k_{t}\right)=y_{t}+\frac{d_{t+1}}{1+r}
$$
where $r>0$ is the interest rate on international debt $d_{t}$ (acquired in period $t-1$ and due in period $t$ ), $k_{t}$ is the stock of physical capital at time $t, q_{t}$ is the price of capital (taking consumption as the numeraire) at time $t$, and $y_{t}$ is output produced according to Cobb-Douglass technology
$$
\left(\mathrm{TECH}_{t}\right) \quad y_{t}=A_{t} k_{t}^{\alpha},
$$
where $A_{t} \geq 0$ is an exogenous deterministic productivity factor, and $\alpha \in(0,1)$ is a technology parameter controlling the marginal product of capital. Borrowing is constrained by a stock collateral constraint according to
$$
\left(\mathrm{CC}_{t}\right) \quad d_{t+1} \leq \kappa q_{t} k_{t+1},
$$
where $\kappa \in(0,1)$ is the fraction of the value of the capital stock pledged as collateral for the debt obligation due the following period. ${ }^{10}$ Households are atomistic, treating the price of capital, $q_{t}$, as exogenous, though endogenous to the economy overall in equilibrium, thus inducing a pecuniary externality. In other words, an increase in market demand for capital serves to boost $q_{t}$, allowing agents to borrow more, while a fall in market demand for capital serves to drive down $q_{t}$, which may force deleveraging. However, households do not internalize this mechanism as they correctly understand their individual demand for capital has a negligible impact on its price.

[^4]The trade balance at time $t, t b_{t}$, is defined as the gap between output and consumption,

$$
t b_{t} \equiv y_{t}-c_{t},
$$

and the current account at time $t, c a_{t}$, as the trade balance less interest paid on debt borrowed in the preceding period,

$$
c a_{t} \equiv t b_{t}-\frac{r}{1+r} d_{t} .
$$

The problem faced by households at time 0 is to choose sequences $c_{t}>0, k_{t+1} \geq 0$, and $d_{t+1}$ to maximize lifetime utility subject to $\mathrm{SBC}_{t}, \mathrm{TECH}_{t}$, and $\mathrm{CC}_{t}$ for all $t \geq 0$, admitting the sequences of prices $q_{t}$ and productivity factors $A_{t}$ along with initial endowments $k_{0} \geq 0$ and $d_{0}$ as given. The Lagrangian for the problem is given by

$$
\mathcal{L}=\sum_{t=0}^{\infty} \beta^{t}\left\{\log c_{t}+\lambda_{t}\left[A_{t} k_{t}^{\alpha}+\frac{d_{t+1}}{1+r}-\left(c_{t}+d_{t}+q_{t}\left(k_{t+1}-k_{t}\right)\right)\right]+\lambda_{t} \mu_{t}\left[\kappa q_{t} k_{t+1}-d_{t+1}\right]\right\},
$$

where $\beta^{t} \lambda_{t}$ and $\beta^{t} \lambda_{t} \mu_{t}$ are Lagrange multipliers respectively associated with $\mathrm{SBC}_{t}$ and $\mathrm{CC}_{t}$. The first-order conditions with respect to $c_{t}, d_{t+1}$, and $k_{t+1}$ are respectively given by

$$
\begin{gathered}
c_{t}^{-1}=\lambda_{t} \\
\lambda_{t}\left[(1+r)^{-1}-\mu_{t}\right]=\beta \lambda_{t+1},
\end{gathered}
$$

and

$$
\lambda_{t} q_{t}\left[1-\kappa \mu_{t}\right]=\beta \lambda_{t+1}\left[\alpha A_{t+1} k_{t+1}^{\alpha-1}+q_{t+1}\right] .
$$

The second first-order condition, the Euler equation for debt, equates the marginal benefit of $d_{t+1}$, which is $\lambda_{t}(1+r)^{-1}$ utility units from increased consumption today, with the marginal cost, which is the sum of $\beta \lambda_{t+1}$ utility units (owing to giving up a unit of consumption tomorrow) and $\mu \lambda_{t}$ utility units (reflecting a shadow punishment for increasing debt when the household is up against the collateral constraint). The third first-order condition, the Euler equation for capital, equates the
marginal cost of a unit of capital, which is $\lambda_{t} q_{t}$ utility units, with the marginal benefit, which is the sum of $\beta \lambda_{t+1}\left[\alpha A_{t+1} k_{t+1}^{\alpha-1}+q_{t+1}\right]$ (the present value of additional output it generates-or marginal product of capital - in the next period and the price at which it can be sold in the next period in utility terms) and $\lambda_{t} \kappa \mu_{t} q_{t}$ (reflecting the shadow benefit from relaxing the collateral constraint when the household is up against it).

Complementary slackness of $\mathrm{CC}_{t}$ implies

$$
\mu_{t} \geq 0, \quad \mu_{t}\left(\kappa q_{t} k_{t+1}-d_{t+1}\right)=0
$$

Further, SGU show that the nonsatiation of preferences in the household's optimization problem implies a terminal optimality condition: ${ }^{11}$

$$
\lim _{t \rightarrow \infty} \frac{d_{t+1}}{(1+r)^{t}}=\kappa \lim _{t \rightarrow \infty} \frac{q_{t} k_{t+1}}{(1+r)^{t}} .
$$

Aggregate supply of capital is assumed to be fixed for all time (e.g. a plot of land) at $k>0$, so that in equilibrium, market clearing requires

$$
k_{t}=k .
$$

Note that under a fixed capital supply, the price of capital directly reflects the market demand for capital. We also restrict ourselves to equilibria where the price of capital, $q_{t} \geq 0$, obeys a no-bubble condition,

$$
(\mathrm{NOBUB}) \quad \lim _{t \rightarrow \infty} \frac{q_{t}}{(1+r)^{t}}=0
$$

so that the terminal optimality condition and constant capital supply imply a no-Ponzi game condition,

[^5]$$
\text { (NPG) } \quad \lim _{t \rightarrow \infty} \frac{d_{t+1}}{(1+r)^{t}}=0
$$

The NPG condition along with forward iteration of $\mathrm{SBC}_{t}$ imply an intertemporal resource constraint, whereby exogenous initial debt is covered by the net present value of future trade balances:

$$
d_{0}=\sum_{t=0}^{\infty} \frac{t b_{t}}{(1+r)^{t}} .
$$

We define the relative patience ratio as $\delta:=\beta(1+r)$, which is the ratio of household-to-market discount factors. We focus on the case where households are impatient relative to the market: $\delta<1$. In fact, several analytic results that follow extend to $\delta \geq 1$ as well, but the economic interpretations are somewhat different. Hence, to ease exposition, we will maintain $\delta<1$ as the default assumption throughout sections 1.2-2.2 of this paper unless otherwise specified, leaving the discussion of the case where $\delta \geq 1$ to section 2.3. Also, since the relative patience ratio enters quite naturally into various expressions and since the restriction $\delta<1$ must be accounted for, we typically express $\beta$ in terms of $\delta$, with only a few exceptions for symbolic conciseness. From the above relations, we define the equilibrium:

Definition 1 A competitive bubble-free equilibrium (or equilibrium) is the sequence of vectors $\left(\left(c_{t}, \mu_{t}, q_{t}, d_{t+1}\right)\right)_{t \geq 0}$ having $c_{t}>0, \mu_{t} \geq 0, q_{t} \geq 0$, satisfying

$$
\begin{gather*}
d_{0}=\sum_{t=0}^{\infty} \frac{y_{t}-c_{t}}{(1+r)^{t}},  \tag{1.1}\\
(1+r)\left(c_{t}+d_{t}-y_{t}\right)=d_{t+1},  \tag{1.2}\\
\frac{c_{t+1}}{c_{t}}=\frac{\delta}{1-\mu_{t}(1+r)},  \tag{1.3}\\
\frac{1-\kappa \mu_{t}}{1-\mu_{t}(1+r)}(1+r) q_{t}-\frac{\alpha}{k} y_{t+1}=q_{t+1}, \tag{1.4}
\end{gather*}
$$

$$
\begin{gather*}
\mu_{t}\left(\kappa k q_{t}-d_{t+1}\right)=0,  \tag{1.5}\\
d_{t+1} \leq \kappa k q_{t}  \tag{1.6}\\
\lim _{t \rightarrow \infty} \frac{q_{t}}{(1+r)^{t}}=0, \tag{1.7}
\end{gather*}
$$

given $d_{0}$ and the exogenous sequences $\left(A_{t}\right)_{t \geq 0}$ and $\left(y_{t}\right)_{t \geq 0}$ with $y_{t}=A_{t} k^{\alpha}$.

In the absence of the collateral constraint, we refer to the model as the unconstrained model, in contrast to the general constrained model. We refer to an equilibrium in the unconstrained model as an unconstrained equilibrium, which obeys the exact same equilibrium conditions (1.1)-(1.7) excluding (1.6) and with $\mu_{t}=0 \quad \forall t \geq 0$. For brevity, we also refer to $\left\{\mu_{\tau}\right\}_{\tau \geq 0}$ variously as the Lagrange multipliers (or simply multipliers) or shadow values of the collateral constraint.

### 1.2.2 $\tilde{\mu}_{t}$ characterization of equilibria

In this section, we characterize equilibria in terms of the path of a single endogenous variable, namely the shadow value of the collateral constraint. We observe that the expression $1-\mu_{t}(1+r)$ appears in Definition 1. Moreover, from (1.3), we see that $c_{t}>0$ and $\mu_{t} \geq 0$ imply $1-\mu_{t}(1+r) \in$ $(0,1]$. Thus, $1-\mu_{t}(1+r)$ is simply a normalization of the multiplier $\mu_{t}$ that is confined to the unit interval and decreases in $\mu_{t}$. We hence define the following to ease the exposition of shadow values in the sequel.

Definition 2 The normalized multiplier in period $t$ is the quantity

$$
\begin{equation*}
\tilde{\mu}_{t} \equiv 1-\mu_{t}(1+r), \tag{1.8}
\end{equation*}
$$

whose range is given by

$$
\begin{equation*}
\tilde{\mu}_{t} \in(0,1] . \tag{1.9}
\end{equation*}
$$

The collateral constraint is tight in period $t$ when $\tilde{\mu}_{t}<1$ (equivalently, $\mu_{t}>0$ ). The collateral
constraint is slack in period $t$ when $\tilde{\mu}_{t}=1$ (equivalently, $\mu_{t}=0$ ).

With slight abuse, we will often refer to an equilibrium in terms of the normalized multipliers, i.e. as a sequence $\left(\left(c_{t}, \tilde{\mu}_{t}, q_{t}, d_{t+1}\right)\right)_{t \geq 0}$. Note that a slack constraint as given in Definition 2 permits the collateral constraint in (1.6) to either bind or hold with strict inequality in accordance with (1.5). We adopt this nomenclature throughout this paper for simplicity since the positivity of the shadow value $\mu_{t}$, rather than inequality of the constraint, is ultimately what materially affects equilibrium dynamics. Complementing Definition 2, we also describe the collateral constraint using qualitative terms such as "tighter" (resp. "less tight") to describe a higher (resp. lower) $\mu_{t}$, or equivalently, a lower (resp. higher) $\tilde{\mu}_{t}$.

We now proceed to characterize all endogenous variables in terms of $\left\{\tilde{\mu}_{t}\right\}_{t \geq 0}$. Recursive dependencies are eliminated by backward iteration of (1.3) (using (1.1) to determine $c_{0}$ ), and forward iteration of (1.2) and (1.4) (availing of the NPG and NOBUB terminal optimality conditions). Backward iteration of (1.3) yields ${ }^{12} \forall t \geq 0$

$$
\begin{equation*}
c_{t}=\left(\delta^{t} \prod_{\tau=0}^{t-1} \tilde{\mu}_{\tau}^{-1}\right) c_{0} \tag{1.10}
\end{equation*}
$$

By the same method used to obtain the equilibrium condition (1.1), NPG and forward iteration of (1.2) applied to debt at any time $t \geq 0$ imply

$$
\begin{equation*}
d_{t}=\sum_{\tau=0}^{\infty} \frac{y_{t+\tau}-c_{t+\tau}}{(1+r)^{\tau}} . \tag{1.11}
\end{equation*}
$$

Since $c_{\tau}>0 \forall \tau \geq 0$ implies $d_{t}<\sum_{\tau=0}^{\infty} \frac{y_{t+\tau}}{(1+r)^{\tau}}$ when the latter is well-defined, we define

$$
\begin{equation*}
d_{t}^{\mathrm{NDL}} \equiv \sum_{\tau=0}^{\infty} \frac{y_{t+\tau}}{(1+r)^{\tau}} \tag{1.12}
\end{equation*}
$$

as the natural debt limit at time $t$, an exogenous quantity that is simply the net present value of

[^6]output from time $t$ onward. We thus implicitly assume throughout this paper that the exogenous initial debt is less than the initial natural debt limit that is well-defined $\left(d_{0}<d_{0}^{\mathrm{DNL}}<\infty\right)$ to ensure a valid equilibrium. ${ }^{13}$ Note that we make no restriction on the sign of $d_{0}$; if $d_{0}>0$, then households are initially endowed as net debtors, while if $d_{0}<0$, they are initially endowed as net lenders. Substituting (1.10) into (1.1) determines initial consumption according to
\[

$$
\begin{equation*}
c_{0}=\left(d_{0}^{\mathrm{NDL}}-d_{0}\right)\left[\sum_{t=0}^{\infty} \beta^{t} \prod_{\tau=0}^{t-1} \tilde{\mu}_{\tau}^{-1}\right]^{-1} \tag{1.13}
\end{equation*}
$$

\]

Thus, consumption and debt are expressible in terms of $\left\{\tilde{\mu}_{t}\right\}_{t \geq 0}$. Moreover, since $d_{0}<d_{0}^{\mathrm{DNL}}$, we see the condition $\tilde{\mu}_{t}>0$ (equivalently, $\mu_{t}<(1+r)^{-1}$ ) holding in all periods is in fact equivalent to consumption being positive for all periods.

Likewise, forward iteration of (1.4) until some time $T \geq t$ implies

$$
\begin{equation*}
q_{t}=\left[\frac{\alpha}{k} \sum_{\tau=0}^{T-1} y_{t+1+\tau} \prod_{j=0}^{\tau} \frac{\tilde{\mu}_{t+j}}{1+r-\kappa+\kappa \tilde{\mu}_{t+j}}\right]+q_{t+T} \prod_{j=0}^{T-1} \frac{\tilde{\mu}_{t+j}}{1+r-\kappa+\kappa \tilde{\mu}_{t+j}} . \tag{1.14}
\end{equation*}
$$

Since the product $\prod_{j=0}^{T-1} \frac{\tilde{\mu}_{t+j}}{1+r-\kappa+\kappa \tilde{\mu}_{t+j}}$ is positive by (1.9) and strictly increasing in the normalized multipliers, ${ }^{14}$ it is maximized when the normalized multipliers are set to unity, and hence bounded above by $\prod_{j=1}^{T} \frac{1}{(1+r)}=(1+r)^{-T}$. Consequently, the NOBUB condition (1.7) implies the last term in (1.14) vanishes in the limit as $T \rightarrow \infty$, yielding

$$
\begin{equation*}
q_{t}=\frac{\alpha}{k} \sum_{\tau=0}^{\infty} y_{t+1+\tau} \prod_{j=0}^{\tau} \frac{\tilde{\mu}_{t+j}}{1+r-\kappa+\kappa \tilde{\mu}_{t+j}} . \tag{1.15}
\end{equation*}
$$

An equilibrium may have the collateral constraint slack for all periods, tight for all periods, or

[^7]tight for some periods. Having expressed all quantities in terms of the normalized multipliers, we may in principle solve for equilibria in terms of fundamentals by considering generic equilibrium candidates that feature tightness in various periods. If, for instance, we consider a candidate equilibrium with a tight collateral constraint for $M$ specifically chosen periods, then complementary slackness condition (1.5) applied to those periods gives $M$ binding constraints through which the $M$ positive Lagrange multipliers for those periods may be solved for, with the multipliers for the remaining periods set to nil (i.e. normalized multipliers set to unity). Once the multipliers and other endogenous quantities are solved for in terms of fundamentals, they will collectively comprise an equilibrium if the tight multipliers satisfy their presupposed constraints, namely that their normalized counterparts are in the open unit interval $(0,1)$, and if the collateral constraint is confirmed to hold for the periods that were presumed to have a slack constraint.

Analytically implementing this procedure for a generic equilibrium candidate to obtain closedform equilibrium expressions is unfortunately not tractable in general, as the equations that determine the multipliers comprise an $M$-dimensional nonlinear system. This nonlinearity may generate multiple equilibria having the same temporal pattern of constraint tightness, something we show in section 2.2. In general, however, the system is infinite-dimensional if we assume the constraint is tight infinitely often. ${ }^{15}$ Still, we can use this characterization to explicitly solve for simple kinds of equilibria or otherwise obtain insights on general properties of any equilibrium, which is our approach for the remainder of the paper.

In the sequel, the preceding equilibrium characterization is referred to as the $\tilde{\mu}_{t}$-characterization. Below are some useful facts and other terminology that will facilitate our discussion.

Fact 1 Given $\rho \in(0,1)$, a nonnegative real sequence $\left(x_{t}\right)_{t \geq 0}$ is bounded if and only if the sequence of its net present values $\left(\sum_{\tau=0}^{\infty} \rho^{\tau} x_{t+\tau}\right)_{t \geq 0}$ is bounded.

Proof.

$$
x_{t} \leq \bar{x} \Longrightarrow \sum_{\tau=0}^{\infty} \rho^{\tau} x_{t+\tau} \leq \sum_{\tau=0}^{\infty} \rho^{\tau} \bar{x}=\frac{1}{1-\rho} \bar{x}
$$

[^8]$$
\sum_{\tau=0}^{\infty} \rho^{\tau} x_{t+\tau} \leq \bar{X} \Longrightarrow x_{t} \leq x_{t}+\rho \sum_{\tau=0}^{\infty} \rho^{\tau} x_{t+1+\tau}=\sum_{\tau=0}^{\infty} \rho^{\tau} x_{t+\tau} \leq \bar{X}
$$

Fact 2 Given $\rho \in(0,1)$, a real sequence $\left(x_{t}\right)_{t \geq 0}$ converges to $x$ if and only if the sequence of its net present values $\left(\sum_{\tau=0}^{\infty} \rho^{\tau} x_{t+\tau}\right)_{t \geq 0}$ converges to $\sum_{\tau=0}^{\infty} \rho^{\tau} x=x /(1-\rho)$.

Proof. Define $X_{t}:=\sum_{\tau=0}^{\infty} \rho^{\tau} x_{t+\tau}$. If $x_{t} \rightarrow x$, then $\forall \epsilon>0 \exists T \geq 0$ s.t. $\forall t \geq T,\left|x_{t}-x\right|<(1-\rho) \epsilon$ so

$$
\left|X_{t}-\sum_{\tau=0}^{\infty} \rho^{\tau} x\right|=\left|\sum_{\tau=0}^{\infty} \rho^{\tau}\left(x_{t+\tau}-x\right)\right| \leq \sum_{\tau=0}^{\infty} \rho^{\tau}\left|x_{t+\tau}-x\right|<\sum_{\tau=0}^{\infty} \rho^{\tau}(1-\rho) \epsilon=\epsilon .
$$

If $X_{t} \rightarrow \sum_{\tau=0}^{\infty} \rho^{\tau} x$, then $\forall \epsilon>0 \exists T \geq 0$ s.t. $\forall t \geq T,\left|X_{t}-\sum_{\tau=0}^{\infty} \rho^{\tau} x\right|<\epsilon /(1+\rho)$ so $\left|x_{t}-x\right|=\left|X_{t}-\sum_{\tau=0}^{\infty} \rho^{\tau} x+\rho\left(\sum_{\tau=0}^{\infty} \rho^{\tau} x-X_{t+1}\right)\right| \leq\left|X_{t}-\sum_{\tau=0}^{\infty} \rho^{\tau} x\right|+\rho\left|\sum_{\tau=0}^{\infty} \rho^{\tau} x-X_{t+1}\right|<\frac{\epsilon}{1+\rho}+\rho \frac{\epsilon}{1+\rho}=\epsilon$.

Definition 3 Given sequence $\left(x_{t}\right)_{t \geq 0}$, define $x_{\infty} \equiv \lim _{t \uparrow \infty} x_{t}$ when the limit is well-defined. An equilibrium features eventual starvation whenever $c_{\infty}=0$ (equivalently $d_{\infty}=d_{\infty}^{\mathrm{NDL}}$ ). A constant output regime has positive constant output from period 1 onward: $\left\{y_{t+1}\right\}_{t \geq 0}=y>0$.

### 1.3 Steady-state equilibrium

As a first step to understanding equilibrium dynamics, we consider the steady-state equilibrium (SSE).

Definition 4 A steady-state equilibrium (SSE) is a constant equilibrium where $\left(c_{t}, \tilde{\mu}_{t}, q_{t}, d_{t+1}\right)=$ $\left(c^{*}, \tilde{\mu}^{*}, q^{*}, d^{*}\right) \forall t \geq 0$.

An SSE in the unconstrained model is naturally defined identically to that of the constrained model, with $\mu^{*}=0$ and excepting (1.6). Note that unless $\delta=1$, there can be no SSE in the unconstrained setting, since (1.10) implies any steady-state consumption must have $c^{*}=0$ when $\delta \neq 1$, which violates the requirement that $c^{*}>0$.

At this point, we solve for the SSE and obtain its existence conditions. While we may use the $\tilde{\mu}_{t}$-characterization to do so, it is straightforward to directly appeal to Definition 1. In an SSE, observe that (1.2) implies $y_{t+1}$ must be constant for $t \geq 0$ :

$$
\begin{equation*}
y_{t+1}=c^{*}+\frac{r}{1+r} d^{*}=: y, t \geq 0 . \tag{1.16}
\end{equation*}
$$

Then (1.1) implies

$$
d_{0}=y_{0}-c^{*}+\sum_{t=1}^{\infty} \frac{y-c^{*}}{(1+r)^{t}},
$$

and by (1.16) we have

$$
\begin{equation*}
y_{0}=d_{0}+c^{*}-\frac{1}{1+r} d^{*} . \tag{1.17}
\end{equation*}
$$

The simultaneous equations (1.16) and (1.17) give SSE values of consumption and debt in terms of output and initial debt:

$$
\begin{gather*}
d^{*}=y-y_{0}+d_{0},  \tag{1.18}\\
c^{*}=\frac{y+r\left(y_{0}-d_{0}\right)}{1+r} . \tag{1.19}
\end{gather*}
$$

Thus SSE debt is simply the sum of initial debt and the change in output from period 0 to period 1 , while SSE consumption is a weighted average between the output from period 1 onward, $y$, and the surfeit in period 0 after the initial debt obligation is satisfied, $y_{0}-d_{0}$. Note that from (1.16), we may equivalently write $c^{*}=y-\frac{r}{1+r} d^{*}$, a typical characteristic of open economy models in the steady state, which conveys that households consume their permanent income, given by the sum of nonfinancial income, $y$, and interest income, $-r d^{*} /(1+r)$.

Observe from (1.3), we obtain $\tilde{\mu}^{*}$ :

$$
\begin{equation*}
\tilde{\mu}^{*}=\delta, \tag{1.20}
\end{equation*}
$$

which implies that households must be tightly borrowing-constrained in an SSE. This result stands in contrast to the SGU model when $\delta=1$, under which the SSE has a slack constraint ( $\tilde{\mu}^{*}=1$ ).

However, this result is to be expected in an economy in which absent any collateral constraint, impatient households would frontload consumption, incur eventual starvation, and approach their natural debt limit (as detailed in section 1.4).

Substituting (1.20) into (1.4) yields the SSE price of capital:

$$
\begin{equation*}
q^{*}=\frac{\delta}{r+(1-\delta)(1-\kappa)} \frac{\alpha y}{k} . \tag{1.21}
\end{equation*}
$$

Note that since $\tilde{\mu}^{*}<1$, the complementary slackness condition (1.5) implies the collateral constraint binds, which, by (1.18), restricts the level of initial debt that can sustain an SSE:

$$
\begin{equation*}
d_{0}^{\mathrm{SSE}}=y_{0}-\left(1-\frac{\alpha \kappa \delta}{r+(1-\delta)(1-\kappa)}\right) y . \tag{1.22}
\end{equation*}
$$

Recall that $c^{*}>0$ requires that initial debt respects its natural debt limit, which will, in fact, automatically be satisfied under our parameter restrictions so long as $y>0$ :

$$
d_{0}^{\mathrm{SSE}}<y_{0}+\frac{1-r}{r} y<y_{0}+\frac{y}{r}=d_{0}^{\mathrm{NDL}},
$$

the first inequality following from the fact that $d_{0}^{\mathrm{SSE}}$ is strictly increasing in $\alpha, \kappa$, and $\delta$ and hence has a supremum corresponding to when these parameters are set to unity.

The SSE values of consumption and debt in terms of output are then given as follows:

$$
\begin{gather*}
d^{*}=\kappa k q^{*}=\frac{\alpha \kappa \delta}{r+(1-\delta)(1-\kappa)} y .  \tag{1.23}\\
c^{*}=y-\frac{r}{1+r} d^{*}=\left(1-\frac{\alpha r \kappa \delta}{(1+r)(r+(1-\delta)(1-\kappa))}\right) y . \tag{1.24}
\end{gather*}
$$

Finally, the SSE values of the trade balance and current account are given by

$$
t b_{t}^{*}=\left\{\begin{array}{ll}
\frac{r}{1+r} d^{*}-\left(y-y_{0}\right) & t=0  \tag{1.25}\\
\frac{r}{1+r} d^{*} & t \geq 1
\end{array},\right.
$$

$$
c a_{t}^{*}=\left\{\begin{array}{ll}
-\frac{1}{1+r}\left(y-y_{0}\right) & t=0  \tag{1.26}\\
0 & t \geq 1
\end{array} .\right.
$$

In other words, the trade balance is simply the interest obligations on external debt for all periods, except in period 0 , where there is a downward adjustment if there is a negative shock to output $\left(y_{0}<y\right)$. Likewise, the current account is initially commensurate with any such initial shock in output, and is nil thereafter since the net debt position is constant in the steady state.

Thus, an SSE can be supported by any constant output regime so long as initial debt is given by (1.22) (admitting all other fundamentals as given) so that the collateral constraint always binds, with the corresponding SSE values given by (1.20), (1.21), (1.23), and (1.24). The fact that initial borrowing is restricted is in fact shared by any equilibrium featuring regular cycles, as discussed in section 2.1. Note that $c^{*}$ decreases in $\delta$, while $d_{0}, d^{*}, \tilde{\mu}^{*}$, and $q^{*}$ increase in $\delta$. That is, the more impatient households are, a valid SSE must feature greater consumption and a greater shadow price of borrowing against a tight constraint (lower $\tilde{\mu}^{*}$ ), with consumers borrowing less (given smaller future trade balances to borrow against) and consequently requiring less capital value to collateralize their debt. It is important to interpret these comparative statics results with some caution. In particular, it may appear that more patient agents enjoy less welfare in an SSE (lower $\left.c^{*}\right)$. However, comparing welfare of SSEs under differing patience is meaningless since different SSEs can only be supported under differential initial debts. Indeed, more patient agents must be endowed with a greater debt obligation in a valid SSE since they are less tightly borrowingconstrained (higher $\tilde{\mu}^{*}$ ). The intuition regarding this latter relation will become more clear in the context of consumption growth in section 1.5.2. Essentially, more patient agents wish to frontload their consumption less, while being more tightly constrained (lower $\tilde{\mu}^{*}$ ) corresponds with a higher shadow cost of borrowing and induces agents to push forward their consumption, the latter effect thus complementing the former. Hence, the more patient agents are, the higher $\tilde{\mu}^{*}$ must be to maintain a constant consumption path. Still, when agents are impatient relative to the market, $\tilde{\mu}^{*}$ must be less than unity to induce some counteracting forward-pushing behavior.

### 1.4 Unconstrained equilibrium

In this section, we solve for the unique equilibrium of the unconstrained model. This analysis will serve as a useful benchmark against which to compare the constrained model. Moreover, solving for the unconstrained equilibrium allows us to find when the constrained model can support an eternally slack equilibrium (ESE).

Definition 5 A eternally slack equilibrium (ESE) is an equilibrium where $\tilde{\mu}_{t}=1 \quad \forall t \geq 0$.

Clearly, the ESE -if it exists under appropriate restrictions on the exogenous variables of the constrained model— must concur with the unconstrained equilibrium by definition.

To solve for the unconstrained equilibrium, we may use the $\tilde{\mu}_{t}$-characterization, taking the case where $\tilde{\mu}_{t}=1 \quad \forall t \geq 0$. From (1.10), the unconstrained equilibrium consumption is given by

$$
\begin{equation*}
c_{t}^{\mathrm{UE}}=c_{0}^{\mathrm{UE}} \delta^{t} \tag{1.27}
\end{equation*}
$$

with (1.13) determining initial consumption as

$$
\begin{equation*}
c_{0}^{\mathrm{UE}}=(1-\beta)\left(d_{0}^{\mathrm{NDL}}-d_{0}\right) \tag{1.28}
\end{equation*}
$$

Substituting (1.27) and (1.28) into (1.11) yields the debt as

$$
\begin{equation*}
d_{t}^{\mathrm{UE}}=d_{t}^{\mathrm{NDL}}-\left(d_{0}^{\mathrm{NDL}}-d_{0}\right) \delta^{t} \tag{1.29}
\end{equation*}
$$

Finally, (1.15) determines the price of capital as

$$
\begin{equation*}
q_{t}^{\mathrm{UE}}=\frac{\alpha}{k(1+r)} d_{t+1}^{\mathrm{NDL}} \tag{1.30}
\end{equation*}
$$

Thus, (1.27) -(1.30) give the unconstrained equilibrium. ${ }^{16}$ Moreover, the trade balance and current

[^9]account in equilibrium are given by
\[

$$
\begin{gather*}
t b_{t}^{\mathrm{UE}}=y_{t}-c_{0}^{\mathrm{UE}} \delta^{t}  \tag{1.31}\\
c a_{t}^{\mathrm{UE}}=y_{t}-\frac{r}{1+r} d_{t}^{\mathrm{NDL}}-\frac{1-\delta}{1+r}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right) \delta^{t} . \tag{1.32}
\end{gather*}
$$
\]

Note that the unconstrained equilibrium expressions apply even when $\delta=1$, in which case they coincide with the steady-state equilibrium in a constant output regime. Also note that while $q_{t}^{\mathrm{UE}}$ does not depend on $\beta$, we see that $c_{0}^{\mathrm{UE}}$ and $\left\{d_{\tau+1}^{\mathrm{UE}}\right\}_{\tau \geq 0}$ decrease in $\beta$. Thus, absent any collateral constraint, the more impatient households are, the more they borrow to frontload consumption, allowing their future consumption to diminish exponentially at rate $\delta$ and approaching their natural debt limit over time. In the long run, we have the following when $y_{\infty}$ is well-defined:

$$
\begin{gathered}
c_{\infty}^{\mathrm{UE}}=0, \\
d_{\infty}^{\mathrm{UE}}=d_{\infty}^{\mathrm{NDL}}=\frac{1+r}{r} y_{\infty}, \\
q_{\infty}^{\mathrm{UE}}=\frac{\alpha}{k(1+r)} d_{\infty}^{\mathrm{NDL}}=\frac{\alpha}{r k} y_{\infty}, \\
t b_{\infty}=y_{\infty}, \\
c a_{\infty}=0 .
\end{gathered}
$$

Thus, we see that the unconstrained equilibrium features eventual starvation.
Now consider whether the constrained model can support an ESE. A unique ESE specified by (1.27) -(1.30) exists so long as $d_{t+1}^{\mathrm{UE}} \leq \kappa k q_{t}^{\mathrm{UE}} \quad \forall t \geq 0$, or equivalently,

$$
\begin{equation*}
\text { ESE existence condition: } \frac{1+r-\alpha \kappa}{1+r} d_{t+1}^{\mathrm{NDL}} \leq\left(d_{0}^{\mathrm{NDL}}-d_{0}\right) \delta^{t+1} \quad \forall t \geq 0 \tag{1.33}
\end{equation*}
$$

This inequality gives the necessary and sufficient condition for existence of an ESE and is a generalization of the "regular shocks" condition in SGU (which is obtained by setting $\delta=1$, $\left\{y_{\tau+1}\right\}_{\tau \geq 0}=y$ ). The left-hand side of (1.33) is equivalently $d_{t+1}^{\mathrm{NDL}}-\kappa k q_{t}^{\mathrm{UE}}$ and the right-hand side is $\delta^{t+1} \sum_{\tau=0}^{\infty} \frac{c_{\tau}}{(1+r)^{\tau}}$, so that in words, this restriction asserts that the extent to which the natural debt limit for the debt obligation due at time $t+1$ exceeds the collateral constraint is no more than the time $t+1$ value (discounting under relative patience ratio $\delta$ ) of the net present value of the entire consumption stream (equivalently, $d_{0}^{\mathrm{NDL}}-d_{0}$, the extent to which the initial natural debt limit exceeds the initial debt obligation). Note that (1.33) requires that $y_{t}=O\left(\delta^{t}\right)$, and thus a necessary condition for (1.33) is $y_{\infty}=0$. Consequently, the ESE is only sustainable in an economy that faces depressionary production in the long run, with all endogenous ESE quantities thus vanishing in the limit. Moreover, this observation implies that the ESE and SSE can never coexist since an SSE requires a constant output regime while an ESE requires that any constant output path from period 1 onward must be a zero sequence.

An extreme example of an economy that can support an ESE is indeed one with no production from period 1 onward $\left(\left\{y_{t+1}\right\}_{t \geq 0}=0\right)$ with initial debt strictly less than initial output ( $d_{0}<d_{0}^{\mathrm{NDL}}=$ $y_{0}$ ). In fact, regardless of impatience, the ESE is the unique equilibrium that can be supported by such constant zero output since (1.15) implies the price of capital in this case would always be zero, while a binding collateral constraint in any period would imply a zero debt obligation in the following period, which would violate the zero natural debt limit. In effect, impatient agents in this setting initially consume a fraction $1-\beta$ (commensurate with their degree of impatience) of their surfeit of output net of their debt obligations $\left(y_{0}-d_{0}\right)$ in period 0 , and lend the remaining surfeit to generate income in period 1. Without production thereafter, their income in subsequent periods comes exclusively from the principal and interest payments on the debt lent out in previous periods. However, since consumption diminishes at the rate of the relative patience ratio, the lending diminishes (equivalently, their debt position increases) at the same rate, with the agents eventually approaching their limiting natural debt limit of zero (the net present value of future output) over time. Thus, so long as households initially have enough output to fund their initial
debt obligation, having no output thereafter ensures agents will always face a net lending position and thus be free from the collateral constraint. More generally, we see from (1.33) that the closer the initial debt is to the initial natural debt limit or the more impatient the agents are, the stronger the restriction that the economy should produce less output from period 1 onward for the constraint to be slack. In other words, when agents are initially endowed less as net debtors or more as net lenders (i.e. smaller $d_{0}$ ) or when they are more patient (i.e. higher $\delta$ ), the greater flexibility afforded to them to sustain greater positive output after time 0 while never facing a tight collateral constraint.

Example 1 ESE in an eternal depression. A less extreme example of an economy that would satisfy (1.33) is one where output geometrically declines at least as fast as the relative patience ratio, according to $y_{t}=y_{0} \gamma^{t}$ for $y_{0}>0, \gamma \in(0, \delta]$. Unlike in the previous toy example, this economy always produces positive output, but does so exponentially less over time, so that households face a permanent depression. This permanent depression is compatible with the preferences of impatient agents, who wish to consume their most at time 0 and sacrifice consumption in later periods absent any collateral constraint. We can see what parameter restrictions would suffice for such an output rule to guarantee the existence of the ESE by substituting the rule into (1.33):

$$
\frac{1+r-\alpha \kappa}{1+r} y_{0}\left(1-\frac{\gamma}{1+r}\right)^{-1} \gamma^{t+1} \leq\left(y_{0}\left(1-\frac{\gamma}{1+r}\right)^{-1}-d_{0}\right) \delta^{t+1} \quad \forall t \geq 0
$$

so that it suffices if the parameters satisfy

$$
\frac{1+r-\alpha \kappa}{1+r} y_{0}\left(1-\frac{\gamma}{1+r}\right)^{-1} \leq y_{0}\left(1-\frac{\gamma}{1+r}\right)^{-1}-d_{0}
$$

or equivalently,

$$
\frac{d_{0}}{y_{0}} \leq \frac{\alpha \kappa}{1+r-\gamma} \Longleftrightarrow \gamma d_{0} \leq \kappa k q_{0}^{\mathrm{UE}} .
$$

We see that it suffices if the initial debt-to-output ratio is accordingly bounded. The lattermost inequality affords another interpretation: it suffices if $\gamma d_{0}$, the period 1 value of the initial debt
obligation when discounted at rate $\gamma$, obeys the initial collateral constraint. Figure 1.2 illustrates the equilibrium paths in an ESE for such an economy with $\gamma=0.75 \delta, \frac{d_{0}}{y_{0}}=\frac{\alpha \kappa}{1+r-\gamma}$. We see the value of collateral mimics the behavior of output since the former is proportional to next period's natural debt limit, the net present value of future output. Consumption declines geometrically but not as fast as output does since it only falls at the rate of the relative patience ratio. Debt, on the other hand, behaves non-monotonically. Households are initially endowed as net borrowers $\left(d_{0}>0\right)$, but they soon react to the rapidly declining output and transition to a lending position to raise additional income, increasing their lending for some finite time. They remain lenders thereafter, but reduce their lending position after some time as they only need fund lower consumption in the long run. In the long run, they approach their zero long-term natural debt limit. If output declines at a sufficiently smaller rate (higher $\gamma$, very close to $\delta$ ), households tend to take longer to transition to lending (Figure 1.3, $\gamma=0.95 \delta$ ). If output declines at the rate of the relative patience ratio itself, then households never lend if they are initial borrowers, and simply diminish their borrowing at the rate of the relative patience ratio $\left(d_{t}^{\mathrm{UE}}=d_{0} \delta^{t}\right)$. The trade balance and current account mimic the debt behavior.

As a final remark for this section, note it is generally possible for $d_{0}>y_{0}$, with households borrowing against future income to fund consumption, so long as the initial debt is strictly less than the net present value of all future output (the initial natural debt limit). However, certain parameter values may require $d_{0} \leq y_{0}$ to sustain an ESE. To see this, note that at time 0 , the inequality of (1.33) is

$$
\begin{equation*}
(1+r-\alpha \kappa)\left(d_{0}^{\mathrm{NDL}}-y_{0}\right) \leq\left(d_{0}^{\mathrm{NDL}}-d_{0}\right) \delta, \tag{1.34}
\end{equation*}
$$

and we see that in the event $1+r \geq \alpha \kappa+\delta$, it is required that $d_{0} \leq y_{0}$ to ensure the existence of the ESE. That is, if the gross interest rate on debt obligations is sufficiently high relative to the combined impact of the household's relative patience ratio and the importance of capital ( $\alpha \kappa$ ) in the economy, then agents cannot afford to have an initial debt obligation exceeding initial production if they wish to never face a tight collateral constraint. This observation will be pertinent to our


Figure 1.2: ESE under output rule $y_{t}=y_{0} \gamma^{t}$ with $\gamma=0.75 \delta$. Parameters are set as $y_{0}=1, r=$ $0.01, \delta=0.95, \kappa=0.5, \alpha=0.3, d_{0}=\frac{\alpha \kappa}{1+r-\gamma} y_{0}$.
discussion of multiplicity of equilibria in section 2.2.1.

### 1.5 Characteristics of equilibria

In the previous two sections, we have shown that under appropriate restrictions on exogenous parameters, there is an equilibrium where the collateral constraint is always tight (SSE) and one where the constraint is always slack (ESE). In general, there may be other kinds of equilibria, including those with an occasionally tight constraint. In this section, we discuss general properties featured in any equilibrium using the $\tilde{\mu}_{t}$-characterization. Moreover, the properties in sections


Figure 1.3: ESE under output rule $y_{t}=y_{0} \gamma^{t}$ with $\gamma=0.95 \delta$. Parameters are set as $y_{0}=1, r=$ $0.01, \delta=0.95, \kappa=0.5, \alpha=0.3, d_{0}=\frac{\alpha \kappa}{1+r-\gamma} y_{0}$
1.5.1-1.5.3 apply for any $\delta$, hence subsuming the SGU model.

### 1.5.1 Fisherian deflation relative to unconstrained equilibrium

Our first object is to show how any equilibrium features relative Fisherian deflation, or a strictly lower price of capital in a period when the collateral constraint is tight compared to the unconstrained equilibrium (or equivalently the ESE when it exists). Note that (1.15) shows that $q_{t}$ is positive if output is positive at least once in a future period, and is strictly increasing in $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \geq t}$, and is thus maximized when these normalized multipliers are set to unity (corresponding with the unconstrained equilibrium). Moreover, demand for capital does not depend on past multipliers.

These observations give the following proposition:

Proposition 1 In equilibrium, $q_{t}>0$ if $\exists \tau \geq t \geq 0$ where $y_{\tau+1}>0$; further,

$$
\begin{equation*}
q_{t} \leq q_{t}^{\mathrm{UE}} \quad \forall t \geq 0 \tag{1.35}
\end{equation*}
$$

and $q_{t}=q_{t}^{\mathrm{UE}}$ if and only if $\tilde{\mu}_{\tau}=1 \quad \forall \tau \geq t \geq 0$.

Proposition 1 gives a stronger result than Fisherian deflation relative to the unconstrained equilibrium, since it asserts that all periods contemporaneous to and prior to a period facing a tight constraint feature a lower price of capital compared to the unconstrained equilibrium. More generally, (1.15) shows that tighter contemporaneous or future collateral constraints (lower $\tilde{\mu}_{\tau} \quad \forall \tau \geq t$ ) yield a lower equilibrium price of capital.

The intuition behind this result can be gleaned from the Euler equations for capital and debt discussed in section 1.2.1:

$$
\begin{gathered}
\lambda_{t} q_{t}\left[1-\kappa \mu_{t}\right]=\beta \lambda_{t+1}\left[\alpha A_{t+1} k_{t+1}^{\alpha-1}+q_{t+1}\right] \\
\lambda_{t}\left[(1+r)^{-1}-\mu_{t}\right]=\beta \lambda_{t+1} .
\end{gathered}
$$

One one hand, we see from the first Euler equation that the tighter the collateral constraint is (higher $\mu_{t}$ ), the higher the marginal benefit of an additional unit of capital is since the shadow benefit from relaxing the constraint, $\lambda_{t} \kappa \mu_{t} q_{t}$, is higher. This contribution from a tighter constraint should increase demand for capital, inducing an upward effect on the equilibrium price of capital. At the same time, from the second Euler equation, we see that the tighter the constraint is, the higher the marginal cost of an additional unit of debt is since the shadow punishment from incurring additional debt, $\lambda_{t} \mu$, is higher. This latter contribution should decrease demand for debt, reducing the demand for collateral value (since debt and collateral are effectively complements), inducing a downward effect on the equilibrium price of capital. Since the fraction of capital posted
as collateral is small relative to the gross interest rate $(\kappa<1+r)$, the second effect (the complementary debt channel) dominates the first, leading to this "tightness"-induced Fisherian deflation. Stated differently, the equilibrium price of capital responds negatively to the shadow value of the constraint since the negative effect from the shadow cost of borrowing along with complementarity of debt and collateral value outweigh the positive effect of the shadow benefit of relaxing the constraint. The fact that the price of capital depends on its next period's realization leads to a similar Fisherian deflation induced by future tightness, a consequence of the forward-planning nature of agents. Nonetheless, since agents always consume a positive amount in equilibrium, so long as there is some positive output to be produced in a future period, there will be positive demand for capital to collateralize borrowing.

While any equilibrium features Fisherian deflation relative to the unconstrained equilibrium, a natural question to consider is whether any equilibrium features absolute Fisherian deflation, or a drop in the price of capital when transitioning from a period with a slack constraint to one with a tight constraint. In general, absolute Fisherian deflation is not guaranteed in our model, although the next claim provides sufficient conditions for this behavior.

Claim 1 In equilibrium, if $y_{t+1+\tau} \geq$ (resp. $\left.\leq\right) y_{t+2+\tau}$ and $\tilde{\mu}_{t+\tau} \geq($ resp. $\leq) \tilde{\mu}_{t+1+\tau} \forall \tau \geq 0$, then $q_{t} \geq($ resp. $\leq) q_{t+1}$ for $t \geq 0 .{ }^{17}$

Proof. From (1.15), we have

$$
\begin{gathered}
q_{t}-q_{t+1}=\frac{\alpha}{k}\left[\sum_{\tau=0}^{\infty} y_{t+1+\tau} \prod_{j=0}^{\tau} \frac{\tilde{\mu}_{t+j}}{1+r-\kappa+\kappa \tilde{\mu}_{t+j}}-\sum_{\tau=0}^{\infty} y_{t+2+\tau} \prod_{j=0}^{\tau} \frac{\tilde{\mu}_{t+1+j}}{1+r-\kappa+\kappa \tilde{\mu}_{t+1+j}}\right] \\
\geq(\text { resp. } \leq) \frac{\alpha}{k}\left[\sum_{\tau=0}^{\infty}\left(y_{t+1+\tau}-y_{t+2+\tau}\right) \prod_{j=0}^{\tau} \frac{\tilde{\mu}_{t+j}}{1+r-\kappa+\kappa \tilde{\mu}_{t+j}}\right] \geq(\text { resp. } \leq) 0,
\end{gathered}
$$

the first inequality following from the monotonic chain of inequalities for the normalized multipliers, and the second inequality following from the monotonic chain of inequalities for output.

[^10]That is, the price of capital falls if future output monotonically falls and the constraint monotonically tightens. Conversely, monotonic growth in future output and monotonic weakening of the constraint ensure inflation in the price of capital. This result is quite intuitive in a perfect-foresight economy; though a tight constraint would lead agents to demand less capital than they would in the unconstrained equilibrium, they need not reduce their demand for capital from the preceding period unless they expect an even more dismal economic situation in the future. Indeed, if households expect future productivity to improve and to be less borrowing-constrained with a lower shadow value for the constraint over time, they may possibly increase their demand for capital despite transitioning to a period where they are transitorily more tightly constrained. Thus, the growth in the price of capital responds not just to the contemporary shadow value of the constraint, but future shadow values as well. This observation stands in contrast to consumption growth, discussed in section 1.5.2.

### 1.5.2 Consumption rationing

As with the price of capital, we determine how consumption in any equilibrium depends on the tightness of the collateral constraint. An initial observation we make concerns consumption growth. From the Euler equation for debt (1.3), we see $c_{t+1} \gtreqless c_{t}$ whenever $\tilde{\mu}_{t} \lesseqgtr \tilde{\mu}^{*}$; that is, consumption growth is positive (resp. negative) from $t$ to $t+1$ so long as the collateral constraint is tighter (resp. less tight) in period $t$ relative to its steady-state tightness. Moreover, $c_{t+1} / c_{t} \geq \delta$, so that consumption growth is always at least as great as that of the unconstrained equilibrium, wherein consumption decreases at the rate of the relative patience ratio. The intuition here is that with a less tight contemporaneous constraint (relative to the SSE), the impatient agent prefers frontloading consumption, and thus plans for less consumption in the next period. However, a tighter contemporaneous constraint induces a higher marginal cost of raising an additional unit of debt to fund consumption in that period, so households are willing to frontload less and possibly push forward consumption, consuming more in the next period when the constraint is sufficiently tight $\left(\tilde{\mu}_{t}<\tilde{\mu}^{*}\right)$.

While we thus appreciate the dynamics of consumption growth, we also consider the dynamics of consumption levels. Observe from (1.13) that initial consumption is strictly increasing in $\left\{\tilde{\mu}_{t}\right\}_{t \geq 0}$. Moreover, substituting initial consumption (1.13) into (1.10) yields

$$
\begin{equation*}
c_{t}=\delta^{t}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right)\left[\sum_{\tau=0}^{t-1} \beta^{\tau}\left(\Pi_{j=\tau}^{t-1} \tilde{\mu}_{j}\right)+\beta^{t}+\sum_{\tau=t+1}^{\infty} \beta^{\tau}\left(\Pi_{j=t}^{\tau-1} \tilde{\mu}_{j}^{-1}\right)\right]^{-1}, \tag{1.36}
\end{equation*}
$$

which shows that $c_{t}$ is strictly decreasing in $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \leq t-1}$ and strictly increasing in $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \geq t}$. That is, tighter collateral constraints prior to a period serve to boost consumption in that period, while tighter constraints contemporaneous to or after a period serve to diminish consumption in that period. Households thus display this kind of rationing property, whereby the less tightly constrained they were in the past or the more tightly constrained they expect to be from today onward, the more they plan to ration today. Conversely, the more tightly constrained they were in the past or the less tightly constrained they expect to be from today onward, the more they plan to enjoy a glut today. From these relations, we obtain an upper bound on consumption by setting past normalized multipliers to zero, and setting the contemporaneous and future normalized multipliers to unity, giving the next proposition and corollary.

Proposition 2 In equilibrium,

$$
\begin{equation*}
c_{t} \leq \delta^{t}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right)\left[\sum_{\tau=t}^{\infty} \beta^{\tau}\right]^{-1}=(1+r)^{t} c_{0}^{\mathrm{UE}} \quad \forall t \geq 0 \tag{1.37}
\end{equation*}
$$

where the inequality binds only in an ESE at $t=0$.

Corollary 1 If the collateral constraint is tight for at least one period, then an equilibrium has

$$
\begin{equation*}
c_{0}<c_{0}^{\mathrm{UE}} . \tag{1.38}
\end{equation*}
$$

Proposition 2 states that in any equilibrium, no matter if consumption grows during very tight periods as described before, consumption at any time $t$ cannot exceed the period $t$ future value of the unconstrained initial consumption. Corollary 1 is obtained in SGU and asserts that when
facing the eventuality of a tight constraint, households always ration initial consumption relative to its unconstrained counterpart. ${ }^{18}$

### 1.5.3 Debt deleveraging relative to unconstrained equilibrium

An immediate consequence of Corollary 1 is that the household deleverages (reduces debt) in period 1 relative to the unconstrained equilibrium if the collateral constraint is tight at least once. This result follows from the sequential budget constraint (1.2) at time 0 , by which $c_{0}<c_{0}^{\mathrm{UE}}$ implies $d_{1}<d_{1}^{\mathrm{UE}}$. Naturally this result also implies $c a_{0}>c a_{0}^{\mathrm{UE}}$ and $t b_{0}>t b_{0}^{\mathrm{UE}}$ when the constraint is tight at least once.

In fact, we can strengthen this deleveraging result by seeing how equilibrium debt depends on all the normalized multipliers. As discussed in section 1.5.1, the Euler equation for debt suggests that a tighter contemporaneous constraint increases the shadow punishment from incurring additional debt, which should lead to decreased borrowing in equilibrium. Moreover, given the complementary relationship between debt and collateral value, one would expect the former to respond negatively to tighter future constraints as does the latter. Additionally, tighter past constraints incentivize agents to plan for a glut in future consumption, leaving less trade balances to borrow against, implying a negative impact on equilibrium debt. We appeal to the $\tilde{\mu}_{t}$-characterization to see these dependencies more completely. For convenience, define $M_{t}:=\prod_{\tau=0}^{t-1} \tilde{\mu}_{\tau}$ (with the normalization $M_{0}=1$ as per footnote 12). $M_{t}$ measures "aggregate tightness" of the collateral constraint in the periods preceding $t$ (being smaller when the constraint is more tight in these periods). Then

[^11]by (1.10), (1.11), and (1.13), we have
\[

$$
\begin{gather*}
d_{t+1}=d_{t+1}^{\mathrm{NDL}}-\delta^{t+1}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right) \sum_{\tau=0}^{\infty} w_{\tau}\left(\prod_{j=\tau}^{t+\tau} \tilde{\mu}_{j}\right)^{-1}  \tag{1.39}\\
w_{\tau}:=\frac{\beta^{\tau} M_{\tau}^{-1}}{\sum_{j=0}^{\infty} \beta^{j} M_{j}^{-1}}
\end{gather*}
$$
\]

Recall from (1.11) that the extent to which the natural debt limit at any time exceeds borrowing is the net present value of consumption from that time onward: $d_{t}^{\text {NDL }}-d_{t}=\sum_{\tau=0}^{\infty} \frac{c_{t+\tau}}{(1+r)^{\tau}}$. Thus, in words, (1.39) states that the net present value of consumption from time $t+1$ onward is the $\delta$-discounted time $t+1$ future value of the net present value of the entire consumption stream adjusted by a particular weighted average of $\left(\prod_{j=\tau}^{t+\tau} \tilde{\mu}_{j}\right)^{-1}$ across $\tau \geq 0$. This representation seems to concisely suggest a positive association between the normalized multipliers and debt, suggesting that tighter collateral constraints (lower $\tilde{\mu}_{t}$ ) should reduce debt. In fact, this positive association is true but care needs to be taken since the weights of the weighted average depend positively on some of the normalized multipliers.

Proposition 3 In equilibrium,

$$
\begin{equation*}
d_{t+1} \leq d_{t+1}^{\mathrm{UE}} \quad \forall t \geq 0 \tag{1.40}
\end{equation*}
$$

with equality in any period holding only in an ESE.

Proof. It suffices to show $d_{t+1}$ is strictly increasing in $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \geq 0}$.
We proceed in two steps: first, we show $d_{t+1}$ is strictly increasing in $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \leq t}=\left\{\tilde{\mu}_{0}, \ldots, \tilde{\mu}_{t}\right\}$ and separately we show that it is strictly increasing in $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \geq t}=\left\{\tilde{\mu}_{t}, \tilde{\mu}_{t+1}, \ldots\right\}$.

1. By (1.11), we have that $d_{t+1}$ strictly decreases in $\left\{c_{\tau}\right\}_{\tau \geq t+1}$. From section 1.5.2, we know $c_{t}$ is strictly decreasing in $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \leq t-1}$ and strictly increasing in $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \geq t}$. Consequently, $\left\{c_{\tau}\right\}_{\tau \geq t+1}$ strictly decreases in $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \leq t}$, so $d_{t+1}$ strictly increases in $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \leq t}$.
2. The weighted average in (1.39) can be expressed as

$$
\sum_{\tau=0}^{\infty} w_{\tau}\left(\prod_{j=\tau}^{t+\tau} \tilde{\mu}_{j}\right)^{-1}=\beta^{-(t+1)}\left(1-\frac{\sum_{\tau=0}^{t} \beta^{\tau} M_{\tau}^{-1}}{\sum_{\tau=0}^{\infty} \beta^{\tau} M_{\tau}^{-1}}\right)
$$

and the right-hand side is strictly decreasing in $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \geq t}$.
Both Propositions 1 and 3 have immediate implications on the collateral constraint. For instance, we see that a sufficient condition for the collateral constraint to be satisfied in any equilibrium is $d_{t+1}^{\mathrm{UE}} \leq \kappa k q_{t}$. Moreover, equilibria where the constraint is slack after some transition time yield a relatively easy way to check if the collateral constraint holds from the transition time onward:

Corollary 2 If for time $T \geq 0, \tilde{\mu}_{t}=1 \quad \forall t \geq T$, then the inequality of the ESE existence condition (1.33) holding for $t \geq T$ is sufficient for the collateral constraint to hold for all $t \geq T$ in equilibrium.

Proof. For $t \geq T, d_{t+1} \underbrace{\leq}_{\text {Prop. } 3} d_{t+1}^{U E} \underbrace{\leq}_{\text {(1.33) }} \kappa k q_{t}^{U E} \underbrace{=}_{\text {Prop. 1 }} \kappa k q_{t}$.
Having established deleveraging relative to the unconstrained equilibrium, another point to consider is when households deleverage from one period to the next. The next claim addresses this question.

Claim 2 In equilibrium, if $y_{t+\tau} \geq($ resp. $\leq) y_{t+1+\tau}$ and $\tilde{\mu}_{t+\tau} \leq($ resp. $\geq) \tilde{\mu}^{*} \forall \tau \geq 0$, then $d_{t} \geq$ $($ resp. $\leq) d_{t+1}$ for $t \geq 0 .{ }^{19}$

Proof. (1.11) and (1.3) imply

$$
\begin{gathered}
d_{t}-d_{t+1}=\sum_{\tau=0}^{\infty} \frac{y_{t+\tau}-c_{t+\tau}}{(1+r)^{\tau}}-\sum_{\tau=0}^{\infty} \frac{y_{t+1+\tau}-c_{t+1+\tau}}{(1+r)^{\tau}} \\
=\sum_{\tau=0}^{\infty} \frac{y_{t+\tau}-y_{t+1+\tau}+c_{t+1+\tau}-c_{t+\tau}}{(1+r)^{\tau}}
\end{gathered}
$$

[^12]$$
=\sum_{\tau=0}^{\infty} \frac{y_{t+\tau}-y_{t+1+\tau}+\left(\tilde{\mu}^{*} / \tilde{\mu}_{t+\tau}-1\right) c_{t+\tau}}{(1+r)^{\tau}} \geq(\text { resp. } \leq) 0
$$

That is, households deleverage from one period to the next if they expect output to monotonically fall and the constraint to be tighter than the steady state. Conversely, monotonic growth in output and sub-steady-state tightness of the constraint drives increased borrowing.

The reader may wonder whether the conditions of Claims 1 and 2 may coincide to ensure a concurrent decrease (resp. increase) in borrowing and collateral value, i.e. $d_{t+1} \geq$ (resp. $\leq$ ) $d_{t+2}, q_{t} \geq($ resp. $\leq) q_{t+1}$. Our discussion in section 1.5 . 4 will clarify when these conditions can overlap. We shall see that if future output is not constantly zero, the sufficient conditions of Claim 1 for $q_{t} \geq($ resp. $\leq) q_{t+1}$-namely $y_{\tau+1} \geq($ resp. $\leq) y_{\tau+2}$ and $\tilde{\mu}_{\tau} \geq($ resp. $\leq) \tilde{\mu}_{\tau+1}$ for all $\tau \geq t$ can only be met if $\tilde{\mu}_{\infty} \geq$ (resp. $\leq$ ) $\tilde{\mu}^{*} .{ }^{20}$ Meanwhile, the sufficient conditions of Claim 2 for $d_{t+1} \geq($ resp. $\leq) d_{t+2}$-namely $y_{\tau+1} \geq($ resp. $\leq) y_{\tau+2}$ and $\tilde{\mu}_{\tau+1} \leq($ resp. $\geq) \tilde{\mu}^{*}$ for all $\tau \geq t$-can only be met if $\tilde{\mu}_{\infty}=\tilde{\mu}^{*}$. The conditions of both claims then overlap only if $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \geq t+1}=\tilde{\mu}^{*}$. Thus, a monotonic fall (resp. rise) in future output along with future steady-state tightness is sufficient-though not necessary-to guarantee a concurrent decrease (resp. increase) in borrowing and collateral value.

### 1.5.4 Long-run behavior

Our preceding analyses were based on how equilibrium quantities depend on the multipliers. As discussed in section 1.2.2, analytically solving for the equilibrium multipliers themselves is generally cumbrous. Nonetheless, it is possible to determine the kinds of behavior of the equilibrium multipliers that are not permissible in the long run. To facilitate such discussion, we define the following.

Definition 6 An equilibrium features stabilizing tightness when $\tilde{\mu}_{\infty}$ exists. .

[^13]Note the SSE and ESE feature stabilizing tightness, with $\tilde{\mu}_{\infty}=\tilde{\mu}^{*}=\delta$ and $\tilde{\mu}_{\infty}=1$ respectively. Now consider the long-run behavior of consumption and debt. From our discussion of consumption growth in section 1.5.2, we see that if the constraint is weakly less tight relative to the SSE after some transition time and nonvanishingly less tight than the SSE infinitely often in the sense that there is a constant $\tilde{\mu}$ so that $\tilde{\mu}_{\tau} \geq \tilde{\mu}>\tilde{\mu}^{*}$ for infinitely many $\tau \geq 0$, then the long-run growth rate of consumption is negative and nonvanishing, implying such an equilibrium must feature eventual starvation. Conversely, if the constraint is weakly more tight relative to the SSE after some transition time and nonvanishingly more tight than the SSE infinitely often so that $\tilde{\mu}_{\tau} \leq \tilde{\mu}<\tilde{\mu}^{*}$ for infinitely many $\tau \geq 0$ for some constant $\tilde{\mu}$, then the long-run growth rate of consumption is positive and nonvanishing, implying consumption grows unbounded. In the latter scenario, if households are not endowed with unboundedly increasing output, they must eventually take an unbounded lending position to fund their future consumption growth. However, this behavior implies the collateral constraint is eventually slack, contradicting the presumed long-run tightness of the constraint. Even if output is permitted to grow unboundedly but has well-defined net present value, the constraint still cannot be so tight in the long run to induce long-run consumption growth greater than rate $1+r$, or else the net present value of trade balances-and hence equilibrium debt-would be ill-defined. These observations suggest the kinds of behavior that are permissible in the long run in any equilibrium.

Lemma 1 An equilibrium that features eventual starvation exists only if $y_{\infty}=0$.
Proof. From (1.11), the collateral constraint, and Proposition 1, we have

$$
d_{t+1}^{\mathrm{NDL}}-\sum_{\tau=0}^{\infty} \frac{c_{t+1+\tau}}{(1+r)^{\tau}}=d_{t+1} \leq \kappa k q_{t} \leq \kappa k q_{t}^{\mathrm{UE}}=\frac{\alpha \kappa}{1+r} d_{t+1}^{\mathrm{NDL}},
$$

so that

$$
\frac{1+r-\alpha \kappa}{1+r} d_{t+1}^{\mathrm{NDL}} \leq \sum_{\tau=0}^{\infty} \frac{c_{t+1+\tau}}{(1+r)^{\tau}}
$$

Thus, $c_{t} \rightarrow 0 \Longleftrightarrow \sum_{\tau=0}^{\infty} \frac{c_{t+1+\tau}}{(1+r)^{\tau}} \rightarrow 0 \Longrightarrow d_{t+1}^{\text {NDL }} \rightarrow 0 \Longleftrightarrow y_{t} \rightarrow 0$ (both $\Longleftrightarrow$ are by Fact 2 ).

## Proposition 4

1. Under a well-defined $d_{0}^{\mathrm{NDL}}<\infty$, an equilibrium never has $\tilde{\mu}_{\tau} \leq \beta \quad \forall \tau \geq T$ for some time $T \geq 0$.
2. Under a bounded output path, an equilibrium never has $\tilde{\mu}_{\hat{\tau}} \leq \tilde{\mu}^{*} \forall \hat{\tau} \geq T$ for some time $T \geq 0$ with $\tilde{\mu}_{\tau} \leq \tilde{\mu}<\tilde{\mu}^{*}$ infinitely often for constant $\tilde{\mu}$.
3. An equilibrium where $\tilde{\mu}_{\hat{\tau}} \geq \tilde{\mu}^{*} \quad \forall \hat{\tau} \geq T$ for some time $T \geq 0$ with $\tilde{\mu}_{\tau} \geq \tilde{\mu}>\tilde{\mu}^{*}$ infinitely often for constant $\tilde{\mu}$ always features eventual starvation and hence exists only if $y_{\infty}=0$.

Proof. Note that by (1.3) and (1.11), whenever $\tilde{\mu}_{\tau} \leq \overline{\tilde{\mu}} \quad \forall \tau \geq T \geq 0$ for some constant $\overline{\tilde{\mu}}$, then

$$
d_{t+1} \leq d_{t+1}^{\mathrm{NDL}}-\sum_{j=0}^{\infty} \frac{c_{t+1}}{(1+r)^{j}}\left(\tilde{\mu}^{*} / \overline{\tilde{\mu}}\right)^{j} \forall t \geq T .
$$

1. Under well-defined $d_{0}^{\mathrm{NDL}}<\infty$, we have $d_{t+1}^{\mathrm{NDL}}<\infty \forall t \geq 0$. If there is some time $T$ where $\tilde{\mu}_{\tau} \leq \beta \quad \forall \tau \geq T$, then letting $\overline{\tilde{\mu}}=\beta$ in the above inequality implies the right-hand side is $-\infty$, a contradiction.
2. Under a bounded path of $y_{t}$, or equivalently a bounded path of $d_{t}^{\text {NDL }}$ (c.f. Fact 1 ) with $d_{t}^{\mathrm{NDL}} \leq \bar{d}<\infty \quad \forall t \geq 0$, if there is some $\tilde{\mu}$ and time $T$ where $\tilde{\mu}_{\hat{\tau}} \leq \tilde{\mu}^{*} \quad \forall \hat{\tau} \geq T$ with $\tilde{\mu}_{\tau} \leq \tilde{\mu}<\tilde{\mu}^{*}$ infinitely often, then letting $\overline{\tilde{\mu}}=\tilde{\mu}^{*}$ in the above inequality implies $d_{t+1}<$ $\bar{d}-c_{t+1} \sum_{j=0}^{\infty} \frac{1}{(1+r)^{i}} .^{21}$ Since $c_{t} \rightarrow \infty$, then $d_{t} \rightarrow-\infty$, giving an eventually slack constraint, contradicting the presupposed tightness.
3. Suppose there is some time $T$ where $\tilde{\mu}_{\hat{\tau}} \geq \tilde{\mu}^{*} \forall \hat{\tau} \geq T$ with $\tilde{\mu}_{\tau} \geq \tilde{\mu}>\tilde{\mu}^{*}$ infinitely often. Then $c_{\infty}=0$ and hence $y_{\infty}=0$ by Lemma 1 .

## Corollary 3

[^14]1. Under a well-defined $d_{0}^{\mathrm{NDL}}<\infty$, any equilibrium with stabilizing tightness has $\tilde{\mu}_{\infty} \geq \beta$.
2. Under a bounded output path, any equilibrium with stabilizing tightness has $\tilde{\mu}_{\infty} \geq \tilde{\mu}^{*}=\delta$.
3. Under a nonvanishing output path, any equilibrium with stabilizing tightness has $\tilde{\mu}_{\infty} \leq \tilde{\mu}^{*}=$ $\delta$.

We can see how the requirement $y_{\infty}=0$ in an ESE is a special case of statement 3 of Proposition 4. When output is bounded and nonvanishing (e.g. a constant output regime), Proposition 4 implies any equilibrium cannot have a shadow value of the collateral constraint that is nonvanishingly persistently lower or persistently higher than the steady-state level in the long run. Thus, when agents are impatient relative to the market, an equilibrium in a regime of bounded and nonvanishing output must either have $\tilde{\mu}_{\infty}=\tilde{\mu}^{*}$ or otherwise feature infinite vacillation between very tight periods ( $\tilde{\mu}_{t}<\tilde{\mu}^{*}$ ) and less tight periods ( $\tilde{\mu}_{t}>\tilde{\mu}^{*}$ ). The long-run behavior of $\tilde{\mu}_{t}$ relates to long-run behavior of other endogenous variables: For instance, if $y_{\infty}>0$ exists, then $\tilde{\mu}_{\infty}=\tilde{\mu}^{*}$ is necessary for $c_{\infty}$ and $d_{\infty}$ to exist. ${ }^{22}$ In contrast, the case of vacillating $\tilde{\mu}_{t}$ motivates the possibility of deterministic cycles, which we explore in the sequel.

[^15]
# Chapter 2: Deterministic Cycles and Multiple Equilibria in Open Economies with Stock Collateral Constraints 

### 2.1 Deterministic cycles

Thus far, we have looked at properties featured in any equilibrium. In this section, we draw our attention to a broad class of equilibria featuring regular cycles. The benchmark equilibria we have discussed at this point-namely the SSE and ESE- feature a constant pattern of constraint tightness. In this respect, it is of particular interest to determine whether our parameter assumptions permit the existence of other kinds of equilibria with an always tight constraint, or alternatively, feature persistent vacillation between periods of tightness and slackness. Indeed, we show such equilibria exist under plausible parametrizations.

### 2.1.1 $N$-cyclic equilibria

To motivate this discussion, we define the notion of an N -cyclic quantity.

Definition 7 Given integers $N>0$ and $\tau \geq 0$, a sequence $\left(x_{t+\tau}\right)_{t \geq 0}$, is $N$-cyclic when

$$
t \equiv t^{\prime}(\bmod N) \Longrightarrow x_{t+\tau}=x_{t^{\prime}+\tau} \forall t, t^{\prime} \geq 0 .
$$

A cycle of $x_{t+\tau}$ is any $N$ consecutive realizations of $x_{t+\tau}$.

An $N$-cyclic quantity is thus one that repeats itself every $N$ periods (at least from some period $\tau$ onward), and an $N$-cyclic equilibrium is simply an equilibrium that is $N$-cyclic where the constraint is tight at least once in a cycle. ${ }^{1}$ Of course, if $x_{t}$ is $N$-cyclic, then it is also $N \tau$-cyclic for any integer $\tau \geq 1$, although our general focus will be on $N$-cyclic quantities where $N$ is minimal. The

[^16]possibility of $N$-cyclic equilibria arises due to competition between two forces: the friction of the collateral constraint and impatience relative to the market. As the Euler equation for debt (1.3) shows, the former force induces households to push forward consumption due to the shadow cost of borrowing today, while the latter force induces households to frontload consumption. The SSE that we have examined is the unique 1-cyclic equilibrium. The next proposition characterizes the more general N -cyclic equilibria.

Proposition 5 Suppose $\left(\varepsilon_{t}\right)_{t \geq 0}$ is an equilibrium, $\varepsilon_{t}=\left(c_{t}, \tilde{\mu}_{t}, q_{t}, d_{t+1}\right)$. Then $\varepsilon_{t}$ is $N$-cyclic if and only if $\tilde{\mu}_{t}$ and $y_{t+1}$ are $N$-cyclic with $y_{n^{\prime}+1}>0, \tilde{\mu}_{n}<1$ for some $n, n^{\prime} \in \mathcal{N} \equiv\{0,1, \ldots, N-1\}$. Moreover, if $\varepsilon_{t}$ is an $N$-cyclic equilibrium, it can only be supported under unique $d_{0}$ given $\left(\tilde{\mu}_{n}\right)_{n \in \mathcal{N}}$ and other parameters, ${ }^{2}$ and satisfies

$$
\begin{equation*}
\left(\prod_{\tau \in \mathcal{N}} \tilde{\mu}_{\tau}\right)^{1 / N}=\tilde{\mu}^{*} \tag{2.1}
\end{equation*}
$$

Proof. See Appendix A.1.
It is easily verified that the SSE satisfies Proposition 5. Proposition 5 reveals that cyclicity of the shadow values and output are necessary and sufficient conditions for cyclicity of an equilibrium. Of course, cyclic output subsumes the case of constant output. Moreover, as with the SSE, an equilibrium of a given sequence of multipliers that features regular cycles can only exist under unique initial borrowing (given other parameters), ${ }^{3}$ which, by the sequential budget constraint (1.2), is on the path of the debt cycle adjusting for any output shock: $d_{0}=d_{N}-y_{N}+y_{0}$. This restriction emerges from an overdetermined system; in addition to the binding collateral constraint that determines the shadow values in the tight periods, an additional restriction is imposed by cyclicity. Particularly, by our discussion of consumption growth in section 1.5.2, the fact that N cyclic consumption has zero growth over any $N$ consecutive periods implies the geometric average of the normalized multipliers in a cycle must be their steady-state value, $\tilde{\mu}^{*}=\delta$. This averaging

[^17]condition is also appreciated in the context of our discussion of long-run behavior, since a bounded nonvanishing output path requires stabilizing tightness at the steady-state level or else vacillation between super-steady-state tightness ( $\tilde{\mu}_{t}<\tilde{\mu}^{*}$ ) and sub-steady-state tightness ( $\tilde{\mu}_{t}>\tilde{\mu}^{*}$ ) infinitely often.

In general, the $\tilde{\mu}_{t}$-characterization gives an infinite-dimensional system that determines the shadow values if the constraint is tight infinitely often. However, cyclicity greatly simplifies the analysis; Appendix A. 2 shows all cyclic equilibria are characterized by an explicit finitedimensional system, with as many dimensions as there are assumed tight periods in the cycle. Cyclicity also strengthens infinite-horizon properties of general equilibria, as seen in the following analogues of Claims 1 and 2.

Claim 3 In an $N$-cyclic equilibrium, if $y_{n+1+\tau} \geq($ resp. $\leq) y_{n+2+\tau}$ and $\tilde{\mu}_{n+\tau} \geq($ resp. $\leq) \tilde{\mu}_{n+1+\tau}$ $\forall \tau \in\{0, \ldots, N-2\}$, then $q_{n} \geq($ resp. $\leq) q_{n+1}$ for $n \geq 0 .{ }^{4}$

Proof. By cyclicity, $q_{t+N}=q_{t} \forall t \geq 0$, so that rearranging (1.14) implies

$$
\begin{equation*}
q_{t}=\frac{\alpha}{k}\left(1-\prod_{j \in \mathcal{N}} \frac{\tilde{\mu}_{j}}{1+r-\kappa+\kappa \tilde{\mu}_{j}}\right)^{-1} \sum_{\tau \in \mathcal{N}} y_{t+1+\tau} \prod_{j=0}^{\tau} \frac{\tilde{\mu}_{t+j}}{1+r-\kappa+\kappa \tilde{\mu}_{t+j}} . \tag{2.2}
\end{equation*}
$$

Consequently, we have

$$
\begin{aligned}
& q_{n}-q_{n+1}=\frac{\alpha}{k}\left(1-\prod_{j \in \mathcal{N}} \frac{\tilde{\mu}_{j}}{1+r-\kappa+\kappa \tilde{\mu}_{j}}\right)^{-1}\left[\sum_{\tau \in \mathcal{N}} y_{n+1+\tau} \prod_{j=0}^{\tau} \frac{\tilde{\mu}_{n+j}}{1+r-\kappa+\kappa \tilde{\mu}_{n+j}}-\sum_{\tau \in \mathcal{N}} y_{n+2+\tau} \prod_{j=0}^{\tau} \frac{\tilde{\mu}_{n+1+j}}{1+r-\kappa+\kappa \tilde{\mu}_{n+1+j}}\right] \\
& \geq(\text { resp. } \leq) \frac{\alpha}{k}\left(1-\prod_{j \in \mathcal{N}} \frac{\tilde{\mu}_{j}}{1+r-\kappa+\kappa \tilde{\mu}_{j}}\right)^{-1}\left[\sum_{\tau \in \mathcal{N}}\left(y_{n+1+\tau}-y_{n+2+\tau}\right) \prod_{j=0}^{\tau} \frac{\tilde{\mu}_{n+j}}{1+r-\kappa+\kappa \tilde{\mu}_{n+j}}\right] \\
& =\frac{\alpha}{k}\left(1-\prod_{j \in \mathcal{N}} \frac{\tilde{\mu}_{j}}{1+r-\kappa+\kappa \tilde{\mu}_{j}}\right)^{-1}\left[-\left(y_{n+1}-y_{n+N}\right) \prod_{j \in \mathcal{N}} \frac{\tilde{\mu}_{j}}{1+r-\kappa+\kappa \tilde{\mu}_{j}}+\sum_{\tau=0}^{N-2}\left(y_{n+1+\tau}-y_{n+2+\tau}\right) \prod_{j=0}^{\tau} \frac{\tilde{\mu}_{n+j}}{1+r-\kappa+\kappa \tilde{\mu}_{n+j}}\right] \\
& \geq(\text { resp. } \leq) \frac{\alpha}{k}\left(1-\prod_{j \in \mathcal{N}} \frac{\tilde{\mu}_{j}}{1+r-\kappa+\kappa \tilde{\mu}_{j}}\right)^{-1}\left(\prod_{j \in \mathcal{N}} \frac{\tilde{\mu}_{j}}{1+r-\kappa+\kappa \tilde{\mu}_{j}}\right)\left[-\left(y_{n+1}-y_{n+N}\right)+\sum_{\tau=0}^{N-2} y_{n+1+\tau}-y_{n+2+\tau}\right]=0 .
\end{aligned}
$$

[^18]The first inequality follows from the fact that the monotonic chain of inequalities for the normalized multipliers implies $\prod_{j=0}^{\tau} \frac{\tilde{\mu}_{n+j}}{1+r-\kappa+\kappa \tilde{\mu}_{n+j}} \geq$ (resp. $\leq$ ) $\prod_{j=0}^{\tau} \frac{\tilde{\mu}_{n+1+j}}{1+r-\kappa+\kappa \tilde{\mu}_{n+1+j}}$. The second inequality follows from the monotonic chain of inequalities for output and the fact that $\prod_{j=0}^{\tau} \frac{\tilde{\mu}_{n+j}}{1+r-\kappa+\kappa \tilde{\mu}_{n+j}}$ is a cumulative discount factor for part of the cycle ( $\tau+1$ periods), which is greater than $\prod_{j \in \mathcal{N}} \frac{\tilde{\mu}_{j}}{1+r-\kappa+\kappa \tilde{\mu}_{j}}$ -the cumulative discount factor for the entire cycle.

Claim 4 In an $N$-cyclic equilibrium, if $y_{n+1+\tau} \geq$ (resp. $\left.\leq\right) y_{n+2+\tau}$ and $\tilde{\mu}_{n+1+\tau} \leq($ resp. $\geq) \tilde{\mu}^{*}$ $\forall \tau \in\{0, \ldots, N-2\}$, then $d_{n+1} \geq($ resp. $\leq) d_{n+2}$ for $n \geq 0 .{ }^{5}$

Proof. From (1.11), cyclicity implies that $\forall t \geq 0$,

$$
\begin{align*}
& d_{t+1}=\sum_{\tau \in \mathcal{N}} \frac{y_{t+1+\tau}-c_{t+1+\tau}}{(1+r)^{\tau}}+(1+r)^{-N} d_{t+1} \\
& \Longrightarrow d_{t+1}=\frac{1}{1-(1+r)^{-N}} \sum_{\tau \in \mathcal{N}} \frac{y_{t+1+\tau}-c_{t+1+\tau}}{(1+r)^{\tau}} \tag{2.3}
\end{align*}
$$

By (1.3), the condition on normalized multipliers gives $c_{n+1+j} \leq($ resp. $\geq) c_{n+2+j}, j \in\{0, \ldots, N-2\}$, so

$$
\begin{gathered}
d_{n+1}-d_{n+2}=\frac{1}{1-(1+r)^{-N}} \sum_{\tau \in \mathcal{N}} \frac{y_{n+1+\tau}-c_{n+1+\tau}}{(1+r)^{\tau}}-\frac{1}{1-(1+r)^{-N}} \sum_{\tau \in \mathcal{N}} \frac{y_{n+2+\tau}-c_{n+2+\tau}}{(1+r)^{\tau}} \\
=\frac{1}{1-(1+r)^{-N}}\left[-(1+r)^{-(N+1)}\left(y_{n+1}-y_{n+N}+c_{n+N}-c_{n+1}\right)+\sum_{\tau=0}^{N-2} \frac{y_{n+1+\tau}-y_{n+2+\tau}+c_{n+2+\tau}-c_{n+1+\tau}}{(1+r)^{\tau}}\right] \\
\geq(\text { resp. } \leq) \frac{(1+r)^{-(N+1)}}{1-(1+r)^{-N}}\left[-\left(y_{n+1}-y_{n+N}+c_{n+N}-c_{n+1}\right)+\sum_{\tau=0}^{N-2} y_{n+1+\tau}-y_{n+2+\tau}+c_{n+2+\tau}-c_{n+1+\tau}\right]=0 .
\end{gathered}
$$

That is, under cyclicity, monotonically falling output and monotonic tightening of the constraint for the duration of a cycle guarantees absolute Fisherian deflation. Likewise, super-steady-state tightness for all periods but the final period in the cycle (which exhibits sub-steady-state tightness by (2.1)) and monotonically falling output for the duration of a cycle guarantees deleveraging from one period to the next. Intuitively, it suffices to examine a single cycle in an N -cyclic equilibrium

[^19]since subsequent cycles are discounted iterations of the contemporaneous one.

### 2.1.2 2-cyclic equilibria

We now consider the most elementary kind of cyclic equilibria beyond the SSE, namely 2cyclic equilibria. In such equilibria, the geometric average condition (2.1) requires $\tilde{\mu}_{1}=\delta^{2} / \tilde{\mu}_{0}$, and hence their period 0 multiplier characterizes their entire behavior. In the special case of 2cyclic equilibria, Claims 3 and 4 reduce to the following corollary.

Corollary 4 In a 2-cyclic equilibrium, if $y_{1} \geq($ resp. $\leq) y_{2}$ and $\tilde{\mu}_{0} \geq($ resp. $\leq) \tilde{\mu}^{*}$, then $q_{0} \geq$ $($ resp. $\leq) q_{1}$ and $d_{1} \geq($ resp. $\leq) d_{2} .{ }^{6}$

In other words, a 2-cyclic equilibrium features a concurrent decline (resp. rise) in borrowing and collateral value from period 0 to period 1 if the period 0 constraint is less (resp. more) tight than in the steady state and future output falls (resp. rises) in the course of a cycle.

We classify 2-cyclic equilibria into three types, which are studied in detail in Appendix A.3:
Definition 8 A 2-cyclic equilibrium is one of three types:

1. Slack-tight : Slack constraint in period $0\left(\tilde{\mu}_{0}=1, \tilde{\mu}_{1}=\delta^{2}\right)$
2. Tight-slack: Slack constraint in period $1\left(\tilde{\mu}_{0}=\delta^{2}, \tilde{\mu}_{1}=1\right)$
3. Tight-tight : Always tight constraint $\left(\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right), \tilde{\mu}_{1}=\delta^{2} / \tilde{\mu}_{0}\right)$

The SSE is one tight-tight equilibrium that exists in a constant output regime, where $\tilde{\mu}_{0}=\delta$. In fact, any 2-cyclic equilibrium exists in a constant output regime under fairly plausible parametrizations:

Proposition 6 In a constant output regime under the unique $d_{0}$ required to support the 2 -cyclic equilibrium of given $\tilde{\mu}_{0}$,

1. For $r$ sufficiently small and either i) $\delta$ large enough, or ii) $\alpha \kappa$ large enough, a slack-tight equilibrium and a tight-slack equilibrium exist.
2. For $r$ sufficiently small, and either i) $\delta$ sufficiently large or ii) $\kappa$ sufficiently large, for any $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right)$, there is an $\alpha \in(0,1)$ such that the corresponding tight-tight equilibrium exists.
[^20]Proof. See Appendix A.3.1-A.3.3.
Thus, a 2-cyclic equilibrium corresponding to any kind of feasible $\tilde{\mu}_{0} \in\left[\delta^{2}, 1\right]$ can be constructed under the required $d_{0}$ when parameters are tuned accordingly. The initial borrowing restriction ensures the collateral constraint is met in one tight period in the cycle, while the other parameter assumptions ensure the constraint is met in the other period in the cycle. Note that the slack-tight and tight-slack equilibria exist under the same conditions (so long as their respective initial borrowing requisites are met); as we show in Appendix A.3, this concurrence arises from the fact that in a constant output regime, both equilibria have the exact same restriction on fundamentals for the constraint to be met in the slack period. Indeed by symmetry, the slack-tight and tight-slack equilibria are mirror images of one another in a constant output regime, with period 0 equilibrium values for one swapped with period 1 equilibrium values for the other and vice versa. For these equilibria, when $r$ is sufficiently small, it suffices if $\delta$ is approximately greater than $\frac{1-\alpha \kappa}{1+\alpha \kappa}$; in other words, under a low interest rate, it suffices if agents are either sufficiently patient ( $\delta$ large) or capital is very important to the economy ( $\alpha \kappa$ large). In contrast, constructing an arbitrary tight-tight equilibrium requires ensuring the constraint binds in both periods of the cycle. Our construction uses the intermediate value theorem to show for any $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right)$, there is an $\alpha \in(0,1)$ under which the constraint binds for the tight-tight equilibrium when the interest rate is sufficiently small and either households are sufficiently patient ( $\delta \approx 1$ ) or the fraction of capital that is collateralized is sufficiently large ( $\kappa \approx 1$ ).

Although the sufficient conditions for existence of the different kinds of 2-cyclic equilibria appear to overlap in Proposition 6, the proposition does not guarantee that different 2-cyclic equilibria can coexist, since each particular equilibrium has its own requisite $d_{0}$. However, Table 2.1 gives a parametrization consistent with Proposition 6 that supports both the SSE and a tight-slack equilibrium, illustrated in Figure 2.1. In this tight-slack equilibrium, we observe an initial expansionary phase, wherein the tight constraint at the outset induces households to plan for a boost in consumption for the next period, yielding a decline in the current account and a trade deficit. In accord with Corollary 4, to fund next period's increased domestic absorption, households raise their

| $r=.01, \delta=0.95, \kappa=0.18272563, \alpha=0.3, k=1,\left\{y_{t}\right\}_{t \geq 1}=y>0$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Period | $\tilde{\mu}_{t}$ | $c_{t} / y$ | $d_{t+1} / y$ | $q_{t} / y$ | $\kappa k q_{t} / y$ |
| Tight-slack |  |  |  |  |  |
| $t=0$ | 0.9025 | 0.9646 | 0.9983 | 5.4636 | 0.9983 |
| $t=1$ | 1.0000 | 1.0154 | 1.0238 | 5.7065 | 1.0427 |
| SSE |  |  |  |  |  |
| $t \geq 0$ | 0.9500 | 0.9899 | 1.0238 | 5.6032 | 1.0238 |

Table 2.1: 2-cyclic equilibria $\left(d_{0}=d_{2}-y+y_{0}\right)$
external borrowing and demand for collateral, resulting in capital appreciation. The constraint now slackened, household impatience dominates, rendering a contractionary phase featuring deleveraging, capital depreciation, and a current account reversal. We revisit the example in Table 2.1 and the more general possibility of multiple equilibria in section 2.2.

### 2.1.3 Cycles of higher periodicity and chaos

Appendix A. 2 shows how a general $N$-cyclic equilibrium is obtained as the solution (if it exists) of an explicit finite-dimensional nonlinear system. From this characterization, a specific 3-cyclic equilibrium has been obtained, shown in Table 2.2 and illustrated in Figure 2.2. This equilibrium has a tight-tight-slack pattern and exists in a constant output regime under a parametrization that is fairly consistent with the kind suggested in Proposition 6 (particularly, low $r$ and high $\delta$ ). In this cycle, we observe that though households are tightly constrained at the outset, their impatience dominates, rendering an initial contractionary phase. The following expansion and contraction of the cycle after a respective tight and slack constraint mimic those seen in Figure 2.1. In contrast to Figure 2.1, the economy displayed in Figure 2.2 always enjoys a trade surplus. The existence of a 3-cyclic equilibrium is significant: if there is an appropriate equilibrium selection rule under which a debt policy function exists, ${ }^{7}$ i.e. $d_{t+1}=D\left(d_{t}\right)$ for a function $D: \mathbb{R} \rightarrow \mathbb{R}$, then the LiYorke theorem (Li and Yorke 1975) implies the existence of debt cycles of any periodicity as well as chaos, a result shown in the context of flow collateral constraints (Schmitt-Grohé and Uribe 2021). Characterizing such a debt policy function is elusive at the moment but is attractive for

[^21]

Figure 2.1: Tight-slack equilibrium (blue) and SSE (red) given in Table $2.1\left(y_{0}=y\right)$
future research.

### 2.2 Self-fulfilling financial crises

In this section, we discuss whether a given set of fundamentals can support multiple equilibria. The key finding here is that weak economic fundamentals may give rise to multiple equilibria, with agents thus possibly choosing a welfare-inferior equilibrium driven by nonfundamental pessimistic expectations of collateral value as part of a self-fulfilling financial crisis. The pecuniary externality in the economy allows for coexistence with such a "bad" equilibrium. We consider multiple equilibria in two cases: 1) a variable output regime, and 2) a constant output regime.

| $r=.01, \delta=0.95, \kappa=0.65, \alpha=0.3, k=1,\left\{y_{t}\right\}_{t \geq 1}=y>0$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Period | $\tilde{\mu}_{t}$ | $c_{t} / y$ | $d_{t+1} / y$ | $q_{t} / y$ | $\kappa k q_{t} / y$ |
| $t=0$ | 0.9569 | 0.9203 | 6.6452 | 10.2234 | 6.6452 |
| $t=1$ | 0.8960 | 0.9137 | 6.6245 | 10.1915 | 6.6245 |
| $t=2$ | 1.0000 | 0.9687 | 6.6591 | 10.4192 | 6.7725 |

Table 2.2: 3-cyclic equilibrium $\left(d_{0}=d_{3}-y+y_{0}\right)$

### 2.2.1 Variable output regime

To motivate the kind of equilibrium we consider in this subsection, we define the following:

Definition 9 A $\tau$-spot tight equilibrium ( $\tau-S T E$ ) is an equilibrium where the collateral constraint is tight only in period $\tau \geq 0$.

The SGU model with $\delta=1$ and constant output shows that it is possible for weak fundamentals to support both the SSE-for which the collateral constraint is always slack under $\delta=1$ - and a 0-STE that transitions to a steady state after period 0 . We will show the direct extension of this result for $\delta \leq 1$ under variable output, namely that it is possible to sustain both the ESE and a 0-STE.

As a first step, we invoke the $\tilde{\mu}_{t}$-characterization to write a 0 -STE. For $t \geq 1$, a 0 -STE, expressed in terms of its period 0 normalized multiplier, $\tilde{\mu}_{0}$, is given by

$$
\begin{gather*}
c_{t}^{0-\mathrm{STE}}=\left((1-\beta) \tilde{\mu}_{0}+\beta\right)^{-1} c_{t}^{\mathrm{UE}},  \tag{2.4}\\
c_{0}^{0-\mathrm{STE}}=\left(1-\beta+\tilde{\mu}_{0}^{-1} \beta\right)^{-1} c_{0}^{\mathrm{UE}},  \tag{2.5}\\
d_{t}^{0-\mathrm{STE}}=d_{t}^{\mathrm{UE}}-\delta^{t}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right)\left(\left((1-\beta) \tilde{\mu}_{0}+\beta\right)^{-1}-1\right),  \tag{2.6}\\
q_{t}^{0-\mathrm{STE}}=q_{t}^{\mathrm{UE}},  \tag{2.7}\\
q_{0}^{0-\mathrm{STE}}=\frac{1+r}{\kappa+(1+r-\kappa) \tilde{\mu}_{0}^{-1}} q_{0}^{\mathrm{UE}}, \tag{2.8}
\end{gather*}
$$

where $\tilde{\mu}_{0} \in(0,1)$ solves $d_{1}^{0-S T E}=\kappa k q_{0}^{0-S T E}$, the binding constraint in period 0 . The expressions (2.4)-(2.8) clearly illustrate our observations on the comparison between the 0-STE and ESE from


Figure 2.2: 3-cyclic equilibrium given in Table $2.2\left(y_{0}=y\right)$
section 1.5, shown in Figure 2.3. In the $0-$ STE, pessimistic expectations for the value of collateral in period 0 induce households to substantially contract their initial consumption and plan a slight glut in later periods relative to the ESE (in accord with our discussion in section 1.5.2) and deleverage in all periods relative to the ESE (as per Proposition 3). Their expectations are fulfilled when their behavior in aggregate deflates the price of capital in period 0 relative to the ESE (as per Proposition 1), rendering a tight constraint in period 0 . We note how the 0 -STE exhibits signs associated with a sudden stop, including the initial increased current account and contracted domestic absorption. The next result reveals when the 0-STE coexists with the ESE.

Proposition 7 Consider the following conditions:


Figure 2.3: ESE (blue) and 0-STE (red) under same parameters as those of Figure 1.3. Note $q_{t}$ coincides for both equilibria for $t \geq 1$.

1. The ESE existence condition (1.33) holds for all periods with strict inequality in period 0 ,
2. $d_{0}>y_{0}$.

The ESE and 0-STE coexist if both 1 . and 2 . hold, and only if 2 . holds.

Proof. By Corollary 2, the inequality of (1.33) applied to all times from period 1 onward is sufficient for the collateral constraint to be satisfied in the $0-$ STE for these periods. The constraint in period $0, d_{1}^{0-S T E}=\kappa k q_{0}^{0-S T E}$, determines $\tilde{\mu}_{0} \in(0,1)$ as the root of a parabola:

$$
\Lambda\left(\tilde{\mu}_{0}\right)=0
$$

$$
\begin{gathered}
\Lambda(\tilde{\mu}):=A \tilde{\mu}^{2}+B \tilde{\mu}+C, \\
A=d_{1}^{\mathrm{NDL}} \kappa(1-\alpha)(1-\beta) \\
B=d_{1}^{\mathrm{NDL}}[(1-\beta)(1+r)-\kappa(1-\beta(1-\alpha))]+\kappa \delta\left(d_{0}-y_{0}\right) \\
C=\delta(1+r-\kappa)\left(d_{0}-y_{0}\right) .
\end{gathered}
$$

Observe that

$$
\begin{gathered}
\Lambda(1)=A+B+C \\
=(1+r)\left(\frac{1+r-\alpha \kappa}{1+r} d_{1}^{\mathrm{NDL}}-\left(d_{0}^{\mathrm{NDL}}-d_{0}\right) \delta\right),
\end{gathered}
$$

and by inspection, the condition of (1.33) applied to period 0 is equivalent to $\Lambda(1) \leq 0$. Note that since $A>0, \Lambda(\cdot)$ has positive end behavior and thus can have a root in $(0,1)$ only if

$$
\Lambda(0)=C>0 \Longleftrightarrow d_{0}>y_{0} .
$$

This shows necessity of condition 2 . Sufficiency of condition 1 and 2 follows from the fact that $\Lambda(\cdot)$ has a root in $(0,1)$ if $\Lambda(1)<0$ and $\Lambda(0)>0$.

Proposition 7 shows that weak fundamentals, including high foreign debt, make economies more vulnerable to financial crises driven by nonfundamental shocks. Note that from our discussion in section 1.4, the gross interest rate must be small relative to the combined impact of patience and the importance of capital (in the sense that $1+r<\alpha \kappa+\delta$ ) if $d_{0}>y_{0}$ and the ESE existence condition is to be satisfied.

Note the 0-STE is welfare-inferior relative to the ESE since it results in a reallocation of consumption relative to the latter. In particular, the consumption paths in both equilibria have the same lifetime net present value, but the initial rationing of the 0-STE is substantial enough to render it welfare-inferior (see Figure 2.3). Moreover, the welfare gain of the ESE over the 0-STE is commensurate with the shadow value of the initial collateral constraint in the 0-STE. Formally, the

| Equilibrium | Constraint tightness | Conditions for existence (N=necessary, S=sufficient) |
| :---: | :---: | :---: |
| SSE | always tight | N and $\mathrm{S}:\left\{y_{t+1}\right\}_{t \geq 0}=y>0, d_{0}=\kappa k q^{*}-\left(y-y_{0}\right)$ |
| ESE | never tight | N and S: ESE existence condition (1.33) |
| 0 -STE | tight only initially | $\mathrm{S}:$ ESE existence condition (with strict inequality at $t=0), d_{0}>y_{0}$ |
| $N$-cyclic equilibrium | repeats every $N$ periods | $\mathrm{N}: N$-cyclic $\tilde{\mu}_{t}$ and $y_{t+1}, \Pi_{\tau \in \mathcal{N}} \tilde{\mu}_{\tau}=\delta^{N}$, unique $d_{0}$ (see Appendix A.2 for S$)$ |

Table 2.3: Types of equilibria
welfare gain of the ESE over the 0-STE is given by

$$
W\left(\tilde{\mu}_{0}\right) \equiv \sum_{t=0}^{\infty} \beta^{t} \log c_{t}^{\mathrm{UE}}-\sum_{t=0}^{\infty} \beta^{t} \log c_{t}^{0-\mathrm{STE}}
$$

which, by (2.4)-(2.5), simplifies to

$$
\begin{equation*}
W\left(\tilde{\mu}_{0}\right)=\frac{1}{1-\beta} \log \left((1-\beta) \tilde{\mu}_{0}^{\beta}+\beta \tilde{\mu}_{0}^{-(1-\beta)}\right) \tag{2.9}
\end{equation*}
$$

which is positive and strictly decreasing in $\tilde{\mu}_{0}$, since $W(1)=0$ and

$$
\operatorname{sign}\left\{\partial_{\tilde{\mu}_{0}} W\left(\tilde{\mu}_{0}\right)\right\}=\operatorname{sign}\left\{\partial_{\tilde{\mu}_{0}} \exp \left\{(1-\beta) W\left(\tilde{\mu}_{0}\right)\right\}\right\}=\operatorname{sign}\left\{-\beta(1-\beta) \tilde{\mu}_{0}^{-(1-\beta)}\left(\tilde{\mu}_{0}^{-1}-1\right)\right\}<0
$$

Thus, under weak fundamentals of the kind in Proposition 7, households effectively under-borrow in the 0-STE , such that a benevolent social planner would always prefer the ESE to the 0-STE. Various kinds of equilibria that we have discussed thus far and their characteristics are summarized in Table 2.3.

### 2.2.2 Constant output regime

Coexistence between the ESE and a welfare-inferior equilibrium requires a long-run depression. However, it is of interest to determine whether self-fulfilling financial crises can emerge in constant output regimes. Since such regimes necessarily lead to a tight collateral constraint infinitely often (Proposition 4), the simplest kind of equilibria to consider in this respect are 2-cyclic equilibria, which include the SSE. Since such equilibria exist under knife-edge initial borrowing,
coexistence between two such equilibria requires that the initial borrowing requisites coincide. Interestingly, under some of the same conditions from Proposition 6 that guarantee existence, parameters can be tuned to ensure the concurrence of the required $d_{0}$, and hence guarantee coexistence.

Lemma 2 For the set of 2-cyclic equilibria, in which each member is uniquely characterized by its period 0 normalized multiplier $\tilde{\mu}_{0} \in\left[\delta^{2}, 1\right]$, the following hold in a constant output regime:

1. The required $d_{0}$ is strictly decreasing in $\tilde{\mu}_{0} \in[\delta, 1]$.
2. The required $d_{0}$ is strictly decreasing in $\tilde{\mu}_{0}$ if either i) $\delta$ is sufficiently small, ii) r is sufficiently large, or iii) $\kappa$ is sufficiently small.The required $d_{0}$ is strictly decreasing in $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right]$ if $\delta \leq \frac{1+r-\kappa}{\sqrt{\kappa(1+\kappa)}}$.
3. The slack-tight equilibrium requires lower $d_{0}$ than any other 2 -cyclic equilibrium.
4. A tight-tight equilibrium has the same required $d_{0}$ as at most one other tight-tight equilibrium. For $r$ sufficiently small, and either i) $\delta$ large enough and $\kappa>\frac{1}{2}$ or ii) $\kappa$ sufficiently large, there are values $v_{1} \in\left(\delta^{2}, \delta\right)$ and $v_{2} \in(\delta, 1)$ such that any tight-tight equilibrium with $\tilde{\mu}_{0} \in\left(\delta^{2}, v_{1}\right)$ (resp. $\left.\tilde{\mu}_{0} \in\left(v_{1}, v_{2}\right)\right)$ has the same required $d_{0}$ as exactly one other tight-tight equilibrium with $\tilde{\mu}_{0} \in\left(v_{1}, v_{2}\right)\left(\right.$ resp. $\left.\tilde{\mu}_{0} \in\left(\delta^{2}, v_{1}\right)\right) .{ }^{8}$.
5. For $r$ sufficiently small, $\forall \tilde{\mu}_{0} \in\left(\delta^{2}, 1\right), \exists \kappa \in(0,1)$ such that the tight-tight equilibrium corresponding to $\tilde{\mu}_{0}$ has the same required $d_{0}$ as the tight-slack equilibrium.

Proof. See Appendix A.3.4.
Existence conditions in Proposition 6 combined with requisite initial borrowing properties in Lemma 2 immediately give the following result on coexistence.

Proposition 8 In a constant output regime,

1. No two distinct 2-cyclic equilibria both having $\tilde{\mu}_{0} \in[\delta, 1]$ can coexist.
2. For either i) $\delta$ sufficiently small, ii) r sufficiently large, or iii) $\kappa$ sufficiently small, no two distinct 2-cyclic equilibria can coexist. No two distinct 2-cyclic equilibria both having $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right]$ can coexist if $\delta \leq \frac{1+r-\kappa}{\sqrt{\kappa(1+\kappa)}}$.

[^22]3. The slack-tight equilibrium cannot coexist with any other 2-cyclic equilibrium.
4. A tight-tight equilibrium can coexist with at most one other tight-tight equilibrium. For $r$ sufficiently small, and either i) $\delta$ large enough and $\kappa>\frac{1}{2}$ or ii) $\kappa$ sufficiently large, there is an $\alpha \in(0,1)$ such that the SSE coexists with a tight-tight equilibrium having $\tilde{\mu}_{0}<\delta$.
5. For $r$ sufficiently small and $\delta$ large enough, $\forall \tilde{\mu}_{0} \in\left(\delta^{2}, 1\right), \exists \alpha, \kappa \in(0,1)$ such that the tight-tight equilibrium corresponding to $\tilde{\mu}_{0}$ coexists with the tight-slack equilibrium.

Note a 2 -cyclic equilibrium is completely characterized by $\tilde{\mu}_{0}$ (recall $\tilde{\mu}_{1}=\delta^{2} / \tilde{\mu}_{0}$ by Proposition 5 ). Lemma 2 is obtained by studying the shape of the requisite $d_{0}$ for 2 -cyclic equilibria as a function of $\tilde{\mu}_{0}$ in a constant output regime, visualized in Figure 2.4. Our proof of Lemma 2 in Appendix A.3.4 is based on studying the required $d_{0}$ (as a function of $\tilde{\mu}_{0}$ ) imputed by the binding constraint in period 1 for tight-tight and slack-tight equilibria (red curve in Figure 2.4), and separately studying the required $d_{0}$ imputed by the binding constraint in period 0 for tight-slack equilibria (blue dot in in Figure 2.4). The first three results of Lemma 2 and Proposition 8 give uniqueness conditions within subclasses of 2-cyclic equilibria based purely on the requisite $d_{0}$; they assert that two distinct 2-cyclic equilibria cannot coexist if one of them is the slack-tight equilibrium, or if they both have a period 0 constraint that is less tight than steady-state level (i.e. $\tilde{\mu}_{0} \geq \delta$ ), or if households are sufficiently impatient relative to the spread between gross interest and the fraction of collateralized capital, as these conditions induce incompatible initial borrowing restrictions whereby the requisite $d_{0}$ is greater for the equilibrium with the tighter period 0 constraint.

The last two results of Lemma 2 and Proposition 8 give conditions for multiplicity of equilibria. These conditions guarantee nonmonotonic behavior of the requisite $d_{0}$ as a function of $\tilde{\mu}_{0}$. We show in Appendix A.3.4 that for $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right)$, the required $d_{0}$ as a function of $\tilde{\mu}_{0}$ has a derivative with monotonically decreasing sign, implying a tight-tight equilibrium can have compatible requisite $d_{0}$ with at most one other tight-tight equilibrium. Clearly, by the second statement of Lemma 2 , a necessary condition for such compatibility is that $\delta>\frac{1+r-\kappa}{\sqrt{\kappa(1+\kappa)}}$, while the fourth statement of Lemma 2 gives sufficient conditions. We show that under a sufficiently small interest rate and $\delta$ approximately greater than $\frac{1-\kappa}{\kappa}$, the requisite $d_{0}$ in the limit as $\tilde{\mu}_{0} \rightarrow \delta^{2}$ is at a lower level than that
corresponding to the SSE and increases in the interval $\tilde{\mu}_{0} \in\left(\delta^{2}, v_{1}\right)$ (attaining a maximum when $\left.\tilde{\mu}_{0}=v_{1}<\delta\right)$ and decreases in the interval $\tilde{\mu}_{0} \in\left(v_{1}, 1\right)$ (attaining a minimum when $\tilde{\mu}_{0}=1$, corresponding to the slack-tight equilibrium), as seen in Figure 2.4. Thus, for a low enough interest rate, there is a window of infinitely many pairs of possible tight-tight equilibria having compatible initial borrowing when a sufficiently large amount of capital is collateralized, or at least half of capital is collateralized and households are sufficiently patient. While this result allows for the possibility of infinitely many coexisting pairs of tight-tight equilibria, obtaining sufficient conditions for such coexistence requires the existence of such equilibria in the first place, and is hence more delicate than simply ensuring compatible $d_{0}$ requirements. Statement 2 of Proposition 6 ensures existence of an arbitrary candidate tight-tight equilibrium by tuning $\alpha$ to ensure the collateral constraint binds for the particular $\tilde{\mu}_{0}$ considered. However, $\alpha$ cannot generally be tuned to ensure the constraint binds for two distinct candidate equilibria, while other parameters are restricted to give sufficient conditions for compatible $d_{0}$ or existence. Nonetheless, the SSE has a weak existence requirement since it exists under arbitrary parameters in a constant output regime subject to its $d_{0}$ requirement. Thus, statement 4 of Proposition 8 gives a more conservative result of ensuring coexistence of the SSE with a tight-tight equilibrium having a period 0 constraint tighter than the steady-state level.

Table 2.4 shows various parametrizations supporting 2-cyclic equilibria in a constant output regime. Parametrizations 1-2 show how a slack-tight (ST) equilibrium is supported when $\alpha \kappa$ is large enough (No. 1) or $\delta$ is large enough (No. 2) in accordance with statement 1 of Proposition 6. Under the same parameters, a tight-slack equilibrium that is a mirror image of the slack-tight equilibrium exists; of course, a slack-tight equilibrium requires lower $d_{0}$ than the tightslack equilibrium (statement 3 of Lemma 2). ${ }^{9}$ In accordance with statements 4-5 of Proposition 8, parametrizations 3-5 show how $\alpha$ or $\kappa$ can be calibrated to ensure coexistence of the SSE with another tight-tight (TT) equilibrium having $\tilde{\mu}_{0}<\delta$, particularly under sufficiently large $\kappa$ (No. 3) or large enough $\delta$ (No. 4), or coexistence of a tight-tight equilibrium with the tight-slack equilibrium

[^23]

Figure 2.4: Required $\frac{d_{0}-y_{0}}{y}$ of 2-cyclic equilibria as a function of $\tilde{\mu}_{0}$ in a constant output regime imputed by a binding period 0 constraint and alternatively a binding period 1 constraint. Parameters here are set as $r=0.01, \delta=0.95, \kappa=0.65, \alpha=0.3,\left\{y_{t}\right\}_{t \geq 1}=y>0$. The blue point corresponds with the requisite $d_{0}$ that sustains a tight-slack (TS) equilibrium, while the red point corresponds with the requisite $d_{0}$ that sustains a slack-tight (ST) equilibrium. If the blue and red curves intersect at an interior $\tilde{\mu}_{0}$, then a tight-tight equilibrium associated with that $\tilde{\mu}_{0}$ can be sustained at the corresponding requisite $d_{0}$ (they always intersect at $\tilde{\mu}_{0}=\tilde{\mu}^{*}$ ). If the blue and red curve intersect at two distinct interior $\tilde{\mu}_{0}$ at the same level of $\frac{d_{0}-y_{0}}{y}$, then two distinct tight-tight equilibria coexist.
(No.5, also in Table 2.1). ${ }^{10}$
Between two coexisting 2-cyclic equilibria under general fundamentals (including variable output regime), the one with the tighter period 0 constraint (smaller $\tilde{\mu}_{0}$ ) is welfare-inferior. To see this result formally, we obtain the welfare gap between two 2-cyclic equilibria in terms of their period 0 normalized multipliers ( $\tilde{\mu}_{0}^{2-c y c, 1}, \tilde{\mu}_{0}^{2-c y c, 2}$ ) in Appendix A.3.5:

$$
W\left(\tilde{\mu}_{0}^{2-\mathrm{cyc}, 1}, \tilde{\mu}_{0}^{2-\mathrm{cyc}, 2}\right) \equiv \sum_{t=0}^{\infty} \beta^{t} \log c_{t}^{2-\mathrm{cyc}, 1}-\sum_{t=0}^{\infty} \beta^{t} \log c_{t}^{2-\mathrm{cyc}, 2}
$$

[^24]| No. | $r$ | $\delta$ | $\kappa$ | $\alpha$ | Equilibrium | $\frac{c_{0}}{y}, \frac{c_{1}}{y}$ | $\frac{d_{1}}{y}, \frac{d_{2}}{y}$ | $\frac{\kappa k q_{0}}{y}, \frac{\kappa k q_{1}}{y}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0.01 | 0.65 | 0.95 | 0.30 | ST | 1.1441, 0.7437 | 5.7665, 5.5653 | 5.7924, 5.5653 |
| 2 | 0.06 | 0.95 | 0.30 | 0.30 | ST | 0.9738, 0.9251 | 0.9053, 0.8802 | 0.9153, 0.8802 |
| 3 | 0.01 | 0.65 | 0.95 | 0.23502055 | TT ( $\tilde{\mu}_{0}=0.62105263$ ) | 0.9263, 0.9694 | 5.2556, 5.2773 | 5.2556, 5.2773 |
|  |  |  |  |  | SSE | 0.9477 | 5.2773 | 5.2773 |
| 4 | 0.01 | 0.95 | 0.65 | 0.05960432 | TT ( $\tilde{\mu}_{0}=0.90769231$ ) | 0.9644, 1.0093 | 1.3158, 1.3384 | 1.3158, 1.3384 |
|  |  |  |  |  | SSE | 0.9867 | 1.3384 | 1.3384 |
| 5 | 0.01 | 0.95 | 0.18272563 | 0.30 | TS | 0.9646, 1.0154 | 0.9983, 1.0238 | 0.9983, 1.0427 |
|  |  |  |  |  | SSE | 0.9899 | 1.0238 | 1.0238 |

Table 2.4: Various parametrizations supporting 2-cyclic equilibria $\left(\left\{y_{t}\right\}_{t \geq 1}=y>0, d_{0}=d_{2}-y+\right.$ $y_{0}$ ).

$$
=\frac{1}{1-\beta^{2}}\left[\log \left(\frac{\tilde{\mu}_{0}^{2-\mathrm{cyc}, 1}}{\left(\tilde{\mu}_{0}^{2-\mathrm{cyc}, 1}+\beta\right)^{1+\beta}}\right)-\log \left(\frac{\tilde{\mu}_{0}^{2-\mathrm{cyc}, 2}}{\left(\tilde{\mu}_{0}^{2-\mathrm{cyc}, 2}+\beta\right)^{1+\beta}}\right)\right] .
$$

Thus,

$$
\left.\left.W\left(\tilde{\mu}_{0}^{2-c y c, 1}, \tilde{\mu}_{0}^{2-c y c}, 2\right)>0 \Longleftrightarrow \frac{\tilde{\mu}_{0}^{2-c y c}, 1}{\left(\tilde{\mu}_{0}^{2-c y c}, 1\right.}+\beta\right)^{1+\beta}>\frac{\tilde{\mu}_{0}^{2-c y c, 2}}{\left(\tilde{\mu}_{0}^{2-c y c}, 2\right.}+\beta\right)^{1+\beta} \Longleftrightarrow \tilde{\mu}_{0}^{2-c y c, 1}>\tilde{\mu}_{0}^{2-c y c, 2}
$$

where the lattermost implication follows from the fact that

$$
\operatorname{sign}\left\{\partial_{\tilde{\mu}_{0}} \frac{\tilde{\mu}_{0}}{\left(\tilde{\mu}_{0}+\beta\right)^{1+\beta}}\right\}=\operatorname{sign}\left\{\left(\tilde{\mu}_{0}+\beta\right)^{1+\beta}-(1+\beta) \tilde{\mu}_{0}\left(\tilde{\mu}_{0}+\beta\right)^{\beta}\right\}=\operatorname{sign}\left\{\beta\left(\tilde{\mu}_{0}+\beta\right)^{\beta}\left(1-\tilde{\mu}_{0}\right)\right\} \geq 0,
$$

with the inequality binding only when $\tilde{\mu}_{0}=1$. We also see in parametrizations 3-5 of Table 2.4 how households underborrow in the welfare-inferior 2-cyclic equilibrium relative to the SSE. Note that all examples of multiple equilibria that we have looked at occur when $d_{0}>y_{0}$, which was a necessary feature in Proposition 7. It is of interest in subsequent research to consider whether $d_{0}>y_{0}$ is indeed always necessary for multiple equilibria.

### 2.3 The case when $\delta \geq 1$

As a final discussion point, it may be of interest to the theoretically-minded reader see how robust the findings of this paper are under a more general patience assumption. This paper has
largely been framed in the context of $\delta<1$ due to both theoretical and empirical interest and to ease exposition, but many analytic results are not sensitive to this assumption, though the interpretations may be different. The $\tilde{\mu}_{t}$-characterization primarily shows the dependency of equilibria on the shadow values of the collateral constraint and was obtained free of assumptions on $\delta$. Consequently, the ESE and ESE existence condition obtained in section 1.4 still apply under $\delta \geq 1$. Likewise, the main results of Fisherian deflation, consumption rationing, deleveraging, and multiplicity of equilibria under variable output-as given in Propositions 1, 2, 3, and 7 along with their associated corollaries, in addition to Claims 1 and 2—still apply. ${ }^{11}$ It is worth noting that the ESE existence condition is far more permissive when $\delta>1$, since output can grow unboundedly over time, though it cannot persistently grow faster than the relative patience ratio. Thus, the allocation of the unconstrained equilibrium can be supported under eventually constant output or indeed an eternal economic expansion, an interesting counterpart to the eternal depression example discussed in section 1.4.

The primary difference when $\delta \geq 1$ is in deterministic cycles and long-run behavior. When $\delta=1$ the SSE has an always slack constraint $\left(\tilde{\mu}^{*}=1\right)$ and coincides with the ESE in a regime with constant output from period 1 onward, while when $\delta>1$ there can be no SSE since the equilibrium normalized multiplier given by (1.20) would exceed unity-instead, consumption would always grow. More generally, it is clear that an $N$-cyclic equilibrium (for $N \geq 2$ ) cannot exist for $\delta \geq 1$ since it is impossible for the geometric average condition (2.1) of Proposition 5 to be met when the constraint is tight at least once. These observations have immediate implications on the long-run behavior discussed in section 1.5.4. Statement 1 of Proposition 4 (and relatedly statement 1 of Corollary 3), whereby a well-defined natural debt limit prohibits the normalized multiplier from being too small in the long run (weakly smaller than $\beta$ ), still applies when $\delta \geq 1$. Moreover,

[^25]Lemma 1 and statement 3 of Proposition 4 concerning equilibria that are less tight than the SSE in the long run do not apply when $\delta \geq 1$ since an equilibrium can neither have $c_{t} \rightarrow 0$ nor a negative shadow value of the constraint. Correspondingly, statement 3 of Corollary 3 is trivially true regardless of output since $\tilde{\mu}_{t} \leq 1 \leq \delta$. Statement 2 of Proposition 4 and statement 2 of Corollary 3 are slightly different, however, when $\delta \geq 1$ :

## Case 1

$\delta=1$ : Here, $\tilde{\mu}^{*}=1$ and from (1.3), consumption growth is always non-negative in equilibrium. The second statement of Proposition 4 still holds; that is, an equilibrium can never have a nonvanishingly tight constraint infinitely often under a bounded output trajectory. The second statement of Corollary 3 then amounts to asserting that under a path of bounded output, an equilibrium must feature stabilizing tightness with $\tilde{\mu}_{\infty}=1$.

## Case 2

$\delta>1$ : Here, $\tilde{\mu}^{*}>1$ and from (1.3), consumption growth is always nonvanishingly positive in equilibrium. Consequently, unless output is permitted to grow unboundedly, an equilibrium necessarily features agents eventually assuming a permanent unboundedly growing lending position, implying that the collateral constraint eventually slackens. Thus, the second statement of Proposition 4 and the second statement of Corollary 3 are both replaced by the following statement: if the output path is bounded, an equilibrium has some threshold time $T$ so that $\tilde{\mu}_{\tau}=1 \forall \tau \geq T$ and hence must feature stabilizing tightness with $\tilde{\mu}_{\infty}=1$.

These observations on stabilizing tightness are summarized in Figure 2.5, where the bold and shaded regions indicate possible (for $\delta \in(0,1)$ ) and necessary (for $\delta \in[1,1+r$ )) values of the limiting normalized multiplier as a function of the relative patience ratio (fixing $r$ ) when the output path is bounded. Though Figure 2.5 formally shows the limiting normalized multiplier under stabilizing tightness, it may be understood more loosely as conveying the feasible long-run behavior of the normalized multiplier in the sense that it cannot remain below the bolded boundary line


Figure 2.5: Stabilizing tightness under bounded output trajectory
persistently in the long run (and nonvanishingly so infinitely often) in the way described in Proposition 4 . We see that the greater $\delta$ is, the less flexibility afforded to the long-run behavior of the normalized multiplier. The intuition is as follows: The more patient agents are, the less they prefer frontloading consumption. Absent an unboundedly growing output path, less of such consumption frontloading (more forward-pushing behavior) implies households do not have as much in future trade balances to borrow against in the long run. Since a lower normalized multiplier makes the shadow cost of borrowing higher, it exacerbates the behavior from higher $\delta$ by inducing even more forward-pushing behavior. Forward-looking agents who are thus more patient would then account for this behavior by borrowing less, thereby facing a larger long-term normalized multiplier. When households are so patient that they never frontload consumption and in fact would only push forward their consumption $(\delta \geq 1)$, their shadow cost of borrowing must vanish in the long run ( $\mu_{\infty}=1$ ), or else they would not have enough in future trade balances to borrow against under a persistently non-negligible shadow cost of borrowing that would push forward consumption even more.

### 2.4 Conclusion

This paper has analytically established how stock collateral constraints affect open economies when agents have general subjective discounting and face variable productivity. We have seen that it is possible for such economies to enjoy the unconstrained allocation under a condition on output that relates to their patience. When agents are impatient relative to the market, this condition requires that agents face a long-run depressionary scenario that would incentivize them to deleverage over time and even possibly switch to an asset position to protect themselves from the constraint. This condition is more permissive and allows for an economic expansion when agents are more patient relative to the market. When this condition is not satisfied, the collateral constraint binds at least once, aggravating the business cycle. In particular, households deleverage in all periods and experience deflation of asset prices in all periods up to any constrained period relative to what would have been experienced absent the collateral constraint. Moreover, the more tightly constrained households expect to be in the future, the more they ration their consumption.

While collateral constraints exacerbate the business cycle, they also may be its engine and induce nonfundamental instability. When households are impatient relative to the market, under plausible fundamentals in a constant output regime, it is possible for households to face deterministic 2-cycles and even 3-cycles, periodically vacillating between more constrained and less constrained periods, and these cycles can also coexist. In variable output regimes, a welfareinferior equilibrium with an initially binding constraint can coexist with one that always yields the unconstrained allocation.

All instances of multiple equilibria that we have looked at in our model feature an initial debt-to-output ratio greater than one, and it would be of interest in future work to determine whether this condition is necessary for multiplicity. It would also be of interest to more fully characterize a debt policy function under an appropriate equilibrium selection rule. The existence of such a function would imply existence of debt cycles of any periodicity and chaos. Further, the properties of such a function would allow an analysis of the stability of different debt cycles.

# Chapter 3: Robust Pricing Mediation in Bargaining 

### 3.1 Introduction

Price contracting between a buyer and seller under mutual uncertainty of their reservation values appears in economic and legal applications. For instance, in a corporate acquisition, the acquirer is a buyer and the acquiree is a seller, both of whom may be uncertain about the value of the acquiree. In the legal domain, a defendant and plaintiff may be a buyer and seller respectively of an out-of-court settlement claim, both of whom may be uncertain of the potential damages that would be mandated and legal costs that would be incurred if the damage suit was taken to trial. ${ }^{1}$ Such settings are often amenable to intervention by a mediator who has general historical information on which to recommend a price contract, even if not possessing particular information on the reservation values of the parties involved. The mediator may manifest as an online trading platform, which is able to efficiently gather information and make such a recommendation. Such a platform may be endowed with a historical repository of quantifiable information on the distribution of reservation values for the transacting parties. Our interest, however, is in a much weaker informational setting where no one is equipped with such quantifiable information of risks. ${ }^{2}$ We focus on risk neutral trading parties who must contract on a transaction price before themselves realizing their values and thus seek the help of an uncertainty-averse mediator who is fundamentally benevolent. The benevolence is reflected in the mediator's preferences in that the mediator would choose a price contract identical to what the parties themselves would choose if all were equipped with complete information on reservation values. We also allow for parties to have asymmetric bargaining power, reflecting exogenous institutional features. In the legal and corporate acquisition

[^26]examples, the transacting parties may be risk neutral firms having different powers (e.g. powers in attorney access or market powers) who hire an arbitrator, whose compensation may be tied with the ultimate success of the price contract when values are realized. In this setting, how should the mediator choose a price contract when only equipped with sparse summary statistics on the reservation values of the parties? How does this contract compare with what the parties would choose if equipped with the same information? Moreover, how resilient are these price contracts to changes in bargaining power?

In our model, a mediator chooses a price at which a buyer and seller trade an indivisible good. The unknown reservation values are the valuation that the buyer gains and the cost that the seller incurs, both drawn from some unknown joint distribution (independently so in the baseline model). With complete information on their ex post reservation values, the buyer and seller negotiate a price a price as per the Nash bargaining solution (Nash 1953; Nash 1950), maximizing the Nash product of payoffs under exogenous bargaining powers $\alpha, 1-\alpha \in(0,1)$ afforded to the buyer and seller respectively. This choice, the complete information price (CIP), represents a simple weighted average of the ex post buyer's valuation and seller's cost, the weighting determined by the bargaining powers. Absent complete information, the trading parties recruit the mediator, who only has aggregate historical information-captured in our model as means and a feasible range of the buyer and seller values- and understands there is surplus to be enjoyed from trade in expectation. Under such limited information, the ambiguity-averse mediator chooses the incomplete information price (IIP) to maximize the distributionally worst-case expected Nash product by playing against an adversarial Nature in a Stackelberg game, an approach we discuss more in the sequel.

The object of this paper is to explore the analytic properties of the IIP and determine how the IIP compares with the mean CIP, which is what the trading parties themselves would negotiate if symmetrically equipped with the mediator's information. We first explicitly obtain worst-case distributions that Nature would choose in response to the mediator in the determination of the IIP, both when buyer and seller values are independent (Proposition 9) and when they are permitted to be dependent (Proposition 13). The admission of dependence always makes the mediator worse
off, and in fact, may prevent trade from ever occurring in the worst case regardless of the mediator's price strategy when the expected surplus from trade is small enough-we refer to such a scenario as a dismal equilibrium, and fully characterize its necessary and sufficient conditions in Corollary 5. Nonetheless, regardless of the prospect of dependence, it is always optimal for Nature in equilibrium to ensure the marginal distribution of each party's value has two-point support containing the boundary point of the feasible range at which that party would trade and the mediator's price (at which that party would not trade), the masses accordingly chosen to respect the mean constraints. We establish in Propositions 10 and 14 how the mean CIP and the IIP-both in the cases of independent and dependent buyer and seller values-compare in terms of bargaining power. We show that while the mean CIP is fully sensitive to bargaining power, the IIP has a more shallow dependency on bargaining power, the shallowness only exacerbated when the mediator faces the prospect of dependent values. That is, there is a unique critical bargaining power level where the two price objects will concur, but the mediator does not incorporate the full effect of the bargaining power in the IIP relative to the mean CIP at other bargaining power levels. Further, the critical bargaining power is more in favor of the buyer (resp. seller) if the feasible range of the buyer's valuation above the mean valuation is greater (resp. less) than the feasible range of the seller's cost below the mean cost.

We then pursue comparative statics results for both the IIP and this critical bargaining power at which IIP and mean CIP coincide. Propositions 11 and 15 show how IIP depends upon the model fundamentals-namely the means, feasible range bounds, and bargaining power. While the IIP and mean CIP both increase in the means and seller's bargaining power, the IIP, somewhat counterintuitively, is set lower when the bounds of feasible ranges are higher in the independent values case in order to offset the party made more likely to trade by Nature's mean-preserving strategy. The prospect of dependence complicates this relationship in potentially nonmonotonic ways, particularly when bargaining power is not extreme. However, Proposition 15 shows that when a party's bargaining power is small enough, the IIP decreases in the bound of the feasible range at which that party would trade (if their value realized that bound). Proposition 12 obtains how the critical
bargaining power at which IIP and mean CIP intersect depends upon model fundamentals in the independent values case. Here, we show the critical bargaining power of the buyer increases in the feasible range bounds but behaves nonmonotonically in the mean buyer and seller values due to competing effects from the mean CIP and IIP; however, when the expected surplus is large enough, the critical buyer bargaining power decreases in the means.

The remainder of this paper is organized as follows: Section 3.2 presents and solves the baseline model where buyer and seller values are independent, and discusses model assumptions in detail. Section 3.3 discusses the comparison between the IIP and mean CIP. Section 3.4 discusses comparative statics results. Section 3.5 extends results of the baseline model obtained in the preceding sections to allow for dependent values. Section 3.6 concludes.

## Related literature

The distinction between quantifiable "risk" and unquantifiable "uncertainty" goes back to at least Knight 1921. While problems of risk are often approached in a Bayesian setup under the assumption of a known prior distribution (either objective or subjective), problems of Knightian uncertainty are not immediately amenable to an obvious solution. Apart from the issue of how to decide on an appropriate prior, the Bayesian approach to model such uncertainty suffers from compelling experimental critiques pointed out by Ellsberg 1961. ${ }^{3}$ The problem of modeling decision making under Knightian uncertainty found a rigorous solution concept in the theory of statistical decision functions (c.f. Wald 1949, Blackwell and Girshick 1954), which regards the problem of a decision maker faced with distributional uncertainty as a zero-sum game with Nature. This approach of a maxmin game with Nature found a rigorous economic footing in the seminal paper Gilboa and Schmeidler 1989 that axiomatized the preferences of a maxmin expected utility decision maker in a way that is consistent with the critique of Ellsberg 1961. ${ }^{4}$

[^27]The approach of robust decision making in a game with Nature is congruent with Wilson's doctrine (Wilson 1987) in not relying on complete information of prior distributions and has also enjoyed recent application in mechanism design and operations research among others. One strand in these literatures takes a minimax regret approach, which minimizes the worst-case shortfall of the chosen mechanism relative to the complete-information optimal mechanism. This approach can be traced back to at least Savage 1951, 1954 and was independently suggested by Niehans 1948 (c.f. Bergemann and Schlag 2008, Guo and Shmaya 2019, and Koçyiğit, Rujeerapaiboon, and Kuhn 2021 among some recent applications). Another strand adopts a maxmin payoff approach, which is what we pursue. Such work often assumes various moment or boundary conditions are known, thus becoming constraints for adversarial Nature. The methods used in this respect trace back to Scarf 1957, who looks at a worst-case demand distribution under a mean and variance constraint in a problem of inventory management. Delage and Ye 2010 likewise look at distributionally robust stochastic programs under first and second moment constraints, allowing for some degree of moment uncertainty. Blanchet et al. 2020 show how dropout training used in generalized linear models in machine learning is minimax under mean and range constraints. Meanwhile, a similar approach has been adopted in mechanism design applications: Carrasco et al. 2018 consider the monopoly pricing problem in the case of a single buyer in the setup of a simultaneous game with Nature under arbitrarily many moment conditions, and solves the special case of a known mean and range. Che 2022 solves the same special case in the more general setting of multiple buyers and looks at optimal auction format design (c.f. also Brooks and Du 2021). Che and Zhong 2021 broaden the application to selling multiple goods. Carroll 2015, 2019 look at contracting settings. We adapt methods used in this literature in tractably solving Nature's problem (c.f. Wiesemann, Kuhn, and Sim 2014 for discussion on tractability), even while looking at a novel application of distributionally robust decision making to a bargaining setup.

While many works look at a saddle point solution, others, including Wolitzky 2016 and Koçyiğit et al. 2020, look at a Stackelberg game with Nature, which is what we pursue. There are some
compelling model motivations for the sequential approach. Analytically, our sequential maxmin approach renders the model quite tractable, in contrast to solving for a saddle point in a simultaneous game. By the max-min inequality, zero-sum games have a second mover advantage, so in this sense, we exacerbate the worst-case approach that the mediator takes in choosing a price. More fundamentally, however, a sequential maxmin game with Nature can be interpreted metaphorically as representing the preferences of a mediator who simply is a minimal expected utility maximizer, which has an axiomatic basis in uncertainty aversion (c.f. Gilboa and Schmeidler 1989). Thus, the Stackelberg setup allows us to obviate the need for Nature in the first place by simply assuming an uncertainty-averse mediator.

### 3.2 Model

### 3.2.1 Setup

A mediator chooses a transaction price at which a risk neutral buyer and seller trade an indivisible good. If the trade occurs at price $p$, the buyer enjoys valuation $v$ and pays the seller the price $p$, while the seller incurs a cost $c$. If trade does not occur, both buyer and seller attain an outside option payoff of zero. Neither the parties nor mediator observe the values $v$ and $c$ when deciding the price, but the mediator knows that these values are independently drawn from an unknown joint distribution, and that the supports of their marginal distributions are contained in a known range $[\underline{w}, \bar{w}]$, where $\underline{w} \geq 0$. The mean values of $v$ and $c$ are also known by the mediator, where $\mathbb{E}[v]=\mu_{v}, \mathbb{E}[c]=\mu_{c}$, and $\bar{w}>\mu_{v}>\mu_{c}>\underline{w}$. Thus, the mediator knows there is positive surplus to be enjoyed in expectation, but does not know whether this is also the case ex post.

If symmetrically informed of both their values $v$ and $c$, the parties would negotiate a price $p$ without intervention according to the Nash bargaining solution with exogenous bargaining powers $\alpha$ and $1-\alpha$ afforded to the buyer and seller respectively, where $\alpha \in(0,1)$. That is, if there is surplus from trade, the transaction price under complete information, $p^{\mathrm{CIP}}$, is chosen to maximize
the weighted Nash product:

$$
\begin{gathered}
p^{\mathrm{CIP}} \in \underset{p \geq 0}{\arg \max }(v-p)^{\alpha}(p-c)^{1-\alpha} \mathbf{1}_{c<p<v} \\
=(1-\alpha) v+\alpha c .
\end{gathered}
$$

We call the above the complete information price (CIP), which is simply a weighted average of the buyer's valuation and seller's cost, the weights determined by bargaining power, so that $p^{\mathrm{CIP}} \uparrow v$ as $\alpha \downarrow 0$ and $p^{\mathrm{CIP}} \downarrow c$ as $\alpha \uparrow 1$.

In the absence of information of the values $v$ and $c$ as well as their joint distribution, agents act in the following manner: If the buyer and seller were symmetrically informed of both their mean values (which is the information the mediator has), the price they would negotiate is the mean CIP, defined as ${ }^{5}$

$$
\begin{equation*}
\bar{p}^{\mathrm{CIP}} \equiv(1-\alpha) \mu_{v}+\alpha \mu_{c} . \tag{3.1}
\end{equation*}
$$

This choice follows from the assumed risk neutrality of the buyer and seller and the fact that the Nash bargaining solution assumes that payoffs in the Nash product are von Neumann-Morgenstern utilities (c.f. Harsanyi 1956). In constrast, the mediator plays against adversarial Nature by choosing a price (possibly in mixed strategies) in an attempt to maximize the mean Nash product, whereupon Nature, observing a realized price from the mediator's strategy, adversarially chooses a Borel probability measure on $\mathbb{R}_{\geq 0}^{2}$ from the set of all such measures, $\mathcal{P}\left(\mathbb{R}_{\geq 0}^{2}\right)$, subject to known mean and boundary constraints to minimize the mean Nash product. Explicitly, admitting $(v, c)$ as an $\mathbb{R}_{\geq 0}^{2}$-valued random vector, the mediator chooses the incomplete information price (IIP) as ${ }^{6}$

$$
\begin{equation*}
p_{\perp}^{\mathrm{IIP}} \in \underset{p \geq 0}{\arg \max } \min _{H \in \mathcal{P}\left(\mathbb{R}_{\geq 0}^{2}\right)} \mathbb{E}_{(v, c) \sim H}\left[(v-p)^{\alpha}(p-c)^{1-\alpha} \mathbf{1}_{c<p<v}\right] \tag{3.2}
\end{equation*}
$$

[^28]subject to
\[

$$
\begin{equation*}
v \text { and } c \text { independent: } v \perp c \tag{3.3}
\end{equation*}
$$

\]

$$
\begin{gather*}
\mathbb{E}_{(v, c) \sim H}[(v, c)]=\left(\mu_{v}, \mu_{c}\right),  \tag{3.4}\\
\quad \operatorname{supp}[H] \subseteq[\underline{w}, \bar{w}]^{2} . \tag{3.5}
\end{gather*}
$$

While the buyer and seller do not play a strategic role in the above setup, we do consider when their corresponding terms in the Nash product are positive, and occasionally refer to their realized values as inducing buyer-side, seller-side, all-sides, and no-sides trade, corresponding with when $v>p, c<p, c<p<v$, and $v \leq p \leq c$ respectively. In the sequel, I denote a generic probability measure $H$ with finite support $\left\{h_{1}, \ldots, h_{k}\right\}$ having respective probability masses $\left\{\pi_{1}, \ldots, \pi_{k}\right\}$ by $H=\sum_{i=1}^{k} \pi_{i} \delta_{h_{i}}$.

### 3.2.2 Discussion of model assumptions

There are quite a few model features and assumptions that merit discussion. We have made assumptions on the exogenous parameters. First, we have assumed that buyer and seller values are independent. For the applications we consider, such as corporate acquisitions and legal disputes, this is a strong assumption that is likely not to hold. However, independence offers a baseline setting (considered in sections 3.2-3.4) from which we can derive useful insights and to which we compare with the more general setting in which values can be dependent (section 3.5). The possibility of dependence clearly offers more flexibility to Nature and thus contributes to exacerbating the worst-case setup of the mediator's problem. Second, we have assumed a common range for buyer and seller values, along with zero outside options. As will become apparent in the argument that follows, the minimum of the range of $c$ and the maximum of the range of $v$ are the primary relevant features, ${ }^{7}$ so a common range assumption merely dispenses of superfluous information. Moreover, zero outside options is without loss since nonzero outside options can be absorbed in

[^29]$v$ and $c$. Third, we have included a role for bargaining power, a departure from the symmetry assumption of the original Nash bargaining model (Nash 1953; Nash 1950). This inclusion plays an important role in our analysis; if the price is the instrument through which surplus is divided, then bargaining power refines this instrument, and as we shall formally see in section 3.3, this refining power-if appropriately tuned-allows us to consider the coincidence of various price contracts. We focus on interior bargaining powers $(\alpha \in(0,1))$ to simplify discussion, although our results can be extended fairly simply for the extreme cases where $\alpha=0,1$. In these extreme cases, the strategy for Nature that we consider would be valid so long as the mediator's payoff is zero when the price is exactly equal to either $c$ or $v$ (the indicator as written in (3.2)). ${ }^{8}$

It is also worth pointing out the role of implicit preferences and information structure in the game. Ex post, the mediator would choose a price that maximizes the Nash product just as the negotiating parties themselves would if their values were commonly known. In this sense, the mediator is understood to be benevolent in that the Nash product is an ex post utility function for the mediator. Consequently, admitting the mean Nash product as the mediator's payoff in the incomplete information setup assumes the mediator acts as a (worst-case) expected utility maximizer. This benevolence of the mediator can be motivated by augmenting the game to include a third stage (after the mediator and Nature) in which the buyer and seller both symmetrically observe their actual values $v$ and $c$ and the mediator's recommended price contract and compensate the mediator an amount that is commensurate with the ex post Nash product, a proxy for the ultimate success of the price contract. Note that by Jensen's inequality, for any giving pricing scheme, the mediator, endowed with such preferences, would be better off if the mean values of buyer and seller were realized with certainty rather than be subject to the incomplete information setting. However, the substance of Propositions 10 and 14 is that there is a unique bargaining power that would induce the mediator to choose exactly identical price contracts in these two setups, but the mediator would otherwise underweight the effect of bargaining power under incomplete information.

[^30]Our approach to resolving the unquantifiable uncertainty is through a Stackelberg game with Nature under known mean and boundary constraints, where Nature moves after observing a realized price from the mediator's strategy. Our constraints offer a fairly weak informational setting, although the ambiguity set of the mediator may be enriched by including higher-order moment constraints, a feature considered in Carrasco et al. 2018; for instance, we may consider an upper bound constraint on variance. Che and Zhong 2021 point out that such convex moment constraints tend to play similar roles in Nature's strategy as the boundary constraints that we consider. We have discussed some general model motivations for a sequential approach, but it is also worth noting that Nature's equilibrium strategy is robust to the mediator's payoff in the sequential game. For instance, as will become clear, Nature's optimal strategy in the sequential setup would remain unchanged if the mediator's objective was to maximize the probability of trade, in which case the objective in (3.2) would be given by $\mathbb{E}_{(v, c) \sim H}\left[\mathbf{1}_{c<p<v}\right]$; the exercise of obtaining the mediator's price under this objective is pursued in Appendix B.2. The concavity associated with the mediator's objective, rather than the particular form of the objective, is what is material to adversarial Nature in a Stackelberg game. Contrastingly, Nature is sensitive to the form of the objective in a simultaneous game since Nature's strategy is then chosen to make the mediator indifferent across the support of the mediator's mixed strategy. As such, a simpler objective-maximizing worstcase probability of trade, for instance- would make the model more amenable to constructively obtaining a saddle point solution, although this objective abstracts away from bargaining power considerations, which is a key focus of our exercise.

### 3.2.3 Nature's problem

By independence, Nature's problem reduces to choosing $F^{*}, G^{*}$ given price $p$ where

$$
\begin{equation*}
F^{*} \in \underset{F \in \mathcal{P}\left(\mathbb{R}_{\geq 0}\right)}{\arg \min } \mathbb{E}_{v \sim F}\left[(v-p)^{\alpha} \mathbf{1}_{p<v}\right] \tag{3.6}
\end{equation*}
$$

$$
\begin{equation*}
G^{*} \in \underset{G \in \mathcal{P}\left(\mathbb{R}_{\geq 0}\right)}{\arg \min } \mathbb{E}_{c \sim G}\left[(p-c)^{1-\alpha} \mathbf{1}_{c<p}\right], \tag{3.7}
\end{equation*}
$$

subject to (3.4) and (3.5).
If the mediator chooses price $p \geq \mu_{v}$, then it suffices for Nature to choose $F$ having unit mass at $\mu_{v}$ and to choose any $G$ that satisfies its mean constraint. Likewise, if the mediator chooses price $p \leq \mu_{c}$, then it suffices for Nature to choose $G$ having unit mass at $\mu_{c}$ and to choose any $F$ that satisfies its mean constraint. To solve Nature's strategy if the mediator's price is chosen as $p \in\left(\mu_{c}, \mu_{v}\right)$, we appeal to the following lemma, adapted from Lemma 1 in Che 2022:

Lemma 3 Suppose $\mathcal{P}(\Omega, \mu)$ is the set of Borel probability measures on $\mathbb{R}^{n}$ with mean $\mu$ and having support contained in $\Omega$, where $\Omega$ is a compact subset of $\mathbb{R}^{n}$. Given function $\phi: \Omega \rightarrow \mathbb{R}$,

$$
\begin{equation*}
H^{*} \in \underset{H \in \mathcal{P}(\Omega, \mu)}{\arg \min } \mathbb{E}_{z \sim H}[\phi(z)] \tag{3.8}
\end{equation*}
$$

if and only if $H^{*} \in \mathcal{P}(\Omega, \mu)$ and there is an affine function $L: \Omega \rightarrow \mathbb{R}$ such that

1. $L(z) \leq \phi(z) \quad \forall z \in \Omega$
2. $\operatorname{supp}\left[H^{*}\right] \subseteq\{z: L(z)=\phi(z)\}$.

Proof. We show sufficiency (necessity can be shown using the dual program and is found in Che 2022). Observe that for any $H \in \mathcal{P}(\Omega, \mu)$, we have

$$
\mathbb{E}_{z \sim H^{*}}[\phi(z)]=\mathbb{E}_{z \sim H^{*}}[L(z)]=\mathbb{E}_{z \sim H}[L(z)] \leq \mathbb{E}_{z \sim H}[\phi(z)],
$$

the first equality following from condition 2 , the second equality following from the mean constraint and affinity of $L$, and the final inequality following from condition 1 .

A straightforward application of Lemma 3 allows us to solve Nature's problems (3.6) and (3.7):

Proposition 9 If the mediator chooses price $p \in\left(\mu_{c}, \mu_{v}\right)$, then the unique equilibrium strategy for Nature is given by

$$
\begin{gather*}
F^{*}=\left(1-\pi_{\bar{w}}(p)\right) \delta_{p}+\pi_{\bar{w}}(p) \delta_{\bar{w}}  \tag{3.9}\\
G^{*}=\left(1-\pi_{\underline{w}}(p)\right) \delta_{p}+\pi_{\underline{w}}(p) \delta_{\underline{w}},  \tag{3.10}\\
\pi_{\bar{w}}(p)=\frac{\mu_{v}-p}{\bar{w}-p}  \tag{3.11}\\
\pi_{\underline{w}}(p)=\frac{p-\mu_{c}}{p-\underline{w}} . \tag{3.12}
\end{gather*}
$$

Proof. We appeal to Lemma 3. The payoff functions are given by $\phi(v)=(v-p)^{\alpha} \mathbf{1}_{p<v}$ and $\phi(c)=(p-c)^{1-\alpha} \mathbf{1}_{c<p}$, both of which are strictly concave over their positive portions. It is easily verified that $F^{*}$ having $\operatorname{supp}\left[F^{*}\right]=\{p, \bar{w}\}$ and the affine function $L(v)=\frac{v-p}{(\bar{w}-p)^{1-\alpha}}$ satisfy the conditions of the lemma for $\phi(v)$. Likewise, it is easily verified that $G^{*}$ having supp $\left[G^{*}\right]=\{\underline{w}, p\}$ and the affine function $L(c)=\frac{p-c}{(p-\underline{w})^{\alpha}}$ satisfy the lemma for $\phi(c)$. The masses $\pi_{\bar{w}}(p), \pi_{\underline{w}}(p)$ are determined by the mean constraints.

To show uniqueness, we appeal to the necessity of the conditions of the lemma. It is clear that a singleton support at the mean is not optimal as this would ensure certain trade for the corresponding party, so the supports must have at least two points. Each support cannot contain a non-boundary point in the positive portion of the payoff function; otherwise, any affine function intersecting the payoff function at such a point (by condition 2 of the lemma) would violate condition 1 of the lemma for some portion of the feasible range. Each support cannot contain a point in the zero portion of the payoff function other than price $p$; otherwise, any affine function would either violate condition 1 or must be identically zero. The latter cannot be true or else the support would only be contained within the zero portion of the payoff functions (by condition 2) so that supp [ $F$ ] (resp. supp $[G]$ ) would always lie below (resp. above) the mean.

The intuition of Proposition 9 is as follows: Nature chooses $F$ and $G$ to put as much of their mass as possible at values where trade does not occur, subject to obeying the moment conditions. For each distribution, this is done precisely by allocating mass over a two-point support that includes price $p$ - at which trade does not occur- and the boundary point at which trade does occur.

The masses are determined by the moment conditions, so that more mass of a distribution is placed at the transaction price if the price is closer to the distribution's mean.

### 3.2.4 Mediator's problem

Working backwards, the mediator chooses the IIP as

$$
p_{\perp}^{\mathrm{IIP}} \in \underset{p \in\left(\mu_{c}, \mu_{v}\right)}{\arg \max }\left\{\Pi_{\perp}(p) \equiv \mathbb{E}_{v \sim F^{*}}\left[(v-p)^{\alpha} \mathbf{1}_{p<v}\right] \mathbb{E}_{c \sim G^{*}}\left[(p-c)^{1-\alpha} \mathbf{1}_{c<p}\right]\right\}
$$

Substituting in (3.9)-(3.12) yields

$$
\begin{align*}
& p_{\perp}^{\mathrm{IIP}} \in \underset{p \in\left(\mu_{c}, \mu_{v}\right)}{\arg \max }(\bar{w}-p)^{\alpha}(p-\underline{w})^{1-\alpha} \pi_{\bar{w}}(p) \pi_{\underline{w}}(p)  \tag{3.13}\\
& =\underset{p \in\left(\mu_{c}, \mu_{v}\right)}{\arg \max }\left(\mu_{v}-p\right)\left(p-\mu_{c}\right)(\bar{w}-p)^{\alpha-1}(p-\underline{w})^{-\alpha} . \tag{3.14}
\end{align*}
$$

Note that since the objective is zero at $p=\mu_{c}$ and $p=\mu_{\nu}$, Rolle's theorem implies that there is an interior optimal price in the interval $\left(\mu_{c}, \mu_{v}\right)$ where the FOC will be satisfied. The FOC is given by $\Pi_{\perp}^{\prime}(p)=(\bar{w}-p)^{\alpha-2}(p-\underline{w})^{-\alpha-1}\left[2\left(\frac{1}{2}\left(\mu_{v}+\mu_{c}\right)-p\right)(\bar{w}-p)(p-\underline{w})+(p-((1-\alpha) \underline{w}+\alpha \bar{w}))\left(\mu_{v}-p\right)\left(p-\mu_{c}\right)\right]=0$.

Defining

$$
\begin{equation*}
h_{\perp}(p) \equiv(\bar{w}-p)^{2-\alpha}(p-\underline{w})^{1+\alpha} \Pi_{\perp}^{\prime}(p), \tag{3.16}
\end{equation*}
$$

we see $h_{\perp}$ is a cubic polynomial with a positive leading coefficient (of unity), satisfying $h_{\perp}\left(\mu_{c}\right)=$ $\left(\mu_{v}-\mu_{c}\right)\left(\bar{w}-\mu_{c}\right)\left(\mu_{c}-\underline{w}\right)>0$ and $h_{\perp}\left(\mu_{v}\right)=-\left(\mu_{v}-\mu_{c}\right)\left(\bar{w}-\mu_{v}\right)\left(\mu_{v}-\underline{w}\right)<0$, implying there is a unique $p_{\perp}^{\mathrm{IIP}} \in\left(\mu_{c}, \mu_{v}\right)$ that satisfies the FOC.

### 3.3 Comparison between IIP and mean CIP

An immediate question that arises is how the IIP obtained in section 3.2.4, $p_{\perp}^{\mathrm{IIP}}$, compares with the mean CIP obtained in section 3.2.1, $\bar{p}^{\mathrm{CIP}}$. Both prices are in the interval $\left(\mu_{c}, \mu_{v}\right)$, although
the former is chosen by the mediator, while the latter is what would be chosen by the transacting parties when equipped with the mediator's information. The next proposition establishes how the IIP and mean CIP relate to each other based on bargaining power.

Proposition 10 There is a unique $\alpha_{\perp}^{*} \in(0,1)$ such that for $\alpha \lesseqgtr \alpha_{\perp}^{*}$, we have $\bar{p}^{\mathrm{CIP}} \gtreqless p_{\perp}^{\text {IIP }}$. Moreover, $\alpha_{\perp}^{*} \gtreqless \frac{1}{2}$ when $\mu_{c}-\underline{w} \lesseqgtr \overline{>} \bar{w}-\mu_{v}$. Finally, $\lim _{\mu_{v}-\mu_{c} \downarrow 0} \alpha_{\perp}^{*}=\frac{1}{2}, \lim _{\mu_{c}-\underline{w} \downarrow 0} \alpha_{\perp}^{*}=1, \lim _{\bar{w}-\mu_{v} \downarrow 0} \alpha_{\perp}^{*}=$ 0 .

Proof. A useful fact to appeal to is the following:
Fact 3 A generic cubic polynomial in variable $x$ of the form

$$
M(x)=A x^{3}+B x^{2}+C x+D
$$

with $A<0$ (resp. $A>0$ ) will have a unique real root if it is weakly decreasing (resp. increasing) at its inflection point, which occurs if and only if

$$
0 \geq(\text { resp. } \leq) M^{\prime}\left(-\frac{B}{3 A}\right)=-\frac{B^{2}}{3 A}+C \Longleftrightarrow B^{2} \leq 3 A C .
$$

In order to see how bargaining power affects the relation between mean CIP and IIP, treat $\alpha$ as a variable. Define $h_{\perp}^{\mathrm{CIP}}(\cdot)$ as a function of $1-\alpha$ induced by the function $h_{\perp}(\cdot)$ evaluated at $\bar{p}^{\mathrm{CIP}}$ :

$$
\begin{equation*}
h_{\perp}^{\mathrm{CIP}}(1-\alpha) \equiv h_{\perp}\left(\bar{p}^{\mathrm{CIP}}\right) . \tag{3.17}
\end{equation*}
$$

Expanding (3.17), $h_{\perp}^{\mathrm{CIP}}(\cdot)$ is given by

$$
\begin{align*}
& h_{\perp}^{\mathrm{CIP}}(1-\alpha)=-\left(\mu_{v}-\mu_{c}\right)^{2}\left(\bar{w}-\mu_{v}+\mu_{c}-\underline{w}\right)(1-\alpha)^{3}+3\left(\mu_{c}-\underline{w}\right)\left(\mu_{v}-\mu_{c}\right)^{2}(1-\alpha)^{2} \\
& -\left(\mu_{c}-\underline{w}\right)\left(\mu_{v}-\mu_{c}\right)\left(2\left(\bar{w}-\mu_{c}\right)+\mu_{v}-\mu_{c}\right)(1-\alpha)+\left(\mu_{v}-\mu_{c}\right)\left(\mu_{c}-\underline{w}\right)\left(\bar{w}-\mu_{c}\right) . \tag{3.18}
\end{align*}
$$

Recall from section 3.2.4 that $h_{\perp}^{\mathrm{CIP}}(0)=h_{\perp}\left(\mu_{c}\right)=\left(\mu_{v}-\mu_{c}\right)\left(\bar{w}-\mu_{c}\right)\left(\mu_{c}-\underline{w}\right)>0$ and $h_{\perp}^{\mathrm{CIP}}(1)=$ $h_{\perp}\left(\mu_{v}\right)=-\left(\mu_{v}-\mu_{c}\right)\left(\bar{w}-\mu_{v}\right)\left(\mu_{v}-\underline{w}\right)<0$. Thus, $h_{\perp}^{\mathrm{CIP}}$ has at least one root in $(0,1)$. It is easily
verified that $h_{\perp}^{\mathrm{CIP}}$ satisfies the condition of Fact 3 with strict inequality and thus the root is unique, corresponding to the value of $\alpha$ for which the mean CIP satisfies the FOC and is thus equivalent to the IIP. Consequently, there is a unique $\alpha_{\perp}^{*} \in(0,1)$ such that for $\alpha \lesseqgtr \alpha_{\perp}^{*}$, we have $h_{\perp}^{\mathrm{CIP}}(1-\alpha) \lesseqgtr 0$, and thus $\bar{p}^{\text {CIP }} \gtreqless p_{\perp}^{\text {IIP }}$. Moreover, when $\alpha=\frac{1}{2}$, we have

$$
h_{\perp}^{\mathrm{CIP}}\left(\frac{1}{2}\right)=\frac{\left(\mu_{v}-\mu_{c}\right)^{2}}{8}\left[\left(\mu_{c}-\underline{w}\right)-\left(\bar{w}-\mu_{v}\right)\right] .
$$

Thus, we have $\frac{1}{2} \lesseqgtr \alpha_{\perp}^{*}$ when $h_{\perp}^{\text {CIP }}\left(\frac{1}{2}\right) \lesseqgtr 0 \Longleftrightarrow\left(\mu_{c}-\underline{w}\right)-\left(\bar{w}-\mu_{v}\right) \lesseqgtr 0$. Further, as $\left(\mu_{v}-\mu_{c}\right) \downarrow 0, \frac{1}{\left(\mu_{\nu}-\mu_{c}\right)} h_{\perp}^{\mathrm{CIP}}\left(\frac{1}{2}\right) \rightarrow 0 \Longrightarrow \alpha_{\perp}^{*} \rightarrow \frac{1}{2}$; as $\left(\mu_{c}-\underline{w}\right) \downarrow 0, h_{\perp}^{\mathrm{CIP}}(0) \rightarrow 0 \Longrightarrow \alpha_{\perp}^{*} \uparrow 1$; and as $\left(\bar{w}-\mu_{v}\right) \downarrow 0, h_{\perp}^{\mathrm{CIP}}(1) \rightarrow 0 \Longrightarrow \alpha_{\perp}^{*} \downarrow 0$.

Proposition 10 reveals the manner in which uncertainty aversion distorts the mediator's pricing strategy relative to what the buyer and seller would choose with the same information set. There is some critical intermediate level of bargaining power, $\alpha_{\perp}^{*}$, where the two prices coincide. In effect, if the mediator could commit before the game to a contract that endows the buyer with this critical bargaining power, then the mediator would act as the buyer and seller would. However, the IIP has an overall shallower dependency on $\alpha$ than the mean CIP, so that when the seller (resp. buyer) has bargaining power greater than $1-\alpha_{\perp}^{*}$ (resp. greater than $\alpha_{\perp}^{*}$ ), the mediator chooses a lower (resp. higher) price than what the buyer and seller would choose. This critical bargaining power will be more in favor of the buyer $\left(\alpha_{\perp}^{*}>\frac{1}{2}\right)$ if the feasible range of the buyer's valuation above its mean is greater than the feasible range of the seller's cost below its mean. The intuition on this latter result regarding the degree of asymmetry afforded by the critical bargaining power will become more clear in our comparative statics discussion in section 3.4. In Figure 3.1, we see the comparison between the IIP and mean CIP and how the critical bargaining power is thus affected under various parametrizations for the feasible range bounds (fixing the means). It can be seen that while the mean CIP is fully sensitive to bargaining power, IIP has a shallow dependency on bargaining power that ensures that they concur at a unique bargaining power in the way predicted by Proposition 10.


Figure 3.1: IIP vs. mean CIP as a function of bargaining power. Parameters are set so that $\mu_{c}$ and $\mu_{v}$ are fixed, while $\bar{w}$ and $\underline{w}$ vary.

### 3.4 Comparative Statics

It is clear from (3.1) how the mean CIP responds to the underlying parameters. If either the mean seller's cost or mean buyer's valuation increases, the mean CIP-the weighted average of the two- increases as well. Moreover, if the buyer has greater bargaining power, the mean CIP decreases toward the seller's mean cost. The next proposition establishes how these properties extend to the IIP.

Proposition $11 p_{\perp}^{\mathrm{IIP}}$ strictly increases in $\mu_{v}, \mu_{c}$ and strictly decreases in $\underline{w}, \bar{w}, \alpha$.

Proof. With slight abuse of notation, augment $h_{\perp}(\cdot)$ in (3.16) to admit generic parameter $\beta \in$ $\left\{\mu_{v}, \mu_{c}, \bar{w}, \underline{w}, \alpha\right\}$ as an argument. By definition of $p_{\perp}^{\mathrm{IIP}}$ and the implicit function theorem, we have

$$
h_{\perp}\left(p_{\perp}^{\mathrm{IIP}}, \beta\right)=0 \Longrightarrow \partial_{\beta} p_{\perp}^{\mathrm{IP}}=-\left.\frac{\partial_{\beta} h(p, \beta)}{\partial_{p} h(p, \beta)}\right|_{p=p_{\perp}^{\mathrm{IP}} .}
$$

From section 3.2.4, we know $h_{\perp}(\cdot, \beta)$ is strictly decreasing at $p_{\perp}^{\mathrm{IIP}}$. Thus, $\operatorname{sign}\left\{\partial_{\beta} p_{\perp}^{\mathrm{IIP}}\right\}$ $=\operatorname{sign}\left\{\partial_{\beta} h_{\perp}\left(p_{\perp}^{\mathrm{IIP}}, \beta\right)\right\}$. Differentiating yields

$$
\begin{gathered}
\partial_{\alpha} h_{\perp}\left(p_{\perp}^{\mathrm{IIP}}, \alpha\right)=-\left.\left(\mu_{v}-p\right)\left(p-\mu_{c}\right)(\bar{w}-\underline{w})\right|_{p=p_{\perp}^{\mathrm{IP}}}<0, \\
\partial_{\mu_{v}} h_{\perp}\left(p_{\perp}^{\mathrm{IPP}}, \mu_{v}\right)=\left.\frac{1}{\mu_{v}-p}\left[h_{\perp}\left(p, \mu_{v}\right)+\left(p-\mu_{c}\right)(\bar{w}-p)(p-\underline{w})\right]\right|_{p=p_{\perp}^{\mathrm{IP}}>0,} \\
\partial_{\mu_{c}} h_{\perp}\left(p_{\perp}^{\mathrm{IIP}}, \mu_{c}\right)=\left.\frac{1}{p-\mu_{c}}\left[-h_{\perp}\left(p, \mu_{c}\right)+\left(\mu_{v}-p\right)(\bar{w}-p)(p-\underline{w})\right]\right|_{p=p_{\perp}^{\mathrm{IP}}}>0, \\
\partial_{\underline{w}} h_{\perp}\left(p_{\perp}^{\mathrm{IIP}}, \underline{w}\right)=\left.\frac{1}{p-\underline{w}}\left[-h_{\perp}(p, \underline{w})-\alpha\left(\mu_{v}-p\right)\left(p-\mu_{c}\right)(\bar{w}-p)\right]\right|_{p=p_{\perp}^{\mathrm{II}}}<0, \\
\partial_{\bar{w}} h_{\perp}\left(p_{\perp}^{\mathrm{IIP}}, \bar{w}\right)=\left.\frac{1}{\bar{w}-p}\left[h_{\perp}(p, \bar{w})-(1-\alpha)\left(\mu_{v}-p\right)\left(p-\mu_{c}\right)(p-\underline{w})\right]\right|_{p=p_{\perp}^{\mathrm{IP}}}<0 .
\end{gathered}
$$

Proposition 11 thus establishes that the IIP responds to the parameters $\mu_{v}, \mu_{c}$, and $\alpha$ in the same way that the mean CIP does, a fairly intuitive result. The results for $\underline{w}$ and $\bar{w}$, while initially somewhat counterintuitive, are appreciated when the role of Nature is considered. When the lower bound of the feasible cost values, $\underline{w}$, is increased toward $\mu_{c}$, adversarial Nature is compelled by the moment conditions to place more mass of the seller's distribution at $\underline{w}$ for a given price, increasing the seller-side probability of trade. Since the buyer has some positive bargaining power, the mediator is thus afforded the flexibility to favor the buyer more by decreasing the price, which compels Nature to put more mass of the buyer's distribution at $\bar{w}$, thus increasing the buyer-side probability of trade. Conversely, when the upper bound of feasible buyer valuations, $\bar{w}$, is decreased toward $\mu_{\nu}$, Nature places more mass of the buyer's distribution at $\bar{w}$ for a given price, making buyer-side trade more likely. Since the seller has some positive bargaining power, the mediator is afforded the flexibility to favor the seller more by increasing the price, which compels Nature to put more mass of the seller's distribution at $\underline{w}$, thus making seller-side trade more likely.

It is also of interest to see how the critical bargaining power of Proposition 10 at which the mean CIP and IIP coincide responds to the fundamentals. The following proposition establishes


Figure 3.2: $\alpha_{\perp}^{*}$ as a function of various parameters.
these relations.

Proposition 12 1. $\alpha_{\perp}^{*}$ strictly increases in $\underline{w}$ and $\bar{w}$.
2. If $\mu_{c} \geq \frac{1}{2}(\underline{w}+\bar{w})$, then $\alpha_{\perp}^{*}<\frac{1}{2}$ strictly decreases in $\mu_{v}$. Otherwise, given $\bar{w}, \underline{w}, \mu_{c}$, there is a $\mu_{v}^{*} \in\left(\frac{1}{2}(\underline{w}+\bar{w}), \bar{w}+\underline{w}-\mu_{c}\right)$ so that $\alpha_{\perp}^{*}$ strictly increases (resp. decreases) in $\mu_{v}<\left(\right.$ resp. >) $\mu_{v}^{*}$.
3. If $\mu_{v} \leq \frac{1}{2}(\underline{w}+\bar{w})$, then $\alpha_{\perp}^{*}>\frac{1}{2}$ strictly decreases in $\mu_{c}$. Otherwise, given $\bar{w}, \underline{w}, \mu_{v}$, there is a $\mu_{c}^{*} \in\left(\bar{w}+\underline{w}-\mu_{\nu}, \frac{1}{2}(\underline{w}+\bar{w})\right)$ so that $\alpha_{\perp}^{*}$ strictly increases (resp. decreases) in $\mu_{c}>\left(\right.$ resp. <) $\mu_{c}^{*}$. Proof. See Appendix B.1.

The comparative statics results of Proposition 11 helps us better understand both Proposition 12 and the degree of asymmetry of $\alpha_{\perp}^{*}$ specified by Proposition 10 . First let us consider how $\alpha_{\perp}^{*}$
depends on the bounds $\underline{w}$ and $\bar{w}$. If $\mu_{c}-\underline{w}=\bar{w}-\mu_{v}$, then we have a symmetric setup so that the IIP and mean CIP coincide under equitable bargaining power ( $\alpha_{\perp}^{*}=\frac{1}{2}$ ). If the upper bound of feasible buyer valuations, $\bar{w}$, is unilaterally increased from the symmetric setup so that $\mu_{c}-\underline{w}<\bar{w}-\mu_{v}$, then the mean CIP is unchanged, but the IIP decreases for every possible bargaining power allocation as per Proposition 11 (shift from red curve to yellow curve in Figure 3.1); since the IIP has a shallower dependency on bargaining power than the mean CIP, the two prices can only coincide at a higher $\alpha$. Relatedly, if the lower bound of feasible seller costs, $\underline{w}$, is unilaterally decreased from the symmetric setup so that $\mu_{c}-\underline{w}>\bar{w}-\mu_{v}$, the IIP increases for every possible bargaining power allocation by Proposition 11 (shift from red curve to blue curve in Figure 3.1) so that the two prices can only coincide at a lower $\alpha$. These relations produce the monotonic behavior seen in the bottom two panels of Figure 3.2 .

However, the dependency of $\alpha_{\perp}^{*}$ on the mean buyer and seller valuations is nonmonotonic since the means affect both the mean CIP and the IIP. On one hand, if either the mean buyer valuation or mean seller cost is unilaterally increased, this drives an increase in the IIP for any bargaining power allocation, which contributes to decreasing $\alpha_{\perp}^{*}$. On the other hand, an increase in either of the means also drives an increase in the mean CIP for any bargaining power allocation, which contributes to increasing $\alpha_{\perp}^{*}$. If the expected surplus from trade is large (resp. small), in that $\mu_{\nu}$ is large (resp. small) or $\mu_{c}$ is small (resp. large), then if $\mu_{v}$ or $\mu_{c}$ are respectively increased, the resulting positive effect on $\alpha_{\perp}^{*}$ from the mean CIP is outweighed by (resp. outweighs) the negative effect from the IIP, producing the relations seen in the top two panels of Figure 3.2.

### 3.5 Dependent values

### 3.5.1 Setup and equilibrium

It is natural for applications involving uncertainty to consider buyer and seller reservation values that are possibly dependent. This setting is identical to that considered in section 3.2.1 without the independence assumption (3.3). How is Nature's strategy affected in this case? The next proposition asserts that Nature would always exploit the extra freedom afforded to it by rendering values
that are dependent and obtains Nature's resulting strategy.
Proposition 13 In the absence of the independence restriction (3.3), Nature's equilibrium strategy always has $v$ and $c$ dependent. For $p \in\left(\mu_{c}, \mu_{v}\right)$, an equilibrium strategy for Nature, $H^{*}$, is given by the following:

1. If $\pi_{\bar{w}}(p)+\pi_{\underline{w}}(p)<1, H^{*}=\pi_{\bar{w}}(p) \delta_{(\bar{w}, p)}+\pi_{\underline{w}}(p) \delta_{(p, \underline{w})}+\left(1-\pi_{\bar{w}}(p)-\pi_{\underline{w}}(p)\right) \delta_{(p, p)}$
2. If $\pi_{\bar{w}}(p)+\pi_{\underline{w}}(p)>1, H^{*}=\left(1-\pi_{\underline{w}}(p)\right) \delta_{(\bar{w}, p)}+\left(1-\pi_{\bar{w}}(p)\right) \delta_{(p, \underline{w})}+\left(\pi_{\bar{w}}(p)+\pi_{\underline{w}}(p)-1\right) \delta_{(\bar{w}, \underline{w})}$
3. If $\pi_{\bar{w}}(p)+\pi_{\underline{w}}(p)=1, H^{*}=\pi_{\bar{w}}(p) \delta_{(\bar{w}, p)}+\pi_{\underline{w}}(p) \delta_{(p, \underline{w})}$.

The (behavioral) strategy in case 2 is unique.
Proof. Independent values is never optimal for Nature by Lemma 3. If in fact Nature's strategy in Proposition 9 was optimal, then the imputed joint distribution has four-point support given by $\{(p, \underline{w}),(\bar{w}, p),(p, p),(\bar{w}, \underline{w})\}$. Since the mediator's payoff $\phi(v, c)$ is positive at $(\bar{w}, \underline{w})$ and identically zero at the other three points (whenever either the buyer or seller value coincides with the price $p)$, there can never be an affine function (a plane in this case) that coincides with the payoff $\phi(v, c)$ over the four-point support, contradicting the necessity of the conditions of the lemma.

Sufficiency is easily shown by Lemma 1 with the obvious choices of affine functions (in case 3, there are infinitely many affine functions that suffice). The (behavioral) strategy in case 2 is unique as any other points in the support would make any affine function satisfying condition 2 violate condition 1 of the lemma.

Proposition 13 shows that under dependent values, it is still optimal for Nature to choose the same marginal distributions as in the independent values case. However, while the independent values case allowed the mediator to obtain a positive expected payoff when choosing any $p \in\left(\mu_{c}, \mu_{v}\right)$, dependence allows for the possibility that Nature may always achieve its first best of giving a zero payoff to the mediator regardless of the latter's pricing strategy, thus making the mediator's equilibrium strategy ill-defined. We call this a dismal equilibrium, and it occurs if the parameters allow Nature's equilibrium support to always feature either the buyer or seller values coinciding with the mediator's price. The following corollary characterizes when the dismal equilibrium occurs.

Corollary 5 Define the following quadratic function in $p$ :

$$
\begin{equation*}
\Lambda(p) \equiv\left(\mu_{v}-p\right)\left(p-\mu_{c}\right)-\left(\bar{w}-\mu_{v}\right)\left(\mu_{c}-\underline{w}\right) . \tag{3.19}
\end{equation*}
$$

Then a dismal equilibrium occurs if and only if

$$
\begin{equation*}
\Lambda\left(\frac{\mu_{v}+\mu_{c}}{2}\right) \leq 0 \tag{3.20}
\end{equation*}
$$

Moreover,

1. If $\bar{w}-\mu_{v}=\mu_{c}-\underline{w}$, then (3.20) is satisfied if and only if $\mu_{c} \geq \frac{3}{4} \underline{w}+\frac{1}{4} \bar{w}$, or equivalently, $\mu_{v} \leq \frac{1}{4} \underline{w}+\frac{3}{4} \bar{w}$.
2. If $\bar{w}-\mu_{v}>\mu_{c}-\underline{w}$, then (3.20) is satisfied if $\mu_{c} \geq \frac{3}{4} \underline{w}+\frac{1}{4} \bar{w}$ and only if $\mu_{v} \leq \frac{1}{4} \underline{w}+\frac{3}{4} \bar{w}$.
3. If $\bar{w}-\mu_{v}<\mu_{c}-\underline{w}$, then (3.20) is satisfied if $\mu_{v} \leq \frac{1}{4} \underline{w}+\frac{3}{4} \bar{w}$ and only if $\mu_{c} \geq \frac{3}{4} \underline{w}+\frac{1}{4} \bar{w}$.

Proof. By Proposition 13, the dismal equilibrium occurs if and only if for any $p \in\left(\mu_{c}, \mu_{v}\right)$, we have $\pi_{\bar{w}}(p)+\pi_{\underline{w}}(p) \leq 1 \Longleftrightarrow \Lambda(p) \leq 0$. Note that $\Lambda(p)$ is a downward facing parabola in $p$ satisfying $\Lambda\left(\mu_{c}\right)=\Lambda\left(\mu_{v}\right)=-\left(\bar{w}-\mu_{v}\right)\left(\mu_{c}-\underline{w}\right)<0$ and is thus maximized at $p_{\max } \equiv \frac{\mu_{c}+\mu_{v}}{2}$. Consequently, the dismal equilibrium occurs if and only if $\Lambda\left(p_{\max }\right) \leq 0$. Observe that for $\bar{w}-\mu_{v} \gtreqless \mu_{c}-\underline{w}$,

$$
\begin{gathered}
2\left(\frac{\mu_{v}+\mu_{c}}{2}-\underline{w}\right)\left(\frac{3}{4} \underline{w}+\frac{1}{4} \bar{w}-\mu_{c}\right) \\
\gtreqless\left(\frac{\mu_{v}+\mu_{c}}{2}-\underline{w}\right)\left(\frac{1}{2} \mu_{v}+\underline{w}-\frac{3}{2} \mu_{c}\right)=\left(\frac{\mu_{v}-\mu_{c}}{2}\right)^{2}-\left(\mu_{c}-\underline{w}\right)^{2} \\
\gtreqless \Lambda\left(p_{\max }\right)=\left(\frac{\mu_{v}-\mu_{c}}{2}\right)^{2}-\left(\bar{w}-\mu_{v}\right)\left(\mu_{c}-\underline{w}\right) \\
\gtreqless\left(\frac{\mu_{v}-\mu_{c}}{2}\right)^{2}-\left(\bar{w}-\mu_{v}\right)^{2}=\left(\bar{w}-\left(\frac{\mu_{v}+\mu_{c}}{2}\right)\right)\left(\frac{3}{2} \mu_{v}-\bar{w}-\frac{1}{2} \mu_{c}\right) \\
\gtreqless 2\left(\bar{w}-\left(\frac{\mu_{v}+\mu_{c}}{2}\right)\right)\left(\mu_{v}-\frac{1}{4} \underline{w}-\frac{3}{4} \bar{w}\right) .
\end{gathered}
$$

The desired necessary and sufficient conditions follow immediately.
Our object of interest in the dependent value case is thus to look at the mediator's optimal price strategy in a non-dismal equilibrium. By Proposition 13, this equilibrium is valid so long as there is a price for which $\pi_{\bar{w}}(p)+\pi_{\underline{w}}(p)>1$, which holds when the condition of Corollary 5 is violated. Qualitatively, Corollary 5 is violated when the mean buyer and seller values are far apart, so that as long as the expected surplus is sufficiently large, the mediator enjoys the prospect of a non-dismal equilibrium. In this case, the mediator chooses the IIP as

$$
\begin{align*}
& p^{\operatorname{IIP}} \in \underset{p \in\left(\mu_{c}, \mu_{v}\right)}{\arg \max }\left\{\Pi(p) \equiv(\bar{w}-p)^{\alpha}(p-\underline{w})^{1-\alpha}\left(\pi_{\bar{w}}(p)+\pi_{\underline{w}}(p)-1\right)\right\}  \tag{3.21}\\
= & \underset{p \in\left(\mu_{c}, \mu_{v}\right)}{\arg \max }(\bar{w}-p)^{\alpha}(p-\underline{w})^{1-\alpha}\left(\pi_{\bar{w}}(p) \pi_{\underline{w}}(p)-\left(1-\pi_{\bar{w}}(p)\right)\left(1-\pi_{\underline{w}}(p)\right)\right) . \tag{3.22}
\end{align*}
$$

Comparison of (3.22) with (3.13) shows exactly how dependent values makes the mediator worse off relative to independent values for any price strategy. The probability of all-sides trade in the dependent values case is the probability of all-sides trade less the probability of no-sides trade in the independent values case. We discuss the import of this dependence penalty more in the context of Proposition 15.

Substituting in (3.11)-(3.12), the objective in (3.22) simplifies to

$$
\Pi(p)=\Lambda(p)(\bar{w}-p)^{\alpha-1}(p-\underline{w})^{-\alpha},
$$

where $\Lambda(p)$ is as defined in (3.19). By Rolle's theorem, there is an interior optimal price in the interval $\left(\Lambda_{-}, \Lambda_{+}\right)$where the FOC will be satisfied, where $\Lambda_{-}, \Lambda_{+}$are the roots of $\Lambda$ given by

$$
\begin{equation*}
\Lambda_{ \pm} \equiv \frac{1}{2}\left(\mu_{v}+\mu_{c}\right) \pm \sqrt{\frac{1}{4}\left(\mu_{v}-\mu_{c}\right)^{2}-\left(\bar{w}-\mu_{v}\right)\left(\mu_{c}-\underline{w}\right)} . \tag{3.23}
\end{equation*}
$$

It is clear how dependence limits the profitable portion of the mediator's strategy set since $\left(\Lambda_{-}, \Lambda_{+}\right)$
$\subset\left(\mu_{c}, \mu_{v}\right)$. The FOC is then given by

$$
\begin{equation*}
\Pi^{\prime}(p)=(\bar{w}-p)^{\alpha-2}(p-\underline{w})^{-\alpha-1}\left[2\left(\frac{1}{2}\left(\mu_{v}+\mu_{c}\right)-p\right)(\bar{w}-p)(p-\underline{w})+(p-((1-\alpha) \underline{w}+\alpha \bar{w})) \Lambda(p)\right]=0 . \tag{3.24}
\end{equation*}
$$

Defining

$$
\begin{equation*}
h(p) \equiv(\bar{w}-p)^{2-\alpha}(p-\underline{w})^{1+\alpha} \Pi^{\prime}(p) \tag{3.25}
\end{equation*}
$$

(3.24) shows that $h(p)$ is a cubic polynomial in $p$ with leading coefficient of unity satisfying $h\left(\Lambda_{ \pm}\right) \lessgtr 0$, implying a unique solution to the FOC.

### 3.5.2 Comparison between IIP and mean CIP and comparative statics

Proposition 10 shows how the dependency of the price upon $\alpha$ is shallowed by playing against adversarial Nature. The following lemma compares the IIP in the dependent case with that in the independent case and shows how this shallowness is only exacerbated when Nature is made more adversarial in being afforded the freedom of dependence.

Lemma $4(1-\alpha) \underline{w}+\alpha \bar{w} \lesseqgtr p^{\mathrm{IIP}} \lesseqgtr p_{\perp}^{\mathrm{IIP}}$.

Proof. Comparing (3.15) and (3.24), the FOCs in the dependent and independent cases are related in that

$$
\begin{equation*}
h(p)=h_{\perp}(p)-(p-((1-\alpha) \underline{w}+\alpha \bar{w}))\left(\bar{w}-\mu_{v}\right)\left(\mu_{c}-\underline{w}\right) . \tag{3.26}
\end{equation*}
$$

We thus have $p^{\mathrm{IIP}} \lesseqgtr p_{\perp}^{\mathrm{IIP}}$ if and only if

$$
\begin{aligned}
0 \lesseqgtr-h_{\perp}\left(p_{\perp}^{\mathrm{IP}}\right)+h_{\perp}\left(p^{\mathrm{IPP}}\right) & =-h_{\perp}\left(p_{\perp}^{\mathrm{IIP}}\right)+h\left(p^{\mathrm{IIP}}\right)+\left(p^{\mathrm{IPP}}-((1-\alpha) \underline{w}+\alpha \bar{w})\right)\left(\bar{w}-\mu_{v}\right)\left(\mu_{c}-\underline{w}\right) \\
& =\left(p^{\mathrm{IIP}}-((1-\alpha) \underline{w}+\alpha \bar{w})\right)\left(\bar{w}-\mu_{v}\right)\left(\mu_{c}-\underline{w}\right),
\end{aligned}
$$

where the first inequality is from the fact $h_{\perp}$ is decreasing over its domain, the following equality uses (3.26), and the final equality is by definition of $p_{\perp}^{\text {IIP }}$ and $p^{\text {IIP }}$ satisfying their respective FOCs.

We now extend Proposition 10 to see how all three price objects we have discussed-the mean CIP and the IIP in the independent and dependent cases-compare in terms of bargaining power. The comparison is illustrated in Figure 3.3.


Figure 3.3: IIP (independent and dependent) vs. mean CIP as a function of bargaining power.

Proposition 14 For $\mu_{c}-\underline{w} \lesseqgtr \bar{w}-\mu_{v}$, there are unique $\alpha^{*}, \alpha^{* *} \in(0,1)$ satisfying $\alpha^{* *} \lesseqgtr \frac{1}{2} \lesseqgtr \alpha^{*} \lesseqgtr$ $\alpha_{\perp}^{*}$ so that

1) For $\alpha \lesseqgtr \alpha^{* *}$, we have $p^{\text {IIP }} \lesseqgtr p_{\perp}^{\text {IIP }}$
2) For $\alpha \lesseqgtr \alpha^{*}$, we have $p^{\text {IIP }} \lesseqgtr \bar{p}^{C I P}$

Proof. Observe that $\bar{p}^{\mathrm{CIP}}=(1-\alpha) \mu_{v}+\alpha \mu_{c}$ and $\hat{w} \equiv(1-\alpha) \underline{w}+\alpha \bar{w}$ coincide at $\alpha=\hat{\alpha} \equiv \frac{\mu_{v}-\underline{w}}{\mu_{v}-\underline{w}+\overline{\bar{w}}-\mu_{c}}$. For $\mu_{c}-\underline{w} \lesseqgtr \bar{w}-\mu_{v}$, we must have $\hat{\alpha} \lesseqgtr \frac{1}{2}$, and by Proposition 10, since $\frac{1}{2} \lesseqgtr \alpha_{\perp}^{*}$, we have $\hat{w}=\bar{p}^{\mathrm{CIP}} \gtreqless p_{\perp}^{\mathrm{IIP}}$ at $\alpha=\hat{\alpha}$. Since $\hat{w}$ is increasing in $\alpha$ and $p_{\perp}^{\mathrm{IPP}}$ is decreasing in $\alpha$ (Proposition 11), $\hat{w}$ and $p_{\perp}^{\text {IIP }}$ uniquely coincide at $\alpha^{* *} \lesseqgtr \widehat{\lesssim} \lesseqgtr \frac{1}{2}$. Lemma 4 then immediately gives the first result.

To show the second result, note that $\bar{p}^{\mathrm{CIP}}$ and $p^{\text {IIP }}$ coincide when $h\left(\bar{p}^{\mathrm{CIP}}\right)=0$, or by (3.26) when

$$
h_{\perp}\left(\bar{p}^{\mathrm{CIP}}\right)=\left(\bar{p}^{\mathrm{CIP}}-\hat{w}\right)\left(\bar{w}-\mu_{v}\right)\left(\mu_{c}-\underline{w}\right),
$$

for which the left hand side is increasing in $\alpha$ (c.f. proof of Proposition 10) and the right hand side is decreasing in $\alpha$, while $h_{\perp}\left(\bar{p}^{\mathrm{CIP}}\right) \lesseqgtr 0$ whenever $\alpha \lesseqgtr \alpha_{\perp}^{*}$ and $\bar{p}^{\mathrm{CIP}}-\hat{w} \lesseqgtr 0$ whenever $\alpha \gtreqless \hat{\alpha}$. Thus $\bar{p}^{\mathrm{CIP}}$ and $p^{\text {IIP }}$ uniquely coincide at $\alpha=\alpha^{*}$, which must be between $\alpha_{\perp}^{*}$ and $\hat{\alpha}$, and hence between $\alpha_{\perp}^{*}$ and $\alpha^{* *}$. Since $h(\cdot)$ is decreasing in its price argument, it is thus clear how the second result follows.

Finally, observe $\bar{p}^{\mathrm{CIP}}=\frac{1}{2}\left(\mu_{v}+\mu_{c}\right)$ when $\alpha=\frac{1}{2}$. Thus, since $\Lambda\left(\frac{\mu_{v}+\mu_{c}}{2}\right)>0$ in a non-dismal equilibrium (c.f. Corollary 5 ), by (3.24), $\left.h\left(\bar{p}^{\mathrm{CIP}}\right)\right|_{\alpha=\frac{1}{2}}=\left(\frac{1}{2}\left(\mu_{v}+\mu_{c}\right)-\frac{1}{2}(\underline{w}+\bar{w})\right) \Lambda\left(\frac{\mu_{v}+\mu_{c}}{2}\right) \lesseqgtr 0$ when $\mu_{c}-\underline{w} \lesseqgtr \overline{\bar{w}}-\mu_{\nu}$, and since $h\left(\bar{p}^{\text {CIP }}\right)$ is increasing in $\alpha$, this implies $\frac{1}{2} \lesseqgtr \alpha^{*}$.

We also consider the comparative statics of $p^{\text {IIP }}$, extending Proposition 11 to the dependent setup.

Proposition $15 p^{\text {IIP }}$ strictly increases in $\mu_{v}, \mu_{c}$ and strictly decreases in $\alpha$. For $\mu_{c}-\underline{w}$ sufficiently small, $p^{\text {IIP }}$ strictly decreases in $\bar{w}$; for $\bar{w}-\mu_{v}$ sufficiently small, $p^{\text {IIP }}$ strictly decreases in $\underline{w}$. For $\alpha$ sufficiently small (resp. large), $p^{\text {IIP }}$ strictly decreases (resp. increases) in $\bar{w}$ and strictly increases (resp. decreases) in $\underline{w} .{ }^{9}$

[^31]Proof. With slight abuse of notation, augment $h(\cdot)$ to admit generic parameter $\beta \in\left\{\mu_{\nu}, \mu_{c}, \bar{w}, \underline{w}, \alpha\right\}$ as an argument. By definition of $p^{\text {IIP }}$, we have

$$
h\left(p^{\mathrm{IP}}, \beta\right)=0 \Longrightarrow \partial_{\beta} p^{\mathrm{IP}}=-\left.\frac{\partial_{\beta} h(p, \beta)}{\partial_{p} h(p, \beta)}\right|_{p=p^{\mathrm{IIP}} .}
$$

Since $h(\cdot, \beta)$ is strictly decreasing at $p^{\mathrm{IIP}}, \operatorname{sign}\left\{\partial_{\beta} p^{\mathrm{IIP}}\right\}=\operatorname{sign}\left\{\partial_{\beta} h\left(p^{\mathrm{IIP}}, \beta\right)\right\}$. Differentiating yields

$$
\begin{gathered}
\partial_{\alpha} h\left(p^{\mathrm{IIP}}, \alpha\right)=-(\bar{w}-\underline{w}) \Lambda\left(p^{\mathrm{IIP}}\right)<0, \\
\partial_{\mu_{v}} h\left(p^{\mathrm{IIP}}, \mu_{v}\right)=(1-\alpha)\left(p^{\mathrm{IIP}}-\underline{w}\right)(\bar{w}-\underline{w})>0, \\
\partial_{\mu_{c}} h\left(p^{\mathrm{IIP}}, \mu_{c}\right)=\alpha\left(\bar{w}-p^{\mathrm{IIP}}\right)(\bar{w}-\underline{w})>0, \\
\partial_{\bar{w}} h\left(p^{\mathrm{IIP}}, \bar{w}\right)=\left.\frac{1}{\bar{w}-p}\left[h(p, \bar{w})-(1-\alpha)\left(\mu_{v}-p\right)(p-\underline{w})^{2}+\alpha\left(\mu_{c}-\underline{w}\right)(\bar{w}-p)^{2}\right]\right|_{p=p^{\mathrm{IP}}} \\
\partial_{\underline{w}} h\left(p^{\mathrm{IIP}}, \underline{w}\right)=\left.\frac{1}{p-\underline{w}}\left[-h(p, \underline{w})-\alpha\left(p-\mu_{c}\right)(\bar{w}-p)^{2}+(1-\alpha)\left(\bar{w}-\mu_{v}\right)(p-\underline{w})^{2}\right]\right|_{p=p^{\mathrm{IIP}}}
\end{gathered}
$$

Inspection of the last two derivatives makes clear how the desired results for $\bar{w}$ and $\underline{w}$ hold locally, i.e. given particular $\underline{w}$ and $\bar{w}$, the values of $\alpha, \mu_{c}-\underline{w}$, or $\bar{w}-\mu_{v}$ may be chosen to obtain the desired monotonic behavior of $p^{\text {IIP }}$ in $\bar{w}$ and $\underline{w}$. However, allowing $\bar{w}$ or $\underline{w}$ to unilaterally vary is subject to ensuring $\bar{w}$ is bounded above and $\underline{w}$ is bounded below in a way that respects $\Lambda\left(\frac{\mu_{v}+\mu_{c}}{2}\right)>0$, the condition for the non-dismal equilibrium. ${ }^{10}$ Thus, it is clear that the desired results indeed hold globally, i.e. a fixed value of $\alpha, \mu_{c}-\underline{w}$, or $\bar{w}-\mu_{v}$ may be chosen to obtain the desired monotonic behavior of $p^{\text {IIP }}$ in the entire feasible range of $\bar{w}$ or $\underline{w}$.

We see that while the IIP under dependent values responds to the mean reservation values and bargaining power monotonically in a manner analogous to the independent case, its dependency on the bounds of the feasible range is more complicated and can be potentially nonmonotonic. Such nonmonotonicity arises due to a tradeoff induced by Nature's strategy under dependence, $\underline{w}$ over their entire feasible ranges. Likewise, for the same value of $\mu_{c}-\underline{w}$ (resp. $\bar{w}-\mu_{v}$ ) chosen small enough, $p^{\text {IIP }}$ strictly decreases in $\bar{w}$ (resp. $\underline{w}$ ) over its entire feasible range.
${ }^{10}$ Of course, $\underline{w}$ is also bounded below by zero by assumption
although can be tempered to exhibit monotonic behavior at extreme bargaining powers or mean reservation values. Recall that for any price, Nature maintains the same probability of buyer-side and seller-side trade as in the independent case, namely $\pi_{\bar{w}}(p)$ and $\pi_{\underline{w}}(p)$ respectively. However, as per (3.22), Nature penalizes the probability of all-sides trade relative to the independent case when the probability of trade from either side is low (note the probability of buyer-side and seller-side trade cannot both be less than $1 / 2$ ). If the lower bound of the feasible cost values, $\underline{w}$, is increased toward $\mu_{c}$, then for a given price, Nature places more mass of the seller's marginal distribution at $\underline{w}$, thus increasing likelihood of seller-side trade. If the mediator cares about the buyer enough ( $\alpha$ large) or if Nature induces a high enough likelihood of buyer-side trade ( $\bar{w}-\mu_{v}$ small), the mediator would choose to favor the buyer more and decrease the price, just as in the independent values case. However, decreasing the price also renders a lower seller-side likelihood of trade, which aggravates the aforementioned dependence penalty imposed by Nature's strategy. If the mediator cares about the seller enough ( $\alpha$ small), the mediator would then choose to complement Nature's seller-favoring behavior by increasing the price. Similarly, if the upper bound on feasible buyer valuations, $\bar{w}$, is decreased toward $\mu_{v}$, then for a given price, Nature places more mass of the of the buyer's marginal distribution at $\bar{w}$, increasing buyer-side likelihood of trade. If the mediator cares about the seller enough ( $\alpha$ small) or if Nature induces a high enough likelihood of seller-side trade ( $\mu_{c}-\underline{w}$ small), the mediator would choose to favor the seller more and increase the price, just as in the independent values case. However, increasing the price renders a lower buyer-side likelihood of trade, which exacerbates Nature's dependence penalty. If the mediator cares about the buyer enough ( $\alpha$ large), the mediator would choose to complement Nature's buyer-favoring behavior by decreasing the price.

The comparative statics of $\alpha^{* *}$ and $\alpha^{*}$ may also be explored, extending Proposition 12. We do not do so as that is not our primary objective in this paper, but given the nonmonotonic dependency on the bounds of the feasible range in Proposition 15, we expect the results of Proposition 12 to only be further compounded with nonmonotonic behavior.

### 3.6 Conclusion

This paper has analytically considered how an uncertainty-averse mediator motivated by the ultimate success of a transaction would choose a price contract for a risk neutral buyer and seller to trade a good in a setting of unquantifiable uncertainty with sparse information on means and feasible ranges of their reservation values and knowledge of expected surplus. We have studied the role of bargaining power and looked at how the mediator's choice that insures against distributional uncertainty would compare with what the parties themselves would choose when equipped with the mediator's information. We have seen these choices may coincide, but while the parties themselves would decide on a contract that fully responds to bargaining power, the mediator's worst-case approach of playing against Nature produces a shallow dependency of the price contract on bargaining power. If buyer and seller values may be dependent, then when the expected surplus from trade is small, Nature may produce a distribution of buyer and seller values under which trade would never occur, rendering the mediator's price contract indeterminate. However, if trade does occur with positive probability in the worst case under dependent values, this more adversarial game with Nature only aggravates the shallow dependency of the price contract upon bargaining power. Nonetheless, the choices of both the mediator and the transacting parties qualitatively respond similarly in some intuitive ways in that they both increase in the mean buyer and seller values and decrease in the buyer's bargaining power. However, the mediator's price contract responds to the bounds of the feasible range of buyer and seller values in a way that offsets the party that is made more likely to trade by Nature's moment-preserving strategy, a relation that can be nonmonotonic by complementing Nature when buyer and seller values are dependent.

It is of interest to see how our model can be adapted to address other questions. We may extend our model to a multi-party setup where multiple corporate entities aim to contract on transfer prices-we expect our results to qualitatively extend in such an environment. It may also be of interest to compare the mediator's worst-case price contract to other price contracts, such as one that would be decided between buyer and seller in a Bayesian game under various priors. Such
comparisons provide insights into the role of preferences and information structure in contracts. Finally, this paper's study of sensitivity of various price contracts to bargaining power may be of consequence for future policy work seeking to construct contracts that are resilient to institutional changes.

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## Appendix A: Appendix to Chapter 2

We assume throughout this appendix that $\delta \in(0,1)$. To ease notation, we define $Z_{t, \tau}:=$ $\prod_{j=0}^{\tau-1} \frac{\tilde{\mu}_{t+j}}{1+r-\kappa+\kappa \tilde{\mu}_{t+j}}$, and as in section 1.5.3, we define $M_{t}:=\prod_{\tau=0}^{t-1} \tilde{\mu}_{\tau}$ and $w_{\tau}:=\frac{\beta^{\tau} M_{\tau}^{-1}}{\sum_{j=0}^{\infty} \beta^{j} M_{j}^{-1}}$ (with the normalization $Z_{t, 0}=M_{0}=1$ as per footnote 12). Note that from (1.15) in the $\tilde{\mu}_{t}$-characterization, $Z_{t, \tau}$ is the $\tau$-period ahead discount factor of future output that determines the equilibrium price of capital at time $t$; this factor never exceeds $(1+r)^{-\tau}$.

## A. 1 Proof of Proposition 5

First, we show that a necessary condition for an $N$-cyclic equilibrium is that $y_{t+1}$ is $N$-cyclic with $y_{n^{\prime}+1}>0$ for some $n^{\prime} \in \mathcal{N}$. Given $q_{t}$ and $\tilde{\mu}_{t}$ are $N$-cyclic, by (1.15), we have $\forall t \geq 0$,

$$
\begin{gathered}
0=q_{t+N}-q_{t}=\frac{\alpha}{k} \sum_{\tau=1}^{\infty} Z_{t, \tau}\left(y_{t+N+\tau}-y_{t+\tau}\right) \\
=\frac{\alpha}{k} Z_{t, 1}\left(y_{t+N+1}-y_{t+1}\right)+\frac{\alpha}{k} \frac{\tilde{\mu}_{t}}{1+r-\kappa+\kappa \tilde{\mu}_{t}} \sum_{\tau=1}^{\infty} Z_{t+1, \tau}\left(y_{t+N+1+\tau}-y_{t+1+\tau}\right),
\end{gathered}
$$

and we also have

$$
0=q_{t+N+1}-q_{t+1}=\frac{\alpha}{k} \sum_{\tau=1}^{\infty} Z_{t+1, \tau}\left(y_{t+N+1+\tau}-y_{t+1+\tau}\right)
$$

Substituting the latter equality into the former shows $y_{t+1}=y_{t+N+1}$, so that $y_{t+1}$ is $N$-cyclic. Moreover, $y_{n^{\prime}+1}>0$ for some $n^{\prime} \in \mathcal{N}$ since constant zero output uniquely supports the ESE.

Next, we show that an equilibrium having $N$-cyclic $\tilde{\mu}_{t}$ and $y_{t+1}$ with $y_{n^{\prime}+1}>0, \tilde{\mu}_{n}<1$ for some $n, n^{\prime} \in \mathcal{N}$ must be an $N$-cyclic equilibrium with $\left(\prod_{\tau \in \mathcal{N}} \tilde{\mu}_{\tau}\right)^{1 / N}=\delta$. Note that $\forall t \geq 0$, we have $\prod_{\tau \in \mathcal{N}} \tilde{\mu}_{t+\tau}=M_{N}$, while (1.15) shows that $q_{t}$ must be $N$-cyclic. Moreover, by (1.39), a tight
constraint in period $n$ and $n+N$ implies

$$
\begin{gathered}
d_{n+1}=d_{n+1}^{\mathrm{NDL}}-\delta^{n+1}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right) \sum_{\tau=0}^{\infty} w_{\tau}\left(\prod_{j=\tau}^{n+\tau} \tilde{\mu}_{j}\right)^{-1}=\kappa k q_{n}, \\
d_{n+N+1}=d_{n+N+1}^{\mathrm{NDL}}-\delta^{n+N+1}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right) \sum_{\tau=0}^{\infty} w_{\tau}\left(\prod_{j=\tau}^{n+N+\tau} \tilde{\mu}_{j}\right)^{-1}=\kappa k q_{n+N},
\end{gathered}
$$

so that subtracting the two expressions yields

$$
0=d_{n+N+1}-d_{n+1}=\delta^{n+1}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right) \sum_{\tau=0}^{\infty} w_{\tau}\left(\prod_{j=\tau}^{n+\tau} \tilde{\mu}_{j}\right)^{-1}\left(1-\delta^{N} M_{N}^{-1}\right) .
$$

Thus, we have that $M_{N}=\prod_{\tau \in \mathcal{N}} \tilde{\mu}_{\tau}=\delta^{N}$. By (1.10), we then have $\forall t \geq 0$,

$$
c_{t+N}=\delta^{t+N} M_{t+N}^{-1} c_{0}=\left(\delta^{t+N} \prod_{\tau=0}^{t+N-1} \tilde{\mu}_{\tau}^{-1}\right) c_{0}=\delta^{N} M_{N}^{-1} c_{t}=c_{t},
$$

implying $c_{t}$ is $N$-cyclic. Moreover, $d_{t+1}$ is $N$-cyclic by (1.11) since $\forall t \geq 0$,

$$
d_{t+N+1}=\sum_{\tau=0}^{\infty} \frac{y_{t+N+1+\tau}-c_{t+N+1+\tau}}{(1+r)^{\tau}}=\sum_{\tau=0}^{\infty} \frac{y_{t+1+\tau}-c_{t+1+\tau}}{(1+r)^{\tau}}=d_{t+1} .
$$

Finally, observe that collateral constraint binding during all the tight periods along with the restriction $M_{N}=\delta^{N}$ yields an overdetermined system, so that an equilibrium of a given sequence of multipliers can only be supported under unique $d_{0}$ (admitting other parameters as given).

## A. 2 Finite-dimensional characterization of $N$-cyclic equilibria

Suppose an $N$-cyclic equilibrium has a tight constraint in periods $\tilde{\mathcal{N}} \subseteq \mathcal{N}$. We reduce the system of binding constraints that determine the tight multipliers to a $(|\tilde{\mathcal{N}}|-1)$-dimensional system using the cyclicity of the multipliers and the restriction $M_{N}=\delta^{N}$. Note that $Z_{t, N}=Z_{t^{\prime}, N} \forall t, t^{\prime} \geq 0$,
so we let $Z_{N}:=Z_{t, N}$ denote this common value. Recall from (2.2), collateral value is

$$
\kappa k q_{n}=\alpha \kappa \sum_{\tau=0}^{\infty} Z_{n, \tau+1} y_{n+1+\tau}=\frac{\alpha \kappa}{1-Z_{N}} \sum_{\tau \in \mathcal{N}} Z_{n, \tau+1} y_{n+1+\tau} .
$$

From (1.39), debt is given by

$$
\begin{aligned}
& d_{n+1}=d_{n+1}^{\mathrm{NDL}}-\delta^{n+1}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right) \sum_{\tau=0}^{\infty} w_{\tau}\left(\prod_{j=\tau}^{n+\tau} \tilde{\mu}_{j}\right)^{-1} \\
& \quad=d_{n+1}^{\mathrm{NDL}}-\delta^{n+1}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right) \frac{\sum_{\tau=0}^{\infty} \beta^{\tau} M_{n+\tau+1}^{-1}}{\sum_{\tau=0}^{\infty} \beta^{\tau} M_{\tau}^{-1}} .
\end{aligned}
$$

By the geometric average condition (2.1), the numerator of the second term simplifies to

$$
\begin{gathered}
\sum_{\tau=0}^{\infty} \beta^{\tau} M_{n+\tau+1}^{-1}=\sum_{\tau=0}^{N-n-2} \beta^{\tau} M_{n+\tau+1}^{-1}+\beta^{N-n-1} M_{N}^{-1} \sum_{\tau=0}^{\infty} \beta^{\tau} M_{\tau}^{-1} \\
=\beta^{-n-1}\left[\sum_{\tau=n+1}^{N-1} \beta^{\tau} M_{\tau}^{-1}+(1+r)^{-N} \sum_{\tau=0}^{\infty} \beta^{\tau} M_{\tau}^{-1}\right]
\end{gathered}
$$

while the denominator of the second term simplifies to

$$
\sum_{\tau=0}^{\infty} \beta^{\tau} M_{\tau}^{-1}=\frac{1}{1-(1+r)^{-N}} \sum_{\tau \in \mathcal{N}} \beta^{\tau} M_{\tau}^{-1}
$$

so that debt is

$$
d_{n+1}=d_{n+1}^{\mathrm{NDL}}-(1+r)^{n+1}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right)\left[\left(1-(1+r)^{-N}\right) \frac{\sum_{\tau=n+1}^{N-1} \beta^{\tau} M_{\tau}^{-1}}{\sum_{\tau \in \mathcal{N}} \beta^{\tau} M_{\tau}^{-1}}+(1+r)^{-N}\right] .
$$

Finally, the natural debt limits are given by

$$
d_{0}^{\mathrm{NDL}}=d_{N}^{\mathrm{NDL}}+y_{0}-y_{N},
$$

$$
d_{n+1}^{\mathrm{NDL}}=\frac{1}{1-(1+r)^{-N}} \sum_{\tau \in \mathcal{N}} \frac{y_{n+1+\tau}}{(1+r)^{\tau}}
$$

In summary, an $N$-cyclic equilibrium where the constraint is tight in periods $\tilde{\mathcal{N}} \subseteq \mathcal{N}$ exists so long as the following conditions are satisfied:

1. $y_{t+1}$ is $N$-cyclic and strictly positive at least once in its cycle
2. Given that $\tilde{\mu}_{t}$ is $N$-cyclic and $\left\{\tilde{\mu}_{n}\right\}_{n \in \mathcal{N} \backslash \tilde{\mathcal{N}}}=1$, there exists values of $\left\{\tilde{\mu}_{n}\right\}_{n \in \tilde{\mathcal{N}}} \in(0,1)$ and initial debt $d_{0}$ satisfying $\prod_{\tau \in \tilde{\mathcal{N}}} \tilde{\mu}_{\tau}=\delta^{N}$ and the binding collateral constraint $\forall n \in \tilde{\mathcal{N}}:{ }^{1}$

$$
d_{n+1}^{\mathrm{NDL}}-(1+r)^{n+1}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right)\left[\left(1-(1+r)^{-N}\right) \frac{\sum_{\tau=n+1}^{N-1} \beta^{\tau} M_{\tau}^{-1}}{\sum_{\tau \in \mathcal{N}} \beta^{\tau} M_{\tau}^{-1}}+(1+r)^{-N}\right]=\alpha \kappa\left(1-Z_{N}\right)^{-1} \sum_{\tau \in \mathcal{N}} Z_{n, \tau+1} y_{n+1+\tau}
$$

3. The collateral constraint holds $\forall n \in \mathcal{N} \backslash \tilde{\mathcal{N}}$ :

$$
d_{n+1}^{\mathrm{NDL}}-(1+r)^{n+1}\left(d_{0}^{\mathrm{NDL}}-d_{0}\right)\left[\left(1-(1+r)^{-N}\right) \frac{\sum_{\tau=n+1}^{N-1} \beta^{\tau} M_{\tau}^{-1}}{\sum_{\tau \in \mathcal{N}} \beta^{\tau} M_{\tau}^{-1}}+(1+r)^{-N}\right] \leq \alpha \kappa\left(1-Z_{N}\right)^{-1} \sum_{\tau \in \mathcal{N}} Z_{n, \tau+1} y_{n+1+\tau} .
$$

## A. 3 2-cyclic equilibria and proofs of Proposition 6 and Lemma 2

In this appendix section, we discuss 2-cyclic equilibria in detail. Appendices A.3.1-A.3.3 discuss the three kinds of such equilibria using the characterization in Appendix A. 2 and collectively prove Proposition 6, while appendix A.3.4 considers their coexistence in proving Lemma 2. Note that from the characterization in Appendix A.2, in a general $N$-cyclic equilibrium, the debt obligation due in period $N$ is the same as the initial debt obligation adjusted for the difference in output $\left(d_{N}=d_{0}-y_{0}+y_{N}\right),{ }^{2}$ so the constraint in period $N-1$ simplifies to the following restriction on initial borrowing:

$$
\begin{equation*}
d_{0} \leq y_{0}-y_{N}+\alpha \kappa\left(1-Z_{N}\right)^{-1} \sum_{\tau \in \mathcal{N}} Z_{N-1, \tau+1} y_{N+\tau} \tag{A.1}
\end{equation*}
$$

[^32]Also in the special case of a 2-cyclic equilibrium, the natural debt limit in period 1 is given by

$$
\begin{equation*}
d_{1}^{\mathrm{NDL}}=\frac{1}{1-(1+r)^{-2}}\left(y_{1}+\frac{y_{2}}{1+r}\right) . \tag{A.2}
\end{equation*}
$$

## A.3.1 Slack-tight

In this case, $\tilde{\mu}_{0}=1, \tilde{\mu}_{1}=\delta^{2}$, and hence, $Z_{0,1}=\frac{1}{1+r}, Z_{1,1}=\frac{\delta^{2}}{1+r-\kappa+\kappa \delta^{2}}, Z_{2}=\frac{\delta^{2}}{(1+r)\left(1+r-\kappa+\kappa \delta^{2}\right)}$. By (A.1), the binding constraint in period 1 implies initial debt must satisfy

$$
d_{0}=y_{0}-y_{2}+\alpha \kappa \frac{Z_{1,1} y_{2}+Z_{2} y_{1}}{1-Z_{2}}
$$

or equivalently,

$$
\begin{equation*}
d_{0}=y_{0}-y_{2}+\alpha \kappa \delta^{2} \frac{(1+r) y_{2}+y_{1}}{(1+r)\left(1+r-\kappa\left(1-\delta^{2}\right)\right)-\delta^{2}} . \tag{A.3}
\end{equation*}
$$

Moreover, the collateral constraint must hold in period 0 :

$$
d_{1}^{\mathrm{NDL}}-\left(y_{0}+\frac{1}{1+r} d_{1}^{\mathrm{NDL}}-d_{0}\right)\left[\frac{1+\delta(1+r)}{1+r+\delta}\right] \leq \alpha \kappa \frac{Z_{0,1} y_{1}+Z_{2} y_{2}}{1-Z_{2}}
$$

or equivalently, substituting in the initial debt required in (A.3) and natural debt limit in (A.2), we obtain a cubic inequality in $\delta:^{3}$

$$
\begin{gathered}
\Xi\left(\delta \mid y_{1}, y_{2}\right) \geq 0 \\
\Xi\left(\delta \mid y_{1}, y_{2}\right):=A \delta^{3}+B \delta^{2}+C \delta+D, \\
A=-\alpha \kappa(1+r-\kappa) y_{1}+\left(\kappa(1-\alpha)(1+r)^{2}-(1+r-\alpha \kappa)\right) y_{2}, \\
B=(1+r-\alpha \kappa)(1-\kappa(1+r)) y_{1}, \\
C=\alpha \kappa(1+r-\kappa) y_{1}+(1+r)^{2}(1+r-\kappa) y_{2},
\end{gathered}
$$

[^33]$$
D=-(1+r)(1+r-\kappa)(1+r-\alpha \kappa) y_{1} .
$$

Now assume a constant output regime, so that $y_{1}=y_{2}=: y>0$. Then $\Xi(\cdot \mid y, y)$ has a root at $\delta=1$, and the remaining roots can be solved as roots of a quadratic function by polynomial division. To simplify the analysis, note that evaluating at $r=0$ yields

$$
\begin{aligned}
\left.\Xi(\delta \mid y, y)\right|_{r=0}= & y(1-\kappa)\left[-(1+\alpha \kappa) \delta^{3}+(1-\alpha \kappa) \delta^{2}+(1+\alpha \kappa) \delta-(1-\alpha \kappa)\right] \\
& =y(1-\kappa)(1+\alpha \kappa)(1-\delta)(1+\delta)\left(\delta-\frac{1-\alpha \kappa}{1+\alpha \kappa}\right)
\end{aligned}
$$

so that $\left.\Xi(\delta \mid y, y)\right|_{r=0}>0 \forall \delta \in\left(\frac{1-\alpha \kappa}{1+\alpha \kappa}, 1\right)$. Thus, by continuity, for sufficiently low $r>0$ and either i) high enough $\delta<1$ or ii) high enough $\alpha \kappa<1$, the collateral constraint is satisfied in period 0 , and the slack-tight equilibrium exists when initial debt is restricted as per (A.3).

## A.3.2 Tight-slack

In this case, $\tilde{\mu}_{0}=\delta^{2}, \tilde{\mu}_{1}=1$, and hence, $Z_{0,1}=\frac{\delta^{2}}{1+r-\kappa+\kappa \delta^{2}}, Z_{1,1}=\frac{1}{1+r}, Z_{2}=\frac{\delta^{2}}{(1+r)\left(1+r-\kappa+\kappa \delta^{2}\right)}$. In this case, initial borrowing is restricted by the binding collateral constraint in period 0 :

$$
d_{1}^{\mathrm{NDL}}-\left(y_{0}+\frac{1}{1+r} d_{1}^{\mathrm{NDL}}-d_{0}\right)\left[\frac{1+r+\delta}{1+\delta(1+r)}\right]=\alpha \kappa \frac{Z_{0,1} y_{1}+Z_{2} y_{2}}{1-Z_{2}}
$$

or equivalently,

$$
d_{1}^{\mathrm{NDL}}-\left(y_{0}+\frac{1}{1+r} d_{1}^{\mathrm{NDL}}-d_{0}\right)\left[\frac{1+r+\delta}{1+\delta(1+r)}\right]=\alpha \kappa \delta^{2} \frac{(1+r) y_{1}+y_{2}}{(1+r)\left(1+r-\kappa\left(1-\delta^{2}\right)\right)-\delta^{2}}
$$

Substituting in the natural debt limit expression in (A.2), then solving for $d_{0}$ and substituting into the collateral constraint in period 1 implies that the tight-slack equilibrium exists when

$$
\begin{equation*}
d_{0}=y_{0}-\delta \frac{(1+r) y_{1}+y_{2}}{1+r+\delta}\left(1-\frac{\alpha \kappa \delta(1+\delta(1+r))}{(1+r)\left(1+r-\kappa\left(1-\delta^{2}\right)\right)-\delta^{2}}\right) \tag{A.4}
\end{equation*}
$$

$$
\leq y_{0}-y_{2}+\alpha \kappa \frac{Z_{1,1} y_{2}+Z_{2} y_{1}}{1-Z_{2}}
$$

which simplifies to the following cubic inequality in $\delta$ :

$$
\Xi\left(\delta \mid y_{2}, y_{1}\right) \geq 0
$$

In other words, the underlying constraint on fundamentals for the collateral constraint to be met in the non-tight periods in the tight-slack equilibrium is exactly the same as that required in the slacktight equilibrium, except swapping $y_{1}$ and $y_{2}$. As a result, in a constant output regime, excluding the initial borrowing requirement, a tight-slack equilibrium exists under the same conditions discussed in the slack-tight case, as expected.

## A.3.3 Tight-tight

In this case, $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right), \tilde{\mu}_{1}=\delta^{2} / \tilde{\mu}_{0}$, and we have $Z_{0,1}=\frac{\tilde{\mu}_{0}}{1+r-\kappa+\kappa \tilde{\mu}_{0}}, Z_{1,1}=\frac{\delta^{2} / \tilde{\mu}_{0}}{1+r-\kappa+\kappa \delta^{2} / \tilde{\mu}_{0}}, Z_{2}=$ $Z_{0,1} Z_{1,1}$. Such an equilibrium, like the SSE , features a constraint that always binds and consequently, borrowing always perfectly mimics the price of capital.

In a tight-tight equilibrium, we admit initial borrowing to be restricted by the binding collateral constraint in period 1, i.e. ${ }^{4}$

$$
d_{0}=y_{0}-y_{2}+\alpha \kappa \frac{Z_{1,1} y_{2}+Z_{2} y_{1}}{1-Z_{2}}
$$

or explicitly as a function of the period 0 normalized multiplier,

$$
\begin{equation*}
d_{0}^{2-c y c, T \mathrm{~T}}\left(\tilde{\mu}_{0} \mid y_{1}, y_{2}\right)=y_{0}-y_{2}+\alpha \kappa \delta^{2} \frac{\left(\kappa y_{2}+y_{1}\right) \tilde{\mu}_{0}+y_{2}(1+r-\kappa)}{\kappa(1+r-\kappa) \tilde{\mu}_{0}^{2}+\left((1+r-\kappa)^{2}-\delta^{2}\left(1-\kappa^{2}\right)\right) \tilde{\mu}_{0}+\delta^{2} \kappa(1+r-\kappa)}, \tag{A.5}
\end{equation*}
$$

while the binding collateral constraint binding in period 0 yields

$$
d_{1}^{\mathrm{NDL}}-\left(y_{0}+\frac{1}{1+r} d_{1}^{\mathrm{NDL}}-d_{0}\right)\left[\frac{\tilde{\mu}_{0}+\delta(1+r)}{(1+r) \tilde{\mu}_{0}+\delta}\right]=\alpha \kappa \frac{Z_{0,1} y_{1}+Z_{2} y_{2}}{1-Z_{2}} .
$$

Substituting the initial debt restriction, $d_{0}^{2-c y c, T T}\left(\tilde{\mu}_{0} \mid y_{1}, y_{2}\right)$, into the binding constraint in period 0

[^34]determines $\tilde{\mu}_{0}$ according to the following cubic equation:
\[

$$
\begin{gathered}
\Upsilon\left(\tilde{\mu}_{0} \mid y_{1}, y_{2}\right)=0, \\
\Upsilon\left(\tilde{\mu} \mid y_{1}, y_{2}\right):=A \tilde{\mu}^{3}+B \tilde{\mu}^{2}+C \tilde{\mu}+D, \\
A=\kappa(1+r)(1-\alpha)(1+r-\kappa) y_{1} \\
B=\left[(1+r)\left((1+r-\kappa)^{2}-\delta^{2}\left(1-\kappa^{2}\right)\right)-\alpha \delta \kappa(\delta((1+r) \kappa-1)+1+r-\kappa)\right] y_{1}-\delta \kappa(1+r-\kappa)(1+r+\alpha \delta) y_{2} \\
C=\delta^{2} \kappa(1+r-\kappa)(1+r+\alpha \delta) y_{1}-\delta\left[(1+r)\left((1+r-\kappa)^{2}-\delta^{2}\left(1-\kappa^{2}\right)\right)-\alpha \delta \kappa(\delta((1+r) \kappa-1)+1+r-\kappa)\right] y_{2} \\
D=-\kappa \delta^{3}(1+r)(1-\alpha)(1+r-\kappa) y_{2} .
\end{gathered}
$$
\]

Now consider a constant output regime. In this case, clearly $\tilde{\mu}_{0}=\tilde{\mu}^{*}=\delta$ is one valid root of $\Upsilon(\cdot \mid y, y)$ since the SSE is one kind of tight-tight equilibrium in a constant output regime, ${ }^{5}$ and the remaining roots can be found by polynomial division. To simplify the analysis, assume $r$ is sufficiently small. Additionally, assume either i) $\delta$ is sufficiently large, or ii) $\kappa$ is sufficiently large:
i) Evaluating at $r=0$ and $\delta=1$ yields

$$
\left.\Upsilon(\tilde{\mu} \mid y, y)\right|_{r=0, \delta=1}=\kappa(1-\kappa)(1-\tilde{\mu})\left[\tilde{\mu}(\alpha+3)-(1-\alpha)\left(1+\tilde{\mu}+\tilde{\mu}^{2}\right)\right] y
$$

[^35]and evaluating at $\alpha=0$ and $\alpha=1$ yields
$$
\left.\Upsilon(\tilde{\mu} \mid y, y)\right|_{r=0, \delta=1, \alpha=0}=-\kappa(1-\kappa)(1-\tilde{\mu})^{3} y<0<3 \kappa(1-\kappa) \tilde{\mu}(1-\tilde{\mu}) y=\left.\Upsilon(\tilde{\mu} \mid y, y)\right|_{r=0, \delta=1, \alpha=1} .
$$
ii) Evaluating at $r=0$ and taking the limit $\kappa \rightarrow 1$ yields
\[

$$
\begin{gathered}
\left.\lim _{\kappa \rightarrow 1} \frac{\Upsilon(\tilde{\mu} \mid y, y)}{1-\kappa}\right|_{r=0} \\
=\lim _{\kappa \rightarrow 1} \tilde{\mu}(\delta-\tilde{\mu})\left(-(1-\kappa)+\delta^{2}(1+\kappa)+\alpha \delta \kappa(1-\delta)+\delta \kappa(1+\alpha \delta)\right) y-\kappa(1-\alpha)\left(\delta^{3}-\tilde{\mu}^{3}\right) y \\
=(\delta-\tilde{\mu})\left[\tilde{\mu}\left(2 \delta^{2}+\alpha \delta(1-\delta)+\delta(1+\alpha \delta)\right)-(1-\alpha)\left(\delta^{2}+\delta \tilde{\mu}+\tilde{\mu}^{2}\right)\right] y,
\end{gathered}
$$
\]

and observe that for $\tilde{\mu}<($ resp. $>) \delta$, evaluating at $\alpha=0$ and $\alpha=1$ implies

$$
\begin{aligned}
\left.\lim _{\kappa \rightarrow 1} \frac{\Upsilon(\tilde{\mu} \mid y, y)}{1-\kappa}\right|_{r=0, \alpha=0}= & -(\delta-\tilde{\mu})\left[\delta^{2}+\tilde{\mu}^{2}-2 \tilde{\mu} \delta^{2}\right] y<(\text { resp. }>)-(\delta-\tilde{\mu})\left[\delta^{2}+\tilde{\mu}^{2}-2 \tilde{\mu} \delta\right] y=-(\delta-\tilde{\mu})^{3} y<(\text { resp. }>) 0, \\
& \left.\lim _{\kappa \rightarrow 1} \frac{\Upsilon(\tilde{\mu} \mid y, y)}{1-\kappa}\right|_{r=0, \alpha=1}=2 \delta \tilde{\mu}(1+\delta)(\delta-\tilde{\mu})>(\text { resp. < }) 0 .
\end{aligned}
$$

Thus, by continuity and the intermediate value theorem, sufficiently small $r$ and either i) sufficiently large $\delta$ or ii) sufficiently large $\kappa$ ensures there is an $\alpha \in(0,1)$ under which $\Upsilon\left(\tilde{\mu}_{0} \mid y, y\right)=0$ for any $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right)$ and the corresponding tight-tight equilibrium exists.

## A.3.4 Proof of Lemma 2

Throughout this appendix section, unless otherwise specified, assume a constant output regime and denote the resultant required $d_{0}$ in the slack-tight equilibrium, tight-slack equilibrium, and an arbitrary tight-tight equilibrium respectively by $d_{0}^{2-c y c, S T}, d_{0}^{2-c y c, T S}$, and $d_{0}^{2-c y c, T T}\left(\tilde{\mu}_{0} \mid y, y\right)$ (c.f. (A.3)-(A.5) under constant output), where $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right)$. We first study the restricted borrowing function for a tight-tight equilibrium, $d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}(\cdot \mid y, y)$ (c.f. red curve in Figure 2.4). Note that since this function was derived from a binding constraint in period 1, it also applies to a slack-tight equilibrium: $d_{0}^{2-c y c, T T}(1 \mid y, y)=d_{0}^{2-c y c, S T} ;$ in fact, this relation holds under variable output as well.

Now we appeal to the following fact:
Fact 4 A function $f: x \rightarrow \frac{A x+B}{C x^{2}+D x+E}$ is increasing (resp. decreasing) at $\bar{x}$ if

$$
\operatorname{sign}\left\{f^{\prime}(\bar{x})=\frac{A\left(C \bar{x}^{2}+D \bar{x}+E\right)-(A \bar{x}+B)(2 C \bar{x}+D)}{\left(C \bar{x}^{2}+D \bar{x}+E\right)^{2}}\right\}=\operatorname{sign}\left\{-A C \bar{x}^{2}-2 B C \bar{x}+A E-B D\right\}>(\text { resp. }<) 0 .
$$

By Fact 4, the sign of the derivative of $d_{0}^{2-c y c, T T}(\cdot \mid y, y)$ at $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right)$ is given by the sign of

$$
\begin{gather*}
-\kappa(1+\kappa)(1+r-\kappa) \tilde{\mu}_{0}^{2}-2 \kappa(1+r-\kappa)^{2} \tilde{\mu}_{0}+\delta^{2} \kappa(\kappa+1)(1+r-\kappa)-(1+r-\kappa)\left((1+r-\kappa)^{2}-\delta^{2}\left(1-\kappa^{2}\right)\right)  \tag{A.6}\\
=(1+r-\kappa)\left[-\kappa(1+\kappa)\left(\tilde{\mu}_{0}^{2}-\delta^{2}\right)-2 \kappa(1+r-\kappa) \tilde{\mu}_{0}-\left((1+r-\kappa)^{2}-\delta^{2}\left(1-\kappa^{2}\right)\right)\right] \\
<(1+r-\kappa)\left[-\kappa(1+\kappa)\left(\tilde{\mu}_{0}^{2}-\delta^{2}\right)-2 \kappa(1+r-\kappa) \delta^{2}-\left((1+r-\kappa)^{2}-\delta^{2}\left(1-\kappa^{2}\right)\right)\right] \\
=(1+r-\kappa)\left[-\kappa(1+\kappa)\left(\tilde{\mu}_{0}^{2}-\delta^{2}\right)-(1+r-\kappa)^{2}+\delta^{2}\left(1-2 \kappa(1+r)+\kappa^{2}\right)\right] \\
<(1+r-\kappa)\left[-\kappa(1+\kappa)\left(\tilde{\mu}_{0}^{2}-\delta^{2}\right)-(1+r-\kappa)^{2}+\delta^{2}\left((1+r)^{2}-2 \kappa(1+r)+\kappa^{2}\right)\right] \\
=-(1+r-\kappa)\left[\kappa(1+\kappa)\left(\tilde{\mu}_{0}^{2}-\delta^{2}\right)+\left(1-\delta^{2}\right)(1+r-\kappa)^{2}\right], \tag{A.7}
\end{gather*}
$$

where the first inequality comes from $\tilde{\mu}_{0}>\delta^{2}$ and the second inequality comes from $r>0$. Observe that the expression in (A.7) is negative if $\tilde{\mu}_{0} \geq \delta$. Consequently, $d_{0}^{2-c y c, T T}\left(\tilde{\mu}_{0} \mid y, y\right)$ is strictly decreasing in $\tilde{\mu}_{0}$ for 2 -cyclic equilibria having $\tilde{\mu}_{0} \in[\delta, 1]$, proving statement 1 of Lemma 2. Also, since the expression in (A.7) is strictly decreasing in $\tilde{\mu}_{0}$, it is less than its value at $\tilde{\mu}_{0}=\delta^{2}$, which is given by

$$
\begin{gathered}
-(1+r-\kappa)\left[\kappa(1+\kappa)\left(\delta^{4}-\delta^{2}\right)+\left(1-\delta^{2}\right)(1+r-\kappa)^{2}\right] \\
=-(1+r-\kappa)\left(1-\delta^{2}\right)\left((1+r-\kappa)^{2}-\delta^{2} \kappa(1+\kappa)\right),
\end{gathered}
$$

which is nonpositive if $\delta \leq \frac{1+r-\kappa}{\sqrt{\kappa(1+\kappa)}}$, so that when this condition is met, the requisite initial debt
is decreasing in $\tilde{\mu}_{0}$ for 2 -cyclic equilibria having $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right] .{ }^{6}$ Note that this condition is met for either i) $\delta$ sufficiently small, ii) $r$ sufficiently large, or iii) $\kappa$ sufficiently small. Moreover, any of these three conditions would also ensure the requisite initial borrowing for the tight-slack equilibrium (corresponding to $\tilde{\mu}_{0}=\delta^{2}$ ) is greater than that of any other 2-cyclic equilibrium since

$$
\begin{gathered}
d_{0}^{2-\mathrm{cyc}, \mathrm{TS}}-d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}\left(\delta^{2} \mid y, y\right) \\
=\left[y_{0}-\frac{\delta(2+r)}{1+r+\delta}\left(1-\frac{\alpha \kappa \delta(1+\delta(1+r))}{(1+r)\left(1+r-\kappa\left(1-\delta^{2}\right)\right)-\delta^{2}}\right) y\right]-\left[y_{0}-y+\alpha \kappa y \frac{(\kappa+1) \delta^{2}+1+r-\kappa}{(1+r)\left(1+r-\kappa\left(1-\delta^{2}\right)\right)-\delta^{2}}\right],
\end{gathered}
$$

and taking the various parameter limits yields

$$
\begin{gathered}
{\left.\left[d_{0}^{2-\mathrm{cyc}, \mathrm{TS}}-d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}\left(\delta^{2} \mid y, y\right)\right]\right|_{\delta=0}=\frac{1+r-\alpha \kappa}{1+r} y>0} \\
\lim _{r \rightarrow \infty}\left[d_{0}^{2-\mathrm{cyc}, \mathrm{TS}}-d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}\left(\delta^{2} \mid y, y\right)\right]=(1-\delta) y>0 \\
{\left.\left[d_{0}^{2-\mathrm{cyc}, \mathrm{TS}}-d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}\left(\delta^{2} \mid y, y\right)\right]\right|_{\kappa=0}=\frac{(1+r)(1-\delta)}{1+r+\delta} y>0}
\end{gathered}
$$

proving statement 2 of Lemma 2.
To prove statement statement 3 of Lemma 2, we first show a slack-tight equilibrium (corresponding to $\tilde{\mu}_{0}=1$ ) always requires less initial borrowing than any tight-tight equilibrium. Since (A.6) shows the sign of the derivative of $d_{0}^{2-c y c, T T}\left(\tilde{\mu}_{0} \mid y, y\right)$ is monotonically decreasing in $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right]$, it suffices to show that $d_{0}^{2-c y c, T T}\left(\delta^{2} \mid y, y\right)>d_{0}^{2-c y c, T T}(1 \mid y, y)=d_{0}^{2-c y c, S T}$, which is satisfied since

$$
d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}\left(\delta^{2} \mid y, y\right)-d_{0}^{2-\mathrm{cyc}, \mathrm{ST}}
$$

[^36]\[

$$
\begin{gathered}
=\left[y_{0}-y+\alpha \kappa y \frac{(\kappa+1) \delta^{2}+1+r-\kappa}{(1+r)\left(1+r-\kappa\left(1-\delta^{2}\right)\right)-\delta^{2}}\right]-\left[y_{0}-y+\alpha \kappa y \frac{\delta^{2}(2+r)}{(1+r)\left(1+r-\kappa\left(1-\delta^{2}\right)\right)-\delta^{2}}\right] \\
=\alpha \kappa y \frac{(1+r-\kappa)\left(1-\delta^{2}\right)}{(1+r)\left(1+r-\kappa\left(1-\delta^{2}\right)\right)-\delta^{2}}>0
\end{gathered}
$$
\]

Moreover, the slack-tight equilibrium requires less initial borrowing than the tight-slack equilibrium:

$$
\begin{gathered}
d_{0}^{2-\mathrm{cyc}, \mathrm{TS}}-d_{0}^{2-\mathrm{cyc}, \mathrm{ST}} \\
=\left[y_{0}-\frac{\delta(2+r)}{1+r+\delta}\left(1-\frac{\alpha \kappa \delta(1+\delta(1+r))}{(1+r)\left(1+r-\kappa\left(1-\delta^{2}\right)\right)-\delta^{2}}\right) y\right]-\left[y_{0}-\left(1-\frac{\alpha \kappa \delta^{2}(2+r)}{(1+r)\left(1+r-\kappa\left(1-\delta^{2}\right)\right)-\delta^{2}}\right) y\right] \\
=\frac{1-\delta}{1+r+\delta}\left(1+r-\frac{\alpha \kappa \delta^{2}(2+r) r}{(1+r)\left(1+r-\kappa\left(1-\delta^{2}\right)\right)-\delta^{2}}\right) y>\frac{(1-\delta)(1+r-\delta)}{1+r+\delta^{2}} y>0,
\end{gathered}
$$

where the penultimate strict inequality comes from the fact that the preceding expression is decreasing in $\alpha$ and $\kappa$ and hence has an infimum when these values are set to unity. This result can be seen more directly by noting that in a constant output regime, $d_{0}=d_{2}-y+y_{0}$ and symmetry between the slack-tight and tight-slack equilibria (i.e. period 0 equilibrium values for one are swapped with period 1 equilbrium values for the other and vice versa) imply $d_{0}^{2-c y c, T S}>d_{0}^{2-c y c, S T}$ is equivalent to $d_{1}>d_{2}$ in a slack-tight equilibrium, which follows from Corollary 4. This proves statement 3 of Lemma 2.

To show statement 4 of Lemma 2, note that the monotonically decreasing sign of the derivative of $d_{0}^{2-c y c, T T}(\cdot \mid y, y)$ implies that a tight-tight equilibrium can only have the same requisite initial borrowing with at most one other tight-tight equilibrium. To show the second part of the statement, by monotonicity of the derivative sign and the fact that $d_{0}^{2-c y c, T T}(\cdot \mid y, y)$ is decreasing at $\tilde{\mu}_{0}=\delta$, it suffices to show that $d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}\left(\delta^{2} \mid y, y\right)<d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}(\delta \mid y, y)=d_{0}^{\mathrm{SSE}}$ under the proposed parameter limits; then clearly, there will be some $v_{1} \in\left(\delta^{2}, \delta\right)$ that maximizes the required initial borrowing and some $v_{2} \in(\delta, 1)$ such that every tight-tight equilibrium having period 0 normalized multiplier in the interval $\left(\delta, v_{1}\right)$ has exactly one counterpart tight-tight equilibrium with period 0 normalized multiplier in the interval $\left(v_{1}, v_{2}\right)$ with matching requisite initial borrowing, and vice versa. Explic-
itly, $v_{1}$ is computed as the positive root of the expression given in (A.6), i.e.

$$
\begin{gather*}
0=-\kappa(1+\kappa)\left(v_{1}^{2}-\delta^{2}\right)-2 \kappa(1+r-\kappa) v_{1}-\left((1+r-\kappa)^{2}-\delta^{2}\left(1-\kappa^{2}\right)\right) \\
\Longrightarrow v_{1}=-\frac{(1+r-\kappa)}{1+\kappa}+\sqrt{\frac{1}{\kappa}\left(\delta^{2}-\left(\frac{1+r-\kappa}{1+\kappa}\right)^{2}\right)} \tag{A.8}
\end{gather*}
$$

while $\nu_{2}$ is computed as the solution in the interval $(\delta, 1)$ such that

$$
\begin{gather*}
d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}\left(v_{2} \mid y, y\right)=d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}\left(\delta^{2} \mid y, y\right) \\
\Longrightarrow y_{0}-y+\alpha \kappa y \delta^{2} \frac{(\kappa+1) v_{2}+1+r-\kappa}{\kappa(1+r-\kappa) v_{2}^{2}+\left((1+r-\kappa)^{2}-\delta^{2}\left(1-\kappa^{2}\right)\right) v_{2}+\delta^{2} \kappa(1+r-\kappa)} \\
=y_{0}-y+\alpha \kappa y \frac{(\kappa+1) \delta^{2}+1+r-\kappa}{(1+r)\left(1+r-\kappa\left(1-\delta^{2}\right)\right)-\delta^{2}} \\
\Longrightarrow v_{2}=\frac{\delta^{2}(1+\kappa-\kappa(1+r-\kappa))-(1+r-\kappa)^{2}}{\kappa\left(\delta^{2}(1+\kappa)+1+r-\kappa\right)} . \tag{A.9}
\end{gather*}
$$

Now observe that

$$
\begin{gathered}
\left.d_{0}^{\mathrm{SSE}}\right|_{r=0}-\left.d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}\left(\delta^{2} \mid y, y\right)\right|_{r=0} \\
=\left[y_{0}-\left(1-\frac{\alpha \kappa \delta}{(1-\delta)(1-\kappa)}\right) y\right]-\left[y_{0}-\left(1-\alpha \kappa \frac{(1+\kappa) \delta^{2}+1-\kappa}{\left(1-\delta^{2}\right)(1-\kappa)}\right) y\right] \\
=\frac{\alpha \kappa^{2}}{(1+\delta)(1-\kappa)}\left[\delta-\left(\frac{1-\kappa}{\kappa}\right)\right] y,
\end{gathered}
$$

which is positive if $\delta>\frac{1-\kappa}{\kappa}$, which requires $\kappa>\frac{1}{2}$. Continuity completes the argument for statement 4.

For statement 5 of Lemma 2, we appeal to the intermediate value theorem to show there is a $\kappa \in(0,1)$ such that the tight-slack initial borrowing concurs with that of an arbitrary tight-tight
equilibrium when $r$ is small enough. Take any $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right)$. Now note that at $\kappa=0$ we have

$$
\left.d_{0}^{2-c y c, T T}\left(\tilde{\mu}_{0} \mid y, y\right)\right|_{\kappa=0}=y_{0}-y<y_{0}-\frac{\delta(1+r)+\delta}{1+r+\delta} y=\left.d_{0}^{2-\mathrm{cyc}, \mathrm{TS}}\right|_{\kappa=0} .
$$

At $r=0$, taking limits as $\kappa \rightarrow 1$ yields

$$
\begin{gathered}
\left.\lim _{\kappa \rightarrow 1}(1-\kappa) d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}\left(\tilde{\mu}_{0} \mid y, y\right)\right|_{r=0}=\lim _{\kappa \rightarrow 1}\left(y_{0}-y\right)(1-\kappa)+\alpha \kappa y \delta^{2} \frac{(\kappa+1) \tilde{\mu}_{0}+(1-\kappa)}{\kappa \tilde{\mu}_{0}^{2}+\left((1-\kappa)-\delta^{2}(1+\kappa)\right) \tilde{\mu}_{0}+\delta^{2} \kappa} \\
=2 \alpha y \delta^{2} \frac{\tilde{\mu}_{0}}{\tilde{\mu}_{0}^{2}-2 \delta^{2} \tilde{\mu}_{0}+\delta^{2}}>\frac{2 \alpha \delta^{2}}{1-\delta^{2}} y \\
=\lim _{\kappa \rightarrow 1}\left(y_{0}-\frac{2 \delta}{1+\delta} y\right)(1-\kappa)+\frac{2 \alpha \kappa \delta^{2}}{1-\delta^{2}} y=\left.\lim _{\kappa \rightarrow 1}(1-\kappa) d_{0}^{2-c y c, T S}\right|_{r=0},
\end{gathered}
$$

where the inequality is easily seen by noting that $d_{0}^{2-\mathrm{cyc}, \mathrm{TT}}\left(\tilde{\mu}_{0} \mid y, y\right)$ must be strictly greater than $d_{0}^{2-c y c, S T}$ by statement 3 of Lemma 2, and

$$
\left.\lim _{\kappa \rightarrow 1}(1-\kappa) d_{0}^{2-\mathrm{cyc}, \mathrm{ST}}\right|_{r=0}=\frac{2 \alpha \delta^{2}}{1-\delta^{2}} y .
$$

## A.3.5 Welfare

Consider welfare of a 2 -cyclic equilibrium, which by (1.10) of the $\tilde{\mu}_{t}$-characterization simplifies to

$$
\begin{equation*}
\sum_{t=0}^{\infty} \beta^{t} \log c_{t}=\frac{1}{1-\beta^{2}}\left[\log c_{0}+\beta \log \left(\frac{\delta}{\tilde{\mu}_{0}}\right) c_{0}\right] \tag{A.10}
\end{equation*}
$$

By (1.13) of the $\tilde{\mu}_{t}$-characterization, initial consumption is

$$
c_{0}=\left(d_{0}^{\mathrm{NDL}}-d_{0}\right)\left[\sum_{\tau=0}^{\infty} \beta^{\tau} M_{\tau}^{-1}\right]^{-1}=\left(1-(1+r)^{-2}\right)\left(d_{0}^{\mathrm{NDL}}-d_{0}\right)\left(\frac{\tilde{\mu}_{0}}{\tilde{\mu}_{0}+\beta}\right),
$$

and substituting into (A.10) and simplifying yields welfare as

$$
\frac{1}{1-\beta^{2}}\left[(1+\beta) \log \left(1-(1+r)^{-2}\right)\left(d_{0}^{\mathrm{NDL}}-d_{0}\right)+\beta \log \delta+\log \left(\frac{\tilde{\mu}_{0}}{\left(\tilde{\mu}_{0}+\beta\right)^{1+\beta}}\right)\right] .
$$

Thus, under the same fundamentals, the welfare gap between two 2-cyclical equilibria in terms of their period 0 normalized multipliers $\left(\tilde{\mu}_{0}^{2-c y c, 1}, \tilde{\mu}_{0}^{2-c y c, 2}\right)$ is given by

$$
\begin{aligned}
& W\left(\tilde{\mu}_{0}^{2-c y c, 1}, \tilde{\mu}_{0}^{2-c y c, 2}\right)=\sum_{t=0}^{\infty} \beta^{t} \log c_{t}^{2-c y c, 1}-\sum_{t=0}^{\infty} \beta^{t} \log c_{t}^{2-c y c, 2} \\
= & \frac{1}{1-\beta^{2}}\left[\log \left(\frac{\tilde{\mu}_{0}^{2-c y c}, 1}{\left(\tilde{\mu}_{0}^{2-c y c, 1}+\beta\right)^{1+\beta}}\right)-\log \left(\frac{\tilde{\mu}_{0}^{2-c y c}, 2}{\left(\tilde{\mu}_{0}^{2-c y c, 2}+\beta\right)^{1+\beta}}\right)\right] .
\end{aligned}
$$

## Appendix B: Appendix to Chapter 3

## B. 1 Proof of Proposition 12

Proof. Define

$$
\begin{equation*}
\delta \equiv \mu_{v}-\mu_{c}, \underline{\delta} \equiv \mu_{c}-\underline{w}, \bar{\delta} \equiv \bar{w}-\mu_{c}, \bar{\delta}^{\prime} \equiv \bar{w}-\mu_{v}, \underline{\delta}^{\prime} \equiv \mu_{v}-\underline{w} . \tag{B.1}
\end{equation*}
$$

It will be more instructive in understanding the symmetries of the problem to work with $h_{\perp}^{\mathrm{CIP}}(\cdot)$ in (3.18) expressed in terms of $\delta, \underline{\delta}, \bar{\delta}$ to understand how $\alpha_{\perp}^{*}$ depends on $\mu_{v}, \underline{w}$ :

$$
\begin{equation*}
h_{\perp}^{\mathrm{CIP}}(1-\alpha)=-\delta^{2}(\bar{\delta}+\underline{\delta}-\delta)(1-\alpha)^{3}+3 \underline{\delta} \delta^{2}(1-\alpha)^{2}-\underline{\delta} \delta(2 \bar{\delta}+\delta)(1-\alpha)+\delta \underline{\delta} \bar{\delta} \tag{B.2}
\end{equation*}
$$

To understand how $\alpha_{\perp}^{*}$ depends on $\mu_{c}, \bar{w}$, we work with a slight redefinition of $h_{\perp}^{\mathrm{CIP}}(\cdot)$ expressed as a function of $\alpha$ in terms of $\delta, \underline{\delta}^{\prime}, \bar{\delta}^{\prime}$ :

$$
\begin{equation*}
\hat{h}_{\perp}^{\mathrm{CIP}}(\alpha) \equiv \delta^{2}\left(\underline{\delta}^{\prime}+\bar{\delta}^{\prime}-\delta\right) \alpha^{3}-3 \bar{\delta}^{\prime} \delta^{2} \alpha^{2}+\bar{\delta}^{\prime} \delta\left(2 \underline{\delta}^{\prime}+\delta\right) \alpha-\delta \bar{\delta}^{\prime} \underline{\delta}^{\prime} \tag{B.3}
\end{equation*}
$$

With slight abuse of notation, augment $h_{\perp}^{\mathrm{CIP}}(\cdot)$ in (B.2) to admit generic parameter $\beta \in\{\underline{\delta}, \delta\}$ as an argument, and augment $\hat{h}_{\perp}^{\mathrm{CIP}}(\cdot)$ in (B.3) to admit generic parameter $\hat{\beta} \in\left\{\bar{\delta}^{\prime}, \delta\right\}$ as an argument. By definition of $\alpha_{\perp}^{*}$, we have

$$
\begin{gathered}
h_{\perp}^{\mathrm{CIP}}\left(1-\alpha_{\perp}^{*}, \beta\right)=0 \Longrightarrow \partial_{\beta} \alpha_{\perp}^{*}=\left.\frac{\partial_{\beta} h_{\perp}^{\mathrm{CIP}}(1-\alpha, \beta)}{\partial_{1-\alpha} h_{\perp}^{\mathrm{CIP}}(1-\alpha, \beta)}\right|_{\alpha=\alpha_{\perp}^{*}} \\
\hat{h}_{\perp}^{\mathrm{CIP}}\left(\alpha_{\perp}^{*}, \hat{\beta}\right)=0 \Longrightarrow \partial_{\hat{\beta}} \alpha_{\perp}^{*}=-\left.\frac{\partial_{\beta} \hat{h}_{\perp}^{\mathrm{CIP}}(\alpha, \hat{\beta})}{\partial_{\alpha} \hat{h}_{\perp}^{\mathrm{CIP}}(\alpha, \hat{\beta})}\right|_{\alpha=\alpha_{\perp}^{*}}
\end{gathered}
$$

From the proof of Proposition 10 , we know $h_{\perp}^{\mathrm{CIP}}(\cdot, \beta)$ is strictly decreasing at $1-\alpha_{\perp}^{*}$, and relatedly, $\hat{h}_{\perp}^{\mathrm{CIP}}(\cdot, \hat{\beta})$ is strictly increasing at $\alpha_{\perp}^{*}$. Thus,

$$
\begin{aligned}
& \operatorname{sign}\left\{\partial_{\beta} \alpha_{\perp}^{*}\right\}=-\operatorname{sign}\left\{\partial_{\beta} h_{\perp}^{\mathrm{CIP}}\left(1-\alpha_{\perp}^{*}, \beta\right)\right\} \\
& \operatorname{sign}\left\{\partial_{\hat{\beta}} \alpha_{\perp}^{*}\right\}=-\operatorname{sign}\left\{\partial_{\hat{\beta}} \hat{h}_{\perp}^{\mathrm{CIP}}\left(\alpha_{\perp}^{*}, \hat{\beta}\right)\right\} .
\end{aligned}
$$

Differentiating $h_{\perp}^{\text {CIP }}$ in (B.2) yields

$$
\begin{gathered}
\partial_{\underline{\delta}} h_{\perp}^{\mathrm{CIP}}\left(1-\alpha_{\perp}^{*}, \underline{\delta}\right)=\frac{1}{\underline{\delta}}\left[h_{\perp}^{\mathrm{CIP}}\left(1-\alpha_{\perp}^{*}, \underline{\delta}\right)+\delta^{2}(\bar{\delta}-\delta)(1-\alpha)^{3}\right]>0, \\
\partial_{\delta} h_{\perp}^{\mathrm{CIP}}\left(1-\alpha_{\perp}^{*}, \delta\right)=\frac{1}{\delta}\left[h_{\perp}^{\mathrm{CIP}}\left(1-\alpha_{\perp}^{*}, \delta\right)+\delta^{2}\left(1-\alpha_{\perp}^{*}\right)\left(-(\bar{\delta}+\underline{\delta}-2 \delta)\left(1-\alpha_{\perp}^{*}\right)^{2}+3 \underline{\delta}\left(1-\alpha_{\perp}^{*}\right)-\underline{\delta}\right)\right] \\
=\delta\left(1-\alpha_{\perp}^{*}\right)\left(-(\bar{\delta}+\underline{\delta}-2 \delta)\left(1-\alpha_{\perp}^{*}\right)^{2}+3 \underline{\delta}\left(1-\alpha_{\perp}^{*}\right)-\underline{\delta}\right) .
\end{gathered}
$$

Since $\partial_{\underline{\underline{\delta}}} \alpha_{\perp}^{*}=-\partial_{\underline{w}} \alpha_{\perp}^{*}$, the desired result for $\underline{w}$ is immediate. To examine the sign of $\partial_{\delta} \alpha_{\perp}^{*}=$ $\partial_{\mu_{\nu}} \alpha_{\perp}^{*}$, define the function $\Delta_{\delta}\left(1-\alpha_{\perp}^{*}\right) \equiv \frac{\partial_{\delta} h_{\perp}^{\mathrm{CIP}}\left(1-\alpha_{\perp}^{*}, \delta\right)}{\delta\left(1-\alpha_{\perp}^{*}\right)}$, which is quadratic in the argument $1-\alpha_{\perp}^{*}$ and satisfies

$$
\Delta_{\delta}(0)=-\underline{\delta}<0,4 \Delta_{\delta}\left(\frac{1}{2}\right)=\Delta_{\delta}(1)=\underline{\delta}+2 \delta-\bar{\delta}=2\left(\mu_{v}-\frac{1}{2}(\underline{w}+\bar{w})\right) .
$$

We consider various cases:

Case 3 Suppose $\bar{w}-\mu_{v} \leq \mu_{c}-\underline{w}$ (equivalently, $1-\alpha_{\perp}^{*} \geq \frac{1}{2}$ by Proposition 10). This condition is met if $\mu_{c} \geq \frac{1}{2}(\underline{w}+\bar{w})$ and only if $\mu_{v}>\frac{1}{2}(\underline{w}+\bar{w})$, and the latter implies that $\Delta_{\delta}\left(1-\alpha_{\perp}^{*}\right)>0$ for $1-\alpha_{\perp}^{*} \geq \frac{1}{2}$. Thus, $\alpha_{\perp}^{*} \leq \frac{1}{2}$ strictly decreases in $\mu_{v}$.

Case 4 Suppose $\bar{w}-\mu_{v}>\mu_{c}-\underline{w}$ (equivalently, $1-\alpha_{\perp}^{*}<\frac{1}{2}$ by Proposition 10). This condition is met only if $\mu_{c}<\frac{1}{2}(\underline{w}+\bar{w})$.
i. If $\mu_{v} \leq \frac{1}{2}(\underline{w}+\bar{w})$, then $\Delta_{\delta}\left(1-\alpha_{\perp}^{*}\right)$ is a downward parabola (since in this case we have $\underline{\delta}+\delta=\mu_{v}-\underline{w} \leq \bar{w}-\mu_{v}=\bar{\delta}-\delta$, so that the leading coefficient is $\left.-(\bar{\delta}+\underline{\delta}-2 \delta) \leq-2 \underline{\delta}<0\right)$.

We wish to show $\Delta_{\delta}\left(1-\alpha_{\perp}^{*}\right)<0$ for $1-\alpha_{\perp}^{*}<\frac{1}{2}$. If $\mu_{v}=\frac{1}{2}(\underline{w}+\bar{w})$, then $\Delta_{\delta}\left(1-\alpha_{\perp}^{*}\right)$ has roots at $1-\alpha_{\perp}^{*}=\frac{1}{2}, 1$ giving the desired result. If $\mu_{v}<\frac{1}{2}(\underline{w}+\bar{w})$, it suffices to show that either $\Delta_{\delta}\left(1-\alpha_{\perp}^{*}\right)$ has negative discriminant (and hence is always negative) or is maximized at $1-\alpha_{\perp}^{*} \geq \frac{1}{2}$. Suppose by contradiction that $\Delta_{\delta}\left(1-\alpha_{\perp}^{*}\right)$ has both nonnegative discriminant and a vertex at some $1-\alpha_{\perp}^{*} \leq \frac{1}{2} . \Delta_{\delta}\left(1-\alpha_{\perp}^{*}\right)$ has nonnegative discriminant when $9 \underline{\delta}^{2}-4(\bar{\delta}+\underline{\delta}-2 \delta) \underline{\delta} \geq$ $0 \Longleftrightarrow 5 \underline{\delta}-4(\bar{\delta}-2 \delta) \geq 0$. The vertex of $\Delta_{\delta}\left(1-\alpha_{\perp}^{*}\right)$ occurs at some $1-\alpha_{\perp}^{*} \leq \frac{1}{2}$ when $\frac{3 \underline{\delta}}{2(\bar{\delta}+\underline{\delta}-2 \delta)} \leq \frac{1}{2} \Longleftrightarrow-2 \underline{\delta}+\bar{\delta}-2 \delta \geq 0$. Summing the two inequalities yields $3 \underline{\delta}-3(\bar{\delta}-2 \delta) \geq$ $0 \Longleftrightarrow 0 \leq \underline{\delta}-\bar{\delta}+2 \delta=-2\left(\frac{1}{2}(\underline{w}+\bar{w})-\mu_{v}\right)$, a contradiction. Thus, $\alpha_{\perp}^{*}>\frac{1}{2}$ strictly increases in $\mu_{v} \leq \frac{1}{2}(\underline{w}+\bar{w})$.
ii. If instead $\mu_{v}>\frac{1}{2}(\underline{w}+\bar{w})$,then $\Delta_{\delta}\left(1-\alpha_{\perp}^{*}\right)$ has a unique root at some $1-\tilde{\alpha} \in\left(0, \frac{1}{2}\right)$ (dependent upon the other parameters) so that $\Delta_{\delta}\left(1-\alpha_{\perp}^{*}\right)<($ resp. $>) 0$ whenever $1-\alpha_{\perp}^{*}<($ resp. $>) 1-\tilde{\alpha}$. That is, $\alpha_{\perp}^{*}>$ (resp. <) $\tilde{\alpha}$ is strictly increasing (resp. decreasing) in $\mu_{\nu}$.

From Case 4 i , at $\mu_{v}=\frac{1}{2}(\underline{w}+\bar{w}), \alpha_{\perp}^{*}$ is increasing in $\mu_{v}$ and thus $\alpha_{\perp}^{*}>\tilde{\alpha}$ at this point. Also observe

$$
\partial_{\mu_{v}} \tilde{\alpha}=\partial_{\delta} \tilde{\alpha}=\left.\frac{\partial_{\delta} \Delta_{\delta}(1-\alpha)}{\partial_{1-\alpha} \Delta_{\delta}(1-\alpha)}\right|_{\alpha=\tilde{\alpha}},
$$

and since $\Delta_{\delta}$ is strictly increasing at $1-\tilde{\alpha}$, $\operatorname{sign}\left\{\partial_{\mu_{v}} \tilde{\alpha}\right\}=\operatorname{sign}\left\{\partial_{\delta} \Delta_{\delta}(1-\tilde{\alpha})=2(1-\tilde{\alpha})^{2}\right\}>0$, and thus $\tilde{\alpha}$ is strictly increasing in $\mu_{v}$. Since $\alpha_{\perp}^{*}=\frac{1}{2}$ when $\mu_{v}=\bar{w}+\underline{w}-\mu_{c}$, it is thus clear that there is a $\mu_{v}^{*} \in\left(\frac{1}{2}(\underline{w}+\bar{w}), \bar{w}+\underline{w}-\mu_{c}\right)$ at which $\alpha_{\perp}^{*}=\tilde{\alpha}$ (attaining a maximum) so that $\alpha_{\perp}^{*}>$ (resp. <) $\tilde{\alpha}$ is strictly increasing (resp. decreasing) in $\mu_{v}<($ resp. $>) \mu_{v}^{*}$.

The results for $\bar{w}, \mu_{c}$ are obtained analogously by appealing to the symmetry of the problem, observing that $\hat{h}_{\perp}^{\mathrm{CIP}}(\cdot)$ has the same form as $-h_{\perp}^{\mathrm{CIP}}(\cdot)$, except that $1-\alpha, \bar{\delta}, \underline{\delta}$ in the latter are respectively replaced by $\alpha, \underline{\delta}^{\prime}, \bar{\delta}^{\prime}$ in the former, and noting that $\partial_{\bar{\delta}^{\prime}} \alpha_{\perp}^{*}=\partial_{\bar{w}} \alpha_{\perp}^{*}, \partial_{\delta} \alpha_{\perp}^{*}=-\partial_{\mu_{c}} \alpha_{\perp}^{*}$.

## B. 2 Pricing to achieve best worst-case probability of trade

In this section, we solve for the mediator's pricing strategy that maximizes the worst-case probability of trade, namely the price that solves

$$
\begin{equation*}
\max _{p \geq 0} \min _{H \in \mathcal{P}\left(\mathbb{R}_{\geq 0}^{2}\right)} \mathbb{E}_{(v, c) \sim H}\left[\mathbf{1}_{c<p<v}\right] \tag{B.4}
\end{equation*}
$$

subject to (3.4) and (3.5). We consider the problem both with and without the independence constraint (3.3), and denote the resulting price strategies by $\tilde{p}_{\perp}^{\mathrm{IP}}$ and $\tilde{p}^{\text {IIP }}$ respectively. We also obtain comparative statics results. As mentioned in section 3.2.2, Nature's optimal strategy is unchanged for the new objective.

## B.2. 1 Independent values

Under the independence constraint, the mediator chooses a price

$$
\tilde{p}_{\perp}^{\mathrm{IIP}} \in \underset{p \in\left(\mu_{c}, \mu_{v}\right)}{\arg \max }\left\{\tilde{\Pi}_{\perp}(p) \equiv \mathbb{E}_{v \sim F^{*}}\left[\mathbf{1}_{p<v}\right] \mathbb{E}_{c \sim G^{*}}\left[\mathbf{1}_{c<p}\right]\right\}
$$

and substituting in Nature's strategy, summarized by (3.9)-(3.12), we see the mediator's price is given by

$$
\begin{aligned}
& \tilde{p}_{\perp}^{\mathrm{IP}} \in \arg \max _{p} \pi_{\bar{w}}(p) \pi_{\underline{w}}(p) \\
& =\arg \max _{p} \frac{\mu_{v}-p}{\bar{w}-p} \frac{p-\mu_{c}}{p-\underline{w}} .
\end{aligned}
$$

The FOC is given by

$$
\begin{align*}
0 & =\tilde{\Pi}_{\perp}^{\prime}(p)=(\bar{w}-p)^{-2}(p-\underline{w})^{-2}\left[2\left(\frac{1}{2}\left(\mu_{v}+\mu_{c}\right)-p\right)(\bar{w}-p)(p-\underline{w})+2\left(p-\frac{1}{2}(\underline{w}+\bar{w})\right)\left(\mu_{v}-p\right)\left(p-\mu_{c}\right)\right] \\
& =(\bar{w}-p)^{-2}(p-\underline{w})^{-2}\left[\left(\left(\mu_{c}-\underline{w}\right)-\left(\bar{w}-\mu_{v}\right)\right) p^{2}+2\left(\bar{w} \underline{w}-\mu_{v} \mu_{c}\right) p+(\underline{w}+\bar{w}) \mu_{v} \mu_{c}-\left(\mu_{v}+\mu_{c}\right) \bar{w} \underline{w}\right] . \tag{B.5}
\end{align*}
$$

Defining

$$
\begin{equation*}
\tilde{h}_{\perp}(p) \equiv(\bar{w}-p)^{2}(p-\underline{w})^{2} \tilde{\Pi}_{\perp}^{\prime}(p), \tag{B.6}
\end{equation*}
$$

we see from (B.5) that $\tilde{h}_{\perp}$ is a linear or quadratic polynomial, satisfying $\tilde{h}_{\perp}\left(\mu_{c}\right)=\left(\mu_{v}-\mu_{c}\right)(\bar{w}-$ $\left.\mu_{c}\right)\left(\mu_{c}-\underline{w}\right)>0$ and $\tilde{h}_{\perp}\left(\mu_{v}\right)=-\left(\mu_{v}-\mu_{c}\right)\left(\bar{w}-\mu_{v}\right)\left(\mu_{v}-\underline{w}\right)<0$, implying there is a unique $\tilde{p}_{\perp}^{\mathrm{IIP}} \in\left(\mu_{c}, \mu_{v}\right)$ that satisfies the FOC. Consider three cases:

Case $5 \mu_{c}-\underline{w}<\bar{w}-\mu_{\nu}$. Then $\tilde{h}_{\perp}$ is a downward parabola, of which $\tilde{p}_{\perp}^{\text {IIP }}$ is its upper root:

$$
\tilde{p}_{\perp}^{\mathrm{IP}}=\frac{\left(\bar{w} \underline{w}-\mu_{v} \mu_{c}\right)+\sqrt{\left(\bar{w}-\mu_{v}\right)\left(\mu_{c}-\underline{w}\right)\left(\mu_{v}-\underline{w}\right)\left(\bar{w}-\mu_{c}\right)}}{\left(\bar{w}-\mu_{v}\right)-\left(\mu_{c}-\underline{w}\right)} .
$$

Case $6 \mu_{c}-\underline{w}>\bar{w}-\mu_{\nu}$. Then $\tilde{h}_{\perp}$ is an upward parabola, of which $\tilde{p}_{\perp}^{\text {IIP }}$ is its lower root:

$$
\tilde{p}_{\perp}^{\mathrm{IP}}=\frac{-\left(\bar{w} \underline{w}-\mu_{v} \mu_{c}\right)-\sqrt{\left(\bar{w}-\mu_{v}\right)\left(\mu_{c}-\underline{w}\right)\left(\mu_{v}-\underline{w}\right)\left(\bar{w}-\mu_{c}\right)}}{\left(\mu_{c}-\underline{w}\right)-\left(\bar{w}-\mu_{v}\right)} .
$$

Case $7 \mu_{c}-\underline{w}=\bar{w}-\mu_{v}$. Then $\tilde{h}_{\perp}$ is a line and $\tilde{p}_{\perp}^{\text {IIP }}=\frac{1}{2}\left(\mu_{v}+\mu_{c}\right)$.

Moreover, we have the direct analogue of Proposition 11:

Proposition $16 \tilde{p}_{\perp}^{\text {IIP }}$ strictly increases in $\mu_{v}, \mu_{c}$ and strictly decreases in $\underline{w}, \bar{w}$.
Proof. With slight abuse of notation, augment $\tilde{h}_{\perp}(\cdot)$ in (B.6) to admit generic parameter $\beta \in$ $\left\{\mu_{v}, \mu_{c}, \bar{w}, \underline{w}\right\}$ as an argument. By definition of $\tilde{p}_{\perp}^{\text {IIP }}$, we have

$$
\tilde{h}_{\perp}\left(\tilde{p}_{\perp}^{\mathrm{IIP}}, \beta\right)=0 \Longrightarrow \partial_{\beta} \tilde{p}_{\perp}^{\mathrm{IP}}=-\left.\frac{\partial_{\beta} \tilde{h}(p, \beta)}{\partial_{p} \tilde{h}(p, \beta)}\right|_{p=\tilde{p}_{\perp}^{\mathrm{IP}} .}
$$

Since $\tilde{h}_{\perp}(\cdot, \beta)$ is strictly decreasing at $\tilde{p}_{\perp}^{\text {IIP }}, \operatorname{sign}\left\{\partial_{\beta} \tilde{p}_{\perp}^{\text {IIP }}\right\}=\operatorname{sign}\left\{\partial_{\beta} \tilde{h}_{\perp}\left(\tilde{p}_{\perp}^{\text {IIP }}, \beta\right)\right\}$. Differentiating yields

$$
\partial_{\mu_{v}} \tilde{h}_{\perp}\left(\tilde{p}_{\perp}^{\mathrm{IIP}}, \mu_{v}\right)=\left.\frac{1}{\mu_{v}-p}\left[\tilde{h}_{\perp}\left(p, \mu_{v}\right)+\left(p-\mu_{c}\right)(\bar{w}-p)(p-\underline{w})\right]\right|_{p=\tilde{p}_{\perp}^{\mathrm{IP}}}>0,
$$

$$
\begin{aligned}
\partial_{\mu_{c}} \tilde{h}_{\perp}\left(\tilde{p}_{\perp}^{\mathrm{IIP}}, \mu_{c}\right) & =\left.\frac{1}{p-\mu_{c}}\left[-\tilde{h}_{\perp}\left(p, \mu_{c}\right)+\left(\mu_{v}-p\right)(\bar{w}-p)(p-\underline{w})\right]\right|_{p=\tilde{p}_{\perp}^{\mathrm{IP}}>0} \\
\partial_{\underline{w}} \tilde{h}_{\perp}\left(\tilde{p}_{\perp}^{\mathrm{IPP}}, \underline{w}\right) & =\left.\frac{1}{p-\underline{w}}\left[-\tilde{h}_{\perp}(p, \underline{w})-\left(\mu_{v}-p\right)\left(p-\mu_{c}\right)(\bar{w}-p)\right]\right|_{p=\tilde{p}_{\perp}^{\mathrm{IP}}}<0, \\
\partial_{\bar{w}} \tilde{h}_{\perp}\left(\tilde{p}_{\perp}^{\mathrm{IIP}}, \bar{w}\right) & =\left.\frac{1}{\bar{w}-p}\left[\tilde{h}_{\perp}(p, \bar{w})-\left(\mu_{v}-p\right)\left(p-\mu_{c}\right)(p-\underline{w})\right]\right|_{p=\tilde{p}_{\perp}^{\mathrm{IP}}}<0 .
\end{aligned}
$$

## B.2.2 Dependent values

Absent the independence constraint, the mediator chooses the price as

$$
\begin{equation*}
\tilde{p}^{I I P} \in \underset{p \in\left(\mu_{c}, \mu_{v}\right)}{\arg \max }\left(\tilde{\Pi}(p) \equiv \pi_{\bar{w}}(p) \pi_{\underline{w}}(p)-\left(1-\pi_{\bar{w}}(p)\right)\left(1-\pi_{\underline{w}}(p)\right)\right) . \tag{B.7}
\end{equation*}
$$

Substituting (3.11)-(3.12), the objective in (B.7) simplifies to

$$
\tilde{\Pi}(p)=[(\bar{w}-p)(p-\underline{w})]^{-1} \Lambda(p)
$$

where $\Lambda(p)$ is as defined in (3.19). The FOC is given by

$$
\begin{gathered}
0=\tilde{\Pi}^{\prime}(p)=(\bar{w}-p)^{-2}(p-\underline{w})^{-2}\left[2\left(\frac{1}{2}\left(\mu_{v}+\mu_{c}\right)-p\right)(\bar{w}-p)(p-\underline{w})+2\left(p-\frac{1}{2}(\bar{w}+\underline{w})\right) \Lambda(p)\right] \\
=(\bar{w}-p)^{-2}(p-\underline{w})^{-2}\left[\left(\left(\mu_{c}-\underline{w}\right)-\left(\bar{w}-\mu_{v}\right)\right) p^{2}+2\left(\underline{w}\left(\bar{w}-\mu_{v}\right)-\bar{w}\left(\mu_{c}-\underline{w}\right)\right) p+\bar{w}^{2}\left(\mu_{c}-\underline{w}\right)-\underline{w}^{2}\left(\bar{w}-\mu_{v}\right)\right] .
\end{gathered}
$$

Defining

$$
\tilde{h}(p) \equiv(\bar{w}-p)^{2}(p-\underline{w})^{2} \tilde{\Pi}^{\prime}(p),
$$

we see that $\tilde{h}$ is a linear or quadratic polynomial, satisfying $\tilde{h}\left(\Lambda_{ \pm}\right) \lessgtr 0$, where $\Lambda_{ \pm}$is as defined in (3.23), implying there is a unique $\tilde{p}^{\mathrm{IIP}} \in\left(\Lambda_{-}, \Lambda_{+}\right)$that solves the FOC. Consider three cases:

Case $8 \mu_{c}-\underline{w}<\bar{w}-\mu_{v}$. Then $\tilde{h}$ is a downward parabola, of which $\tilde{p}^{\text {IIP }}$ is its upper root:

$$
\tilde{p}^{\mathrm{IP}}=\frac{\underline{w}\left(\bar{w}-\mu_{v}\right)-\bar{w}\left(\mu_{c}-\underline{w}\right)+\sqrt{\left(\bar{w}-\mu_{v}\right)\left(\mu_{c}-\underline{w}\right)(\bar{w}-\underline{w})^{2}}}{\left(\bar{w}-\mu_{v}\right)-\left(\mu_{c}-\underline{w}\right)} .
$$

Case $9 \mu_{c}-\underline{w}>\bar{w}-\mu_{v}$. Then $\tilde{h}$ is an upward parabola, of which $\tilde{p}^{\text {IIP }}$ is its lower root:

$$
\tilde{p}^{\mathrm{IP}}=\frac{\bar{w}\left(\mu_{c}-\underline{w}\right)-\underline{w}\left(\bar{w}-\mu_{v}\right)-\sqrt{\left(\bar{w}-\mu_{v}\right)\left(\mu_{c}-\underline{w}\right)(\bar{w}-\underline{w})^{2}}}{\left(\mu_{c}-\underline{w}\right)-\left(\bar{w}-\mu_{v}\right)} .
$$

Case $10 \mu_{c}-\underline{w}=\bar{w}-\mu_{v}$. Then $\tilde{h}$ is a line and $\tilde{p}^{\text {IIP }}=\frac{1}{2}\left(\mu_{v}+\mu_{c}\right)$.

Moreover, we have the direct analogue of Proposition 15:
Proposition $17 \tilde{p}^{\text {IIP }}$ strictly increases in $\mu_{v}, \mu_{c}$. For $\mu_{c}-\underline{w}$ sufficiently small, $\tilde{p}^{\text {IIP }}$ strictly decreases in $\bar{w}$; for $\bar{w}-\mu_{v}$ sufficiently small, $\tilde{p}^{\text {IIP }}$ strictly decreases in $\underline{w} .{ }^{1}$

Proof. With slight abuse of notation, augment $\tilde{h}(\cdot)$ to admit generic parameter $\beta \in\left\{\mu_{\nu}, \mu_{c}, \bar{w}, \underline{w}\right\}$ as an argument. By definition of $\tilde{p}^{\text {IIP }}$, we have

$$
\tilde{h}\left(\tilde{p}^{\mathrm{IIP}}, \beta\right)=0 \Longrightarrow \partial_{\beta} \tilde{p}^{\mathrm{IIP}}=-\left.\frac{\partial_{\beta} \tilde{h}(p, \beta)}{\partial_{p} \tilde{h}(p, \beta)}\right|_{p=\tilde{p}^{\mathrm{IP}} .}
$$

Since $\tilde{h}(\cdot, \beta)$ is strictly decreasing at $\tilde{p}^{\text {IIP }}, \operatorname{sign}\left\{\partial_{\beta} \tilde{p}^{\text {IPP }}\right\}=\operatorname{sign}\left\{\partial_{\beta} \tilde{h}\left(\tilde{p}^{\text {IIP }}, \beta\right)\right\}$. Differentiating yields

$$
\begin{gathered}
\partial_{\mu_{v}} \tilde{h}\left(\tilde{p}^{\mathrm{IP}}, \mu_{v}\right)=\left(\tilde{p}^{\mathrm{IIP}}-\underline{w}\right)^{2}>0, \\
\partial_{\mu_{c}} \tilde{h}\left(\tilde{p}^{\mathrm{IP}}, \mu_{c}\right)=\left(\bar{w}-\tilde{p}^{\mathrm{IIP}}\right)^{2}>0, \\
\partial_{\bar{w}} \tilde{h}\left(\tilde{p}^{\mathrm{IIP}}, \bar{w}\right)=\left.\frac{1}{\bar{w}-p}\left[\tilde{h}(p, \bar{w})-\left(\mu_{v}-p\right)(p-\underline{w})^{2}+\left(\mu_{c}-\underline{w}\right)(\bar{w}-p)^{2}\right]\right|_{p=\tilde{p}^{\mathrm{IP}},}, \\
\partial_{\underline{w}} \tilde{h}\left(\tilde{p}^{\mathrm{IIP}}, \underline{w}\right)=\left.\frac{1}{p-\underline{w}}\left[-\tilde{h}(p, \underline{w})-\left(p-\mu_{c}\right)(\bar{w}-p)^{2}+\left(\bar{w}-\mu_{v}\right)(p-\underline{w})^{2}\right]\right|_{p=\tilde{p}^{\mathrm{II}}} .
\end{gathered}
$$

[^37]Inspection of the last two derivatives makes clear how the desired results for $\bar{w}$ and $\underline{w}$ hold. As with Proposition 15, a fixed value of $\mu_{c}-\underline{w}$ or $\bar{w}-\mu_{v}$ may be chosen to obtain the desired monotonic behavior of $\tilde{p}^{\text {IIP }}$ in the entire feasible range of $\bar{w}$ or $\underline{w}$.


[^0]:    ${ }^{1}$ The idea of a debt-deflation channel goes back to Fisher 1933. See also Bernanke and Gertler 1989, 1990 and Kiyotaki and Moore 1997.

[^1]:    ${ }^{2}$ See also Bernanke and Gertler 1989 for a closed economy "financial accelerator" model.
    ${ }^{3}$ See, for instance, Paasche 2001, which studies a three-country version of Kiyotaki and Moore 1997. See also Iacoviello 2005, which starts with a variant of the Bernanke and Gertler 1989 model and incorporates a stock collateral constraint in the spirit of Kiyotaki and Moore 1997.
    ${ }^{4}$ Jeanne and Korinek 2019 explain how multiplicity of equilibria comes from a self-reinforcing loop that links consumption to the price of collateral. They focus on obtaining a sufficient condition for equilibrium uniqueness in the special case of constant output and $\delta=1$ and offer intuitive guidance on how an extension of this condition may apply under a more general output trend and impatience assumption.
    ${ }^{5}$ See, for instance, Bernanke and Gertler 1989, Mendoza 2002, and Aghion, Bacchetta, and Banerjee 2004.

[^2]:    ${ }^{6}$ There is also a growing closed economy literature showing how endogenous cycles can arise from financial frictions. See, for instance, Woodford 1989, Suarez and Sussman 1997, Matsuyama 2007, Martin and Ventura 2012, Gu et al. 2013, Gorton and Ordoñez 2014, 2020, Benhabib, Miao, and Wang 2016, Azariadis, Kaas, and Wen 2016, Benhabib, Dong, and Wang 2018, Miao and Wang 2018, Beaudry, Galizia, and Portier 2020, Dong and Xu 2020, Chousakos, Gorton, and Ordoñez 2020, and Cui and Kaas 2021, among others.
    ${ }^{7}$ In contrast to our model, Mendoza works with a particular class of preferences with an endogenous subjective discount factor that enables the model to support a unique, invariant limiting distribution of foreign assets. However, he later points out in the context of calibrating the model that any "impatience effects" introduced by endogenous discounting have negligible quantitative implications. Our perfect-foresight model with exogenous discounting abstracts away from these considerations.

[^3]:    ${ }^{8}$ As described in the sequel, this externality results from agents not internalizing general equilibrium effects of their decisions on asset prices.
    ${ }^{9}$ See, for instance, Uribe 2006a,b, Lorenzoni 2008, Korinek 2011, Bianchi 2011, Bianchi and Mendoza 2015, Benigno et al. 2013, 2016, Schmitt-Grohé and Uribe 2017b, Dávila and Korinek 2018, and Jeanne and Korinek 2019, among others.

[^4]:    ${ }^{10}$ An alternative specification would be to assume that borrowing is constrained according to the expected value of capital at the time of maturation, in which case the right-hand side of the constraint would be $\kappa q_{t+1} k_{t+1}$. This kind of constraint is adopted in Devereux, Young, and Yu 2016, Kiyotaki and Moore 1997, and Iacoviello 2005, whereas our timing convention is more in tune with Bianchi and Mendoza 2015. Devereux, Young, and Yu 2016 observe that either timing convention can be appropriately microfounded, but empirically, it is an open question what type of constraint fits the data best.

[^5]:    ${ }^{11}$ Effectively, if this condition was not satisfied, households can issue additional debt to fund an increase in consumption while still meeting the collateral constraint, improving lifetime utility in the process.

[^6]:    ${ }^{12}$ In general we adopt the convention that for a generic sequence $\left\{a_{\tau}\right\}_{\tau \geq 0}$, we have $\sum_{\tau=0}^{-1} a_{\tau}=0$, and $\prod_{\tau=0}^{-1} a_{\tau}=1$, which are natural extensions of summation and product notation.

[^7]:    ${ }^{13}$ Note that $d_{0}^{\mathrm{DNL}}$ is well-defined if and only if $d_{t}^{\mathrm{NDL}}$ is well-defined. Moreover, since output is nonnegative, welldefinedness of $d_{0}^{\mathrm{NDL}}$ is equivalent to the series $\sum_{\tau=0}^{\infty} \frac{y_{\tau}}{(1+r)^{\tau}}$ not diverging to infinity (which we write as $d_{0}^{\mathrm{DNL}}<\infty$ ). This assumption does not hold if, for instance, output geometrically grows faster than rate $1+r$. The well-definedness of the natural debt limit is not required to ensure (1.11) gives a valid equilibrium debt, which only requires a welldefined net present value of trade balances. Nonetheless, the well-definedness of the natural debt limit will avoid indeterminacies associated with the characterization that follows.
    ${ }^{14}$ To see the product is strictly increasing in the normalized multipliers, note that the function characterized by $g(\tilde{\mu}):=\frac{\tilde{\mu}}{1+r-\kappa+\kappa \tilde{\mu}}$ can be expressed as $g(\tilde{\mu})=\frac{1}{\kappa}\left(1-\frac{1+r-\kappa}{1+r-\kappa+\kappa \tilde{\mu}}\right)$ and $1+r>\kappa$.

[^8]:    ${ }^{15}$ As we discuss in section 2.1, an exception here is in solving for $N$-cyclic equilibria, which feature a constraint that binds infinitely often but whose characterization can nonetheless be reduced to solving a finite-dimensional system.

[^9]:    ${ }^{16}$ Note that the assumption that the natural debt limit is well-defined is clearly required to support the unconstrained equilibrium under general $\delta$; otherwise, the price of capital (among other things) would be ill-defined.

[^10]:    ${ }^{17}$ The inequality for the price of capital is strict if at least one of the chain of inequalities for the trajectory of output or normalized multipliers is strict.

[^11]:    ${ }^{18}$ In fact, while we obtained Corollary 1 using the $\tilde{\mu}_{t}$-characterization, we can obtain it more directly using the equilibrium conditions, as done in SGU. To see how, recall consumption growth is bounded below according to $c_{t+1} / c_{t} \geq \delta$. Moreover, if there is at least one period with a tight constraint, then that period experiences consumption growth strictly greater than $\delta$. However, the intertemporal resource constraint (1.1) asserts that initial debt is the net present value of all future trade balances, so that any equilibrium must have a fixed net present value of consumption given an exogenous initial debt and output path. Since the unconstrained equilibrium quantities themselves were solved to satisfy this restriction on the net present value of consumption, then it follows that any equilibrium with a tight constraint for some period has initial consumption strictly less than that of the unconstrained counterpart ( $c_{0}<c_{0}^{\mathrm{UE}}$ ) to ensure both consumption paths have the same net present value.

[^12]:    ${ }^{19}$ The inequality for debt is strict if at least one of collection of inequalities for the normalized multipliers or the trajectory of output is strict.

[^13]:    ${ }^{20}$ If future output is zero, i.e. $\left\{y_{\tau+1}\right\}_{\tau \geq t}=0$, then $\tilde{\mu}_{\tau} \leq \tilde{\mu}_{\tau+1}$ for all $\tau \geq t$ does not require $\tilde{\mu}_{\infty} \leq \tilde{\mu}^{*}$. A simple counterexample is the ESE.

[^14]:    ${ }^{21}$ The inequality can be made strict here since $\tilde{\mu}_{\tau}<\tilde{\mu}^{*}$ infinitely often.

[^15]:    ${ }^{22}$ If $d_{\infty}$ exists, then $c_{\infty}=y_{\infty}-\frac{r}{1+r} d_{\infty}$ by (1.2). If $c_{\infty}$ exists and $y_{\infty}>0$, then $c_{\infty}>0$ (Lemma 1 ), so $c_{t+1} / c_{t} \rightarrow 1$, which is equivalent to $\tilde{\mu}_{\infty}=\tilde{\mu}^{*}$ by (1.3). Note that (1.3) does not ensure $\tilde{\mu}_{\infty}=\tilde{\mu}^{*}$ is sufficient for $c_{\infty}$ to exist since a vanishing growth rate does not imply convergence; a counterexample is the sequence $2,1,3 / 2,2,5 / 3,4 / 3,1,5 / 4,6 / 4,7 / 4,2 \ldots$

[^16]:    ${ }^{1}$ Note that the ESE cannot be $N$-cyclic since all endogenous ESE quantities vanish in the limit as $t \uparrow \infty$.

[^17]:    ${ }^{2}$ Note the unique $d_{0}$ satisfies its natural debt limit: $d_{0}=d_{N}-y_{N}+y_{0}<d_{N}^{\text {NDL }}-y_{N}+y_{0}=d_{0}^{\text {NDL }}$.
    ${ }^{3}$ This result contrasts with the ESE, for instance, where a continuum of values of $d_{0}$ is permissible (given other parameters) so long as the ESE existence condition is met.

[^18]:    ${ }^{4}$ The inequality for the price of capital is strict if at least one of the chain of inequalities for the trajectory of output or normalized multipliers is strict.

[^19]:    ${ }^{5}$ The inequality for debt is strict if at least one of collection of inequalities for the normalized multipliers or the trajectory of output is strict.

[^20]:    ${ }^{6}$ The inequalities for the price of capital and debt are strict if either the inequality for output or normalized multiplier is strict.

[^21]:    ${ }^{7}$ In general, the prospect of multiple equilibria discussed in section 2.2 gives rise to a debt policy correspondence. A debt policy function gives rise to a univariate difference equation to which the Li-Yorke theorem may be applied.

[^22]:    ${ }^{8}$ The explicit values of $v_{1}$ and $v_{2}$ are given in Appendix A.3.4

[^23]:    ${ }^{9}$ In a constant output regime, since $d_{0}=d_{2}-y+y_{0}$ and the slack-tight and tight-slack equilibria are mirror images of one another, an equivalent statement is that $d_{2}<d_{1}$ in a slack-tight equilibrium, which follows from Corollary 4.

[^24]:    ${ }^{10}$ Note that No. 5 does not fine-tune $\alpha$ (a condition of statement 2 of Proposition 6, and relatedly statement 5 of Proposition 8) since the tight-tight equilibrium in question is the SSE, which exists under arbitrary parameters (subject to its initial debt requirement).

[^25]:    ${ }^{11}$ Due to long-run behavior, the claims are simplified as follows when $\delta \geq 1$ :
    Claim 1: In equilibrium, if there is a period $t \geq 0$ where $y_{t+1} \geq y_{t+2} \geq y_{t+3} \ldots$ and $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \geq t}=1$, then $q_{t} \geq q_{t+1}$; if $y_{t+1} \leq y_{t+2} \leq y_{t+3} \ldots$ and $\tilde{\mu}_{t} \leq \tilde{\mu}_{t+1} \leq \tilde{\mu}_{t+2} \ldots$, then $q_{t} \leq q_{t+1}$. In both cases, the inequality for the price of capital is strict if at least one of the chain of inequalities for the trajectory of output or normalized multipliers is strict.

    Claim 2: In equilibrium, if there is a period $t \geq 0$ where $y_{t} \geq y_{t+1} \geq y_{t+2} \ldots$, then $d_{t} \geq d_{t+1}$, and the inequality for debt is strict if at least one of the chain of inequalities for the output trajectory is strict or the collateral constraint is tight at any time from $t$ onward; if $y_{t} \leq y_{t+1} \leq y_{t+2} \ldots$ and $\left\{\tilde{\mu}_{\tau}\right\}_{\tau \geq t}=1$, then $d_{t} \leq d_{t+1}$, and the inequality for debt is strict if at least one of the chain of inequalities for the output trajectory is strict

[^26]:    ${ }^{1}$ Damage suit settlement is also a motivating application in bargaining under private information, studied in Kennan and Wilson 1993.
    ${ }^{2}$ Indeed, even with such information, one may not have good reason to believe a historical distribution of reservation values reflects the present distribution of values.

[^27]:    ${ }^{3}$ A particularly stark example is used to illustrate the Ellsberg paradox: if an urn A has 50 red and 50 black balls, and an urn B has an unknown mixture of red and black balls, people typically strictly prefer taking a bet that they would draw a red ball from urn A over one where they would draw one from urn B , even while strictly preferring to take a bet that they would draw a black ball from urn A over one where they would draw one from urn B.
    ${ }^{4}$ The behavioral literature has ripened with models that aim to resolve the Ellsberg paradox; see Gilboa and Marinacci 2013 for a review. Leading approaches include Schmeidler 1989, Segal 1987, 1990, Epstein 1999, Klibanoff,

[^28]:    ${ }^{5}$ The mean CIP is so named since it satisfies $\bar{p}{ }^{\mathrm{CIP}}=\mathbb{E}\left[p^{\mathrm{CIP}} \mid v>c\right]$. Thus, the mean CIP can be thought of as what would be chosen (either by the mediator or negotiated by the transacting parties absent a mediator) on average in a setting of complete information given there is surplus from trade.
    ${ }^{6}$ Since Nature observes a realized price, we may restrict our focus to pure strategies for the mediator.

[^29]:    ${ }^{7}$ Section 3.2.3 shows the lower bound on $v$ can be as high as $\mu_{c}$ and the upper bound on $c$ can be as low as $\mu_{v}$. Further flexibility on the bounds can only improve the prospects for the mediator.

[^30]:    ${ }^{8}$ Such a payoff can be endogenously motivated by augmenting the game to include a third stage (after the mediator and Nature) where buyer and seller would decide to accept or reject the price contract after observing their actual private values, in which case a well-defined equilibrium would have the buyer and seller breaking ties in favor of Nature.

[^31]:    ${ }^{9}$ In fact, the proof shows that for the same value of $\alpha$ chosen extreme enough, $p^{\text {IIP }}$ is strictly monotonic in $\bar{w}$ and

[^32]:    ${ }^{1}$ The system may be effectively characterized as $(|\tilde{\mathcal{N}}|-1)$ - dimensional in determining $|\tilde{\mathcal{N}}|-1$ of the tight multipliers. The requisite initial debt and remaining tight multiplier are then trivially determined by the remaining binding collateral constraint and geometric average condition respectively.
    ${ }^{2}$ This also follows directly from the sequential budget constraint (1.2).

[^33]:    ${ }^{3}$ We condition the cubic inequality on output in the functional form to more easily compare it with the condition for the tight-slack equilibrium in the sequel.

[^34]:    ${ }^{4}$ One could also consider $d_{0}$ restricted by the binding collateral constraint in period 0 .

[^35]:    ${ }^{5}$ In fact, a tight-tight equilibrium with $\tilde{\mu}_{0}=\delta$ must have constant output in order for the collateral constraint to hold and hence uniquely corresponds to the SSE. To see this, note that since $r>0$, we have $\delta((1+r) \kappa-1)+1+r-\kappa>$ $(1-\kappa)(1-\delta)>0$, and thus the coefficient of $y_{1}$ in $B$ is decreasing in $\alpha$. Consequently, for $y_{1}>$ (resp. <) $y_{2}$, we have

    $$
    \begin{gathered}
    \Upsilon\left(\delta \mid y_{1}, y_{2}\right)=\delta^{3} \kappa(1+r-\kappa)(2(1+r)-\alpha(1+r-\delta))\left(y_{1}-y_{2}\right) \\
    +\delta^{2}\left[(1+r)\left((1+r-\kappa)^{2}-\delta^{2}\left(1-\kappa^{2}\right)\right)-\alpha \delta \kappa(\delta((1+r) \kappa-1)+1+r-\kappa)\right]\left(y_{1}-y_{2}\right) \\
    >(\text { resp. }<) \delta^{2}(1+r-\kappa)[\delta(r \kappa-\delta(1-\kappa))+(1+r)(1+r-\kappa)]\left(y_{1}-y_{2}\right)>(\text { resp. < }) 0
    \end{gathered}
    $$

    The first inequality comes from setting $\alpha=1$ in the first expression since it is decreasing (resp. increasing) in $\alpha$. The last inequality comes from noting that either $r \kappa-\delta(1-\kappa)>0$ and we are done, or else, if $r \kappa-\delta(1-\kappa)<0$, then $\delta(r \kappa-\delta(1-\kappa))$ is decreasing in $\delta$ and hence minimized at $\delta=1$, and $\delta(r \kappa-\delta(1-\kappa))+\left.(1+r)(1+r-\kappa)\right|_{\delta=1}=$ $r(2+r)>0$. Thus, $\Upsilon\left(\delta \mid y_{1}, y_{2}\right)=0$ if and only if $y_{1}=y_{2}$.

[^36]:    ${ }^{6}$ In fact, these properties extend for nonconstant output. Going through the same steps shows that the sign of the derivative of $d_{0}^{2-c y c, T T}\left(\cdot \mid y_{1}, y_{2}\right)$ at $\tilde{\mu}_{0}$ is less than $-(1+r-\kappa)\left[\kappa\left(y_{1}+\kappa y_{2}\right)\left(\tilde{\mu}_{0}^{2}-\delta^{2}\right)+y_{2}\left(1-\delta^{2}\right)(1+r-\kappa)^{2}\right]$, which is negative for $\tilde{\mu}_{0}>\delta$ or $y_{2}>0$; in the case where $\tilde{\mu}_{0}=\delta$ and $y_{2}=0$, the derivative is zero, implying $d_{0}^{2-\text { cyc,TT }}\left(\cdot \mid y_{1}, y_{2}\right)$ is maximized at $\tilde{\mu}_{0}=\delta$. Moreover, $d_{0}^{2-\text { cyc,TT }}\left(\tilde{\mu}_{0} \mid y, y\right)$ is monotonically decreasing in $\tilde{\mu}_{0}$ for all $\tilde{\mu}_{0} \in\left(\delta^{2}, 1\right)$ if $\delta \leq(1+r-\kappa) \sqrt{\frac{y_{2}}{\kappa\left(y_{1}+\kappa y_{2}\right)}}$.

[^37]:    ${ }^{1}$ As with Proposition 15, for the same value of $\mu_{c}-\underline{w}$ (resp. $\bar{w}-\mu_{v}$ ) chosen small enough, $\tilde{p}^{\text {IIP }}$ strictly decreases in $\bar{w}$ (resp. $\underline{w}$ ) over its entire feasible range.

