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BIFURCATION VALUES OF POLYNOMIAL FUNCTIONS AND PERVERSE SHEAVES

by Kiyoshi TAKEUCHI

ABSTRACT. — We characterize bifurcation values of polynomial functions by using the theory of perverse sheaves and their vanishing cycles. In particular, by introducing a method to compute the jumps of the Euler characteristics with compact support of their fibers, we confirm the conjecture of Némethi–Zaharia in many cases.

RÉSUMÉ. — Nous caractérisons les valeurs de bifurcation de fonctions polynomiales en utilisant la théorie des faisceaux pervers et leurs cycles évanescents. En particulier, en introduisant une méthode pour calculer les sauts de caractéristiques d'Euler à support compact de leurs fibres, nous confirmons la conjecture de Némethi–Zaharia dans de nombreux cas.

1. Introduction

For a polynomial function $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ it is well-known that there exists a finite subset $B \subset \mathbb{C}$ such that the restriction

$$\mathbb{C}^n \setminus f^{-1}(B) \longrightarrow \mathbb{C} \setminus B \tag{1.1}$$

of f is a C^∞ locally trivial fibration. We denote by B_f the smallest subset $B \subset \mathbb{C}$ satisfying this condition. Let $\operatorname{Sing} f \subset \mathbb{C}^n$ be the set of the critical points of $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$. Then by the definition of B_f, obviously we have $f(\operatorname{Sing} f) \subset B_f$. The elements of B_f are called *bifurcation values* of f. The determination of the bifurcation set $B_f \subset \mathbb{C}$ is a fundamental problem and was studied by many mathematicians and from several viewpoints, e.g. [3, 4, 9, 10, 19, 20, 23, 24, 29] and [30]. The essential difficulty consists in the fact that in general f has a lot of singularities at infinity. Here we study B_f via the Newton polyhedron of f. We denote by $\Gamma_\infty(f)$ the convex

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hull of the Newton polytope $NP(f)$ of f and the origin in $\mathbb{R}^n$. We call it the Newton polyhedron at infinity of f. Throughout this paper we assume that $\dim \Gamma_\infty(f) = n$. Recall that f is said to be *convenient* if $\Gamma_\infty(f)$ intersects the positive part of each coordinate axis. Kouchnirenko [13] proved that if f is convenient and non-degenerate at infinity (for the definition see Section 3) then $B_f = f(\operatorname{Sing} f)$. However, in the non-convenient case, Némethi and Zaharia [19] showed that more bifurcation values may occur due to so-called "bad faces". Let us explain this phenomenon here and refer for details to Section 3.

Definition 1.1 ([26]). — *We say that a face $\gamma \prec \Gamma_\infty(f)$ is* atypical *if $0 \in \gamma$, $\dim \gamma \geqslant 1$ and the cone $\sigma(\gamma) \subset \mathbb{R}^n$ which corresponds it in the dual fan of $\Gamma_\infty(f)$ (for the definition see Section 3) is not contained in the first quadrant $\mathbb{R}^n_+ := \mathbb{R}^n_{\geqslant 0}$ of $\mathbb{R}^n$.*

This definition is closely related to that of the bad faces of $NP(f - f(0))$ in Némethi–Zaharia [19]. See Section 3 for the details and examples. In this paper, we consider the case where f is not convenient. Let $\gamma_1, \dots, \gamma_m$ be the atypical faces of $\Gamma_\infty(f)$. As we see in Theorem 1.2 below, in the generic case where f is non-degenerate at infinity, the singularities at infinity of f are produced only from γ_i. For $1 \leqslant i \leqslant m$ let $K_i = f_{\gamma_i}(\operatorname{Sing} f_{\gamma_i}) \subset \mathbb{C}$ be the set of the critical values of the γ_i-part

$$f_{\gamma_i} : T = (\mathbb{C}^*)^n \longrightarrow \mathbb{C} \tag{1.2}$$

of f. Let us set

$$K_f = f(\operatorname{Sing} f) \cup \{f(0)\} \cup \left(\bigcup_{i=1}^{m} K_i \right). \tag{1.3}$$

Then Némethi–Zaharia [19] proved the following fundamental result.

Theorem 1.2 (Némethi–Zaharia [19]). — *Assume that f is non-degenerate at infinity. Then we have $B_f \subset K_f$.*

Moreover they proved the equality $B_f = K_f$ for $n = 2$ and conjectured its validity in higher dimensions. The essential problem is to prove the inverse inclusion $K_f \subset B_f$. This has been a long standing conjecture until now. Later Zaharia [30] proved $K_f \setminus \{f(0)\} \subset B_f$ for $n \geqslant 2$ under some additional assumptions. In particular, he assumed that f has isolated singularities at infinity on a fixed smooth toric compactification of $\mathbb{C}^n$. We can easily see that even if f is non-degenerate at infinity this condition is not satisfied in general. See (4.13) in the proof of Theorem 4.3 below. Namely his assumption is very strong and moreover depends on the choice of a particular smooth toric compactification of $\mathbb{C}^n$. In this paper, we overcome this problem by introducing the following intrinsic definition. For $1 \leqslant i \leqslant m$ let $L_{\gamma_i} \simeq \mathbb{R}^{\dim \gamma_i}$ be the linear subspace of $\mathbb{R}^n$ spanned by γ_i and set $T_i = \mathrm{Spec}(\mathbb{C}[L_{\gamma_i} \cap \mathbb{Z}^n]) \simeq (\mathbb{C}^*)^{\dim \gamma_i}$. We regard f_{γ_i} as a regular function on T_i.

Definition 1.3. — *We say that* f has isolated singularities at infinity over $b \in K_f \setminus [f(\mathrm{Sing}\, f) \cup \{f(0)\}]$ *if for any* $1 \leqslant i \leqslant m$ *the hypersurface* $f_{\gamma_i}^{-1}(b) \subset T_i \simeq (\mathbb{C}^*)^{\dim \gamma_i}$ *in* T_i *has only isolated singular points. We simply say that* f has isolated singularities at infinity *if it is so over any value* $b \in K_f \setminus [f(\mathrm{Sing}\, f) \cup \{f(0)\}]$.

With this new definition at hand, by using also the more sophisticated machinary of vanishing cycle functors for constructible sheaves we can eventually work on a singular toric variety. Then we use the theory of perverse sheaves to improve Zaharia's result. In this way, we prove the inverse inclusion $K_f \setminus \{f(0)\} \subset B_f$ and confirm the conjecture of [19] in many cases. In particular, for $n = 3$ we obtain the following result.

Theorem 1.4. — *Let* $f : \mathbb{C}^3 \longrightarrow \mathbb{C}$ *be a non-degenerate polynomial at infinity such that* $\dim \Gamma_\infty(f) = 3$. *Then, if* f *has isolated singularities at infinity over* $b \in K_f \setminus [f(\mathrm{Sing}\, f) \cup \{f(0)\}]$, *we have* $b \in B_f$. *In particular, if* f *has isolated singularities at infinity, we have* $K_f \setminus \{f(0)\} \subset B_f$.

For $n = 3$ in the generic case, we thus confirm the conjecture of [19]. In fact, to prove Theorem 1.4 we show moreover that the Euler characteristics with compact support of the fibers of $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ jump at the point b. The jump of Euler characteristics was used as a test for the bifurcation locus in case of "isolated singularities at infinity" (defined in various ways) in many other articles and from different points of view (see [1, 2, 3, 9, 10, 23, 24, 25, 27] etc.). To introduce our results in higher dimensions, we need also the following definition.

Definition 1.5. — *We say that an atypical face* $\gamma_i \prec \Gamma_\infty(f)$ *is* relatively simple *if the cone* $\sigma_i := \sigma(\gamma_i) \subset \mathbb{R}^n$ *which corresponds to it in the dual fan of* $\Gamma_\infty(f)$ *is simplicial or satisfies the condition* $\dim \sigma_i \leqslant 3$.

This condition implies that the constant sheaf on the affine toric variety associated to the cone σ_i such that $\dim \sigma_i = n - \dim \gamma_i$ is perverse (up to some shift). If σ_i is simplicial, then the affine toric variety associated to it is an orbifold and the perversity follows. If $\dim \sigma_i \leqslant 3$ we can show the corresponding perversity by a result of Fieseler [7] on the intersection cohomology complexes of toric varieties. See Lemma 2.5 below. In higher dimensions, this perversity is essential in our proof of the inverse inclusion $K_f \setminus \{f(0)\} \subset B_f$. Note that if $\dim \gamma_i \geqslant n-3$ we have $\dim \sigma_i \leqslant 3$ and the atypical face γ_i is relatively simple. In particular, if $n \leqslant 4$ this condition is always satisfied. Now we define a function $\chi_c : \mathbb{C} \longrightarrow \mathbb{Z}$ on $\mathbb{C}$ by

$$\chi_c(t) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H_c^j(f^{-1}(t); \mathbb{C}) \qquad (t \in \mathbb{C}). \tag{1.4}$$

Let us fix a point $b \in K_f \setminus [f(\operatorname{Sing} f) \cup \{f(0)\}] \subset \bigcup_{i=1}^m K_i$ and define the jump $E_f(b) \in \mathbb{Z}$ of the function χ_c at b by

$$E_f(b) = (-1)^{n-1} \{\chi_c(b+\varepsilon) - \chi_c(b)\} \in \mathbb{Z}, \tag{1.5}$$

where $\varepsilon > 0$ is sufficiently small. Recall that for a polytope Δ in $\mathbb{R}^n$ its relative interior $\operatorname{rel.int}(\Delta)$ is the interior of Δ in its affine span $\operatorname{Aff}(\Delta) \simeq \mathbb{R}^{\dim \Delta}$ in $\mathbb{R}^n$. Then we have the following result.

Theorem 1.6. — *Assume that* $\dim \Gamma_\infty(f) = n$, f *is non-degenerate at infinity and has isolated singularities at infinity over* $b \in K_f \setminus [f(\operatorname{Sing} f) \cup \{f(0)\}]$ *and for any* $1 \leqslant i \leqslant m$ *such that* $b \in K_i$ *we have* $\operatorname{rel.int}(\gamma_i) \subset \operatorname{Int}(\mathbb{R}^n_+)$. *Assume also that there exists* $1 \leqslant i \leqslant m$ *such that* $b \in K_i$ *and* $\gamma_i \prec \Gamma_\infty(f)$ *is relatively simple. Then we have* $E_f(b) > 0$ *and hence* $b \in B_f$.

If $n = 4$, all the atypical faces γ_i are relatively simple and we obtain the following corollary.

Corollary 1.7. — *Let* $f : \mathbb{C}^4 \longrightarrow \mathbb{C}$ *be a non-degenerate polynomial at infinity such that* $\dim \Gamma_\infty(f) = 4$. *Then, if* f *has isolated singularities at infinity over* $b \in K_f \setminus [f(\operatorname{Sing} f) \cup \{f(0)\}]$ *and for any* $1 \leqslant i \leqslant m$ *such that* $b \in K_i$ *the condition* $\operatorname{rel.int}(\gamma_i) \subset \operatorname{Int}(\mathbb{R}^4_+)$ *is satisfied, then we have* $E_f(b) > 0$. *In particular, if* f *has isolated singularities at infinity and* $\Gamma_\infty(f) \setminus \{0\} \subset \operatorname{Int}(\mathbb{R}^4_+)$, *we have* $K_f \setminus \{f(0)\} \subset B_f$.

For general $n \geqslant 2$ we have also the following corollary.

Corollary 1.8. — *Assume that* $\dim \Gamma_\infty(f) = n$ *and* f *is non-degenerate at infinity. Then, if moreover* f *has isolated singularities at infinity,* $\Gamma_\infty(f) \setminus \{0\} \subset \operatorname{Int}(\mathbb{R}^n_+)$ *and all the atypical faces* γ_i $(1 \leqslant i \leqslant m)$ *are relatively simple, then we have* $K_f \setminus \{f(0)\} \subset B_f$.

Since γ_i is relatively simple if $\dim \gamma_i \geqslant n-3$, Theorem 1.6 extends the result of Zaharia [30]. Indeed, he assumed the much stronger condition that for any $1 \leqslant i \leqslant m$ such that $b \in K_i$ we have $\dim \gamma_i = n-1$ (which implies also $\operatorname{rel.int}(\gamma_i) \subset \operatorname{Int}(\mathbb{R}^n_+)$). His assumption means that on a fixed smooth toric compactification of $\mathbb{C}^n$ compatible with $\Gamma_\infty(f)$ the function f has isolated singular points only on T-orbits at infinity of dimension $n-1$

over the point $b \in K_f \setminus [f(\operatorname{Sing} f) \cup \{f(0)\}]$. However under our weaker assumption, in the proof of Theorem 1.6 we encounter non-isolated singular points of f at infinity on such a smooth compactification (see (4.13)). We overcome this difficulty by reducing the problem to the case of isolated singular points. To this end, we consider the direct image of the vanishing cycle of a constructible sheaf by a special morphism

$$\pi : X = X_{\Sigma'_C} \longrightarrow X_{\Sigma_C} \tag{1.6}$$

of toric varieties. In this way, we can eventually work on the singular toric variety X_{Σ_C} canonically associated to $\Gamma_\infty(f)$. This is the reason why we can employ our intrinsic definition in Definition 1.3. Then, on X_{Σ_C} the function f has only isolated singular points at infinity (over the point $b \in K_f \setminus [f(\operatorname{Sing} f) \cup \{f(0)\}]$). Finally, to finish the proofs of Theorems 1.4 and 1.6, we apply the theory of perverse sheaves and their vanishing cycles. Here we use the perversity of the constant sheaf on the toric variety associated to the cone σ_i to obtain the positivity $E_f(b) > 0$. For the moment, it is not clear if we can further relax the assumption on σ_i by using the very general formula for vanishing cycle sheaves in Massey [15, Lemma 2.2] etc. Note also that our condition rel. $\operatorname{int}(\gamma_i) \subset \operatorname{Int}(\mathbb{R}^n_+)$ in Theorem 1.6 is equivalent to the one $\sigma_i \cap \mathbb{R}^n_+ = \{0\}$. However in higher dimensions, there still remain some atypical faces for which this condition is not satisfied (see Example 3.5 below). So it is desirable to relax the condition $\sigma_i \cap \mathbb{R}^n_+ = \{0\}$. In this direction, we have only a partial answer in Theorem 4.5 which extends Theorems 1.4 and 1.6 in a unified manner. We hope that we can drop some of the conditions in it in the future.

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2. Review on constructible and perverse sheaves

In this section, we recall some results on constructible and perverse sheaves. In this paper, we essentially follow the terminology of [5], [11] and [12]. For example, for a topological space X we denote by $\mathbf{D}^b(X)$ the derived category whose objects are bounded complexes of sheaves of $\mathbb{C}_X$-modules on X. Denote by $\mathbf{D}^b_c(X)$ the full subcategory of $\mathbf{D}^b(X)$ consisting of constructible objects.

Definition 2.1. — *Let X be an algebraic variety over $\mathbb{C}$. Then we say that a $\mathbb{Z}$-valued function $\psi\colon X \longrightarrow \mathbb{Z}$ on X is* constructible *if there exists a stratification $X = \bigsqcup_\alpha X_\alpha$ of X such that $\psi|_{X_\alpha}$ is constant for any α. We denote by $F_{\mathbb{Z}}(X)$ the abelian group of constructible functions on X.*

Let $\mathcal{F} \in \mathbf{D}^b_c(X)$ be a constructible sheaf (complex of sheaves) on an algebraic variety X over $\mathbb{C}$. Then we can naturally associate to it a constructible function $\chi(\mathcal{F}) \in F_{\mathbb{Z}}(X)$ on X defined by

$$\chi(\mathcal{F})(x) = \sum_{j\in\mathbb{Z}} (-1)^j \dim H^j(\mathcal{F})_x \qquad (x \in X). \tag{2.1}$$

For a constructible function $\psi\colon X \longrightarrow \mathbb{Z}$, we take a stratification $X = \bigsqcup_\alpha X_\alpha$ of X such that $\psi|_{X_\alpha}$ is constant for any α as above. We denote the Euler characteristic of X_α by $\chi(X_\alpha)$. Then we set

$$\int_X \psi := \sum_\alpha \chi(X_\alpha)\cdot\psi(x_\alpha) \in \mathbb{Z}, \tag{2.2}$$

where x_α is a reference point in X_α. Then we can easily show that $\int_X \psi \in \mathbb{Z}$ does not depend on the choice of the stratification $X = \bigsqcup_\alpha X_\alpha$ of X. Hence we obtain a homomorphism

$$\int_X : F_{\mathbb{Z}}(X) \longrightarrow \mathbb{Z} \tag{2.3}$$

of abelian groups. For $\psi \in F_{\mathbb{Z}}(X)$, we call $\int_X \psi \in \mathbb{Z}$ the topological (Euler) integral of ψ over X. More generally, to a morphism $f : X \longrightarrow Y$ of algebraic varieties over $\mathbb{C}$ we can associate a homomorphism $\int_f : F_{\mathbb{Z}}(X) \longrightarrow F_{\mathbb{Z}}(Y)$ of abelian groups as follows. For $\psi \in F_{\mathbb{Z}}(X)$ we define $\int_f \psi \in F_{\mathbb{Z}}(Y)$ by

$$\left(\int_f \psi\right)(y) = \int_{f^{-1}(y)} \psi \in \mathbb{Z} \qquad (y \in Y). \tag{2.4}$$

Then for any constructible sheaf $\mathcal{F} \in \mathbf{D}^b_c(X)$ on X we have the equality

$$\int_f \chi(\mathcal{F}) = \chi(Rf_*(\mathcal{F})). \tag{2.5}$$

Now we recall the following well-known property of Deligne's vanishing cycle functors. Let X be an algebraic variety over $\mathbb{C}$ and $f : X \longrightarrow \mathbb{C}$ a non-constant regular function on X and set $X_0 = \{x \in X \mid f(x) = 0\} \subset X$. Then we denote Deligne's vanishing cycle functor associated to f by

$$\varphi_f : \mathbf{D}^b_c(X) \longrightarrow \mathbf{D}^b_c(X_0) \tag{2.6}$$

(see [5, Section 4.2] and [12, Section 8.6] etc. for the details).

Proposition 2.2 (cf. [5, Proposition 4.2.11] and [12, Exercise VIII.15] etc.). — *Let $\pi : Y \longrightarrow X$ be a proper morphism of algebraic varieties over $\mathbb{C}$ and $f : X \longrightarrow \mathbb{C}$ a non-constant regular function on X. Set $g = f \circ \pi : Y \longrightarrow \mathbb{C}$, $X_0 = \{x \in X \mid f(x) = 0\}$ and $Y_0 = \{y \in Y \mid g(y) = 0\}$. Then for any $\mathcal{G} \in \mathbf{D}^b_c(Y)$ we have an isomorphism*

$$\varphi_f(R\pi_*\mathcal{G}) \simeq R(\pi|_{Y_0})_*\varphi_g(\mathcal{G}), \tag{2.7}$$

where the morphism $\pi|_{Y_0} : Y_0 \longrightarrow X_0$ is induced by π.

Recall that for an algebraic variety X over $\mathbb{C}$ the category $\mathrm{Perv}(X)$ of perverse sheaves on it is a full subcategory of $\mathbf{D}^b_c(X)$. Here we use the convention that for smooth X the shifted constant sheaf $\mathbb{C}_X[\dim X] \in \mathbf{D}^b_c(X)$ is perverse. The following result is a very special case of [5, Corollary 5.2.17].

Lemma 2.3. — *Let X be an algebraic variety X over $\mathbb{C}$ and $Y \subset X$ a hypersurface in it. Set $U = X \setminus Y$ and let $j : U \hookrightarrow X$ be the inclusion map. Then the functors*

$$j_!, Rj_* : \mathbf{D}^b_c(U) \longrightarrow \mathbf{D}^b_c(X) \tag{2.8}$$

preserve the perversity.

Now for $\mathcal{F} \in \mathbf{D}^b_c(X)$ let $\mathcal{S} : X = \sqcup_{\alpha \in A} X_\alpha$ be a Whitney stratification of X adapted to it. Then for a non-constant regular function $f : X \longrightarrow \mathbb{C}$ on X we define a subset $\mathrm{Sing}_{\mathcal{S}}(f) \subset X$ of X by

$$\mathrm{Sing}_{\mathcal{S}}(f) = \bigsqcup_{\alpha \in A} \mathrm{Sing}(f|_{X_\alpha}) \subset X. \tag{2.9}$$

We call it the stratified singular locus of f with respect to $\mathcal{S}$ (see [5, Definition 4.2.7]). By the Whitney condition on $\mathcal{S}$ it is a closed algebraic subset of X. By [5, Propositon 4.2.8] we have

$$\mathrm{supp}\, \varphi_f(\mathcal{F}) \subset X_0 \cap \mathrm{Sing}_{\mathcal{S}}(f). \tag{2.10}$$

Recall also that the shifted vanishing cycle functor

$${}^p\varphi_f(\,\cdot\,) := \varphi_f(\,\cdot\,)[-1] : \mathbf{D}^b_c(X) \longrightarrow \mathbf{D}^b_c(X_0) \tag{2.11}$$

preserves the perversity. Then we obtain the following result (see the proofs of [5, Propositions 6.1.1 and 6.1.2]).

Lemma 2.4. — *Assume that $\mathcal{F}$ is perverse and the dimension of $X_0 \cap \mathrm{Sing}_{\mathcal{S}}(f)$ is zero. Then we have the concentration*

$$H^l\{{}^p\varphi_f(\mathcal{F})\} \simeq 0 \qquad (l \neq 0). \tag{2.12}$$

Proof. — By our assumption the perverse sheaf ${}^p\varphi_f(\mathcal{F}) \in \mathrm{Perv}(X_0)$ is supported on some points in X_0. Then the desired concentration follows immediately from the perversity of ${}^p\varphi_f(\mathcal{F})$ (see [11, Proposition 8.1.22]). □

The following lemma will be used in the proofs of our main theorems. Let τ be a strictly convex rational polyhedral cone in $\mathbb{R}^n$ and Σ_τ the fan in $\mathbb{R}^n$ formed by all its faces. Denote by X_{Σ_τ} the (n-dimensional) toric variety associated to Σ_τ (see [8] and [21] etc.).

LEMMA 2.5. — *In the above situation, assume also that τ is simplicial or satisfies the condition $\dim \tau \leqslant 3$. Then the constant sheaf $\mathbb{C}_{X_{\Sigma_\tau}}$ on X_{Σ_τ} is perverse (up to some shift).*

Proof. — If τ is simplicial, then X_{Σ_τ} is an orbifold (see [8, p. 34]) and the assertion follows from [11, Proposition 8.2.21]. It is the case when $\dim \tau \leqslant 2$. Assume that $\dim \tau = 3$. Let $T_\tau \simeq (\mathbb{C}^*)^{n-\dim \tau} \subset X_{\Sigma_\tau}$ be the (minimal) T-orbit in X_{Σ_τ} associated to $\tau \in \Sigma_\tau$ and $i_\tau : T_\tau \hookrightarrow X_{\Sigma_\tau}$, $j_\tau : X_{\Sigma_\tau} \setminus T_\tau \hookrightarrow X_{\Sigma_\tau}$ the inclusion maps. Then by Fiesler [7, Theorems 1.1 and 1.2] we obtain

$$H^l i_\tau^{-1} R(j_\tau)_* \mathbb{C}_{X_{\Sigma_\tau} \setminus T_\tau} \simeq \begin{cases} \mathbb{C}_{T_\tau} & (l=0), \\ 0 & (l=1). \end{cases} \tag{2.13}$$

This implies that we have

$$H^l i_\tau^! \mathbb{C}_{X_{\Sigma_\tau}} \simeq 0 \qquad (l < 3 = \operatorname{codim} T_\tau). \tag{2.14}$$

Then the assertion follows from [11, Proposition 8.1.22]. □

3. Some compactifications of $\mathbb{C}^n$

In this section, we recall the constructions of some smooth compactifications of $\mathbb{C}^n$ in Zaharia [30] and Takeuchi–Tibăr [26]. Let $f(x) = \sum_{v \in \mathbb{Z}_+^n} a_v x^v$ be a polynomial on $\mathbb{C}^n$ ($a_v \in \mathbb{C}$).

DEFINITION 3.1.

(1) *We call the convex hull of* $\mathrm{supp}(f) := \{v \in \mathbb{Z}_+^n \mid a_v \neq 0\} \subset \mathbb{Z}_+^n \subset \mathbb{R}_+^n$ *in $\mathbb{R}^n$ the Newton polytope of f and denote it by $NP(f)$.*

(2) *(see [14] etc.) We call the convex hull of $\{0\} \cup NP(f)$ in $\mathbb{R}^n$ the Newton polyhedron at infinity of f and denote it by $\Gamma_\infty(f)$.*

For an element $u \in \mathbb{R}^n$ of (the dual vector space of) $\mathbb{R}^n$ define the supporting face $\gamma_u \prec \Gamma_\infty(f)$ of u in $\Gamma_\infty(f)$ by

$$\gamma_u = \left\{ v \in \Gamma_\infty(f) \,\middle|\, \langle u, v\rangle = \min_{w \in \Gamma_\infty(f)} \langle u, w\rangle \right\}. \tag{3.1}$$

Then we introduce an equivalence relation $\sim$ on (the dual vector space of) $\mathbb{R}^n$ by $u \sim u' \iff \gamma_u = \gamma_{u'}$. We can easily see that for any face $\gamma \prec \Gamma_\infty(f)$ of $\Gamma_\infty(f)$ the closure of the equivalence class associated to γ in $\mathbb{R}^n$ is an $(n - \dim\gamma)$-dimensional rational convex polyhedral cone $\sigma(\gamma)$ in $\mathbb{R}^n$. Moreover the family $\{\sigma(\gamma) \mid \gamma \prec \Gamma_\infty(f)\}$ of cones in $\mathbb{R}^n$ thus obtained is a subdivision of $\mathbb{R}^n$. We call it the dual subdivision of $\mathbb{R}^n$ by $\Gamma_\infty(f)$. If $\dim \Gamma_\infty(f) = n$ it satisfies the axiom of fans (see [8] and [21] etc.). We call it the dual fan of $\Gamma_\infty(f)$.

We have the following two classical definitions due to Kouchnirenko:

Definition 3.2 ([13]). — *Let $\partial f\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the map defined by $\partial f(x) = (\partial_1 f(x), \ldots, \partial_n f(x))$. Then we say that f is* tame at infinity *if the restriction $(\partial f)^{-1}(B(0;\varepsilon)) \longrightarrow B(0;\varepsilon)$ of ∂f to a sufficiently small ball $B(0;\varepsilon)$ centered at the origin $0 \in \mathbb{C}^n$ is proper.*

Definition 3.3 ([13]). — *We say that the polynomial*

$$f(x) = \sum_{v \in \mathbb{Z}^n_+} a_v x^v \quad (a_v \in \mathbb{C})$$

is non-degenerate at infinity *if for any face γ of $\Gamma_\infty(f)$ such that $0 \notin \gamma$ the complex hypersurface $\{x \in (\mathbb{C}^*)^n \mid f_\gamma(x) = 0\}$ in $(\mathbb{C}^*)^n$ is smooth and reduced, where we defined the γ-part f_γ of f by $f_\gamma(x) = \sum_{v \in \gamma \cap \mathbb{Z}^n_+} a_v x^v$.*

Broughton showed in [3] that if f is non-degenerate at infinity and convenient then it is tame at infinity. This implies that the reduced homology of the general fiber of f is concentrated in dimension $n-1$. The concentration result was later extended to polynomial functions with isolated singularities with respect to some fiber-compactifying extension of f by Siersma and Tibăr [24] and by Tibăr [28, Theorem 4.6, Corollary 4.7]. In this paper we mainly consider non-convenient polynomials.

Definition 3.4 ([26]). — *We say that a face $\gamma \prec \Gamma_\infty(f)$ is* atypical *if $0 \in \gamma$, $\dim \gamma \geqslant 1$ and the cone $\sigma(\gamma) \subset \mathbb{R}^n$ which corresponds it in the dual subdivision of $\Gamma_\infty(f)$ is not contained in the first quadrant $\mathbb{R}^n_+$ of $\mathbb{R}^n$.*

This definition is related to that of the bad faces of $NP(f - f(0))$ in Némethi–Zaharia [19] as follows. If $\Delta \prec NP(f - f(0))$ is a bad face of $NP(f - f(0))$, then the convex hull γ of $\{0\} \cup \Delta$ in $\mathbb{R}^n$ is an atypical one of $\Gamma_\infty(f)$. Conversely, if $\gamma \prec \Gamma_\infty(f)$ is an atypical face and $\Delta = \gamma \cap NP(f - f(0)) \prec NP(f - f(0))$ satisfies the condition $\dim \Delta = \dim \gamma$ then Δ is a bad face of $NP(f - f(0))$.

Example 3.5. — Let $n = 3$ and consider a non-convenient polynomial $f(x, y, z)$ on $\mathbb{C}^3$ whose Newton polyhedron at infinity $\Gamma_\infty(f)$ is the convex hull of the points $(2, 0, 0), (2, 2, 0), (2, 2, 3) \in \mathbb{R}^3_+$ and the origin $0 = (0, 0, 0) \in \mathbb{R}^3$. Then the line segment connecting the point $(2, 2, 0)$ (resp. $(2, 0, 0)$) and the origin $0 \in \mathbb{R}^3$ is an atypical face of $\Gamma_\infty(f)$. However the triangle whose vertices are the points $(2, 0, 0)$, $(2, 2, 0)$ and the origin $0 \in \mathbb{R}^3$ is not so. Note that for the line segment γ connecting $(2, 0, 0)$ and the origin we have $\dim \sigma(\gamma) \cap \mathbb{R}^3_+ = 2$.

From now we recall the smooth compactifications of $\mathbb{C}^n$ in [26] and [30] (for their applications to monodromies at infinity see [6, 17, 18] and [26]). Assume that the polynomial $f(x) = \sum_{v \in \mathbb{Z}^n_+} a_v x^v \in \mathbb{C}[x_1, \dots, x_n]$ is "non-convenient" and $\dim \Gamma_\infty(f) = n$. Let Σ_0 be the dual fan of $\Gamma_\infty(f)$. Assume also that f is non-degenerate at infinity. We consider $\mathbb{C}^n$ as a toric variety associated with the fan Ξ in $\mathbb{R}^n$ formed by all the faces of the first quadrant $\mathbb{R}^n_+ \subset \mathbb{R}^n$. Denote by $T \simeq (\mathbb{C}^*)^n$ the open dense torus in it. Let Σ_1 be a subdivision of the dual fan Σ_0 of $\Gamma_\infty(f)$ which contains Ξ as its subfan. Then we can construct a smooth subdivision Σ of Σ_1 without subdividing the cones in Ξ (see e.g. [22, Lemma (2.6), Chapter II, p. 99]). This implies that the toric variety X_Σ associated with Σ is a smooth compactification of $\mathbb{C}^n$. This construction of X_Σ coincides with the one in Zaharia [30]. Recall that T acts on X_Σ and the T-orbits are parametrized by the cones in Σ. For a cone $\sigma \in \Sigma$ denote by $T_\sigma \simeq (\mathbb{C}^*)^{n - \dim \sigma}$ the corresponding T-orbit. If $\sigma^\perp \simeq \mathbb{R}^{n - \dim \sigma}$ is the orthogonal complement of (the affine span of) σ we have $T_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\perp \cap \mathbb{Z}^n])$. There exist also natural affine open subsets $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n$ of X_Σ associated to n-dimensional cones σ in Σ as follows. Let σ be an n-dimensional (smooth) cone in Σ and $\{w_1, \dots, w_n\} \subset \mathbb{Z}^n$ the set of the (non-zero) primitive vectors on the edges of σ. Let σ° be the dual cone of σ. Then by the smoothness of σ the semigroup ring $\mathbb{C}[\sigma^\circ \cap \mathbb{Z}^n]$ is isomorphic to the polynomial ring $\mathbb{C}[y_1, \dots, y_n]$. This implies that the affine open subset $\mathbb{C}^n(\sigma) := \mathrm{Spec}(\mathbb{C}[\sigma^\circ \cap \mathbb{Z}^n])$ of X_Σ is isomorphic to $\mathbb{C}^n_y$. Moreover, on $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n_y$ the function $f(x) = \sum_{v \in \mathbb{Z}^n_+} a_v x^v$ has the following form:

$$f(y) = \sum_{v \in \mathbb{Z}^n_+} a_v y_1^{\langle w_1, v \rangle} \cdots y_n^{\langle w_n, v \rangle} = y_1^{b_1} \cdots y_n^{b_n} \times f_\sigma(y), \tag{3.2}$$

where we set

$$b_i = \min_{v \in \Gamma_\infty(f)} \langle w_i, v \rangle \leqslant 0 \qquad (i = 1, 2, \ldots, n) \tag{3.3}$$

and $f_\sigma(y)$ is a polynomial on $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n_y$. In $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n_y$ the hypersurface $Z := \overline{f^{-1}(0)} \subset X_\Sigma$ is explicitly written as $\{y \in \mathbb{C}^n(\sigma) \mid f_\sigma(y) = 0\}$. By (3.2) we see that f is extended to a meromorphic function on $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n_y$. The variety X_Σ is covered by such affine open subsets. Let τ be a d-dimensional face of the n-dimensional cone $\sigma \in \Sigma$. For simplicity, assume that $w_1, \ldots, w_d$ generate τ. Then in the affine chart $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n_y$ the T-orbit T_τ associated to τ is explicitly defined by

$$\begin{aligned} T_\tau &= \{(y_1, \ldots, y_n) \in \mathbb{C}^n(\sigma) \mid y_1 = \cdots = y_d = 0,\ y_{d+1}, \ldots, y_n \neq 0\} \\ &\simeq (\mathbb{C}^*)^{n-d}. \end{aligned}$$

Hence we have

$$X_\Sigma = \bigcup_{\dim \sigma = n} \mathbb{C}^n(\sigma) = \bigsqcup_{\tau \in \Sigma} T_\tau. \tag{3.4}$$

Now f extends to a meromorphic function on X_Σ, which may still have points of indeterminacy. For simplicity we denote this meromorphic extension also by f. From now on, we will eliminate its points of indeterminacy by blowing up X_Σ (see [17, Section 3] and [18, Section 3] for the details). For a cone σ in Σ by taking a non-zero vector u in the relative interior $\mathrm{rel.\,int}(\sigma)$ of σ we define a face γ_σ of $\Gamma_\infty(f)$ by $\gamma_\sigma = \gamma_u$. Note that γ_σ does not depend on the choice of $u \in \mathrm{rel.\,int}(\sigma)$. We call it the supporting face of σ in $\Gamma_\infty(f)$. Following Libgober–Sperber [14], we say that a T-orbit T_σ in X_Σ (or a cone $\sigma \in \Sigma$) is at infinity if the supporting face $\gamma_\sigma \prec \Gamma_\infty(f)$ satisfies the condition $0 \notin \gamma_\sigma$. We can easily see that f has poles on the union of T-orbits at infinity as follows. Let $\rho_1, \rho_2, \ldots, \rho_r$ be the 1-dimensional cones at infinity in Σ. Then $T_{\rho_1}, T_{\rho_2}, \ldots, T_{\rho_r}$ are the $(n-1)$-dimensional T-orbits at infinity in X_Σ. For any $i = 1, 2, \ldots, r$ the toric divisor $D_i := \overline{T_{\rho_i}}$ is a smooth hypersurface in X_Σ. Let us denote the (unique non-zero) primitive vector in $\rho_i \cap \mathbb{Z}^n$ by u_i. Then the order $a_i > 0$ of the pole of f along D_i is given by

$$a_i = -\min_{v \in \Gamma_\infty(f)} \langle u_i, v \rangle. \tag{3.5}$$

From this we see that the poles of f are contained in the normal crossing divisor $D := D_1 \cup \cdots \cup D_r$. Moreover by the non-convenience of f, there exist some cones $\sigma \in \Sigma$ such that $\sigma \notin \Xi$ and $0 \in \gamma_\sigma$ i.e. γ_σ is an atypical face of $\Gamma_\infty(f)$. For such σ the function f extends holomorphically to a neighborhood of $T_\sigma \subset X_\Sigma \setminus \mathbb{C}^n$. For this reason we call them *horizontal T-orbits* in X_Σ (in the tame case where f is convenient, they do not appear). Note also that by the non-degeneracy at infinity of f, for any non-empty subset $I \subset \{1, 2, \dots, r\}$ the hypersurface $Z = \overline{f^{-1}(0)}$ in X_Σ intersects $D_I := \bigcap_{i \in I} D_i$ transversally (or the intersection is empty). At such intersection points, f has indeterminacy. We can easily see that the meromorphic extension of f to X_Σ has points of indeterminacy in the subvariety $D \cap Z$ of X_Σ of codimension two. Now, in order to eliminate the indeterminacy of the meromorphic function f on X_Σ, we first consider the blow-up $\pi_1 \colon X_\Sigma^{(1)} \longrightarrow X_\Sigma$ of X_Σ along the $(n-2)$-dimensional smooth subvariety $D_1 \cap Z$. Then the indeterminacy of the pull-back $f \circ \pi_1$ of f to $X_\Sigma^{(1)}$ is improved. If $f \circ \pi_1$ still has points of indeterminacy on the intersection of the exceptional divisor E_1 of π_1 and the proper transform $Z^{(1)}$ of Z, we construct the blow-up $\pi_2 \colon X_\Sigma^{(2)} \longrightarrow X_\Sigma^{(1)}$ of $X_\Sigma^{(1)}$ along $E_1 \cap Z^{(1)}$. By repeating this procedure a_1 times, we obtain a tower of blow-ups

$$X_\Sigma^{(a_1)} \underset{\pi_{a_1}}{\longrightarrow} \cdots\cdots \underset{\pi_2}{\longrightarrow} X_\Sigma^{(1)} \underset{\pi_1}{\longrightarrow} X_\Sigma. \tag{3.6}$$

For the details see the figures in [17, p. 420]. Then the pull-back of f to $X_\Sigma^{(a_1)}$ has no indeterminacy over T_{ρ_1}. It also extends to a holomorphic function on (an open dense subset of) the exceptional divisor of the last blow-up π_{a_1}. For this reason we call it and its proper transform F_1 in the variety $\widetilde{X_\Sigma}$ that we construct below *horizontal exceptional divisors*. Note that for any $t \in \mathbb{C}$ the closure of the hypersurface $f^{-1}(t) \subset \mathbb{C}^n$ in $X_\Sigma^{(a_1)}$ intersects F_1 transversally. Moreover it does not intersect the other exceptional divisors.

Next we apply this construction to the proper transforms of D_2 and Z in $X_\Sigma^{(a_1)}$. Then we obtain also a tower of blow-ups

$$X_\Sigma^{(a_1)(a_2)} \longrightarrow \cdots\cdots \longrightarrow X_\Sigma^{(a_1)(1)} \longrightarrow X_\Sigma^{(a_1)} \tag{3.7}$$

and the indeterminacy of the pull-back of f to $X_\Sigma^{(a_1)(a_2)}$ is eliminated over $T_{\rho_1} \sqcup T_{\rho_2}$. By applying the same construction to (the proper transforms of) $D_3, D_4, \dots, D_r$, we finally obtain a proper morphism $\pi \colon \widetilde{X_\Sigma} \longrightarrow X_\Sigma$ such that $g := f \circ \pi$ has no point of indeterminacy on the whole $\widetilde{X_\Sigma}$. Note that the smooth compactification $\widetilde{X_\Sigma}$ of $\mathbb{C}^n$ thus obtained is not a toric variety any more. By constructing a blow-up $\widetilde{X_\Sigma} \longrightarrow X_\Sigma$ of X_Σ to eliminate the indeterminacy of f we thus obtain a commutative diagram:

(3.8)
$$\begin{array}{ccc} \mathbb{C}^n & \overset{\iota}{\longrightarrow} & \widetilde{X_\Sigma} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{C} & \underset{j}{\longrightarrow} & \mathbb{P}^1 \end{array}$$

of holomorphic maps, where $\iota : \mathbb{C}^n \hookrightarrow \widetilde{X_\Sigma}$ and $j : \mathbb{C} \hookrightarrow \mathbb{P}^1$ are the inclusion maps and g is proper. On $\widetilde{X_\Sigma}$ we have constructed also r (smooth) horizontal exceptional divisors $F_1, F_2, \dots, F_r$. The other exceptional divisors in $\widetilde{X_\Sigma}$ are called *intermediate exceptional divisors*. By our construction of the blow-up $\pi \colon \widetilde{X_\Sigma} \longrightarrow X_\Sigma$, $F_1 \cup F_2 \cup \cdots \cup F_r$ is a normal crossing divisor in $\widetilde{X_\Sigma}$ and for any non-empty subset $I \subset \{1, 2, \dots, r\}$ and $t \in \mathbb{C}$ the hypersurface $g^{-1}(t) \subset \widetilde{X_\Sigma}$ intersects $F_I := \cap_{i \in I} F_i$ transversally. Moreover $g^{-1}(t)$ does not intersect intermediate exceptional divisors. For a point $b \in \mathbb{C}$ define a function $h : \mathbb{C} \longrightarrow \mathbb{C}$ on $\mathbb{C}$ by $h(t) = t - b$ so that we have $h^{-1}(0) = \{b\}$. Then by the above-mentioned property of F_i and (2.10) the support of the constructible sheaf $\varphi_{h \circ g}(\iota_! \mathbb{C}_{\mathbb{C}^n})$ does not intersect the union of the exceptional divisors in $\pi \colon \widetilde{X_\Sigma} \longrightarrow X_\Sigma$. Moreover, for the pole divisor $D = D_1 \cup \cdots \cup D_r \subset X_\Sigma$ of (the meromorphic extension of) f to X_Σ, the support does not intersect $\pi^{-1}(D)$.

4. Bifurcation sets of polynomial functions

In this section we study the bifurcation values of polynomial functions. Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial function. Throughout this section we assume that f is non-degenerate at infinity and $\dim \Gamma_\infty(f) = n$. Let Σ_0 be the dual fan of $\Gamma_\infty(f)$. Let $\gamma_1, \dots, \gamma_m$ be the atypical faces of $\Gamma_\infty(f)$. For $1 \leqslant i \leqslant m$ let $K_i \subset \mathbb{C}$ be the set of the critical values of the γ_i-part

(4.1)
$$f_{\gamma_i} : T = (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$$

of f. We denote by $\operatorname{Sing} f \subset \mathbb{C}^n$ the set of the critical points of $f \colon \mathbb{C}^n \longrightarrow \mathbb{C}$ and set

(4.2)
$$K_f = f(\operatorname{Sing} f) \cup \{f(0)\} \cup \left(\bigcup_{i=1}^m K_i \right).$$

Then the following result was obtained by Némethi–Zaharia [19].

Theorem 4.1 (Némethi–Zaharia [19]). — *In the situation above, we have* $B_f \subset K_f$.

Remark 4.2. — If for an atypical face γ_i of $\Gamma_\infty(f)$ the face $\Delta = \gamma_i \cap NP(f - f(0)) \prec NP(f - f(0))$ of $NP(f - f(0))$ is not bad in the sense of Némethi–Zaharia [19], then $\dim NP(f_{\gamma_i} - f(0)) = \dim \Delta < \dim \gamma_i$, $f_{\gamma_i} - f(0)$ is a positively homogeneous Laurent polynomial on $T = (\mathbb{C}^*)^n$ and we have $K_i = \{f(0)\}$. Therefore the above inclusion $B_f \subset K_f$ coincides with the one in [19].

Moreover the authors of [19] proved the equality $B_f = K_f$ for $n = 2$ and conjectured its validity in higher dimensions. Later Zaharia [30] proved it for any $n \geqslant 2$ but under some supplementary assumptions on f. By using the definitions and the notations in Section 1 we can improve his result as follows.

THEOREM 4.3. — *Assume that f has isolated singularities at infinity over $b \in K_f \setminus [f(\operatorname{Sing} f) \cup \{f(0)\}]$ and for any $1 \leqslant i \leqslant m$ such that $b \in K_i$ the relative interior $\operatorname{rel.int}(\gamma_i)$ of $\gamma_i \prec \Gamma_\infty(f)$ is contained in $\operatorname{Int}(\mathbb{R}^n_+)$. Assume also that there exists $1 \leqslant i \leqslant m$ such that $b \in K_i$ and $\gamma_i \prec \Gamma_\infty(f)$ is relatively simple. Then we have $E_f(b) > 0$ and hence $b \in B_f$.*

Proof. — By our assumption, for any $1 \leqslant i \leqslant m$ the hypersurface $f_{\gamma_i}^{-1}(b) \subset T_i \simeq (\mathbb{C}^*)^{\dim \gamma_i}$ in $T_i = \operatorname{Spec}(\mathbb{C}[L_{\gamma_i} \cap \mathbb{Z}^n])$ has only isolated singular points at $p_{i,1}, \ldots, p_{i,n_i}$. Here some n_i can be zero. Obviously we have $n_i > 0$ if and only if $b \in K_i$. From now we shall freely use the smooth compactification $\widetilde{X_\Sigma}$ of $\mathbb{C}^n$ and the notations related to it in Section 3. Let $\operatorname{Cone}_\infty(f) \subset \mathbb{R}^n_v$ be the cone generated by $\Gamma_\infty(f)$. We define its dual cone $C \subset \mathbb{R}^n_u$ by

$$C = \{u \in \mathbb{R}^n \mid \langle u, v\rangle \geqslant 0 \ \text{ for any } v \in \operatorname{Cone}_\infty(f)\}. \tag{4.3}$$

Then a cone $\sigma \in \Sigma$ is at infinity if and only if it is not contained in C. We shall prove that the jump $E_f(b) \in \mathbb{Z}$ of the constructible function on $\mathbb{C}$

$$\chi_c(t) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j_c(f^{-1}(t); \mathbb{C}) \quad (t \in \mathbb{C}) \tag{4.4}$$

at the point $b \in K_f \setminus [f(\operatorname{Sing} f) \cup \{f(0)\}]$ is positive. For the point $b \in \mathbb{C}$ define a function $h : \mathbb{C} \longrightarrow \mathbb{C}$ on $\mathbb{C}$ by $h(t) = t - b$ so that we have $h^{-1}(0) = \{b\}$. Then we have

$$E_f(b) = (-1)^{n-1} \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j \varphi_h(Rf_! \mathbb{C}_{\mathbb{C}^n})_b, \tag{4.5}$$

where $\varphi_h : \mathbf{D}^b_c(\mathbb{C}) \longrightarrow \mathbf{D}^b_c(\{b\})$ is Deligne's vanishing cycle functor associated to h. Since we have $f = g\circ\iota$ on a neighborhood of $b \in K_f \setminus [f(\operatorname{Sing} f)\cup \{f(0)\}]$ and g is proper, by Proposition 2.2 we obtain an isomorphism

$$\varphi_h(Rf_!\mathbb{C}_{\mathbb{C}^n}) \simeq R(g|_{g^{-1}(b)})_*\varphi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n}). \tag{4.6}$$

This implies that for the constructible function $\chi\{\varphi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n})\} \in F_{\mathbb{Z}}(g^{-1}(b))$ on $g^{-1}(b) = (h\circ g)^{-1}(0) \subset \widetilde{X_\Sigma}$ we have

$$\sum_{j\in\mathbb{Z}}(-1)^j \dim H^j\varphi_h(Rf_!\mathbb{C}_{\mathbb{C}^n})_b = \int_{g^{-1}(b)} \chi\{\varphi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n})\}. \tag{4.7}$$

Hence for the calculation of $E_f(b)$, it suffices to calculate

$$\chi\{\varphi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n})\}(p) = \sum_{j\in\mathbb{Z}}(-1)^j \dim H^j\varphi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n})_p \tag{4.8}$$

at each point p of $g^{-1}(b)$. Let Σ_C (resp. Σ'_C) be the fan formed by all the faces of the cone C (resp. by all the cones in Σ contained in C) and denote by X_{Σ_C} (resp. $X_{\Sigma'_C}$) the possibly singular (resp. smooth) toric variety associated to it. Then $X := X_{\Sigma'_C} = \sqcup_{\sigma\subset C}T_\sigma$ is an open subset of X_Σ and there exists a natural proper morphism

$$\pi : X = X_{\Sigma'_C} \longrightarrow X_{\Sigma_C} \tag{4.9}$$

of toric varieties. Note that for the pole divisor D of (the meromorphic extension of) f to X_Σ (see Section 3) we have $X = X_\Sigma \setminus D$. Recall also that the centers of the blow-ups in the construction of $\widetilde{X_\Sigma} \longrightarrow X_\Sigma$ are above $D = X_\Sigma \setminus X$. Hence we can consider X also as an open subset of $\widetilde{X_\Sigma}$. Since the Newton polytope $NP(f)$ of f is contained in the dual cone $C^\circ = \operatorname{Cone}_\infty(f)$ of C and

$$X_{\Sigma_C} = \operatorname{Spec}(\mathbb{C}[C^\circ \cap \mathbb{Z}^n]), \tag{4.10}$$

we can naturally regard f as regular functions on X_{Σ_C} and $X = X_{\Sigma'_C}$. This implies that $X = X_{\Sigma'_C}$ is an open subset of $g^{-1}(\mathbb{C})\cap\widetilde{X_\Sigma}$. In particular, if $\sigma \in \Sigma'_C$ is not contained in $\mathbb{R}^n_+$ then $T_\sigma \subset X\setminus\mathbb{C}^n$ and f extends holomorphically to T_σ. Namely T_σ is a horizontal T-orbit in $X \setminus \mathbb{C}^n$. By our assumption $b \notin f(\operatorname{Sing} f)$ and the result at the end of Section 3, we see also that the support of the constructible sheaf $\varphi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n}) \in \mathbf{D}^b_c(g^{-1}(b))$ is contained in $(X \setminus \mathbb{C}^n) \cap g^{-1}(b)$. We thus obtain an equality

$$E_f(b) = (-1)^{n-1} \int_{(X\setminus\mathbb{C}^n)\cap g^{-1}(b)} \chi\{\varphi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n})\}. \tag{4.11}$$

Namely, for the calculation of $E_f(b)$ it suffices to calculate the constructible function $\chi\{\varphi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n})\}$ only on T-orbits in $X\setminus\mathbb{C}^n$ associated to the cones

$\sigma \in \Sigma'_C \subset \Sigma$ such that $\mathrm{rel.int}(\sigma) \subset C \setminus \mathbb{R}^n_+$. For $\sigma \in \Sigma'_C \subset \Sigma$ such that $\mathrm{rel.int}(\sigma) \subset \mathrm{Int}(C) \setminus \mathbb{R}^n_+$ we have $\gamma_\sigma = \{0\} \prec \Gamma_\infty(f)$ and the restriction of $g|_X : X \longrightarrow \mathbb{C}$ to the T-orbit $T_\sigma \subset X$ is the constant function $f(0) \in \mathbb{C}$. Hence we get $g^{-1}(b) \cap T_\sigma = \emptyset$ for the point $b \in K_f \setminus [f(\mathrm{Sing}\, f) \cup \{f(0)\}]$. For $1 \leqslant i \leqslant m$ let $\sigma_i = \sigma(\gamma_i) \in \Sigma_0$ be the cone which corresponds to γ_i in the dual fan Σ_0 of $\Gamma_\infty(f)$. Recall that by the definition of atypical faces we have $0 \in \gamma_i$ and the face $\sigma_i \prec C$ of C is not contained in $\mathbb{R}^n_+$. For $\sigma \in \Sigma'_C \subset \Sigma$ such that $\mathrm{rel.int}(\sigma) \subset \partial C \setminus \mathbb{R}^n_+$ there exists unique $1 \leqslant i \leqslant m$ for which we have $\mathrm{rel.int}(\sigma) \subset \mathrm{rel.int}(\sigma_i)$. If $\dim \sigma = \dim \sigma_i$ we have an isomorphism $T_\sigma \simeq T_i = \mathrm{Spec}(\mathbb{C}[L_{\gamma_i} \cap \mathbb{Z}^n]) \simeq (\mathbb{C}^*)^{\dim \gamma_i}$ and the restriction of $g|_X : X \longrightarrow \mathbb{C}$ to $T_\sigma \subset X$ is naturally identified with $f_{\gamma_i} : T_i \longrightarrow \mathbb{C}$. This implies that the hypersurface $g^{-1}(b) \cap T_\sigma \subset T_\sigma \simeq T_i$ has only isolated singular points $p_{i,1}, \dots, p_{i,n_i} \in T_\sigma \simeq T_i$ and

$$T_\sigma \cap \ \mathrm{supp}\ \varphi_{h\circ g}(\iota_! \mathbb{C}_{\mathbb{C}^n}) \subset \{p_{i,1}, \dots, p_{i,n_i}\} \tag{4.12}$$

in this case. On the other hand, if $\dim \sigma < \dim \sigma_i$ we have $\dim T_\sigma > \dim T_i$ and for the hypersurface $g^{-1}(b) \cap T_\sigma \subset T_\sigma$ there exists an isomorphism

$$g^{-1}(b) \cap T_\sigma \simeq f_{\gamma_i}^{-1}(b) \times (\mathbb{C}^*)^{\dim T_\sigma - \dim T_i}. \tag{4.13}$$

This implies that $g^{-1}(b) \cap T_\sigma \subset T_\sigma$ has non-isolated singular points if $n_i > 0$. From now on, we shall overcome this difficulty by using Proposition 2.2. For $1 \leqslant i \leqslant m$ let Σ_i be the fan in $\mathbb{R}^n$ formed by all the faces of σ_i and denote by X_{Σ_i} the (possibly singular) toric variety associated to it. Then X_{Σ_i} is an open subset of X_{Σ_C}. Let $\sigma_i^\circ \subset \mathbb{R}^n$ be the dual cone of σ_i in $\mathbb{R}^n$. Then $\sigma_i^\circ \simeq C_i \times \mathbb{R}^{\dim \gamma_i}$ for a proper convex cone C_i in $\mathbb{R}^{n - \dim \gamma_i}$ and we have an isomorphism

$$X_{\Sigma_i} \simeq \mathrm{Spec}(\mathbb{C}[\sigma_i^\circ \cap \mathbb{Z}^n]). \tag{4.14}$$

Note that the (minimal) T-orbit T_{σ_i} in X_{Σ_i} which corresponds to $\sigma_i \in \Sigma_i$ is naturally identified with $T_i = \mathrm{Spec}(\mathbb{C}[L_{\gamma_i} \cap \mathbb{Z}^n]) \simeq (\mathbb{C}^*)^{\dim \gamma_i}$. More precisely X_{Σ_i} is the product $X_i \times T_{\sigma_i}$ of the $(n - \dim \gamma_i)$-dimensional affine toric variety $X_i = \mathrm{Spec}(\mathbb{C}[C_i \cap \mathbb{Z}^{n-\dim \gamma_i}])$ and $T_{\sigma_i} \simeq T_i \simeq (\mathbb{C}^*)^{\dim \gamma_i}$. Since $NP(f) \subset \sigma_i^\circ$ and $f \in \mathbb{C}[\sigma_i^\circ \cap \mathbb{Z}^n]$, we can naturally regard f as a regular function on X_{Σ_i}. We denote it by $f_i : X_{\Sigma_i} \longrightarrow \mathbb{C}$. For $1 \leqslant i \leqslant m$ let $\Sigma'_i \subset \Sigma$ be the subfan of Σ consisting of the cones in Σ contained in σ_i and denote by $X_{\Sigma'_i}$ the smooth toric variety associated to it. Then $X_{\Sigma'_i}$ is an open subset of $X \subset \widetilde{X_\Sigma}$ and there exists a proper morphism

$$\pi_i : X_{\Sigma'_i} \longrightarrow X_{\Sigma_i} \tag{4.15}$$

of toric varieties. Moreover we have a commutative diagram

$$\begin{array}{ccc} X_{\Sigma_i'} & \longrightarrow & X = X_{\Sigma_C'} \\ {\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle \pi} \\ X_{\Sigma_i} & \longrightarrow & X_{\Sigma_C} \end{array} \tag{4.16}$$

such that $\pi^{-1}X_{\Sigma_i} = X_{\Sigma_i'} \subset X$, where the horizontal arrows are the inclusion maps. It is also easy to see that the closed subset $(X \setminus \mathbb{C}^n) \cap g^{-1}(b)$ of X is covered by the affine open subvarieties $X_{\Sigma_1'}, \dots, X_{\Sigma_m'} \subset X$. Note that for the restriction $g_i = g|_{X_{\Sigma_i'}} : X_{\Sigma_i'} \longrightarrow \mathbb{C}$ of $g|_X$ we have $g_i = f_i \circ \pi_i$. Then by applying Proposition 2.2 to the proper morphism $\pi_i : X_{\Sigma_i'} \longrightarrow X_{\Sigma_i}$ we obtain an isomorphism

$$R(\pi_i|_{g_i^{-1}(b)})_* \varphi_{h\circ g_i}(\iota_! \mathbb{C}_{\mathbb{C}^n}|_{X_{\Sigma_i'}}) \simeq \varphi_{h\circ f_i} \left\{ R(\pi_i)_*(\iota_! \mathbb{C}_{\mathbb{C}^n}|_{X_{\Sigma_i'}}) \right\}. \tag{4.17}$$

The advantage to consider $\varphi_{h\circ f_i}\{R(\pi_i)_*(\iota_!\mathbb{C}_{\mathbb{C}^n}|_{X_{\Sigma_i'}})\}$ instead of $\varphi_{h\circ g_i}(\iota_!\mathbb{C}_{\mathbb{C}^n}|_{X_{\Sigma_i'}})$ is that its support is a discrete subset of $f_i^{-1}(b) \subset X_{\Sigma_i} \subset X_{\Sigma_C}$ by our assumption that f has isolated singularities at infinity over $b \in K_f \setminus [f(\operatorname{Sing} f) \cup \{f(0)\}]$. Set

$$\mathcal{F}_i = R(\pi_i)_*(\iota_!\mathbb{C}_{\mathbb{C}^n}|_{X_{\Sigma_i'}}) \simeq R(\pi_i)_! \mathbb{C}_{\mathbb{C}^n \cap X_{\Sigma_i'}} \in \mathbf{D}_c^b(X_{\Sigma_i}). \tag{4.18}$$

Then the topological integral

$$\int_{g^{-1}(b)} \chi\{\varphi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n})\} = \int_{(X\setminus\mathbb{C}^n)\cap g^{-1}(b)} \chi\{\varphi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n})\} \tag{4.19}$$

is equal to

$$\sum_{i=1}^{m}\sum_{j=1}^{n_i} \chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\}. \tag{4.20}$$

If $b \notin K_i$ ($\Longleftrightarrow n_i = 0$) we have $\varphi_{h\circ f_i}(\mathcal{F}_i) \simeq 0$ on a neighborhood of $T_{\sigma_i} \subset X_{\Sigma_i}$. Let us consider the remaining case where $b \in K_i$ ($\Longleftrightarrow n_i > 0$). Then by our assumption $\operatorname{rel.int}(\gamma_i) \subset \operatorname{Int}(\mathbb{R}^n_+)$ we have $\sigma_i \cap \mathbb{R}^n_+ = \{0\}$. This implies that for the embedding $\iota_i : T = (\mathbb{C}^*)^n \hookrightarrow X_{\Sigma_i}$ there exists an isomorphism $\mathcal{F}_i \simeq (\iota_i)_!\mathbb{C}_T$. Hence by Lemma 2.3, $\mathcal{F}_i$ is a perverse sheaf on X_{Σ_i} (up to some shift). Since the support of $\varphi_{h\circ f_i}(\mathcal{F}_i)$ is discrete, by Lemma 2.4 we thus obtain the concentration

$$H^l\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}} \simeq 0 \quad (l \neq n-1) \tag{4.21}$$

for any $1 \leqslant j \leqslant n_i$. Set $\mu_{i,j} = \dim H^{n-1}\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}} \geqslant 0$. Then $E_f(b)$ can be expressed as a sum of non-negative integers as follows:

$$E_f(b) = (-1)^{n-1}\int_{(X\setminus\mathbb{C}^n)\cap g^{-1}(b)} \chi\{\varphi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n})\} = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\mu_{i,j}. \tag{4.22}$$

By our assumption there exists $1 \leqslant i \leqslant m$ such that $n_i > 0$ ($\Longleftrightarrow b \in K_i$) and $\gamma_i \prec \Gamma_\infty(f)$ is relatively simple. Then the cone $\sigma_i \in \Sigma_0$ satisfies the condition $\sigma_i \cap \mathbb{R}^n_+ = \{0\}$. For a face $\tau \prec \sigma_i$ of σ_i we set $Y_\tau = \overline{T_\tau} \subset X_{\Sigma_i}$ and $f_\tau = f_i|_{Y_\tau} : Y_\tau \longrightarrow \mathbb{C}$. Note that we have $T_{\sigma_i} = Y_{\sigma_i}$. Then for any $1 \leqslant j \leqslant n_i$ we can easily show that $(-1)^{n-1}\mu_{i,j} = \chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} = \chi\{\varphi_{h\circ f_i}((\iota_i)_!\mathbb{C}_T)_{p_{i,j}}\}$ is equal to the alternating sum

$$\sum_{\tau\prec\sigma_i}(-1)^{\dim\tau}\chi\{\varphi_{h\circ f_\tau}(\mathbb{C}_{Y_\tau})_{p_{i,j}}\}. \tag{4.23}$$

Here we used the additivity of the vanishing cycle functor $\varphi_{h\circ f_i}(\,\cdot\,)$. Since γ_i is relatively simple, by Lemma 2.5 for any face $\tau \prec \sigma_i$ of σ_i the constant sheaf $\mathbb{C}_{Y_\tau}$ on Y_τ is perverse (up to some shift). Moreover by our assumption that f has isolated singularities at infinity over $b \in K_f\setminus[f(\mathrm{Sing}\, f)\cup\{f(0)\}]$, the support of $\varphi_{h\circ f_\tau}(\mathbb{C}_{Y_\tau})$ is discrete on a neighborhood of $T_{\sigma_i} \subset X_{\Sigma_i}$. By Lemma 2.4 we thus obtain the concentration

$$H^l\varphi_{h\circ f_\tau}(\mathbb{C}_{Y_\tau})_{p_{i,j}} \simeq 0 \qquad (l \neq \dim Y_\tau - 1 = n - \dim\tau - 1) \tag{4.24}$$

for any $1 \leqslant j \leqslant n_i$ and $\tau \prec \sigma_i$. Set

$$\mu_{i,j,\tau} = \dim H^{n-\dim\tau-1}\varphi_{h\circ f_\tau}(\mathbb{C}_{Y_\tau})_{p_{i,j}} \geqslant 0. \tag{4.25}$$

Then $\mu_{i,j} = (-1)^{n-1}\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} \geqslant 0$ is expressed as a sum of non-negative integers as follows:

$$\mu_{i,j} = \sum_{\tau\prec\sigma_i}\mu_{i,j,\tau} \geqslant 0. \tag{4.26}$$

Moreover the integer μ_{i,j,σ_i} is positive by the smoothness of $T_{\sigma_i} = Y_{\sigma_i}$. Consequently we get $E_f(b) > 0$. This completes the proof. □

In the generic (Newton non-degenerate) case, for any $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n_i$ we can explicitly calculate the above integer $\mu_{i,j} \geqslant 0$ by [16, Theorem 3.4, Corollary 3.6 and Remark 4.3] as follows. First by multiplying a monomial on $T_{\sigma_i} \simeq (\mathbb{C}^*)^{\dim\gamma_i}$ to f_i we may assume that f_i is a regular function on $X_i\times\mathbb{C}^{\dim\gamma_i}$. Next by a translation in $\mathbb{C}^{\dim\gamma_i}$ we reduce the problem to the case $p_{i,j} = 0 \in \mathbb{C}^{\dim\gamma_i}$. Then we can apply [16, Theorem 3.4 and Corollary 3.6] to $\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}} \simeq \psi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}$ if $f_i : (X_i \times \mathbb{C}^{\dim\gamma_i}, 0) \longrightarrow (\mathbb{C}, 0)$ is Newton non-degenerate at $p_{i,j} = 0 \in \mathbb{C}^{\dim\gamma_i}$. In this way, even if σ_i is not simplicial we can express the integer $\mu_{i,j} \geqslant 0$ as an alternating sum of the normalized volumes of polytopes in $\mathbb{R}^n_+ \setminus \Gamma_+(f)_{i,j}$, where $\Gamma_+(f)_{i,j} \subset \mathbb{R}^n_+$ is the (local) Newton polyhedron of f_i at $p_{i,j}$. See [16, Corollary 3.6] for the details. We conjecture that it is positive in our situation. In the case where $n = 3$ we have the following stronger result.

THEOREM 4.4. — *Assume that $n = 3$ and f has isolated singularities at infinity over $b \in K_f \setminus [f(\mathrm{Sing}\, f) \cup \{f(0)\}]$. Then we have $E_f(b) > 0$ and hence $b \in B_f$.*

Proof. — The proof is similar to that of Theorem 4.3. We shall use the notations in it. For any $1 \leqslant i \leqslant m$ the dimension of the atypical face $\gamma_i \prec \Gamma_\infty(f)$ is 1 or 2. If $\dim \gamma_i = 2$ and $n_i > 0$ we have $\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} > 0$ for any $1 \leqslant j \leqslant n_i$ by the result of Zaharia [30]. If $\dim \gamma_i = 1$ and $n_i > 0$ the two-dimensional cone σ_i is simplicial but $\sigma_i \cap \mathbb{R}^3_+$ can be bigger than $\{0\}$. Nevertheless we can show the positivity $\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} > 0$ for any $1 \leqslant j \leqslant n_i$ by calculating $\mathcal{F}_i \in \mathbf{D}^b_c(X_{\Sigma_i})$ very explicitly depending on how σ_i intersects $\mathbb{R}^3_+$. First we consider the case where $\dim \sigma_i = 2$, $\dim \sigma_i \cap \mathbb{R}^3_+ = 1$ and $\mathrm{rel.int}(\sigma_i \cap \mathbb{R}^3_+) \subset \mathrm{rel.int}(\sigma_i)$. Then for any point $q \in T_{\sigma_i} \subset X_{\Sigma_i}$ its fiber of the map

$$\pi_i|_{\mathbb{C}^3 \cap X_{\Sigma'_i}} : \mathbb{C}^3 \cap X_{\Sigma'_i} \longrightarrow X_{\Sigma_i} \tag{4.27}$$

is isomorphic to $\mathbb{C}^*$. For its cohomology groups with compact support $H^l_c(\mathbb{C}^*; \mathbb{C})$ $(l \in \mathbb{Z})$ we have

$$H^l_c(\mathbb{C}^*; \mathbb{C}) \simeq \begin{cases} \mathbb{C} & (l = 1, 2), \\ 0 & (l \neq 1, 2). \end{cases} \tag{4.28}$$

Hence for the point $q \in T_{\sigma_i}$ we have

$$H^l(\mathcal{F}_i)_q \simeq \begin{cases} \mathbb{C} & (l = 1, 2), \\ 0 & (l \neq 1, 2) \end{cases} \tag{4.29}$$

and $\chi(\mathcal{F}_i)(q) = 0$. Since the two one-dimensional faces $\rho_{i,1}, \rho_{i,2}$ of σ_i are not contained in $\mathbb{R}^3_+$ there exists also an isomorphism

$$\mathcal{F}_i|_{X_{\Sigma_i} \setminus T_{\sigma_i}} \simeq (\iota_i)_! \mathbb{C}_T|_{X_{\Sigma_i} \setminus T_{\sigma_i}} = \mathbb{C}_T|_{X_{\Sigma_i} \setminus T_{\sigma_i}}. \tag{4.30}$$

It follows from (4.29) and (4.30) we have an equality

$$\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} = \chi\{\varphi_{h\circ f_i}((\iota_i)_! \mathbb{C}_T)_{p_{i,j}}\} = \chi\{\varphi_{h\circ f_i}(\mathbb{C}_T)_{p_{i,j}}\} \tag{4.31}$$

for any $1 \leqslant j \leqslant n_i$. Then for any $1 \leqslant j \leqslant n_i$ we obtain the positivity

$$\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} = \chi\{\varphi_{h\circ f_i}(\mathbb{C}_T)_{p_{i,j}}\} > 0 \tag{4.32}$$

by the proof of Theorem 4.3. Next we consider the case where $\dim \sigma_i = 2$ and $\sigma_i \cap \mathbb{R}^3_+$ is one of the two one-dimensional faces $\rho_{i,1}, \rho_{i,2}$ of σ_i. We may assume that $\sigma_i \cap \mathbb{R}^3_+ = \rho_{i,1}$. For $1 \leqslant j \leqslant 2$ we denote by $T_{i,j} \simeq (\mathbb{C}^*)^2$ the T-orbit in X_{Σ_i} associated to $\rho_{i,j} \prec \sigma_i$. Then for $Y_{\{2\}} = \overline{T_{i,2}}$ we have an isomorphism $\mathcal{F}_i \simeq \mathbb{C}_{X_{\Sigma_i} \setminus Y_{\{2\}}}$. Since $\mathbb{C}_{X_{\Sigma_i}}$ is a perverse sheaf (up to some shift) and the two-dimensional variety $Y_{\{2\}} = \overline{T_{i,2}}$ is smooth, for any $1 \leqslant j \leqslant n_i$ we obtain the positivity

$$\begin{aligned}\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} &= \chi\{\varphi_{h\circ f_i}(\mathbb{C}_{X_{\Sigma_i}})_{p_{i,j}}\} - \chi\{\varphi_{h\circ f_i}(\mathbb{C}_{Y_{\{2\}}})_{p_{i,j}}\} \\ &\geqslant -\chi\{\varphi_{h\circ f_i}(\mathbb{C}_{Y_{\{2\}}})_{p_{i,j}}\} > 0.\end{aligned} \tag{4.33}$$

Finally, let us treat the case where $\dim \sigma_i = \dim \sigma_i \cap \mathbb{R}^3_+ = 2$. Since the face γ_i is atypical, its dual cone σ_i is not contained in $\mathbb{R}^3_+$ and hence we have $\sigma_i \cap \mathbb{R}^3_+ \neq \sigma_i$ in this case. Assume also that $\operatorname{rel.int}(\sigma_i \cap \mathbb{R}^3_+) \subset \operatorname{rel.int}(\sigma_i)$. Then for any point $q \in T_{\sigma_i} \subset X_{\Sigma_i}$ its fiber of the map

$$\pi_i|_{\mathbb{C}^3 \cap X_{\Sigma'_i}} : \mathbb{C}^3 \cap X_{\Sigma'_i} \longrightarrow X_{\Sigma_i} \tag{4.34}$$

is isomorphic to the singular algebraic curve $\{(x_1, x_2) \in \mathbb{C}^2 \mid x_1 x_2 = 0\} \subset \mathbb{C}^2$. By calculating its Euler characteristic with compact support, we obtain $\chi(\mathcal{F}_i)(q) = 1$. Moreover we have the isomorphism (4.30) in this case. We thus obtain the positivity

$$\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} = \chi\{\varphi_{h\circ f_i}(\mathbb{C}_T)_{p_{i,j}}\} + \chi\{\varphi_{h\circ f_i}(\mathbb{C}_{T_{\sigma_i}})_{p_{i,j}}\} > 0 \tag{4.35}$$

for any $1 \leqslant j \leqslant n_i$. Similarly we can prove the non-negativity and the positivity also in the remaining case. This completes the proof. □

We thus confirm the conjecture of [19] for $n = 3$ in the generic case. Similarly, we can improve Theorem 4.3 as follows. In fact, Theorem 4.5 below extends Theorems 4.3 and 4.4 in a unified manner. Note that the condition $\operatorname{rel.int}(\gamma_i) \subset \operatorname{Int}(\mathbb{R}^n_+)$ is equivalent to the one $\sigma_i \cap \mathbb{R}^n_+ = \{0\}$ for the cone $\sigma_i = \sigma(\gamma_i) \in \Sigma_0$.

Theorem 4.5. — *Assume that f has isolated singularities at infinity over $b \in K_f \setminus [f(\operatorname{Sing} f) \cup \{f(0)\}]$ and for any $1 \leqslant i \leqslant m$ such that $b \in K_i$ the set $\sigma_i \cap \mathbb{R}^n_+$ is a face of $\mathbb{R}^n_+$ of dimension $\leqslant 2$. Assume also that there exists $1 \leqslant i \leqslant m$ such that $b \in K_i$, $\gamma_i \prec \Gamma_\infty(f)$ is relatively simple and moreover in the case $\dim \sigma_i \cap \mathbb{R}^n_+ = 2$ the number of the common edges of $\sigma_i \cap \mathbb{R}^n_+$ and σ_i is $\leqslant 1$. Then we have $E_f(b) > 0$ and hence $b \in B_f$.*

Proof. — The proof is similar to those of Theorems 4.3 and 4.4. We shall use the notations in them. In the proof of Theorem 4.3 we proved for $1 \leqslant i \leqslant m$ such that $\sigma_i \cap \mathbb{R}^n_+ = \{0\}$ (resp. $\sigma_i \cap \mathbb{R}^n_+ = \{0\}$ and γ_i is relatively simple) we have $(-1)^{n-1}\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} \geqslant 0$ (resp. > 0) for any $1 \leqslant j \leqslant n_i$. Let us consider the remaining cases where $1 \leqslant \dim \sigma_i \cap \mathbb{R}^n_+ \leqslant 2$. For a face $\tau \prec \sigma_i$ of such σ_i, by taking a reference point $q \in T_\tau \subset X_{\Sigma_i}$

of the T-orbit T_τ associated to it we set $e(\tau) = \chi(\mathcal{F}_i)(q)$. Then as in the proof of Theorem 4.4 we can easily show that

$$e(\tau) = \begin{cases} 1 & (\dim \tau \cap \mathbb{R}^n_+ = \dim \tau), \\ 0 & (\dim \tau \cap \mathbb{R}^n_+ < \dim \tau). \end{cases} \tag{4.36}$$

In particular, for the zero-dimensional face $\{0\} \prec \sigma_i$ of σ_i we have $T_{\{0\}} = T$, $\mathcal{F}_i|_T \simeq \mathbb{C}_T$ and $e(\{0\}) = 1$. We thus obtain an equality

$$(-1)^{n-1}\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} = (-1)^{n-1} \sum_{\tau : e(\tau)=1} \chi\{\varphi_{h\circ f_i}(\mathbb{C}_{T_\tau})_{p_{i,j}}\} \tag{4.37}$$

for any $1 \leqslant j \leqslant n_i$. First let us consider the case where $\dim \sigma_i \cap \mathbb{R}^n_+ = 1$. If $\sigma_i \cap \mathbb{R}^n_+$ is not an edge of the cone σ_i, by (4.37) we have

$$(-1)^{n-1}\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} = (-1)^{n-1}\chi\{\varphi_{h\circ f_i}(\mathbb{C}_T)_{p_{i,j}}\} \tag{4.38}$$

for any $1 \leqslant j \leqslant n_i$. By the proof of Theorem 4.3 this integer is non-negative. Moreover it is positive if γ_i is relatively simple. Let $\rho_{i,1}, \rho_{i,2}, \ldots, \rho_{i,d_i} \prec \sigma_i$ be the edges of σ_i. For $1 \leqslant j \leqslant d_i$ we denote by $T_{i,j} \simeq (\mathbb{C}^*)^{n-1}$ the T-orbit in X_{Σ_i} associated to $\rho_{i,j} \prec \sigma_i$. If $\sigma_i \cap \mathbb{R}^n_+$ is an edge ρ of σ_i, by (4.37) we can easily see that for the remaining edges $\rho_{i,j}$ $(1 \leqslant j \leqslant d_i)$ of σ_i satisfying $\rho_{i,j} \neq \rho$ and the hypersurface $Z_i := \bigcup_{j:\rho_{i,j}\neq\rho} \overline{T_{i,j}} \subset X_{\Sigma_i}$ defined by them there exists an isomorphism $\mathcal{F}_i \simeq \mathbb{C}_{X_{\Sigma_i}\setminus Z_i}$. Since the hypersurface complement $X_{\Sigma_i} \setminus Z_i$ is an affine open subset of X_{Σ_i}, $\mathcal{F}_i$ is perverse (up to some shift) and we obtain the non-negativity

$$(-1)^{n-1}\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} = (-1)^{n-1}\chi\{\varphi_{h\circ f_i}(\mathbb{C}_{X_{\Sigma_i}\setminus Z_i})_{p_{i,j}}\} \geqslant 0 \tag{4.39}$$

for any $1 \leqslant j \leqslant n_i$. Moreover we can rewrite this integer as follows:

$$(-1)^{n-1}\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} = (-1)^{n-1} \sum_{\tau : \rho \not\prec \tau} (-1)^{\dim \tau}\chi\{\varphi_{h\circ f_i}(\mathbb{C}_{\overline{T_\tau}})_{p_{i,j}}\}. \tag{4.40}$$

If γ_i is relatively simple, the right hand side is a sum of non-negative integers and for a facet τ of σ_i such that $\rho \not\prec \tau$ the closure $\overline{T_\tau}$ of T_τ is smooth and we have the positivity

$$(-1)^{n-1+\dim \tau}\chi\{\varphi_{h\circ f_i}(\mathbb{C}_{\overline{T_\tau}})_{p_{i,j}}\} > 0. \tag{4.41}$$

Finally let us consider the case where $\dim \sigma_i \cap \mathbb{R}^n_+ = 2$. Assume that $(\sigma_i \cap \mathbb{R}^n_+) \setminus \{0\} \subset \mathrm{rel.int}(\sigma_i)$. Since the case where $\dim \sigma_i = \dim \sigma_i \cap \mathbb{R}^n_+ = 2$ was already treated in the proof of Theorem 4.4, here we treat only the case where $\dim \sigma_i > \dim \sigma_i \cap \mathbb{R}^n_+ = 2$. Then by (4.37) we obtain the non-negativity

$$(-1)^{n-1}\chi\{\varphi_{h\circ f_i}(\mathcal{F}_i)_{p_{i,j}}\} = (-1)^{n-1}\chi\{\varphi_{h\circ f_i}(\mathbb{C}_T)_{p_{i,j}}\} \geqslant 0 \tag{4.42}$$

for any $1 \leqslant j \leqslant n_i$. Moreover it is positive if γ_i is relatively simple. Similarly we can prove the non-negativity and the positivity also in the remaining cases. We omit the details. This completes the proof. □

In the case $n = 4$ we can also partially verify the conjecture of [19] as follows.

Theorem 4.6. — *Assume that $n = 4$, f has isolated singularities at infinity over $b \in K_f \setminus [f(\operatorname{Sing} f) \cup \{f(0)\}]$ and for any $1 \leqslant i \leqslant m$ such that $b \in K_i$ and $\dim \sigma_i = \dim \sigma_i \cap \mathbb{R}^4_+ = 3$ there exists no common edge of σ_i and $\sigma_i \cap \mathbb{R}^4_+$. Assume also that there exists $1 \leqslant i \leqslant m$ such that $b \in K_i$ and in the case $\dim \sigma_i = 3$ and $\dim \sigma_i \cap \mathbb{R}^4_+ = 2$ the number of the common edges of σ_i and $\sigma_i \cap \mathbb{R}^4_+$ is $\leqslant 1$. Then we have $E_f(b) > 0$ and hence $b \in B_f$.*

Corollary 4.7. — *Assume that $n = 4$, f has isolated singularities at infinity over $b \in K_f \setminus [f(\operatorname{Sing} f) \cup \{f(0)\}]$ and for any $1 \leqslant i \leqslant m$ such that $b \in K_i$ we have $\dim \sigma_i \cap \mathbb{R}^4_+ \leqslant 1$ or $\dim \sigma_i \leqslant 2$. Then we have $E_f(b) > 0$ and hence $b \in B_f$.*

Since the proof of Theorem 4.6 is similar to those of Theorems 4.3, 4.4 and 4.5, we omit it here.

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