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X-RAY TRANSFORM AND BOUNDARY RIGIDITY FOR ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract. — We consider the boundary rigidity problem for asymptotically hyperbolic manifolds. We show injectivity of the X-ray transform in several cases and consider the non-linear inverse problem which consists of recovering a metric from boundary measurements for the geodesic flow.

Résumé. — On considère le problème de rigidité du bord pour les variétés asymptotiquement hyperboliques. Nous montrons l'injectivité de la transformée en rayons X dans plusieurs cas et considérons le problème inverse non-linéaire qui consiste en la détermination de la métrique à partir de données au bord sur le flot géodésique.

1. Introduction

In this work, we consider the problem of the geodesic X-ray transform on asymptotically hyperbolic manifolds, and some applications to the boundary rigidity problem in that non-compact setting.

Let $\overline{M}$ be a compact connected smooth manifold-with-boundary of dimension $n+1$ with $n \geqslant 1$. A smooth metric g on the interior M of $\overline{M}$ is said to be *asymptotically hyperbolic* if $\bar{g}_0 := \rho_0^2 g$ extends to a smooth metric on $\overline{M}$ with $|\mathrm{d}\rho_0|_{\bar{g}_0} = 1$ at $\partial\overline{M}$, where $\rho_0 \in C^\infty(\overline{M}; \mathbb{R}_{\geqslant 0})$ is a smooth defining function for $\partial\overline{M}$, i.e. $\{\rho_0 = 0\} = \partial\overline{M}$ with $\mathrm{d}\rho_0$ not vanishing at $\partial\overline{M}$. The boundary $\partial\overline{M}$ equipped with the conformal class of $\bar{g}_0|_{T\partial\overline{M}}$ is called the *conformal boundary*, or *conformal infinity*, of (M, g). It follows from [16] that for each metric h in the conformal infinity, there exists a smooth boundary defining function ρ so that $|\mathrm{d}\rho|_{\rho^2 g} = 1$ near $\partial\overline{M}$ and $\rho^2 g|_{T\partial\overline{M}} = h$; this function is uniquely determined near $\partial\overline{M}$ by h. Such

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a function ρ is called a *geodesic boundary defining function* associated to the conformal representative h. The flow of the gradient of ρ with respect to the metric $\bar{g} := \rho^2 g$ induces a product decomposition $(0,\epsilon)_\rho \times \partial\overline{M}$ of a collar neighborhood $\mathcal{C}_\epsilon$ near $\partial\overline{M}$ in which the metric has the form

$$g = \frac{\mathrm{d}\rho^2 + h_\rho}{\rho^2} \quad \text{on } (0,\epsilon)_\rho \times \partial\overline{M},$$

with h_ρ a smooth 1-parameter family of metrics on $\partial\overline{M}$ which extends smoothly to $\rho \in [0,\epsilon)$ and satisfies $h_0 = h$. For convenience, we can extend freely ρ as a smooth positive function to M so that $\rho \geqslant \epsilon$ in $M \setminus \mathcal{C}_\epsilon$. The metric g is a complete metric with sectional curvatures tending to -1 at $\partial\overline{M}$; it has infinite volume and all convex co-compact hyperbolic manifolds are particular cases of asymptotically hyperbolic manifolds. (Recall that a convex co-compact hyperbolic manifold is a smooth complete Riemannian manifold with constant curvature -1, finite geometry (finitely many ends), which is realized as a quotient $\Gamma\backslash\mathbb{H}^{n+1}$ of the hyperbolic space by a discrete group Γ of isometries containing only hyperbolic transformations.) Other interesting examples are Poincaré–Einstein manifolds, which appear in the AdS-CFT correspondence and were introduced as a tool to study conformal geometry; see [13].

Geodesics of g can be viewed as integral curves of the Hamiltonian vector field X of $|\xi|_g^2/2$ on the unit cotangent bundle $S^*M := \{(x,\xi) \in T^*M; |\xi|_g = 1\}$ of M, projected to M by $\pi : S^*M \to M$ the projection on the base. Geodesics approach $\partial\overline{M}$ normally and are determined by their second order deviation from the normal. In order to encode this, we introduce an extension $\overline{S^*M}$ of S^*M to $\overline{M}$. Recall, from e.g. [34], that the b-cotangent bundle ${}^bT^*\overline{M}$ is a smooth vector bundle on $\overline{M}$ isomorphic to T^*M over M and with local smooth sections $\{\mathrm{d}\rho/\rho, \mathrm{d}y_1, \dots, \mathrm{d}y_n\}$ near $\partial\overline{M}$, if $(\rho, y_1, \dots, y_n)$ are local coordinates near $\partial\overline{M}$. The dual metric to g, viewed as a metric on ${}^bT^*\overline{M}|_M$, extends smoothly to $\overline{M}$ but degenerates over $\partial\overline{M}$. The extension $\overline{S^*M}$ is defined to be the unit cosphere bundle in ${}^bT^*\overline{M}$ with respect to the quadratic form g. It takes the form

$$\overline{S^*M} = S^*M \sqcup \partial_-S^*M \sqcup \partial_+S^*M,$$

where each of $\partial_\pm S^*M$ is a canonical subset of ${}^bT^*\overline{M}|_{\partial M}$ independent of g which can be identified with $T^*\partial\overline{M}$ upon choosing a metric h in the conformal infinity of g. Elements of $\partial_\pm S^*M$ correspond by duality using g to second order tangential deviations from the normal at a boundary point. ∂_-S^*M is regarded as the incoming boundary and ∂_+S^*M as the outgoing boundary. Each point $z_\pm \in \partial_\pm S^*M$ is the limit of a unique integral curve of X as $t \to \pm\infty$.

The *trapped set* K of the flow $\varphi_t : S^*M \to S^*M$ of X is the set of points $z \in S^*M$ for which the integral curve $\{\varphi_t(z); t \in \mathbb{R}\}$ remains in a compact set; since the regions $\{\rho \geqslant \epsilon\}$ are strictly convex for small ϵ, this can alternatively be defined by (here ρ is lifted to S^*M by π)

$$K := \Big\{z \in S^*M; \inf_{t\in\mathbb{R}} \rho(\varphi_t(z)) > 0\Big\}.$$

This is a compact set that is globally invariant by φ_t. We say that g is *non-trapping* if $K = \emptyset$. In this work, we will consider either the non-trapping case or the case where K is a *hyperbolic set* for the flow in the following sense: there is a continuous, flow-invariant splitting of $T_K(S^*M) := T(S^*M)|_K$ of the form

$$T_K(S^*M) = \mathbb{R}X \oplus E_s \oplus E_u$$

where E_s, E_u are subbundles over K satisfying that there is $\nu > 0$ and $C > 0$ such that for all $z = (x, \xi) \in K$ and

$$\begin{aligned} &\forall\, \zeta \in E_s(z), \quad \|\mathrm{d}\varphi_t(z).\zeta\|_G \leqslant Ce^{-\nu t}\|\zeta\|_G, \qquad \forall\ t \geqslant 0,\\ &\forall\, \zeta \in E_u(z), \quad \|\mathrm{d}\varphi_t(z).\zeta\|_G \leqslant Ce^{-\nu |t|}\|\zeta\|_G, \quad \forall\ t \leqslant 0. \end{aligned} \tag{1.1}$$

(Here G denotes the Sasaki metric for g, see (2.27).) The incoming and outgoing trapped sets are defined by

$$\Gamma_\pm := \{z \in S^*M; \inf_{t\in\mathbb{R}^+} \rho(\varphi_{\mp t}(z)) > 0\},$$

they correspond to geodesics trapped in the past $(+)$ or in the future $(-)$. When K is hyperbolic, then $\Gamma_\pm$ and K have zero Liouville measure; see (2.12). Each untrapped geodesic $\gamma(t)$ of g converges to a point $y_- \in \partial\overline{M}$ in the past and $y_+ \in \partial\overline{M}$ in the future, and the corresponding integral curve on S^*M converges to some $z_- \in \partial_- S^*M$ in the past and $z_+ \in \partial_+ S^*M$ in the future. The set of untrapped geodesics is parametrized by $\partial_- S^*M \setminus \overline{\Gamma_-}$, corresponding to the backward limit of the integral curve. In the non-trapping case, $\Gamma_\pm$ are empty.

Our first result concerns the X-ray transform on symmetric m-tensors, which can be defined as the operator

$$I_m : C_c^\infty(M; \otimes_S^m T^*M) \to C^\infty(\partial_- S^*M \setminus \overline{\Gamma_-}),$$
$$I_m(f)(z) = \int_{\mathbb{R}} f(\gamma_z(t))(\otimes^m \dot\gamma_z(t))\mathrm{d}t$$

where $\gamma_z(t)$ is the geodesic with backward limit $z \in \partial_- S^*M$ (here $\otimes_S^m T^*M$ denotes the bundle of symmetric tensors of rank m on M). This operator extends continuously to the space $\rho^{1-m}C^\infty(\overline{M}; \otimes_S^m T^*\overline{M})$; see (3.19).

Theorem 1.1. — *Let (M, g) be an asymptotically hyperbolic manifold such that g has no conjugate points and the trapped set is either empty or a hyperbolic set. Let $f \in \rho^{1-m}C^\infty(\overline{M}; \otimes_S^m T^*\overline{M})$ satisfy $I_m f = 0$.*

(1) *If $m = 0$ then $f = 0$.*
(2) *If $m = 1$, there exists $q \in \rho C^\infty(\overline{M})$ such that $f = \mathrm{d}q$.*
(3) *If $m > 1$ and if the curvature of g is non-positive, then there exists a symmetric tensor $q \in \rho^{2-m}C^\infty(\overline{M}, \otimes_S^{m-1} T^*\overline{M})$ such that $f = Dq$, where D denotes the symmetrized covariant derivative.*

We note that if f is compactly supported, we can find a large convex region in M that contains the support of f and the problem reduces to the case of a compact manifold with strictly convex boundary. Known results resolve the problem in that case; see the discussion of the literature below.

Corollary 1.2. — *Let (M, g) be an asymptotically hyperbolic manifold with negative curvature and $m \geqslant 0$. If $f \in \rho^{1-m}C^\infty(\overline{M}; \otimes_S^m T^*\overline{M})$ satisfies $I_m f = 0$, then there exists $q \in \rho^{2-m}C^\infty(\overline{M}, \otimes_S^{m-1} T^*\overline{M})$ such that $f = Dq$. (In particular, if $m = 0$, then $f = 0$.)*

Corollary 1.2 follows from Theorem 1.1 since manifolds with negative curvature cannot have conjugate points and the trapped set, if nonempty, is hyperbolic.

The X-ray transform for functions was studied on the hyperbolic space $\mathbb{H}^{n+1}$ by Helgason and Berenstein-Casadio Tarabusi: injectivity is proved in [22] for functions decaying like $e^{-d_g(\cdot, o)}$ for $o \in \mathbb{H}^{n+1}$ fixed (this corresponds exactly to the decay condition in Theorem 1.1), and an inversion formula is given in [3, 21]. For Cartan–Hadamard manifolds, recent work by Lehtonen [30] shows injectivity of I_0 in dimension 2 and then Lehtonen–Railo–Salo [31] extended the result to higher dimensions and tensors. In comparison to [30, 31], we allow hyperbolic trapping, we do not require M to be simply connected, and for $m \in \{0, 1\}$ we allow some positive curvature, but our assumption about the geometry at infinity is stronger. We have not tried to obtain the sharpest regularity assumptions on f and it can easily be seen from the proof that the regularity assumptions can be relaxed (we refer to [31] for sharper regularity conditions).

The study of the geodesic X-ray transform on compact domains has a long history. *Simple metrics* are metrics on domains with strictly convex boundaries for which the exponential map is a diffeomorphism at each point. Injectivity of the X-ray transform goes back to Mukhometov [38] for functions, then to Anikonov–Romanov [2] for 1-forms, while Pestov–Sharafutdinov [42] proved injectivity for all tensors in negative curvature (see also Paternain–Salo–Uhlmann [41] for more general results on tensors). Similar results for tensors of rank $m \leqslant 2$ were shown for analytic simple metrics and for generic simple metrics by Stefanov–Uhlmann [50]. For simple metrics in dimension 2, injectivity for 2-tensors was first shown by Sharafutdinov [49] and has been proved recently by Paternain–Salo–Uhlmann [40] for tensor fields of all ranks. For manifolds with strictly convex foliations, injectivity is shown in Uhlmann–Vasy [57] for functions and in Stefanov–Uhlmann–Vasy [56] for 2-tensors. Injectivity for all tensors for all metrics with negative curvature and strictly convex boundary is proved in Guillarmou [18], without simplicity assumptions. Microlocal analysis of the X-ray transform for some cases with conjugate points was done in [23, 37, 52, 53] with generic uniqueness and stability results for a certain class of non-simple metrics in [52].

To prove Theorem 1.1, we need to do a careful analysis of the geodesic flow near infinity. We show that the X-ray transform determines the function (or tensor modulo Dq terms) up to $\mathcal{O}(\rho^\infty)$ at the boundary by using the "short geodesics", i.e. those geodesics staying in regions $\{\rho \leqslant \epsilon\}$ for small $\epsilon > 0$. We then conclude by using Pestov identities on large regions $\{\rho \geqslant \epsilon\}$, with $\epsilon \to 0$. We also use the results of [18] to deal with the trapped case. We observe that our assumptions in (1) and (2) of Theorem 1.1 allow conjugate points at infinity, in the sense that there could be Jacobi fields vanishing at the endpoints y_-, y_+ at infinity along a non-trapped geodesic. This is a true generalization: Eptaminitakis–Graham [12] have constructed examples of non-trapping asymptotically hyperbolic manifolds with no conjugate points which do have conjugate points at infinity.

The *boundary rigidity* problem for simple metrics on compact domains asks if one can recover a simple metric from its boundary distance function (the set of distances between boundary points). Many results are known on the boundary and lens rigidity problems in the compact setting, we refer to the surveys [7, 24, 48, 51] and to the introduction of [55] for references. Here, we consider an analogue of the boundary rigidity problem for asymptotically hyperbolic metrics. First, for each $z_- \in \partial_- S^*M$, there is a unique geodesic γ_{z_-} with backward limit z_-. If γ_{z_-} is not trapped in the future, we denote its forward limit by $z_+ \in \partial_+ S^*M$. Thus we can define a map

$$S_g : \partial_- S^*M \setminus \overline{\Gamma_-} \to \partial_+ S^*M \setminus \overline{\Gamma_+}, \quad S_g(z_-) = z_+$$

called the *scattering map* for the geodesic flow. It is a symplectic map with respect to the canonical symplectic structures on $\partial_\pm S^*M$ induced by their identifications with $T^*\partial\overline{M}$. For such a geodesic γ_{z_-}, given a defining function ρ, we define the *renormalized length* relative to ρ by

$$L_g(z_-) := \lim_{\epsilon\to 0}\big(\ell_g(\gamma_{z_-}\cap\{\rho\geqslant\epsilon\}) + 2\log\epsilon\big)$$

where ℓ_g denotes the length for the metric g. We show that L_g is a well-defined function on $\partial_- S^*M\setminus\overline{\Gamma_-}$ which depends on the choice of ρ in a simple explicit fashion (see (4.2)). We may also view L_g as determined by a choice of representative metric h in the conformal infinity by taking ρ to be the corresponding geodesic defining function. The functions L_g and S_g are closely related to the sojourn time and scattering relation appearing in Sa Barreto–Wang [47]. Renormalized volumes, areas and lengths already appeared quite naturally when analyzing the geometry of asymptotically hyperbolic Einstein manifolds and in the AdS/CFT correspondence (see for example [1, 15]). Boundary rigidity and integral geometry appear in the physics literature concerning the AdS/CFT duality and holography as well, see [8, 44].

We first show that the renormalized length data determine the metric to infinite order at the boundary.

Theorem 1.3. — *Let $\overline{M}$ be a compact connected manifold-with-boundary and let g, g' be two asymptotically hyperbolic metrics on M. Suppose for some choices h and h' of conformal representatives in the conformal infinities of g and g', the renormalized lengths agree for the two metrics, i.e. $L_g = L_{g'}$. Then there exists a diffeomorphism $\psi:\overline{M}\to\overline{M}$ which is the identity on $\partial\overline{M}$ and such that $\psi^*g' - g = \mathcal{O}(\rho^\infty)$ at $\partial\overline{M}$.*

As a consequence of Theorem 1.3, we deduce boundary rigidity for real-analytic metrics under a topological hypothesis. If $\overline{M}$ is a real-analytic manifold-with-boundary, we say that a metric g on M is a real-analytic asymptotically hyperbolic metric if g is real-analytic, asymptotically hyperbolic, and $\bar{g} = \rho^2 g$ is real-analytic up to $\partial\overline{M}$, where ρ is a real-analytic defining function for $\partial\overline{M}$.

Theorem 1.4. — *Let $\overline{M}$ be a compact connected real-analytic manifold-with-boundary such that $\pi_1(\overline{M},\partial\overline{M}) = 0$. Let g, g' be two real-analytic asymptotically hyperbolic metrics on M. If $L_g = L_{g'}$ for some real-analytic metrics h and h' in the conformal infinities of g and g', then there exists a real-analytic diffeomorphism $\psi:\overline{M}\to\overline{M}$ which is the identity on $\partial\overline{M}$ and such that $\psi^*g' = g$.*

In Theorems 1.3 and 1.4, we only require that $L_g = L_{g'}$ on a neighborhood of infinity in $\partial_- S^*M$, corresponding to short geodesics. We will show that $\partial_- S^*M \cap \overline{\Gamma_-}$ is a compact set, so the domain of L_g always contains a full neighborhood of infinity.

In the case of compact simple metrics, the determination of the metric and the curvature at $\partial\overline{M}$ was proved by Michel [36], and the result corresponding to Theorem 1.3 was shown by Lassas–Sharafutdinov–Uhlmann [29] (see also Stefanov–Uhlmann [52] for non-simple metrics). A rigidity result for real-analytic metrics on compact manifolds-with-boundary is proved in [58].

Finally, we prove a deformation rigidity result for the boundary rigidity problem. We define a non-trapping asymptotically hyperbolic manifold to be *simple* if it has no conjugate points at infinity in the sense stated above. This holds in particular for non-trapping metrics when the sectional curvature is non-positive. This definition of simple in the asymptotically hyperbolic case is a weaker starting point than the requirement that the exponential map be a diffeomorphism in the compact case. We show that if (M, g) is simple, then its geodesic flow is hyperbolic with respect to the Sasaki metric. This together with a recent result of Knieper [27] giving sufficient conditions for no conjugate points on a complete non-compact Riemannian manifold with hyperbolic geodesic flow imply that a simple asymptotically hyperbolic manifold has no conjugate points. Other consequences of the hyperbolicity of the geodesic flow are that for each pair of points $y_- \neq y_+ \in \partial\overline{M}$, there is a unique geodesic with endpoints $y_\pm$, and that the exponential map extends smoothly to the boundary as a diffeomorphism in an appropriate sense (Propositions 5.12 and 5.15). The fact that there is a unique geodesic joining any two boundary points enables us to define the renormalized boundary distance relative to a defining function ρ by

$$d_g^R : \partial\overline{M} \times \partial\overline{M} \setminus \mathrm{diag} \to \mathbb{R}, \quad d_g^R(y_-, y_+) := L_g(z_-)$$

where $z_- \in \partial_- S^*_{y_-} M$ is defined by the equation $S_g(z_-) = z_+$ for some $z_+ \in \partial_+ S^*_{y_+} M$.

Theorem 1.5. — *Let $\overline{M}$ be a compact connected manifold-with-boundary and suppose that for $s \in [0,1]$, $g(s)$ is a smooth family of non-trapping asymptotically hyperbolic metrics with non-positive sectional curvature. Assume that for some smooth family $h(s)$ of representatives of the conformal infinities of $g(s)$, one of the following two conditions holds:*

(1) *The renormalized length functions* $L_{g(s)}$ *and scattering maps* $S_{g(s)}$ *are constant in s. (Here* $S_{g(s)}$ *is a viewed as a map* : $T^*\partial\overline{M} \to T^*\partial\overline{M}$ *via the identifications induced by* $h(s)$*).*

(2) *The renormalized boundary distance functions* $d^R_{g(s)}$ *are constant in* s*.*

Then there is a smooth family of diffeomorphisms $\psi(s) : \overline{M} \to \overline{M}$ *for* $s \in [0,1]$ *which satisfies* $\psi(s)^*g(s) = g(0)$ *and* $\psi(s)|_{\partial\overline{M}} = \mathrm{Id}$.

To prove Theorem 1.5 under hypothesis (1), we first use Theorem 1.3 to arrange that the metrics agree to infinite order at the boundary. Then we use Theorem 1.1 after proving that the linearization of the pair $(L_{g(s)}, S_{g(s)})$ reduces to the description of the kernel of the X-ray transform on symmetric 2-tensors. We reduce (2) to (1) by showing in Proposition 5.24 that if two simple metrics have the same renormalized boundary distance functions, then they have the same scattering maps and renormalized length functions. As one of the steps in doing this, we show in Proposition 5.19 that if (M,g) is simple, then $d_g(p,q)+\log\rho(p)+\log\rho(q)$ extends smoothly to $\overline{M}\times\overline{M}\backslash\mathrm{diag}$, where $d_g(p,q)$ denotes the distance function in the metric g.

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2. Asymptotically hyperbolic manifolds and their geodesic flow

For an asymptotically hyperbolic manifold (M,g), the sectional curvatures of g tend to -1 uniformly at the boundary, and more precisely the curvature tensor $\mathcal{R}_g$ of g is of the form

$$\mathcal{R}_g = -g\circ g + \rho^{-3}Q \tag{2.1}$$

where $\circ$ denotes the Kulkarni–Nomizu product and $Q \in C^\infty(\overline{M}; \otimes^4 T^*\overline{M})$.

By abuse of notation, we will sometimes write also g and h_ρ for the metrics induced by g and h_ρ on the cotangent bundles T^*M and $T^*\partial\overline{M}$.

2.1. The geodesic flow on the unit cotangent bundle

Let us describe the geodesic flow near the boundary $\partial\overline{M}$. We work on the unit cotangent bundle

$$S^*M = \{(x,\xi) \in T^*M; |\xi|_g^2 = 1\}$$

and denote by $\pi : S^*M \to M$ the projection to the base. The Liouville 1-form is denoted α and the Hamilton vector field of $\frac{1}{2}|\xi|_g^2$ is the generator X of the geodesic flow φ_t on S^*M. If $(\rho, y^1, \dots, y^n)$ are local coordinates near $\partial\overline{M}$ with ρ a defining function for $\partial\overline{M}$, we use dual coordinates $(\xi_0, \eta_1, \dots, \eta_n)$ on $T^*\overline{M}$ so that $\xi = \xi_0 \mathrm{d}\rho + \eta.\mathrm{d}y$. We begin by considering an extension of the g-cosphere bundle S^*M to $\overline{M}$.

Recall that ${}^bT^*\overline{M}$ denotes the b-cotangent bundle of $\overline{M}$, a smooth vector bundle on $\overline{M}$. A basis for its fibers near $\partial\overline{M}$ consists of $\{\rho^{-1}\mathrm{d}\rho, \mathrm{d}y^1, \dots, \mathrm{d}y^n\}$, and we use dual coordinates $(\bar{\xi}_0, \eta)$ so that $\xi = \bar{\xi}_0 \rho^{-1}\mathrm{d}\rho + \sum_i \eta_i \mathrm{d}y^i$. It is easily verified that the function $\xi \mapsto \overline{\xi_0}$ is an invariant on ${}^bT^*\overline{M}|_{\partial\overline{M}}$, i.e. it is independent of the choice of coordinates (ρ, y). In particular, the subsets $\{\bar{\xi}_0 = \pm 1\}$ of ${}^bT^*\overline{M}|_{\partial\overline{M}}$ are invariantly defined independently of any choices.

Observe that g defines a smooth quadratic form on ${}^bT^*\overline{M}$ all the way up to $\partial\overline{M}$, which however degenerates on $\partial\overline{M}$. We denote by $\overline{S^*M} = \{(x,\xi) \in {}^bT^*\overline{M} : |\xi|_g = 1\}$ the unit cosphere bundle in ${}^bT^*\overline{M}$ with respect to g. Choose a representative metric h for the conformal structure at infinity and use the induced product decomposition near $\partial\overline{M}$. If $y = (y^1, \dots, y^n)$ are local coordinates on $\partial\overline{M}$, we obtain coordinates (ρ, y) on $\overline{M}$. We have for $x \in \overline{M}$ near $\partial\overline{M}$:

$$\overline{S_x^*M} = \{(x,\xi) : \bar{\xi}_0{}^2 + \rho^2|\eta|_{h_\rho}^2 = 1\}.$$

It follows that $\overline{S^*M}$ is a smooth non-compact submanifold-with-boundary of ${}^bT^*\overline{M}$ which is naturally identified with S^*M over M. We define $\partial_\pm S^*M = \{(x,\xi); x \in \partial\overline{M}, \overline{\xi_0} = \mp 1\}$; as noted above, these subsets of ${}^bT^*\overline{M}|_{\partial\overline{M}}$ are independent of the choice of g and of the local coordinates. Thus we have

$$\overline{S^*M} = S^*M \sqcup \partial_-S^*M \sqcup \partial_+S^*M.$$

Given g and the choice of conformal representative metric h, we can identify each of $\partial_\pm S^*_x M$ with $T^*_x\partial\overline{M}$ via the identifications

$$\mp\,\rho^{-1}\mathrm{d}\rho+\sum_i \eta_i \mathrm{d}y^i \mapsto \sum_i \eta_i \mathrm{d}y^i. \tag{2.2}$$

We view $\partial_- S^*M$ as the incoming boundary and $\partial_+ S^*M$ as the outgoing boundary. We will denote by $\iota_\partial : \partial_- S^*M \cup \partial_+ S^*M \to \overline{S^*M}$ the smooth inclusion map. The projection $\pi : S^*M \to M$ extends as a smooth map

$$\pi : \overline{S^*M} \to \overline{M}.$$

We define the vertical bundle $\mathcal{V} = \ker \mathrm{d}\pi$, a smooth subbundle of $T\overline{S^*M}$ of rank n.

Since g degenerates at $\partial\overline{M}$ as a metric on ${}^bT^*\overline{M}$, it does not induce an isomorphism between ${}^bT^*\overline{M}$ and ${}^bT\overline{M}$ over $\partial\overline{M}$. Instead, it induces an isomorphism globally with another natural bundle extending the tangent bundle. Suppose $\overline{M}$ is a manifold-with-boundary equipped with a line subbundle $\mathcal{L} \subset T\overline{M}|_{\partial\overline{M}}$ which is transverse to $T\partial\overline{M}$. In our setting, $\mathcal{L}$ is the orthogonal complement to $T\partial\overline{M}$ with respect to $\bar{g} = \rho^2 g$. Consider the space of smooth vector fields ${}^{\mathcal{L}}\mathcal{V}$ on $\overline{M}$ defined by

$${}^{\mathcal{L}}\mathcal{V} = \{V \in C^\infty(\overline{M}; T\overline{M}) : V|_{\partial\overline{M}} = 0 \text{ and } (\rho^{-1}V)(x) \in \mathcal{L}_x,\ x \in \partial\overline{M}\}.$$

In the usual way, ${}^{\mathcal{L}}\mathcal{V}$ can be regarded as the space of smooth sections of a smooth vector bundle ${}^{\mathcal{L}}T\overline{M}$ on $\overline{M}$. If $(\rho, y = y^1, \dots, y^n)$ are any local coordinates near a point of $\partial\overline{M}$ so that $\mathcal{L} = \operatorname{span}\{\partial_\rho\}$, then $\{\rho\partial_\rho, \rho^2\partial_{y^1}, \dots, \rho^2\partial_{y^n}\}$ is a basis for ${}^{\mathcal{L}}T_x\overline{M}$ for any $x \in \overline{M}$ near $\partial\overline{M}$. For an asymptotically hyperbolic metric g in normal form, the induced isomorphism $T^*M \to TM$ maps $\bar{\xi}_0\rho^{-1}\mathrm{d}\rho + \sum_i \eta_i \mathrm{d}y^i \mapsto \bar{\xi}_0\rho\partial_\rho + \sum_{i,j} h_\rho^{ij}\eta_i\rho^2\partial_{y^j}$, where (h_ρ^{ij}) denotes the matrix of the metric induced by h_ρ on $T^*\partial\overline{M}$ in the coordinates η_i. Clearly this isomorphism extends to the boundary as a smooth isomorphism of vector bundles ${}^bT^*\overline{M} \to {}^{\mathcal{L}}T\overline{M}$ which pulls back the degenerate metric induced by g on ${}^{\mathcal{L}}T\overline{M}$ to that on ${}^bT^*\overline{M}$. The bundle ${}^bT^*\overline{M} \cong {}^{\mathcal{L}}T\overline{M}$ is a natural extension of the (co)tangent bundle for the study of geodesics of an AH metric. For instance, the tangent vector field of a geodesic γ is a smooth nonvanishing section of ${}^{\mathcal{L}}T\overline{M}|_\gamma$ all the way up to the boundary, and, as we will see, the geodesics emanating from or ending on a boundary point x are parametrized by the fibers $\partial_\mp S^*_x M$. As a comparison, recall that the 0-cotangent bundle is the smooth bundle ${}^0T^*\overline{M}$ over $\overline{M}$ whose fibers near the boundary have basis $\{\frac{\mathrm{d}\rho}{\rho}, \frac{\mathrm{d}y^i}{\rho}\}$. The 0-unit cotangent bundle is ${}^0S^*\overline{M} := \{(x,\xi) \in {}^0T^*\overline{M}; |\xi|_g = 1\}$; this is a compact manifold-with-boundary. The bundle ${}^0T^*\overline{M}$ is the natural bundle for analysis of differential operators defined in terms of an asymptotically hyperbolic metric (see [33]); we will use it only mildly in Section 3.

Lemma 2.1. — *The Hamiltonian vector field X on S^*M has the form $X = \rho\overline{X}$, where $\overline{X}$ is a smooth vector field on $\overline{S^*M}$ which is transverse to the boundary $\partial\overline{S^*M} = \partial_- S^*M \sqcup \partial_+ S^*M$.*

Proof. — As a vector field on T^*M, we know that X is tangent to S^*M, so it suffices to analyze X in coordinates on ${}^bT^*\overline{M}$. Since $H = \frac{1}{2}\rho^2(\xi_0^2 + |\eta|^2_{h_\rho})$, we have in coordinates $(\rho, y, \xi = \xi_0 \mathrm{d}\rho + \eta.\mathrm{d}y)$

$$X = \rho^2\xi_0\partial_\rho + \rho^2\sum_{i,j} h_\rho^{ij}\eta_i\partial_{y^j} - \left[\rho(\xi_0^2 + |\eta|^2_{h_\rho}) + \frac{1}{2}\rho^2\partial_\rho|\eta|^2_{h_\rho}\right]\partial_{\xi_0} - \frac{1}{2}\rho^2\sum_k \partial_{y^k}|\eta|^2_{h_\rho}\partial_{\eta_k}. \tag{2.3}$$

Smooth coordinates $(\bar\rho, \bar y, \bar\xi_0, \bar\eta)$ on ${}^bT^*\overline{M}$ are given by

$$\bar\rho = \rho, \quad \bar y = y, \quad \bar\xi_0 = \rho\xi_0, \quad \bar\eta = \eta. \tag{2.4}$$

So

$$\partial_\rho = \partial_{\bar\rho} + \xi_0\partial_{\bar\xi_0}, \quad \partial_y = \partial_{\bar y}, \quad \partial_{\xi_0} = \rho\partial_{\overline{\xi_0}}, \quad \partial_\eta = \partial_{\bar\eta}.$$

Substituting into (2.3), one finds $X = \rho\overline{X}$, with

$$\overline{X} = \bar\xi_0\partial_{\bar\rho} + \bar\rho\sum_{i,j} h_\rho^{ij}\bar\eta_i\partial_{\overline{y^j}} - \left[\bar\rho|\bar\eta|^2_{h_\rho} + \frac{1}{2}\bar\rho^2\partial_\rho|\bar\eta|^2_{h_\rho}\right]\partial_{\overline{\xi_0}} - \frac{1}{2}\bar\rho\sum_k \partial_{y^k}|\bar\eta|^2_{h_\rho}\partial_{\overline{\eta_k}}. \tag{2.5}$$

The result is now clear, since $\partial\overline{S^*M}$ is given by $\bar\rho = 0$, and $\overline{\xi_0} = \pm 1$ on $\partial\overline{S^*M}$. □

We notice that a similar observation was made in [35, Lemma 2.6]. For simplicity, in what follows we will use the notation $(\rho, y, \bar\xi_0, \eta)$ for the coordinates on ${}^bT^*\overline{M}$, instead of $(\bar\rho, \bar y, \bar\xi_0, \bar\eta)$.

Recall that we identify each of $\partial_\mp S^*M$ with $T^*\partial\overline{M}$ via (2.2). This identification depends on the product decomposition induced by the choice of conformal representative h. If $\widehat{h} = e^{2u}h$ is another choice, with $u \in C^\infty(\partial\overline{M})$, and $\widehat\rho$, $\widehat{y}^i$ denote the corresponding coordinates, then $\widehat\rho = e^u\rho + \mathcal{O}(\rho^2)$, $\widehat{y}^i = y^i + \mathcal{O}(\rho)$. An easy calculation shows that

$$\pm\widehat\rho^{-1}\mathrm{d}\widehat\rho + \sum_i \widehat\eta_i\mathrm{d}\widehat{y}^i = \pm\rho^{-1}\mathrm{d}\rho + \sum_i \widehat\eta_i\mathrm{d}y^i \pm \mathrm{d}u$$

as elements of $\partial_\mp S^*M$. So the identification (2.2) is determined up to the map $(y,\eta)\mapsto(y,\eta\mp \mathrm{d}u(y))$ of $T^*\partial\overline{M}$. This is a symplectomorphism of $T^*\partial\overline{M}$ for each u, so it follows that each of $\partial_\mp S^*M$ has a canonical structure as a symplectic manifold, with symplectic form $\sum_i \mathrm{d}\eta_i\wedge \mathrm{d}y^i$.

The Liouville 1-form α on T^*M is given by $\alpha=\xi_0\mathrm{d}\rho+\eta.\mathrm{d}y$ near the boundary and the symplectic form on T^*M is $\mathrm{d}\alpha=\mathrm{d}\xi_0\wedge\mathrm{d}\rho+\mathrm{d}\eta\wedge\mathrm{d}y$. The form α restricts to S^*M as a contact form, satisfying $\alpha(X)=1$ and $\iota_X\mathrm{d}\alpha=0$. The associated volume form is $\mu=\alpha\wedge(\mathrm{d}\alpha)^n$. We call Liouville symplectic form on $\partial_\pm S^*M$ the symplectic form $\sum_i\mathrm{d}\eta_i\wedge\mathrm{d}y^i$ described in the previous paragraph. We call Liouville volume form on $\partial_\pm S^*M$ the volume form $\mu_\partial=(\sum_i\mathrm{d}\eta_i\wedge\mathrm{d}y^i)^n$ obtained from the Liouville symplectic form. The volume forms μ and μ_∂ induce densities $|\mu|$ and $|\mu_\partial|$ on S^*M and $\partial_\pm S^*M$ called Liouville measures. The flow $\varphi_t: S^*M\to S^*M$ of X preserves the Liouville measure.

Lemma 2.2. — *The Liouville* 1*-form α on S^*M is such that $\rho\alpha$ and $\mathrm{d}\alpha$ extend smoothly to $\overline{S^*M}$ and $\iota_\partial^*(\mathrm{d}\alpha)$ is the symplectic form on $\partial\overline{S^*M}$. The volume form $\mu=\alpha\wedge(\mathrm{d}\alpha)^n$ on S^*M is such that $\rho\mu$ and $\iota_X\mu$ extend smoothly to $\overline{S^*M}$, and $\iota_\partial^*\iota_X\mu$ is equal to the Liouville volume form μ_∂ on $\partial\overline{S^*M}$.*

Proof. — We work on ${}^bT^*\overline{M}$ in the coordinates $(\rho,y,\bar\xi_0,\eta)$. We have

$$\alpha=\xi_0\mathrm{d}\rho+\sum_i\eta_i\mathrm{d}y^i=\rho^{-1}\bar\xi_0\mathrm{d}\rho+\sum_i\eta_i\mathrm{d}y^i. \tag{2.6}$$

Clearly $\rho\alpha$ extends smoothly to all of ${}^bT^*\overline{M}$. Now $\mathrm{d}\alpha=\rho^{-1}\mathrm{d}\bar\xi_0\wedge\mathrm{d}\rho+\sum_i\mathrm{d}\eta_i\wedge\mathrm{d}y^i$. But differentiating $\bar\xi_0{}^2+\rho^2|\eta|^2_{h_\rho}=1$ shows that $\bar\xi_0\mathrm{d}\bar\xi_0=-\rho|\eta|_h^2\mathrm{d}\rho+\mathcal{O}(\rho^2)$ on $T\overline{S^*M}$. Hence

$$\mathrm{d}\alpha=\sum_i\mathrm{d}\eta_i\wedge\mathrm{d}y^i+\mathcal{O}(\rho)\quad\text{on }\ T\overline{S^*M}. \tag{2.7}$$

In particular, $\mathrm{d}\alpha$ extends smoothly to $\overline{S^*M}$ and $\iota_\partial^*(\mathrm{d}\alpha)=\sum_i\mathrm{d}\eta_i\wedge\mathrm{d}y^i$ as claimed. It follows also that $\rho\mu=(\rho\alpha)\wedge(\mathrm{d}\alpha)^n$ and $\iota_X\mu=\alpha(X)(\mathrm{d}\alpha)^n+0=(\mathrm{d}\alpha)^n$ extend smoothly to $\overline{S^*M}$, and $\iota_\partial^*\iota_X\mu=\iota_\partial^*((\mathrm{d}\alpha)^n)=(\iota_\partial^*(\mathrm{d}\alpha))^n=(\sum_i\mathrm{d}\eta_i\wedge\mathrm{d}y^i)^n$. □

Observe from (2.6), (2.7) that

$$\rho\mu=\bar\xi_0\mathrm{d}\rho\wedge\left(\sum_i\mathrm{d}\eta_i\wedge\mathrm{d}y^i\right)^n+\mathcal{O}(\rho).$$

Since $\bar\xi_0=\pm1$ on $\partial_\mp S^*M$, it follows that the orientations induced by $\rho\mu$ and $(\sum_i\mathrm{d}\eta_i\wedge\mathrm{d}y^i)^n$ agree on ∂_+S^*M, but are opposite on ∂_-S^*M.

The boundary behavior of the geodesics of a conformally compact metric was analyzed in [32], where in particular it was proved that the flow φ_t is complete. The following lemma describing the trajectories of the flow lines of X near the boundary is essentially contained in [32]. We formulate the result in terms of $\overline{S^*M}$, and for completeness and for use in our intended applications, we give a proof. Note from (2.3) that Hamilton's equations for the integral curves of X on the level set S^*M are given near $\partial\overline{M}$ by

$$\begin{gathered}\dot\rho = \rho^2\xi_0, \quad \dot y^j = \rho^2\sum_i h_\rho^{ij}\eta_i,\\ \dot\xi_0 = -\frac{1}{\rho} - \frac{\rho^2}{2}\partial_\rho|\eta|^2_{h_\rho}, \quad \dot\eta_j = -\frac{\rho^2}{2}\partial_{y^j}|\eta|^2_{h_\rho}.\end{gathered} \tag{2.8}$$

Lemma 2.3. — *There is $\epsilon > 0$ small enough so that for each $(x,\xi) \in S^*M$ with $\rho(x) < \epsilon$, and $\xi = \xi_0 \mathrm{d}\rho + \eta.\mathrm{d}y$ with $\xi_0 \leqslant 0$, the flow trajectory $\varphi_t(x,\xi)$ converges to a point $z_+ \in \partial_+S^*M$ with rate $\mathcal{O}(e^{-t})$ as $t \to +\infty$ and $\rho(\varphi_t(x,\xi)) \leqslant \rho(x,\xi)$ for all $t \geqslant 0$. In addition, if $A \subset S^*M \cap \{\rho \in (0,\epsilon), \xi_0 \leqslant 0\}$ is a compact set, then the set $\{\varphi_t(x,\xi); (x,\xi) \in A, t \geqslant 0\}$ is contained in a compact set of $\overline{S^*M} \cap \{\rho \in [0,\epsilon), \xi_0 \leqslant 0\}$. The same results hold with $\xi_0 \geqslant 0$ and backward time, with limit $z_- \in \partial_-S^*M$.*

Proof. — First note that for any $z = (x,\xi) = (\rho, y, \xi_0, \eta) \in S^*M$, the trajectory $\varphi_t(x,\xi) = (\rho(t), y(t), \xi_0(t), \eta(t))$ satisfies $\rho(t)^2\xi_0(t)^2 + \rho(t)^2|\eta(t)|^2_{h_{\rho(t)}} = 1$. In particular, $\rho(t)|\eta(t)|_{h_{\rho(t)}}$ is bounded. From (2.8), we see that if $\epsilon > 0$ is small enough and $\xi_0 \leqslant 0$, $\rho \leqslant \epsilon$, then

$$\forall\, t \geqslant 0, \quad \dot\xi_0(t) < -\frac{1}{4\rho(t)} \quad \text{and} \quad \dot\rho(t) \leqslant 0.$$

Thus $u(t) := \rho(t)^{-1}$ satisfies

$$\ddot u = -\dot\xi_0 \geqslant \frac{1}{4}u, \quad u(0) \geqslant \epsilon^{-1}, \quad \dot u(0) \geqslant 0.$$

It follows that $u(t) \geqslant \epsilon^{-1}\cosh(t/2)$, so

$$\rho(t) \leqslant \frac{\epsilon}{\cosh(t/2)}. \tag{2.9}$$

This preliminary decay estimate will be improved below.

Now, differentiating $\bar\xi_0(t) := \rho(t)\xi_0(t)$ by using (2.8), we get

$$\dot{\bar\xi}_0(t) = \bar\xi_0(t)^2 - 1 + \mathcal{O}(\rho(t)^3|\eta(t)|^2_{h_{\rho(t)}}) = (\bar\xi_0(t)^2 - 1)(1 + \mathcal{O}(\rho(t))),$$

where we used $\rho(t)^2|\eta(t)|^2_{h_{\rho(t)}} = 1 - \bar\xi_0(t)^2$ and the remainder is uniform in z. Thus there exists $C > 0$ uniform in (x,ξ) such that

$$\partial_t(F(\bar\xi_0(t))) \leqslant -1 + C\rho(t)$$

where $F(v) = \frac{1}{2}\log\frac{1+v}{1-v}$. Now (2.9) shows that $\int_0^\infty \rho(t)\,\mathrm{d}t < \infty$, so $F(\bar\xi_0(t)) \leqslant -t + C'$. Since $v+1 = 2e^{2F(v)}/\big(e^{2F(v)}+1\big)$, it follows that there is $C>0$ uniform such that for all $t\geqslant 0$

$$0 \leqslant \bar\xi_0(t) + 1 \leqslant Ce^{-2t}. \tag{2.10}$$

This implies that $\rho(t)|\eta(t)|_{h_{\rho(t)}} = \mathcal{O}(e^{-t})$ as $t\to+\infty$ uniformly in (x,ξ), thus from (2.8) we have $y(t)$ and $\eta(t)$ converging exponentially fast to limits for each (x,ξ) and moreover $(y(t),\eta(t))$ stays in a compact set if (x,ξ) is in a fixed compact set of $S^*M\cap\{\rho\in(0,\epsilon),\xi_0\leqslant 0\}$. Now we deduce from this and from (2.10), (2.8) that

$$0\leqslant \dot\rho/\rho+1\leqslant Ce^{-2t}, \quad \rho(0)e^{-t}\leqslant\rho(t)\leqslant C\rho(0)e^{-t} \tag{2.11}$$

where $C>0$ is uniform with respect to the initial condition (x,ξ). Since all the ${}^bT^*\overline{M}$-coordinates $(\rho(t),y(t),\bar\xi_0(t),\eta(t))$ of $\varphi_t(x,\xi)$ converge exponentially with $\rho(t)\to 0$ and $\bar\xi_0(t)\to -1$, it follows that $\varphi_t(x,\xi)$ converges to some point $z_+\in\partial_+S^*M$ as $t\to\infty$. The same argument works in backward time with initial conditions such that $\xi_0\geqslant 0$. □

We remark that one can give an alternate proof of Lemma 2.3 by analyzing the flow of the vector field $\overline{X}$ defined in Lemma 2.1, which is smooth up to $\partial\overline{S^*M}$. We will use such an approach in further analysis of the geodesic flow below.

Lemma 2.3 implies via the duality isomorphism ${}^bT^*\overline{M}\cong{}^{\mathcal{L}}T\overline{M}$ that the tangent vector to the geodesic $\gamma(t)=\pi(\varphi_t(x,\xi))$ has the form $\bar\xi_0(t)\rho(t)\partial_\rho+\rho(t)^2\sum_{ij}h^{ij}_{\rho(t)}\eta_i(t)\partial_{y^j}$ with $\bar\xi_0(t)\to -1$ and $\eta_i(t)$ convergent as $t\to\infty$. Also, as a consequence of Lemma 2.3, we see that the regions $\{\rho\geqslant\epsilon\}$ are strictly convex with respect to the flow for $\epsilon>0$ small enough.

We define the *incoming* (−) *and outgoing* (+) *tails* of the flow by

$$\Gamma_\mp = \{(x,\xi)\in S^*M;\rho(\varphi_t(x,\xi))\not\to 0 \text{ as } t\to\pm\infty\}.$$

These are closed flow-invariant sets in S^*M. By Lemma 2.3, there is $\epsilon>0$ such that

$$\Gamma_-\cap\{\rho<\epsilon,\xi_0\leqslant 0\}=\emptyset,\quad \Gamma_+\cap\{\rho<\epsilon,\xi_0\geqslant 0\}=\emptyset.$$

We define the *trapped set of the flow* to be the compact flow-invariant set

$$K:=\Gamma_-\cap\Gamma_+.$$

Notice that $K\cap\{\rho\leqslant\epsilon\}=\emptyset$ for some $\epsilon>0$ small enough, by Lemma 2.3. We say that (M,g) is *non-trapping* if $K=\emptyset$.

Lemma 2.4. — *(M,g) is non-trapping if and only if $\Gamma_+=\emptyset$ if and only if $\Gamma_-=\emptyset$.*

Proof. — We show that $\Gamma_- \neq \emptyset$ implies $K \neq \emptyset$; the argument for Γ_+ is the same with the direction of time reversed. If $z \in \Gamma_-$, we can choose $t_n \to \infty$ so that $z_n := \varphi_{t_n}(z) \to y$ for some $y \in S^*M$. Then $y \in \Gamma_-$ since Γ_- is closed. But we also have $y \in \Gamma_+$, since otherwise by Lemma 2.3 there would be a small ball B containing y and $\epsilon > 0$, $T > 0$ so that $\varphi_{-t}(B) \subset \{\rho < \epsilon\}$ for all $t > T$. But $z_n \in B$ for large n, and $\varphi_{-t_n}(z_n) = z \notin \{\rho < \epsilon\}$ if ϵ is small enough. Since $t_n \to \infty$ as $n \to \infty$, this is a contradiction. □

Observe that if (M, g) is non-trapping, then M is necessarily simply connected, as otherwise there would be a closed geodesic, and g would have a non-empty trapped set. Indeed, M is diffeomorphic to the subset $\{\rho > \epsilon\}$ when $\epsilon > 0$ is small enough, and its closure is a manifold with strictly convex boundary; if $\pi_1(M)$ is non trivial, we can find a closed geodesic by in each free-homotopy class by the argument of Lemma 2.2 in [19] (by minimizing the energy functional).

Later, we will deal with the two cases where either $K = \emptyset$, or the trapped set K is a hyperbolic set in the sense defined in the introduction. It is shown in Proposition 2.4 of [18] that if K is a hyperbolic set, then for all $\epsilon > 0$ small, $\mathrm{Vol}_\mu(\Gamma_\pm \cap \{\rho \geqslant \epsilon\}) = 0$ (here we can use the results of [18] since $\{\rho \geqslant \epsilon\}$ is a strictly convex set in S^*M). In particular this implies that

$$\mathrm{Vol}_\mu(K) = \mathrm{Vol}_\mu(\Gamma_\pm) = 0 \tag{2.12}$$

in S^*M. We can also define the dual decomposition

$$T^*_K(S^*M) = \mathbb{R}\alpha \oplus E_s^* \oplus E_u^*$$

where $E_u^*(E_u \oplus \mathbb{R}X) = 0$, $E_s^*(E_s \oplus \mathbb{R}X) = 0$, and α is the contact form. As explained in [18, Section 2.3] (see also [9, Lemma 2.10]), the bundle E_s has a continuous extension to Γ_-, denoted E_-, and E_u has a continuous extension to Γ_+, denoted E_+, in a way that the hyperbolicity estimates (1.1) still hold. The dual bundles also have extensions E_-^* over Γ_- and E_+^* over Γ_+, and $E_\pm^*$ are globally invariant by the symplectic flow Φ_t on $T^*(S^*M)$. Here Φ_t is the symplectic lift of the flow φ_t to $T^*(S^*M)$ given by

$$\Phi_t(z, \zeta) = (\varphi_t(z), (\mathrm{d}\varphi_t(z)^{-1})^T.\zeta), \quad \zeta \in T_z^*(S^*M). \tag{2.13}$$

As a consequence of Lemma 2.3, we have the

Corollary 2.5. — *The following maps are well-defined and smooth*

$$B_\pm : S^*M \setminus \Gamma_\mp \to \partial_\pm S^*M, \quad B_\pm(x, \xi) := \lim_{t \to \pm\infty} \varphi_t(x, \xi).$$

Moreover, they extend smoothly to $\overline{S^*M} \setminus \overline{\Gamma_\mp}$, where $\overline{\Gamma_\mp}$ denotes the closure of $\Gamma_\mp$ in $\overline{S^*M}$, and $B_\pm(z) = z$ for each $z \in \partial_\pm S^*M$.

Proof. — If $z_0 := (x_0, \xi_0) \notin \Gamma_-$, for $\epsilon_0 > 0$ as in Lemma 2.3, there is $T > 0$ large enough so that we have $\rho(t) := \rho(\varphi_t(z_0)) < \epsilon_0$ for all $t > T$. There is necessarily an open interval $A \subset [T, \infty)$ where $\rho(t)$ is decreasing, thus $\xi_0(t) < 0$ on $A = (a, b)$ and by Lemma 2.3, $\rho(t)$ is actually decreasing on $[a, +\infty)$ and $\varphi_t(z_0)$ converges to a point in $\partial_+ S^*M$ as $t \to +\infty$; this point is denoted by $B_+(z_0)$. Extend ρ from a neighborhood of $\partial\overline{M}$ to all of $\overline{M}$ so that $\rho > 0$ on M, and write $X = \rho\overline{X}$ as in Lemma 2.1. The flow lines of $\overline{X}$ in S^*M are the same as the flow lines of X, only the parametrization changes: if $\overline{\varphi}_\tau(z)$ is the flow of $\overline{X}$, then

$$\forall\, z \in S^*M, \quad \overline{\varphi}_\tau(z) = \varphi_{t(\tau,z)}(z) \text{ with } t(\tau, z) := \int_0^\tau \frac{1}{\rho(\overline{\varphi}_s(z))}\mathrm{d}s. \tag{2.14}$$

Since $\overline{X}$ is smooth on $\overline{S^*M}$, does not vanish and is transverse to $\partial\overline{S^*M}$, the implicit function theorem gives that there is a finite time $\tau_+(z)$ smooth in z near z_0 such that $\rho(\overline{\varphi}_{\tau_+(z)}(z)) = 0$ for all z near z_0. The map $B_+(z)$ is simply $\overline{\varphi}_{\tau_+(z)}(z)$ and thus is smooth and extends smoothly to $\partial\overline{S^*M} \setminus \overline{\Gamma_-}$ with $B_+(z) = z$ when $z \in \partial_+ S^*M$. The same argument works with B_-. □

As in the proof of Corollary 2.5, we will always denote by $\tau_\pm(z) \geqslant 0$ the time so that

$$\overline{\varphi}_{\pm\tau_\pm(z)}(z) = B_\pm(z), \quad z \notin \overline{\Gamma_\mp}.$$

We also note that the closures can be described by

$$\overline{\Gamma_\pm} = \Gamma_\pm \cup \{z \in \partial_\pm S^*M; \overline{\varphi}_\tau(z) \in \Gamma_\pm, \forall\, \tau, \mp\tau > 0\}. \tag{2.15}$$

We now define the *scattering map* $S_g : \partial_- S^*M \setminus \overline{\Gamma_-} \to \partial_+ S^*M \setminus \overline{\Gamma_+}$ for the flow by

$$S_g(z) = B_+(z) = \overline{\varphi}_{\tau_+(z)}(z). \tag{2.16}$$

Corollary 2.5 shows that S_g is well-defined and smooth.

Proposition 2.6. — *The scattering map $S_g : \partial_- S^*M \setminus \overline{\Gamma_-} \to \partial_+ S^*M \setminus \overline{\Gamma_+}$ is a symplectic map.*

Proof. — Recall from Lemma 2.2 that the symplectic form on $\partial\overline{S^*M}$ is $\iota_\partial(\mathrm{d}\alpha)$. Observe that for each τ, $\overline{\varphi}_\tau^*\mathrm{d}\alpha = \mathrm{d}\alpha$, since $\mathcal{L}_{\overline{X}}(\mathrm{d}\alpha) = d(\iota_{\overline{X}}\mathrm{d}\alpha) = 0$ by the fact that $\iota_{\overline{X}}\mathrm{d}\alpha = \rho^{-1}\iota_X\mathrm{d}\alpha = 0$. Now we can write $S_g(z) = \overline{\varphi}_{\tau_+(z)}(z)$ for each $z \in \partial_- S^*M \setminus \overline{\Gamma_-}$. We thus get for each $v \in T_z(\partial_- S^*M)$

$$dS_g(z).v = \mathrm{d}\overline{\varphi}_{\tau_+(z)}(z).v + \overline{X}(S_g(z))\mathrm{d}\tau_+(z).v.$$

Therefore we get for each $v, w \in T_z(\partial_- S^*M)$

$$\begin{aligned} &S_g^*(\mathrm{d}\alpha)_z(v,w) \\ &= \mathrm{d}\alpha_{S_g(z)}(\mathrm{d}\overline{\varphi}_{\tau_+(z)}(z).v, \mathrm{d}\overline{\varphi}_{\tau_+(z)}(z).w) \\ &\quad + (\mathrm{d}\tau_+(z).v)\mathrm{d}\alpha_{S_g(z)}(\overline{X}(S_g(z)), w) + (\mathrm{d}\tau_+(z).w)\mathrm{d}\alpha_{S_g(z)}(v, \overline{X}(S_g(z))) \\ &= \mathrm{d}\alpha_z(v,w) \end{aligned}$$

since $\iota_{\overline{X}}\mathrm{d}\alpha = 0$ and $\mathrm{d}\alpha_{\overline{\varphi}_\tau(z)}(\mathrm{d}\overline{\varphi}_\tau(z).v, \mathrm{d}\overline{\varphi}_\tau(z).w) = \mathrm{d}\alpha_z(v,w)$ for each $\tau \leqslant \tau_+(z)$. □

Recall that a choice of representative metric h in the conformal infinity of g induces the identifications (2.2) between $\partial_\pm S^*M$ and $T^*\partial\overline{M}$. When comparing S_g for different metrics, we will view S_g as mapping $T^*\partial\overline{M}$ to itself via such identifications. The main reason for this is that then $S_g = S_{\psi^* g}$ if $\psi : \overline{M} \to \overline{M}$ is a diffeomorphism restricting to the identity on the boundary, so long as the identifications between $\partial_\pm S^*M$ and $T^*\partial\overline{M}$ for g and $\psi^* g$ are both with respect to the same metric h. Since $\mathrm{d}\psi$ is generally nontrivial on $\partial_\pm S^*M$, it is not typically the case that $S_g = S_{\psi^* g}$ when S_g and $S_{\psi^* g}$ are viewed as maps from $\partial_- S^*M$ to $\partial_+ S^*M$.

Next we describe the pull-back by the flow.

Lemma 2.7. — *Let $f \in C^\infty(\overline{S^*M})$, then the function $(t,z) \mapsto f(\varphi_t(z))$ is a smooth function on $\mathbb{R} \times S^*M$ which can be written for $t \geqslant 0$ in the form $f(\varphi_{\pm t}(z)) = F_\pm(e^{-t}, z)$ for some function $F_\pm \in C^\infty\big([0,1) \times (\overline{S^*M} \setminus (\partial_\mp S^*M \cup \Gamma_\mp))\big)$ satisfying*

$$F_\pm(e^{-t}, z) = f(B_\pm(z)) + \mathcal{O}(\tau_\pm(z)e^{-t}) \tag{2.17}$$

and the remainder is uniform for z in compact sets of $\overline{S^*M} \setminus (\partial_\mp S^*M \cup \Gamma_\mp)$.

Proof. — The flow $\overline{\varphi}_\tau$ of $\overline{X}$ is a non-complete flow satisfying (2.14). Since $\overline{X}$ is smooth down to $\rho = 0$ and since near each $(y_0, \eta_0) \in \partial_+ S^*M$ we have $\overline{X} = -\partial_\rho + \rho Y$ in the coordinates (ρ, y, η) for some smooth vector field Y near (y_0, η_0), we obtain that $\rho(\overline{\varphi}_\tau(y,\eta))$ for small $\tau \leqslant 0$ is a smooth function of (τ, y, η) such that

$$\rho(\overline{\varphi}_\tau(y,\eta)) = -\tau + \mathcal{O}(\tau^2), \tag{2.18}$$

with remainder uniform for (y,η) in compact sets. Now take $z_0 \notin \Gamma_- \cup \partial_- S^*M$, we can write each point z in a small enough neighborhood of z_0 in $\overline{S^*M} \setminus \partial_- S^*M$ as $z = \overline{\varphi}_{-\tau_+(z)}(B_+(z))$ with $\tau_+(z)$ smooth in z and we get that for $\tau \in [0, \tau_+(z))$, the function $t(\tau, z)$ defined by (2.14) is given by

$$\begin{aligned} t(\tau, z) &= \int_{-\tau_+(z)}^{\tau-\tau_+(z)} \frac{1}{\rho(\bar{\varphi}_s(B_+(z)))} \mathrm{d}s = \int_{\tau_+(z)-\tau}^{\tau_+(z)} \frac{1}{s} \mathrm{d}s + G(\tau, z) \\ &= -\log\left(1 - \frac{\tau}{\tau_+(z)}\right) + G(\tau, z) \end{aligned} \tag{2.19}$$

where G is a smooth function of τ, z, for $\tau \in [0, \tau_+(z)]$ and z in a neighborhood of z_0 in $\overline{S^*M} \setminus (\partial_- S^*M \cup \Gamma_-)$. This implies in particular that for $t \geqslant 0$, $\tau = \tau_+(z) - e^{-t}\tau_+(z)H(e^{-t}, z)$ for some smooth positive function H on $[0, 1) \times \overline{S^*M} \setminus (\partial_- S^*M \cup \Gamma_-)$, thus for $t \geqslant 0$

$$\varphi_t(z) = \bar{\varphi}_{-e^{-t}\tau_+(z)H(e^{-t}, z)}(B_+(z)). \tag{2.20}$$

Thus if $f \in C^\infty(\overline{S^*M})$, then $f(\varphi_t(z))$ is equal to $F_+(e^{-t}, z)$ for some smooth function F_+ in $[0, 1) \times \overline{S^*M} \setminus (\partial_- S^*M \cup \Gamma_-)$, satisfying (2.17). The same argument works with $f(\varphi_{-t}(z))$ for $t \geqslant 0$. $\square$

2.2. Short geodesics

In asymptotically hyperbolic manifolds, there are geodesics that are arbitrarily small when viewed in the conformally compactified manifold. For example, in hyperbolic space viewed as the unit ball, half-circles orthogonal to the unit sphere $\mathbb{S}^n = \partial \mathbb{H}^{n+1}$ with endpoints arbitrarily close to one another are such geodesics. In order to prove the existence of and analyze these geodesics in general, we introduce two types of local coordinates near the boundary $\partial \overline{S^*M}$ and describe the form of $\overline{X}$ in each of them.

Fix $0 < \epsilon$ small. We cover the region $\rho \in [0, \epsilon)$ of $\overline{S^*M}$ by the two types of neighborhoods

$$U_1 := \left\{ (\rho, y, \bar{\xi}_0, \eta) \in \overline{S^*M}; \rho < \epsilon,\ \rho |\eta|_{h_\rho} < \frac{1}{2} \right\},$$

$$U_2 := \left\{ (\rho, y, \bar{\xi}_0, \eta) \in \overline{S^*M}; \rho < \epsilon,\ |\eta|_{h_\rho} > \frac{1}{2} \right\},$$

and we use the coordinates (ρ, y, η) on U_1 and (θ, y, η) on U_2, where $\theta \in [0, \pi]$ is defined by

$$\sin(\theta) = \rho |\eta|_{h_\rho}, \quad \cos(\theta) = \bar{\xi}_0.$$

Notice that U_1 has two connected components $U_1^\pm$ corresponding to $\mathrm{sign}(\bar{\xi}_0) = \pm 1$. For U_2, if ϵ is small enough, the function $\rho \to \rho |\eta|_{h_\rho}$ has positive derivative for $\rho < \epsilon$ so is invertible; the limit $\rho \to 0$ when η is in a compact set corresponds to either $\theta \to 0$ or $\theta \to \pi$. We can recover the coordinate ξ_0 by the expression $\xi_0 = |\eta|_{h_\rho} \cot(\theta)$. A function $f : \overline{S^*M} \to \mathbb{R}$ is smooth if it is smooth on S^*M and if $f|_{U_1}$ viewed in the coordinates (ρ, y, η) extends as a smooth function to $\{\rho = 0\}$, and $f|_{U_2}$ viewed in the coordinates (θ, y, η) extends as a smooth function down to $\{\theta = 0\}$ and up to $\{\theta = \pi\}$. The vector field $\overline{X}$ can be written in the coordinates (ρ, y, η) in U_1 as

$$\overline{X} = \operatorname{sign}(\xi_0)\sqrt{1 - \rho^2|\eta|^2_{h_\rho}}\partial_\rho + \rho\sum_{i,j} h^{ij}_\rho \eta_i \partial_{y^j} - \frac{1}{2}\rho\sum_k \partial_{y^k}|\eta|^2_{h_\rho}\partial_{\eta_k} \tag{2.21}$$

and in the coordinates (θ, y, η) in U_2 as

$$\begin{aligned} Y := |\eta|^{-1}_{h_\rho}\overline{X} = (1 + Q)\partial_\theta \\ + \sin(\theta)\sum_{i,j}\frac{h^{ij}_\rho \eta_i}{|\eta|^2_{h_\rho}}\partial_{y^j} - \frac{1}{2}\sin(\theta)\sum_k \frac{\partial_{y^k}|\eta|^2_{h_\rho}}{|\eta|^2_{h_\rho}}\partial_{\eta_k} \end{aligned} \tag{2.22}$$

with $Q := \frac{\sin\theta}{2|\eta|^3_{h_\rho}}\partial_\rho|\eta|^2_{h_\rho}$. For instance, to evaluate the coefficient of ∂_θ in (2.22), from $\theta = \cos^{-1}\overline{\xi_0}$ one has $\partial_{\overline{\xi_0}}\theta = -(1 - \overline{\xi_0}^2)^{-1/2} = -(\rho|\eta|_{h_\rho})^{-1}$, so from (2.5) there follows

$$|\eta|^{-1}_{h_\rho}\overline{X}\theta = (\rho|\eta|^2_{h_\rho})^{-1}\left[\rho|\eta|^2_{h_\rho} + \frac{1}{2}\rho^2\partial_\rho|\eta|^2_{h_\rho}\right] = 1 + Q.$$

The existence and asymptotics of short geodesics is the content of the following lemma.

Lemma 2.8. — *There exists $R_0 > 0$ so that if $z = (y_0, \eta_0) \in \partial_- S^*M$ with $|\eta_0|_{h_0} > R_0$, then $z \notin \overline{\Gamma}_-$. If we set $R = |\eta_0|_{h_0} > R_0$ and $\delta = R^{-1}$, the integral curves $(\theta(s), y(s), \eta(s))$ of Y have the property that θ, y, and $\delta\eta$ extend smoothly in δ down to $\delta = 0$. Moreover,*

$$\tau_+(z) = \delta\pi + \mathcal{O}(\delta^2), \quad \rho(\overline{\varphi}_\tau(z)) = \delta\sin(\alpha_z(\tau)) + \mathcal{O}(\delta^2)$$

where $\alpha_z : [0, \tau_+(z)] \to [0, \pi]$ is a diffeomorphism depending smoothly on z and satisfying $\partial_\tau\alpha_z(\tau) = R + \mathcal{O}(1)$.

Proof. — We write $X = \sin(\theta)Y$ in the region U_2 of S^*M, where $Y = |\eta|^{-1}_{h_\rho}\overline{X}$ is given by (2.22) and we recall $\sin(\theta) = \rho|\eta|_{h_\rho}$. Denote by $(\theta(s), y(s), \eta(s))$ the integral curve of Y with initial condition $(0, y_0, \eta_0)$ and set $R = |\eta_0|_{h_0}$ with $R > R_0$. Rescale the integral curve equations: set $\delta = R^{-1}$, $u = (y - y_0)/\delta$, $\omega = \delta\eta$. Then $\omega_0 = \delta\eta_0$ has $|\omega_0|_{h_0} = 1$ and the integral curve equations for Y become

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = 1 + \delta\widetilde{Q}, \quad \frac{\mathrm{d}u^i}{\mathrm{d}s} = \sin\theta\,\frac{\sum_j h_\rho^{ij}\omega_j}{|\omega|^2_{h_\rho}}, \quad \frac{\mathrm{d}\omega_i}{\mathrm{d}s} = -\delta\sin\theta\,\frac{\partial_{y^i}|\omega|^2_{h_\rho}}{2|\omega|^2_{h_\rho}} \tag{2.23}$$

with initial conditions $\theta(0) = 0$, $u(0) = 0$, $\omega(0) = \omega_0$, where $\widetilde{Q} = Q/\delta = \frac{\sin\theta}{2|\omega|^3_{h_\rho}}\partial_\rho|\omega|^2_{h_\rho}$. Everywhere the argument of h_ρ and its derivatives is $y_0 + \delta u$ and ρ is determined implicitly as the solution of $\rho|\omega|_{h_\rho} = \delta\sin\theta$. The right-hand sides of these equations are smooth in all arguments $(\theta, u, \omega, y_0, \delta)$, including down to $\delta = 0$. The solution for $\delta = 0$ is

$$\theta = s, \qquad \omega = \omega_0, \qquad u = (1 - \cos s)\omega_0^\sharp,$$

where $\omega_0^\sharp$ is the dual vector to ω_0 using $h_0(y_0)$. This corresponds to the geodesic on the hyperbolic space defined by the metric $\rho^{-2}\big(\mathrm{d}\rho^2 + h_0(y_0)\big)$ with coefficients frozen at y_0. By a standard result [6, Theorem 7.4], there is a solution smooth in $\delta \geqslant 0$ small and for all s up to $\theta(s) = \pi$, whereupon $\rho = 0$ (one may continue slightly further by choosing some smooth extension of h_ρ to $\rho < 0$). Hence the geodesic reaches ∂_+S^*M, so $z \notin \overline{\Gamma}_-$. The implicit function theorem implies that there is a uniquely defined smooth function s_0 for $0 \leqslant \delta$ small for which $\theta(s_0) = \pi$ and $s_0 = \pi$ for $\delta = 0$. We view $s_0(z)$ as a function of $z = (y_0, \eta_0)$ for $R = |\eta_0|_{h_0}$ large. So $s_0(z) = \pi + \mathcal{O}(\delta)$. Since $|\omega|_{h_\rho} = 1$ for all s when $\delta = 0$, it follows that $|\omega|_{h_\rho} = 1 + \mathcal{O}(\delta)$ for $s \in [0, s_0(z)]$, which becomes in terms of the original variables

$$\delta|\eta|_{h_\rho} = 1 + \mathcal{O}(\delta). \tag{2.24}$$

We also have $\rho/(\delta\sin\theta) = |\omega|^{-1}_{h_\rho} = 1 + \mathcal{O}(\delta)$, so $\rho = \delta\sin\theta + \mathcal{O}(\delta^2)$ uniformly for $s \in [0, s_0(z)]$.

Since $Y = |\eta|^{-1}_{h_\rho}\overline{X}$, we also can write the flow $\bar{\varphi}_\tau(z)$ as a reparametrization of the flow of Y (just like in (2.14)) and viewing s as a function of (τ, z) we get $s(\tau, z) = R\tau + \mathcal{O}(\tau)$, $\tau_+(z) = \delta\pi + \mathcal{O}(\delta^2)$ and

$$\partial_\tau s(\tau, z) = |\eta(s(\tau, z))|_{h_\rho} = R + \mathcal{O}(1).$$

Since $\dot{\theta}(s) = 1 + \mathcal{O}(\delta)$, we obtain that $\alpha_z(\tau) := \theta(s(\tau, z))$ satisfies $\partial_\tau\alpha_z(\tau) = R + \mathcal{O}(1)$ and this achieves the proof. □

2.3. Splitting of TS^*M, Sasaki metric, conjugate points

As we discussed previously, for any Riemannian manifold (M, g), the cosphere bundle S^*M is a contact manifold with contact splitting $TS^*M =$

$\mathbb{R}X \oplus \ker \alpha$. Moreover, we have the further splitting

$$\ker \alpha = \mathcal{H} \oplus \mathcal{V}, \tag{2.25}$$

where $\mathcal{V} := \ker \mathrm{d}\pi$ is the vertical bundle and $\mathcal{H} = \ker \alpha \cap \ker \mathcal{K}$ is the horizontal bundle. Here $\mathcal{K} : TT^*M \to TM$ is the connection map, defined by $\mathcal{K}(\zeta) = D_t z(0)^\sharp$, where $\zeta \in T_{(x,\xi)}T^*M$, $z(t)$ is a curve in T^*M with $z(0) = (x, \xi)$, $\dot{z}(0) = \zeta$, D_t is the covariant derivative along the curve $\pi(z(t))$ in M, and $^\sharp$ denotes the canonical isomorphism $T^*M \to TM$ induced by g (see [39, Chapter 1.3.1] for details about $\mathcal{K}$). If $z = (x, \xi) \in S^*M$, then any $\zeta \in \ker \mathcal{K}$, a priori only assumed to be in $T_z T^*M$, is actually already in $T_z S^*M$. If $\mathcal{Z} \to S^*M$ is the bundle whose fibers are $\mathcal{Z}_{(x,\xi)} = \{v \in T_x M : \xi(v) = 0\}$, the maps $\mathrm{d}\pi|_{\mathcal{H}} : \mathcal{H} \to \mathcal{Z}$ and $\mathcal{K}|_{\mathcal{V}} : \mathcal{V} \to \mathcal{Z}$ are isomorphisms. We denote by $\mathcal{L}$ the isomorphism

$$\mathcal{L} : \ker \alpha \to \mathcal{Z} \oplus \mathcal{Z}, \qquad \mathcal{L}(\zeta) = \big(\mathrm{d}\pi(\zeta), \mathcal{K}(\zeta)\big). \tag{2.26}$$

The Sasaki metric G on S^*M is defined by

$$G(\zeta, \zeta') = g(\mathrm{d}\pi(\zeta), \mathrm{d}\pi(\zeta')) + g(\mathcal{K}(\zeta), \mathcal{K}(\zeta')), \quad \zeta, \zeta' \in T_z(S^*M). \tag{2.27}$$

If $z \in S^*M$, the space of normal Jacobi fields along the geodesic $\gamma_z(t) := \pi(\varphi_t(z))$ is isomorphic to $\ker \alpha_z = \mathcal{H}_z \oplus \mathcal{V}_z$. For $\zeta = h + v \in \mathcal{H}_z \oplus \mathcal{V}_z$, the corresponding Jacobi field $Y(t)$ is determined by the initial conditions $\big(Y(0), D_t Y(0)\big) = \mathcal{L}(\zeta) = \big(\mathrm{d}\pi(h), \mathcal{K}(v)\big)$. Two points $p, q \in M$ are said to be *conjugate points* if there exist $z \in S_p^*M$ and $T > 0$ so that $\varphi_T(z) \in S_q^*M$ and

$$\mathrm{d}\varphi_T(z).\mathcal{V}(z) \cap \mathcal{V}(\varphi_T(z)) \neq \{0\}. \tag{2.28}$$

This is equivalent to the statement that there is a normal Jacobi field along γ which vanishes at both 0 and T.

Lemma of [17, p. 201] asserts that if p is a point in a simply connected Riemannian manifold M such that $\exp_p$ is everywhere defined and a local diffeomorphism, then the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism. Since $\exp_p$ is everywhere defined and a local diffeomorphism for each p in a complete manifold with no conjugate points, it follows that the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism at each point in a non-trapping asymptotically hyperbolic manifold with no conjugate points.

3. Boundary value problem and X-ray transform

We first consider the non-trapping case, i.e. $\Gamma_- \cup \Gamma_+ = \emptyset$.

3.1. Resolvent in the non-trapping case

The first boundary value problem we consider is the following:

Lemma 3.1. — *For each $\lambda \in \mathbb{C}$, for each $f \in C_c^\infty(S^*M)$, there is a unique $u_\pm(\lambda) \in C^\infty(S^*M)$ such that*

$$(-X \pm \lambda)u_\pm(\lambda) = f, \ \text{ with } u_\pm(\lambda) = 0 \text{ near } \partial_\pm S^*M.$$

and the operator $R_\pm(\lambda) : C_c^\infty(S^*M) \to C^\infty(SM)$ defined by $R_\pm(\lambda)f = u_\pm(\lambda)$ is continuous and holomorphic in λ.

Proof. — The operator $R_\pm(\lambda)$ is simply given by

$$\begin{aligned} R_+(\lambda)f(z) &= \int_0^\infty e^{-\lambda t} f(\varphi_t(z))\mathrm{d}t, \\ R_-(\lambda)f(z) &= -\int_{-\infty}^0 e^{\lambda t} f(\varphi_t(z))\mathrm{d}t, \end{aligned} \tag{3.1}$$

its continuity and uniqueness are clear. □

We want to extend the action of these operators to $C^\infty(\overline{S^*M})$.

Lemma 3.2. — *The operators of Lemma 3.1 extend to holomorphic families of operators $R_\pm(\lambda) : C^\infty(\overline{S^*M}) \to C^\infty(S^*M)$ for $\mathrm{Re}(\lambda) > 0$ with meromorphic extensions to $\mathbb{C}$ with first order poles at $-\mathbb{N}_0$. The residue of $R_\pm(\lambda)$ at $\lambda = 0$ is the operator $P_\pm$ defined by*

$$P_\pm f = \pm f \circ B_\pm.$$

Proof. — We just consider $R_+(\lambda)$ as $R_-(\lambda)$ is similar. First, we notice that from Lemma 2.3, for z in any compact set $B \subset S^*M$, the curves $(\varphi_t(z))_{t\in\mathbb{R}, z\in B}$ lie in a compact region of $\overline{S^*M}$, thus

$$R_+(\lambda)f(z) = \int_0^\infty e^{-\lambda t} f(\varphi_t(z))\mathrm{d}t$$

converges uniformly on compact sets of S^*M for $\mathrm{Re}(\lambda) > 0$; it is smooth and holomorphic in λ there, and also bounded on $A \cap S^*M$ for each compact set $A \subset \overline{S^*M}$ (by a constant depending on $\mathrm{Re}(\lambda)$). Next, by Lemma 2.7, we can write for z in each open set $A \subset S^*M$ with compact closure in S^*M

$$R_+(\lambda)f(z) = \int_0^\infty e^{-\lambda t} F_+(e^{-t}, z)\mathrm{d}t$$

for some F_+ smooth in $[0,1) \times \overline{S^*M} \setminus \partial_- S^*M$ with $F_+(e^{-t}, z) = f(B_+(z)) + \mathcal{O}(e^{-t}\tau_+(z))$, thus we have by Taylor expansion of $F_+(u, z)$ at $u = 0$ that for $\mathrm{Re}(\lambda) > 0$ and each $N \in \mathbb{N}$

$$R_+(\lambda)f(z) = \sum_{j=0}^{N} \int_0^\infty e^{-\lambda t} F_{+,j}(z) e^{-jt}\mathrm{d}t + \int_0^\infty e^{-\lambda t} e^{-t(N+1)} r_N(t,z)\mathrm{d}t$$
$$= \sum_{j=0}^{N} \frac{F_{+,j}(z)}{\lambda + j} + \int_0^\infty e^{-\lambda t} e^{-t(N+1)} r_N(t,z)\mathrm{d}t$$

for some r_N bounded in $[0,\infty) \times \bar{A}$. The last integral is holomorphic in $\mathrm{Re}(\lambda) > -N-1$ and the first terms admit a meromorphic extension with poles at $-\mathbb{N}_0$. We notice that the residue at 0 is given by the operator $P_+ f := f \circ B_+$. □

We can then define the operators $R_\pm(0) := \lim_{\lambda \to 0^+}(R_\pm(\lambda) - \lambda^{-1} P_\pm)$ acting on $C^\infty(\overline{S^*M})$, which by using (2.17) can also be written (for $z \in S^*M$) as the converging integrals

$$(3.2)\qquad \begin{aligned} R_+(0)f(z) &= \int_0^\infty (f(\varphi_t(z)) - f(B_+(z)))\mathrm{d}t, \\ R_-(0)f(z) &= \int_0^\infty (f(B_-(z)) - f(\varphi_{-t}(z)))\mathrm{d}t. \end{aligned}$$

We define $\rho C^\infty(\overline{S^*M})$ to be the subspace of $C^\infty(\overline{S^*M})$ consisting of smooth functions on $\overline{S^*M}$ which vanish at $\partial \overline{S^*M}$ (such functions f can be factorized as $f = \rho \widetilde{f}$ for some smooth $\widetilde{f}$).

Lemma 3.3. — *The operators $R_\pm(0)$ defined by* (3.2) *extend as continuous operators*

$$R_\pm(0) : C^\infty(\overline{S^*M}) \to C^\infty(\overline{S^*M} \setminus \partial_\mp S^*M)$$

satisfying $(R_\pm(0)f)|_{\partial_\pm S^*M} = 0$ and

$$(3.3)\qquad -XR_\pm(0) = \mathrm{Id} \mp P_\pm.$$

If $f \in C^\infty(\overline{S^*M})$ vanishes at $\partial_\pm S^*M$, then $u_\pm := R_\pm(0)f$ is the unique smooth solution in $\overline{S^*M} \setminus \partial_\mp S^*M$ of the boundary value problem

$$-Xu_\pm = f, \quad u_\pm|_{\partial_\pm S^*M} = 0.$$

Finally $R_\pm(0)$ extend as continuous operators

$$(3.4)\qquad R_\pm(0) : \rho C^\infty(\overline{S^*M}) \to C^\infty(\overline{S^*M}).$$

Proof. — From Lemma 2.7 and (2.17), we notice that $R_\pm(0)f$ extend smoothly to $\partial_\pm S^*M$ and we get near $\partial_\pm S^*M$ (uniformly on compact sets of $\overline{S^*M} \setminus \partial_- S^*M$)

$$(3.5)\qquad R_\pm(0)f(z) = \mathcal{O}(\tau_\pm(z)),$$

thus vanishing at $\partial_\pm S^*M$. The second statement is clear since the difference u of two solutions would be constant along flow lines of X, and thus equal to $u\circ B_\pm = 0$. For the last part, we use (2.14) to get by a change of variable $\tau\mapsto t(\tau,z)$

$$R_+(0)f(z) = \int_0^{\tau_+(z)} f(\bar\varphi_\tau(z))\frac{1}{\rho(\bar\varphi_\tau(z))}\mathrm{d}\tau = \int_0^{\tau_+(z)} \widetilde{f}(\bar\varphi_\tau(z))\mathrm{d}\tau \tag{3.6}$$

if $f=\rho\widetilde{f}$ for some smooth $\widetilde{f}\in C^\infty(\overline{S^*M})$. This proves the last claim of the Lemma and the same argument works with $R_-(0)$. □

3.2. Extension operator and X-ray transform in the non-trapping case

The next boundary value problem for the flow we consider is the extension problem. We have the

Lemma 3.4. — *For each $\omega\in C^\infty(\partial_-S^*M)$, there is a unique $w\in C^\infty(\overline{S^*M})$ such that*

$$Xw=0,\quad w|_{\partial_-S^*M}=\omega$$

and it is given by $w(z)=\omega(B_-(z))$. Its value at ∂_+S^*M is $w|_{\partial_+S^*M}=\omega\circ S_g^{-1}$ where S_g is the scattering map defined by (2.16).

Proof. — The solution w has to satisfy $\overline{X}w=0$ and w is constant along flow lines of $\overline{X}$, thus $w(z)$ is given by $\omega(B_-(z))$. The other part is clear. □

We point out that for compact simple manifolds, Pestov–Uhlmann [43] gives a characterisation of the initial data of smooth solutions of $Xw=0$ on SM in terms of the scattering map. Here, in contrast, the smoothness of the boundary value $w|_{\partial_-S^*M}$ is sufficient. This difference is a consequence of the fact that for us, ∂_-S^*M and ∂_+S^*M are disjoint subsets of $\overline{S^*M}$, whereas they intersect in the tangent directions in the case of compact simple manifolds.

We define the *extension operator* using this Lemma by

$$\mathcal{E}: C^\infty(\partial_-S^*M)\to C^\infty(\overline{S^*M}),\quad \mathcal{E}\omega(z)=\omega(B_-(z)). \tag{3.7}$$

By Lemma 2.3, we also see that $\mathcal{E}: C_c^\infty(\partial_-S^*M)\to C_c^\infty(\overline{S^*M})$.

We can now define the *X-ray transform* operator I by

$$I:\rho C^\infty(\overline{S^*M})\to C^\infty(\partial_-S^*M),\quad If(z):=(R_+(0)f)|_{\partial_-S^*M}. \tag{3.8}$$

We can relate $\mathcal{E}$ to the operator I by the

Lemma 3.5. — *The extension operator* $\mathcal{E}$ *defined by* (3.7) *is the adjoint of* I *with respect to the scalar product induced by the Liouville measures* $|\mu|$ *on* S^*M *and* $|\mu_\partial|$ *on* ∂_-S^*M*.*

Proof. — Let $f \in C_c^\infty(S^*M)$ and $\omega \in C^\infty(\partial_-S^*M)$, then by using $\mathcal{L}_X\mu = 0$, $X\mathcal{E}(\omega) = 0$, $R_+(0)f|_{\partial_+S^*M} = 0$ and Lemma 2.2, we get

$$\begin{aligned}\int_{S^*M}&(f\cdot\mathcal{E}(\omega))|\mu| \\ &= \int_{S^*M} -X(R_+(0)f\cdot\mathcal{E}(\omega))\mu = -\int_{S^*M}\mathcal{L}_X(R_+(0)f\cdot\mathcal{E}(\omega)\mu) \\ &= -\int_{S^*M} d(R_+(0)f\cdot\mathcal{E}(\omega)\iota_X\mu) = \int_{\partial_-S^*M} If\cdot\omega\,|\mu_\partial|\end{aligned}$$

which gives the desired property. The same argument works with ω smooth compactly supported and $f \in \rho C^\infty(\overline{S^*M})$. □

In view of this Lemma, we will instead write I^* instead of $\mathcal{E}$ for what follows when $\mathcal{E}$ acts on $C_c^\infty(\partial_-S^*M)$. Using a similar argument, we also get a Santaló formula

Lemma 3.6. — *Let* $f \in C_c^\infty(S^*M)$*, we have the identity*

$$\int_{S^*M} f|\mu| = \int_{\partial_-S^*M} If(z)|\mu_\partial(z)|.$$

Consequently I *extends to a bounded operator*

$$I : L^1(S^*M, |\mu|) \to L^1(\partial_-S^*M, |\mu_\partial|).$$

Proof. — We just use Stokes formula like in the proof of Lemma 3.5 to get

$$\int_{S^*M} f\mu = -\int_{S^*M} X(R_+(0)f)\mu = \int_{\partial_-S^*M} If\,|\mu_\partial|.$$

The boundedness of I on L^1 just follows by density. □

We next relate the operator I^*I to the resolvents $R_\pm(0)$. First, define the operator

$$\Pi : C^\infty(\overline{S^*M}) \to C^\infty(S^*M), \quad \Pi := R_+(0) - R_-(0). \tag{3.9}$$

It satisfies for each $f \in C^\infty(\overline{S^*M})$, $X\Pi f = P_+f + P_-f$ and thus

$$X\Pi f = 0 \;\text{ if }\; f|_{\partial\overline{S^*M}} = 0.$$

If $f \in \rho C^\infty(\overline{S^*M})$, we can actually write Πf as the converging integral

$$\forall\, z \in S^*M, \quad \Pi f(z) = \int_{-\infty}^{\infty} f(\varphi_t(z))\mathrm{d}t \tag{3.10}$$

(notice from (2.20) that (3.10) also converges if f is any continuous functions on S^*M which is $\mathcal{O}(1/|\log(\rho)|^{\alpha})$ for $\alpha > 1$ near $\partial\overline{S^*M}$). Then we get

Lemma 3.7. — *We have that $\Pi = I^*I$ as operators mapping $C_c^\infty(S^*M)$ to $C^\infty(\overline{S^*M})$, and this extends to the identity $\Pi = \mathcal{E}I$ as operators mapping $\rho C^\infty(\overline{S^*M})$ to $C^\infty(\overline{S^*M})$.*

Proof. — First by (3.2), we have for each $f, f' \in C_c^\infty(S^*M)$ real valued,

$$\langle R_+(0)f, f'\rangle = -\langle f, R_-(0)f'\rangle, \tag{3.11}$$

that is $R_+(0)^* = -R_-(0)$. If $u := R_+(0)f$, we have $u|_{\partial_- S^*M} = If$ and by Stokes formula

$$\langle R_+(0)f, f\rangle = -\int_{S^*M} Xu.u\mu = -\frac{1}{2}\int_{S^*M} X(u^2)\mu = \frac{1}{2}\int_{\partial_- S^*M} (If)^2|\mu_\partial|$$

which shows $\Pi f = I^*If$ using (3.11). By Lemma 3.5 and since $If \in C_c^\infty(\partial_- S^*M)$ if $f \in C_c^\infty(S^*M)$, we get $\Pi = \mathcal{E}I$ on $C_c^\infty(S^*M)$ and thus $\Pi = \mathcal{E}I$ on $\rho C^\infty(\overline{S^*M})$ by density and boundedness of $\mathcal{E}$ on $C^\infty(\overline{S^*M})$. □

We can extend this identity to weighted L^2 spaces by using

Lemma 3.8. — *For $\beta > 1/2$, the operator $I : |\log\rho|^{-\beta}L^2(S^*M, |\mu|) \to L^2(\partial_- S^*M, |\mu_\partial|)$ is bounded and we have $\Pi = I^*I$ as a bounded operator $\Pi : |\log\rho|^{-\beta}L^2(S^*M, |\mu|) \to |\log\rho|^{\beta}L^2(S^*M, |\mu|)$.*

Proof. — Let $\beta > 1$. By Lemma 2.8 and a change of variable $\tau \mapsto \alpha_z(\tau)$ (with $\alpha_z(\tau)$ the function of Lemma 2.8), there is R_0 so that for each $R > R_0$ and all $z = (y, \eta) \in \partial_- S^*M$ with $|\eta|_h = R$, there are $C, C' > 0$ so that

$$\begin{aligned}
&\int_0^{\tau_+(z)} \rho^{-1}|\log\rho|^{-\beta}(\bar\varphi_\tau(z))\mathrm{d}\tau \\
&\qquad\qquad \leqslant CR\int_0^{\tau_+(z)} \sin(\alpha_z(\tau))^{-1}\left|\log\frac{\sin(\alpha_z(\tau))}{R}\right|^{-\beta}\mathrm{d}\tau \\
&\qquad\qquad \leqslant C\int_0^{\pi} \sin(\alpha)^{-1}\left|\log\frac{\sin(\alpha)}{R_0}\right|^{-\beta}(1+\mathcal{O}(1/R))\mathrm{d}\alpha \\
&\qquad\qquad \leqslant C'.
\end{aligned}$$

If $z = (y, \eta)$ is such that $|\eta|_h \leqslant R_0$, the trajectory $\bar\varphi_\tau(z)$ stays in a compact set of $\overline{S^*M}$ and using (2.18), one has

$$\int_0^{\tau_+(z)} \frac{\rho^{-1}}{|\log\rho|^{\beta}}(\bar\varphi_\tau(z))\mathrm{d}\tau \leqslant C$$

for some C uniform in z (depending on R_0). We can then write, by using Lemma 3.6, (3.6) and Cauchy–Schwartz, that for f real-valued and $\beta > 1$

$$\begin{aligned}\|If\|_{L^2}^2 &\leqslant \int_{\partial_- S^*M} \left(\int_0^{\tau_+(z)} \frac{1}{\rho|\log\rho|^\beta}(\overline{\varphi}_\tau(z))\mathrm{d}\tau \right.\\ &\qquad\qquad \left. \times \int_0^{\tau_+(z)} \left(\frac{|\log\rho|^\beta}{\rho} f^2 \right)(\overline{\varphi}_\tau(z))\mathrm{d}\tau \right)|\mu_\partial(z)|\\ &\leqslant C \int_{\partial_- S^*M} \int_0^{\tau_+(z)} (\rho^{-1}|\log\rho|^\beta f^2)(\overline{\varphi}_\tau(z))\mathrm{d}\tau|\mu_\partial(z)|\\ &\leqslant C\||\log\rho|^{\beta/2} f\|_{L^2(S^*M,|\mu|)}\end{aligned}$$

for some $C > 0$ uniform. This proves the claim. □

Next we relate the X-ray transform to the scattering map.

Lemma 3.9. — *Let $f \in C^\infty(\overline{S^*M})$, then*

$$IXf = S_g^*(f|_{\partial_+ S^*M}) - f|_{\partial_- S^*M}.$$

Proof. — We have $Xf \in \rho C^\infty(\overline{S^*M})$, thus IXf makes sense as an element in $C^\infty(\partial_- S^*M)$ and by (3.6), we have for $z \in \partial_- S^*M$

$$\begin{aligned}IXf(z) = \int_0^{\tau_+(z)} (\rho^{-1}Xf)(\overline{\varphi}_\tau(z))\mathrm{d}\tau &= \int_0^{\tau_+(z)} (\overline{X}f)(\overline{\varphi}_\tau(z))\mathrm{d}\tau\\ &= f(\overline{\varphi}_{\tau_+(z)}(z)) - f(z)\end{aligned}$$

which completes the proof. □

Now we characterize the kernel of I in the following Lemma.

Lemma 3.10. — *Let $f \in \rho C^\infty(\overline{S^*M})$, then $If = 0$ if and only if there exists $u \in \rho C^\infty(\overline{S^*M})$ such that $Xu = f$.*

Proof. — If we set $u := -R_+(0)f$, we have $Xu = f$ and $u|_{\partial_+ S^*M} = 0$ and $u \in C^\infty(\overline{S^*M})$ by (3.4). By definition of If, if $If = 0$ then $u|_{\partial_- S^*M} = 0$. Notice that we also have $u = -R_-(0)f$. The converse follows from Lemma 3.9. □

3.3. The case with hyperbolic trapping

In this section, we assume that the trapped set K is a hyperbolic set for the geodesic flow. It has zero Liouville measure and $\Gamma_\pm$ also have zero measure by [18, Section 2.4]. The incoming and outgoing resolvents $R_\pm(\lambda)$ can be defined like in the non-trapping case by the expression (3.1) for

$\mathrm{Re}(\lambda) > 0$. These integrals extend analytically to $\lambda \in \mathbb{C}$ continuously as maps

$$R_\pm(\lambda) : C_c^\infty(S^*M) \to C^\infty(S^*M \setminus \Gamma_\mp).$$

Since each compact set of S^*M is included in some strictly convex manifold $\{z \in S^*M; \rho(z) \geqslant \epsilon\}$ with boundary, we can use [18, Propositions 4.2 and 4.3] which say that

$$R_\pm(0) : H^s_{\mathrm{comp}}(S^*M) \to H^{-s}_{\mathrm{loc}}(S^*M) \cap L^p_{\mathrm{loc}}(S^*M) \tag{3.12}$$

for all $s > 0$ and all $p \in [1, \infty)$, and the wave-front sets of $R_\pm(0)f$ satisfy

$$\mathrm{WF}(R_\pm(0)f) \subset E^*_\mp$$

where $E^*_\mp \subset T^*_{\Gamma_\mp}(S^*M)$ are continuous subbundles over $\Gamma_\mp$ satisfying

$$E^*_-(E_s \oplus \mathbb{R}X) = 0, \quad E^*_+(E_u \oplus \mathbb{R}X) = 0$$

on K. Moreover $R_\pm(0)$ satisfy

$$-XR_\pm(0) = \mathrm{Id}$$

in the distribution sense in S^*M and are given by the expressions

$$\begin{aligned} R_+(0)f(z) &= \int_0^{\tau_+(z)} f(\overline{\varphi}_\tau(z)) \frac{1}{\rho(\overline{\varphi}_\tau(z))} \mathrm{d}\tau, \qquad \forall\, z \notin \Gamma_-, \\ R_-(0)f(z) &= -\int_{-\tau_-(z)}^{0} f(\overline{\varphi}_\tau(z)) \frac{1}{\rho(\overline{\varphi}_\tau(z))} \mathrm{d}\tau, \quad \forall\, z \notin \Gamma_+. \end{aligned} \tag{3.13}$$

Proposition 3.11. — *If the trapped set K is a hyperbolic set, the resolvents $R_\pm(0)$ extend as bounded maps*

$$\rho C^\infty(\overline{S^*M}) \to L^p_{\mathrm{loc}}(S^*M)$$

for all $p < \infty$, $R_\pm(0)f$ extend as functions in $C^\infty(\overline{S^*M} \setminus \overline{\Gamma_\mp})$ where $\overline{\Gamma_\mp}$ are given by (2.15), and $(R_\pm(0)f)|_{\partial_\pm S^*M} = 0$. Finally, as distributions on S^*M, we have

$$\mathrm{WF}(R_\pm(0)f) \subset E^*_\mp.$$

Proof. — First, the expression (3.13) for the resolvent and the arguments of Lemma 3.3 show that for $f \in \rho C^\infty(\overline{S^*M})$, $R_\pm(0)f$ extends as a smooth function in the subset of $\overline{S^*M}$ where $\tau_\pm$ is smooth, i.e. on $\overline{S^*M} \setminus \overline{\Gamma_\mp}$. In particular, $R_\pm(0)f$ is smooth near $\partial_\pm S^*M$ and since $\tau_\pm|_{\partial_\pm S^*M} = 0$, we get $R_\pm(0)f|_{\partial_\pm S^*M} = 0$. If in addition $\mathrm{supp}(f) \cap \Gamma_\pm = \emptyset$, then $R_\pm(0)f$ is easily seen to be in $C^\infty(S^*M)$ ([18, Lemma 4.1]). Let us now show that $R_\pm(0)f$ makes sense as a function in $L^p_{\mathrm{loc}}(S^*M)$ for all $p < \infty$. We consider $R_+(0)f$, as the argument is the same for $R_-(0)f$. In view of (3.12) and the discussion above, it suffices to consider $R_+(0)(\rho\widetilde{f})$ with $\widetilde{f} \in C^\infty(\overline{S^*M})$ supported in an arbitrarily small open set U contained in a small neighborhood of $\overline{\Gamma_+}\cap\{z\in\overline{S^*M};\rho(z)\leqslant\epsilon\}$. If U is a small enough open set then $\overline{\varphi}_{-\tau_0}(U)\subset\{\rho>\delta\}$ for some $\tau_0>0$ and $\delta>0$. The following formula holds for $\widetilde{f}\in C^\infty(U)\cap C^\infty(\overline{S^*M})$

$$\overline{\varphi}_{-\tau_0}^*R_+(0)(\rho\overline{\varphi}_{\tau_0}^*\widetilde{f})=R_+(0)(\rho\widetilde{f})$$

as functions on $S^*M\setminus\Gamma_-$. Now we can use $\mathrm{supp}(\rho\overline{\varphi}_{\tau_0}^*\widetilde{f})\subset\{\rho>\delta\}$, and (3.12) shows that $\overline{\varphi}_{-\tau_0}^*R_+(0)(\rho\overline{\varphi}_{\tau_0}^*\widetilde{f})$ makes sense as a function in $L^p_{\mathrm{loc}}(S^*M)$ for all $p<\infty$, giving a sense to $R_+(0)(\rho\widetilde{f})$ as an element in $L^p_{\mathrm{loc}}(S^*M)$ for all $p<\infty$. The wave-front set of $R_+(0)(\rho\widetilde{f})$ is the flowout of the wave-front set $R_+(0)(\rho\overline{\varphi}_{\tau_0}^*\widetilde{f})$ by the map $\overline{\Phi}_{\tau_0}:=(\overline{\varphi}_{\tau_0},(\mathrm{d}\overline{\varphi}_{\tau_0}^{-1})^T)$ on $T^*(S^*M)$, and is thus contained in E_-^* (using that E_-^* is invariant by the flow Φ_t of (2.13), it is easy to check that $\overline{\Phi}_{\tau_0}(E_-^*)\cap T^*(S^*M)\subset E_-^*$). □

The same argument as Lemma 3.6 shows that for each $f\in C_c^\infty(S^*M\setminus(\Gamma_+\cup\Gamma_-))$, we have the identity

$$\int_{S^*M}f|\mu|=\int_{\partial_-S^*M}\int_0^{\tau_+(z)}f(\overline{\varphi}_\tau(z))\frac{1}{\rho(\overline{\varphi}_\tau(z))}\mathrm{d}\tau|\mu_\partial(z)| \tag{3.14}$$

and using that $S^*M\setminus(\Gamma_+\cup\Gamma_-)$ is an open set of full measure, we can use a density argument to deduce that Santalo's formula (3.14) holds for all $f\in L^1(S^*M,|\mu|)$ and the X-ray transform operator

$$I:L^1(S^*M,|\mu|)\to L^1(\partial_-S^*M,|\mu_\partial|),$$
$$If(z):=\int_0^{\tau_+(z)}f(\overline{\varphi}_\tau(z))\frac{1}{\rho(\overline{\varphi}_\tau(z))}\mathrm{d}\tau$$

is bounded and can also be considered as a map $I:\rho C^\infty(\overline{S^*M})\to C^\infty(\partial_-S^*M\setminus\overline{\Gamma_-})$ by setting $If=(R_+(0)f)|_{\partial_-S^*M\setminus\overline{\Gamma_-}}$.

We obtain a Livsic type theorem similar to Lemma 3.10.

Lemma 3.12. — *Assume that the trapped set K is a hyperbolic set. Let $f\in\rho C^\infty(\overline{S^*M})$, then $If=0$ on $\partial_-S^*M\setminus\overline{\Gamma_-}$ if and only if there exists $u\in\rho C^\infty(\overline{S^*M})$ such that $Xu=f$.*

Proof. — By Proposition 3.11, if we set $u:=-R_+(0)f$, we have $Xu=f$ in the distribution sense with $u\in C^\infty(\overline{S^*M}\setminus\overline{\Gamma_-})\cap L^p_{\mathrm{loc}}(S^*M)$ and $u|_{\partial_+S^*M}=0$. By definition of If, $If=0$ implies that

$$R_+(0)f(z)=R_-(0)f(z)\quad\forall\, z\in S^*M\setminus(\Gamma_-\cup\Gamma_+)$$

but this also implies that $u = -R_-(0)f = -R_+(0)f$ as functions in $L^1_{\rm loc}(S^*M)$. Proposition 3.11 shows that $u \in C^\infty(\overline{S^*M} \setminus K)$, and that

$$\mathrm{WF}(u) \subset E_-^* \cap E_+^* = \{0\}$$

thus $u \in C^\infty(\overline{S^*M})$. We also have $u|_{\partial_- S^*M} = 0$ since $u = -R_-(0)f$, thus $u|_{\partial\overline{S^*M}} = 0$. That $If = 0$ if $f = Xu$ with $u \in \rho C^\infty(\overline{S^*M})$ is straightforward and goes like in Lemma 3.9. □

3.4. Injectivity of X-ray transform on tensors

The gradient of a function $f \in C^\infty(S^*M)$ with respect to the Sasaki metric G splits into

$$\nabla f = (Xf)X + \overset{v}{\nabla} f + \overset{h}{\nabla} f, \quad \text{with } \overset{v}{\nabla} f \in C^\infty(S^*M, \mathcal{V}),\ \overset{h}{\nabla} f \in C^\infty(S^*M, \mathcal{H})$$

where $\mathcal{H}, \mathcal{V}$ are the horizontal and vertical bundles defined in Section 2.3. We can then view $\overset{v}{\nabla} f, \overset{h}{\nabla} f$ as elements in $C^\infty(S^*M; \mathcal{Z})$ using $\mathrm{d}\pi$ and $\mathcal{K}$.

We need to describe the behavior of $u = R_+(0)f$ near $\partial\overline{S^*M}$ when f is a function vanishing to high order at the boundary, and similarly for its derivative.

Lemma 3.13. — *Assume that g is either non-trapping or the trapped set K is hyperbolic. Let $f \in C^\infty(S^*M)$ be a function which can be written as $f = \rho^k \widetilde{f}$ for some $\widetilde{f} \in C^\infty(S^*M)$, and some $k \in (1, \infty)$ and such that*

$$\|\widetilde{f}\|_{L^\infty(S^*M)} < \infty, \quad \|\nabla \widetilde{f}\|_{L^\infty(S^*M)} < \infty$$

with respect to the Sasaki metric G. Let $\epsilon > 0$, the function $u_\pm := R_\pm(0)f$ is such that there is $C_{k,\epsilon} > 0$ such that for all $z \in W_\pm^\epsilon := \{\rho \leqslant \epsilon, \pm\xi_0 \leqslant 0\}$

$$|u_\pm(z)| \leqslant C_{k,\epsilon}\rho(z)^k \|\widetilde{f}\|_{L^\infty}, \quad \|\nabla u_\pm(z)\|_G \leqslant C_{k,\epsilon}\rho(z)^k(\|\nabla\widetilde{f}\|_{L^\infty} + \|\widetilde{f}\|_{L^\infty}).$$

More generally, if $z := (\rho, y, \bar{\xi}_0, \overline{\eta})$ are local coordinates near the boundary of ${}^0S^*\overline{M}$ with $\bar{\xi}_0^2 + \sum_{i,j} h_\rho^{ij}\overline{\eta}_i\overline{\eta}_j = 1$ and if $f \in \rho^\infty C^\infty({}^0S^*\overline{M})$, for all multiindices α, all $N \in \mathbb{N}$ and all $z \in W_\pm^\epsilon$

$$|\partial_z^\alpha u_\pm(z)| \leqslant C_{N,\alpha}\rho(z)^N \|\rho^{-N}\partial_z^\alpha f\|_{L^\infty({}^0S^*\overline{M})}.$$

Proof. — We deal with the case u_+ since the u_- case is similar. From (2.11), if $\epsilon > 0$ is small enough, there is $C > 0$ such that for all $z = (x, \xi) \in W_+^\epsilon$ and all $t \geqslant 0$

$$\rho(\varphi_t(z)) \leqslant C\rho(z)e^{-t} \tag{3.15}$$

which implies that for such z

$$|u_+(z)| \leqslant C_k \rho(z)^k \|\widetilde{f}\|_{L^\infty} \int_0^\infty e^{-kt} \mathrm{d}t \leqslant C_k \rho(z)^k \|\widetilde{f}\|_{L^\infty}.$$

To estimate ∇u, it suffices to estimate $\|\overset{v}{\nabla} u\|_G$ and $\|\overset{h}{\nabla} u\|_G$. Using the decomposition of $\mathrm{d}\varphi_t$ in the splitting (2.25) in terms of Jacobi fields [39, Lemma 1.40], we have for each $V \in \mathcal{V}$ of Sasaki norm 1

$$\begin{aligned} &\left| G(\overset{v}{\nabla} u_+(z), V) \right| \\ &\leqslant C \int_0^\infty \rho^k(\varphi_t(z)) \left(\|\widetilde{f}\|_{L^\infty} \left| \frac{\mathrm{d}\rho}{\rho}(\varphi_t(z)) \right|_g + \|\nabla \widetilde{f}\|_{L^\infty} \right) (|Y_t(z)|_g + |Y_t'(z)|_g) \mathrm{d}t \end{aligned}$$

where $Y_t(z)$ is the Jacobi field solving

$$Y_t''(z) + \mathcal{R}(Y_t(z), \dot{x}(t))\dot{x}(t) = 0, \quad Y_0(z) = 0, \quad Y_0'(z) = V$$

if $\mathcal{R}$ denotes the Riemann curvature tensor of g and $x(t) = \pi(\varphi_t(z))$. Since the sectional curvatures at x are uniformly pinched in $(-1 - c\rho(x), -1 + c\rho(x))$ for some c uniform, and since $\rho(\varphi_t(z)) = \mathcal{O}(e^{-t})$ uniformly in z, we get $\mathcal{R}(Y_t(z), \dot{x}(t))\dot{x}(t) = -Y_t(z) + \mathcal{O}(e^{-t}|Y_t(z)|)$, and by Gronwall's inequality we deduce that there is $C > 0$ so that for each t and each $z \in W_+^\epsilon$

$$|Y_t(z)|_g + |Y_t'(z)|_g \leqslant C e^t.$$

One has $|\mathrm{d}\rho/\rho|_g = 1$ in the region $\rho \leqslant \epsilon$ and using the uniform estimates (3.15) and (2.11), we deduce that there is $C_k > 0$ such that

$$\|\overset{v}{\nabla} u_+(z)\|_G \leqslant C_k \rho(z)^k (\|\widetilde{f}\|_{L^\infty} + \|\nabla \widetilde{f}\|_{L^\infty}).$$

The same argument works with $|\overset{h}{\nabla} u_+(z)|$.

To prove the last statement, we first notice that $|\partial_z^\alpha \varphi_t(z)| \leqslant C_\alpha e^{c_0|\alpha|.|t|}$ for some c_0 by using Gronwall's inequality and the fact that the vector field X has Lipschitz constants uniformly bounded on the compact manifold ${}^0S^*\overline{M}$. Then take $k > N + c_0|\alpha|$, we have

$$|\partial_z^\alpha u_\pm(z)| \leqslant C_\alpha \int_0^\infty \rho^N(\varphi_t(z)) \|\rho^{-N}(\partial_z^\alpha f)\|_{L^\infty} e^{c_0|\alpha|t} \mathrm{d}t$$

and the conclusion follows from (3.15). ☐

We can view a symmetric tensor $f \in C^\infty(M; \otimes_S^m T^*M)$ of rank $m \in \mathbb{N}_0$ as a function on S^*M by the map

$$\pi_m^* : C^\infty(M; \otimes_S^m T^*M) \to C^\infty(S^*M), \quad \pi_m^* f(x, \xi) := f(x)(\otimes^m \xi^\sharp).$$

where $\xi^\sharp$ is the dual to ξ through the metric, i.e. $g(\xi^\sharp, \cdot) = \xi$. If $m = 0$, $\pi_0^* = \pi^*$ is simply the pullback by the base projection $\pi : S^*M \to M$.

Notice also that π_m^* maps $C_c^\infty(M;\otimes_S^m T^*M)$ to $C_c^\infty(S^*M)$. We denote by $\mathcal{D}'(S^*M)$ (resp. $\mathcal{D}'(M;\otimes_S^m T^*M)$) the space of distributions on S^*M, i.e. the dual space to the space $C_c^\infty(S^*M)$ of compactly supported smooth functions on S^*M (resp. dual space to $C_c^\infty(M;\otimes_S^m T^*M)$).

It is straightforward to see that π_m^* maps continuously

$$\pi_m^* : \rho^{-m} C^\infty(\overline{M};\otimes_S^m T^*\overline{M}) \to C^\infty(\overline{S^*M}).$$

The dual operator defined by

$$\pi_{m*} : \mathcal{D}'(S^*M) \to \mathcal{D}'(M,\otimes_S^m T^*M), \quad \langle \pi_m^* f, \widetilde{f}\rangle = \langle f, \pi_{m*}\widetilde{f}\rangle$$

(for each $f \in C_c^\infty(M;\otimes_S^m T^*M), \widetilde{f} \in C_c^\infty(S^*M)$) is also continuous as a map

$$\begin{aligned} &\pi_{m*} : C^\infty(S^*M) \to C^\infty(M,\otimes_S^m T^*M),\\ &\pi_{m*} : \rho^N L^\infty(\overline{S^*M}) \to \rho^{N-m} L^\infty(\overline{M},\otimes_S^m T^*\overline{M}),\\ &\pi_{m*} : \rho^N C^\infty({}^0 S^*\overline{M}) \to \rho^{N-m} C^\infty(\overline{M},\otimes_S^m T^*\overline{M}). \end{aligned} \tag{3.16}$$

If $\mathcal{S}$ denotes the symmetrization operator on tensors, we define the symmetrized derivative as

$$D := \mathcal{S}\nabla$$

where $\nabla : C^\infty(M;\otimes_S^m T^*M) \to C^\infty(M; T^*M \otimes (\otimes_S^m T^*M))$ is the Levi-Civita connection for g. It is easy to check that for $f \in C^\infty(M;\otimes_S^m T^*M)$

$$\pi_{m+1}^* Df = X\pi_m^* f. \tag{3.17}$$

Recall also that for $m = 1$ and f a smooth 1-form, $2Df = \mathcal{L}_{f^\sharp} g$ where $\mathcal{L}$ is the Lie derivative and $f^\sharp$ is the dual vector field to f through g. For a tensor $f = \rho^{-m}\widetilde{f}$ with $\widetilde{f} \in C^\infty(\overline{M};\otimes_S^m T^*\overline{M})$, one has for $|\xi|_g = 1$

$$\|\nabla \pi_m^* f(x,\xi)\|_G \leqslant C_m(|\nabla^{\bar{g}} \widetilde{f}(x)|_{\bar{g}} + |\widetilde{f}(x)|_{\bar{g}}) \tag{3.18}$$

for some constant $C_m > 0$ depending on m. We call X-ray transform on symmetric tensors of rank m the map

$$I_m : \rho^{-m+1} C^\infty(\overline{M};\otimes_S^m T^*\overline{M}) \to C^\infty(\partial_- S^*M \setminus \overline{\Gamma_-}), \quad I_m := I\pi_m^*. \tag{3.19}$$

We want to study the kernel of I_m and we follow the presentation of [41, Section 2] in that aim. The generator X of the flow acts also on smooth sections of $\mathcal{Z}$ by using parallel transport along geodesics: if $v \in \Gamma(\mathcal{Z})$, $v(\varphi_t(x,\xi))$ is a vector field along the geodesic $\pi(\varphi_t(x,\xi))$, and we set

$$Xv(x,\xi) = \nabla_{\partial_t} v(\varphi_t(x,\xi))|_{t=0}$$

(here ∇ is the Levi-Civita connection pulled back to the bundle $\mathcal{Z}$ over S^*M). The adjoints of $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$ acting on compactly supported functions are denoted $\overset{v}{\mathrm{div}}$ and $\overset{h}{\mathrm{div}}$. Finally let $\mathcal{R}_{x,\xi} : \mathcal{Z}_{x,\xi} \to \mathcal{Z}_{x,\xi}$ be the operator defined by $\mathcal{R}_{x,\xi} v = \mathcal{R}(v, \xi^\sharp)\xi^\sharp$ where $\mathcal{R}$ is the Riemann curvature tensor of g. By Lemma 2.1 in [41], we have the commutator formulas

$$[X, \overset{v}{\nabla}] = -\overset{h}{\nabla}, \quad [X, \overset{h}{\nabla}] = \mathcal{R}\overset{v}{\nabla}, \quad \overset{h}{\mathrm{div}}\overset{v}{\nabla} - \overset{v}{\mathrm{div}}\overset{h}{\nabla} = -nX \tag{3.20}$$

where we view $\overset{v}{\nabla} u$ and $\overset{h}{\nabla} u$ as sections of the bundle $\mathcal{Z}$ for $u \in C^\infty(S^*M)$. Let us start with a simple

Lemma 3.14. — *Let $f \in \rho^{-m+1} C^\infty(\overline{M}; \otimes_S^m T^*\overline{M})$ for $m \in \mathbb{N}$. Then there exists a tensor $q \in \rho^{-m+2} C^\infty(\overline{M}; \otimes_S^{m-1} T^*\overline{M})$ such that $\iota_{\partial_\rho}(f - Dq) = 0$ near $\partial\overline{M}$.*

Proof. — We write f as

$$f = \sum_{j=0}^{m} \mathcal{S}(f_j \otimes \mathrm{d}\rho^{m-j})$$

where f_j are tangential symmetric tensors of rank j, i.e. $\iota_{\partial_\rho} f_j = 0$ near $\partial\overline{M}$, and $f_j \in \rho^{-m+1} C^\infty(\overline{M}; \otimes_S^j T^*\partial\overline{M})$. First we recall (see [4, Theorem 1.159]) that for $g = \bar{g}/\rho^2$,

$$\nabla^g_X Y = \nabla^{\bar{g}}_X Y - \frac{\mathrm{d}\rho}{\rho}(X) Y - \frac{\mathrm{d}\rho}{\rho}(Y) X + \rho^{-1} \bar{g}(X, Y)\partial_\rho. \tag{3.21}$$

Using this and the Koszul formula, we have

$$\nabla^g \mathrm{d}\rho = \frac{1}{2}\partial_\rho h_\rho + \frac{\mathrm{d}\rho^2}{\rho} - \frac{h_\rho}{\rho}. \tag{3.22}$$

If α is a smooth tangential 1-form in a collar $(0, \epsilon)_\rho \times \partial\overline{M}$ near $\partial\overline{M}$ (that is $\iota_{\partial\rho}\alpha = 0$), we have from the Koszul formula that $(\nabla^g \alpha)(\partial_\rho, \partial_\rho) = 0$ near $\partial\overline{M}$. If in addition α is smooth up to $\partial\overline{M}$, we also get from (3.21)

$$\nabla^g \alpha = \frac{\mathrm{d}\rho}{\rho} \otimes \alpha + \alpha \otimes \frac{\mathrm{d}\rho}{\rho} + \alpha', \quad \alpha' \in C^\infty(\overline{M}; \otimes^2 T^*\overline{M}). \tag{3.23}$$

We also have for q_0 a smooth function near $\partial\overline{M}$

$$D(q_0 \mathrm{d}\rho^{m-1})(\partial_\rho, \dots, \partial_\rho) = \partial_\rho q_0 + (m-1)\rho^{-1} q_0.$$

To eliminate the $\mathrm{d}\rho^m$ term in f, we have to solve $\partial_\rho(\rho^{m-1} q_0) = \rho^{m-1} f_0$ and, assuming that $f_0 = \mathcal{O}(\rho^{-m+1})$, we thus set (for some $\chi \in C_c^\infty([0, \epsilon))$ equal to 1 near 0)

$$q_0(\rho, y) = \chi(\rho)\rho^{-m+1} \int_0^\rho s^{m-1} f_0(s, y)\mathrm{d}s \in \rho^{-m+2} C^\infty(\overline{M})$$

so that $(f - D(q_0 \mathrm{d}\rho^{m-1}))(\partial_\rho, \ldots, \partial_\rho) = 0$ near $\partial\overline{M}$. This means that $f - D(q_0\mathrm{d}\rho^{m-1}) = \sum_{j=0}^{m-1} \mathcal{S}(\widetilde{f}_j \otimes \mathrm{d}\rho^{m-1-j})$ near $\partial\overline{M}$ for some tensors $\widetilde{f}_j \in \rho^{-m+1}C^\infty(\overline{M}; \otimes_S^{j+1} T^*\partial\overline{M})$. We can then proceed by induction. For a tensor $q_j \in C^\infty((0,\epsilon) \times \partial\overline{M}; \otimes_S^j T^*\partial\overline{M})$, we have from (3.22), (3.23)

$$D(\mathcal{S}(q_j \otimes \mathrm{d}\rho^{m-1-j})) \\ = \mathcal{S}((\partial_\rho q_j + (m-1+j)\rho^{-1}q_j + Aq_j) \otimes \mathrm{d}\rho^{m-j}) + T,$$

where A is a smooth section of $\mathrm{End}(\otimes_S^j T^*\partial\overline{M})$ up to $\rho = 0$, and T is a section of $\mathcal{S}(\mathrm{d}\rho^{m-j-1} \otimes (\otimes_S^{j+1} T^*\partial\overline{M})) \oplus \mathcal{S}(\mathrm{d}\rho^{m-j-2} \otimes (\otimes_S^{j+2} T^*\partial\overline{M}))$. Consequently, for $r_j \in C^\infty((0,\epsilon)\times\partial\overline{M}; \otimes_S^j T^*\partial\overline{M})$, the equation $D(\mathcal{S}(q_j \otimes \mathrm{d}\rho^{m-1-j})) = \mathcal{S}(r_j \otimes \mathrm{d}\rho^{m-j})$ modulo terms in $\mathcal{S}(\mathrm{d}\rho^{m-1-j} \otimes (\otimes_S^{j+1} T^*\partial\overline{M})) \oplus \mathcal{S}(\mathrm{d}\rho^{m-j-2} \otimes (\otimes_S^{j+2} T^*\partial\overline{M}))$ becomes an ODE of the form

$$(\partial_\rho + A)(\rho^{m+j-1}q_j) = \rho^{m+j-1}r_j.$$

If $r_j \in \rho^{-m+1}C^\infty([0,\epsilon)\times\partial\overline{M}; \otimes_S^j T^*\partial\overline{M})$, there is $q_j \in \rho^{-m+2}C^\infty([0,\epsilon) \times \partial\overline{M}; \otimes_S^j T^*\partial\overline{M})$ solving this ODE. Therefore, we can inductively construct $q \in \rho^{-m+2}C^\infty(\overline{M}; \otimes_S^{m-1} T^*\overline{M})$ which satisfies $\iota_{\partial\rho}(f - Dq) = 0$ near $\partial\overline{M}$. □

Next we will show the following

Proposition 3.15. — *Let (M,g) be an asymptotically hyperbolic manifold and $m \geqslant 0$. Let $f \in \rho^{1-m}C^\infty(\overline{M}; \otimes_S^m T^*\overline{M})$ be a tensor satisfying $I_m f = 0$. Then there exists a tensor $q \in \rho^{2-m}C^\infty(\overline{M}; \otimes_S^{m-1} T^*\overline{M})$ such that, for all $N \in \mathbb{N}$, $f - Dq \in \rho^N C^\infty(\overline{M}; \otimes_S^m T^*\overline{M})$. (In particular, in the case $m = 0$, this states that $f \in \rho^\infty C^\infty(\overline{M})$.)*

Proof. — Begin with the case $m = 0$. We will show that if $f = \rho^k \widetilde{f}$ with $\widetilde{f} \in C^\infty(\overline{M})$ and $k \geqslant 1$, for $y_0 \in \partial M$ fixed we will have $\widetilde{f}(0, y_0) = 0$ if $I_0 f = 0$. Since this holds for each y_0, we deduce that $f \in \rho^{k+1}C^\infty(\overline{M})$ and by induction it vanishes to all orders at $\partial\overline{M}$.

Let $R > R_0$ be large and set $\delta = 1/R$ as in the proof of Lemma 2.8, so that the geodesics in S^*M with endpoint in the past given by $(y_0, R\omega_0)$ for $\omega_0 \in T^*_{y_0}\partial M$ fixed with length $|\omega_0|_{h_0} = 1$ are contained in a region $\rho \leqslant C\delta$ where we can use the coordinates (θ, y, η). The proof of Lemma 2.8 shows that $\rho = \delta \sin\theta + \mathcal{O}(\delta^2)$, $y = y_0 + \delta u = y_0 + \mathcal{O}(\delta)$, and $\mathrm{d}\theta/\mathrm{d}t = (\sin\theta)(1 + \mathcal{O}(\delta))$ when θ is viewed as a function of t. Now $\theta : \mathbb{R} \to (0,\pi)$ is a diffeomorphism for δ small, so we can change variable in the integral defining $I_0 f(y_0, R\omega_0)$:

$$\begin{aligned} I_0 f(y_0, R\omega_0) &= \int_{-\infty}^{\infty} (\rho^k \widetilde{f})(\pi(\varphi_t(z)))\mathrm{d}t \\ &= \int_0^{\pi} (\delta \sin\theta)^k \widetilde{f}(\gamma(\theta)) \frac{\mathrm{d}\theta}{\sin\theta} + \mathcal{O}(\delta^{k+1}) \\ &= \delta^k \widetilde{f}(0, y_0) \int_0^{\pi} (\sin\theta)^{k-1} \mathrm{d}\theta + \mathcal{O}(\delta^{k+1}) \end{aligned}$$

where $\gamma(\theta) = (\rho(\theta), y(\theta))$ denotes the θ parametrization of the geodesic starting at $(y_0, R\omega_0)$, and z is a point on the corresponding integral curve. Thus if $I_0 f = 0$, we get $\widetilde{f}(0, y_0) = 0$.

Now we show the case $m \geqslant 1$ similarly. We use Lemma 3.14 and since $I_m(Dq) = 0$, we are reduced to analyze the case where $\iota_{\partial_\rho} f = 0$ near $\partial\overline{M}$. We can assume that the tensor f can be written in the decomposition $[0, \epsilon)_\rho \times \partial\overline{M}$ near a point $y_0 \in \partial\overline{M}$ as

$$f(\rho, y) = \rho^{k-m} \widetilde{f}(\rho, y) = \rho^{k-m} \sum_J \widetilde{f}_J(\rho, y) \mathrm{d}y^J$$

for some $\widetilde{f}_J \in C^\infty(\overline{M})$ and some $k \geqslant 1$, where $\mathrm{d}y^J := \mathrm{d}y^{j_1} \cdots \mathrm{d}y^{j_m}$ if $J = (j_1, \dots, j_m)$ with $1 \leqslant j_i \leqslant n$. Since $X = (\sin\theta)Y$, (2.22) shows that

$$\frac{\mathrm{d}y^j}{\mathrm{d}t} = (\sin\theta)^2 \sum_i \frac{h_\rho^{ij} \eta_i}{|\eta|^2_{h_\rho}} = \delta(\sin\theta)^2 (\omega_0^\sharp)^j + O(\delta^2)$$

where $\omega_0^\sharp$ denotes the dual vector using $h_0(y_0)$. As before, for each $\omega_0 \in T^*_{y_0}\partial\overline{M}$, we have

$$\begin{aligned} I_m f(y_0, R\omega_0) &= \int_{-\infty}^{\infty} (\rho^{k-m} \widetilde{f})(\gamma(t))(\otimes^m \dot{y}(t))\mathrm{d}t \\ &= \delta^k \widetilde{f}(0, y_0)(\otimes^m \omega_0^\sharp) \int_0^{\pi} (\sin\theta)^{k+m-1} \mathrm{d}\theta + \mathcal{O}(\delta^{k+1}). \end{aligned}$$

Thus $\widetilde{f}(0, y_0) = 0$ if $I_m(f) = 0$, which shows by induction on k that f vanishes to all orders at $\partial\overline{M}$. □

Now we prove Theorem 1.1 using Proposition 3.15 and energy identities.

Proof of Theorem 1.1. — First, we use Proposition 3.15. In the case $m = 0$, $f \in \rho^N C^\infty(\overline{M})$ for all $N \in \mathbb{N}$; in the case $m \geqslant 1$, we have $f = Dq + f_1$ with $q \in \rho^{2-m} C^\infty(\overline{M}; \otimes_S^{m-1} T^*\overline{M})$ and $f_1 \in \rho^N C^\infty(\overline{M}; \otimes_S^m T^*\overline{M})$ for all $N \in \mathbb{N}$. Then $\pi_m^* f_1 \in \rho^\infty C^\infty({}^0S^*\overline{M})$. Note that $I_m(Dq) = IX\pi_{m-1}^* q = 0$ by Lemma 3.9, thus $I_m(f_1) = 0$. We can thus replace f by f_1 and to avoid too many notations, we will assume $f = f_1$ for the rest of the proof.

The Pestov identity in the strictly convex region $W_\epsilon := \{z \in S^*M; \rho(z) \geqslant \epsilon\}$ is proved for instance in [41, Proposition 2] but we also need to take into account boundary terms. The manifold W_ϵ has boundary denoted by ∂W_ϵ and the natural volume form on it $\mu_\epsilon := \iota_\epsilon^* \iota_X \mu$ if $\iota_\epsilon : W_\epsilon \to S^*M$ is the inclusion map. We write L^2 for the space $L^2(W_\epsilon, |\mu|)$, then using (3.20), we get for each $u \in C^\infty(S^*M)$

$$\begin{aligned}
&\|\overset{v}{\nabla} Xu\|_{L^2}^2 - \|X\overset{v}{\nabla} u\|_{L^2}^2 \\
&\quad = \langle \overset{v}{\mathrm{div}}\overset{v}{\nabla} Xu, Xu\rangle_{L^2} + \langle X^2\overset{v}{\nabla} u, \overset{v}{\nabla} u\rangle_{L^2} - \int_{\partial W_\epsilon} \langle \overset{v}{\nabla} u, X\overset{v}{\nabla} u\rangle \mu_\epsilon \\
&\quad = \langle (\overset{v}{\mathrm{div}} X^2 \overset{v}{\nabla} - X\overset{v}{\mathrm{div}}\overset{v}{\nabla} X)u, u\rangle_{L^2} - \int_{\partial W_\epsilon} \langle \overset{v}{\nabla} u, (X\overset{v}{\nabla} - \overset{v}{\nabla} X)u\rangle \mu_\epsilon \\
&\quad = \langle (\overset{h}{\mathrm{div}}\overset{v}{\nabla} X - \overset{v}{\mathrm{div}} X\overset{h}{\nabla})u, u\rangle_{L^2} + \int_{\partial W_\epsilon} \langle \overset{v}{\nabla} u, \overset{h}{\nabla} u\rangle \mu_\epsilon \\
&\quad = \langle (\overset{h}{\mathrm{div}}\overset{v}{\nabla} X - \overset{v}{\mathrm{div}}\overset{h}{\nabla} X - \overset{v}{\mathrm{div}}\mathcal{R}\overset{v}{\nabla})u, u\rangle_{L^2} + \int_{\partial W_\epsilon} \langle \overset{v}{\nabla} u, \overset{h}{\nabla} u\rangle \mu_\epsilon \\
&\quad = -\langle (nX^2 + \overset{v}{\mathrm{div}}\mathcal{R}\overset{v}{\nabla})u, u\rangle_{L^2} + \int_{\partial W_\epsilon} \langle \overset{v}{\nabla} u, \overset{h}{\nabla} u\rangle \mu_\epsilon
\end{aligned}$$

and

$$\begin{aligned}
&\|\overset{v}{\nabla} Xu\|_{L^2}^2 - \|X\overset{v}{\nabla} u\|_{L^2}^2 \\
&\qquad = n\|Xu\|_{L^2}^2 - \langle \mathcal{R}\overset{v}{\nabla} u, \overset{v}{\nabla} u\rangle_{L^2} + \int_{\partial W_\epsilon} (\langle \overset{v}{\nabla} u, \overset{h}{\nabla} u\rangle - nuXu)\mu_\epsilon.
\end{aligned}$$

By Lemma 3.10 and Lemma 3.12 we have $R_+(0)\pi_m^* f = R_-(0)\pi_m^* f$, which we denote by $-u \in \rho C^\infty(\overline{S^*M})$, and which satisfies $Xu = \pi_m^* f$. By Lemma 3.13 we have estimates in $\{\rho \leqslant \epsilon, \pm\xi_0 \leqslant 0\}$ for $R_\pm(0)\pi_m^* f$ and thus some estimates on u in $\{\rho \leqslant \epsilon\}$: this implies in particular that u can be extended as a smooth function on ${}^0S^*\overline{M}$ which vanishes to all orders at the boundary $\{\rho = 0\}$, i.e. $u \in \rho^\infty C^\infty({}^0S^*\overline{M})$. Using also (3.18) we deduce that $u, \|\overset{v}{\nabla} u\|_G, \|\overset{h}{\nabla} u\|_G$ are in $\rho^N L^\infty \subset L^2(S^*M)$ for all $N > 0$, and the following functions are also in these spaces

$$\|\overset{v}{\nabla} Xu\|_G = \|\overset{v}{\nabla}\pi_m^* f\|_G, \quad \|X\overset{v}{\nabla} u\|_G \leqslant \|\overset{v}{\nabla}\pi_m^* f\|_G + \|\overset{h}{\nabla} u\|_G.$$

A consequence of this is that we can let $\epsilon \to 0$ to obtain the Pestov identity

$$\|\overset{v}{\nabla} Xu\|_{L^2(S^*M)}^2 - \|X\overset{v}{\nabla} u\|_{L^2(S^*M)}^2 = n\|Xu\|_{L^2(S^*M)}^2 - \langle \mathcal{R}\overset{v}{\nabla} u, \overset{v}{\nabla} u\rangle_{L^2(S^*M)}.$$

If $m = 0$ or $m = 1$, we have $\|\overset{v}{\nabla} Xu\|^2 = m(n-1+m)\|Xu\|^2$ and thus

$$0 = \|X\overset{v}{\nabla} u\|^2_{L^2(S^*M)} + n(1-m)\|Xu\|^2_{L^2(S^*M)} - \langle \mathcal{R}\overset{v}{\nabla} u, \overset{v}{\nabla} u\rangle_{L^2(S^*M)}. \tag{3.24}$$

Let $Z \in C^\infty(S^*M; \mathcal{Z}) \cap \rho^N L^\infty$ so that $|XZ|_G \in \rho^N L^\infty$ and $|X^2 Z|_G \in \rho^N L^\infty$, and define the quadratic form

$$A(Z) = \|XZ\|^2_{L^2(S^*M)} - \langle \mathcal{R}Z, Z\rangle_{L^2(S^*M)}.$$

We first claim that $A(Z) \geqslant 0$ for all such Z. Indeed, let $\chi_\epsilon(x) = \chi(\rho(x)/\epsilon) \in C_c^\infty(S^*M)$ with $\chi \in C^\infty([0,\infty))$ equal to 0 in $\rho \in [0, 1/2]$ and 1 near $\rho = 1$; then $\operatorname{supp} X(\chi_\epsilon) \subset \{\rho \in [\epsilon/2, \epsilon]\}$ and $|X(\chi_\epsilon)| \leqslant C\epsilon^{-1}$. Write $Z_\epsilon = Z\chi_\epsilon$, which is compactly supported. Since

$$A(Z_\epsilon) = \int_{S^*M} \chi_\epsilon(|XZ|_g^2 - \langle \mathcal{R}Z, Z\rangle_g)|\mu| \\ + \int_{S^*M} (|X(\chi_\epsilon)Z|_g^2 + X(\chi_\epsilon^2)\langle Z, XZ\rangle_g)|\mu|$$

we deduce that

$$\lim_{\epsilon\to 0} A(Z_\epsilon) = A(Z).$$

Now, we can use the fact that g has no conjugate points, thus for each geodesic $\gamma_{x,\xi}$ with initial condition $(x,\xi) \in S^*M$, the index form satisfies for each smooth vector field Z in $\mathcal{Z}$ along $\gamma_{x,\xi}$ with $Z(0) = Z(T) = 0$

$$\int_0^T \big(|\nabla_{\partial_t} Z(t)|_g^2 - \langle \mathcal{R}_{x(t)}(Z(t), v(t))v(t), Z(t)\rangle_g\big)\mathrm{d}t \geqslant 0,$$

where $x(t) = \pi(\varphi_t(x,\xi))$ and $v(t) = \dot{x}(t)$. Decomposing $A(Z_\epsilon)$ along geodesics using Lemma 3.6 (or (3.14) in case with trapping)

$$\begin{aligned} &A(Z_\epsilon) \\ &= \int_{\partial_- S^*M\setminus\overline{\Gamma_-}} \int_0^\infty \big(|\nabla_{\partial_t} Z_\epsilon(t)|_g^2 - \langle \mathcal{R}_{x(t)}(Z_\epsilon(t), v(t))v(t), Z_\epsilon(t)\rangle_g\big)\mathrm{d}t|\mu_\partial| \\ &\geqslant 0. \end{aligned}$$

We conclude that $A(Z) \geqslant 0$ as announced.

We next claim that if Z is as in the previous paragraph and $A(Z) = 0$, then $Z = 0$. When restricted to a non-trapped geodesic $\{\varphi_t(x,\xi); t \in \mathbb{R}\}$, Z is viewed as a vector field $Z(t)$ in $\mathcal{Z}_{\varphi_t(x,\xi)}$ along $\gamma_{x,\xi} := \pi(\{\varphi_t(x,\xi); t \in \mathbb{R}\})$, and it satisfies $|Z| = \mathcal{O}(e^{-N|t|})$ and $|\nabla_{\partial_t} Z| = |XZ| = \mathcal{O}(e^{-N|t|})$ as $t \to \pm\infty$ by (2.11). We claim that if $A(Z) = 0$, then for each geodesic $\gamma_{x,\xi} := \pi(\{\varphi_t(x,\xi); t \in \mathbb{R}\})$, we have $Z''(t) + \mathcal{R}_{x(t)}(Z(t), v(t))v(t) = 0$, i.e. Z is a Jacobi field which in turn vanishes faster than any exponential as

$t \to \pm\infty$. Indeed, if $A(Z) = 0$, since $A(Z + sY) \geqslant 0$ for all $s \in \mathbb{R}$ and all $Y \in C_c^\infty(S^*M; \mathcal{Z})$, we have by differentiating at $s = 0$ that

$$\int_{S^*M} (\langle XZ, XY\rangle_g - \langle \mathcal{R}Z, Y\rangle_g)|\mu| = 0$$

and thus by integrating by parts, $X^2 Z + \mathcal{R}Z = 0$. Restricting this identity to the geodesic $\gamma_{x,\xi}$ gives that $Z(t)$ is a Jacobi field vanishing faster than any exponential at $\pm\infty$. Any Jacobi field vanishing faster than $e^{-|t|}$ as $t \to \infty$ or $t \to -\infty$ must vanish identically (see Lemma 5.8), which shows that $Z = 0$ on the set of non-trapped geodesics and thus everywhere by density (recall $\mathrm{Vol}(\Gamma_\pm) = 0$).

Now, using this with $Z = \overset{v}{\nabla} u$ in (3.24), we obtain that $\pi_0^* f = Xu = 0$ when $m = 0$, showing (1). When $m = 1$ we get $\overset{v}{\nabla} u = 0$, which means that $u = \pi_0^* q$ for $q = c_n \pi_{0*} u$ with $c_n > 0$ depending only on n. By (3.16) we deduce that $q \in \rho^\infty C^\infty(\overline{M})$, which shows (2).

To conclude, we consider the case (3) of a symmetric tensor f in $\ker I_m$. We assume that the curvature is non-positive, so that the flow is 1-controlled in the sense of [41, Section 4]. We use the proof of [41, Theorem 10.1], which applies almost verbatim in our case. If $u \in \rho^\infty C^\infty({}^0S^*\overline{M})$ satisfies $Xu = \pi_m^* f$ (just as above for $m = 0, 1$), we decompose u into eigenmodes of the vertical Laplacian $\overset{v}{\Delta} := \overset{v}{\mathrm{div}} \overset{v}{\nabla}$, $u = \sum_{k=0}^\infty u_k$. We recall from [41] that these eigenmodes generate subspaces $\Omega_k \subset C^\infty(S^*M)$, and the operator X maps $\Omega_k \to \Omega_{k-1} \oplus \Omega_{k+1}$. Let $\widetilde{u} = u - \sum_{k \leqslant m-1} u_k$; then the same arguments as in the proof of Theorem 10.1 of [41] show that $X(\overset{v}{\Delta}\widetilde{u} + (m(m+1+n) + n)\widetilde{u}) = 0$. But $\overset{v}{\Delta}\widetilde{u}$ and $\widetilde{u}$ decay to all orders at $\partial\overline{S^*M}$, thus $\overset{v}{\Delta}\widetilde{u} + (m(m+1+n)+n)\widetilde{u} = 0$ and hence $\widetilde{u}_k = 0$ for all $k \neq m+1$. Then $X\widetilde{u} = Xu_{m+1} = X_- u_{m+1}$ with $X_\pm : \Omega_k \to \Omega_{k\pm 1}$ being the differential operators so that $Xw = X_+ w + X_- w$ for $w \in \Omega_k$. In particular we obtain $X_+ u_{m+1} = 0$ and that is equivalent to u_{m+1} being the lift by π_{m+1}^* of a trace-free conformal Killing tensor. But since u_{m+1} decays to all orders at $\partial\overline{S^*M}$, the conformal Killing tensor vanishes at the boundary $\partial\overline{M}$ and a standard Weitzenbock formula shows that $u_{m+1} = 0$ (see for example the proof in [20, Proposition 6.6]). By using (3.16), this implies that $u = \sum_{k \leqslant m-1} u_k$ is of the form $\pi_{m-1}^* q$ for some $q \in \rho^\infty C^\infty(\overline{M}; \otimes_S^{m-1} T^*\overline{M})$, which satisfies $Xu = \pi_m^* f$. By (3.17), we have $Dq = f$. □

Remark 3.16. — The proof above actually shows injectivity of I_m on spaces with weaker regularity and decay. For example for I_0, it is only required that the quantities in the Pestov identity are finite, the boundary term at $\rho = \epsilon$ tends to 0 and $A(Z) \geqslant 0$. For Cartan–Hadamard manifolds (complete simply connected manifolds with non-positive curvature), sharp conditions on the decay are given in [30] in dimension 2 and in [31] in dimensions greater than 2.

4. Renormalized length of geodesics

4.1. Renormalized length

First, we define the renormalized length.

Lemma 4.1. — *Let (M,g) be an asymptotically hyperbolic manifold and ρ a geodesic defining function. For each $z \in \partial_- S^*M \setminus \overline{\Gamma_-}$, the function $\lambda \mapsto I_0(\rho^\lambda)(z)$ has a meromorphic extension from* $\mathrm{Re}(\lambda) > 0$ *to* $\mathrm{Re}(\lambda) > -1$, *with only a simple pole at $\lambda = 0$ and residue*

$$\mathrm{Res}_{\lambda=0}\, I_0(\rho^\lambda) = 2.$$

The regular value $L_g(z) := (I_0(\rho^\lambda)(z) - 2/\lambda)|_{\lambda=0}$ for each $z \in \partial_- S^*M \setminus \overline{\Gamma_-}$ is also given by

$$L_g(z) = \lim_{\epsilon \to 0} \Big(\ell_g(\gamma_z \cap \{\rho > \epsilon\}) + 2 \log \epsilon \Big) \tag{4.1}$$

where γ_z is the geodesic $\pi(\overline{\varphi}_\tau(z))_{\tau \in (0,\tau_+(z))}$.

Proof. — (3.6) shows that for $z \in \partial_- S^*M \setminus \overline{\Gamma_-}$ and $\mathrm{Re}(\lambda) > 0$ we can write

$$I_0(\rho^\lambda)(z) = \int_0^{\tau_+(z)} \rho(\overline{\varphi}_\tau(z))^{\lambda-1} \mathrm{d}\tau = \int_0^\delta + \int_\delta^{\tau_+(z)-\delta} + \int_{\tau_+(z)-\delta}^{\tau_+(z)}.$$

The analogue of (2.18) near $\partial_- S^*M$ shows that if $\delta > 0$ is small enough, then there is $f(\tau, z)$ smooth for $\tau \in [0, \delta]$ with $|\tau f(\tau, z)| < 1/2$ so that $\rho(\overline{\varphi}_\tau(z)) = \tau(1 + \tau f(\tau, z))$. Integration by parts shows that

$$\int_0^\delta \rho(\overline{\varphi}_\tau(z))^{\lambda-1} \mathrm{d}\tau = \int_0^\delta \tau^{\lambda-1}(1 + \tau f(\tau, z))^{\lambda-1} \mathrm{d}\tau = \frac{1}{\lambda} + p(\lambda)$$

where $p(\lambda)$ is holomorphic in $\mathrm{Re}(\lambda) > -1$. Likewise $\int_{\tau_+(z)-\delta}^{\tau_+(z)} \rho(\overline{\varphi}_\tau(z))^{\lambda-1} \mathrm{d}\tau$ has the same form. Since $\int_\delta^{\tau_+(z)-\delta} \rho(\overline{\varphi}_\tau(z))^{\lambda-1} \mathrm{d}\tau$ is an entire function of λ, the first statement in the Lemma is proved.

Next, we see that if we set $\rho(t) := \rho(\varphi_t(z_0))$ for z_0 a point on the orbit $\{\overline{\varphi}_\tau(z); \tau \in (0, \tau_+(z))\}$, then for any $\epsilon > 0$

$$\lim_{\lambda \to 0} \left(I_0(\rho^\lambda)(z) - \frac{2}{\lambda} \right) = \int_{\rho(t) > \epsilon} \mathrm{d}t + \lim_{\lambda \to 0} \left(\int_{\rho(t) < \epsilon} \rho(t)^\lambda \mathrm{d}t - \frac{2}{\lambda} \right).$$

By (2.21), we have for $\pm t > 0$ large enough $\partial_t \rho(t) = X(\rho)(\varphi_t(z_0)) = \mp\rho(t) + \mathcal{O}(\rho(t)^2)$, so we can change variable $u := \rho(t)$ and write

$$\int_{\rho(t)<\epsilon} \rho(t)^\lambda \mathrm{d}t = 2\int_0^\epsilon u^\lambda \frac{\mathrm{d}u}{u(1+\mathcal{O}(u))} = \frac{2\epsilon^\lambda}{\lambda} + \int_0^\epsilon u^\lambda a(u)\mathrm{d}u$$

where $a(u)$ is continuous in $[0,1]$. Thus

$$\lim_{\lambda\to 0}\left(\int_{\rho(t)<\epsilon} \rho(t)^\lambda \mathrm{d}t - \frac{2}{\lambda}\right) = 2\log\epsilon + \mathcal{O}(\epsilon)$$

which completes the proof by letting $\epsilon \to 0$. □

Definition 4.2. — *The function $L_g : \partial_- S^*M\backslash\overline{\Gamma_-}$ defined in Lemma 4.1 is called the* renormalized length function *associated to ρ.*

If $\widehat{\rho}$ is an arbitrary boundary defining function (not necessarily geodesic), it can be written as $\widehat{\rho} = \rho e^\omega$ for some geodesic boundary defining function ρ, and the same argument as in Lemma 4.1 shows that $I_0(\widehat{\rho}^\lambda)$ has a meromorphic extension to $\mathrm{Re}(\lambda) > -1$ with a pole at $\lambda = 0$ and residue $\mathrm{Res}_{\lambda=0} I_0(\widehat{\rho}^\lambda) = 2$. Moreover, one has

$$\begin{aligned}\left[I_0(\widehat{\rho}^\lambda)(z) - I_0(\rho^\lambda)(z)\right]\big|_{\lambda=0} &= \lim_{\lambda\to 0} I_0(\rho^\lambda(e^{\lambda\omega}-1))(z) = \lim_{\lambda\to 0} \lambda I_0(\rho^\lambda\omega)(z)\\ &= \mathrm{Res}_{\lambda=0} I_0(\rho^\lambda\omega)(z).\end{aligned}$$

By the same argument as in Lemma 4.1, it is direct to evaluate this residue to obtain

$$\left[I_0(\widehat{\rho}^\lambda)(z) - I_0(\rho^\lambda)(z)\right]\big|_{\lambda=0} = \omega(\pi(z)) + \omega(\pi(S_g(z))).$$

So the renormalized length function associated to any defining function can be defined as in Lemma 4.1, and it satisfies

$$\widehat{L}_g(z) - L_g(z) = \omega(\pi(z)) + \omega(\pi(S_g(z))). \tag{4.2}$$

Note that two defining functions determine the same L_g if they induce the same representative for the conformal infinity. In particular, a general defining function determines the same L_g as the geodesic defining function inducing the same representative for the conformal infinity.

It is evident that if $\psi : \overline{M} \to \overline{M}$ is a diffeomorphism which restricts to the identity on $\partial\overline{M}$, then $L_g = L_{\psi^* g}$, where both renormalized lengths are calculated with respect to the same representative for the conformal infinity.

4.2. Boundary determination

In this section we prove Theorems 1.3 and 1.4. In both proofs we will use the observation that if g, g' are two asymptotically hyperbolic metrics on M and h, h' are representative metrics for their respective conformal infinities, there exists a diffeormorphism $\psi : \overline{M} \to \overline{M}$ equal to the identity on $\partial\overline{M}$ so that in the product decomposition $[0,\epsilon]_\rho \times \partial\overline{M}$ for g induced by h, one has

$$g = \frac{\mathrm{d}\rho^2 + h_\rho}{\rho^2}, \qquad \psi^* g' = \frac{\mathrm{d}\rho^2 + h'_\rho}{\rho^2} \tag{4.3}$$

where h_ρ and h'_ρ are smooth 1-parameter families of metrics on $\partial\overline{M}$ with $h_0 = h$, $h'_0 = h'$. In fact, if $\chi, \chi' : [0,\epsilon]_\rho \times \partial\overline{M} \to \overline{M}$ are the boundary identification maps for g, g' corresponding to h, h', meaning that

$$\chi^* g = \frac{\mathrm{d}\rho^2 + h_\rho}{\rho^2}, \qquad \chi'^* g' = \frac{\mathrm{d}\rho^2 + h'_\rho}{\rho^2},$$

then one can take ψ to be an extension of $\chi' \circ \chi^{-1}$ to $\overline{M}$. Note that if g and g' are real-analytic, then ψ can be taken to be real-analytic near $\partial\overline{M}$.

Proof of Theorem 1.3. — Choose ψ as in (4.3); we will show that $L_g = L_{g'}$ implies that $h_\rho = h'_\rho$ to infinite order.

We work with one metric g in normal form and use the short geodesics derived in Lemma 2.8 to show that L_g determines the Taylor expansion of h_ρ at $\rho = 0$.. Fix $y_0 \in \partial\overline{M}$ and consider the renormalized length $L_g(y_0, \eta_0)$ (using ρ) where we write $\eta_0 = \delta^{-1}\omega_0$ with δ small and $0 \neq \omega_0 \in T^*_{y_0}\partial\overline{M}$ fixed, but, at least to start, not necessarily satisfying $|\omega_0|_{h_0} = 1$. (2.22) implies that $X\theta = \sin\theta(1+Q)$, where $Q = \frac{\sin\theta}{2|\eta|^3_{h_\rho}}\partial_\rho|\eta|^2_{h_\rho}$. So we can rewrite the integral (with z any point on the orbit $\overline{\varphi}_\tau(y_0,\eta_0)$) for $\mathrm{Re}(\lambda) > 0$

$$\int_{\mathbb{R}} \rho(\pi(\varphi_t(z))^\lambda \mathrm{d}t = \int_0^\pi (\sin\theta)^{\lambda-1} |\eta|^{-\lambda}_{h_\rho} \frac{\mathrm{d}\theta}{(1+Q)}.$$

Since $\lim_{\lambda\to 0}\left(\int_0^\pi (\sin\theta)^{\lambda-1}\,\mathrm{d}\theta - 2/\lambda\right) = 2\log 2$, we have

$$\begin{aligned}
&L_g(y_0, \delta^{-1}\omega_0) \\
&= \lim_{\lambda\to 0}\left(\int_0^\pi (\sin\theta)^{\lambda-1}\left(\frac{|\eta|^{-\lambda}_{h_\rho}}{1+Q}\right)\mathrm{d}\theta - 2/\lambda\right) \\
&= \lim_{\lambda\to 0}\int_0^\pi (\sin\theta)^{\lambda-1}\left(\frac{|\eta|^{-\lambda}_{h_\rho}}{1+Q} - 1\right)\mathrm{d}\theta + 2\log 2
\end{aligned}$$

$$= \lim_{\lambda\to 0}\left[\int_0^\pi \frac{(\sin\theta)^{\lambda-1}}{1+Q}\Big(|\eta|_{h_\rho}^{-\lambda}-1\Big)\mathrm{d}\theta + \int_0^\pi (\sin\theta)^{\lambda-1}\Big(\frac{1}{1+Q}-1\Big)\mathrm{d}\theta\right] + 2\log 2$$

$$= \lim_{\lambda\to 0}\int_0^\pi \frac{(\sin\theta)^{\lambda-1}}{1+Q}\Big(|\eta|_{h_\rho}^{-\lambda}-1\Big)\mathrm{d}\theta - \int_0^\pi \frac{Q}{(\sin\theta)(1+Q)}\mathrm{d}\theta + 2\log 2.$$

If $f(\lambda,\theta)$ is a smooth function satisfying $f(0,\theta)=0$, then the same argument as in the proof of Lemma 4.1 shows that

$$\lim_{\lambda\to 0}\int_0^\pi (\sin\theta)^{\lambda-1} f(\lambda,\theta)\,\mathrm{d}\theta = \partial_\lambda f(0,0) + \partial_\lambda f(0,\pi).$$

This can be used to evaluate the limit in the last line above, giving

$$\begin{aligned}(4.4)\quad & L_g(y_0,\delta^{-1}\omega_0)\\ &= -\log(|\eta|_{h_\rho})|_{\theta=0} - \log(|\eta|_{h_\rho})|_{\theta=\pi} + 2\log 2 - \delta\int_0^\pi \frac{\widetilde{Q}}{(\sin\theta)(1+\delta\widetilde{Q})}\mathrm{d}\theta\\ &= 2\log 2\delta - \log(|\omega|_{h_\rho})|_{\theta=0} - \log(|\omega|_{h_\rho})|_{\theta=\pi} - \delta\int_0^\pi \frac{\widetilde{Q}}{(\sin\theta)(1+\delta\widetilde{Q})}\mathrm{d}\theta\end{aligned}$$

where $\widetilde{Q} = Q/\delta = \frac{\sin\theta}{2|\omega|_{h_\rho}^3}\partial_\rho|\omega|_{h_\rho}^2$ as in the proof of Lemma 2.8. Thus

$$L_g(y_0,\delta^{-1}\omega_0) = 2\log 2\delta - 2\log|\omega_0|_{h_0} + O(\delta).$$

Taking $\delta\to 0$, this shows that $L_g(y_0,\eta_0)$ determines $|\omega_0|_{h_0}$, thus the metric h_0.

Henceforth assume that $|\omega_0|_{h_0}=1$. (4.4) shows that $L_g(y_0,\delta^{-1}\omega_0)$ determines

$$(4.5)\qquad F(\delta) := -\log(|\omega|_{h_\rho})|_{\theta=\pi} - \frac{\delta}{2}\int_0^\pi \frac{\partial_\rho|\omega|_{h_\rho}^2}{|\omega|_{h_\rho}^3}(1+\delta\widetilde{Q})^{-1}\mathrm{d}\theta.$$

Now $F(\delta)$ is a smooth function of δ down to $\delta=0$ which satisfies $F(\delta)=\mathcal{O}(\delta)$. We show that the Taylor expansion of F at $\delta=0$ determines the Taylor expansion of h_ρ at $\rho=0$. Denote $'=\partial_\delta|_{\delta=0}$, write $h(\rho,y)=h_\rho(y)$, and recall the solution $u(\theta)=(1-\cos\theta)\omega_0^\sharp$, $\omega(\theta)=\omega_0$ for $\delta=0$ derived in the proof of Lemma 2.8. Differentiation of (4.5) gives

$$\begin{aligned} F'(0) &= -\frac{1}{2}\big(|\omega|^2_{h_\rho}|_{\theta=\pi}\big)' - \frac{\pi}{2}\partial_\rho h(0,y_0)(\omega_0,\omega_0) \\ &= -\frac{1}{2}\Big(h(0,y_0+\delta u(\pi))(\omega(\pi),\omega(\pi))\Big)' - \frac{\pi}{2}\partial_\rho h(0,y_0)(\omega_0,\omega_0) \\ &= -\sum_k (\omega_0)^k(\partial_{y^k}h)(0,y_0)(\omega_0,\omega_0) - h(0,y_0)(\omega(\pi)',\omega_0) \\ &\qquad\qquad - \frac{\pi}{2}\partial_\rho h(0,y_0)(\omega_0,\omega_0). \end{aligned}$$

It is clear that with the possible exception of $\omega(\pi)'$, the first two terms are determined by h_0. This is the case for $\omega(\pi)'$ as well: taking θ as independent variable, (2.23) becomes

$$\text{(4.6)}\quad \frac{\mathrm{d}u^i}{\mathrm{d}\theta} = \sin\theta\,\frac{\sum_j h^{ij}_\rho\omega_j}{|\omega|^2_{h_\rho}}(1+\delta\widetilde{Q})^{-1},\quad \frac{\mathrm{d}\omega_i}{\mathrm{d}\theta} = -\delta\sin\theta\,\frac{\partial_{y^i}|\omega|^2_{h_\rho}}{2|\omega|^2_{h_\rho}}(1+\delta\widetilde{Q})^{-1}.$$

The linearization of the second equation about $\delta = 0$ is

$$\frac{\mathrm{d}\omega_i'}{\mathrm{d}\theta} = -\frac{1}{2}(\sin\theta)(\partial_{y^i}h)(0,y_0)(\omega_0,\omega_0),$$

from which it is clear that $\omega(\pi)'$ also is determined by h_0. Since we have already determined h_0, it follows that $F'(0)$ determines $\partial_\rho h(0,y_0)(\omega_0,\omega_0)$ for all y_0, ω_0. Thus L_g determines $\partial_\rho h_\rho|_{\rho=0}$.

We now claim by induction on k that L_g determines $\partial_\rho^k h_\rho|_{\rho=0}$. Apply $\partial_\delta^k|_{\delta=0}$ to (4.5) and expand using the chain rule (recall that h and its derivatives are evaluated at $(\rho, y_0+\delta u)$, and ρ is determined implicitly as a function of $(\theta,u,\omega,y_0,\delta)$ by $\rho|\omega|_{h_\rho} = \delta\sin\theta$). The derivatives of ω and u which appear are $\partial_\delta^l\omega$ for $0\leqslant l\leqslant k$ and $\partial_\delta^l u$ for $0\leqslant l\leqslant k-1$. Differentiation of (4.6) shows that the pair $(\partial_\delta^k u|_{\delta=0}, \partial_\delta^k\omega|_{\delta=0})$ satisfies an inhomogeneous system of linear differential equations involving $\partial_\delta^l u|_{\delta=0}$ and $\partial_\delta^l\omega|_{\delta=0}$ for $0\leqslant l\leqslant k-1$; the equation for $\partial_\delta^k u|_{\delta=0}$ involves $\partial_\rho^l h_\rho|_{\rho=0}$ for $0\leqslant l\leqslant k$ but the equation for $\partial_\delta^k\omega|_{\delta=0}$ involves only $\partial_\rho^l h_\rho|_{\rho=0}$ for $0\leqslant l\leqslant k-1$ because of the leading factor of δ in the equation for ω. Thus the derivatives $\partial_\delta^l\omega$ for $0\leqslant l\leqslant k$ and $\partial_\delta^l u$ for $0\leqslant l\leqslant k-1$ which appear in $\partial_\delta^k F|_{\delta=0}$ are all determined by $\partial_\rho^l h_\rho|_{\rho=0}$ for $0\leqslant l\leqslant k-1$. The expansion of $\partial_\delta^k F|_{\delta=0}$ via the chain rule also has explicit dependence on derivatives of h_ρ. Since $\rho=0$ when $\theta=\pi$, only tangential derivatives of h_0 appear when $\partial_\delta^k\log(|\omega|_{h_\rho})|_{\theta=\pi}$ is expanded. Hence $\partial_\delta^k\log(|\omega|_{h_\rho})|_{\theta=\pi}$ is determined by $\partial_\rho^l h_\rho|_{\rho=0}$ for $0\leqslant l\leqslant k-1$. On the other hand, since $\partial_\delta\rho|_{\delta=0} = \sin\theta$, we have

$$\partial_\delta^k\left(\delta\int_0^\pi \frac{\partial_\rho|\omega|^2_{h_\rho}}{|\omega|^3_{h_\rho}}(1+\delta\widetilde{Q})^{-1}\mathrm{d}\theta\right) = k\partial_\rho^k h(0,y_0)(\omega_0,\omega_0)\int_0^\pi(\sin\theta)^{k-1}\mathrm{d}\theta+R_k,$$

where R_k depends only on $\partial_\rho^l h_\rho|_{\rho=0}$ for $0\leqslant l\leqslant k-1$. Thus it follows by induction that L_g determines $\partial_\rho^k h(0,y_0)(\omega_0,\omega_0)$ for each y_0, ω_0, so also $\partial_\rho^k h_\rho|_{\rho=0}$. □

We remark that the determination of h_ρ to infinite order in the proof of Theorem 1.3 above is constructive in the sense that it provides an algorithm for calculating the Taylor expansion in ρ. We also remark that it follows from the proof that $h=h'$ under the hypotheses of Theorem 1.3.

Proof of Theorem 1.4. — Let ψ be a real-analytic diffeomorphism defined in a neighborhood of $\partial\overline{M}$, equal to the identity on $\partial\overline{M}$, and for which (4.3) holds. Theorem 1.3 shows that $h'_\rho-h_\rho$ vanishes to infinite order at $\rho=0$. Real-analyticity implies that $\psi^*g'=g$ near $\partial\overline{M}$.

We show that ψ extends to an isometry $\psi:\overline{M}\to\overline{M}$. If $\overline{M}$ is simply connected, this is the standard result ([28, Corollary 6.4, p. 256]) that a local isometry between simply connected, complete, real-analytic Riemannian manifolds extends to a global isometry. We claim that since our ψ is defined in a full neighborhood of $\partial\overline{M}$, the same argument applies under the weaker hypothesis $\pi_1(\overline{M},\partial\overline{M})=0$. The argument goes as follows. Choose a point $p\in\partial\overline{M}$. If $q\in M$, completeness implies that ψ can be extended by analytic continuation along any curve from p to q. It must be shown that the continuation in a neighborhood of q is independent of the curve. If $\overline{M}$ is simply connected, any closed curve based at p can be deformed to the constant curve, and the result follows. Under the hypothesis $\pi_1(\overline{M},\partial\overline{M})=0$, the closed curve can only be deformed into a closed curve in $\partial\overline{M}$. But since ψ is already defined in a full neighborhood of $\partial\overline{M}$, this is sufficient to conclude that the analytic continuation is path-independent. ψ is a diffeomorphism since ψ^{-1} extends by the same argument, and the relation $\psi^*g'=g$ follows by analytic continuation. □

4.3. Deformation rigidity

In this section we prove Theorem 1.5.

Proof of Theorem 1.5. — First suppose (1) holds. There exists a smooth family $\phi(s):\overline{M}\to\overline{M}$ of diffeomorphisms satisfying $\phi(s)|_{\partial\overline{M}}=\mathrm{Id}$ such that $\phi(s)^*g(s)=(\mathrm{d}\rho^2+h_\rho(s))/\rho^2$ in a product decomposition near $\partial\overline{M}$, with

$h_\rho(s)$ a smooth family of metrics on $\partial\overline{M}$ satisfying $h_0(s) = h(s)$. Thus we can assume that $g(s)$ is already reduced to that form. By Theorem 1.3, we also have $h_\rho(s) = h_\rho(0) + \mathcal{O}(\rho^\infty)$ uniformly in s, and in particular $h(s) = h(0)$. Thus the identifications of $\partial_\pm S^*M$ with $T^*\partial\overline{M}$ agree for all s, so in the rest of the proof we view the boundary as $\partial_\pm S^*M$ rather than $T^*\partial\overline{M}$.

Fix $z \in \partial_- S^*M$; the geodesics for $g(s)$ with initial value $z \in \partial_- S^*M$ form a smooth in s family of curves given by $\gamma_s(\tau, z) := \pi(\bar{\varphi}_\tau(s, z))$ if $\bar{\varphi}_\tau(s, \cdot)$ is the flow of $\overline{X}(s) = \rho^{-1}X(s)$ associated to $g(s)$. Since $S_{g(s)} = S_{g(0)}$ by assumption, we have $z' := S_{g(0)}(z) = S_{g(s)}(z) \in \partial_+ S^*M$. We denote by $\tau_+(s, z)$ the time so that $\bar{\varphi}_{\tau_+(s,z)}(s, z) = z'$, and we define $\gamma_s(\tau, z') := \pi(\bar{\varphi}_{-\tau}(s, z'))$ for $0 \leqslant \tau \leqslant \tau_+(s, z)$. Since $\overline{X}(s) = \overline{X}(0) + \mathcal{O}(s\rho^\infty)$ when viewed as smooth vector fields on ${}^bT^*\overline{M}$, we have for all $N \in \mathbb{N}$ and τ small

$$\bar{\varphi}_\tau(s, z) = \bar{\varphi}_\tau(0; z) + \mathcal{O}(s \max_{\sigma \leqslant \tau} \rho(\gamma_s(\sigma, z))^N),$$
$$\bar{\varphi}_{-\tau}(s, z') = \bar{\varphi}_{-\tau}(0; z') + \mathcal{O}(s \max_{\sigma \leqslant \tau} \rho(\gamma_s(\sigma, z'))^N),$$

where here the remainder is uniform in τ (this follows for instance from the formula in [54, Lemma 2.2] relating two flows in terms of the difference of their vector fields). We write dot for the τ derivative and prime for the s derivative at $s = 0$, and we remove the 0 index when $s = 0$ (e.g. $g(0)$ is denoted g, $g'(0)$ is denoted g', etc). We have from the discussion above that the vector fields $\gamma'(\tau)$ and $\dot{\gamma}'(\tau)$ satisfy for all $N \in \mathbb{N}$

$$\gamma'(\tau, z) = \mathcal{O}(\tau^N), \quad \dot{\gamma}'(\tau, z) = \mathcal{O}(\tau^N) \tag{4.7}$$

for τ small, and the same holds with z' replacing z. If $\bar{g}(s) := \rho^2 g(s)$, remark that $\bar{g}(s)_{\gamma_s(\tau,z)}(\dot{\gamma}_s(\tau, z), \dot{\gamma}_s(\tau, z)) = 1$, thus for $\epsilon \in (0, \tau_+(s, z))$ small, we get for $\mathrm{Re}(\lambda) > 0$

$$\begin{aligned}\partial_s\Big[\int_0^\epsilon \rho^{\lambda-1}(\gamma_s(\tau, z))\bar{g}(s)_{\gamma_s(\tau,z)}(\dot{\gamma}_s(\tau, z), \dot{\gamma}_s(\tau, z))\mathrm{d}\tau\Big]\Big|_{s=0} \\ = \int_0^\epsilon \rho^\lambda(\gamma(\tau, z))\partial_s\Big[\rho^{-1}(\gamma_s(\tau, z))\bar{g}(\dot{\gamma}_s(\tau, z), \dot{\gamma}_s(\tau, z))\Big]\Big|_{s=0}\mathrm{d}\tau \\ + \lambda\int_0^\epsilon \rho^{\lambda-2}(\gamma(\tau, z))\mathrm{d}\rho(\gamma(\tau, z)).\gamma'(\tau, z)\mathrm{d}\tau \\ + \int_0^\epsilon \rho^{\lambda-1}(\gamma(\tau, z))\bar{g}'(\dot{\gamma}(\tau, z), \dot{\gamma}(\tau, z))\mathrm{d}\tau.\end{aligned}$$

Due to (4.7) and $g' = \mathcal{O}(\rho^\infty)$, the three integrals extend holomorphically near $\lambda = 0$ and are uniformly $\mathcal{O}(\epsilon^N)$ for all N for λ near 0. Now the same arguments give the same identity with z replaced by z'. Finally we have

$$\partial_s\Big[\int_\epsilon^{\tau_+(s,z)-\epsilon} \rho^{\lambda-1}(\gamma_s(\tau,z))\overline{g}(s)_{\gamma_s(\tau,z)}(\dot\gamma_s(\tau,z),\dot\gamma_s(\tau,z))\mathrm{d}\tau\Big]\Big|_{s=0}$$
$$= \partial_s\Big[\int_\epsilon^{\tau_+(s,z)-\epsilon} \rho^{\lambda}(\gamma(\tau,z))\rho^{-1}(\gamma_s(\tau,z))\overline{g}(\dot\gamma_s(\tau,z),\dot\gamma_s(\tau,z))\mathrm{d}\tau\Big]\Big|_{s=0}$$
$$+\int_\epsilon^{\tau_+(z)-\epsilon}\rho^{\lambda-1}(\gamma(\tau,z))\overline{g}'(\dot\gamma(\tau,z),\dot\gamma(\tau,z))\mathrm{d}\tau$$
$$+\lambda\int_\epsilon^{\tau_+(z)-\epsilon}\rho^{\lambda-2}(\gamma(\tau,z))\mathrm{d}\rho(\gamma(\tau,z)).\gamma'(\tau,z)\mathrm{d}\tau.$$

Summing up, and evaluating at $\lambda=0$, we obtain

$$(4.8)\quad \partial_s[L_{g(s)}(z)]|_{s=0}$$
$$=\int_\epsilon^{\tau_+(z)-\epsilon}\rho^{-1}(\gamma(\tau,z))\overline{g}'(\dot\gamma(\tau,z),\dot\gamma(\tau,z))\mathrm{d}\tau$$
$$+\partial_s\Big[\int_\epsilon^{\tau_+(s,z)-\epsilon}\rho^{-1}(\gamma_s(\tau,z))\overline{g}(\dot\gamma_s(\tau,z),\dot\gamma_s(\tau,z))\mathrm{d}\tau\Big]\Big|_{s=0}+\mathcal{O}(\epsilon^N).$$

As $\epsilon\to 0$, the first term tends to $I_2(g')(z)$ where I_2 is the X-ray transform on symmetric 2-tensors associated to $g=g(0)$. Let $t(\tau,z)=\int_\epsilon^\tau\rho^{-1}(\gamma_s(\sigma,z))\mathrm{d}\sigma$ and make the change of variable $\tau\mapsto t(\tau,z)$ in the second integral of (4.8), which becomes

$$\partial_s\Big[\int_0^{t_s(\epsilon)} g(\dot\alpha_s(t),\dot\alpha_s(t))\mathrm{d}t\Big]\Big|_{s=0}$$

where $\alpha_s(t):=\gamma_s(\tau,z)$ and $t_s(\epsilon):=t(\tau_+(s,z)-\epsilon,z)$. We recognize the energy functional of the curve $\alpha_s(t)$ for $g(0)$, and since $\alpha_0(t)$ is a geodesic of $g(0)$, we get by [14, Theorem 3.31] and (4.7) that

$$\partial_s\Big[\int_0^{t_s(\epsilon)} g(\dot\alpha_s(t),\dot\alpha_s(t))\mathrm{d}t\Big]\Big|_{s=0}$$
$$= g(\gamma'(\epsilon,z),\dot\alpha_0(t_0(\epsilon)))-g(\gamma'(\epsilon,z'),\dot\alpha_0(0))=\mathcal{O}(\epsilon^N)$$

for all $N\in\mathbb{N}$. Therefore, by letting $\epsilon\to 0$ in (4.8), we conclude that

$$\partial_s[L_{g(s)}(z)]|_{s=0}=I_2(g')(z).$$

By Theorem 1.1, we deduce that $g'=Dq$ for some $q\in\rho^N L^\infty\cap C^\infty(M,T^*M)$ for all $N\geqslant 1$. (The proof of Theorem 1.1 shows that q can be chosen to vanish to infinite order if $f=g'$ does.) The same holds by linearizing at any s and by duality we obtain a vector field $q^\sharp(s)\in$

$C^\infty(M,TM)\cap\rho^N L^\infty$ so that $\partial_s g(s)=\mathcal{L}_{\frac{1}{2}q^\sharp(s)}g(s)$, whose dependence is smooth in s since $q^\sharp(s)$ is the dual to $q(s)=\pi_{1*}(R_{s,+}(0)\partial_s g(s))$ where $R_{s,+}(0)$ is the resolvent of the vector field $X(s)$, which is smooth in s when acting on $\pi_2^*(\rho^\infty C^\infty(\overline{M},\otimes_S^2 T^*\overline{M}))$ by the expression (3.1). Integrating the vector field $\frac{1}{2}q^\sharp(s)$, we obtain a smooth 1-parameter family of diffeomorphisms $\psi(s):\overline{M}\to\overline{M}$ equal to the identity at $\partial\overline{M}$ so that $\psi(s)^*g(s)=g(0)$. This concludes the proof under assumption (1).

Proposition 5.24 shows that (2) implies (1), so the result is true under assumption (2) as well. □

5. Simplicity and renormalized distance

In this section we parametrize geodesics on a non-trapping asymptotically hyperbolic manifold by their two endpoints instead of their starting point and direction. To do this, we clearly need assumptions which imply at least that there is a unique geodesic connecting any two points of the boundary. The assumption we make is that there are no conjugate points at infinity, i.e. there are no nonzero Jacobi fields along any geodesic which decay as $t\to\infty$ and as $t\to-\infty$. We will call such asymptotically hyperbolic manifolds simple. As we discuss below, this implies that (M,g) has no conjugate points.

Stable and unstable bundles for the geodesic flow on a complete manifold (M,g) with no conjugate points are defined in [11, Section 2]. An alternative statement of the condition that there are no nonzero Jacobi fields decaying as $t\to\infty$ and as $t\to-\infty$ is that the stable and unstable bundles are everywhere transverse. In [11, Section 3] it is proved that transversality of the stable and unstable bundles implies that the geodesic flow is Anosov in the case that the universal cover of M is compactly homogeneous, i.e. it can be covered by translates by isometries of a fixed compact set. We first need to establish the analogous result for non-trapping asymptotically hyperbolic manifolds. An easy alternate construction of the stable and unstable bundles can be given in this setting using elementary ODE theory which also proves that they are smooth. In terms of this construction, it is not difficult to establish directly that transversality implies that the geodesic flow is hyperbolic. We begin this section by presenting these arguments.

Remark 5.1. — We typically use the terminology “Anosov geodesic flow” in classical settings such as a compact manifold or the universal cover of a compact manifold, and “hyperbolic geodesic flow” otherwise, for example for asymptotically hyperbolic manifolds. But these mean the same thing: the existence of a splitting (5.3) with uniform contraction/expansion estimates (for some $\nu > 0$) as in Proposition 5.5.

We assume throughout this section that (M, g) is a non-trapping asymptotically hyperbolic manifold. Let $z_0 \in S^*M$. Locally near z_0, choose a smooth hypersurface $\mathcal{S} \subset S^*M$ transverse to X with $z_0 \in \mathcal{S}$. For $z \in \mathcal{S}$, choose an orthonormal basis $\{w_1^z, \dots, w_n^z\}$ for $\mathcal{Z}_z$ varying smoothly with z. For each z and each j, extend w_j^z to the geodesic γ_z by parallel translation. This gives an orthonormal frame field $w_j^z(t)$ for $\dot\gamma_z(t)^\perp$, varying smoothly with $(z,t) \in \mathcal{S} \times \mathbb{R}$. A normal vector field along γ_z can be written $Y(t) = \sum_{j=1}^n y^j(t) w_j^z(t)$. The Jacobi equation takes the form

$$\ddot{y} + R^z(t) y = 0 \qquad (\dot{} = \partial_t), \tag{5.1}$$

where $y(t) = (y^1(t), \dots, y^n(t))^T$ and $R^z(t) \in \mathbb{R}^{n\times n}$ is the matrix of the linear transformation $Y \to \mathcal{R}(Y, \dot\gamma_z(t))\dot\gamma_z(t)$ in the frame $(w_1^z(t), \dots, w_n^z(t))$. Certainly $R^z(t)$ is smooth in t and z. Using (2.1) and the arguments of the proof of Lemma 2.7, one sees that $R^z(t)$ has the form

$$R^z(t) = -I - S^z(t)$$

where $S^z(t) \in \mathbb{R}^{n\times n}$ satisfies $|\partial_z^\alpha S^z(t)| \leqslant C_\alpha e^{-|t|}$, $t \in \mathbb{R}$, with C_α independent of $z \in \mathcal{S}$ near z_0. Here ∂_z denotes partial differentiation with respect to some choice of local coordinates on $\mathcal{S}$.

Reduce (5.1) to a first order system in the usual way: introduce

$$x = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \widetilde{S}^z(t) = \begin{pmatrix} 0 & 0 \\ S^z(t) & 0 \end{pmatrix}$$

so that (5.1) becomes

$$\dot{x} = \big(A + \widetilde{S}^z(t)\big) x. \tag{5.2}$$

Let e_j, $1 \leqslant j \leqslant n$ denote the standard basis for $\mathbb{R}^n$ and set

$$\mathbf{e}_j^\pm = \begin{pmatrix} e_j \\ \pm e_j \end{pmatrix}.$$

Then $\{\mathbf{e}_j^-, \mathbf{e}_j^+; 1 \leqslant j \leqslant n\}$ is an orthogonal basis for $\mathbb{R}^{2n}$ satisfying $A\mathbf{e}_j^\pm = \pm \mathbf{e}_j^\pm$.

Proposition 5.2. — *For each j, $1 \leqslant j \leqslant n$ and each choice of $\pm$, there is a solution $x_{\pm,j}^z$ of* (5.2) *satisfying*

$$\lim_{t\to\infty} e^{\mp t} x_{\pm,j}^z(t) = \mathbf{e}_j^\pm.$$

Moreover, $x^z_{-,j}$ is uniquely determined by this condition, and $x^z_{-,j}(t)$ is C^∞ in (z,t).

The existence and uniqueness part of Proposition 5.2 is a special case of Problem 29 of [6, p. 104], which applies to an ode system of the form (5.2) with A diagonalizable and $\widetilde{S}^z(t)$ integrable. An argument similar to that outlined in the solution to this problem in [6] proves smooth dependence on the parameter z.

Note that $\{x^z_{\pm,j}\}$ is a basis for the solutions of (5.2). Their first components thus form a basis for the solutions of (5.1), and the corresponding $Y^z_{\pm,j}$ form a basis for the space of normal Jacobi fields with the property that $Y^z_{\pm,j}$ is asymptotic to $e^{\pm t}w^z_j(t)$ as $t \to \infty$. Under the isomorphism $\mathcal{L}$ defined in (2.26), the stable bundle corresponds to initial data of the Jacobi fields which decay like e^{-t} as $t \to \infty$:

Definition 5.3. — *The stable bundle E_s is the subbundle of* $\ker \alpha$ *defined by*

$$\mathcal{L}\big(E_s(\varphi_t(z))\big) = \operatorname{span}\{\big(Y^z_{-,j}(t), (Y^z_{-,j})'(t)\big) : 1 \leqslant j \leqslant n\}, \quad t \in \mathbb{R}.$$

Here $' = D_t$. When viewed as a function of the point on a geodesic rather than on the time parameter, the decaying Jacobi fields are independent of which point is labeled as the initial point. Consequently E_s is well-defined independent of the choice of initial point z on the geodesic. Since the $Y^z_{-,j}(t)$ are smooth functions of (z,t), E_s is a smooth subbundle of TS^*M. The unstable bundle E_u is defined analogously in terms of the initial data of the Jacobi fields which decay as $t \to -\infty$. The bundles E_s and E_u are invariant under the geodesic flow since Jacobi fields along a geodesic in M correspond to flow-invariant vector fields along the corresponding integral curve in S^*M.

Definition 5.4. — *A non-trapping asymptotically hyperbolic manifold (M,g) is said to be* simple *if $E_s(z) \cap E_u(z) = \{0\}$ for each $z \in S^*M$.*

A non-trapping asymptotically hyperbolic manifold with non-positive curvature is simple. That there are no solutions to (5.1) which decay exponentially as $t \to \pm\infty$ follows by taking the inner product with $y(t)$ and integrating by parts.

When $E_s(z) \cap E_u(z) = \{0\}$ for all $z \in S^*M$, we have a hyperbolic splitting for the flow:

$$T(S^*M) = \mathbb{R}X \oplus E_s \oplus E_u. \tag{5.3}$$

The next proposition asserts that the decay estimates are uniform in this case.

Proposition 5.5. — *Let (M,g) be a simple asymptotically hyperbolic manifold. Then its geodesic flow is hyperbolic in the following sense: for any $0<\nu<1$, there exists a constant $C>0$ so that:*

(1) *If $z\in S^*M$ and $\zeta\in E_s(z)$, then*

$$\|\mathrm{d}\varphi_t(z).\zeta\|_G \leqslant Ce^{-\nu t}\|\zeta\|_G, \qquad t\geqslant 0,$$
$$\text{and}\quad \|\mathrm{d}\varphi_t(z).\zeta\|_G \geqslant C^{-1}e^{-\nu t}\|\zeta\|_G, \quad t\leqslant 0.$$

(2) *If $z\in S^*M$ and $\zeta\in E_u(z)$, then*

$$\|\mathrm{d}\varphi_t(z).\zeta\|_G \leqslant Ce^{\nu t}\|\zeta\|_G, \qquad t\leqslant 0,$$
$$\text{and}\quad \|\mathrm{d}\varphi_t(z).\zeta\|_G \geqslant C^{-1}e^{\nu t}\|\zeta\|_G, \quad t\geqslant 0.$$

The proof of Proposition 5.5 will be given after Proposition 5.11 below. We remark that Nikolas Eptaminitakis has shown that Proposition 5.5 holds also with $\nu=1$.

It is a consequence of Proposition 5.5 and the following result of Gerhard Knieper that simple asymptotically hyperbolic manifolds have no conjugate points.

Proposition 5.6 ([27]). — *Let (M,g) be a complete connected non-compact Riemannian manifold with sectional curvature bounded below by $-\beta^2$ and with hyperbolic geodesic flow with constants C, ν (as in the statement of Proposition 5.5). There is a constant $\sigma(\beta,\nu,C)>0$ so that if (M,g) satisfies the following three conditions:*

(1) *For any $z\in S^*M$, there is an open neighborhood $\mathcal{U}\subset S^*M$ of z such that $\lim_{t\to\infty} d_g(\gamma(0),\gamma(t))=\infty$ uniformly for all geodesics γ with $\dot\gamma(0)^\flat\in\mathcal{U}$.*
(2) *There exists a compact set $K\subset M$ such that for all $p\in M\setminus K$ and for all geodesics γ with $\gamma(0)=p$, $\gamma(t)|_{t\in[-1,\sigma]}$ has no conjugate points.*
(3) *(M,g) has at least one geodesic without conjugate points,*

then (M,g) has no conjugate points.

It is not hard to verify that (1) and (2) hold for any nontrapping asymptotically hyperbolic manifold, using Lemma 2.3 and the fact that there are no conjugate points on any geodesic segment sufficiently near infinity, where the curvature is negative. Condition (3) also holds since the short geodesics described in Lemma 2.8 have no conjugate points.

Proposition 5.6 is not necessary for the purposes of this paper: if preferred, one can just add the assumption that there are no conjugate points in the definition of a simple asymptotically hyperbolic manifold.

We will prove Proposition 5.5 by dividing the set of all orbits of the geodesic flow (i.e. lifted unparametrized geodesics) into two subsets, one a compact set of orbits, and the other consisting of orbits, each of whose projection to M stays in a fixed small neighborhood of ∂M (short geodesics). A different argument is used to establish the bounds when z lies in either of the two sets of orbits.

We begin by establishing uniform bounds for any compact set of orbits. This is done by deriving uniform bounds locally in the set of orbits, and for this case we obtain the estimates (1), (2) with $\nu = 1$. Unless explicitly stated otherwise, (M, g) is assumed only to be non-trapping.

Let $z_0 \in S^*M$, choose a transverse hypersurface $\mathcal{S} \subset S^*M$, and rewrite the equation for normal Jacobi fields as a $\mathbb{R}^{2n}$-valued first order system via a choice of parallel orthonormal frame as above. We have the following two lemmas, the first asserting uniform upper bounds and the second uniform lower bounds on solutions.

LEMMA 5.7. — *There is a constant $K > 0$ independent of $z \in \mathcal{S}$ near z_0, so that for all $t \in \mathbb{R}$ and $1 \leqslant j \leqslant n$:*

$$|x^z_{-,j}(t)| \leqslant Ke^{-t}, \qquad |x^z_{+,j}(t)| \leqslant Ke^{|t|}.$$

LEMMA 5.8. — *There is a constant k independent of $z \in \mathcal{S}$ near z_0 such that if $\lambda = (\lambda_1, \dots, \lambda_n)^T \in \mathbb{R}^n$, $\mu = (\mu_1, \dots, \mu_n)^T \in \mathbb{R}^n$ and $t \in \mathbb{R}$, then*

$$\left| \sum_{j=1}^n \left(\lambda_j x^z_{+,j}(t) + \mu_j x^z_{-,j}(t) \right) \right| \geqslant k|(\lambda, \mu)|e^{-|t|}$$

and

$$\left| \sum_{j=1}^n \lambda_j x^z_{+,j}(t) \right| \geqslant k|\lambda|e^t.$$

If $E_s(z) \cap E_u(z) = \{0\}$ for $z \in \mathcal{S}$ near z_0, then also

$$\left| \sum_{j=1}^n \mu_j x^z_{-,j}(t) \right| \geqslant k|\mu|e^{-t}. \tag{5.4}$$

The construction of the solutions $x^z_{\pm,j}$ outlined in [6] gives explicit estimates for t large. These estimates can be extended to all $t \in \mathbb{R}$ using Gronwall's inequality. The arguments are standard, so the proofs of Lemmas 5.7 and 5.8 are omitted.

LEMMA 5.9. — *Suppose that $E_s(z) \cap E_u(z) = \{0\}$ for $z \in \mathcal{S}$ near z_0. There is a constant C independent of $z \in \mathcal{S}$ near z_0 so that if $x(t) =$*

$\sum_{j=1}^{n}\mu_j x^z_{-,j}(t)$ *with* $\mu=(\mu_1,\dots,\mu_n)^T\in\mathbb{R}^n$*, then*

$$|x(t+s)|\leqslant Ce^{-t}|x(s)|,\qquad s,t\in\mathbb{R}.$$

Proof. — The first estimate of Lemma 5.7 implies

$$|x(t)|\leqslant K\sqrt{n}|\mu|e^{-t}.$$

Applying this with t replaced by $t+s$ and then (5.4) with t replaced by s gives

$$|x(t+s)|\leqslant K\sqrt{n}|\mu|e^{-t}e^{-s}\leqslant\frac{K\sqrt{n}}{k}e^{-t}|x(s)|.\qquad\square$$

Lemma 5.9 can be reformulated and extended as follows.

Proposition 5.10. — *Suppose that* $E_s(z)\cap E_u(z)=\{0\}$ *for* $z\in S^*M$*. Let* $\mathcal{C}$ *be a compact set of orbits of the geodesic flow on* S^*M*. There exists a constant* $C>0$ *so that if* $z\in o$ *for some* $o\in\mathcal{C}$ *and* $\zeta\in E_s(z)$*, then*

$$\|\mathrm{d}\varphi_t(z).\zeta\|_G\leqslant Ce^{-t}\|\zeta\|_G,\qquad t\geqslant 0.$$

Recall that each orbit intersects ∂_-S^*M exactly once. So we identify the space of orbits with ∂_-S^*M, with the induced topology.

Proof. — First note that Lemma 5.9 gives a uniform bound for all points in the orbit o through z. Namely, $x(0)$ corresponds to an element of $E_s(z)$, $x(s)$ corresponds to an element $\zeta\in E_s(\varphi_s(z))$, and $x(t+s)$ corresponds to $\mathrm{d}\varphi_t(\varphi_s(z)).\zeta$. As s varies over $\mathbb{R}$, $\varphi_s(z)$ varies over all points on o, and as μ varies over $\mathbb{R}^n$, ζ varies over all elements of $E_s(\varphi_s(z))$. So, restricting to $t\geqslant 0$, Lemma 5.9 asserts that there is a uniform upper bound locally in a neighborhood of any orbit in the space of orbits. Hence there is a uniform bound on any compact subset of the space of orbits. $\square$

Geodesics which stay in a set where the curvature is negative are hyperbolic. The following proposition is a special case of [26, Theorem 3.2.17].

Proposition 5.11. — *Suppose* (M,g) *is asymptotically hyperbolic. Let* $0<\epsilon<1$*. If* $z\in S^*M$ *and the geodesic* $\{\pi\big(\varphi_t(z)\big):t\in\mathbb{R}\}$ *is contained in the set where all sectional curvatures* K *satisfy*

$$-(1+\epsilon)^2\leqslant K\leqslant-(1-\epsilon)^2,\tag{5.5}$$

then

$$\|\mathrm{d}\varphi_t(z).\zeta\|_G\leqslant Ce^{-(1-\epsilon)t}\|\zeta\|_G,\qquad t\geqslant 0,\qquad \zeta\in E_s(z)$$

with $C=\frac{1+\epsilon}{1-\epsilon}$.

Proof of Proposition 5.5. — Given $0 < \nu < 1$, set $\epsilon = 1 - \nu$. Lemma 2.8 shows that the set of orbits which lie in the region where (5.5) holds is contained in the complement of a compact set $\mathcal{C}$ of orbits. So the inequality for $t \geqslant 0$ in (1) follows immediately from Propositions 5.10 and 5.11.

The inequality for $t \leqslant 0$ in (1) follows from the inequality for $t \geqslant 0$ upon replacing ζ by $\mathrm{d}\varphi_{-t}(\varphi_t(z)).\zeta$ in the inequality for $t \geqslant 0$, and then replacing t by $-t$.

The inequalities in (2) follow from those in (1) upon noting that the stable and unstable spaces are related by $E_u(z) = \mathrm{d}S\big(E_s(Sz)\big)$, where $S : S^*M \to S^*M$ is the involution $S(x,\xi) = (x,-\xi)$. □

Proposition 5.12. — *Let (M,g) be a simple asymptotically hyperbolic manifold. If $p,q \in \overline{M}$, $p \neq q$, there is a unique geodesic (viewed as an unparametrized curve) connecting p and q. In the case that p and/or q is in $\partial\overline{M}$, this is interpreted to mean that the geodesic approaches the point as $t \to \pm\infty$.*

Proof. — If both p and $q \in M$, this follows from the fact that the exponential map is a diffeomorphism at each point. If one or both of p, q is in $\partial\overline{M}$, it follows from results in [10, 25] as follows. First, the conformal compactification $\overline{M}$ agrees with the compactification used in [10, 25] where the boundary at infinity consists of equivalence classes of geodesics that are asymptotic; two oriented geodesics γ_1 and γ_2 are said to be asymptotic if for any given unit speed parametrization $\gamma_i(t)$ of γ_i there is $C > 0$ such that $d_g(\gamma_1(t), \gamma_2(t)) \leqslant C$ for all $t \geqslant 0$. In fact, it is easily seen that this notion of asymptotic is equivalent to the condition that the two geodesics have the same endpoint in $\overline{M}$ as $t \to \infty$. The appendix to [25] is formulated for the universal cover of a compact manifold with Anosov geodesic flow, but the arguments apply to any complete, simply connected manifold with no conjugate points for which the geodesic flow is Anosov and for which there is a uniform lower bound on sectional curvatures. This appendix proves, first, that any such manifold satisfies the uniform visibility axiom of [10]. Now Proposition 1.5 of [10] asserts that any complete, simply connected manifold with no conjugate points satisfying the uniform visibility axiom has the property that there exists a unique geodesic connecting a point of M and a point of $\partial\overline{M}$, and Proposition 1.7 of [10] asserts that there exists a geodesic connecting any two distinct points of $\partial\overline{M}$. Moreover, the appendix to [25] additionally proves that there exists a unique geodesic connecting any two distinct points of $\partial\overline{M}$. □

We next study boundary mapping properties of the extended exponential map for simple asymptotically hyperbolic manifolds. The behavior of E_s and E_u near $\partial\overline{S^*M}$ plays an important role.

Proposition 5.13. — *On a non-trapping asymptotically hyperbolic manifold, each of the subbundles E_s and E_u of TS^*M extends smoothly to a subbundle of $T\overline{S^*M}$. Moreover, the extensions satisfy $\mathcal{V}_z = E_s(z)$ for $z \in \partial_+ S^*M$ and $\mathcal{V}_z = E_u(z)$ for $z \in \partial_- S^*M$.*

Proof. — Let $z_0 \in S^*M$ and choose a basis $\{\zeta_j(z); 1 \leqslant j \leqslant n\}$ for $E_s(z)$ for z near z_0, depending smoothly on z. Since E_s is invariant under the geodesic flow, it follows that $\{\mathrm{d}\varphi_t(z).\zeta_j(z); 1 \leqslant j \leqslant n\}$ is a basis for $E_s(\varphi_t(z))$ for all $t \in \mathbb{R}$. Now write $\bar{\varphi}_\tau(z) = \varphi_{t(\tau,z)}(z)$ as in (2.14). We will show that $\mathrm{d}\varphi_{t(\tau,z)}(z).\zeta_j(z)$, $1 \leqslant j \leqslant n$, extend smoothly up to $\tau = \tau_+(z)$ as a function of (τ, z) for z near z_0, and that they remain linearly independent. From this it follows that the bundle E_s extends smoothly to $\partial_+ S^*M$.

The chain rule gives

$$\begin{aligned} \mathrm{d}\varphi_{t(\tau,z)}(z).\zeta &= \mathrm{d}\bar{\varphi}_\tau(z).\zeta - X(\bar{\varphi}_\tau(z))\mathrm{d}_z t(\tau,z).\zeta \\ &= \mathrm{d}\bar{\varphi}_\tau(z).\zeta - \rho(\bar{\varphi}_\tau(z))\overline{X}(\bar{\varphi}_\tau(z))\mathrm{d}_z t(\tau,z).\zeta. \end{aligned} \tag{5.6}$$

Certainly $\mathrm{d}\bar{\varphi}_\tau(z).\zeta_j(z)$ and $\overline{X}(\bar{\varphi}_\tau(z))$ extend smoothly up to $\tau = \tau_+(z)$. Differentiation of (2.19) shows that

$$\mathrm{d}_z t(\tau,z) = \eta(\tau,z)\big(\tau - \tau_+(z)\big)^{-1}$$

where $\eta(\tau,z)$ is smooth up to $\tau = \tau_+(z)$. But $\rho(\bar{\varphi}_\tau(z))$ is smooth and vanishes when $\tau = \tau_+(z)$. Thus $\rho(\bar{\varphi}_\tau(z))\mathrm{d}_z t(\tau,z).\zeta_j(z)$ is smooth up to $\tau = \tau_+(z)$, and so also is $\mathrm{d}\varphi_{t(\tau,z)}(z).\zeta_j(z)$.

Now $\mathrm{d}\bar{\varphi}_\tau(z)$ is an isomorphism up to $\tau = \tau_+(z)$ since $\bar{\varphi}_\tau(z)$ is a smooth flow. Since $\{\zeta_1(z), \dots, \zeta_n(z), \overline{X}(z)\}$ is a linearly independent set of vectors at z, it follows that $\{\mathrm{d}\bar{\varphi}_{\tau_+(z)}(z).\zeta_1(z), \dots, \mathrm{d}\bar{\varphi}_{\tau_+(z)}(z).\zeta_n(z), \mathrm{d}\bar{\varphi}_{\tau_+(z)}(z).\overline{X}(z)\}$ is a linearly independent set of vectors at $\bar{\varphi}_{\tau_+(z)}(z)$. But $\mathrm{d}\bar{\varphi}_{\tau_+(z)}(z).\overline{X}(z) = \overline{X}(\bar{\varphi}_{\tau_+(z)}(z))$ since $\bar{\varphi}_\tau$ is the flow of $\overline{X}$. Therefore $\{\mathrm{d}\bar{\varphi}_{\tau_+(z)}(z).\zeta_j(z) + \alpha_j\overline{X}(\bar{\varphi}_{\tau_+(z)}(z)); 1 \leqslant j \leqslant n\}$ is linearly independent for any $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. It thus follows from (5.6) that $\{\mathrm{d}\varphi_{t(\tau,z)}(z).\zeta_j(z); 1 \leqslant j \leqslant n\}$ remains linearly independent up to $\tau = \tau_+(z)$ as claimed.

The same argument letting $\tau \to -\tau_-(z)$ shows that E_s extends smoothly to $\partial_- S^*M$. Then the same argument, taking the $\zeta_j(z)$ to be a basis for $E_u(z)$, shows that E_u extends smoothly to $\overline{S^*M}$.

Near the boundary, we use the smooth coordinates (2.4) for ${}^bT^*\overline{M}$. So $\zeta \in T({}^bT^*\overline{M})$ can be written

$$\zeta = \bar{a}^0\partial_{\bar{\rho}} + \sum_j \bar{a}^j\partial_{\bar{y}^j} + \bar{b}_0\partial_{\bar{\xi}_0} + \sum_j \bar{b}_j\partial_{\bar{\eta}_j}.$$

The $(\bar{\rho}, \bar{y}^j, \bar{\xi}_0, \bar{\eta}_j, \bar{a}^0, \bar{a}^j, \bar{b}_0, \bar{b}_j)$ are smooth coordinates for $T({}^bT^*\overline{M})$ near $\rho = 0$. The fiber of $T\overline{S^*M} \subset T({}^bT^*\overline{M})$ is given by $\bar{b}_0 = 0$ over $\partial\overline{S^*M}$. The condition $\alpha(\zeta) = 0$ reads $\bar{\xi}_0\bar{a}^0 = -\rho\sum_j \bar{\eta}_j\bar{a}^j$, so the fiber of the smooth extension of $\ker\alpha$ is given by $\bar{a}^0 = 0$ over $\partial\overline{S^*M}$. The fiber of the vertical bundle of $T({}^bT^*\overline{M})$ is given by $\bar{a}^0 = 0$, $\bar{a}^j = 0$, and to obtain the fiber of $\mathcal{V} \subset T\overline{S^*M}$ one adds the condition $\bar{\xi}_0\bar{b}_0 = -\rho^2\sum_{i,j} h^{ij}\bar{\eta}_i\bar{b}_j$ of tangency to $\overline{S^*M}$. In particular, the fiber of $\mathcal{V}$ over $\partial\overline{S^*M}$ is given by $\bar{a}^0 = 0$, $\bar{a}^j = 0$, $\bar{b}_0 = 0$.

To identify the fiber of E_s over ∂_+S^*M, we consider the asymptotics of the Sasaki metric. Let $\sigma(\tau)$ be a smooth curve in $\overline{S^*M}$ with $\sigma(\tau_0) \in \partial_+S^*M$ such that σ is transverse to ∂_+S^*M at $\sigma(\tau_0)$, and let $\zeta(\tau)$ be a smooth section of $\ker\alpha \cap T\overline{S^*M}$ along $\sigma(\tau)$ with $\zeta(\tau) \to \zeta(\tau_0) \in T_{\sigma(\tau_0)}\overline{S^*M}$. We claim that if $\zeta(\tau_0) \notin \mathcal{V}_{\sigma(\tau_0)}$, then there is $C > 0$ so that $\|\zeta(\tau)\|_G \geqslant C\rho(\sigma(\tau))^{-1}$ for τ sufficiently close to τ_0. In fact, the coordinates of $\zeta(\tau_0)$ must satisfy $\bar{a}^0 = \bar{b}_0 = 0$ since $\zeta(\tau) \in \ker\alpha \cap T\overline{S^*M}$. Since $\zeta(\tau_0) \notin \mathcal{V}_{\sigma(\tau_0)}$, it must be that $\bar{a}^j(\tau_0) \neq 0$ for some j. But

$$G(\zeta,\zeta) \geqslant g(\mathrm{d}\pi(\zeta), \mathrm{d}\pi(\zeta)) = \rho^{-2}\left[(\bar{a}^0)^2 + \sum_{i,j} h_{ij}\bar{a}^i\bar{a}^j\right] \geqslant \rho^{-2}\sum_{i,j} h_{ij}\bar{a}^i\bar{a}^j,$$

so the claimed inequality follows upon taking $\zeta = \zeta(\tau)$ with τ close to τ_0.

To apply this observation to E_s, choose $z \in S^*M$ and take $\sigma(\tau) = \overline{\varphi}_\tau(z) = \varphi_{t(\tau,z)}(z)$, so $\tau_0 = \tau_+(z)$. Choose a basis $\zeta_j(z)$, $1 \leqslant j \leqslant n$ for $E_s(z)$ and take $\zeta(\tau) = \mathrm{d}\varphi_{t(\tau,z)}(z).\zeta_j(z)$ for some j. (2.11) together with Lemma 5.7 show that $\|\zeta(\tau)\|_G \leqslant Ce^{-t} \leqslant C\rho$. The above observation implies that it must be that $\zeta(\tau_0) \in \mathcal{V}_{B_+(z)}$. But the argument earlier in this proof establishing the smoothness of E_s up to the boundary shows that the $\zeta(\tau_+(z))$ obtained by varying j form a basis for $E_s(B_+(z))$. Thus the fibers of E_s and $\mathcal{V}$ coincide on ∂_+S^*M. The same argument applies to E_u at ∂_-S^*M. □

Remark 5.14. — By analyzing the behavior of the connection map $\mathcal{K}$ near $\partial\overline{S^*M}$, it can be shown that also the horizontal bundle $\mathcal{H}$ extends smoothly to a subbundle of $T\overline{S^*M}$, and $\mathcal{H}_z = \mathcal{V}_z$ for $z \in \partial\overline{S^*M}$. The details are omitted since we will not use this fact.

Set

$$\mathcal{T}^+\overline{S^*M} = \{(\tau, z) \in \mathbb{R}\times\overline{S^*M} : 0 < \tau \leqslant \tau_+(z)\}.$$

Proposition 5.15. — *If (M,g) is simple, then the following map is a diffeomorphism*

$$\Psi : \mathcal{T}^+\overline{S^*M} \to \overline{M}\times\overline{M}\setminus \mathrm{diag}, \qquad \Psi(\tau,z) = \big(\pi(z), \pi(\overline{\varphi}_\tau(z))\big).$$

Proof. — The bijectivity of Ψ is an immediate consequence of Proposition 5.12. Namely, given $(p,q)\in\overline{M}\times\overline{M}\setminus\Delta$, there is a unique geodesic connecting p and q; z corresponds to the initial point and direction, and τ the time when the geodesic is parametrized as $\tau\to\pi(\overline{\varphi}_\tau(z))$.

The map Ψ is clearly smooth, so it suffices to show that $\mathrm{d}\Psi$ is injective at each point. Write

$$\Psi(\tau,z) = \big(\Psi_1(\tau,z), \Psi_2(\tau,z)\big), \qquad \Psi_1(\tau,z) = \pi(z), \quad \Psi_2(\tau,z) = \pi(\overline{\varphi}_\tau(z))$$

so that $\ker\mathrm{d}\Psi = \ker\mathrm{d}\Psi_1\cap\ker\mathrm{d}\Psi_2 \subset \mathbb{R}\partial_\tau\oplus T_z\overline{S^*M}$. Clearly $\ker\mathrm{d}\Psi_1 = \mathbb{R}\partial_\tau\oplus\mathcal{V}_z$. For $a\in\mathbb{R}$ and $\zeta\in T_z\overline{S^*M}$, we have

$$\mathrm{d}\Psi_2(a\partial_\tau+\zeta) = \mathrm{d}\pi\big(\mathrm{d}\overline{\varphi}_\tau(z).\zeta + a\overline{X}(\overline{\varphi}_\tau(z))\big).$$

So injectivity of $\mathrm{d}\Psi(\tau,z)$ is equivalent to the statement that if $\zeta\in\mathcal{V}_z$ and $\mathrm{d}\overline{\varphi}_\tau(z).\zeta\in\mathcal{V}_{\overline{\varphi}_\tau(z)}\oplus\mathbb{R}\overline{X}(\overline{\varphi}_\tau(z))$, then $\zeta=0$.

If both $\pi(z)$ and $\pi(\overline{\varphi}_\tau(z))$ are in M, this follows from the fact that there are no conjugate points: by (5.6), we deduce that $\mathrm{d}\varphi_{t(\tau,z)}(z).\zeta\in\mathcal{V}_{\overline{\varphi}_\tau(z)}\oplus\mathbb{R}\overline{X}(\overline{\varphi}_\tau(z))$. But $\mathrm{d}\varphi_{t(\tau,z)}(z).\zeta$ is already in $\ker\alpha$, so $\mathrm{d}\varphi_{t(\tau,z)}(z).\zeta\in\mathcal{V}_{\overline{\varphi}_\tau(z)}$ and (2.28) implies that $\zeta=0$.

Suppose next that $\pi(z)\in\partial\overline{M}$ (so $z\in\partial_-S^*M$) and $\pi(\overline{\varphi}_\tau(z))\in M$. If $\zeta\in\mathcal{V}_z$, then $\zeta\in E_u(z)$ by Proposition 5.13. Since $E_u\oplus\mathbb{R}\overline{X}$ is invariant under the flow $\overline{\varphi}_\tau$, it follows that $\mathrm{d}\overline{\varphi}_\tau(z).\zeta\in E_u(\overline{\varphi}_\tau(z))\oplus\mathbb{R}\overline{X}(\overline{\varphi}_\tau(z))$. Proposition 2.11 of [11] implies that if (M,g) is complete with no conjugate points and sectional curvatures bounded from below, then E_u and E_s each intersect $\mathcal{V}$ only in $\{0\}$. So if also $\mathrm{d}\overline{\varphi}_\tau(z).\zeta\in\mathcal{V}_{\overline{\varphi}_\tau(z)}\oplus\mathbb{R}\overline{X}(\overline{\varphi}_\tau(z))$, then $\mathrm{d}\overline{\varphi}_\tau(z).\zeta\in\mathbb{R}\overline{X}(\overline{\varphi}_\tau(z))$. Since $\overline{X}$ is invariant under the flow $\overline{\varphi}_\tau$, it follows that $\zeta\in\mathbb{R}\overline{X}(z)\cap\mathcal{V}_z=\{0\}$, so $\zeta=0$ as desired. The argument if $\pi(z)\in M$ and $\pi(\overline{\varphi}_\tau(z))\in\partial\overline{M}$ (so $\overline{\varphi}_\tau(z)\in\partial_+S^*M$) is similar.

The argument if $z\in\partial_-S^*M$ and $\overline{\varphi}_\tau(z)\in\partial_+S^*M$ follows the same idea. In this case $\zeta\in E_u(z)$, so $\mathrm{d}\overline{\varphi}_\tau(z).\zeta\in E_u(\overline{\varphi}_\tau(z))\oplus\mathbb{R}\overline{X}(\overline{\varphi}_\tau(z))$. If also $\mathrm{d}\overline{\varphi}_\tau(z).\zeta\in\mathcal{V}_{\overline{\varphi}_\tau(z)}\oplus\mathbb{R}\overline{X}(\overline{\varphi}_\tau(z)) = E_s(\overline{\varphi}_\tau(z))\oplus\mathbb{R}\overline{X}(\overline{\varphi}_\tau(z))$, then $\mathrm{d}\overline{\varphi}_\tau(z).\zeta\in\mathbb{R}\overline{X}(\overline{\varphi}_\tau(z))$ since $E_u\cap E_s=\{0\}$ (this transversality holds at $\partial\overline{S^*M}$ too by Proposition 5.13 and invariance under the geodesic flow). Once again, translating back to z shows that $\zeta\in\mathbb{R}\overline{X}(z)$, so $\zeta=0$. □

Remark 5.16. — Simplicity is a necessary condition for Ψ to be a local diffeomorphism everywhere on $\mathcal{T}^+\overline{S^*M}$. If (M,g) is non-trapping, $z \in \partial_- S^*M$, and $E_s \cap E_u \neq \{0\}$ along the integral curve $\overline{\varphi}_\tau(z)$, then Ψ is not a local diffeomorphism near $(\tau_+(z), z)$. In fact, as the proof above shows, for any $\zeta \in E_s(z) \cap E_u(z)$, there is $a_\zeta \in \mathbb{R}$ so that $\mathrm{d}\Psi(a_\zeta \partial_\tau + \zeta) = 0$.

Remark 5.17. — The analogue of Proposition 5.15 obtained by replacing $\mathcal{T}^+\overline{S^*M}$ by the set $\mathcal{T}^-\overline{S^*M} = \{(\tau, z) \in \mathbb{R} \times \overline{S^*M} : -\tau_-(z) \leqslant \tau < 0\}$ is also true, with the same proof.

Corollary 5.18. — *If (M,g) is simple, the following map is a diffeomorphism*

$$B : \partial_- S^*M \to \partial\overline{M} \times \partial\overline{M} \setminus \mathrm{diag}, \quad B(z) := (\pi(z), \pi(S_g(z))).$$

Proof. — B is the restriction of Ψ to $\{(\tau_+(z), z) : z \in \partial_- S^*M\}$. □

Let $d_g : M \times M \setminus \mathrm{diag} \to (0, \infty)$ denote the distance function in the metric g. If (M,g) has no conjugate points, then d_g is smooth. Let ρ be a smooth defining function for $\partial\overline{M}$, and define

$$\widetilde{d}_g(p,q) := d_g(p,q) + \log \rho(p) + \log \rho(q).$$

Proposition 5.19. — *If (M,g) is simple, then $\widetilde{d}_g$ extends smoothly to $\overline{M} \times \overline{M} \setminus \mathrm{diag}$.*

Proof. — It suffices to prove the result for geodesic defining functions. First use Proposition 5.15. Given $p, q \in M$, we can uniquely write $p = \pi(z)$, $q = \pi(\overline{\varphi}_\tau(z))$ for $z \in S^*M$ and $0 < \tau < \tau_+(z)$. We need to show that $\widetilde{d}_g(p,q)$ extends smoothly to $\mathcal{T}^+\overline{S^*M}$ as a function of (τ, z).

Now $d_g(p,q)$ is the elapsed time $t(\tau, z)$ for the geodesic segment joining p to q, given by (2.14). As in (2.18), (2.19), for $z \in \overline{S^*M}$ and s near $\tau_+(z)$ we have $\rho(\overline{\varphi}_s(z)) = (\tau_+(z) - s)A_+(s,z)$ for A_+ smooth satisfying $A_+(\tau_+(z), z) = 1$, and for $z \in \overline{S^*M}$ and s near $-\tau_-(z)$ we have $\rho(\overline{\varphi}_s(z)) = (s + \tau_-(z))A_-(s,z)$ for A_- smooth satisfying $A_-(-\tau_-(z), z) = 1$. Consequently

$$\rho(\overline{\varphi}_s(z))^{-1} = \big(s + \tau_-(z)\big)^{-1} + \big(\tau_+(z) - s\big)^{-1} + B(s,z)$$

with $B(s,z)$ smooth for $z \in \overline{S^*M}$, $s \in [-\tau_-(z), \tau_+(z)]$. Carrying out the integration in (2.14), one deduces that $t(\tau, z) + \log(\tau_+(z) - \tau) + \log \tau_-(z)$ extends smoothly to $\mathcal{T}^+\overline{S^*M}$. Since $\log \rho(\overline{\varphi}_\tau(z)) - \log(\tau_+(z) - \tau)$ and $\log \rho(z) - \log \tau_-(z)$ both extend smoothly to $\mathcal{T}^+\overline{S^*M}$, the result follows. □

Remark 5.20. — Theorem 1.2 of [45] and Theorem 6.4 of [46] assert that the conclusion of Proposition 5.19 holds under the assumption that (M,g) is geodesically convex, which is equivalent to assuming that it is non-trapping with no conjugate points. However, the proofs appear to be incomplete. Our proof above uses the additional assumption that there are no conjugate points at infinity.

Remark 5.21. — A study of d_g has been carried out for certain perturbations of hyperbolic space in [35]. This has been extended to non-trapping asymptotically hyperbolic manifolds in a neighborhood of the boundary diagonal by Chen–Hassell [5] and Sá Barreto–Wang [45, 46, 47]: the function $\beta^* d_g + \log \rho_L + \log \rho_R$ extends to a smooth function on $\mathcal{U} \setminus \mathrm{diag}_0$, where $\mathcal{U}$ is a neighborhood of the front face in the Mazzeo–Melrose stretched product space $\overline{M} \times_0 \overline{M}$, diag_0 denotes the closure of the lift of the interior diagonal, ρ_R and ρ_L are defining functions for the right and left faces in $\overline{M} \times_0 \overline{M}$, and β denotes the blow-down map. Combining this with Proposition 5.19, it follows that $\beta^* d_g + \log \rho_L + \log \rho_R \in C^\infty(\overline{M} \times_0 \overline{M} \setminus \mathrm{diag}_0)$ for simple asymptotically hyperbolic metrics. The analysis in these papers of the short geodesics is closely related to Lemma 2.8 above.

For a simple asymptotically hyperbolic metric and a choice of defining function ρ, we define the *renormalized boundary distance* $d_g^R \in C^\infty(\partial\overline{M} \times \partial\overline{M} \setminus \mathrm{diag})$ by

$$d_g^R := \widetilde{d}_g|_{\partial\overline{M} \times \partial\overline{M} \setminus \mathrm{diag}}. \tag{5.7}$$

The realization (4.1) of L_g shows that $d_g^R(p,q) = L_g(B^{-1}(p,q))$ with B defined in Corollary 5.18. Either using (4.2) or directly from the definition, it follows that if $\widehat{\rho} = \rho e^\omega$ is another choice of boundary defining function (with $\omega \in C^\infty(\overline{M})$), and if $\widehat{d_g^R}(p,q)$ denotes the renormalized distance associated to $\widehat{\rho}$, then

$$\widehat{d_g^R}(p,q) - d_g^R(p,q) = \omega(p) + \omega(q), \qquad p, q \in \partial\overline{M}.$$

The renormalized distance can be defined assuming only that there is a unique geodesic joining any two points of $\partial\overline{M}$. But if (M,g) is not simple, the map B in Corollary 5.18 is not a local diffeomorphism, and we do not how to prove Proposition 5.24 below.

Remark 5.22. — Recall that a Busemann function for a point $p \in \partial\overline{M}$ is defined (typically for a Hadamard manifold) as follows. Choose a geodesic $\gamma(t)$ for which $\lim_{t\to\infty} \gamma(t) = p$. The Busemann function associated to γ is the function $B_\gamma : M \to \mathbb{R}$ defined by $B_\gamma(q) = \lim_{t\to\infty} \big(d_g(q,\gamma(t)) - t\big)$. Observe that it follows from Proposition 5.19 that the function $d_g^1(p,q) = d_g(p,q) + \log\rho(p) = \widetilde{d}_g(p,q) - \log\rho(q)$ extends smoothly to $\overline{M} \times M \setminus \mathrm{diag}$.

If $p \in \partial\overline{M}$, the function $q \to d_g^1(p,q)$ on M is a Busemann function for p, depending on the choice of defining function ρ. In fact, it is clear that if γ is a geodesic and ρ is any defining function such that $\lim_{t\to\infty} \rho(\gamma(t))/e^{-t} = 1$, then $B_\gamma(q) = d_g^1(p,q)$. In particular, it is a consequence of Proposition 5.19 that on a simple asymptotically hyperbolic manifold, any Busemann function is in $C^\infty(M)$.

In the next two propositions we fix a representative h for the conformal infinity of a simple asymptotically hyperbolic manifold (M,g), thus determining a geodesic defining function ρ. d_g^R and L_g will denote the corresponding renormalized boundary distance and renormalized length function. The product identification associated to h induces the identification (2.2) of each of $\partial_\pm S^*M$ with $T^*\partial\overline{M}$. We thereby view L_g as defined on $T^*\partial\overline{M}$, and S_g as mapping $T^*\partial\overline{M}$ to itself.

Proposition 5.23. — *Let (M,g) be a simple asymptotically hyperbolic manifold and let h be a representative metric for the conformal infinity. If p, $q \in \partial\overline{M}$, $p \neq q$, then*

$$S_g\big(p, -\mathrm{d}_p(d_g^R(p,q))\big) = \big(q, \mathrm{d}_q(d_g^R(p,q))\big).$$

Here $d_g^R(p,q)$ is the renormalized distance function determined by h, $\mathrm{d}_p(d_g^R(p,q)) \in T_p^*\partial\overline{M}$ denotes its exterior derivative with respect to p, and $\mathrm{d}_q(d_g^R(p,q)) \in T_q^*\partial\overline{M}$ its exterior derivative with respect to q.

Proof. — Define $d_g^1 \in C^\infty(\overline{M} \times M \setminus \mathrm{diag})$ by

$$d_g^1(p',q') = d_g(p',q') + \log\rho(p') = -\log\rho(q') + \widetilde{d}_g(p',q').$$

If $p', q' \in M$, $p' \neq q'$, then $\operatorname{grad}_{q'} d_g^1(p',q') = \operatorname{grad}_{q'} d_g(p',q')$ is the unit tangent vector at q' to the geodesic joining p' and q', oriented to point away from p'. Here $\operatorname{grad}_{q'}$ denotes the gradient with respect to g in the second argument. Since (M,g) is simple and $d_g^1(p',q')$ is smooth in p' up to $\partial\overline{M}$, for $p \in \partial\overline{M}$ we can let $p' \to p$ along the geodesic joining p to q' to deduce that $\operatorname{grad}_{q'} d_g^1(p,q')$ is the unit tangent vector at q' to the geodesic joining p and q', oriented to point away from p. The corresponding dual element of $S_{q'}^*M$ is $\mathrm{d}_{q'}(d_g^1(p,q'))$.

Now fix $p, q \in \partial\overline{M}$, $p \neq q$, and consider the asymptotics of $\mathrm{d}_{q'}(d_g^1(p,q'))$ as q' approaches q along the geodesic joining p to q. Write $q' = (\rho, y)$ in the boundary identification induced by h. Recalling (5.7), we have

$$\begin{aligned}\mathrm{d}_{q'}(d^1_g(p,q')) &= \mathrm{d}_{q'}(-\log\rho + \widetilde{d}_g(p,q'))\\ &= -\rho^{-1}\mathrm{d}\rho + \mathrm{d}_\rho(\widetilde{d}_g(p,q')) + \mathrm{d}_y(\widetilde{d}_g(p,q'))\\ &= -\rho^{-1}\mathrm{d}\rho + \mathcal{O}(1)\mathrm{d}\rho + \mathrm{d}_q(d^R_g(p,q)) + \mathcal{O}(\rho)\mathrm{d}y.\end{aligned}$$

According to the identification (2.2), the limiting point in $\partial_+S^*M \cong T^*\partial\overline{M}$ is therefore $(q, \mathrm{d}_q(d^R_g(p,q)))$, as claimed.

The same argument interchanging the roles of p and q shows that the beginning point in $\partial_-S^*M \cong T^*\partial\overline{M}$ for the geodesic from p to q is $(p, -\mathrm{d}_p(d^R_g(p,q)))$. □

The relation $d^R_g(p,q) = L_g(B^{-1}(p,q))$ and the definition of B in terms of S_g in Corollary 5.18 show that the pair (L_g, S_g) determines the renormalized length d^R_g. We conclude by showing that the converse is true in the following sense:

Proposition 5.24. — *Let $\overline{M}$ be a compact connected manifold-with-boundary and let g_1 and g_2 be simple asymptotically hyperbolic metrics on M. Let h_1 and h_2 be representative metrics in the respective conformal infinities. If $d^R_{g_1} = d^R_{g_2}$, then $L_{g_1} = L_{g_2}$ and $S_{g_1} = S_{g_2}$.*

Proof. — Given $(p,\eta) \in T^*\partial\overline{M}$, let $q = \pi(S_{g_1}(p,\eta))$ be the ending point of the geodesic for g_1 starting from p with initial direction η. By Proposition 5.12, there is a unique geodesic for g_2 starting at p and ending at q. Proposition 5.23 shows that the starting and ending directions for a geodesic are determined by the endpoints and the renormalized distance function d^R_g. Since $d^R_{g_1} = d^R_{g_2}$, one concludes that $S_{g_1} = S_{g_2}$. Since $L_g(p,\eta) = d^R_g(B(p,\eta))$ and B is determined by S_g, it follows that also $L_{g_1} = L_{g_2}$. □

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