ANNALES

DE

L'INSTITUT FOURIER

Manuel HERBST & Karl-Hermann NEEB

On the First Order Cohomology of Infinite-Dimensional Unitary Groups

ON THE FIRST ORDER COHOMOLOGY OF INFINITE-DIMENSIONAL UNITARY GROUPS

by Manuel HERBST & Karl-Hermann NEEB

————

ABSTRACT. — We determine precisely for which irreducible unitary highest weight representation of the group $U(\infty)$, the countable direct limit of the finite-dimensional unitary groups $U(n)$, the corresponding 1-cohomology space H^1 does not vanish. This occurs in particular if a highest weight, viewed as an integer-valued function on $\mathbb{N}$, is finitely supported. In a second step, we extend the finitely supported highest weight representations to norm-continuous unitary representations of the Banach-completions $U_p(\ell^2)$ of the direct limit $U(\infty)$ with respect to the pth Schatten norm for $1 \leqslant p \leqslant \infty$. For $p < \infty$, the corresponding 1-cohomology spaces H^1 do not vanish either, except in three cases. We conclude that these groups do not have Kazhdan's Property (T). On the other hand, for $p = \infty$, the first cohomology spaces all vanish because $U_\infty(\ell^2)$ has property (FH) as a bounded topological group.

RÉSUMÉ. — Parmi les représentations unitaires irréductibles du groupe $U(\infty)$, la limite directe dénombrable des groupes unitaires de dimension finie $U(n)$, qui admettent un plus haut poids, nous déterminons précisément celles qui n'ont pas une 1-cohomologie triviale. Cela se produit en particulier si un plus haut poids, considéré comme une fonction de valeur entière sur $\mathbb{N}$, est une fonction à support fini. De plus, nous étendons les représentations admettant un plus haut poids à support fini en des représentations unitaires irréductibles des complétés de Banach $U_p(\ell^2)$ de la limite directe $U(\infty)$ par rapport à la norme p de Schatten pour $1 \leqslant p \leqslant \infty$. Si $p < \infty$, alors la 1-cohomologie n'est pas triviale non plus avec l'exception de trois cas particuliers. Nous en déduisons que ces groupes n'ont pas la propriété (T) de Kazhdan. De l'autre part, en cas de $p = \infty$, la 1-cohomologie est triviale puisque le groupe topologique $U_\infty(\ell^2)$ possède la propriété (FH).

1. Introduction

For a continuous unitary or orthogonal representation $(\pi, \mathcal{H})$ of the topological group G we call a map $\beta : G \to \mathcal{H}$ a *1-cocycle* if it satisfies the

1-cocycle relation

(1.1) $\beta(gh) = \beta(g) + \pi(g)\beta(h)$, for all $g, h \in G$.

A 1-cocycle of the form $\beta(g) = \partial_v(g) := \pi(g)v - v$ for some $v \in \mathcal{H}$ is called a *1-coboundary* or a *trivial* 1-cocycle. The vector space of all continuous 1-cocycles associated to $(\pi, \mathcal{H})$ is denoted by $Z^1(G, \pi, \mathcal{H})$, its subspace of 1-coboundaries by $B^1(G, \pi, \mathcal{H})$. The quotient space

$$H^1(G, \pi, \mathcal{H}) := Z^1(G, \pi, \mathcal{H})/B^1(G, \pi, \mathcal{H})$$

is called the *1-cohomology space*.

Interest in the first order cohomology of Lie groups has grown in the last decades since it is related to mathematical problems occurring in a wide range of mathematical disciplines such as geometric group theory, unitary representations, ergodic theory, stochastic processes and theoretical physics. Its motivation comes from the construction of irreducible unitary representations in Fock spaces (cf. [14, 21]), the study of unitary representations of mapping groups/current groups (cf. [1, 2, 14, 21]), the classification theory of negative definite functions (cf. [7]), Lévy processes and infinitely divisible probability distributions (cf. [21]) and continuous tensor products of unitary representations (cf. [11, 20, 25]). The study of group cohomology spaces was notably propagated in the 1970s due to influential papers by J. P. Serre. One of the key results developed during that decade was the Delorme–Guichardet Theorem which states that, for a σ-compact locally compact group G, the following are equivalent (cf. [4, Thm. 2.12.4, Prop. 2.2.10]):

Property (T). — There exists a compact subset $K \subseteq G$ and some positive constant $\varepsilon > 0$ such that every continuous unitary representation $(\pi, \mathcal{H})$, for which there exists a unit vector $v \in \mathcal{H}$ such that $\sup_{g \in K} \|\pi(g)v - v\| < \varepsilon$, has a non-trivial G-fixed vector, i.e. $\mathcal{H}^G \neq \{0\}$.

Property (FH). — $H^1(G, \pi, \mathcal{H}) = 0$ for every continuous orthogonal representation $(\pi, \mathcal{H})$ of G.

More precisely, the implication (T) $\Longrightarrow$ (FH) holds for arbitrary topological groups whereas the σ-compactness is necessary for the converse implication (cf. [4, Rem. 2.12.5]).

It is a well-known fact that a 1-cocycle is trivial if and only if it is bounded (cf. [4, Prop. 2.2.9]). This implies that compact groups have property (FH) (see also [21, Thm. 15.1]). In the realm of irreducible unitary representations of finite dimensional Lie groups, the vanishing of the first cohomology

spaces occurs surprisingly often: In view of the Delorme–Guichardet Theorem, [4, Thm. 3.5.4] states that a connected semi-simple Lie group G has property (FH) if and only if no simple factor of its Lie algebra is isomorpic to $\mathfrak{su}(n,1)$ or $\mathfrak{so}(n,1)$ (for $n \in \mathbb{N}$). Moreover, up to equivalence, there exists only a finite number of (topologically complete) irreducible continuous unitary representations $(\pi, \mathcal{H})$ of a connected semi-simple Lie group, for which $H^1(G, \pi, \mathcal{H}) \neq \{0\}$ (cf. [23, Prop. 11]). For a connected nilpotent Lie group, one can show that $H^1(G, \pi, \mathcal{H}) = \{0\}$ holds for every nontrivial, irreducible continuous unitary representation $(\pi, \mathcal{H})$ (e.g. [21, Thm. 17.4]).

In this article, we turn to infinite-dimensional Lie groups: Let $\mathcal{H} \cong \ell^2(\mathbb{N}, \mathbb{C})$ be an infinite-dimensional complex separable Hilbert space. We consider the unitary groups

$$(1.2) \qquad U(\infty) \subset U_1(\mathcal{H}) \subset \cdots \subset U_p(\mathcal{H}) \subset \cdots \subset U_\infty(\mathcal{H}) \subset U(\mathcal{H}).$$

Here $U(\infty)$ is the increasing union of the compact Lie groups $U(n) := U(n, \mathbb{C})$, endowed with the direct limit Lie group structure (cf. [8, Thm. 4.3]). Its Lie algebra $\mathfrak{u}(\infty) = \varinjlim \mathfrak{u}(n)$ is an increasing union of finite dimensional compact Lie algebras (cf. [8, Thm. 4.3]). The other groups $U_p(\mathcal{H}) := U(\mathcal{H}) \cap (\mathbb{1} + \mathcal{B}_p(\mathcal{H}))$ (for $1 \leqslant p \leqslant \infty$) are Banach–Lie groups whose Lie algebras $\mathfrak{u}_p(\mathcal{H}) = \mathfrak{u}(\mathcal{H}) \cap \mathcal{B}_p(\mathcal{H})$ are determined by the pth Schatten ideals $\mathcal{B}_p(\mathcal{H})$ (cf. [12, §II.5, II.6]). Note that $\mathcal{B}_\infty(\mathcal{H})$ is the space of compact operators on $\mathcal{H}$. In [15], unitary highest weight representations of the group $U(\infty)$ have been classified by the orbits of the weights $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ in $\mathbb{Z}^\mathbb{N}$ under the Weyl group $\mathcal{W} \cong S_{(\mathbb{N})}$. This means that two highest weight representations $(\pi_\lambda, \mathcal{H}_\lambda)$ and $(\pi_\mu, \mathcal{H}_\mu)$ are equivalent if and only if the entries of the weights λ and μ coincide up to a finite permutation. The unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ extends to the group $U_1(\mathcal{H})$ if and only if the weight λ is bounded (cf. Proposition III.7 and Theorem III.4 in [15]). Moreover, the proof of Proposition III.10 in [15] shows that $(\pi_\lambda, \mathcal{H}_\lambda)$ extends to the group $U_p(\mathcal{H})$ for some $p > 1$ if and only if the weight λ has finite support. The problem is to determine for which weights λ, the corresponding first cohomology spaces are trivial. The case $\lambda = 0$ corresponds to the one-dimensional trivial representation and the fact that there is no nonzero continuous group homomorphism $G \to \mathbb{R}$ for each of the unitary groups G from (1.2) implies that the corresponding 1-cohomology spaces vanish (cf. Remark 3.7). For $\lambda \neq 0$, we derive the following solutions:

THEOREM 4.10. — *For the direct limit $U(\infty)$, the first cohomology space $H^1(U(\infty), \pi_\lambda, \mathcal{H}_\lambda)$ vanishes if and only if either all but finitely many entries of λ are positive integers or all but finitely many entries are negative*

*integers. In particular, for finitely supported weights λ, the corresponding
1-cohomology spaces never vanish.*

THEOREM 7.9. — *For $1 \leqslant p < \infty$ and for finitely supported nonzero
weights λ, the first cohomology space $H^1(U_p(\mathcal{H}), \pi_\lambda, \mathcal{H}_\lambda)$ vanishes in exactly two cases, namely either if $U_p(\mathcal{H})$ acts on $\mathcal{H}$ via the identical representation or if $U_p(\mathcal{H})$ acts on $\mathcal{H}^*$ via the dual representation. This corresponds
to the case $\lambda = (1, 0, 0, 0, \dots)$ (natural action) resp. $\lambda = (-1, 0, 0, 0, \dots)$
(conatural action).*

In particular, Theorem 7.9 reveals that the group $U_p(\mathcal{H})$ (with $1 \leqslant p < \infty$) neither has property (FH) nor has property (T). For $p = 2$, the fact
that $U_2(\mathcal{H})$ does not have property (T) has already been shown in [22].
The question whether $U_2(\mathcal{H})$ has property (FH) is however stated there as
an open problem (see [22, 3.5]) which is answered by Theorem 7.9.

[3, Prop. 6.5, Rem. 6.8 (a)] shows that the full unitary group $U(\mathcal{H})$ and
the group $U_\infty(\mathcal{H})$ belong to the class of *bounded* groups. We call a topological group G *bounded* if, for every identity neighborhood $U \subseteq G$, there
exists an integer $n \in \mathbb{N}$ and a finite subset $F \subseteq G$ such that $G \subseteq U^n F$.[1]
That $U(\mathcal{H})$ and $U_\infty(\mathcal{H})$ are bounded groups is also shown in [18] with a
proof relying on the spectral calculus of operators on Hilbert spaces. It is
immediate from the definition that bounded groups have property (FH).[2]
Hence, if $G \in \{U_\infty(\mathcal{H}), U(\mathcal{H})\}$ and $(\pi, \mathcal{H})$ is a continuous unitary representation of G, then $H^1(G, \pi, \mathcal{H}) = \{0\}$ (see also [24]). The unitary groups
$U(\infty)$ and $U_p(\mathcal{H})$ for $1 \leqslant p < \infty$ are not bounded because they have
nontrivial 1-cohomology spaces.

The core ideas for the proof of Theorem 4.10 are the following: Any
1-cocycle of one of the unitary groups above has the property that its
restriction to one of the compact subgroups $U(n)$ yields a 1-coboundary.
This leads us to the notion of a *conditional* 1-cocycle in Section 3. If $(G_j)_{j \in J}$
is a family of subgroups of the group G, then we call a 1-cocycle *conditional*
if its restriction to every subgroup G_j yields a 1-coboundary. A special
case arises if $G = \varinjlim G_n$ is the direct limit of an increasing sequence of
subgroups G_n. The statement that every conditional 1-cocycle (w.r.t. the
family of the subgroups G_n) is a 1-coboundary is then equivalent to the

---

[1] Using the terminology of [3], this can be expressed as G being bounded in the right
uniformity of the topological group G. If G is connected, one can always use $F = \{e\}$
which leads some authors to adopt this as a requirement in their definition.

[2] Indeed, if $\beta : G \to \mathcal{H}$ is a continuous 1-cocycle associated to a continuous unitary
representation $(\pi, \mathcal{H})$ of a bounded group G, then $U := \{g \in G : \|\beta(g)\| < 1\}$ is an
open identity neighborhood and $G \subset U^n F$ for some $n \in \mathbb{N}$ and some finite subset F
implies $\sup_{g \in G} \|\beta(g)\| \leqslant n + \max_{f \in F} \|\beta(f)\| < \infty$.

statement that every G_n-fixed vector is already fixed by the whole group G provided that n is sufficiently large (cf. Proposition 3.6). In Section 4, we introduce the highest weight representations $(\pi_\lambda, \mathcal{H}_\lambda)$ of $U(\infty)$ as a direct limit of highest weight representations of the subgroups $U(n)$ and we directly conclude that the representation $(\pi_\lambda, \mathcal{H}_\lambda)$ admits a $U(n)$-fixed vector if and only if one of the highest weight subrepresentations of $U(k)$ (with $k \geqslant n$) admits a $U(n)$-fixed vector. This observation allows us to reformulate the problem as a matter of branching from $U(k)$ to $U(n)$. Thus, Theorem 4.10 is obtained by applying the classical branching law as stated in Theorem 4.1.

Our strategy to prove Theorem 7.9 can be outlined as follows: For finitely supported λ, we realize the extended highest weight representations $(\pi_\lambda, \mathcal{H}_\lambda)$ of the group $U_p(\mathcal{H})$ (for $1 \leqslant p < \infty$) as subrepresentations of finite tensor product representations that are built from the identical and its dual action of $U_p(\mathcal{H})$ on $\mathcal{H}$ respectively on the dual $\mathcal{H}^*$ (cf. Section 7). For these finite tensor product representations (with at least two factors), it is easy to construct nontrivial 1-cocycles as a countably infinite sum of 1-coboundaries (cf. Section 6). Projecting these 1-cocycles onto the corresponding highest weight submodules gives nontrivial 1-cocycles for the highest weight representations $(\pi_\lambda, \mathcal{H}_\lambda)$ (cf. Proposition 7.8). That the first cohomology spaces for the identical representations on $\mathcal{H}$ and the dual representation on $\mathcal{H}^*$ (cf. Section 5) vanish is shown in Remark 3.2 and Proposition 5.1.

In a subsequent paper, we plan to treat the case where λ is bounded with infinite support and the corresponding highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $U(\infty)$ extends to the group $U_1(\mathcal{H})$. The problem here is that it is much harder to realize the representation $(\pi_\lambda, \mathcal{H}_\lambda)$ concretely than it is in the finitely supported case. If the first cohomology space vanishes for the group $U(\infty)$, then it also does for the group $U_1(\mathcal{H})$ since $U(\infty)$ is dense in $U_1(\mathcal{H})$. But this is not the case for every bounded weight with infinite support and this situation remains, at the present stage, an open problem.

2. Notation, terminology and preliminaries

Throughout this article, we use the symbol $\mathcal{H}$ for a complex Hilbert space. We follow the convention that the scalar product $\langle \cdot, \cdot \rangle_\mathcal{H}$ of a complex Hilbert space $\mathcal{H}$ is linear in the first argument and antilinear in the

second. If V is any complex vector space, we write $\overline{V}$ for the complex vector space with the same addition operation but with scalar multiplication $(\lambda, v) \mapsto \overline{\lambda}v$ for $v \in V$ and $\lambda \in \mathbb{C}$. We use the symbol G for a (topological) group and write e for its neutral element. If not otherwise stated, we always assume that any topological group is a Hausdorff space. A *continuous unitary representation* $(\pi, \mathcal{H})$ of G on a complex Hilbert space $\mathcal{H}$ is a group homomorphism $\pi : G \to U(\mathcal{H})$ which is continuous with respect to the strong operator topology on the unitary group $U(\mathcal{H})$. The unitary representation $(\pi, \mathcal{H})$ is called *norm-continuous*, if π is continuous with respect to the norm-topology on $U(\mathcal{H})$. We stick to the convention of writing $\mathcal{H}^G$ for the subspace of G-fixed vectors.

2.1. On the Lie groups used in this article

Throughout this article, a Lie group G is either a real Banach–Lie group or a direct limit of real finite-dimensional Lie groups. We write $\mathfrak{g} := \mathbf{L}(G) \cong T_e(G)$ for its (real) Lie algebra. The corresponding Lie bracket is denoted by $[\,\cdot\,,\cdot\,]$. Elements $z \in \mathfrak{g}_{\mathbb{C}}$ of the complexified Lie algebra split into $z = x + \mathrm{i}\,y$, where $x, y \in \mathfrak{g}$ are uniquely determined and are therefore called the *real* resp. *imaginary part* of z. We obtain an involution on $\mathfrak{g}_{\mathbb{C}}$ via $z^* := -x + \mathrm{i}\,y$ for $z = x + \mathrm{i}\,y$.

Let G be a Banach–Lie group and $(\pi, \mathcal{H})$ be a norm-continuous unitary representation of G. Every continuous 1-cocycle $\beta : G \to \mathcal{H}$ defines a continuous group homomorphism of Banach–Lie groups

$$\alpha : G \to \mathcal{H} \rtimes U(\mathcal{H}), \quad g \mapsto (\beta(g), \pi(g)).$$

By the Automatic Smoothness Theorem of Banach–Lie groups [3] we thus obtain a smooth map. In particular, both maps

$$\pi : G \to U(\mathcal{H}), \quad \beta : G \to \mathcal{H}$$

are smooth. Taking the derivative at e, we obtain continuous $\mathbb{R}$-linear maps

$$\mathrm{d}\pi : \mathfrak{g} \to \mathfrak{u}(\mathcal{H}), \quad \mathrm{d}\beta : \mathfrak{g} \to \mathcal{H}$$

satisfying, for arbitrary $x, y \in \mathfrak{g}$, the relations

$$\mathrm{d}\pi([x,y]) = \mathrm{d}\pi(x)\mathrm{d}\pi(y) - \mathrm{d}\pi(y)\mathrm{d}\pi(x)$$
$$\mathrm{d}\beta([x,y]) = \mathrm{d}\pi(x)\mathrm{d}\beta(y) - \mathrm{d}\pi(y)\mathrm{d}\beta(x).$$

Complexification yields continuous $\mathbb{C}$-linear maps

$$\mathrm{d}\pi_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathcal{B}(\mathcal{H}), \quad \mathrm{d}\beta_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathcal{H}.$$

---

[3] For more details see [17, Thm. IV.1.18] and the references given there.

2.2. Unitary highest weight representations of the group $U(n)$

Let G be a compact Lie group. Standard references for the classification of irreducible unitary representations of G in terms of highest weights are [27] (for the case $G = U(n)$) and [10] (for the classical groups). For the infinite-dimensional unitary groups, we refer to the survey in [16], where the reader also finds an introduction to Banach–Lie groups and crucial results from [15]. Here, we briefly introduce the terms *highest weight* and *highest weight representation* for the group $U(n)$ because these concepts are mentioned several times throughout this paper.

Let $(\pi, \mathcal{H})$ be an irreducible, continuous unitary representation of $U(n)$. Then, $\mathcal{H}$ is finite–dimensional and the derived Lie algebra representation $(d\pi_{\mathbb{C}}, \mathcal{H})$ of $\mathfrak{gl}(n, \mathbb{C})$ is also irreducible. A *weight* λ of the representation $(d\pi_{\mathbb{C}}, \mathcal{H})$ is a linear functional $\lambda : \mathfrak{h} \to \mathbb{C}$ on the maximal abelian subalgebra $\mathfrak{h}$ of diagonal matrices of the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$. Let $\mathfrak{b}$ be a maximal solvable Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ containing $\mathfrak{h}$. It is a consequence of Lie's Theorem on the finite dimensional representations of solvable Lie algebras that there exists a (unique) one-dimensional $\mathfrak{b}$-eigenspace in $\mathcal{H}$. The corresponding $\mathfrak{b}$-eigenvalue is a Lie algebra homomorphism $\lambda : \mathfrak{b} \to \mathbb{C}$ and its restriction to $\mathfrak{h}$ is a weight which is called a *highest weight (for $\mathfrak{b}$)* of the representation $(\pi, \mathcal{H})$. For instance, we may choose $\mathfrak{b}$ to be the upper triangular matrices in $\mathfrak{gl}(n, \mathbb{C})$. Then, one can show that for $x = \text{diag}(x_1, \ldots, x_n) \in \mathfrak{h}$, we have $\lambda(x) = \sum_{i=1}^{n} \lambda_i x_i$ with decreasingly ordered integer-valued coefficients λ_i. This means that any irreducible, continuous unitary representation of $U(n)$ defines a decreasingly ordered integer-valued n-tuple. Conversely, to every $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ there exists an irreducible unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $U(n)$ with highest weight λ (w.r.t. the solvable subalgebra of upper triangular matrices). We call $(\pi_\lambda, \mathcal{H}_\lambda)$ a *unitary highest weight representation*. Two highest weight representations $(\pi_\lambda, \mathcal{H}_\lambda)$, $(\pi_\mu, \mathcal{H}_\mu)$ are isomorphic if and only if $\lambda = \mu$. Therefore, we obtain a one to one correspondence between the equivalence classes of irreducible, continuous unitary representations of $U(n)$ and the decreasingly ordered vectors in $\mathbb{Z}^n$.

For tuples $\lambda \in \mathbb{Z}^n$ which are not necessarily decreasingly ordered, we write $(\pi_\lambda, \mathcal{H}_\lambda)$ for the unitary highest weight representation corresponding to the decreasingly ordered tuple λ' which is obtained after a suitable permutation of the entries of λ. This has the following background: Every tuple $\lambda \in \mathbb{Z}^n$ occurs as a highest weight of some irreducible unitary representation of $U(n)$ for some solvable subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. It is a

general fact in highest weight theory that two highest weight representations are isomorphic if and only if the corresponding weights belong to the same orbit under the Weyl group action on the set of weights. The Weyl group action on $\mathbb{Z}^n$ can be realized by the natural permutation action of S_n on $\mathbb{Z}^n$.

The polar map

$$p : U(n) \times \mathfrak{u}(n) \to \mathrm{GL}(n, \mathbb{C}), \quad (g, x) \mapsto g \exp(\mathrm{i}\, x)$$

is a diffeomorphism and therefore, every continuous unitary representation $(\pi, \mathcal{H})$ of $U(n)$ extends to a holomorphic[4] representation of $\mathrm{GL}(n, \mathbb{C})$ via $\pi_{\mathbb{C}}(g \exp(\mathrm{i}\, x)) := \pi(g) \exp(\mathrm{i}\, \mathrm{d}\pi_{\mathbb{C}}(x))$. This extension is unique (universal complexification) and, in view of Weyl's Unitarian Trick, there is a one-to-one correspondence between continuous, irreducible unitary representations of $U(n)$ and irreducible, holomorphic representations of $\mathrm{GL}(n, \mathbb{C})$. In particular, the classification of the equivalence classes of holomorphic irreducible representations of $\mathrm{GL}(n, \mathbb{C})$ in terms of highest weights is the same as for the group $U(n)$. In this paper, we will use occasionally some results from the highest weight theory of the group $\mathrm{GL}(n, \mathbb{C})$ (especially from [10]). The preceding remark shows that they may be applied equally well to the group $U(n)$.

3. Conditional 1-cocycles

In this section, we set the foundation for our analysis of the 1-cocycles of the unitary groups $U(\infty)$ and $U_p(\mathcal{H})$ (for $1 \leqslant p < \infty$) in Sections 4 and 5. Assume that a topological group G admits an increasing sequence of subgroups (such that the union is dense in G). If the restriction of a 1-cocycle on G to each of these subgroups yields a 1-coboundary, then we call it a *conditional* 1-cocycle. This concept is based on the observation that the restriction of any 1-cocycle on the unitary groups $U(\infty)$ and $U_p(\mathcal{H})$ (for $1 \leqslant p < \infty$) to any of the compact groups $U(n)$ yields a trivial 1-cocycle. Any conditional 1-cocycle may be viewed as a limit of a sequence of 1-coboundaries and the question whether the coboundary property is preserved under this limit can be translated into handy necessary and sufficient criteria (cf. Lemma 3.3). Proposition 3.6 applies in particular to the group $U(\infty)$ and provides a solution to the problem to decide whether a 1-cohomology space of $U(\infty)$ is trivial.

---

[4] In the sense that the derivative is complex linear.

Let G be a topological group with an ascending sequence of topological subgroups $(G_n)_{n \in \mathbb{N}}$ such that $G_\infty := \bigcup_{n \in \mathbb{N}} G_n$ is a dense subgroup in G.

DEFINITION 3.1. — *Let $(\pi, \mathcal{H})$ be a continuous unitary representation of G. A 1-cocycle $\beta : G \to \mathcal{H}$ is called conditional if its restriction to every subgroup G_n is a 1-coboundary. In particular, every 1-coboundary on G is a conditional 1-cocycle. We write $Z^1_{\text{cond}}(G, \pi, \mathcal{H})$ for the vector space of conditional 1-cocycles and $H^1_{\text{cond}}(G, \pi, \mathcal{H})$ for the quotient $Z^1_{\text{cond}}(G, \pi, \mathcal{H})/B^1(G, \pi, \mathcal{H})$.*

We consider a conditional 1-cocycle $\beta \in Z^1_{\text{cond}}(G, \pi, \mathcal{H})$. Restricting β to the subgroup G_n yields a 1-coboundary by assumption, i.e. $\beta|_{G_n} = \partial_{v_n}$ for some $v_n \in \mathcal{H}$. The vector v_n is unique up to adding a G_n-fixed vector. We shall write $\mathcal{H}_n := \mathcal{H}^{G_n}$ for the subspace of G_n-fixed vectors. Any conditional 1-cocycle β thus defines a unique sequence $(v_n \in \mathcal{H}_n^\perp)_{n \in \mathbb{N}}$ with the following compatibility condition

$$(3.1) \qquad v_n - v_m \in \mathcal{H}_m \text{ if } m \leqslant n.$$

This means that we can write $\beta(g) = \lim_{n \to \infty} \pi(g)v_n - v_n$ for any $g \in G_\infty$. We will reformulate the compatibility condition in two ways:

First, note that the spaces $\mathcal{H}_n$ form a decreasing sequence of Hilbert subspaces and the intersection is given by $\bigcap_{n \in \mathbb{N}} \mathcal{H}_n = \mathcal{H}^G$, the subspace of G-fixed vectors. For $n = 0$ we put $G_0 := \{e\}$, $\mathcal{H}_0 := \mathcal{H}$ and $v_0 := 0$. In view of $\mathcal{H}_n^\perp \cong \bigoplus_{k=1}^n (\mathcal{H}_k^\perp \cap \mathcal{H}_{k-1})$, we can write $v_n = \sum_{k=1}^n w_k$ for uniquely determined $w_k \in W_k := \mathcal{H}_k^\perp \cap \mathcal{H}_{k-1}$. The compatibility condition (3.1) then reads $w_k = v_k - v_{k-1}$, hence every conditional 1-cocycle β defines a unique sequence $(w_n \in W_n)_{n \in \mathbb{N}}$. For $g \in G_\infty$, we then have

$$(3.2) \qquad \beta(g) = \sum_{n \in \mathbb{N}} \pi(g)w_n - w_n.$$

A second way to interpret (3.1) is as follows: For any $v \in \bigcup_{n \in \mathbb{N}} \mathcal{H}_n^\perp$, the limit $\lim_{n \to \infty} \langle v_n, v \rangle$ exists and we obtain an antilinear functional $a : \bigcup_{n \in \mathbb{N}} \mathcal{H}_n^\perp \to \mathbb{C}$, $v \mapsto \lim_{n \to \infty} \langle v_n, v \rangle$ which has the property that its restriction to every subspace $\mathcal{H}_n^\perp$ is continuous. Moreover, if $g \in G_\infty$ and $v \in \mathcal{H}$, then $\pi(g)v - v \in \bigcup_{n \in \mathbb{N}} \mathcal{H}_n^\perp$ and we obtain the relation

$$(3.3) \quad a(\pi(g)v - v) = \lim_{n \to \infty} \langle v_n, \pi(g)v - v \rangle$$
$$= \lim_{n \to \infty} \langle \pi(g^{-1})v_n - v_n, v \rangle = \langle \beta(g^{-1}), v \rangle.$$

Remark 3.2. — In a Lie group context, we assume that G is a Banach–Lie group with an increasing sequence $(G_n)_{n \in \mathbb{N}}$ of Banach–Lie subgroups such that $G_\infty = \bigcup_{n \in \mathbb{N}} G_n$ is a dense subgroup. The corresponding Banach–Lie

algebras $\mathfrak{g}_n$ form an increasing sequence of Lie subalgebras of $\mathfrak{g}$. We write $\mathfrak{g}_\infty := \bigcup_{n \in \mathbb{N}} \mathfrak{g}_n \subseteq \mathfrak{g}$ for the union which is a Lie subalgebra of $\mathfrak{g}$. Now, let $(\pi, \mathcal{H})$ be a norm-continuous unitary representation of G and let $\beta \in Z^1_{\mathrm{cond}}(G, \pi, \mathcal{H})$. Recall that we obtain continuous linear maps $\mathrm{d}\pi_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathcal{B}(\mathcal{H})$ and $\mathrm{d}\beta_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathcal{H}$ via complex linear extension of the derivatives. If $(v_n)_{n \in \mathbb{N}}$ is the sequence defined by β, then the compatibility condition (3.1) implies $\mathrm{d}\beta_{\mathbb{C}}(z) = \lim_{n \to \infty} \mathrm{d}\pi_{\mathbb{C}}(z)v_n$. For any $z \in \mathfrak{g}_{\infty, \mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$ and $v \in \mathcal{H}$, we have $\mathrm{d}\pi_{\mathbb{C}}(z)v \in \bigcup_{n \in \mathbb{N}} \mathcal{H}_n^{\perp}$ and obtain

$$(3.4) \quad a(\mathrm{d}\pi_{\mathbb{C}}(z)v) = \lim_{n \to \infty} \langle v_n, \mathrm{d}\pi_{\mathbb{C}}(z)v \rangle$$
$$= \lim_{n \to \infty} \langle \mathrm{d}\pi_{\mathbb{C}}(z^*)v_n, v \rangle = \langle \mathrm{d}\beta_{\mathbb{C}}(z^*), v \rangle.$$

This fact will be used later in Section 5.

LEMMA 3.3. — *Let β, v_n and w_n be as above. For a vector $v \in (\mathcal{H}^G)^{\perp}$, the following are equivalent:*

(1) *The conditional 1-cocycle β is a 1-coboundary with $\beta = \partial_v$.*
(2) *The sequence $(w_n)_{n \in \mathbb{N}}$ is square-summable and $v = \sum_{n \in \mathbb{N}} w_n$.*
(3) *The sequence $(v_n)_{n \in \mathbb{N}}$ converges in $\mathcal{H}$ with limit v.*
(4) *The antilinear functional a extends to a continuous antilinear functional on $\mathcal{H}$ such that $a(v') = \langle v, v' \rangle$ for all $v' \in \mathcal{H}$.*

Proof.

$(1) \implies (2)$. — For each vector $v_n = \sum_{k=1}^n w_k$, we have $v - v_n \in \mathcal{H}_n = \mathcal{H}^{G_n}$, hence $(v - v_n) \perp v_n$. Since the vectors w_k are mutually orthogonal, we find $\|v\|^2 \geqslant \sum_{k=1}^n \|w_k\|^2$ for all $n \in \mathbb{N}$. This shows that the sequence $(w_k)_{k \in \mathbb{N}}$ is square integrable and the series $\sum_{k=1}^{\infty} w_k$ converges in $\mathcal{H}$. By virtue of (3.2), we have $[\pi(g) - \mathbb{1}]v = [\pi(g) - \mathbb{1}] \sum_{k=1}^{\infty} w_k$ for each $g \in G$ and, since both v and $\sum_{k=1}^{\infty} w_k$ belong to $(\mathcal{H}^G)^{\perp}$, we conclude $v = \sum_{k=1}^{\infty} w_k$.

$(2) \implies (3)$. — This is clear, since the convergence of the series implies that the sequence $(v_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $\mathcal{H}$ whose limit coincides with the limit of the series.

$(3) \implies (4)$. — This follows directly from the definition of a.

$(4) \implies (1)$. — For any $v' \in \mathcal{H}$ and $g \in G$, we have $\langle \pi(g^{-1})v - v, v' \rangle = a(\pi(g)v' - v') = \langle \beta(g^{-1}), v' \rangle$ by equation (3.3). This shows that $\beta = \partial_v$. $\quad \square$

Lemma 3.3 shows that if all but finitely many of the spaces W_n are trivial, then every conditional 1-cocycle is trivial. This occurs in particular, if the Hilbert space $\mathcal{H}$ is finite dimensional.

Remark 3.4. — We have $\left(\mathcal{H}^{G_n}\right)^\perp = \bigoplus_{k=1}^n W_k$ for any $n \in \mathbb{N}$ and $\left(\mathcal{H}^G\right)^\perp = \widehat{\bigoplus}_{k \in \mathbb{N}} W_k$. This shows that, for any $n \in \mathbb{N}$,

$$(\forall\, k > n) \quad W_k = \{0\} \iff \mathcal{H}^{G_n} = \mathcal{H}^G.$$

Thus, we have shown the following corollary.

COROLLARY 3.5. — If $\mathcal{H}^G = \mathcal{H}^{G_n}$ for some $n \in \mathbb{N}$, then

$$H^1_{cond}(G, \pi, \mathcal{H}) = \{0\}.$$

We conclude our discussion on conditional 1-cocycles by specializing to the case where $G = \bigcup_{n \in \mathbb{N}} G_n$. We write $G = \varinjlim G_n$ whenever the group topology on G coincides with the direct limit topology. This requirement ensures that every sequence $(w_n \in W_n)_{n \in \mathbb{N}}$ defines a conditional 1-cocycle via equation (3.2). Thus, the space $Z^1_{cond}(G, \pi, \mathcal{H})$ may be described in terms of the spaces W_n and we conclude that the converse statement of Corollary 3.5 holds.

PROPOSITION 3.6. — For $G = \varinjlim G_n$, we have the following statements
(1) $Z^1_{cond}(G, \pi, \mathcal{H}) \cong \prod_{n \in \mathbb{N}} \left(\mathcal{H}^{G_n}\right)^\perp \cap \mathcal{H}^{G_{n-1}}$.
(2) $H^1_{cond}(G, \pi, \mathcal{H}) = \{0\} \iff (\exists\, n \in \mathbb{N}) \quad \mathcal{H}^{G_n} = \mathcal{H}^G$.

Proof.

(1). — Every sequence $(w_n \in W_n)_{n \in \mathbb{N}}$ defines a map $\beta : G \to \mathcal{H}$ via $\beta(g) := \sum_{n \in \mathbb{N}} \pi(g)w_n - w_n$ for $g \in G$. Then, the restriction to every subgroup G_n yields a 1-coboundary $\beta|_{G_n} = \partial_{v_n}$ with $v_n = \sum_{k=1}^n w_k$. In particular, the restriction of β to G_n is continuous. Therefore, β is continuous with respect to the direct limit topology on G. This shows that β is a conditional 1-cocycle. The above mapping defines an isomorphism of vector spaces $\prod_{n \in \mathbb{N}} W_n \to Z^1_{cond}(G, \pi, \mathcal{H})$.

(2). — In view of Corollary 3.5 and Remark 3.4, it remains to show the implication $H^1_{cond}(G, \pi, \mathcal{H}) = \{0\} \implies (\exists\, n \in \mathbb{N})(\forall\, k > n) \quad W_k = \{0\}$. The requirement $Z^1_{cond}(G, \pi, \mathcal{H}) = B^1(G, \pi, \mathcal{H})$ amounts to saying that every sequence $(w_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} W_n$ has to be square summable (Lemma 3.3 and (1)). This possible only if all but finitely many W_n are trivial. □

Remark 3.7. — Assume that all subgroups G_n are compact. Then, the restriction of every 1-cocycle on G to one of the G_n is a 1-coboundary, hence every 1-cocycle on G is conditional. This means that

$$Z^1(G, \pi, \mathcal{H}) = Z^1_{cond}(G, \pi, \mathcal{H}) \quad \text{and} \quad H^1(G, \pi, \mathcal{H}) = H^1_{cond}(G, \pi, \mathcal{H}).$$

Note, that Corollary 3.5 shows that for $\pi = \mathbb{1}$, one has $\{0\} = H^1(G, \mathbb{1}, \mathcal{H}) \cong \operatorname{Hom}_{Grp}(G, \mathcal{H})$. This is a version of the fact that there is no nontrivial,

continuous group homomorphism of G into the group of additive real numbers. Moreover, the direct limit topology on $G_\infty = \bigcup_{n \in \mathbb{N}} G_n$, i.e. the finest group topology on G_∞ for which all inclusions $G_n \hookrightarrow G_\infty$ are continuous, is a group topology (cf. [26, Thm. 2]). Hence, Proposition 3.6 applies in particular to the group G_∞. The unitary group $U(\infty)$ is a direct limit of compact subgroups which are given by the unitary $n \times n$-matrices $U(n)$. Every unitary group $U_p(\mathcal{H})$ (for $\mathcal{H} = \ell^2(\mathbb{N}, \mathbb{C})$ and $1 \leqslant p < \infty$) contains $U(\infty)$ as a dense subgroup. The one-dimensional trivial representation of the unitary groups $U(\infty)$ and $U_p(\mathcal{H})$ is the unitary highest weight representation that corresponds to the weight $\lambda = 0$ and we have $H^1(G, \pi_0, \mathcal{H}_0) = \{0\}$ for every unitary group G occuring in (1.2).

4. Unitary highest weight representations of $U(\infty)$

In this section, we define for every $\lambda \in \mathbb{Z}^{\mathbb{N}}$ an irreducible unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $U(\infty)$ that we call *unitary highest weight representation (with highest weight λ)*. We determine for which λ the spaces $H^1(U(\infty), \pi_\lambda, \mathcal{H}_\lambda)$ are trivial (cf. Theorem 4.10). The main ingredients are the classical Branching Law for the highest weight representations of the unitary groups $U(n)$ (cf. Theorem 4.1) and Proposition 3.6.

Let $n \in \mathbb{N}$. Recall from Section 2 that for any integer-valued tuple $\lambda \in \mathbb{Z}^n$, one can associate a unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $U(n)$. This representation is irreducible and every irreducible unitary representation of $U(n)$ is isomorphic to some $(\pi_\lambda, \mathcal{H}_\lambda)$. Two highest weight representations $(\pi_\lambda, \mathcal{H}_\lambda)$, $(\pi_\mu, \mathcal{H}_\mu)$ are isomorphic if and only if the entries of the weights λ and μ coincide up to permutation. For $n \geqslant 2$ the subgroup $U(n-1)$ decomposes a highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ into a finite sum of highest weight representations $(\pi_\eta, \mathcal{H}_\eta)$ of $U(n-1)$ and it is natural to ask which weights $\eta \in \mathbb{Z}^{n-1}$ occur in the decomposition. The answer is a classical result in branching theory.

THEOREM 4.1. — *Let $\eta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{Z}^{n+1}$ be two decreasingly ordered integer-valued tuples. Then, the unitary highest weight representation $(\pi_\eta, \mathcal{H}_\eta)$ of $U(n)$ is a subrepresentation of $(\pi_\lambda, \mathcal{H}_\lambda)$ if and only if the tuples η and λ satisfy the interlacing condition*

$$(4.1) \qquad \lambda_1 \geqslant \eta_1 \geqslant \lambda_2 \geqslant \eta_2 \geqslant \cdots \geqslant \lambda_n \geqslant \eta_n \geqslant \lambda_{n+1}.$$

Proof. — A proof can be found e.g. in [10, Thm. 8.1.1]. $\square$

DEFINITION 4.2. — *Let $\eta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{Z}^{n+1}$ be two integer-valued tuples. We say that η interlaces λ and write $\eta \preccurlyeq \lambda$ if condition (4.1) is satisfied after a suitable permutation of the entries of both tuples.*

Now, let $\lambda \in \mathbb{Z}^{\mathbb{N}}$ be an integer-valued sequence. For any $n \in \mathbb{N}$, we define the tuple $\lambda^{(n)} := (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ to be the n-tuple consisting of the first n entries of λ. Then, $\lambda^{(n)}$ interlaces $\lambda^{(n+1)}$ and, according to Theorem 4.1, $(\pi_{\lambda^{(n)}}, \mathcal{H}_{\lambda^{(n)}})$ occurs as a subrepresentation of $(\pi_{\lambda^{(n+1)}}, \mathcal{H}_{\lambda^{(n+1)}})$ (w.r.t. the group $U(n)$). Hence, each $\mathcal{H}_{\lambda^{(n)}}$ is isometrically embedded into $\mathcal{H}_{\lambda^{(n+1)}}$ and we write $V_\lambda := \varinjlim \mathcal{H}_{\lambda^{(n)}}$ for the direct limit in the category of pre-Hilbert spaces. The representations $\pi_{\lambda^{(n)}}$ canonically define a unitary representation of $U(\infty) = \bigcup_{n \in \mathbb{N}} U(n)$ on the pre-Hilbert space V_λ which we denote by π_λ. It is clear that π_λ extends to a unitary representation on the Hilbert space completion $\mathcal{H}_\lambda := \overline{V_\lambda}$ which is again denoted by π_λ.

LEMMA 4.3. — *The unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ is continuous and irreducible.*

Proof. — To see the continuity, it is enough to verify that, for any $v \in V_\lambda \subseteq \mathcal{H}_\lambda$, the orbit map $g \mapsto \pi_\lambda(g)v$ is continuous. For fixed $v \in V_\lambda$ and sufficiently large n, we may assume that $v \in \mathcal{H}_{\lambda^{(n)}}$. Therefore, the restriction of the orbit map to the subgroup $U(n)$ is continuous since the representation $(\pi_{\lambda^{(n)}}, \mathcal{H}_{\lambda^{(n)}})$ is continuous. This is true for all sufficiently large n and the continuity thus follows from the fact that the group topology on $U(\infty)$ is given by the direct limit topology. That $(\pi_\lambda, \mathcal{H}_\lambda)$ is irreducible follows from the fact that direct limits of irreducible representations are irreducible (cf. [5, Prop. A.5]). $\square$

DEFINITION 4.4. — *The unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ is called unitary highest weight representation of $U(\infty)$ with highest weight $\lambda \in \mathbb{Z}^{\mathbb{N}}$.*

Remark 4.5. — In [15, §I,II], the unitary highest weight representations of the direct limit Lie algebra $\mathfrak{gl}(\infty) = \bigcup_{n \in \mathbb{N}} \mathfrak{gl}(n)$ are classified in terms of real-valued sequences $\lambda \in \mathbb{R}^{\mathbb{N}}$. The corresponding highest weight module is denoted by $L(\lambda)$. In Section III of [15] it is shown that the underlying Lie algebra representation of $L(\lambda)$ integrates to a representation $\hat{\varrho}_\lambda : \mathrm{GL}(\infty) \to \mathrm{End}(L(\lambda))$ if and only if $\lambda \in \mathbb{Z}^{\mathbb{N}}$. For any such integer-valued weight λ, the module $L(\lambda)$ may be identified with our pre-Hilbert space V_λ and the restriction of $\hat{\varrho}_\lambda$ to the subgroup $U(\infty)$ is just our π_λ. Theorem I.20 of [15] states that two highest weight modules $L(\lambda)$ and $L(\mu)$ are equivalent if and only if the weights belong to the same orbit under the Weyl group $\mathcal{W}$. As remarked in Section II, the Weyl group for $\mathfrak{gl}(\infty)$ may be identified with

the group $S_{(\mathbb{N})}$ of finite permutations on the entries of the weights. Thus, we conclude that two unitary highest weight representations $(\pi_\lambda, \mathcal{H}_\lambda)$ and $(\pi_\mu, \mathcal{H}_\mu)$ of $U(\infty)$ are equivalent if and only if the entries of the weights λ and μ coincide up to a finite(!) permutation of the entries.

Using Proposition 3.6, we want to determine for which tuples $\lambda \in \mathbb{Z}^\mathbb{N}$, the 1-cohomology space for the corresponding highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ vanishes. This is true for $\lambda = 0$ (cf. Remark 3.7) and from now on, we assume that $\lambda \neq 0$. Note that then $\mathcal{H}_\lambda^{U(\infty)} = \{0\}$ by the irreducibility of the highest weight representation.

We need the following two observations that follow directly from Lemma 4.6 below: Let $\lambda \in \mathbb{Z}^\mathbb{N}$ and $(\pi_\lambda, \mathcal{H}_\lambda)$ be the corresponding unitary highest weight representation. For $n \in \mathbb{N}$, we have

$$(4.2) \qquad \mathcal{H}_\lambda^{U(n)} = \{0\} \Longleftrightarrow (\forall\, k \geqslant n) \quad \mathcal{H}_{\lambda^{(k)}}^{U(n)} = \{0\}.$$

For $k > n$, we have

$$(4.3) \qquad \mathcal{H}_{\lambda^{(k)}}^{U(n)} \neq \{0\} \Longleftrightarrow \mathcal{H}_{\eta^{(k-1)}}^{U(n)} \neq \{0\} \text{ for some } \eta^{(k-1)} \preccurlyeq \lambda^{(k)}.$$

Both observations follow from the following lemma:

LEMMA 4.6. — *Let $(\pi, \mathcal{H})$ be a continuous unitary representation of the topological group G. Assume that, for a subset $J \subseteq \mathbb{N}$, we have a family of G-invariant subspaces $(\mathcal{H}_j)_{j \in J}$ such that one of the following conditions is satisfied:*

(1) $\mathcal{H} = \widehat{\bigoplus}_{j \in J} \mathcal{H}_j$

(2) $\mathcal{H} = \overline{\bigcup_{j \in J} \mathcal{H}_j}$

Let $(\rho, \mathcal{K})$ be an irreducible representation of G. Then, $(\rho, \mathcal{K})$ occurs as a subrepresentation in $(\pi, \mathcal{H})$ if and only if it occurs as a subrepresentation in $(\pi, \mathcal{H}_j)$ for some $j \in J$.

Proof. — We denote by P_j the orthogonal projection onto the subspace $\mathcal{H}_j$. We may assume that $\mathcal{K} \subset \mathcal{H}$ is a Hilbert subspace of $\mathcal{H}$. If $(\rho, \mathcal{K}) \subset (\pi, \mathcal{H})$, then, in both cases, we find an index $j \in J$ for which $P_j(\mathcal{K}) \neq \{0\}$. The projection operator $P_j|_\mathcal{K} : \mathcal{K} \to \mathcal{H}_j$ intertwines ρ and π. Therefore, $P_j(\mathcal{K}) \subset \mathcal{H}_j$ is a G-invariant subspace. By Schur's Lemma, we further conclude that $(P_j|_\mathcal{K})^* P_j|_\mathcal{K} = c \mathbb{1}|_\mathcal{K}$. Since the operator $(P_j|_\mathcal{K})^* P_j|_\mathcal{K}$ is positive, the constant c is real-valued and nonngeative. The case $c = 0$ is excluded by $P_j(\mathcal{K}) \neq \{0\}$. Therefore, $A := \frac{1}{\sqrt{c}} P_j|_\mathcal{K} : \mathcal{K} \to P_j(\mathcal{K})$ is a linear isometry intertwining ρ and π. This shows $(\rho, \mathcal{K}) \subset (\pi, \mathcal{H}_j)$. The converse statement is trival. $\qquad\square$

The equivalence (4.3) motivates the following definition.

DEFINITION 4.7. — *Assume that a property (P) is defined for all tuples $\lambda \in \mathbb{Z}^k$ of length $k \geqslant n$. We say that (P) is interlacing-inheritable if a tuple λ has property (P) if and only if there exists a tuple $\eta \preccurlyeq \lambda$ with property (P).*

The property that the highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $U(k)$ corresponding to a tuple $\lambda \in \mathbb{Z}^k$ has a nontrivial $U(n)$-fixed vector (where $k \geqslant n$), is interlacing-inheritable. Any two interlacing-inheritable properties (P) and (P') are equivalent if and only if they are equivalent for all tuples $\lambda \in \mathbb{Z}^n$ of length n.

LEMMA 4.8. — *Let $k \geqslant n$. For any $\lambda \in \mathbb{Z}^k$, the property*

$$(4.4) \qquad \#\{j : \lambda_j \geqslant 0\} \geqslant n \quad \text{and} \quad \#\{j : \lambda_j \leqslant 0\} \geqslant n$$

is interlacing-inheritable.

Proof. — Let $\lambda \in \mathbb{Z}^k$ for $k > n$. We may assume w.l.o.g. that λ is decreasingly ordered. The interlacing condition $\eta \preccurlyeq \lambda$ then reads $\lambda_1 \geqslant \eta_1 \geqslant \lambda_2 \geqslant \eta_2 \geqslant \cdots \geqslant \eta_{k-1} \geqslant \lambda_k$. If η has the property (4.4), then the first (last) n entries of η are $\geqslant 0$ ($\leqslant 0$), hence so are the first (last) n entries of λ. Conversely, assume that λ has the required property. We construct $\eta \in \mathbb{Z}^{k-1}$ as follows: If $k > 2n$, choose $\eta \preccurlyeq \lambda$ such that the first (last) n entries of η coincide with the first (last) n entries of λ. If $k \leqslant 2n$, consider the set $J := \{1, 2, \ldots, n\} \cap \{k-1, k-2, \ldots, k-n\}$. Put $\eta_j := \lambda_j$ whenever the index j is smaller than the indices from J and put $\eta_j := \lambda_{j+1}$ whenever the index j is greater than the indices from J. $\qquad \square$

PROPOSITION 4.9. — *Let $k \geqslant n$ and $\lambda \in \mathbb{Z}^k$. Then the unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ (of $U(k)$) admits nonzero $U(n)$-fixed vectors if and only if the following condition is satisfied:*

$$(4.5) \qquad \#\{j \mid \lambda_j \geqslant 0\} \geqslant n \quad \text{and} \quad \#\{j \mid \lambda_j \leqslant 0\} \geqslant n.$$

Proof. — In view of Lemma 4.8, both conditions are interlacing-inheritable and it just remains to check that they are equivalent for tuples $\lambda \in \mathbb{Z}^n$. Indeed, (4.5) is satisfied if and only if $\lambda = 0$. $\qquad \square$

THEOREM 4.10. — *Let $\lambda \in \mathbb{Z}^{\mathbb{N}} \backslash \{0\}$ and $(\pi_\lambda, \mathcal{H}_\lambda)$ be the corresponding unitary highest weight representation of $U(\infty)$. Then $H^1(U(\infty), \pi_\lambda, \mathcal{H}_\lambda) = \{0\}$ if and only if either*

$$\#\{j \mid \lambda_j \leqslant 0\} < \infty \quad \text{or} \quad \#\{j \mid \lambda_j \geqslant 0\} < \infty.$$

Proof. — According to Proposition 3.6, the first cohomology space vanishes if and only if there exists some $n \in \mathbb{N}$ for which $\mathcal{H}_\lambda^{U(n)} = \{0\}$. This is

equivalent to $\mathcal{H}_{\lambda^{(k)}}^{U(n)} = \{0\}$ for all $k \geqslant n$ (cf. (4.2)), where $\lambda^{(k)} \in \mathbb{Z}^k$ consists of the first k entries of λ. By virtue of Proposition 4.9 this can be rewritten as

$$(\forall\, k \geqslant n) \qquad \#\{j \mid \lambda_j^{(k)} \geqslant 0\} < n \quad \text{or} \quad \#\{j \mid \lambda_j^{(k)} \leqslant 0\} < n.$$
$$\Longleftrightarrow \#\{j \mid \lambda_j \geqslant 0\} < n \quad \text{or} \quad \#\{j \mid \lambda_j \leqslant 0\} < n. \qquad \square$$

5. The identical representation of $U_p(\mathcal{H})$ on $\mathcal{H}$

Now, we prove that the natural actions of the group $U_p(\mathcal{H})$ on $\mathcal{H}$ resp. on its topological dual $\mathcal{H}^*$ have trivial 1-cohomology spaces.

Let $1 \leqslant p < \infty$ and $\mathcal{H}$ be a complex separable Hilbert space with ONB $(e_n)_{n\in\mathbb{N}}$. The unitary group $U_p(\mathcal{H}) = U(\mathcal{H}) \cap (\mathbb{1} + \mathcal{B}_p(\mathcal{H}))$ is a Banach–Lie group with Banach–Lie algebra $\mathfrak{u}_p(\mathcal{H}) = \mathfrak{u}(\mathcal{H}) \cap \mathcal{B}_p(\mathcal{H})$, i.e. the skew hermitian operators of pth Schatten class (cf. [12]). Its complexification is given by the pth Schatten operators $\mathfrak{u}_{p,\mathbb{C}}(\mathcal{H}) = \mathcal{B}_p(\mathcal{H})$. For any $z \in \mathfrak{u}_{p,\mathbb{C}}(\mathcal{H})$ the involution z^* coincides with the usual Hilbert adjoint. The inductive limit $U(\infty)$ is a dense subgroup in $U_p(\mathcal{H})$ so that we can apply the results of Section 3 and in particular Remark 3.2 to any norm-continuous unitary representation of $U_p(\mathcal{H})$. The complexification of the Lie algebra of $U(\infty)$ is given by $\mathfrak{gl}(\infty) := \varinjlim \mathfrak{gl}(n, \mathbb{C})$.

The identical action of $U_p(\mathcal{H})$ on $\mathcal{H}$ is given by the prescription

$$U_p(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}, \quad (g, v) \mapsto gv.$$

This action, restricted to $U(\infty)$, defines a unitary highest weight representation corresponding to the tuple $\lambda := (1, 0, 0, \dots)$ which we denote by $(\pi_\lambda, \mathcal{H}_\lambda) = (\mathrm{id}, \mathcal{H})$. It is clear that the identical action defines a norm-continuous unitary representation of the group $U_p(\mathcal{H})$.

PROPOSITION 5.1. — *We have* $H^1(U_p(\mathcal{H}), \mathrm{id}, \mathcal{H}) = \{0\}$.

Proof. — For $u, w \in \mathcal{H}$, we consider the rank-1-operator $w \otimes u^* := \langle \cdot, u \rangle w$ which is an element of $\mathcal{B}_p(\mathcal{H})$ for each $p \in [1, \infty]$ since

$$\|w \otimes u^*\|_p = \|w\|\, \|u\|.$$

For $v \in \mathcal{H}_0 := \langle e_n : n \in \mathbb{N} \rangle_{\mathrm{lin}}$, we have $v \otimes e_1^* \in \mathfrak{gl}(\infty)$. Now, let $\beta \in Z^1(U_p(\mathcal{H}), \mathrm{id}, \mathcal{H})$ and $a := a_\beta$ be the corresponding antilinear functional from Section 3. Using equation (3.4), we find

$$|a(v)| = |a(v \otimes e_1^*(e_1))| = |\langle \mathrm{d}\beta_\mathbb{C}(e_1 \otimes v^*), e_1 \rangle|$$
$$\leqslant \|\mathrm{d}\beta_\mathbb{C}\|\, \|e_1 \otimes v^*\|_p = \|\mathrm{d}\beta_\mathbb{C}\|\, \|v\|$$

which shows that a extends to a continuous linear functional on $\mathcal{H}$. The assertion now follows from Lemma 3.3. $\qquad\square$

The dual representation $(\mathrm{id}^*, \mathcal{H}^*)$ is also a unitary highest weight representation which corresponds to the tuple $\lambda = (-1, 0, 0, \dots)$.

PROPOSITION 5.2. — We have $H^1(U_p(\mathcal{H}), \mathrm{id}^*, \mathcal{H}^*) = \{0\}$.

Proof. — This immediately follows from Proposition 5.1 and the fact that the first order cohomology space for the dual representation vanishes if and only if it vanishes for the original representation. $\qquad\square$

6. A simple way to construct unbounded 1-cocycles in finite tensor products

This section should be viewed as a preparation for Section 7. We consider a countably infinite sum of 1-coboundaries that converges pointwise and ask whether we thus obtain an unbounded 1-cocycle. To make life easier, we assume the underlying group G to be completely metrizable since then, the pointwise converging sum is automatically continuous. This observation is based on a Baire category argument which is carried out in Appendix A. Note that the unitary groups $U_p(\mathcal{H})$ are completely metrizable for any $p \in [1, \infty]$. We derive a simple sufficient criterion for the unboundedness of the sum when $(\pi, \mathcal{H})$ is an arbitrary continuous unitary representation (cf. Lemma 6.1) and focus afterwards on the case of a finite tensor product representation (cf. Proposition 6.4).

Let G be a completely metrizable group and $(\pi, \mathcal{H})$ be a continuous unitary representation of G.

LEMMA 6.1. — Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal sequence in $\mathcal{H}$ such that, for all n, we have

$$e_n \in \langle \pi(g)v - v \ : \ g \in G, v \in \mathcal{H} \rangle_{lin}.$$

Further, let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{C}$ for which the sum $\beta(g) := \sum_{n \in \mathbb{N}} a_n[\pi(g)e_n - e_n]$ converges in $\mathcal{H}$ for every $g \in G$. Then $\beta : G \to \mathcal{H}$ defines a (continuous) 1-cocycle which is a 1-coboundary if and only if the sequence $(a_n)_{n \in \mathbb{N}}$ is square summable.

Proof. — That β is a continuous 1-cocycle follows from Corollary A.5. If $\sum_{n \in \mathbb{N}} |a_n|^2 < \infty$, then we have

$$\tag{6.1} \sum_{n \in \mathbb{N}} a_n[\pi(g)e_n - e_n] = \pi(g)v - v$$

for the vector $v := \sum_{n \in \mathbb{N}} a_n e_n$. Conversely, assume that β is a 1-coboundary. Then, we find some $v \in \mathcal{H}$ which satisfies (6.1) for all $g \in G$. Let $(e_j)_{j \in J}$ be a complete orthonormal system in $\mathcal{H}$ containing the e_n, i.e. we have $\mathbb{N} \subseteq J$. We expand v w.r.t. the system (e_j) and obtain the coefficients $b_j := \langle v, e_j \rangle$. For $j \in J \backslash \mathbb{N}$, we put $a_j := 0$. Our assumption then leads to

$$\sum_{j \in J} (a_j - b_j)[\pi(g)e_j - e_j] = 0 \quad \text{for all } g \in G.$$

Therefore, we have $\sum_{j \in J} (a_j - b_j)\langle e_j, w \rangle = 0$ for any $w \in D := \langle \pi(g)v - v : g \in G, v \in \mathcal{H} \rangle_{\text{lin}}$. Since $e_n \in D$ by assumption, we obtain that

$$a_n - b_n = \sum_{j \in J} (a_j - b_j)\langle e_j, e_n \rangle = 0$$

and we conclude that the sequence $(a_n)_{n \in \mathbb{N}}$ is square integrable. $\qquad \square$

Now, we turn to unbounded 1-cocycles of finite tensor products: Assume that we are given $m \geqslant 2$ continuous unitary representations $(\pi_i, \mathcal{H}_i)_{i=1,\dots,m}$ of G. We form the Hilbert tensor product $(\pi, \mathcal{H}) := \widehat{\bigotimes}_{i=1,\dots,m}(\pi_i, \mathcal{H}_i)$.

LEMMA 6.2. — *Let $(e_n^{(i)})_{n \in \mathbb{N}}$ be an orthonormal sequence in $\mathcal{H}_i$. For simplicity we will write $g.e_n^{(i)}$ instead of $\pi_i(g)e_n^{(i)}$. Assume that $(a_n)_{n \in \mathbb{N}}$ is a sequence in $\mathbb{C}$ such that, for all i and all $g \in G$,*

$$\sum_{n \in \mathbb{N}} |a_n|^2 \left\| g.e_n^{(i)} - e_n^{(i)} \right\|_{\mathcal{H}_i}^2 < \infty.$$

Then

$$\beta(g) := \sum_{n \in \mathbb{N}} a_n \left[g.e_n^{(1)} \widehat{\otimes} \dots \widehat{\otimes} g.e_n^{(m)} - e_n^{(1)} \widehat{\otimes} \dots \widehat{\otimes} e_n^{(m)} \right]$$

defines a 1-cocycle w.r.t. the tensor product representation. Moreover, we have the estimate

$$(6.2) \quad \sum_{n \in \mathbb{N}} |a_n|^2 \left\| g.e_n^{(1)} \widehat{\otimes} \dots \widehat{\otimes} g.e_n^{(m)} - e_n^{(1)} \widehat{\otimes} \dots \widehat{\otimes} e_n^{(m)} \right\|^2$$

$$\leqslant m \cdot \sum_{i=1}^{m} \sum_{n \in \mathbb{N}} |a_n|^2 \left\| g.e_n^{(i)} - e_n^{(i)} \right\|_{\mathcal{H}_i}^2.$$

Proof. — It is enough to verify that the sum converges for all $g \in G$ (Corollary A.5). Choose some $g \in G$ and unitary operators $U_i \in U(\mathcal{H}_i)$.

For fixed $j \in \{1, \ldots, m\}$ and arbitrary $M > N \in \mathbb{N}$ we calculate

$$\left\| \sum_{n=N}^{M} a_n \cdot U_1 e_n^{(1)} \widehat{\otimes} \ldots \widehat{\otimes} (g-\mathbb{1}).e_n^{(j)} \widehat{\otimes} \ldots \widehat{\otimes} U_m e_n^{(m)} \right\|^2$$

$$= \sum_{n,n'=N}^{M} a_n \overline{a_{n'}} \cdot \prod_{i \neq j} \langle U_i e_n^{(i)}, U_i e_{n'}^{(i)} \rangle_{\mathcal{H}_i} \cdot \langle (g-\mathbb{1}).e_n^{(j)}, (g-\mathbb{1}).e_{n'}^{(j)} \rangle_{\mathcal{H}_j}$$

$$= \sum_{n=N}^{M} |a_n|^2 \left\| (g-\mathbb{1}).e_n^{(j)} \right\|^2 \xrightarrow[N,M \to \infty]{} 0.$$

We conclude that the sum $\sum_{n=1}^{\infty} a_n \cdot U_1 e_n^{(1)} \widehat{\otimes} \ldots \widehat{\otimes} (g-\mathbb{1}).e_n^{(j)} \widehat{\otimes} \ldots \widehat{\otimes} U_m e_n^{(m)}$ converges for every j and g. This shows that

$$\sum_{n \in \mathbb{N}} a_n \cdot g.e_n^{(1)} \widehat{\otimes} \ldots \widehat{\otimes} g.e_n^{(m)} - e_n^{(1)} \widehat{\otimes} \ldots \widehat{\otimes} e_n^{(m)}$$

$$= \sum_{j=1}^{m} \sum_{n \in \mathbb{N}} a_n \cdot g.e_n^{(1)} \widehat{\otimes} \ldots \widehat{\otimes} g.e_n^{(j-1)} \widehat{\otimes} (g-\mathbb{1}).e_n^{(j)} \widehat{\otimes} e_n^{(j+1)} \widehat{\otimes} \ldots \widehat{\otimes} e_n^{(m)}$$

converges for every $g \in G$. The estimate (6.2) follows from

$$\left\| g.e_n^{(1)} \widehat{\otimes} \ldots \widehat{\otimes} g.e_n^{(m)} - e_n^{(1)} \widehat{\otimes} \ldots \widehat{\otimes} e_n^{(m)} \right\|^2$$

$$= \left\| \sum_{j=1}^{m} g.e_n^{(1)} \widehat{\otimes} \ldots \widehat{\otimes} g.e_n^{(j-1)} \widehat{\otimes} (g-\mathbb{1}).e_n^{(j)} \widehat{\otimes} e_n^{(j+1)} \widehat{\otimes} \ldots \widehat{\otimes} e_n^{(m)} \right\|^2$$

$$\leqslant m \sum_{j=1}^{m} \left\| (g-\mathbb{1}).e_n^{(j)} \right\|_{\mathcal{H}_j}^2 . \qquad \square$$

For our purposes, tensor products of the form $(\pi_\mu^*, \mathcal{H}_m^*) \widehat{\otimes} (\pi_\lambda, \mathcal{H}_\lambda)$ are of particular interest, where $(\pi_\lambda, \mathcal{H}_\lambda)$ and $(\pi_\mu, \mathcal{H}_\mu)$ are two irreducible unitary representations of G. The canonical isomorphism $\mathcal{H}_\mu^* \widehat{\otimes} \mathcal{H}_\lambda \cong \mathcal{B}_2(\mathcal{H}_\mu, \mathcal{H}_\lambda)$ induces a unitary representation on $\mathcal{B}_2(\mathcal{H}_\mu, \mathcal{H}_\lambda)$ which is equivalent to the tensor product $\pi_\mu^* \widehat{\otimes} \pi_\lambda$. It is given by the conjugation action

$$(g, A) \mapsto \pi_\lambda(g) A \pi_\mu(g^{-1}), \text{ for all } g \in G, A \in \mathcal{B}_2(\mathcal{H}_\mu, \mathcal{H}_\lambda).$$

In particular, if $(\pi_\lambda, \mathcal{H}_\lambda)$ and $(\pi_\mu, \mathcal{H}_\mu)$ are isomorphic unitary representations, then $\pi_\mu^* \widehat{\otimes} \pi_\lambda$ is equivalent to the conjugation representation on $\mathcal{B}_2(\mathcal{H}_\lambda)$.

LEMMA 6.3. — *Let $A \in \mathcal{B}(\mathcal{H}_\mu, \mathcal{H}_\lambda)$ be a bounded linear operator such that*

$$\beta(g) := \pi_\lambda(g) A \pi_\mu(g^{-1}) - A \in \mathcal{B}_2(\mathcal{H}_\mu, \mathcal{H}_\lambda)$$

for every $g \in G$.

(1) *Assume that $(\pi_\lambda, \mathcal{H}_\lambda)$ and $(\pi_\mu, \mathcal{H}_\mu)$ are non-isomorphic representations. Then, β is a 1-coboundary if and only if $A \in \mathcal{B}_2(\mathcal{H}_\mu, \mathcal{H}_\lambda)$.*

(2) *Assume that $(\pi_\lambda, \mathcal{H}_\lambda) = (\pi_\mu, \mathcal{H}_\mu)$. Then, β is a 1-coboundary if and only if $A \in \mathcal{B}_2(\mathcal{H}_\lambda) + \mathbb{C}\mathbb{1}$.*

Proof. — In both cases, β is a 1-coboundary if and only if there exists $B \in \mathcal{B}_2(\mathcal{H}_\mu, \mathcal{H}_\lambda)$ such that $\pi_\lambda(g)(A - B)\pi_\mu(g^{-1}) = A - B$ for all $g \in G$. We denote by $\mathcal{B}^G(\mathcal{H}_\mu, \mathcal{H}_\lambda)$ the space of linear bounded operators intertwining the representations π_μ and π_λ. If the representation are non-isomorphic, then Schur's Lemma implies that $\mathcal{B}^G(\mathcal{H}_\mu, \mathcal{H}_\lambda) = \{0\}$. This shows the first assertion. In the case $\pi_\mu = \pi_\lambda$, Schur's Lemma states that $\mathcal{B}^G(\mathcal{H}_\lambda) = \mathbb{C} \cdot \mathbb{1}$ which proves the second assertion. $\qquad\square$

PROPOSITION 6.4. — *Let $(\pi_\mu, \mathcal{H}_\mu)$ and $(\pi_\lambda, \mathcal{H}_\lambda)$ be two infinite-dimensional, irreducible representations of the completely metrizable group G. Assume that there exists a bounded but not square summable sequence $(a_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathbb{C}) \setminus \ell^2(\mathbb{N}, \mathbb{C})$ such that, for some orthonormal sequences $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{H}_\lambda$ resp. $\mathcal{H}_\mu$,*

$$(6.3) \quad \sum_{n\in\mathbb{N}} |a_n|^2 \|(\pi_\lambda(g) - \mathbb{1})e_n\|^2_{\mathcal{H}_\lambda} + \sum_{n\in\mathbb{N}} |a_n|^2 \|(\pi_\mu(g) - \mathbb{1})f_n\|^2_{\mathcal{H}_\mu} < \infty$$

holds for all $g \in G$. Then $H^1(G, \pi_\mu^* \widehat{\otimes} \pi_\lambda, \mathcal{H}_\mu^* \widehat{\otimes} \mathcal{H}_\lambda) \neq \{0\}$.

Proof. — First, we note that $\|(\pi_\mu(g) - \mathbb{1})f_n\|_{\mathcal{H}_\mu} = \|(\pi_\mu^*(g) - \mathbb{1})f_n^*\|_{\mathcal{H}_\mu^*}$. By Lemma 6.2, we obtain a 1-cocycle

$$\beta(g) := \sum_{n\in\mathbb{N}} a_n[\pi_\mu^*(g)f_n^* \widehat{\otimes} \pi_\lambda(g)e_n - f_n^* \widehat{\otimes} e_n]$$

for the tensor product. With respect to the conjugation action on $\mathcal{B}_2(\mathcal{H}_\mu, \mathcal{H}_\lambda)$ this 1-cocycle has the form $\beta(g) = \pi_\lambda(g)A\pi_\mu(g^{-1}) - A$, where $A = \sum_{n\in\mathbb{N}} a_n e_n \otimes f_n^* = \sum_{n\in\mathbb{N}} a_n \langle \cdot, f_n \rangle_{\mathcal{H}_\mu} e_n$. Since the sequence (a_n) is bounded but not square summable, the operator A is bounded but not Hilbert–Schmidt. If π_μ and π_λ are not isomorphic, then Lemma 6.3 implies that β is unbounded. If $\pi_\mu \cong \pi_\lambda$ are isomorphic, we may assume w.l.o.g. that $\pi_\mu = \pi_\lambda$. In this case, Lemma 6.3 implies that β is unbounded unless $A \in \mathcal{B}_2(\mathcal{H}_\lambda) + \mathbb{C}\mathbb{1}$. If this is the case, then (a_n) is the sum of a constant and a square summable sequence. It is clear that condition (6.3) then holds for all bounded sequences (a_n) because the constant part is nonzero. In

particular, we may choose the sequence $a_n := (-1)^n$. The corresponding diagonal operator $A \in \mathcal{B}(\mathcal{H}_\lambda)$ cannot be written as a linear combination of the identity $\mathbb{1}$ and a Hilbert–Schmidt operator. In particular, the 1-cocycle β is unbounded. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7. Unitary highest weight representations of $U_p(\mathcal{H})$

For finitely supported weights $\lambda \in \mathbb{Z}^\mathbb{N}$, the corresponding unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $U(\infty)$ extends to a norm-continuous unitary representation of $U_p(\mathcal{H})$ (for $1 \leqslant p < \infty$ and $\mathcal{H} = \ell^2(\mathbb{N}, \mathbb{C})$). We realize these representations as subrepresentations in finite tensor products of the identical representation and its dual representation of $U_p(\mathcal{H})$ on $\mathcal{H}$ resp. on $\mathcal{H}^*$ (cf. Definition 7.5). Our results of the preceding section allow us to construct unbounded 1-cocycles for almost all finitely supported λ. There are only three exceptional cases where the first cohomology spaces are trivial (cf. Theorem 7.9).

We denote by $(\mathbb{N}_0)_\downarrow^{(\mathbb{N})}$ the set of all decreasingly ordered non-negative integer-valued tuples with a finite number of positive entries. To any $\lambda \in (\mathbb{N}_0)_\downarrow^{(\mathbb{N})}$, we associate a *Young diagram* D_λ (also called *Ferrers diagram*) which consists of $\ell_\lambda := \max\{j \mid \lambda_j > 0\}$ rows and the jth row has λ_j boxes, so that the whole diagram consists of $|\lambda| := \sum_{i=1}^{\ell_\lambda} \lambda_i$ boxes. Conversely, any Young diagram D (with row (and column) length weakly decreasing) defines a unique $\lambda \in (\mathbb{N}_0)_\downarrow^{(\mathbb{N})}$ by counting the row boxes. Given a Young diagram D, we obtain the *conjugate* (or *transposed*) Young diagram D' by switching rows and columns. Thus, for any $\lambda \in (\mathbb{N}_0)_\downarrow^{(\mathbb{N})}$, we define the *conjugate* tuple $\lambda' \in (\mathbb{N}_0)_\downarrow^{(\mathbb{N})}$ via the relation $D_{\lambda'} := D'_\lambda$. Note that $\ell_{\lambda'} = \lambda_1$ and the entry j occurs $\lambda_j - \lambda_{j+1}$ times in λ'.

Example 7.1. — For $\lambda := (3, 2, 2, 1)$ we have

$$D_\lambda = \quad \text{and} \quad D'_\lambda =$$
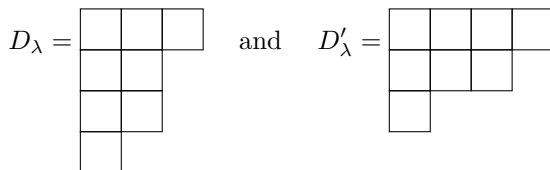
and the conjugate tuple is given by $\lambda' = (4, 3, 1)$.

The corresponding *Young tableau* T_λ is obtained by filling in the boxes of the young diagram D_λ with the numbers $1, 2, \ldots, |\lambda|$ in the following manner: The number 1 is placed in the top box of the first column. The

number $k+1$ is placed in the box directly below k, if it exists and otherwise in the top box of the next column.

Example 7.2. — For $\lambda = (3, 2, 2, 1)$ this yields

$$D_\lambda = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \qquad \longrightarrow \qquad T_\lambda = \begin{array}{|c|c|c|} \hline 1 & 5 & 8 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline 4 \\ \hline \end{array}$$

Let S_λ be the permutation group of the set $\{1, 2, \ldots, |\lambda|\}$. Denote by R_λ the subgroup of permutations leaving all subsets defined by the rows of T_λ invariant and accordingly by C_λ the subgroup leaving all subsets defined by the columns invariant.

Put $\mathcal{H} := \ell^2(\mathbb{N}, \mathbb{C})$ and consider the $|\lambda|$-fold tensor product $\mathcal{H}^{\widehat{\otimes}|\lambda|}$. The permutation group S_λ acts unitarily on the tensor product via

$$(\sigma, \widehat{\otimes}_{j=1}^{|\lambda|} v_j) \mapsto \widehat{\otimes}_{j=1}^{|\lambda|} v_{\sigma^{-1}(j)}$$

for $\sigma \in S_\lambda$ and $v_j \in \mathcal{H}$. We denote this representation by $(\rho, \mathcal{H}^{\widehat{\otimes}|\lambda|})$. The group $U_p(\mathcal{H})$ (for $p \in [1, \infty]$) also acts on the tensor product space via the tensor product representation $(\mathrm{id}, \mathcal{H})^{\widehat{\otimes}|\lambda|}$. Both representations commute, i.e. for $\sigma \in S_\lambda$ and $g \in U_p(\infty)$, we have $\rho(\sigma) \circ \mathrm{id}^{\widehat{\otimes}|\lambda|}(g) = \mathrm{id}^{\widehat{\otimes}|\lambda|}(g) \circ \rho(\sigma)$. Therefore, the linear operator

$$P_\lambda := \sum_{r \in R_\lambda, c \in C_\lambda} \mathrm{sgn}(c) \rho(cr),$$

where $\mathrm{sgn}(c)$ denotes the signum of the permutation c, commutes with $\mathrm{id}^{\widehat{\otimes}|\lambda|}$ and its image

$$\mathcal{H}_\lambda := P_\lambda(\mathcal{H}^{\widehat{\otimes}|\lambda|})$$

is a $U_p(\mathcal{H})$-invariant subspace. We write π_λ for the corresponding continuous unitary representation on $\mathcal{H}_\lambda$. Note that P_λ is not necessarily an orthogonal projection, but the operators

$$P_{C_\lambda} := \frac{1}{f_{C_\lambda}} \sum_{c \in C_\lambda} \mathrm{sgn}(c) \rho(c) \quad \text{and} \quad P_{R_\lambda} := \frac{1}{f_{R_\lambda}} \sum_{r \in R_\lambda} \rho(r)$$

with constants $f_{C_\lambda} := \prod_{j=1}^{\ell_{\lambda'}} (\lambda_j'!)$ and $f_{R_\lambda} := \prod_{j=1}^{\ell_\lambda} (\lambda_j!)$, are orthogonal projections and we have $P_\lambda = (f_{C_\lambda} f_{R_\lambda}) \cdot P_{C_\lambda} P_{R_\lambda}$. Let $(e_n)_{n \in \mathbb{N}}$ denote an ONB for $\mathcal{H}$. For $n \in \mathbb{N}$, we define the vectors

$$(7.1) \quad e_n^{(\lambda)} := \sqrt{f_{C_\lambda}} \cdot \widehat{\otimes}_{m=1}^{\ell_{\lambda'}} \wedge_{i=1}^{\lambda_m'} e_{n+i} = \sqrt{f_{C_\lambda}} \cdot P_{C_\lambda}(\widehat{\otimes}_{m=1}^{\ell_{\lambda'}} \widehat{\otimes}_{i=1}^{\lambda_m'} e_{n+i}).$$

By construction, the vectors $e_n^{(\lambda)}$ are mutually orthogonal. Since

$$P_{R_\lambda}\left(\overset{\ell_{\lambda'}}{\underset{m=1}{\bigotimes}}\overset{\lambda'_m}{\underset{i=1}{\bigotimes}}e_{n+i}\right) = \overset{\ell_{\lambda'}}{\underset{m=1}{\bigotimes}}\overset{\lambda'_m}{\underset{i=1}{\bigotimes}}e_{n+i},$$

we have constructed an orthonormal sequence in $\mathcal{H}_\lambda$. In particular, if $|\lambda| \geqslant 1$, the space $\mathcal{H}_\lambda$ is infinite-dimensional.

Example 7.3. — For $\lambda = (3,2,2,1)$ the orthonormal sequence $(e_n^{(\lambda)})_{n\in\mathbb{N}}$ in $\mathcal{H}_\lambda$ is given by

$$e_n^{(\lambda)} = \underbrace{\sqrt{4!3!1!}}_{=12}\cdot(e_{n+1}\wedge e_{n+2}\wedge e_{n+3}\wedge e_{n+4})\widehat{\otimes}(e_{n+1}\wedge e_{n+2}\wedge e_{n+3})\widehat{\otimes}e_{n+1}.$$

LEMMA 7.4. — *For any $\lambda \in (\mathbb{N}_0)_\downarrow^{(\mathbb{N})}$, the continuous unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $U_p(\mathcal{H})$ extends the highest weight representation of $U(\infty)$ with highest weight λ from Definition 4.4. In particular, it is irreducible.*

Proof. — For any $n \in \mathbb{N}$ with $n \geqslant \ell_\lambda$, consider the canonical embedding $(\mathbb{C}^n)^{\widehat{\otimes}|\lambda|} \hookrightarrow \mathcal{H}^{\widehat{\otimes}|\lambda|}$. The operator P_λ leaves the subspace $(\mathbb{C}^n)^{\widehat{\otimes}|\lambda|}$ invariant as well as the restriction of $\mathrm{id}^{\widehat{\otimes}|\lambda|}$ to the subgroup $U(n) \subset U_p(\mathcal{H})$. Therefore, the subspace $P_\lambda\big((\mathbb{C}^n)^{\widehat{\otimes}|\lambda|}\big)$ is $U(n)$-invariant and, according to Theorem 9.3.9 in [10], this is a unitary highest weight module of $U(n)$ with highest weight $\lambda^{(n)}$. This shows that $\pi_\lambda|_{U(\infty)} = \varinjlim \pi_{\lambda^{(n)}}$ is the direct limit representation on $P_\lambda\big((\mathbb{C}^{(\mathbb{N})})^{\widehat{\otimes}|\lambda|}\big)$ which is dense in $\mathcal{H}_\lambda$. $\qquad\square$

Now, let $\lambda \in \mathbb{Z}^{(\mathbb{N})}$ be an integer-valued tuple such that all but finitely many entries are zero. For any such $\lambda \in \mathbb{Z}^{(\mathbb{N})}$, we define $\lambda^\pm \in (\mathbb{N}_0)_\downarrow^{(\mathbb{N})}$ to be those tuples for which the entries of the finitely supported tuples $\max(\pm\lambda, 0)$ are decreasingly ordered. We put $|\lambda| := |\lambda^+| + |\lambda^-| = \sum_{i=1}^\infty |\lambda_i|$.

DEFINITION 7.5. — *For any $\lambda \in \mathbb{Z}^{(\mathbb{N})}$, we call the continuous unitary representation $(\pi_\lambda, \mathcal{H}_\lambda) := (\pi_{\lambda^-}^*, \mathcal{H}_{\lambda^-}^*)\widehat{\otimes}(\pi_{\lambda^+}, \mathcal{H}_{\lambda^+})$ unitary highest weight representation of the group $U_p(\mathcal{H})$ with heighest weight λ.*

We briefly comment on this definition.

Remark 7.6. — That the representation $(\pi_\lambda, \mathcal{H}_\lambda)$ is irreducible follows from Section 2.17 of [19]. Using Corollary I.14 of [15], one finds that $(\pi_\lambda, \mathcal{H}_\lambda)$ is a unitary highest weight representation of $U_p(\mathcal{H})$ in the sense of Definition III.6 in [15]. For the case $p = \infty$, we refer to [5, Thm. 2.2].

In the remainder of this section we are going to show that, for $|\lambda| \geqslant 2$, the first cohomology space $H^1(U_p(\mathcal{H}), \pi_\lambda, \mathcal{H}_\lambda)$ never vanishes.

LEMMA 7.7. — *Put $q := \frac{2p}{p-2}$ if $2 < p < \infty$ and $q := \infty$ if $1 \leqslant p \leqslant 2$. For every sequence $a = (a_n)_{n \in \mathbb{N}} \in \ell^q(\mathbb{N}, \mathbb{C})$, we have*

$$\sum_{n \in \mathbb{N}} |a_n|^2 \, \|(g - \mathbb{1})e_n\|_{\mathcal{H}}^2 < \infty$$

for every $g \in U_p(\mathcal{H})$. Moreover, this expression depends continuously on g in the $U_p(\mathcal{H})$-topology.

Proof. — For $A \in \mathcal{B}_r(\mathcal{H})$ and $r \in [1, \infty]$, we denote the rth Schatten norm by

$$\|A\|_r := \begin{cases} (\mathrm{Tr}(|A|^r))^{\frac{1}{r}} & \text{if } r < \infty \\ \|A\|_{\mathrm{op}} & \text{if } r = \infty. \end{cases}$$

We remind the reader of the generalized Hölder inequality (cf. [9, Thm. IV.11.2])

(7.2) $$\|BC\|_r \leqslant \|B\|_s \cdot \|C\|_t$$

for $B \in \mathcal{B}_s(\mathcal{H}), C \in \mathcal{B}_t(\mathcal{H})$ and $\frac{1}{r} \leqslant \frac{1}{s} + \frac{1}{t}$ with $s, t \in [1, \infty]$.

For $1 \leqslant p \leqslant 2$, the statement is clear, since $g - \mathbb{1} \in \mathcal{B}_2(\mathcal{H})$ and $\sum_{n \in \mathbb{N}} \|(g - \mathbb{1})e_n\|^2 = \|g - \mathbb{1}\|_2^2$. For $p > 2$, we consider the diagonal operator $Ae_n = a_n e_n$ and note that $A \in \mathcal{B}_q(\mathcal{H})$ and $\sum_{n \in \mathbb{N}} |a_n|^2 \|(g - \mathbb{1})e_n\|^2 = \|(g - \mathbb{1})A\|_2^2$. Applying (7.2) with $r = 2$, $s = p$ and $t = q$ yields the assertion. $\square$

PROPOSITION 7.8. — *Let $\lambda \in \mathbb{Z}^{(\mathbb{N})}$ such that $|\lambda| \geqslant 2$. Then, for $p \in [1, \infty)$, we have*

$$H^1(U_p(\mathcal{H}), \pi_\lambda, \mathcal{H}_\lambda) \neq \{0\}.$$

Proof. — Put $a_n := \frac{1}{\sqrt{n}}$. Then, the sequence $a = (a_n)_{n \in \mathbb{N}}$ is in $\ell^q(\mathbb{N}, \mathbb{C})$ for all $q > 2$ but not in $\ell^2(\mathbb{N}, \mathbb{C})$ and we know from the previous lemma that the sum $\sum_{n \in \mathbb{N}} \frac{1}{n} \|(g - \mathbb{1})e_n\|^2$ is finite for each $g \in G$.

The case $|\lambda^-| = 0$. — We first treat the case $\lambda \in (\mathbb{N}_0)_\downarrow^{(\mathbb{N})}$. Applying Lemma 6.2, we obtain a 1-cocycle

$$\widetilde{\beta}(g) := \sum_{n \in \mathbb{N}} \frac{1}{\sqrt{n}} \left[\overset{\ell_{\lambda'}}{\underset{m=1}{\bigotimes}} \overset{\lambda'_m}{\underset{i=1}{\bigotimes}} g e_{k+i} - \overset{\ell_{\lambda'}}{\underset{m=1}{\bigotimes}} \overset{\lambda'_m}{\underset{i=1}{\bigotimes}} e_{k+i} \right]$$

for the tensor product representation $(\mathrm{id}, \mathcal{H})^{\widehat{\otimes}|\lambda|}$. Projecting onto $\mathcal{H}_\lambda$ and using (7.1) gives

$$\beta(g) := \sqrt{f_{C_\lambda}} \cdot P_{C_\lambda}(\widetilde{\beta}(g)) = \sum_{n \in \mathbb{N}} \frac{1}{\sqrt{n}} [g.e_n^{(\lambda)} - e_n^{(\lambda)}].$$

Here, we have used (7.1) and the fact that P_{C_λ} commutes with the representation $\mathrm{id}^{\widehat{\otimes}|\lambda|}$. Hence, we obtain a 1-cocycle for the highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ and Lemma 6.1 (applied to the sequence $(e_n^{(\lambda)})_{n\in\mathbb{N}}$) tells us that β is in fact unbounded. The condition on the $e_n^{(\lambda)}$ is satisfied, since each $e_n^{(\lambda)}$ is a nontrivial eigenvector of all diagonal operators $t - \mathbb{1}$ for $t \in T_p(\mathcal{H})$, the maximal torus in $U_p(\mathcal{H})$.

The case $|\lambda^+| = 0$. — In this case $(\pi_\lambda, \mathcal{H}_\lambda) = (\pi_{\lambda^-}^*, \mathcal{H}_{\lambda^-}^*)$ and the assertion follows from the fact that the 1-cohomology of the dual representation vanishes if and only if it vanishes for the original representation.

The remaining case $|\lambda^+|, |\lambda^-| \geqslant 1$. — We want to apply Proposition 6.4. It merely remains to verify that

$$\sum_{n\in\mathbb{N}} \frac{1}{n} \left\| (g - \mathbb{1}).e_n^{(\lambda^\pm)} \right\|_{\mathcal{H}_{\lambda^\pm}}^2 < \infty$$

for each $g \in U_p(\mathcal{H})$. Indeed, we can estimate for $\lambda \in \{\lambda^\pm\}$ as follows:

$$\sum_{n\in\mathbb{N}} \frac{1}{n} \left\| (g - \mathbb{1}).e_n^{(\lambda)} \right\|_{\mathcal{H}_\lambda}^2$$

$$= f_{C_\lambda} \sum_{n\in\mathbb{N}} \frac{1}{n} \left\| P_{C_\lambda}[\widehat{\otimes}_{m=1}^{\ell_{\lambda'}} \widehat{\otimes}_{i=1}^{\lambda'_m} g e_{k+i} - \widehat{\otimes}_{m=1}^{\ell_{\lambda'}} \widehat{\otimes}_{i=1}^{\lambda'_m} e_{k+i}] \right\|_{\mathcal{H}_\lambda}^2$$

$$\leqslant f_{C_\lambda} \sum_{n\in\mathbb{N}} \frac{1}{n} \left\| \widehat{\otimes}_{m=1}^{\ell_{\lambda'}} \widehat{\otimes}_{i=1}^{\lambda'_m} g e_{k+i} - \widehat{\otimes}_{m=1}^{\ell_{\lambda'}} \widehat{\otimes}_{i=1}^{\lambda'_m} e_{k+i} \right\|_{\mathcal{H}_\lambda}^2$$

$$\overset{(6.2)}{\leqslant} f_{C_\lambda} |\lambda|^2 \sum_{n\in\mathbb{N}} \frac{1}{n} \| (g - \mathbb{1}) e_n \|_{\mathcal{H}}^2 < \infty. \qquad \square$$

Proposition 7.8 together with Remark 3.7, Propositions 5.1 and 5.2 yields the following theorem.

THEOREM 7.9. — *Put $\mathcal{H} := \ell^2(\mathbb{N}, \mathbb{C})$. Let $\lambda \in \mathbb{Z}^{(\mathbb{N})}$ be a finitely supported integer-valued sequence and $(\pi_\lambda, \mathcal{H}_\lambda)$ the corresponding unitary highest weight representation of $U_p(\mathcal{H})$. Then,*

$$H^1(U_p(\mathcal{H}), \pi_\lambda, \mathcal{H}_\lambda) = \{0\} \iff |\lambda| \leqslant 1.$$

One immediate consequence of Theorem 7.9 is that the group $U_p(\mathcal{H})$ does not have property (FH). Since property (T) implies property (FH) for arbitrary topological groups, this entails that $U_p(\mathcal{H})$ does not have property (T).

COROLLARY 7.10. — *For $1 \leqslant p < \infty$, the group $U_p(\mathcal{H})$ neither has property (FH) nor has property (T).*

Appendix A. Infinite sums of 1-coboundaries

Let $(\pi, \mathcal{H})$ be a continuous unitary representation of the topological group G. In general, a 1-cocycle $\beta : G \to \mathcal{H}$ need not be continuous. Here, we show that, if β is the pointwise limit of a sequence of continuous 1-cocycles and G is completely metrizable, then a Baire category argument shows that β is continuous.

DEFINITION A.1. — *Let X be a topological Hausdorff space. A subset $A \subseteq X$ is called* of first (Baire) category *if it can be written as a countable union of nowhere dense subsets. Otherwise, A is called a subset of* second (Baire) category. *The space X is called* Baire space, *if every nonempty open subset is of second category. A function $f : X \to X'$ with values in topological Hausdorff space X' is called a* function of first class *or a* Baire one function *if it is the pointwise limit of a sequence $(f_n)_{n \in \mathbb{N}}$ of continuous functions $f_n : X \to X'$.*

Example A.2. — According to the Baire Category Theorem, every complete metric space (X, d) is a Baire space (cf. [6, §5, Thm. 1]). In particular, the subset $A = X$ is of second category.

THEOREM A.3. — *Let (X, d), (X', d') be metric spaces and $f : X \to X'$ be a first class function. Then the points of discontinuity of f form a subset of first category.*

Proof. — See e.g. [13, §45.3].   □

PROPOSITION A.4. — *Let $(\pi, \mathcal{H})$ be a continuous unitary representation of the completely metrizable group G and let $\beta : G \to \mathcal{H}$ be the pointwise limit of a sequence of continuous 1-cocycles $(\beta_n)_{n \in \mathbb{N}}$. Then β is a continuous 1-cocycle.*

Proof. — The pointwise convergence implies that the map β is a first class function. It is straightforward to check that the 1-cocycle relation (1.1) is satisfied, hence β is a 1-cocycle. Moreover, the 1-cocycle relation shows that the points of discontinuity of β are either the empty set or coincide with the whole group G. In the latter case, the group G would have to be of first category since β is a first class function (Theorem A.3). This contradicts the fact that G is of second category (Baire Category Theorem).   □

COROLLARY A.5. — *Let $(\pi, \mathcal{H})$ be a continuous unitary representation of the completely metrizable group G and let $(v_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{H}$ for which the sum $\sum_{n \in \mathbb{N}} \pi(g)v_n - v_n$ converges for every $g \in G$. Then $\beta(g) := \sum_{n \in \mathbb{N}} \pi(g)v_n - v_n$ defines a continuous 1-cocycle.*

BIBLIOGRAPHY

[1] S. Albeverio, R. Høegh-Krohn, J. A. Marion, D. H. Testard & B. S. Torrésani, *Noncommutative Distributions: Unitary representations of Gauge Groups and Algebras*, Pure and Applied Mathematics, vol. 175, Marcel Dekker, 1993.

[2] H. Araki, "Factorizable representation of current algebra. Non commutative extension of the Lévy-Kinchin formula and cohomology of a solvable group with values in a Hilbert space", *Publ. Res. Inst. Math. Sci., Kyoto Univ.* **5** (1969/70), p. 361-422.

[3] C. J. Atkin, "Boundedness in uniform spaces, topological groups, and homogeneous spaces", *Acta Math. Hung.* **57** (1991), no. 3-4, p. 213-232.

[4] B. Bekka, P. de la Harpe & A. Valette, *Kazhdan's property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, 2008.

[5] D. Belțiță & K.-H. Neeb, "Schur–Weyl Theory for C^*-algebras", *Math. Nachr.* **285** (2012), no. 10, p. 1170-1198.

[6] N. Bourbaki, "Topologie Générale", chap. 9: Utilisation des nombres réels en topologie générale, Hermann, 1958.

[7] J. Faraut & K. Harzallah, "Distances hilbertiennes invariants sur un espace homogène", *Ann. Inst. Fourier* **24** (1974), no. 3, p. 171-225.

[8] H. Glöckner, "Fundamentals of direct limit Lie Theory", *Compos. Math.* **141** (2005), no. 6, p. 1551-1577.

[9] I. Gohberg, S. Goldberg & N. Krupnik, *Traces and Determinants of Linear Operators*, Operator Theory, Advances and Applications, vol. 116, Birkhäuser, 2000.

[10] R. Goodman & N. R. Wallach, *Representations and Invariants of the Classical Groups*, Encyclopedia of mathematics and its applications, vol. 68, Cambridge University Press, 1998.

[11] A. Guichardet, *Symmetric Hilbert Spaces and Related Topics*, Lecture Notes in Math., vol. 261, Springer, 1972.

[12] P. de la Harpe, *Classical Banach–Lie Algebras and Banach–Lie Groups of Operators in Hilbert space*, Lecture Notes in Math., vol. 285, Springer, 1972.

[13] F. Hausdorff, *Set Theory*, second ed., Chelsea Publishing Company, 1962.

[14] R. S. Ismagilov, *Representations of Infinite-Dimensional Groups*, Translations of Mathematical Monographs, vol. 152, American Mathematical Society, 1996.

[15] K.-H. Neeb, "Holomorphic highest weight representations of infinite-dimensional complex classical groups", *J. Reine Angew. Math.* **497** (1998), p. 171-222.

[16] ———, "Infinite-Dimensional Groups and Their Representations", in *Lie Theory: Lie Algebras and Representations*, Progress in Mathematics, vol. 228, Birkhäuser, 2004, p. 213-328.

[17] ———, "Towards a Lie theory of locally convex groups", *Jpn. J. Math.* **1** (2006), no. 2, p. 291-468.

[18] ———, "Unitary Representation of Unitary Groups", in *Developments and Retrospectives in Lie Theory*, Developments in Mathematics, vol. 37, Springer, 2014, p. 197-243.

[19] G. I. Olshanskii, "Unitary representations of infinite dimensional pairs (G,K) and the formalism of R. Howe", in *Representation of Lie groups and related topic*, Advanced Studies in Contemporary Mathematics, vol. 7, Gordon and Breach Sience Publishers, 1990, p. 269-463.

[20] K. R. Parthasarathy & K. Schmidt, "Factorizable representations of current groups and the Araki–Woods embedding theorem", *Acta Math.* **128** (1972), p. 53-71.

[21] ———, *Positive definite kernels, continuous tensor products, and central limit theorems of probability theory*, Lecture Notes in Math., vol. 272, Springer, 1972.

[22] V. G. PESTOV, "Amenability versus property (T) for non locally compact topological groups", *Trans. Am. Math. Soc.* **370** (2018), no. 10, p. 7417-7436.

[23] G. PINCZON & J. SIMON, "On the 1-cohomology of Lie groups", *Lett. Math. Phys.* **1** (1975), p. 83-91.

[24] C. ROSENDAL, "A topological version of the Bergman property", *Forum Math.* **21** (2009), no. 2, p. 299-332.

[25] R. F. STREATER, "Current commutation relations, continuous tensor products and infinitely divisible group representations", *Rend. Sc. Intern. E. Fermi* **11** (1969), p. 247-263.

[26] A. YAMASAKI, "Inductive limit of general linear groups", *J. Math. Kyoto Univ.* **38** (1998), no. 4, p. 769-779.

[27] D. P. ZELOBENKO, *Compact Lie Groups and their Representations*, Translations of Mathematical Monographs, vol. 40, American Mathematical Society, 1973.

Manuel HERBST
Department of Mathematics
FAU Erlangen-Nuernberg
Cauerstrasse 11
91058 Erlangen (Germany)
herbst@math.fau.de

Karl-Hermann NEEB
Department of Mathematics
FAU Erlangen-Nuernberg
Cauerstrasse 11
91058 Erlangen (Germany)
neeb@math.fau.de