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ON HOMOMORPHISMS BETWEEN CREMONA GROUPS

by Christian URECH (*)

Abstract. — We look at algebraic embeddings of the complex Cremona group in n variables Cr_n to the group of birational transformations $\mathrm{Bir}(M)$ of an algebraic variety M. First we study geometrical properties of an example of an embedding of Cr_2 into Cr_5 that is due to Gizatullin. In a second part, we give a full classification of all algebraic embeddings of Cr_2 into $\mathrm{Bir}(M)$, where M is a variety of dimension 3 and generalize this result partially to algebraic embeddings of Cr_n into $\mathrm{Bir}(M)$, where the dimension of M is $n+1$, for arbitrary n. In particular, this yields a classification of all algebraic $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions on smooth projective varieties of dimension $n+1$ that can be extended to rational actions of Cr_n.

Résumé. — On s'intéresse aux plongements algébriques du groupe de Cremona complexe à n variables Cr_n dans des groupes de transformations birationnelles $\mathrm{Bir}(M)$ d'une varété algébrique M. D'abord on regarde un plongement de Cr_2 dans Cr_5 qui était découvert par Gizatullin. Puis on donne une classification de tous les plongements algébriques de Cr_2 dans $\mathrm{Bir}(M)$ pour des variétés M de dimension 3 et on généralise partiellement ce résultat aux plongements algébriques de Cr_n dans $\mathrm{Bir}(M)$, où la dimension de M est $n+1$ (pour tout n). On obtient notamment une classification de toutes les action régulières de $\mathrm{PGL}_{n+1}(\mathbb{C})$ sur des variétés projectives lisses de dimension $n+1$ qui s'étendent vers des actions rationnelles de Cr_n.

1. Introduction and statement of the results

1.1. Cremona groups

Let M be a complex algebraic variety and $\mathrm{Bir}(M)$ the group of birational transformations of M. Denote by $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$ the complex projective space of

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dimension n. The group

$$\mathrm{Cr}_n := \mathrm{Bir}(\mathbb{P}^n)$$

is called the *Cremona group*. In this paper we are interested in group homomorphisms from Cr_n to $\mathrm{Bir}(M)$. In particular, we will study an embedding of Cr_2 into Cr_5 that was described by Gizatullin [29] and consider the case, where $\dim(M) = n+1$.

A birational transformation $A\colon M \dashrightarrow N$ between varieties M and N induces an isomorphism $\mathrm{Bir}(M) \to \mathrm{Bir}(N)$ by conjugating elements of $\mathrm{Bir}(M)$ with A. A homomorphism $\Phi\colon \mathrm{Bir}(M) \to \mathrm{Bir}(N_1)$ and a homomorphism $\Psi\colon \mathrm{Bir}(M) \to \mathrm{Bir}(N_2)$ are called *conjugate* if there exists a birational transformation $A\colon N_1 \dashrightarrow N_2$ such that $\Psi(g) = A \circ \Phi(g) \circ A^{-1}$ for all $g \in \mathrm{Bir}(M)$.

Example 1.1. — Assume that a variety M is birationally equivalent to $\mathbb{P}^n \times N$ for some variety N. The standard action on the first factor yields an injective homomorphism of Cr_n into $\mathrm{Bir}(\mathbb{P}^n \times N)$ and therefore also into $\mathrm{Bir}(M)$. We call embeddings of this type *standard embeddings*. In particular, we obtain in that way for all nonnegative integers m a standard embedding $\mathrm{Cr}_n \to \mathrm{Bir}(\mathbb{P}^n \times \mathbb{P}^m)$.

Example 1.2. — A variety M is called *stably rational* if there exists a n such that $M \times \mathbb{P}^n$ is rational. There exist varieties of dimension larger than or equal to 3 that are stably rational but not rational (see [4]). We will see that two standard embeddings $f_1\colon \mathrm{Cr}_n \to \mathrm{Bir}(\mathbb{P}^n \times N)$ and $f_2\colon \mathrm{Cr}_n \to \mathrm{Bir}(\mathbb{P}^n \times M)$ are conjugate if and only if N and M are birationally equivalent (Lemma 3.3). So every class of birationally equivalent stably rational varieties of dimension k defines a different conjugacy class of embeddings $\mathrm{Cr}_n \to \mathrm{Bir}(\mathbb{P}^m)$ for $m = n+k$.

If we fix homogeneous coordinates $[x_0 : \cdots : x_n]$ of $\mathbb{P}^n$, every element $f \in \mathrm{Cr}_n$ can be described by homogeneous polynomials of the same degree $f_0, \dots, f_n \in \mathbb{C}[x_0, \dots, x_n]$ without non-constant common factor, such that

$$f([x_0 : \cdots : x_n]) = [f_0 : \cdots : f_n].$$

The *degree* of f is the degree of the f_i. With respect to affine coordinates $[1 : X_1 : \cdots : X_n] = (X_1, \dots, X_n)$, we have

$$f(X_1, \dots, X_n) = (F_1, \dots, F_n),$$

where the $F_i(X_1, \dots, X_n) \in \mathbb{C}(X_1, \dots, X_n)$ are given by the quotients $f_i(1, X_1, \dots, X_n)/f_0(1, X_1, \dots, X_n)$. The subgroup of Cr_n consisting of elements F such that all the F_i are polynomials as well as all the entries of F^{-1}, is exactly $\mathrm{Aut}(\mathbb{A}^n)$, the group of polynomial automorphisms of the affine space $\mathbb{A}^n$. When we look at $\mathrm{Aut}(\mathbb{A}^n)$ as a subgroup of Cr_n, we will always consider the embedding given by the affine coordinates $x_0 \neq 0$.

An important subgroup of Cr_n is the automorphism group

$$\mathrm{Aut}(\mathbb{P}^n) \simeq \mathrm{PGL}_{n+1}(\mathbb{C}).$$

The n-dimensional subgroup of $\mathrm{Aut}(\mathbb{P}^n)$ consisting of diagonal automorphisms will be denoted by D_n. It is the *maximal torus* of Cr_n in the following sense: all diagonalizable subgroups of Cr_n are of rank $\leqslant n$ and all diagonalizable subgroups of rank n in Cr_n are conjugate to D_n ([5]).

Let $A = (a_{ij}) \in M_n(\mathbb{Z})$ be a matrix of integers. The matrix A determines a rational self map of the affine space

$$f_A = (x_1^{a_{11}} x_2^{a_{12}} \dots x_n^{a_{1n}}, x_1^{a_{21}} x_2^{a_{22}} \dots x_n^{a_{2n}}, \dots, x_1^{a_{n1}} x_2^{a_{n2}} \dots x_n^{a_{nn}}).$$

We have $f_A \circ f_B = f_{AB}$ for $A, B \in M_n(\mathbb{Z})$. One observes that f_A is a birational transformation if and only if $A \in \mathrm{GL}_n(\mathbb{Z})$. This yields an injective homomorphism $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{Cr}_n$ whose image we call the *Weyl group* and denote it by W_n. This terminology is justified by the fact that the normalizer of D_n in Cr_n is the semidirect product $\mathrm{Norm}_{\mathrm{Cr}_n}(D_n) = D_n \rtimes \mathrm{W}_n$. Note that $D_n \rtimes \mathrm{W}_n$ is the automorphism group of $(\mathbb{C}^*)^n$. Sometimes, W_n is also called the *group of monomial transformations*.

The well known theorem of Noether and Castelnuovo (see for example [2]) states that over an algebraically closed field k the Cremona group in two variables is generated by $\mathrm{PGL}_3(k)$ and the standard quadratic involution

$$\sigma := [x_1 x_2 : x_0 x_2 : x_0 x_1] \in \mathrm{W}_2.$$

Results of Hudson and Pan ([32], [36]) show that for $n \geqslant 3$ the Cremona group Cr_n is not generated by $\mathrm{PGL}_{n+1}(\mathbb{C})$ and W_n. Let

$$\mathrm{H}_n := \langle \mathrm{PGL}_{n+1}(\mathbb{C}), \mathrm{W}_n \rangle.$$

Blanc and Hedén studied the subgroup G_n of Cr_n generated by $\mathrm{PGL}_{n+1}(\mathbb{C})$ and the element $\sigma_n := [x_0^{-1} : \dots : x_n^{-1}]$ ([11]). In particular, they show that G_n is a finite index subgroup of H_n and that it is strictly contained in H_n if and only if n is odd. Further results about the group structure of G_n can be found in [23].

1.2. The case $\dim(M) \leqslant n$

Let M be a complex projective variety of dimension n and

$$\rho\colon \mathrm{PGL}_{r+1}(\mathbb{C}) \to \mathrm{Bir}(M)$$

an embedding. Then $n \geqslant r$ and if $n = r$ it follows that M is rational and that up to a field homomorphism, ρ is the standard embedding (see [16] and [22]). This implies in particular that there are no embeddings of Cr_n into $\mathrm{Bir}(M)$ if $\dim(M) < n$. In the Appendix we recall these results and show that the restriction of an automorphism of Cr_n to the subgroup H_n is inner up to a field automorphism.

1.3. Algebraic homomorphisms

We call a group homomorphism $\Psi\colon \mathrm{Cr}_n \to \mathrm{Bir}(M)$ *algebraic* if its restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is an algebraic morphism. The algebraic structure of $\mathrm{Bir}(M)$ and some properties of algebraic homomorphisms will be discussed in Section 2. Recall that an element $f \in \mathrm{Cr}_n$ is called *algebraic*, if the sequence $\{\deg(f^n)\}_{n\in\mathbb{Z}^+}$ is bounded.

Definition 1.3. — *Let M be a variety and $\varphi_M\colon \mathrm{Cr}_n \to \mathrm{Bir}(M)$ a non-trivial algebraic group homomorphism. We say that φ_M is* reducible *if there exists a variety N such that $0 < \dim(N) < \dim(M)$ and an algebraic homomorphism $\varphi_N\colon \mathrm{Cr}_n \to \mathrm{Bir}(N)$ together with a dominant rational map $\pi\colon M \dashrightarrow N$ that is Cr_n-equivariant with respect to the rational actions induced by φ_M and φ_N respectively, i.e. $\pi \circ \varphi_M(g) = \varphi_N(g) \circ \pi$ for all $g \in \mathrm{Cr}_n$.*

Remark 1.4. — In [50], Zhang uses the terminology *primitive action* for irreducible actions in the sense of Definition 1.3; in [15], Cantat says that an action *admits a non-trivial factor* if it is reducible.

Note that if we look at the induced action of Cr_n on the function field $\mathbb{C}(M)$ of M, reducibility is equivalent to the existence of a non-trivial Cr_n-invariant function field $\mathbb{C}(N) \subset \mathbb{C}(M)$.

1.4. An example by Gizatullin

In [29], Gizatullin looks at the following question: Let $\psi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_{n+1}(\mathbb{C})$ be a linear representation. Does ψ extend to a homomorphism $\Psi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_n$? He shows that the linear representations given by the action of $\mathrm{PGL}_3(\mathbb{C})$ on conics, cubics and quartics can be extended to homomorphisms from Cr_2 to Cr_5, Cr_9 and Cr_{14}, respectively. These homomorphisms are related to the rational action of Cr_2 on moduli spaces of certain vector bundles on $\mathbb{P}^2$ that were discovered by Artamkin ([3]).

In Section 3 we study in detail some geometrical properties of the homomorphism

$$\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$$

that was described by Gizatullin; by construction, the restriction of Φ to $\mathrm{PGL}_3(\mathbb{C})$ yields the linear representation $\varphi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ given by the action of $\mathrm{PGL}_3(\mathbb{C})$ on plane conics. Among other things, we prove the following:

Theorem 1.5. — *Let $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ be the Gizatullin homomorphism. Then the following is true:*

(1) *The group homomorphism Φ is injective and irreducible.*
(2) *The rational action of Cr_2 on $\mathbb{P}^5$ that is induced by Φ preserves the Veronese surface V and its secant variety $S \subset \mathbb{P}^5$ and induces rational actions of Cr_2 on V and S.*
(3) *The Veronese embedding $v\colon \mathbb{P}^2 \to \mathbb{P}^5$ is Cr_2-equivariant with respect to the standard rational action on $\mathbb{P}^2$.*
(4) *The dominant secant rational map $s\colon \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow S \subset \mathbb{P}^5$ (see Section 3.4) is Cr_2-equivariant with respect to the diagonal action of Cr_2 on $\mathbb{P}^2 \times \mathbb{P}^2$.*
(5) *The rational action of Cr_2 on $\mathbb{P}^5$ preserves a volume form on $\mathbb{P}^5$ with poles of order three along the secant variety S.*
(6) *The group homomorphism Φ sends the group of polynomial automomorphisms $\mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Cr}_2$ to $\mathrm{Aut}(\mathbb{A}^5)$.*

Note that the injectivity of Φ follows from (3); in Section 3.8 irreducibility is proved. Part (2)–(4) of Theorem 1.5 will be proved in Section 3.4, part (5) in Section 3.6 and part (6) in Section 3.7.

The representation $\varphi^\vee$ of $\mathrm{PGL}_3(\mathbb{C})$ into $\mathrm{PGL}_6(\mathbb{C})$ given by $\psi \circ \alpha$, where α is the algebraic homomorphism $g \mapsto {}^t g^{-1}$, is conjugate in Cr_5 to the representation φ. This conjugation yields the embedding $\Phi^\vee\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$, whose image preserves the secant variety S as well and induces a rational action on it. As the secant variety S is rational, Φ and $\Phi^\vee$ induce two non-standard homomorphisms from Cr_2 to Cr_4, which we denote by Ψ_1 and Ψ_2 respectively. In Section 3.5 we prove the following:

Proposition 1.6. — *The two homomorphisms $\Psi_1, \Psi_2\colon \mathrm{Cr}_2 \to \mathrm{Cr}_4$ are not conjugate in Cr_4; moreover they are irreducible and therefore not conjugate to the standard embedding.*

Remark 1.7. — The homomorphism Ψ_1 is injective, since it restricts to the standard action on the Veronese surface. However, it seems to be unclear, whether Ψ_2 is injective as well. Since the restriction of Ψ_2 to $\mathrm{PGL}_3(\mathbb{C})$ is injective, it seems unlikely that Ψ_2 is not injective. But it is not clear how to prove that.

Since Φ is algebraic, the images of algebraic elements under Φ are algebraic again (see Proposition 2.4). Calculation of the degrees of some examples suggests that Φ might even preserve the degrees of all elements in Cr_2. However, we were only able to prove the following (Section 3.7):

Theorem 1.8. — *Let* $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ *be the Gizatullin-embedding. Then*

(1) *for all elements* $f \in \mathrm{Cr}_2$ *we have* $\deg(f) \leqslant \deg(\Phi(f))$,
(2) *for all* $g \in \mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Cr}_2$ *we have* $\deg(g) = \deg(\Phi(g))$.

The image of the Weyl group W_2 under Φ is not contained in the Weyl group W_5. *More generally, it can be shown that there exists no algebraic homomorphism from* Cr_2 *to* Cr_5 *that preserves automorphisms, diagonal automorphisms and the Weyl group* (see [46]).

1.5. Algebraic embeddings in codimension 1

In Section 4 and Section 5 we look at algebraic homomorphisms $\mathrm{Cr}_n \to \mathrm{Bir}(M)$ in the case where M is a smooth projective variety of dimension $n+1$ for $n \geqslant 2$.

Example 1.9. — For all curves C of genus $\geqslant 1$, the variety $\mathbb{P}^n \times C$ is not rational and there exists the standard embedding $\Psi_C\colon \mathrm{Cr}_n \to \mathrm{Bir}(\mathbb{P}^n \times C)$.

Example 1.10. — Cr_n acts rationally on the total space of the canonical bundle of $\mathbb{P}^n$

$$K_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-(n+1)) \simeq \bigwedge^n (T\mathbb{P}^n)^\vee$$

by $f(p,\omega) = (f(p), \omega \circ (df_p)^{-1})$, where $p \in \mathbb{P}^n$ and $\omega \in \bigwedge^n (T_p\mathbb{P}^n)^\vee$. More generally, we obtain a rational action of Cr_n on the total space of the bundle $K_{\mathbb{P}^n}^{\otimes l} \simeq \mathcal{O}_{\mathbb{P}^n}(-(n+1)l)$ and on its projective completion

$$F_l := \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-l(n+1))$$

for all $l \in \mathbb{Z}_{\geqslant 0}$. This yields a countable family of injective homomorphisms

$$\Psi_l\colon \mathrm{Cr}_n \to \mathrm{Bir}(F_l).$$

Note that the restriction of this rational action to $\mathrm{PGL}_n(\mathbb{C})$ is regular, hence these embeddings are algebraic.

We can choose affine coordinates $(x_1,\dots,x_n,x_{n+1})$ of F_l such that Ψ_l is given by

$$\Psi_l(f)(x_1,\dots,x_n,x_{n+1}) = (f(x_1,\dots,x_n), J(f(x_1,\dots,x_n))^{-l}x_{n+1}),$$

where, $J(f(x_1,\dots,x_n))$ denotes the determinant of the Jacobian of f at the point $(x_1,\dots,x_n)$. Observe that Ψ_0 is conjugate to the standard embedding.

Example 1.11. — Let $\mathbb{P}(T\mathbb{P}^2)$ be the total space of the fiberwise projectivisation of the tangent bundle over $\mathbb{P}^2$. Then $\mathbb{P}(T\mathbb{P}^2)$ is rational and there is an injective group homomorphism

$$\Psi_B\colon \mathrm{Cr}_2 \to \mathrm{Bir}(\mathbb{P}(T\mathbb{P}^2))$$

defined by $\Psi_B(f)(p,v) := (f(p), \mathbb{P}(df_p)(v))$. Here, $\mathbb{P}(df_p)\colon \mathbb{P}T_p \to \mathbb{P}T_{f(p)}$ defines the projectivisation of the differential df_p of f at the point $p\in\mathbb{P}^2$.

Example 1.12. — The Grassmannian of lines in the projective 3-space $\mathbb{G}(1,3)$ is a rational variety of dimension 4 with a transitive algebraic $\mathrm{PGL}_4(\mathbb{C})$-action. This action induces an algebraic embedding of $\mathrm{PGL}_4(\mathbb{C})$ into Cr_4. In Proposition 5.2 we will show that the image of this embedding does not lie in any subgroup isomorphic to Cr_3. *So no group action of* $\mathrm{PGL}_4(\mathbb{C})$ *on* $\mathbb{G}(1,3)$ *by automorphisms can be extended to a rational action of* Cr_3.

The classification of $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions on smooth projective varieties of dimension $n+1$ is well known to the experts; in Section 4 we study their conjugacy classes. We will see that Examples 1.9 to 1.12 describe up to birational conjugation and up to algebraic automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$ all possible $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions on smooth projective varieties of dimension $n+1$ and that these actions are not birationally conjugate to each other. This yields a classification of algebraic homomorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$ to $\mathrm{Bir}(M)$ up to birational conjugacy, for smooth projective M of dimension $n+1$. We will study in Section 5 how these actions extend to rational actions of Cr_n on M. Denote by $\alpha\colon \mathrm{PGL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$ the algebraic automorphism given by $g \mapsto {}^t(g^{-1})$.

Theorem 1.13. — *Let $n \geqslant 2$, let M be a complex projective variety of dimension $n+1$ and let $\varphi\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Bir}(M)$ be a non-trivial algebraic homomorphism. Then*

(1) *φ is conjugate, up to the automorphism α, to exactly one of the embeddings described in Example 1.9 to 1.12.*

(2) *If $n=3$ and φ is conjugate to the action described in Example 1.12, then neither φ nor $\varphi\circ\alpha$ can be extended to a homomorphism of* H_3 *to* $\mathrm{Bir}(M)$.
(3) *If φ is conjugate to one of the embeddings described in Example 1.9 to 1.11 then exactly one of the embeddings φ or $\varphi\circ\alpha$ extends to a homomorphism of* Cr_n *to* $\mathrm{Bir}(M)$.
(4) *φ extends to H_n if and only if it extends to Cr_n; moreover, in this case the extension to H_n is unique.*

Theorem 1.13 classifies all group homomorphisms $\Psi\colon \mathrm{H}_n\to\mathrm{Bir}(M)$ for projective varieties M of dimension $n+1$ such that the restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is an algebraic morphism. By the theorem of Noether and Castelnuovo, we obtain in particular a full classification of all algebraic homomorphisms from Cr_2 to $\mathrm{Bir}(M)$ for projective varieties M of dimension 3:

Corollary 1.14. — *Let M be a projective variety of dimension 3 and $\Psi\colon \mathrm{Cr}_2\to\mathrm{Bir}(M)$ a non-trivial algebraic group homomorphism. Then Ψ is conjugate to exactly one of the homomorphisms described in Example 1.9 to 1.11.*

The following observations are now immediate:

Corollary 1.15. — *Let M be a projective variety of dimension 3 and $\Psi\colon \mathrm{Cr}_2\to\mathrm{Bir}(M)$ a non-trivial algebraic homomorphism. Then*

(1) *Ψ is injective.*
(2) *There exists a Cr_2-equivariant rational map $f\colon M\dashrightarrow\mathbb{P}^2$ with respect to the rational action induced by Ψ and the standard action respectively. In particular, all algebraic homomorphisms from Cr_2 to $\mathrm{Bir}(M)$ are reducible.*
(3) *There exists an integer $C_\Psi\in\mathbb{Z}$ such that*

$$1/C_\Psi\deg(f)\leqslant\deg(\Psi(f))\leqslant C_\Psi\deg(f).$$

Note that Part (3) of Corollary 1.15 resembles in some way Theorem 1.8 and leads to the following question:

Question 1.16. — *Let $\Phi\colon \mathrm{Cr}_2\to\mathrm{Cr}_n$ be an algebraic embedding. Does there always exist a constant C depending only on Φ such that $1/C\deg(f)\leqslant\deg(\Phi(f))\leqslant C\deg(f)$ for all $f\in\mathrm{Cr}_2$?*

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2. Algebraic homomorphisms

In this section we recall some results on the algebraic structure of $\mathrm{Bir}(M)$ and of some of its subgroups and we discuss our notion of algebraic homomorphisms.

2.1. The Zariski topology

We can equip $\mathrm{Bir}(M)$ with the so-called Zariski topology. Let A be an algebraic variety and

$$f\colon A\times M \dashrightarrow A\times M$$

an A-birational map (i.e. a map of the form $(a,x)\mapstochar\dashrightarrow (a,f(a,x))$ that induces an isomorphism between open subsets U and V of $A\times M$ such that the projections from U and from V to A are both surjective). For each $a\in A$ we obtain therefore an element of $\mathrm{Bir}(M)$ defined by $x\mapsto p_2(f(a,x))$, where p_2 is the second projection. Such a map $A\to \mathrm{Bir}(M)$ is called a *morphism* or *family of birational transformations parametrized by A*.

Definition 2.1. — *The Zariski topology on* $\mathrm{Bir}(M)$ *is the finest topology such that all morphisms $f\colon A\to \mathrm{Bir}(M)$ for all algebraic varieties A are continuous (with respect to the Zariski topology on A).*

The map $\iota\colon \mathrm{Bir}(M)\to\mathrm{Bir}(M)$, $x\mapsto x^{-1}$ is continuous as well as the maps $x\mapsto g\circ x$ and $x\mapsto x\circ g$ for any $g\in\mathrm{Bir}(M)$. This follows from the fact that the inverse of an A-birational map as above is again an A-birational map as is the right/left-composition with an element of $\mathrm{Bir}(M)$. The Zariski topology was introduced in [20] and [40] and studied in [10].

2.2. Algebraic subgroups

An *algebraic subgroup* of $\mathrm{Bir}(M)$ is the image of an algebraic group G by a morphism $G \to \mathrm{Bir}(M)$ that is also an injective group homomorphism. It can be shown that algebraic groups are closed in the Zariski topology and of bounded degree in the case of $\mathrm{Bir}(M) = \mathrm{Cr}_n$. Conversely, closed subgroups of bounded degree in Cr_n are always algebraic subgroups with a unique algebraic group structure that is compatible with the Zariski topology (see [10]).

Let N be a smooth projective variety that is birationally equivalent to M. Let G be an algebraic group acting regularly and faithfully on N. This yields a morphism $G \to \mathrm{Bir}(M)$, so G is an algebraic subgroup of $\mathrm{Bir}(M)$. On the other hand, a theorem by Weil states that all algebraic subgroups of $\mathrm{Bir}(M)$ have this form.

Theorem 2.2 ([42, 47, 49]). — *Let $G \subset \mathrm{Bir}(M)$ be an algebraic subgroup, where M is a smooth projective variety over $\mathbb{C}$. Then there exists a smooth projective variety N and a birational map $f\colon M \to N$ that conjugates G to a subgroup of $\mathrm{Aut}(N)$ such that the induced action on N is algebraic.*

It can be shown (see for example, [10]) that the sets $(\mathrm{Cr}_n)_{\leqslant d} \subset \mathrm{Cr}_n$ consisting of all birational transformations of degree $\leqslant d$ are closed with respect to the Zariski topology. So the closure of a subgroup of bounded degree in Cr_n is an algebraic subgroup and can therefore be regularized in the sense of the above theorem. We obtain:

Corollary 2.3. — *Let $G \subset \mathrm{Cr}_n$ be a subgroup that is contained in some $(\mathrm{Cr}_n)_{\leqslant d}$, then there exists a smooth projective variety N and a birational transformation $f\colon \mathbb{P}^n \dashrightarrow N$ such that $fGf^{-1} \subset \mathrm{Aut}(N)$.*

The maximal algebraic subgroups of Cr_2 have been classified together with the rational surfaces on which they act as automorphisms ([7, 27]). In dimension 3, a classification for maximal connected algebraic subgroups exists: [43, 44, 45].

2.3. Algebraic homomorphisms

We defined a group homomorphism from Cr_n to $\mathrm{Bir}(M)$ to be algebraic if its restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is a morphism. Note that this is a priori a weaker notion than being continuous with respect to the Zariski topology.

It is not clear, whether algebraic homomorphisms are always continuous. However, for dimension 2 we have the following partial result, which will proved in Section 2.5:

PROPOSITION 2.4. — *Let* $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Bir}(M)$ *be a homomorphism of groups. The following are equivalent:*

(1) Φ *is algebraic.*
(2) *The restriction of* Φ *to any algebraic subgroup of* Cr_2 *is algebraic.*
(3) *The restriction of* Φ *to one positive dimensional algebraic subgroup of* Cr_2 *is algebraic.*

2.4. One-parameter subgroups

A one-parameter subgroup is a connected linear algebraic group of dimension 1. It is well known (see for example [33]) that all one-parameter subgroups are isomorphic to either $\mathbb{C}$ or $\mathbb{C}^*$. The group $\mathbb{C}$ is unipotent, the group $\mathbb{C}^*$ semi-simple.

Proposition 2.5 shows that, up to conjugation by birational maps, there exists only one birational action of $\mathbb{C}$ and only one of $\mathbb{C}^*$ on $\mathbb{P}^2$:

PROPOSITION 2.5. — *In* Cr_2 *all one-parameter subgroups isomorphic to* $\mathbb{C}$ *are conjugate and all one-parameter subgroups isomorphic to* $\mathbb{C}^*$ *are conjugate.*

The first part of Proposition 2.5 follows from results in [9] and [6] (see also [14]). The second part is a special case of Theorem 2.6. A detailed explanation of the proof can be found in [46].

THEOREM 2.6 ([5, 37]). — *In* Cr_n *all tori of dimension* $\geqslant n-2$ *are conjugate to a subtorus of* D_n. *Moreover, two subtori of* D_n *are conjugate in* Cr_n *to each other if and only if they are isomorphic.*

Let G be a connected linear algebraic group and $\{U_i\}_{i\in I}$ the set of one-parameter subgroups of G. Then the subgroup $H \subset G$ generated by all the U_i is closed and connected and there exist one-parameter subgroups $U_1, U_2, \ldots, U_n$ such that $U_1 \cdot U_2 \ldots U_n = H$ ([33, Proposition 7.5]). On the other hand, if $\mathfrak{g}$ is the Lie algebra of G, then the exponential map $\exp\colon \mathfrak{g} \to G$ induces a diffeomorphism from an analytically open set of $\mathfrak{g}$ to an analytically open neigborhood V of the identity in G. For all elements $A \in \mathfrak{g}$, the closure of the abelian subgroup $\{\exp(tA) \mid t \in \mathbb{C}\}$ in G is connected and therefore contained in H. We obtain that V is contained in

G and hence that H is open in the analytic topology. This yields $H = G$ and thus

$$U_1 \cdot U_2 \dots U_n = G.$$

The following Lemma is a classical result (see for example [41]):

Lemma 2.7. — *Let G be a linear algebraic group and $U_1, \dots, U_n$ be algebraic subgroups such that $U_1 \cdot U_2 \dots U_n = G$. Let H be a linear algebraic group and $\varphi\colon G \to H$ a homomorphism of abstract groups such that $\varphi|_{U_i}$ is a homomorphism of algebraic groups for all i. Then φ is a homomorphism of algebraic groups.*

2.5. Algebraic and abstract group homomorphisms

Let G and H be algebraic groups that are isomorphic as abstract groups. The question whether G and H are also isomorphic as algebraic groups has been treated in detail in [12] (see also [24] and [21]). We will use the following result:

Proposition 2.8. — *Let G be an algebraic group that is isomorphic to $\mathrm{PGL}_n(\mathbb{C})$ as an abstract group. Then G is isomorphic to $\mathrm{PGL}_n(\mathbb{C})$ as an algebraic group. Moreover, for every abstract isomorphism*

$$\rho\colon \mathrm{PGL}_n(\mathbb{C}) \to G$$

there exists an automorphism of fields $\tau\colon \mathbb{C} \to \mathbb{C}$ such that $\rho \circ \tau$ is an algebraic isomorphism.

Remark 2.9. — It is well known that the automorphisms of $\mathrm{PGL}_n(\mathbb{C})$ as an algebraic group are compositions of inner automorphisms and the automorphism

$$\alpha\colon \mathrm{PGL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C}), \quad g \mapsto {}^t g^{-1}.$$

Proof of Proposition 2.4. — We first show how (1) implies (2). Let G be an algebraic subgroup of Cr_2. We can assume that G is connected. By the above remark, there exist one parameter subgroups $U_1, \dots, U_k \subset G$ such that $U_1 \dots U_k = G$. Since, by Proposition 2.5, the group U_i is conjugate to a one parameter subgroup of $\mathrm{PGL}_3(\mathbb{C})$ for all i, we obtain that the restriction of φ to any of the U_i is an algebraic homomorphism of groups and that $\varphi(G) \subset \mathrm{Cr}_n$ is of bounded degree. Then $\overline{\varphi(G)} \subset \mathrm{Cr}_n$ is an algebraic group. We can now apply Lemma 2.7 and conclude that the restriction of φ to G is a homomorphism of algebraic groups.

Statement (3) follows immediately from statement (2), so it only remains to prove that (3) implies (1). Let $\varphi\colon \mathrm{Cr}_2 \to \mathrm{Bir}(M)$ be a homomorphism of abstract groups and let $G \subset \mathrm{Cr}_2$ be a positive dimensional algebraic subgroup such that the restriction of φ to G is a morphism. Since G is infinite, it contains a one parameter subgroup $U \subset G$.

Let $U_1, \ldots, U_n \subset \mathrm{PGL}_3(\mathbb{C})$ be unipotent one parameter subgroups such that $U_1 \ldots U_n = \mathrm{PGL}_3(\mathbb{C})$. If U is unipotent, all the subgroups U_i are conjugate to U. Hence the restriction of φ to U_i is a morphism for all i. The image $\varphi(\mathrm{PGL}_3(\mathbb{C})) \subset \mathrm{Cr}_n$ is of bounded degree, so $\overline{\varphi(\mathrm{PGL}_3(\mathbb{C}))} \subset \mathrm{Cr}_n$ is an algebraic group and with Lemma 2.7 it follows that the restriction of φ to $\mathrm{PGL}_3(\mathbb{C})$ is a morphism.

Denote by $D_1 \subset \mathrm{PGL}_3(\mathbb{C})$ the subgroup given by elements of the form $[cx_0 : x_1 : x_2]$, $c \in \mathbb{C}^*$ and by $T \subset \mathrm{PGL}_3(\mathbb{C})$ the subgroup of all elements of the form $[x_0 : x_1 + cx_0 : x_2]$, $c \in \mathbb{C}$; we have $D_1 \simeq \mathbb{C}^*$ and $T \simeq \mathbb{C}$. If U is semi-simple, it is, again by Proposition 2.5, conjugate to D_1, hence the restriction of φ to D_1 is a morphism as well. Note that

$$T = \{[x_0 : x_1 + cx_0 : x_2] \mid c \in \mathbb{C}\} = \{dgd^{-1} \mid d \in D_1\} \cup \{\mathrm{id}\}$$

where $g = [x_0 : x_1 + x_0 : x_2]$. We obtain that $\varphi(T)$ is of bounded degree and contained in the algebraic group $\overline{\varphi(T)} \subset \mathrm{Cr}_n$. As $\varphi(T)$ consists of two $\varphi(D_1)$-orbits, it is constructible and therefore closed. We obtain that the images of all unipotent subgroups of Cr_2 under φ are algebraic subgroups.The map $\varphi(U_1) \times \cdots \times \varphi(U_n) \to \mathrm{Cr}_n$ is a morphism, so its image is a constructible set and therefore closed since it is a group. Hence $\varphi(\mathrm{PGL}_3(\mathbb{C})) = \varphi(U_1) \ldots \varphi(U_n)$ is an algebraic subgroup. By Proposition 2.8 it is isomorphic as an algebraic group to $\mathrm{PGL}_3(\mathbb{C})$ and there exists an automorphism of fields $\tau\colon \mathbb{C} \to \mathbb{C}$ such that $\varphi \circ \tau\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C})$ is an isomorphism of algebraic groups. But since the restriction of φ to T is already an algebraic homomorphism, it follows that τ is the identity. □

Remark 2.10. — Proposition 2.4 shows in particular that algebraic homomorphisms $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Bir}(M)$ send algebraic elements to algebraic elements. This result follows also directly from the fact that a birational transformation $f \in \mathrm{Cr}_2$ of degree d can be written as the product of at most $4d$ linear maps and $4d$ times the standard quadratic involution σ (see for example [2]); we therefore obtain that the sequence $\{\deg(\Phi(f)^n)\}$ is bounded if $\{\deg(f^n)\}$ is bounded.

3. An example by Gizatullin

3.1. Projective representations of the projective linear group

The results from representation theory of linear algebraic groups that we use in this section can be found, for example, in [28, 38].

Proposition 3.1. — *There is a bijection between homomorphisms of algebraic groups from* $\mathrm{SL}_n(\mathbb{C})$ *to* $\mathrm{SL}_m(\mathbb{C})$ *such that the image of the center is contained in the center and homomorphisms of algebraic groups from* $\mathrm{PGL}_n(\mathbb{C})$ *to* $\mathrm{PGL}_m(\mathbb{C})$.

From Proposition 3.1 and some elementary representation theory of $\mathrm{SL}_3(\mathbb{C})$ it follows that $n = 6$ is the smallest number such that there exist non-trivial and non-standard homomorphisms of algebraic groups from $\mathrm{PGL}_3(\mathbb{C})$ to $\mathrm{PGL}_n(\mathbb{C})$. In fact, up to automorphisms of $\mathrm{PGL}_3(\mathbb{C})$ there are exactly two non-trivial representations from $\mathrm{PGL}_3(\mathbb{C})$ to $\mathrm{PGL}_6(\mathbb{C})$.

The first one is reducible. Let $\psi'\colon \mathrm{GL}_3 \to \mathrm{GL}_6$ be the linear representation given by the diagonal action on $\mathbb{C}^3 \times \mathbb{C}^3$; we denote by $\psi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ its projectivisation.

The second one is given by the action of $\mathrm{PGL}_3(\mathbb{C})$ on the space of conics. The latter one can be parametrized by the space $\mathbb{P}M_3$ of symmetric 3×3-matrices up to scalar multiple and is isomorphic to $\mathbb{P}^5$. Let $g \in \mathrm{PGL}_3(\mathbb{C})$, we define $\varphi(g) \in \mathrm{PGL}_6(\mathbb{C})$ by $(a_{ij}) \mapsto g(a_{ij})({}^tg)$.

In this section we identify the space of conics with $\mathbb{P}^5$ in the following way:

$$(a_{ij}) \mapsto [a_{00} : a_{11} : a_{22} : a_{12} : a_{02} : a_{01}]$$

In other words, the conic C given by the zeroes of the equation

$$F = a_{00}X^2 + a_{11}Y^2 + a_{22}Z^2 + 2a_{12}YZ + 2a_{02}XZ + 2a_{01}XY$$

is identified with the point $[a_{00} : a_{11} : a_{22} : a_{12} : a_{02} : a_{01}] \in \mathbb{P}^5$.

Observe that with our definition, $\varphi(g)$ sends the conic C to the conic given by the zero set of the polynomial $F \circ ({}^tg)$.

Let

$$\alpha\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C})$$

be the algebraic automorphism $g \mapsto ({}^tg)^{-1}$. Then $\varphi(\alpha(g))$ maps the conic C to $g(C)$, which is the conic given by the zero set of the polynomial $F \circ g^{-1}$. Accordingly, $\varphi(\alpha(g)) \in \mathrm{PGL}_6(\mathbb{C})$ maps the matrix $(a_{ij}) \in M_3$ to $({}^tg)^{-1}(a_{ij})g^{-1}$.

The action of $\mathrm{PGL}_3(\mathbb{C})$ on $\mathbb{P}^5$ induced by φ has exactly three orbits that are characterized by the rank of the corresponding symmetric matrix in M_3.

Geometrically they correspond to the sets of smooth conics, pairs of distinct lines and double lines. The set of double lines is a surface isomorphic to $\mathbb{P}^2$ and called the *Veronese surface*; we denote it by V. The set of singular conics S is the secant variety of V and has dimension 4.

To describe the $\mathrm{PGL}_3(\mathbb{C})$-orbits with respect to the action induced by ψ, consider a point $p = [x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \in \mathbb{P}^5$. Then p can either be mapped by an element of $\psi(\mathrm{PGL}_3(\mathbb{C}))$ to a point of the form $[a : 0 : 0 : b : 0 : 0]$, where $[a : b] \in \mathbb{P}^1$, or to the point $[1 : 0 : 0 : 0 : 0 : 1]$ and these points are all in different $\psi(\mathrm{PGL}_3(\mathbb{C}))$-orbits. The stabilizer of $[1 : 0 : 0 : 0 : 0 : 1]$ in $\psi(\mathrm{PGL}_3(\mathbb{C}))$ is the subgroup of matrices of the form

$$\begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}, \text{ where } g \in \mathrm{PGL}_3(\mathbb{C}) \text{ has the form } \begin{bmatrix} 1 & a & 0 \\ 0 & b & 0 \\ 0 & c & 1 \end{bmatrix}.$$

Therefore, the orbit of $[1 : 0 : 0 : 0 : 0 : 1]$ under $\psi(\mathrm{PGL}_3(\mathbb{C}))$ has dimension 5. The orbit of a point of the form $[a : 0 : 0 : b : 0 : 0]$, on the other hand, has dimension 2. So we have a family parametrized by $\mathbb{P}^1$ of orbits of dimension 2 and one orbit of dimension 5. In particular, there is no $\psi(\mathrm{PGL}_3(\mathbb{C}))$-invariant subset of dimension 4.

The following observation is easy but useful. We leave its proof to the reader.

Lemma 3.2. — *Let X and Y be two projective varieties with biregular actions of a group G and let $f\colon X \dashrightarrow Y$ be a G-equivariant rational map. Then the indeterminacy locus $I_f \subset X$ and the exceptional locus $\mathrm{Exc}(f) \subset X$ are G-invariant sets.*

Note that Lemma 3.2 implies in particular that all equivariant rational maps with respect to actions without orbits of codimension $\geqslant 2$ are morphisms.

Lemma 3.3. — *Let M and M' be irreducible complex projective varieties such that $M \times \mathbb{P}^n$ et $M' \times \mathbb{P}^n$ are birationally equivalent. Then the standard embeddings*

$$\Psi\colon\ \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^n \times M) \text{ and } \Psi'\colon\ \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}^n \times M')$$

are conjugate if and only if M and M' are birationally equivalent.

Proof. — If M and M' are birationally equivalent it follows directly that Ψ and Ψ' are conjugate. On the other hand, assume that there exists a birational map $A\colon \mathbb{P}^n \times M \dashrightarrow \mathbb{P}^n \times M'$ that conjugates Ψ to Ψ', i.e. $A \circ \Psi(g) = \Psi'(g) \circ A$ for all $g \in \mathrm{PGL}_{n+1}(\mathbb{C})$. The images $\Psi(\mathrm{PGL}_{n+1}(\mathbb{C}))$ and

$\Psi'(\mathrm{PGL}_{n+1}(\mathbb{C}))$ permute the fibers $\{p\} \times M$, $p \in \mathbb{P}^n$ and $\{p\} \times M'$, $p \in \mathbb{P}^n$ respectively. By Lemma 3.2, no fiber is fully contained in the exceptional locus of A.

The fiber

$$F := [1 : 1 : \cdots : 1] \times M \subset \mathbb{P}^n \times M$$

consists of all fixed points of the image of the subgroup of coordinate permutations $\Psi(\mathcal{S}_{n+1})$ and it is isomorphic to M. Correspondingly, the fiber

$$F' := [1 : 1 : \cdots : 1] \times M' \subset \mathbb{P}^n \times M'$$

consists of all fixed points of $\Psi'(\mathcal{S}_{n+1})$ and is isomorphic to M'. Hence the strict transform of F under A is F' and we obtain that M and M' are birationally equivalent. □

Proposition 3.4. — *Let $\varphi, \psi \colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ be the homomorphisms defined in Section 3.1. The two subgroups $\varphi(\mathrm{PGL}_3(\mathbb{C}))$ and $\psi(\mathrm{PGL}_3(\mathbb{C}))$ are not conjugate in Cr_5.*

Proof. — Assume that there is an element $f \in \mathrm{Cr}_5$ that conjugates $\varphi(\mathrm{PGL}_3(\mathbb{C}))$ to $\psi(\mathrm{PGL}_3(\mathbb{C}))$. Note that $\mathbb{P}^5$ has no $\psi(\mathrm{PGL}_3(\mathbb{C}))$-invariant subset of dimension 4. Hence, by Lemma 3.2, f must be a birational morphism and therefore an automorphism. But this isn't possible since the action of $\varphi(\mathrm{PGL}_3(\mathbb{C}))$ has an orbit of dimension 4 and the action of $\psi(\mathrm{PGL}_3(\mathbb{C}))$ does not. □

3.2. A rational action on the space of plane conics

Our goal is to extend the group homomorphism $\varphi\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ to a group homomorphism

$$\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5 .$$

A first naive idea is to check whether the map $\Psi\colon \{\mathrm{PGL}_3(\mathbb{C}), \sigma\} \to \mathrm{Cr}_5$ defined by $\Psi(g) = \varphi(g)$ for $g \in \mathrm{PGL}_3(\mathbb{C})$ and $\Psi(\sigma) = [x_0^{-1} : x_2^{-1} : \cdots : x_5^{-1}]$ extends to a group homomorphism $\mathrm{Cr}_2 \to \mathrm{Cr}_5$. However, $\Psi(\sigma)$ and $\Psi(h)$ do not satisfy relation (3) of Lemma A.4. Let $h = [Z - X : Z - Y : Z] \in \mathrm{Cr}_2$, then

$$([x_0^{-1} : x_1^{-1} : \cdots : x_5^{-1}] \circ \varphi(h))^3 \neq \mathrm{id}\,.$$

In [29], Gizatullin constructs an extension $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ of φ, defined by $\Phi|_{\mathrm{PGL}_3(\mathbb{C})} = \varphi$ and

$$\Phi(\sigma) = [x_1x_2 : x_0x_2 : x_0x_1 : x_3x_0 : x_4x_1 : x_5x_2]\,.$$

He shows the following:

Proposition 3.5 ([29]). — *The map* $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ *is a group homomorphism.*

3.3. The dual action

Consider the representation $\varphi^\vee\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_6(\mathbb{C})$ that is defined by

$$\varphi^\vee(g) := {}^t\varphi(g)^{-1}.$$

In other words, $\varphi^\vee = \varphi \circ \alpha$, where $\alpha\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C})$ is the algebraic automorphism $g \mapsto ({}^tg)^{-1}$. Let $A = (a_{ij})$ be a 3×3 matrix. The cofactor matrix $C(A)$ of A is given by

$$C_{ij}(A) = (-1)^{i+j} A_{ij}\,,$$

where A_{ij} is the i,j-minor of A, i.e. the determinant of the 2×2-matrix obtained by removing the i-th row and j-th column of A. We denote by

$$Ad(A) := {}^tC(A)$$

the *adjugate* matrix of A. This is a classical construction and it is well known that $Ad(AB) = Ad(B)Ad(A)$ and that if A is invertible, then $Ad(A) = \det(A)A^{-1}$. In particular, $Ad\colon \mathbb{P}M_3 \dashrightarrow \mathbb{P}M_3$ is a birational map. The conic corresponding to the symmetric matrix A is the dual of the conic corresponding to the symmetric matrix A. This is one of the birational maps that A. R. Williams described in 1938 in his paper "Birational transformations in 4-space and 5-space" ([48]).

Lemma 3.6. — *We identify* $\mathbb{P}^5$ *with the projectivized space of symmetric* 3×3 *matrices* $\mathbb{P}M_3$*. The birational transformation* $Ad \in \mathrm{Cr}_5$ *is given by*

$$[x_1x_2 - x_3^2 : x_0x_2 - x_4^2 : x_0x_1 - x_5^2 : x_4x_5 - x_0x_3 : x_3x_5 - x_1x_4 : x_3x_4 - x_2x_5]\,.$$

Moreover, Ad conjugates φ to $\varphi^\vee$.

Proof. — It is a straightforward calculation that the rational map Ad from $\mathbb{P}^5$ to itself that corresponds to Ad is given by

$$[x_1x_2 - x_3^2 : x_0x_2 - x_4^2 : x_0x_1 - x_5^2 : x_4x_5 - x_0x_3 : x_3x_5 - x_1x_4 : x_3x_4 - x_2x_5]\,.$$

The actions of $\mathrm{PGL}_3(\mathbb{C})$ on $\mathbb{P}M_3$ induced by φ and $\varphi^\vee$ are given by $\varphi(g)(X) = gX({}^tg)$ and $\varphi^\vee(g)X = {}^t(g^{-1})Xg^{-1}$ respectively, for all $X \in \mathbb{P}M_3$. We obtain

$$\begin{aligned} Ad(\varphi(g)(X)) = Ad({}^tg)Ad(X)Ad(g) &= ({}^tg)^{-1}Ad(X)g^{-1} \\ &= \varphi^\vee(g)Ad(X)\,. \end{aligned}$$ □

Remark 3.7. — The blow-up Q of $\mathbb{P}^5$ along the Veronese surface is the so called *space of complete conics*. Let $U \subset \mathbb{P}^5$ be the open orbit of the $\mathrm{PGL}_3(\mathbb{C})$-action on $\mathbb{P}^5$ given by φ, i.e. $U = \mathbb{P}^5 \setminus S$. Then U can be embedded into $\mathbb{P}(\mathbb{C}^6) \times \mathbb{P}((\mathbb{C}^6)^\vee)$ by sending a conic $C \in U$ to the pair $(C, C^\vee)$, where $C^\vee$ denotes the dual conic of C. It turns out that Q is isomorphic to the closure of U in $\mathbb{P}(\mathbb{C}^6) \times \mathbb{P}((\mathbb{C}^6)^\vee)$. Moreover, the $\mathrm{PGL}_3(\mathbb{C})$-action on $\mathbb{P}^5$ given by φ lifts to an algebraic action on Q and the birational map ad to an automorphism of Q. More details on this subject can be found for example in [13].

Lemma 3.6 shows that the representations φ and $\varphi^\vee$ are conjugate to each other in Cr_5 by the birational transformation Ad. By conjugating $\Phi(\sigma)$ with Ad we can extend $\varphi^\vee$ to the *dual embedding* $\Phi^\vee \colon \mathrm{Cr}_2 \to \mathrm{Cr}_5$ and obtain

$$\Phi^\vee(\sigma) = [p_0^2 x_0 : p_1^2 x_1 : p_2^2 x_2 : p_1 p_2 x_3 : p_0 p_2 x_4 : p_0 p_1 x_5],$$

where $p_0 = (x_1 x_2 - x_3^2), p_1 = (x_0 x_2 - x_4^2)$ and $p_2 = (x_0 x_1 - x_5^2)$.

3.4. Geometry of Φ

The embedding Φ induces a rational action of Cr_2 on the space of conics on $\mathbb{P}^2$. The action of $\Phi(\sigma)$ can be seen geometrically as follows (compare with [29, Introduction]): Let $Q_0 := [1 : 0 : 0]$, $Q_1 := [0 : 1 : 0]$ and $Q_2 := [0 : 0 : 1]$. Let $C \subset \mathbb{P}^2$ be a conic that doesn't pass through any of the points Q_i. Write

$$C = \{a_{00}X^2 + a_{11}Y^2 + a_{22}Z^2 + 2a_{12}YZ + 2a_{02}XZ + 2a_{01}XY = 0\} \subset \mathbb{P}^2$$

Denote by P_{ij}, $i \in \{1,2,3\}$, $j \in \{1,2\}$ the points of intersection of C with the lines l_i, where $l_0 := \{X = 0\}$, $l_1 := \{Y = 0\}$ and $l_2 := \{Z = 0\}$. Denote by f_{ij} the line passing through Q_i and P_{ij}. The images $\sigma(f_{ij})$ are again lines passing through the point Q_i. Let P'_{ij} be the intersection of $\sigma(f_{ij})$ with l_i. One checks that the conic D given by the zero set of the polynomial

$$a_{11}a_{22}x_0^2 + a_{00}a_{22}x_1^2 + a_{00}a_{11}x_2^2 + 2a_{00}a_{12}x_1x_2 + 2a_{11}a_{02}x_0x_2 + 2a_{22}a_{01}x_0x_2$$

passes through the points P'_{ij}. Since no 4 of the 6 points P'_{ij} lie on the same line, D is the unique conic through the points P'_{ij}. We have thus proven the following:

Proposition 3.8. — *For a general conic $C \subset \mathbb{P}^2$ there exists a unique conic D through the six points P'_{ij} and D is the image of C under $\Phi(\sigma)$.*

Notice as well that the indeterminacy points of $\Phi(\sigma)$ in $\mathbb{P}^5$ correspond to the subspace of dimension 2 of conics passing through the points Q_1, Q_2, Q_3 and the subspaces of dimension 2 of conics consisting of one l_i and any other line. The three subspaces of dimension 4 of conics passing through one of the points Q_i are contracted by the action of $\Phi(\sigma)$ and form the exceptional divisor.

In homogeneous coordinates of $\mathbb{P}^5$, the four planes of the indeterminacy locus of $\Phi(\sigma)$ can be described as follows

$$E_0 = \{x_1 = x_2 = x_3 = 0\}, \qquad E_1 = \{x_0 = x_2 = x_4 = 0\},$$
$$E_2 = \{x_0 = x_1 = x_5 = 0\}, \qquad F = \{x_1 = x_2 = x_3 = 0\}.$$

The exceptional divisor of $\Phi(\sigma)$ consists of the three hyperplanes

$$H_0 = \{x_0 = 0\}, \quad H_1 = \{x_1 = 0\}, \quad H_2 = \{x_2 = 0\},$$

The hyperplanes H_0, H_1 and H_2 are contracted by $\Phi(\sigma)$ onto the planes E_0, E_1 and E_2 respectively. Note as well that E_0, E_1 and E_2 are contained in the secant variety $S \subset \mathbb{P}^5$ of the Veronese surface V and that they are tangent to V.

The geometrical description of the rational action of $\Phi^\vee(\sigma)$ on the space of conics is the dual of the construction described above. If C is a conic not passing through any of the points Q_0, Q_1, Q_2, we get $\Phi^\vee(\sigma)(C)$ in the following way: let $l_{i,1}, l_{i,2}$ be the tangents of C passing through the point Q_i. Then the images of the $l_{i,1}$ and $l_{i,2}$ under σ are lines again. There exists a unique conic having all the lines $\sigma(l_{i,1})$ and $\sigma(l_{i,2})$ for all i as tangents.

These geometrical constructions show that $\Phi(\mathrm{Cr}_2)$ preserves the space of conics consisting of double lines and therefore the Veronese surface V in $\mathbb{P}^5$. The injective morphism

$$v\colon \mathbb{P}^2 \to \mathbb{P}^5,\ [X:Y:Z] \mapsto [X^2 : Y^2 : Z^2 : YZ : XZ : XY]$$

is called the *Veronese morphism*. It is an isomorphism onto its image, which is V. It is well known that v is $\mathrm{PGL}_3(\mathbb{C})$-equivariant with respect to the standard action and the action induced by Φ respectively. The restriction of $\Phi(\sigma)$ to V is a birational transformation. We therefore obtain a rational action of Cr_2 on $V \simeq \mathbb{P}^2$. Since the restriction of this rational action to $\mathrm{PGL}_3(\mathbb{C})$ is the standard action, we obtain by Corollary A.3 that v is Cr_2-equivariant.

We observe as well that $\Phi(\mathrm{Cr}_2)$ preserves the secant variety $S \subset \mathbb{P}^5$ of V. Note that S is the image of the rational map:

$$s\colon \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow S \subset \mathbb{P}^5,$$

that maps the point $[X:Y:Z],[U:V:W] \in \mathbb{P}^2 \times \mathbb{P}^2$ to the point

$$[XU : YV : ZW : 1/2(YW + UZ) : 1/2(XW + ZU) : 1/2(XV + YU)].$$

Note that s is generically $2:1$. Again, the geometrical construction above shows that s is Cr_2-equivariant with respect to the diagonal action on $\mathbb{P}^2 \times \mathbb{P}^2$ and the action given by Φ on $\mathbb{P}^5$ respectively.

We obtain the following sequence of Cr_2-equivariant rational maps:

$$\mathbb{P}^2 \xrightarrow{\Delta} \mathbb{P}^2 \times \mathbb{P}^2 \overset{s}{\dashrightarrow} \mathbb{P}^5,$$

where Δ is the diagonal embedding. This proves part (2) to (4) of Theorem 1.5.

Let $f \in \mathrm{Cr}_n$ be a birational transformation and let $l \subset \mathbb{P}^n$ be a general line and $H \subset \mathbb{P}^5$ a general hyperplane. Then, $f^{-1}(H)$ intersects l in $\deg(f)$ points, which is equivalent to $f(l)$ intersecting H in $\deg(f)$ points. More generally, if $C \subset \mathbb{P}^n$ is a general curve of degree d, then $f(C)$ intersects H in $d \cdot \deg(f)$ points. If C and H are not in general position, but C is not contained in the exceptional locus of f and $f(C)$ is not contained in H, we only have that $f(C)$ and H intersect in $\leqslant d \cdot \deg(f)$ points. With this and the observation that $\Phi(\mathrm{Cr}_2)$ preserves the Veronese surface and extends the canonical rational action of Cr_2 we are able to prove part (1) of Theorem 1.8:

Proposition 3.9. — *Let $f \in \mathrm{Cr}_2$. Then* $\deg(f) \leqslant \deg(\Phi(f))$.

Proof. — Denote by $v \colon \mathbb{P}^2 \to \mathbb{P}^5$ the Veronese embedding. Let $C \subset \mathbb{P}^2$ be a general conic. The image $v(C) \subset \mathbb{P}^5$ is a curve of degree 4 given by the intersection of a hyperplane $H \subset \mathbb{P}^5$ and the Veronese surface. Let $f \in \mathrm{Cr}_2$ be a birational transformation of degree d. The strict transform $f(C')$ of a general conic $C' \subset \mathbb{P}^2$ intersects C in $4d$ different points. By the above results (namely, (3) in Theorem 1.5) we know that $v(f(C')) = \Phi(f)(v(C'))$. The curve $v(C')$ is a curve of degree 4 and since $v(f(C')) = \Phi(f)(v(C'))$ intersects the hyperplane H in $4d$ points we obtain by the above remark that $\deg(\Phi(f)) \geqslant d$. □

3.5. Two induced homomorphisms from Cr_2 into Cr_4

The birational map $Ad \in \mathrm{Cr}_5$ contracts the secant variety $S \subset \mathbb{P}^5$ onto the Veronese surface $V \subset \mathbb{P}^5$. However, the exceptional locus of $\Psi^\vee(\sigma) = Ad\Phi(\sigma)Ad$ consists of the three hyperplanes

$$G_0 = \{z_1z_2 - z_3^2 = 0\}, \quad G_1 = \{z_0z_2 - z_4^2 = 0\}, \quad G_2 = \{z_0z_1 - z_5^2 = 0\},$$

with respect to homogeneous coordinates $[z_0 : z_1 : z_2 : z_3 : z_4 : z_5]$ of $\mathbb{P}^5$.

This implies in particular that the restriction of $\Phi^\vee(\sigma)$ to S induces a birational map of S and therefore that any element in $\Phi^\vee(\mathrm{Cr}_2)$ restricts to a birational map of S.

Since S is a cubic hypersurface and contains the two disjoint planes

$$E_1 = \{z_1 = z_2 = z_3 = 0\}, \quad E_2 = \{z_0 = z_4 = z_5 = 0\},$$

it is rational. Explicitely, projection onto E_1 and E_2 yields the birational map $A\colon S \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \mapsto [z_1 : z_2 : z_3], [z_0 : z_4 : z_5].$$

The inverse transformation A^{-1} is given by

$$[x_0 : x_1 : x_2], [y_0 : y_1 : y_2] \mapsto [p_2y_0, p_1x_0, p_1x_1, p_1x_2, p_2y_1, p_2y_2],$$

where $p_1 = (x_0y_1^2 + x_1y_2^2 - 2x_2y_1y_2)$ and $p_2 = y_0(x_0x_1 - x_2^2)$.

Let $f \in \mathrm{Cr}_2$. As seen above, both images $\Phi(f)$ and $\Phi^\vee(f)$ restrict to a birational map of S. So conjugation of Φ and $\Phi^\vee$ by A yields two embeddings from Cr_2 into $\mathrm{Bir}(\mathbb{P}^2 \times \mathbb{P}^2) \simeq \mathrm{Cr}_4$, which we denote by Ψ_1 and Ψ_2 respectively.

Proof of Proposition 1.6. — Irreducibility is proved in Section 3.8.

By Theorem 2.6, all tori $D_2 \subset \mathrm{Cr}_4$ are conjugate to the standard torus $D_2 \subset \mathrm{Cr}_4$. We calculate the map that conjugates $\Psi_1(D_2) = \Psi_2(D_2)$ to the image of the standard embedding of D_2 explicitely. Let $\rho\colon \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ be the birational transformation defined by

$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([x_2y_0 : x_0y_1 : x_2y_1], [x_0y_1^2 : x_1y_2^2 : x_2y_1y_2]).$$

The inverse map ρ^{-1} is given by

$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([x_1^2y_2^2 : x_2^2y_0y_1 : x_1x_2y_2^2], [x_0y_0 : x_2y_0 : x_1y_2]).$$

One calculates that $\rho A \Psi_1([aX : bY : cZ])A^{-1}\rho^{-1}$ maps the point $([x_0 : x_1 : x_2], [y_0 : y_1 : y_2])$ to $([ax_0 : bx_1 : cx_2], [y_0 : y_1 : y_2])$. Correspondingly, $\rho A \Psi_2([aX : bY : cZ])A^{-1}\rho^{-1}$ maps $([x_0 : x_1 : x_2], [y_0 : y_1 : y_2])$ to $([a^{-1}x_0 : b^{-1}x_1 : c^{-1}x_2], [y_0 : y_1 : y_2])$. So the second coordinates parametrize the closures of the D_2-orbits. Since W_2 normalizes D_2, its image preserves the D_2-orbits. We thus obtain two homomorphisms

$$\chi_1\colon \mathrm{W}_2 \to \mathrm{Cr}_2, \quad \chi_2\colon \mathrm{W}_2 \to \mathrm{Cr}_2$$

by just considering the rational action of W_2 on the second coordinate.

Assume that there exists an element $A \in \mathrm{Bir}(\mathbb{P}^2 \times \mathbb{P}^2)$ that conjugates Ψ to $\Psi^\vee$. As A normalizes $\Psi_1(D_2) = \Psi_2(D_2)$, it preserves the $\Psi_1(D_2)$-orbits as well. Hence by restriction on the second coordinate, it conjugates χ_1 to χ_2. It therefore suffices to show that χ_1 and χ_2 are not conjugate.

In Cr_2 we have

$$f := [XY : YZ : Z^2] = \tau_1 g_0 \sigma g_0 \sigma g_0 \tau_2\,,$$

where $\tau_1 = [Z : Y : X], \tau_2 = [Y : Z : X]$ and $g_0 = [Y - X : Y : Z]$. By calculating the corresponding images under Φ we obtain

$$\Phi(f) = \Phi(\tau_1 g_0 \sigma g_0 \sigma g_0 \tau_2) = [x_0x_1 : x_1x_2 : x_2^2 : x_2x_3 : -x_2x_5 + 2x_3x_4 : x_1x_4]$$

and $\Phi^\vee(f) = [g_0 : g_1 : g_2 : g_3 : g_4 : g_5]$, where

$$\begin{aligned}
g_0 &= (x_0x_1 - x_5^2)^2 x_0\,,\\
g_1 &= x_0^2x_1^2x_2 - 2x_0x_1x_2x_5^2 - 4x_0x_1x_3x_4x_5 + 4x_0x_3^2x_5^2\\
&\qquad\qquad + 4x_1x_4^2x_5^2 + x_2x_5^4 - 4x_3x_4x_5^3\,,\\
g_2 &= (x_0x_2 - x_4^2)^2 x_1\,,\\
g_3 &= (x_0x_2 - x_4^2)(x_0x_1x_3 - 2x_1x_4x_5 + x_3x_5^2)\,,\\
g_4 &= -(x_0x_2 - x_4^2)(x_0x_1 - x_5^2)x_5\,,\\
g_5 &= (x_0x_1 - x_5^2)(x_0x_1x_4 - 2x_0x_3x_5 + x_4x_5^2)\,.
\end{aligned}$$

This yields

$$\chi_1(f) = [(y_1 - 2y_2)^2 : y_0y_1 : -y_2(y_1 - 2y_2)]$$

and

$$\begin{aligned}
\chi_2(f) = [y_0^2y_1 + 4y_0y_1^2 - 6y_0y_1y_2 - 3y_1y_2^2 + 4y_2^3 : y_0(y_0 + 2y_1 - 3y_2)^2 :\\
(2y_0y_1 - y_0y_2 - y_2^2)(y_0 + 2y_1 - 3y_2)]\,.
\end{aligned}$$

We show that these two transformations are not conjugate in Cr_2. With respect to affine coordinates $[y_0 : y_1 : 1]$ one calculates

$$\chi_1(f)^2 = \left(\frac{y_0y_1 - 2y_1 + 4}{y_1 - 2}, y_1\right).$$

From this we see that the integer sequence $\deg(\chi_1(f)^n)$ grows linearly in n and is, in particular, not bounded.

Let $A = [y_0 - y_2 : y_1 - y_2 : y_2]$. Then $A\chi_2(f)^2A^{-1}$ is given by

$$[-y_0^2y_1^2(2y_1 + y_0) : y_0^2y_1^2(3y_1 + 2y_0) : p(y_0, y_1, y_2)(3y_1 + 2y_0)(2y_1 + y_0)]\,,$$

where $p(y_0, y_1, y_2) = (6y_1^2y_2 + 7y_2y_0y_1 + 6y_0y_1^2 + 2y_0^2y_2 + 2y_0^2y_1)$. We claim that

$$\begin{aligned}
f_A^n &= A\chi_2(f)^{2n}A^{-1}\\
&= [-y_0^2y_1^2(2ny_1 + (2n-1)y_0) : y_0^2y_1^2((2n+1)y_1 + 2ny_0) : f_n]\,,
\end{aligned}$$

where $f_n = (2ny_1 + (2n-1)y_0)((2n+1)y_1 + 2ny_0)p_n(y_0, y_1, y_2)$ for some homogeneous $p_n \in \mathbb{C}[y_0, y_1, y_2]$ of degree 3. Note that this claim implies in particular that $\deg(\chi_2(f)^n)$ is bounded for all n and hence that $\chi_1(f)$ and $\chi_2(f)$ are not conjugate.

To prove the claim we proceed by induction. Assume that f_A^n has the desired form. One calculates that the first coordinate of $f_A^{n+1} = A\chi_2(f)^2A^{-1} \circ f_A^n$ is

$$-ry_0^2y_1^2((2n+2)y_1 + (2n+1)y_0)\,,$$

the second coordinate is

$$ry_0^2y_1^2((2n+3)y_1 + (2n+1)y_0)$$

and the third coordinate

$$r((2n+2)y_1 + (2n+1)y_0)((2n+3)y_1 + (2n+1)y_0)p_{n+1}(x_0, x_1, x_2)\,,$$

where $r = y_0^4y_1^4(2ny_0+(2n-1)y_1)^2((2n+1)y_0+2ny_1)^2$ and $p_n \in \mathbb{C}[x_0, x_1, x_2]$ is homogeneous of degree 3. This proves the claim. $\square$

3.6. A volume form

Let M be a complex projective variety. It is sometimes interesting to study subgroups of $\mathrm{Bir}(M)$ that preserve a given form. In [8] and [25] the authors study for example birational maps of surfaces that preserve a meromorphic symplectic form (see [19] for the 3-dimensional case). In [30] and [18] Cremona transformations in dimension 3 preserving a contact form are studied.

Define

$$F := \det \begin{pmatrix} x_0 & x_5 & x_4 \\ x_5 & x_1 & x_3 \\ x_4 & x_3 & x_2 \end{pmatrix}$$

and let

$$\Omega := \frac{x_5^6}{F^2} \cdot dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4\,.$$

Then Ω is a 5-form on $\mathbb{P}^5$ with a double pole along the secant variety of the Veronese surface. Note that the total volume of $\mathbb{P}^5$ is infinite.

Proposition 3.10. — *All elements in $\Phi(\mathrm{Cr}_2)$ preserve Ω.*

Proof. — We show that $\Phi(\mathrm{PGL}_3(\mathbb{C}))$ and $\Phi(\sigma)$ preserve Ω.

Let $g = [-X : -Y : Z] \in \Phi(\mathrm{PGL}_3(\mathbb{C}))$. One checks that $\Phi(g)$ preserves Ω. Since $\Phi(\mathrm{PGL}_3(\mathbb{C}))$ preserves F, we have that $\Phi(fgf^{-1})$ preserves Ω as well. As $\Phi(\mathrm{PGL}_3(\mathbb{C}))$ is simple, the whole group preserves Ω.

With respect to affine coordinates given by $x_5=1$, we have

$$\Phi(\sigma)=(x_1,x_0,x_0x_1x_2^{-1},x_0x_3x_2^{-1},x_1x_4x_2^{-1}).$$

A direct calculation yields $\Omega\circ\Phi(\sigma)=\Omega$. □

3.7. Polynomial automorphisms

In this section we will prove Claim (6) of Theorem 1.5 as well as Theorem 1.8. Let $\mathrm{Aut}(\mathbb{A}^2)\subset\mathrm{Cr}_2$ be the subgroup of automorphisms of the affine plane with respect to the affine coordinates $[1:X:Y]$. By the theorem of Jung and van der Kulk (see for example [34]), $\mathrm{Aut}(\mathbb{A}^2)$ has the following amalgamated product structure

$$\mathrm{Aut}(\mathbb{A}^2)=\mathrm{Aff}_2*_{\cap}\mathcal{J}_2,$$

where $\mathcal{J}_2$ denotes the subgroup of elementary automorphisms, which is the subgroup of all elements of the form

$$\{(c_1X+b,c_2Y+p(X))\mid c_1,c_2,b\in\mathbb{C},p(X)\in\mathbb{C}[X]\}.$$

Let $f\in\mathrm{Aut}(\mathbb{A}^2)$ and assume that $f=a_1j_1a_2j_2\dots j_{n-1}a_n$, where $a_1,a_n\in\mathrm{Aff}_2$, $a_i\in\mathrm{Aff}_2\setminus\mathcal{J}_2$ for $2\leqslant i\leqslant n-1$ and $j_i\in\mathcal{J}_2\setminus\mathrm{Aff}_2$. It is well known that $\deg(f)=\deg(j_1)\deg(j_2)\dots\deg(j_{n-1})$.

Let $\mathrm{Aut}(\mathbb{A}^5)\subset\mathrm{Cr}_5$ be given by the affine coordinates $[1:x_1:\cdots:x_5]$. Lemma 3.11 follows from a direct calculation.

Lemma 3.11. — *The image* $\Phi(\mathrm{Aff}_2)$ *is contained in* Aff_5.

We consider the following elements in $\mathcal{J}_2$:

$$f_n^\lambda:=(X,Y+\lambda X^n),$$

where $n\in\mathbb{Z}_{\geqslant 0}$ and $\lambda\in\mathbb{C}$.

Lemma 3.12. — *For all* $n\in\mathbb{Z}_{\geqslant 0}$ *we have*

$$\Phi(f_n^\lambda)=(x_1,x_2+\lambda^2x_1^n+\lambda x_3A_n-\lambda x_4x_1A_{n-1},x_3+\lambda x_1B_{n-1},x_4+\lambda B_n,x_5),$$

where

$$A_n=2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k+1}x_5^{n-2k-1}(x_5^2-x_1)^k$$

and

$$B_n=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}x_5^{n-2k}(x_5^2-x_1)^k.$$

Moreover, the following recursive identities hold:

$$A_n = 2x_5 A_{n-1} - x_1 A_{n-2}\,,$$
$$B_n = 2x_5 B_{n-1} - x_1 B_{n-2}\,.$$

Proof. — For $n = 0$ and $n = 1$ the claim follows from a direct calculation.

Let $s := (X, XY) \in \mathrm{Cr}_2$. Then we have $f_{n+1}^{\lambda} = s f_n^{\lambda} s^{-1}$. In Cr_2 the identity $s = \tau_1 g_0 \sigma g_0 \sigma g_0 \tau_2$ holds, where $\tau_1 = (XY^{-1}, Y^{-1}), \tau_2 = (Y^{-1}, XY^{-1})$ and $g_0 = (X, X_Y)$. Note that τ_1 and τ_2 are elements of $\mathrm{PGL}_3(\mathbb{C})$. If we calculate the corresponding images under Φ we obtain

$$\Phi(s) = \Phi(\tau_1 g_0 \sigma g_0 \sigma g_0 \tau_2) = (x_1, x_1x_2, x_1x_4, 2x_4x_5 - x_3, x_5)$$

and

$$\Phi(s^{-1}) = (x_1, x_2x_1^{-1}, 2x_3x_5x_1^{-1} - x_4, x_3x_1^{-1}, x_5)\,.$$

One calculates

$$\begin{aligned} s f_n^{\lambda} s^{-1} = (x_1, x_2 + \lambda^2 x_1^{n+1} + \lambda x_3(2x_5 - x_1)A_{n-1} - \lambda x_4 x_1 A_n,& \\ x_3 + \lambda x_1 B_n, x_4 - \lambda(2x_5 B_n - x_1 B_{n-1})&\,. \end{aligned}$$

This shows by induction that

$$\Phi(f_n^{\lambda}) = (x_1, x_2 + \lambda^2 x_1^n + \lambda x_3 A_n - \lambda x_4 x_1 A_{n-1}, x_3 + \lambda x_1 B_{n-1}, x_4 + \lambda B_n, x_5),$$

where

$$\begin{aligned} A_n &= 2x_5 A_{n-1} - x_1 A_{n-2}\,, \qquad A_0 = 0\,, \qquad A_1 = 2\,; \\ B_n &= 2x_5 B_{n-1} - x_1 B_{n-2}\,, \qquad B_0 = 1\,, \qquad B_1 = x_5\,. \end{aligned}$$

These recursive formulas have the following closed form:

$$\begin{aligned} A_n &= \frac{\left(x_5 + \sqrt{{x_5}^2 - x_1}\right)^n - \left(x_5 - \sqrt{{x_5}^2 - x_1}\right)^n}{\sqrt{{x_5}^2 - x_1}}\,, \\ B_n &= \frac{1}{2}\left(x_5 - \sqrt{{x_5}^2 - x_1}\right)^n + 1/2\,\left(x_5 + \sqrt{{x_5}^2 - x_1}\right)^n\,. \end{aligned}$$

The claim follows. □

Since Aff_n together with all the elements f_n^{λ}, $n \in \mathbb{Z}^+$, $\lambda \neq 0$ generates $\mathrm{Aut}(\mathbb{A}^2)$, Lemma 3.12 shows that $\Phi(\mathrm{Aut}(\mathbb{A}^2))$ is contained in $\mathrm{Aut}(\mathbb{A}^5)$ and thus Claim (6) of Theorem 1.5.

Lemma 3.13. — *Let n and m be positive integers and A_n, B_m as in Lemma 3.12. Then*

$$A_n B_{m-1} - A_{n-1} B_m = P(x_1, x_5)\,,$$

where $P \in \mathbb{C}[x_1, x_5]$ is a polynomial of degree $< \max\{m, n\}$.

Proof. — If $n = 1$ or $m = 1$ the claim is true, since $A_0 = 0, A_1 = 2, B_0 = 1, B_1 = x_5$ and $\deg(A_k) = k - 1, \deg(B_k) = k$. By the identities from Lemma 3.12, one obtains

$$\begin{aligned} A_n B_{m-1} - A_{n-1} B_m &= (2x_5 A_{n-1} - x_1 A_{n-2}) B_{m-1} \\ &\qquad - A_{n-1}(2x_5 B_{m-1} - x_1 B_{m-2}) \\ &= x_1(A_{n-1} B_{m-2} - A_{n-2} B_{m-1}). \end{aligned}$$

The claim follows by induction on m and n. $\square$

Lemma 3.14. — *Let $f = f_1^{\lambda_1} f_2^{\lambda_2} \dots f_n^{\lambda_n}$, where $\lambda_n \neq 0$. Then*

$$\Phi(f) = (x_1, x_2 + F, x_3 + p_3(x_1, x_5) + \lambda_n x_1 B_{n-1}, x_4 + p_4(x_1, x_5) + \lambda_n B_n, x_5),$$

where $F = p_2(x_1, x_5) + x_3(\lambda_1 A_1 + \dots + \lambda_n A_n) - x_4 x_1(\lambda_1 A_{n-1} + \dots + \lambda_n A_n)$ and $p_2, p_3, p_4 \in \mathbb{C}[x_1, x_5]$ are polynomials of degree $\leqslant n$. In particular, $\deg(\Phi(f)) = \deg(f)$.

Proof. — It is easy to see that the third and fourth coordinate of $\Phi(f)$ have the claimed form. The more difficult part is the second coordinate.

For $n = 1$ the claim follows directly from Lemma 3.12. We proceed now by induction. Let $\lambda_{n+1} \neq 0$ and m be the largest number, such that $m \leqslant n$ and $n \neq 0$. By the induction hypothesis we may assume that the second coordinate of $\Phi(f_1^{\lambda_1} f_2^{\lambda_2} \dots f_m^{\lambda_m})$ has the form

$$x_2 + p_2(x_1, x_5) + x_3(\lambda_1 A_1 + \dots + \lambda_m A_m) - x_4 x_1(\lambda_1 A_0 + \dots + \lambda_m A_{m-1}).$$

The second coordinate of $\Phi(f_1^{\lambda_1} f_2^{\lambda_2} \dots f_m^{\lambda_m}) \circ \Phi(f_n^{\lambda_n})$ is therefore

$$\begin{aligned} &x_2 + p_2(x_1, x_5) + x_3(\lambda_1 A_1 + \dots + \lambda_m A_m + \lambda_n A_n) \\ &- x_4 x_1(\lambda_1 A_0 + \dots + \lambda_m A_{m-1} + \lambda_n A_n) + x_1 \sum_{k=1}^{m} \lambda_k (A_k B_{n-1} - A_{k-1} B_n). \end{aligned}$$

By Lemma 3.13, $x_1 \sum_{k=1}^{m} \lambda_k (A_k B_{n-1} - A_{k-1} B_n)$ is a polynomial in x_1 and x_5 of degree $\leqslant n$. $\square$

Proof of Theorem 1.8. — The first claim was proved in Proposition 3.9.

To prove the second part, we show in a first step that $\deg(\Phi(f)) = \deg(f)$ for all elements $f \in \mathcal{J}_2$. Composition with an element in Aff_2 doesn't change the degree, by Lemma 3.11. So it is enough to consider elements in $\mathcal{J}_2$ of the form $f = (X, Y + P(X)), P \in \mathbb{C}[X]$. For suitable $\lambda_i \in \mathbb{C}$ we have $f = f_1^{\lambda_1} f_2^{\lambda_2} \dots f_n^{\lambda_n}$, where $\lambda_n \neq 0$. In Lemma 3.14 we've seen that Φ preserves the degree of these elements.

Let now $f \in \mathrm{Aut}(\mathbb{A}^2)$ be an arbitrary automorphism and assume that $f = a_1 j_1 a_2 j_2 \dots j_{n-1} a_n$, where $a_1, a_n \in \mathrm{Aff}_2$, $a_i \in \mathrm{Aff}_2 \setminus \mathcal{J}_2$ for $2 \leqslant i \leqslant n-1$

and $j_i \in \mathcal{J}_2 \setminus \mathrm{Aff}_2$. So $\deg(f) = \deg(j_1)\deg(j_2)\ldots\deg(j_{n-1})$ and

$$\Phi(f) = \Phi(a_1)\Phi(j_1)\Phi(a_2)\Phi(j_2)\ldots\Phi(j_{n-1})\Phi(a_n).$$

This implies in particular that

$$\deg(\Phi(f)) \leqslant \deg(\Phi(a_1))\deg(\Phi(j_1))\ldots\deg(\Phi(j_{n-1}))\deg(\Phi(a_n)) = \deg(f).$$

With this and part (1) of Theorem 1.8, part (2) follows. □

3.8. Irreducibility of Φ, Ψ_1 and Ψ_2

First we show that Φ is irreducible. Assume that there is a rational dominant map $\pi\colon \mathbb{P}^5 \dashrightarrow M$ to a variety M with an algebraic embedding $\varphi_M\colon \mathrm{Cr}_2 \to \mathrm{Bir}(M)$ such that A is Cr_2-equivariant. Since φ_M is algebraic, we may assume that $\mathrm{PGL}_3(\mathbb{C})$ acts regularly on M. We obtain that the restriction of A to the open $\mathrm{PGL}_3(\mathbb{C})$-invariant subset $U \subset \mathbb{P}^5$ consisting of all smooth conics is a $\mathrm{PGL}_3(\mathbb{C})$-equivariant morphism, whose image is an open dense subset of M on which $\mathrm{PGL}_3(\mathbb{C})$ acts transitively. Note that this implies $\dim(M) > 1$.

If $\dim M = 2$, we obtain by Theorem A.1 that $M \simeq \mathbb{P}^2$ with the standard action of $\mathrm{PGL}_3(\mathbb{C})$. The stabilizer in $\mathrm{PGL}_3(\mathbb{C})$ of a point in $U \subset \mathbb{P}^5$ is isomorphic to $\mathrm{SO}_3(\mathbb{C})$. On the other hand the stabilizer in $\mathrm{PGL}_3(\mathbb{C})$ of a point in $\mathbb{P}^2$ is isomorphic to the group of affine transformations $\mathrm{Aff}_2 = \mathrm{GL}_2(\mathbb{C}) \ltimes \mathbb{C}^2$. Since $\mathrm{SO}_3(\mathbb{C})$ can not be embedded into Aff_2, the case $\dim(M) = 2$ is not possible.

If $\dim(M) = 3$, we find, by Theorem 4.1, a $\mathrm{PGL}_3(\mathbb{C})$-equivariant projection $M \dashrightarrow \mathbb{P}^2$ and are again in the case $\dim(M) = 2$.

If $\dim(M) = 4$, let $p \in M$ be a general point and $F_p := A^{-1}(p) \subset \mathbb{P}^5$ the fiber of A. Let $q \in F_p$ be a point that is only contained in one connected component C of F_p. Again, the stabilizer of q is isomorphic to $\mathrm{SO}_3(\mathbb{C})$. This implies that $\mathrm{SO}_3(\mathbb{C})$ acts regularly on the curve C with a fixpoint. The neutral comoponent of the group of birational transformations of C is isomorphic to $\mathrm{PGL}_2(\mathbb{C})$, is abelian or is finite. In all cases we obtain that $\mathrm{SO}_3(\mathbb{C})$ fixes C pointwise. In other words, the group $\mathrm{SO}_3(\mathbb{C})$ preserves each conic of the family of conics in $\mathbb{P}^2$ parametrized by C. This is not possible.

The proof that Ψ_1 and Ψ_2 are irreducible is done analogously.

4. $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions in codimension 1

In this section we look at algebraic embeddings of $\mathrm{PGL}_{n+1}(\mathbb{C})$ into $\mathrm{Bir}(M)$ for complex projective varieties M of dimension $n+1$. Our aim is to prove Theorem 4.1.

Theorem 4.1. — *Let $n \geqslant 2$ and let M be a smooth projective variety of dimension $n+1$ with a rational non-trivial $\mathrm{PGL}_{n+1}(\mathbb{C})$-action. Then, up to birational conjugation and automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$, we have one of the following:*

1. *$M \simeq F_l = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-l(n+1))$ for a unique element $l \in \mathbb{Z}_{\geqslant 0}$ and $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts as in Example 1.10.*
2. *$M \simeq \mathbb{P}^n \times C$ for a unique smooth curve C and $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts on the first factor as in Example 1.1.*
3. *$M \simeq \mathbb{P}(T\mathbb{P}^2)$ and $\mathrm{PGL}_3(\mathbb{C})$ acts as in Example 1.11.*
4. *$M \simeq \mathbb{G}(1,3)$ and $\mathrm{PGL}_4(\mathbb{C})$ acts as in Example 1.12.*

Moreover, these actions are not birationally conjugate to each other.

Remark 4.2. — If M is rational and of dimension 2 or 3, this result can be deduced directly from the classification of maximal algebraic subgroups of Cr_2 and Cr_3 by Enriques, Umemura and Blanc ([7, 27, 43, 44, 45]).

4.1. Classification of varieties and groups of automorphisms

With some geometric invariant theory and using results of Freudenthal about topological ends, the following classification can be made (see [17, Theorem 4.8] and the references in there):

Theorem 4.3. — *Let M be a smooth projective variety of dimension $n+1$ with a non-trivial regular action of $\mathrm{PGL}_{n+1}(\mathbb{C})$, where $n \geqslant 2$. Then we are in one of the following cases:*

1. *$M \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))$ for some $k \in \mathbb{Z}_{\geqslant 0}$.*
2. *$M \simeq \mathbb{P}^n \times C$ for a curve C of genus $\geqslant 1$.*
3. *$M \simeq \mathbb{P}(T\mathbb{P}^2) \simeq \mathrm{PGL}_3(\mathbb{C})/B$, where $B \subset \mathrm{PGL}_3(\mathbb{C})$ is a Borel subgroup.*
4. *$M \simeq \mathbb{G}(1,3) \simeq \mathrm{PGL}_4(\mathbb{C})/P$, where $P \subset \mathrm{PGL}_4(\mathbb{C})$ is the parabolic subgroup consisting of matrices of the form*

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

The neutral components $\mathrm{Aut}^0(M)$ of the automorphism groups of the varieties M that appear in Theorem 4.3 are well known. Proofs of the following Proposition can be found in [1, Proposition 2.4.1, 2.4.2, Example 2.4.2, and Theorem 3.3.2].

Proposition 4.4. — *We have*

- $\mathrm{Aut}^0(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k)) \simeq (\mathrm{GL}_{n+1}(\mathbb{C})/\mu_k) \ltimes \mathbb{C}[x_0, \ldots, x_n]_k$, *where* $\mathbb{C}[x_0, \ldots, x_n]_k$ *denotes the additive group of homogeneous polynomials of degree* k *and* $\mu_k \subset \mathbb{C}^*$ *the group of all elements* $c \in \mathbb{C}^*$ *satisfying* $c^k = 1$,
- $\mathrm{Aut}^0(\mathbb{P}^n \times C) \simeq \mathrm{PGL}_{n+1}(\mathbb{C}) \times \mathrm{Aut}^0(C)$,
- $\mathrm{Aut}^0(\mathbb{P}(T\mathbb{P}^2)) \simeq \mathrm{PGL}_3(\mathbb{C})$,
- $\mathrm{Aut}^0(\mathbb{G}(1,3)) \simeq \mathrm{PGL}_4(\mathbb{C})$.

To describe the $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions on these varieties we recall some results about group cohomology.

4.2. Group cohomology

Let H be a group that acts by automorphisms on a group N. A *cocycle* is a map $\tau\colon H \to N$ such that $\tau(gh) = \tau(g)(g \cdot \tau(h))$ for all $g, h \in H$. Two cocycles τ and ν are *cohomologous* if there exists an $a \in N$ such that

$$\tau(g) = a^{-1}\nu(g)(g \cdot a) \text{ for all } g \in H\,.$$

The set of cocycles up to cohomology will be denoted by $H^1(H, N)$. If H acts trivially on N, the set $H^1(H, N)$ corresponds to the set of group homomorphisms $H \to N$ up to conjugation. The following lemma is well known.

Lemma 4.5. — *Let* $G := N \rtimes H$ *be a semi direct product of groups and* $\pi\colon G \to H$ *the canonical projection on* H. *Then there exists a bijection between* $H^1(H, N)$ *and the sections of* π *up to conjugation in* N.

There always exists the trivial cocycle $\tau_0\colon H \to N$, $g \mapsto e_N$. The set $H^1(G, N)$ is therefore a pointed set with basepoint τ_0. Assume that G acts on two groups A and B by automorphisms. A G-homomorphism $\phi\colon A \to B$ induces a homomorphism of pointed sets

$$\phi_*\colon H^1(G, A) \to H^1(G, B)$$

given by $\phi_*(\tau) = \phi \circ \tau$.

Proposition 4.6 ([39], p. 125, Proposition 1). — *Let G be a group that acts by automorphisms on groups A, B and C. Every exact sequence of G-homomorphisms*

$$1 \to A \to B \to C \to 1$$

induces an exact sequence of pointed sets

$$H^1(G,A) \to H^1(G,B) \to H^1(G,C).$$

4.3. Proof of Theorem 4.1

We use the classification of smooth projective varieties of dimension $n+1$ with a regular $\mathrm{PGL}_{n+1}(\mathbb{C})$ action of Theorem 4.3. We have to show that case (1) appears only if $k = l(n+1)$ for some integer l. Examples 1.9 to 1.12 show that for all the other varieties there exist $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions. So we have to show that these actions are unique and not birationally conjugate to each other.

We start by showing that the actions are unique. By Proposition 4.4, $\mathrm{Aut}^0(\mathbb{P}(T\mathbb{P}^2)) \simeq \mathrm{PGL}_3(\mathbb{C})$ and $\mathrm{Aut}(\mathbb{G}(1,3)^0) \simeq \mathrm{PGL}_4(\mathbb{C})$. The uniqueness of the embedding is clear in these cases since $\mathrm{PGL}_{n+1}(\mathbb{C})$ is a simple group. If $M \simeq \mathbb{P}^n \times C$ uniqueness follows directly from the fact that $\mathrm{PGL}_{n+1}(\mathbb{C})$ does not embed into $\mathrm{Aut}(C)$.

Now we show that $\mathrm{PGL}_{n+1}(\mathbb{C})$ can be embedded into $\mathrm{Aut}^0(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))$ if and only if $n \mid k$. Then we show that in this case, up to conjugation and algebraic automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$, the embedding is unique.

Lemma 4.7. — *Let $\mu_k = \{\lambda \,\mathrm{id} \mid \lambda \in \mathbb{C}, \lambda^k = 1\}$. There exists a non-trivial algebraic group homomorphism $\mathrm{PGL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})/\mu_k$ if and only if $n \mid k$.*

Proof. — If $n \mid k$, we have

$$\mathrm{PGL}_n(\mathbb{C}) \simeq \mathrm{SL}_n(\mathbb{C})/(\mu_k \cap \mathrm{SL}_n(\mathbb{C})) \subset \mathrm{GL}_n(\mathbb{C})/\mu_k.$$

On the other hand, assume that there exists a non-trivial algebraic homomorphism $\phi\colon \mathrm{PGL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})/\mu_k$. Let ξ_n be a primitive n-th root of unity and define

$$J := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \xi_n & 0 & \dots & 0 \\ 0 & 0 & \xi_n^2 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & \xi_n^{n-1} \end{bmatrix} \in \mathrm{PGL}_n(\mathbb{C}).$$

Let $\pi\colon \mathrm{GL}_n(\mathbb{C})/\mu_k \to \mathrm{PGL}_n(\mathbb{C})$ be the standard projection and $\tau \in \mathrm{PGL}_n(\mathbb{C})$ be the permutation $(1, 2, \dots, n)$. The composition $\pi \circ \phi$ is conjugate to the identity or to the automorphism $\alpha\colon g \mapsto {}^t g^{-1}$ by Proposition 2.8. So after conjugation we can assume that $\pi \circ \phi|_{\mathcal{S}_n} = \mathrm{id}_{\mathcal{S}_n}$ and that $\pi \circ \phi|_{D_n}$ is either the identity or given by $d \mapsto d^{-1}$. Hence, we may assume that $\phi(\tau)$ is the class $[c \cdot \tau]$ of the class $c \cdot \tau$ in the quotient $\mathrm{GL}_n(\mathbb{C})/\mu_k$ for some $c \in \mathbb{C}^*$. The image $\phi(J) = [D]$ can be represented by the matrix

$$D = \begin{bmatrix} d_1 & 0 & 0 & \dots \\ 0 & d_2 & 0 & \dots \\ & & \dots & \\ 0 & 0 & \dots & d_n \end{bmatrix},$$

where $[D]^n = [\mathrm{id}]$ and $d_i \neq d_j$ for $i \neq j$. Observe that $\tau J \tau^{-1} = J$ in $\mathrm{PGL}_n(\mathbb{C})$. So $[\tau D \tau^{-1}] = [D]$ in GL_n/μ_k and therefore there exists an $a \in \mathbb{C}^*$ such that $a^k = 1$ and such that

$$\begin{bmatrix} d_1 & 0 & 0 & \dots \\ 0 & d_2 & 0 & \dots \\ & & \dots & \\ 0 & 0 & \dots & d_n \end{bmatrix} = \begin{bmatrix} ad_n & 0 & 0 & \dots \\ 0 & ad_1 & 0 & \dots \\ & & \dots & \\ 0 & 0 & \dots & ad_{n-1} \end{bmatrix}.$$

In other words, $d_{i+1} = ad_i$ for $1 \leqslant i \leqslant n-1$ and $d_1 = ad_n$. This implies $d_n = ad_{n-1} = a^2 d_{n-2} = \dots = a^{n-1} d_1 = a^n d_n$. We obtain $a^n = 1$ and $a^l \neq 1$ for $1 \leqslant l < n$ since $d_i \neq d_j$ for $i \neq j$, hence $n \mid k$. □

Let n and k be positive integers such that $(n+1) \mid k$. Denote by $\mathbb{C}[x_0, \dots, x_n]_k$ the vector space of homogeneous polynomials of degree k. We define

$$G := \mathbb{C}[x_0, \dots, x_n]_k \rtimes \mathrm{PGL}_{n+1}(\mathbb{C}),$$

where the semi direct product is taken with respect to the action $g \cdot p = p \circ g^{-1}$. Here we look at $\mathrm{PGL}_{n+1}(\mathbb{C}) \subset \mathrm{GL}_{n+1}(\mathbb{C})/\mu_k$ as described in Lemma 4.7. Let $\pi\colon G \to \mathrm{PGL}_{n+1}(\mathbb{C})$ be the standard projection and $\iota\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to G$ the standard section of π.

Lemma 4.8. — *Up to conjugation, $\iota\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to G$ is the unique section of π.*

Proof. — Let $\varphi\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to G$ be an arbitrary section of π. We show that φ is conjugate to ι. It is enough to show that $\varphi(\mathrm{PGL}_{n+1}(\mathbb{C})) = \iota(\mathrm{PGL}_{n+1}(\mathbb{C}))$.

The image $\varphi(\mathrm{PGL}_{n+1}(\mathbb{C}))$ acts on $\mathbb{C}[x_0,\dots,x_n]_k$ by conjugation. Let $H \subset D_{n-1}$ be the subgroup consisting of all elements of order d for some d large enough and not divisible by k. Then the action of $\varphi(H)$ on $\mathbb{C}[x_0,\dots,x_n]_k$ has a fixed point p. After conjugation, we can assume that $p=0$. Observe that $\iota(\mathrm{PGL}_{n+1}(\mathbb{C}))$ is the subgroup of G consisting of all elements that fix 0. Hence we may assume $\varphi(H) \subset \iota(\mathrm{PGL}_{n+1}(\mathbb{C}))$ and therefore $\varphi(H) = \iota(H)$.

The centralizer of H in $\mathrm{PGL}_{n+1}(\mathbb{C})$ is the diagonal subgroup D_n. The centralizer of $\iota(H)$ in G is $\iota(D_n)$, since, by assumption, d is no multiple of k. This implies $\varphi(D_n) = \iota(D_n)$. The normalizer of D_n in $\mathrm{PGL}_{n+1}(\mathbb{C})$ is $D_n \rtimes \mathcal{S}_{n+1}$ and the normalizer of $\iota(D_n)$ in G is $\iota(D_n \rtimes \mathcal{S}_{n+1})$. Hence we obtain $\varphi(D_n \rtimes \mathcal{S}_{n+1}) = \iota(D_n \rtimes \mathcal{S}_{n+1})$ and since both ι and φ are sections,

$$\varphi|_{D_n \rtimes \mathcal{S}_{n+1}} = \iota|_{D_n \rtimes \mathcal{S}_{n+1}}\,.$$

Let $g := (x_0+x_1, x_1, \dots, x_n) \in \mathrm{PGL}_{n+1}(\mathbb{C})$. Let $E \subset D_n$ be the centralizer of g in D_n. So E is the subgroup of elements of the form

$$(c_0x_0, c_0x_1, c_2x_2, \dots, c_nx_n)$$

with $c_i \in \mathbb{C}^*$ such that $c_0^2c_2c_3\dots c_n = 1$. Denote $\varphi(g) = (v,g)$, with $v \in \mathbb{C}[x_0,\dots,x_n]_k$. Take a $d \in E$. Then $\iota(d) = (0,d)$ and $(0,d)(v,g)(0,d^{-1}) = (v\circ d^{-1}, g)$ yields $v \circ d^{-1} = v$. Therefore, all summands of v are of the form $x_0^{r_0}x_1^{r_1}(x_0\dots x_n)^s$, where $r_0+r_1 = s$. Assume that

$$v = \sum_{s, r_0+r_1=s} a_{r_0r_1s}x_0^{r_0}x_1^{r_1}(x_0\dots x_n)^s.$$

We calculate

$$(v,g)^2 = (v + v\circ g^{-1}, g^2),$$

and

$$\begin{aligned} v + v\circ g^{-1} = &\sum_{s, r_0+r_1=s} a_{r_0r_1s}x_0^{r_0}x_1^{r_1}(x_0\dots x_n)^s \\ &+ \sum_{s, r_0+r_1=s} a_{r_0r_1s}(x_0-x_1)^{r_0}x_1^{r_1}((x_0-x_1)x_1\dots x_n)^s\,. \end{aligned}$$

On the other hand, for $f = (x_0, 1/2x_1, 2x_2, x_3, \dots, x_n)$ we obtain $g^2 = f \circ g \circ f^{-1}$. Hence

$$(v, g)^2 = (0, f)(v, g)(0, f^{-1}) = (v + v \circ g^{-1}, g^2),$$

and therefore

$$v + v \circ g^{-1} = \sum_{s, r_0 + r_1 = s} 2^{r_1} a_{r_0 r_1 s} x_0^{r_0} x_1^{r_1} (x_0 \dots x_n)^s.$$

This yields $a_{r_0 r_1 s} = 0$ for all r_1, r_2 and s and thus $\varphi(g) \in \iota(\mathrm{PGL}_{n+1}(\mathbb{C}))$.

Since $\mathrm{PGL}_{n+1}(\mathbb{C})$ is generated by $D_n \rtimes \mathcal{S}_{n+1}$ and the element g, this yields $\varphi(\mathrm{PGL}_{n+1}(\mathbb{C})) = \iota(\mathrm{PGL}_{n+1}(\mathbb{C}))$, which finishes the prove. □

Lemma 4.9. — $\mathrm{PGL}_{n+1}(\mathbb{C})$ *acts non-trivially on the fibration* $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))$ *with basis* $\mathbb{P}^n$ *if and only if* $k = l(n+1)$ *for some nonnegative* l. *Moreover, in this case the action is unique up to conjugation and up to algebraic automorphisms of* $\mathrm{PGL}_{n+1}(\mathbb{C})$.

Proof. — Let $\phi\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Aut}^0(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k)))$ be an algebraic embedding. By Proposition 4.4, there exists an exact sequence of algebraic homomorphisms

$$1 \to \mathbb{C}[x_0, \dots, x_n]_k \to \mathrm{Aut}^0(\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k))) \to \mathrm{GL}_{n+1}(\mathbb{C})/\mu_k \to 1.$$

If ϕ is non-trivial, this induces a non-trivial algebraic homomorphism from $\mathrm{PGL}_{n+1}(\mathbb{C})$ into $\mathrm{GL}_{n+1}(\mathbb{C})/\mu_k$ and by Lemma 4.7 this is possible if and only if $(n+1) \mid k$. So assume that $k = l(n+1)$ for an integer l. It remains to show that in this case ϕ is unique up to conjugation and up to algebraic automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$. Let

$$F_l := \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k)).$$

We look at F_l as a $\mathbb{P}^1$-fibration over the basis $\mathbb{P}^n$. So there is an exact sequence

$$1 \to \mathrm{Aut}^0_{\mathbb{P}^n}(F_l) \to \mathrm{Aut}^0(F_l) \xrightarrow{\pi} \mathrm{PGL}_n(\mathbb{C}) \to 1.$$

Here, $\mathrm{Aut}^0_{\mathbb{P}^n}(F_l) \simeq \mathbb{C}^* \ltimes \mathbb{C}[x_0, \dots, x_n]_k$ denotes the subgroup of automorphisms of F_l that fix the basis $\mathbb{P}^n$ pointwise.

Let $H := \mathrm{PGL}_{n+1}(\mathbb{C})$. By Lemma 4.5, the sections of π up to conjugation are in bijection with

$$H^1(H, \mathrm{Aut}^0_{\mathbb{P}^n}(F_l)) = H^1(H, \mathbb{C}[x_0, \dots, x_n]_k \rtimes \mathbb{C}^*/\mu_k).$$

By Proposition 4.6, there is an exact sequence of pointed sets

$$H^1(H, \mathbb{C}[x_0, \dots, x_n]_k) \to H^1(H, \mathrm{Aut}_{\mathbb{P}^n}(F_l)) \to H^1(H, \mathbb{C}^*/\mu_k).$$

The action of H on $\mathbb{C}^*/\mu_k$ is trivial, so $H^1(H, \mathbb{C}^*/\mu_k)$ is the set of homomorphisms $H \to \mathbb{C}^*/\mu_k$. Hence $H^1(H, \mathbb{C}^*/\mu_m) = \{1\}$. By Lemma 4.8, we obtain $H^1(H, \mathbb{C}[x_0,\dots,x_n]_k) = \{1\}$ and thus $H^1(H, \mathrm{Aut}_{\mathbb{P}^n}(F_l)) = \{1\}$. So all sections of π are conjugate.

Now, since H is simple and not contained in $\mathrm{Aut}^0_{\mathbb{P}^n}(F_l)$, we obtain $\pi \circ \phi(H) \subset H$. Both ϕ and π are algebraic morphisms, so $\pi \circ \phi(H) = H$. Therefore, up to the algebraic automorphism $\pi \circ \phi$, the homomorphism ϕ is a section of π. $\square$

It remains to show that the actions from Theorem 4.1 are not birationally conjugate. Let M be a variety of dimension $n+1$ on which $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts faithfully. If M is not rational, then M is isomorphic to $\mathbb{P}^n \times C$ for some smooth curve C. Recall that $\mathbb{P}^n \times C$ is birationally equivalent to $\mathbb{P}^n \times C'$ for smooth curves C and C' if and only if C and C' are birationally equivalent which again implies that C and C' are isomorphic. So if $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts rationally and non trivially on a non rational variety M of dimension $n+1$, then this one is uniquely determined up to algebraic automorphisms of $\mathrm{PGL}_{n+1}(\mathbb{C})$ and up to birational conjugation in $\mathrm{Bir}(M)$.

In the case that M is rational, we have to show that the $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions listed in Theorem 4.1 are not conjugate to each other. For this, note that none of them has an orbit of codimension $\geqslant 1$. Lemma 3.2 implies therefore that any birational transformation conjugating one action to another one must be an isomorphism. As the varieties listed in Theorem 4.1 are not isomorphic we conclude that the actions are not conjugate.

5. Extension to Cr_n and H_n

In this section we study how the $\mathrm{PGL}_{n+1}(\mathbb{C})$-actions described in the above section extend to rational Cr_n-actions. Our goal is to prove Theorem 1.13. We proceed case by case.

5.1. The case $\mathbb{G}(1,3)$

Let

$$s_1 := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{ and } s_2 := \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Z}).$$

Lemma 5.1. — *Let G be a group. There exists no group homomorphism $\rho\colon \mathrm{GL}_3(\mathbb{Z}) \to G$ such that $\rho(s_1)$ has order 3 and $s_2 \in \ker(\rho)$.*

Proof. — Assume that such a ρ exists. We define the following elements in $\mathrm{GL}_3(\mathbb{Z})$:

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},\ B := \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\ T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

One calculates $(A(s_2(Bs_2B^{-1}))A^{-1}) = s_1T$. So s_1T is contained in the kernel of ρ and we get $\rho(T) = \rho(s_1^{-1})$. But this is a contradiction since the order of T is 2. $\square$

The following construction comes up in the context of tetrahedral line complexes (see [26]). Consider the 4 hyperplanes in $\mathbb{P}^3$

$$E_0 := \{x_0 = 0\},\ E_1 := \{x_1 = 0\},\ E_2 := \{x_2 = 0\},\ E_3 := \{x_3 = 0\}.$$

A line $l \in \mathbb{G}(1,3)$ that is not contained in any of the E_i, intersects each plane E_i in one point p_i. We thus obtain a rational surjective map

$$cr\colon \mathbb{G}(1,3) \dashrightarrow \mathbb{P}^1$$

that is defined by associating to the line l the cross ratio between the points p_i.

The closure $\overline{cr^{-1}([a:b])}$ in $\mathbb{G}(1,3)$ is irreducible if and only if $[a:b] \in \mathbb{P}^1 \setminus \{[0:1],[1:0],[1:1]\}$, whereas $\overline{cr^{-1}([a:b])}$ consists of two irreducible components in all the other cases ([26, Chapter 10.3.6]).

Recall that α is the automorphism of $\mathrm{PGL}_4(\mathbb{C})$ given by $g \mapsto {}^tg^{-1}$.

Proposition 5.2. — *There exists no non-trivial group homomorphism*

$$\Phi\colon \langle \mathrm{PGL}_4(\mathbb{C}), \mathrm{W}_3\rangle \to \mathrm{Bir}(\mathbb{G}(1,3))$$

such that $\Phi(\mathrm{PGL}_4(\mathbb{C})) \subset \mathrm{Aut}(\mathbb{G}(1,3))$.

In particular, neither the action of $\mathrm{PGL}_4(\mathbb{C})$ *on* $\mathbb{G}(1,3)$ *given by the embedding* φ_G *(see Example 1.12) nor the action given by* $\varphi_G \circ \alpha$ *can be extended to a rational action of* Cr_4*.*

Proof. — The proof of Corollary A.2 implies that if $\mathrm{PGL}_4(\mathbb{C})$ is contained in the kernel of a homomorphism $\Phi\colon \langle \mathrm{PGL}_4(\mathbb{C}), \mathrm{W}_3\rangle \to \mathrm{Bir}(\mathbb{G}(1,3))$, then Φ is trivial. So we may assume that Φ embeds $\mathrm{PGL}_4(\mathbb{C})$ into the group $\mathrm{Aut}^0(\mathbb{G}(1,3))$. By Theorem 4.1 it is therefore enough to show that φ_G and $\varphi_G \circ \alpha$ do not extend to a homomorphism of $\langle \mathrm{PGL}_4(\mathbb{C}), \mathrm{W}_3\rangle$.

The $\varphi_G(D_3)$-orbit of a line that is not contained in one of the planes E_i and that does not pass through any of the coordinate points $[1:0:0:0],[0:1:0:0],[0:0:1:0],[0:0:0:1]$, has dimension 3 and these are all $\varphi_G(D_3)$-orbits of dimension 3.

Since $\varphi_G(D_3)$ stabilizes the hyperplanes E_i and since the cross ratio is invariant under linear transformations, we obtain that cr is $\varphi_G(D_3)$-invariant. By the above remark, the rational map cr therefore parametrizes all but finitely many $\varphi_G(D_3)$-orbits of dimension 3 by $\mathbb{P}^1 \setminus \{[0:1],[1:0],[1:1]\}$.

The image $\varphi_G(\mathcal{S}_4)$, where $\mathcal{S}_4 \subset \mathrm{PGL}_4(\mathbb{C})$ is the subgroup of coordinate permutations, normalizes $\varphi_G(D_3)$ and therefore it permutes its 3-dimensional orbits. Since $\mathcal{S}_4$ permutes the hyperplanes E_i, we can describe its action on the 3-dimensional $\varphi_G(D_3)$-orbits by its action on the cross ratio of the intersection of general lines with the planes E_i.

Let r be the cross ratio between the points p_0, p_1, p_2, p_3 on a line. One calculates that the cross ratio between p_3, p_1, p_2, p_0 is again r and that the cross ratio between the points p_2, p_0, p_1, p_3 is $\frac{1}{1-r}$. Hence the image of $\tau_1 := [x_3 : x_1 : x_2 : x_0]$ leaves cr invariant, whereas for the permutation $\tau_2 := [x_2 : x_0 : x_1 : x_3]$ we have $cr \circ \varphi(\tau_2) \neq cr$ and $cr \circ \varphi(\tau_2)^2 \neq cr$.

Let $f\colon \mathbb{G}(1,3) \dashrightarrow \mathbb{P}^4$ be a birational transformation and let $\varphi'_G := f \circ \varphi_G \circ f^{-1}$. The image $\varphi'_G(D_3) \subset \mathrm{Cr}_4$ is an algebraic torus of rank 3 and therefore, by Proposition 2.6, conjugate to the standard subtorus $D_3 \subset D_4$ of rank 3. In other words, there exists a rational map $\mathbb{P}^4 \dashrightarrow \mathbb{P}^1$ whose fibers consist of the closure of the $\varphi'_G(D_3)$-orbits. The image $\varphi'_G(\mathcal{S}_4)$ permutes the torus orbits, hence we obtain a homomorphism $\rho\colon \mathcal{S}_4 \to \mathrm{PGL}_2(\mathbb{C})$. By what we observed above, the permutation τ_1 is contained in the kernel of ρ, whereas the image $\rho(\tau_2)$ has order 3. The matrix representation in $\mathrm{GL}_3(\mathbb{Z})$ of τ_1 corresponds to s_1 and the matrix representation of τ_2 corresponds to s_2.

It follows now from Lemma 5.1 that ρ can not be extended to a homomorphism from $\mathrm{GL}_3(\mathbb{Z}) \simeq \mathrm{W}_3$ to $\mathrm{PGL}_2(\mathbb{C})$, which implies that there exists no homomorphism $\Phi\colon \langle \mathrm{PGL}_4(\mathbb{C}), \mathrm{W}_3\rangle \to \mathrm{Cr}_4$ such that $\Phi(\mathrm{PGL}_4(\mathbb{C})) = \varphi'_G(\mathrm{PGL}_4(\mathbb{C}))$, since W_3 normalizes the torus and its image would therefore permute the torus orbits as well. The statement follows. □

5.2. The case $\mathbb{P}(T\mathbb{P}^2)$

Recall that matrices of order two in $\mathrm{PGL}_2(\mathbb{C})$ have the form

$$\begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 & b \\ c & -1 \end{bmatrix}, \text{ where } a \in \mathbb{C}^*,\ b,c \in \mathbb{C},\ bc \neq -1. \tag{5.1}$$

Proposition 5.3. — *The embedding $\varphi_B\colon \mathrm{PGL}_3(\mathbb{C}) \to \mathrm{Bir}(\mathbb{P}(T\mathbb{P}^2))$ extends uniquely to an embedding*

$$\Phi_B\colon \mathrm{Cr}_2 \to \mathrm{Bir}(\mathbb{P}(T\mathbb{P}^2)).$$

Proof. — We show that every extension coincides with the one given in Example 1.11. For this it is enough to show that the image of σ is uniquely determined. Assume that there is an extension $\psi\colon \mathrm{Cr}_2 \to \mathrm{Bir}(\mathbb{P}(T\mathbb{P}^2))$ of φ_B. Our goal is to show $\psi(\sigma) = \Psi_B(\sigma)$.

Let $d \in D_2$, $d = (ax_1, bx_2)$ with respect to affine coordinates given by $x_0 = 1$. Then $\varphi_B(d) = (ax_1, bx_2, (b/a)x_3)$, with respect to suitable local affine coordinates of $\mathbb{P}(T\mathbb{P}^2)$. Let $\phi\colon \mathbb{P}(T\mathbb{P}^2) \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^1$ be the birational map given by

$$\phi\colon (x_1, x_2, x_3) \mapsto \left(x_1, x_2, \frac{x_1}{x_2}x_3\right),$$

with respect to local affine coordinates.

Let $\psi_1\colon \mathrm{Cr}_2 \to \mathrm{Bir}(\mathbb{P}^2 \times \mathbb{P}^1)$ be the algebraic embedding $\psi_1 = \phi \circ \psi \circ \phi^{-1}$. This gives us a $\mathbb{P}^2$-fibration, which we call the horizontal fibration, and a $\mathbb{P}^1$-fibration, which we call the vertical fibration. The image $\psi_1(D_2)$ acts canonically on the first factor and leaves the second one invariant. The horizontal fibers thus consist of the closures of D_2-orbits. Since W_2 normalizes D_2, the image $\psi_1(\mathrm{W}_2)$ permutes the orbits of $\psi_1(D_2)$. Hence it preserves the horizontal fibration and we obtain a homomorphism

$$\rho\colon W_2 \simeq \mathrm{GL}_2(\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C}).$$

In what follows we identify W_2 with $\mathrm{GL}_2(\mathbb{Z})$.

The images of the three transpositions in $\mathcal{S}_3 = \mathrm{W}_2 \cap \mathrm{PGL}_3(\mathbb{C})$ under ρ are:

$$\rho\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \rho\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$
$$\text{and } \rho\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}.$$

The image $\rho(\sigma)$ is either the identity or it has order 2. The elements of the form (5.1) do not commute with the images of $\mathcal{S}_3$ described above. Since σ is contained in the center of W_2, we obtain $\rho(\sigma) = \mathrm{id}$.

It remains to show that the action of $\psi_1(\sigma)$ on the first factor of $\mathbb{P}^2 \times \mathbb{P}^1$ is the standard action. Let $M = \mathbb{P}^2$ be a horizontal fiber. It is stabilized by $\psi(D_2)$ and $\psi(\sigma)$, so we obtain a homomorphism

$$\gamma\colon \langle D_2, \sigma\rangle \to \mathrm{Bir}(M) = \mathrm{Cr}_2\,.$$

Since $\sigma d\sigma^{-1} = d^{-1}$ for all $d \in D_2$, there exists a $d \in D_2$ such that $\gamma(\sigma) = d\sigma$. This is true for all horizontal fibers, so $\psi_1(\sigma)$ induces an automorphism of $U \times \mathbb{P}^1$, where

$$U = \{[x_0 : x_1 : x_2] \mid x_0, x_1, x_2 \neq 0\} \subset \mathbb{P}^2.$$

Let $S \simeq \mathbb{P}^1 \subset U \times \mathbb{P}^1$ be a vertical fiber and $\pi\colon U \times \mathbb{P}^1 \to U$ the projection onto the first factor. Then $\pi \circ \psi_1(\sigma)(S)$ is a regular map from $\mathbb{P}^1$ to the affine set U and is therefore constant. We obtain that $\psi_1(\sigma)$ preserves the vertical fibration.

The image $\psi_1(\mathrm{PGL}_3(\mathbb{C}))$ preserves the vertical fibration as well and projection onto $\mathbb{P}^2$ yields a homomorphism from $\mathrm{PGL}_3(\mathbb{C})$ to Cr_2 that is the standard embedding. Hence $\psi_1(\mathrm{Cr}_2)$ preserves the vertical fibration and we obtain an algebraic homomorphism from Cr_2 to Cr_2, which is uniquely determined by its restriction to $\mathrm{PGL}_3(\mathbb{C})$. So the image $\psi_1(\sigma)$ is uniquely determined by its restriction to $\mathrm{PGL}_3(\mathbb{C})$ (see Appendix). □

Proposition 5.4. — *There exists no homomorphism* $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_3$ *such that*

$$\Phi|_{\mathrm{PGL}_3(\mathbb{C})} = \varphi_B \circ \alpha,$$

where φ_B *denotes the embedding of* PGL_3 *into* Cr_3 *from Example 1.11 and* α *the algebraic automorphism of* PGL_3 *given by* $g \mapsto {}^t g^{-1}$.

Proof. — Assume that such an extension $\Phi\colon \mathrm{Cr}_2 \to \mathrm{Cr}_3$ of $\varphi_B \circ \alpha$ exists.

Observe that $\alpha(D_2) = D_2$ and that $\alpha|_{\mathcal{S}_3} = \mathrm{id}_{\mathcal{S}_3}$. Therefore, we can repeat the same argument as in the proof of Proposition 5.3 to obtain $\Psi(\sigma) = \Phi_B(\sigma)$. But we have

$$\Psi(\sigma)\Psi(g)\Psi(\sigma)\Psi(g)\Psi(\sigma)\Psi(g) \neq \mathrm{id}$$

for $g = [z-x : z-y : z]$. This contradicts the relations in Cr_2 (Proposition A.4). □

5.3. The case $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k(n+1)))$

Proposition 5.5. — *The algebraic homomorphism* $\varphi_l\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Bir}(F_l)$ *extends uniquely to the embedding*

$$\Psi_l\colon \mathrm{H}_n \to \mathrm{Bir}(F_l) \quad \textit{(see Example 1.10)}.$$

Proof. — Suppose that there is an extension $\psi\colon \mathrm{H}_n \to \mathrm{Bir}(F_l)$ of φ_l. We will show that ψ is unique and therefore that $\psi = \Psi_l$.

Let $(x_1, \dots, x_{n-1}, w)$ be local affine coordinates of F_l such that for every $g \in \mathrm{PGL}_{n+1}(\mathbb{C})$ the image $\varphi_l(g)$ acts by

$$(x_1, \dots, x_n, w) \mapsto (g(x_1, \dots, x_n), J(g(x_1, \dots, x_n))^{-l} w).$$

In particular, the image under ψ of $(d_1x_1, \dots, d_nx_n) \in D_n$ acts by

$$(x_1, \dots, x_n, w) \mapsto (d_1x_1, \dots, d_nx_n, (d_1 \dots d_n)^{-l} w).$$

Define $\phi\colon F_l \to \mathbb{P}^n \times \mathbb{P}^1$ by

$$\phi\colon (x_1, \dots, x_n, w) \mapsto (x_1, \dots, x_n, (x_1 \dots x_n)^l w)$$

with respect to local affine coordinates. Let $\psi_1\colon \mathrm{Cr}_n \to \mathrm{Bir}(\mathbb{P}^n \times \mathbb{P}^1)$ be the algebraic embedding $\psi_1 := \phi \circ \psi \circ \phi^{-1}$. Then the image $\psi_1(D_n)$ acts canonically on the first factor and leaves the second one invariant. Since W_n normalises D_n, the image $\psi_1(\mathrm{W}_n)$ permutes the orbits of $\psi_1(D_n)$. Hence $\psi_1(\mathrm{W}_n)$ preserves the horizontal fibration. This induces a homomorphism

$$\rho\colon \mathrm{W}_n \simeq \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{PGL}_2(\mathbb{C}).$$

In what follows, we identify W_n with $\mathrm{GL}_n(\mathbb{Z})$. Define $\mathcal{A}_{n+1} \subset \mathcal{S}_{n+1} \subset \mathrm{PGL}_{n+1}(\mathbb{C})$ to be the subgroup of coordinate permutations $s \in \mathcal{S}_{n+1}$ such that $J(s) = 1$. Hence $\mathcal{A}_{n+1} \in \ker(\rho)$. Note that the fixed point set of $\psi_1(\mathcal{A}_{n+1})$ is the vertical fiber

$$L := [1 : \cdots : 1] \times \mathbb{P}^1 \subset \mathbb{P}^n \times \mathbb{P}^1.$$

Since σ_n commutes with $\mathcal{A}_{n+1}$, the image $\psi_1(\sigma_n)$ stabilises L. The group $\psi_1(D_n)$ acts transitively on an open dense subset of vertical fibers that contains L. Since $\psi_1(\sigma_n)$ normalizes $\psi_1(D_n)$, we obtain that $\psi_1(\sigma_n)$ preserves the vertical fibration. Therefore $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n\rangle$ preserves the vertical fibration. We obtain a homomorphism $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n\rangle \to \mathrm{Cr}_n$, which is, by Corollary A.3 and its proof, the standard embedding.

Let

$$f_A = \left(\frac{1}{x_1}, x_2, \dots, x_n\right).$$

In [11] it is shown that f_A is contained in $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n\rangle$, which implies that $\psi_1(f_A)$ preserves the vertical fibration and that its action on $\mathbb{P}^n$ is the standard action.

Recall that $(hf_A)^3 = \mathrm{id}$ for $h = (1 - x_1, x_2, \dots, x_{n-1}) \in \mathrm{Cr}_n$. The image $\psi_1(h)$ is

$$\psi_1(h)\colon (x_1, \dots, x_n, z) \mapsto (1 - x_1, x_2, \dots, x_n, (-1)^l z).$$

Denote by $A \in \mathrm{GL}_n(\mathbb{Z})$ the integer matrix corresponding to f_A. We have $\rho(A) = \mathrm{id}$ or $\rho(A)$ is of order two, i.e. it has the form (5.1).

Suppose that $\rho(A) = \mathrm{id}$. Then

$$\psi_1(f_A)\colon (x_1, \dots x_n, z) \mapsto \left(\frac{1}{x_1}, x_2 \dots x_n, z\right).$$

The relation $(hf_A)^3 = \mathrm{id}$ then implies that l is even.

Suppose that

$$\rho(f_A) = \begin{bmatrix} 1 & b \\ c & -1 \end{bmatrix}, \text{ where } b, c \in \mathbb{C},\ bc \neq -1,$$

hence

$$\psi_1(f_A)\colon (x_1, \dots x_n, z) \mapsto \left(\frac{1}{x_1}, x_2 \dots x_n, \frac{z+b}{cz-1}\right)$$

and therefore

$$\psi_1(hf_A) = (x_1, \dots x_n, z) \mapsto \left(1 - \frac{1}{x_1}, \dots x_n, \frac{(-1)^l z + (-1)^l b}{cz-1}\right),$$

One calculates that if l is even, then the relation $(hf_A)^3 = \mathrm{id}$ is not satisfied. So assume that l is odd. This gives

$$\psi_1(hf_A)^3 = (x_1, \dots x_n, z) \mapsto \left(x_1, \dots x_n, \frac{a_1 z + a_2}{a_3 z - a_4}\right),$$

where $a_1 = 3bc - 1, a_2 = (bc-1)b - 2b, a_3 = (1-bc)c + 2c$ and $a_4 = 3bc - 1$. So $(hf_A)^3 = \mathrm{id}$ yields either l odd and $b = c = 0$ or l odd and $bc = 3$. However, the latter is not possible. Consider the transformation

$$\tau = (x_1, \dots, x_{n-2}, x_n, x_{n-1}) \in \mathcal{S}_n\,.$$

We have $f_A\tau = \tau f_A$. Note that

$$\psi_1(\tau)\colon (x_1, \dots, x_n, z) \mapsto (x_1, \dots, x_{n-2}, x_n, x_{n-1}, \dots, x_n, (-1)^l z)$$

and this transformation does not commute with $\left(x_1, \dots x_n, \frac{a_1 z + a_2}{a_3 z - a_4}\right)$ in the second case. Hence $c = b = 0$ and l is odd.

Finally, assume that

$$\rho(f_A) = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \text{ where } a \in \mathbb{C}^*.$$

This implies

$$\psi_1(f_A)\colon (x_1, \dots x_n, z) \mapsto \left(\frac{1}{x_1}, x_2 \dots x_n, \frac{1}{az}\right)$$

and hence $\psi(hf_A)^3 \neq \mathrm{id}$.

We conclude that

$$\rho(f_A) = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^l \end{bmatrix}$$

and therefore that the action of $\psi(f_A)$ is uniquely determined by l. Hence

$$\psi|_{\langle \mathrm{PGL}_n(\mathbb{C}), \sigma_{n-1}\rangle} = \Psi_l|_{\langle \mathrm{PGL}_n(\mathbb{C}), \sigma_{n-1}\rangle}\,.$$

Let f_B, f_C, f_D and $f_E \in \mathrm{Cr}_n$ be as in the proof of Corollary A.2. By Lemma A.5 it remains to show that the image $\psi(f_B)$ is uniquely determined. We use once more the relation

$$f_B = f_D f_C f_E f_D^{-1}.$$

Since $\rho(CE) = \mathrm{id}$ and since f_D has order two, we obtain $\rho(B) = \mathrm{id}$.

Let $c \in \mathbb{P}^1$ such that the restriction of $\psi_1(f_B)$ to the hyperplane

$$\{c\} \times \mathbb{P}^n \subset \mathbb{P}^1 \times \mathbb{P}^n$$

is a birational map. Then the restriction of $\psi_1(f_B)$ to $\{c\} \times \mathbb{P}^n$ has to fulfill the relations with the group $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. By Corollary A.3 we obtain that this restriction has to be f_B. Hence the image $\psi_1(f_B)$ is unique. ☐

Proposition 5.6. — *There exists no group homomorphism* $\psi\colon \mathrm{H}_n \to \mathrm{Bir}(F_l)$ *such that* $\psi|_{\mathrm{PGL}_{n+1}(\mathbb{C})} = \varphi_l \circ \alpha$.

Proof. — Assume that such an extension $\psi\colon \mathrm{H}_n \to \mathrm{Cr}_n$ exists. Let ϕ be as in Proposition 5.5 and $\psi_2\colon \mathrm{H}_n \to \mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^n)$,

$$\psi_2 := \phi \circ \varphi_l \circ \alpha \circ \phi^{-1}.$$

Similarly as in the proof of Proposition 5.5 one can show that $\psi_2(\sigma_n)$ preserves the vertical fibration. In that way we obtain an algebraic homomorphism

$$A\colon \langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma \rangle \to \mathrm{Cr}_n$$

such that $A|_{\mathrm{PGL}_{n+1}(\mathbb{C})} = \alpha$. Such a homomorphism does not exist by Corollary A.3. ☐

5.4. The case $C \times \mathbb{P}^n$

Throughout this section, C denotes a projective curve. For the proof of Theorem 4.1 it is enough to consider non rational curves, however, the following propositions hold in the more general case.

Proposition 5.7. — *The embedding* $\varphi_C\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Bir}(C \times \mathbb{P}^n)$ *extends uniquely to the standard embedding*

$$\Phi_C\colon \mathrm{H}_n \to \mathrm{Bir}(C \times \mathbb{P}^n) \quad \text{(see Example 1.9)}.$$

Proof. — Let $\pi\colon C \times \mathbb{P}^n \to C$ be the first projection. Suppose that there is an extension $\Psi\colon \mathrm{H}_n \to \mathrm{Bir}(C \times \mathbb{P}^n)$ of φ_C. By definition, $\Psi(\mathrm{PGL}_{n+1}(\mathbb{C}))$ fixes the fibers of π. Moreover, each fiber of π is a closure of a $\Psi(D_n)$-orbit. Since the elements of W_n commute with D_n, we conclude that $\Psi(\mathrm{W}_n)$

preserves the fibration given by π. Hence H_n preserves the fibration given by π and we obtain a homomorphism

$$\rho\colon \mathrm{H}_n \to \mathrm{Bir}(C)$$

such that $\mathrm{PGL}_{n+1}(\mathbb{C}) \subset \ker(\rho)$. In the Appendix it is shown that the normal subgroup generated by $\mathrm{PGL}_{n+1}(\mathbb{C})$ in H_n is all of H_n. Hence ρ is trivial and $\Psi(\mathrm{H}_n)$ preserves every fiber of π. The restriction $\Psi(\mathrm{H}_n)|_{c\times\mathbb{P}^n}$ for any $c \in C$ defines a homomorphism from H_n to Cr_n such that the restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is the standard embedding. By Corollary A.3, this is the standard embedding. Hence Ψ is unique. ☐

Proposition 5.8. — *There exists no group homomorphism $\Psi\colon \mathrm{H}_n \to \mathrm{Bir}(\mathbb{C} \times \mathbb{P}^n)$ such that $\Psi|_{\mathrm{PGL}_{n+1}(\mathbb{C})} = \varphi_C \circ \alpha$.*

Proof. — Assume there exists such a Ψ. As in the proof of Proposition 5.7 one can show that $\Psi(\mathrm{H}_n)$ fixes the horizontal fibration. The restriction $\Psi(\mathrm{H}_n)|_{c\times\mathbb{P}^{n-1}}$ defines for each $c \in C$ a homomorphism from H_n to Cr_n such that the restriction to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is given by $g \mapsto \alpha(g)$. By Corollary A.3, there exists no such homomorphism. ☐

5.5. Proof of Theorem 1.13

Statement (1) is the content of Theorem 4.1. Statement (2) follows from Proposition 5.2. Statement (3) and (4) follow from the Propositions 5.3, 5.4, 5.5, 5.6, 5.7 and 5.8. Note that the Propositions 5.2, 5.4, 5.6 and 5.8 show that the regular actions of $\mathrm{PGL}_{n+1}(\mathbb{C})$ that do not extend to a rational action of Cr_n also do not extend to a rational action of H_n.

Appendix

Let $\gamma\colon \mathbb{C} \to \mathbb{C}$ be a field homomorphism. By acting on the coordinates, γ induces a bijective map $\Gamma\colon \mathbb{P}^n \to \mathbb{P}^n$. Conjugation with Γ yields a group homomorphism of Cr_n that preserves degrees. Observe that we obtain the image of $g \in \mathrm{Cr}_n$ by letting γ operate on the coefficients of g. By abuse of notation we denote this group homomorphism by γ as well. In [22] Déserti showed that all automorphisms of Cr_2 are inner up to field automorphisms of that type. A generalization of this result is the following theorem by Cantat:

Theorem A.1 ([16]). — *Let M be a smooth projective variety of dimension n and $r \in \mathbb{Z}^+$. Let $\rho\colon \mathrm{PGL}_{r+1}(\mathbb{C}) \to \mathrm{Bir}(M)$ be a non-trivial group homomorphism. Then $n \geqslant r$ and if $n = r$ then M is rational and there exists a homomorphism of fields $\gamma\colon \mathbb{C} \to \mathbb{C}$ such that ρ is up to conjugation the standard embedding of $\mathrm{PGL}_{n+1}(\mathbb{C})$ into Cr_n followed by the group homomorphism $\gamma\colon \mathrm{Cr}_n \to \mathrm{Cr}_n$.*

The goal of this appendix is to prove the following two corollaries of Theorem A.1:

Corollary A.2. — *Let $n > m$ and let $\Phi\colon \mathrm{Cr}_n \to \mathrm{Cr}_m$ be a group homomorphism. Then the normal subgroup of Cr_n containing H_n is contained in the kernel of Φ.*

No such non-trivial homomorphism is known so far. In fact, it is an open question, whether Cr_n is simple for $n \geqslant 3$.

Let $\alpha\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{PGL}_{n+1}(\mathbb{C})$ be the algebraic automorphism defined by $g \mapsto {}^t g^{-1}$.

Corollary A.3. — *Let $\Psi\colon \mathrm{H}_n \to \mathrm{Cr}_n$ be a non-trivial group homomorphism. Then there exists a homomorphism of fields γ of $\mathbb{C}$ and an element $g \in \mathrm{Cr}_n$ such that $g\Psi g^{-1}$ is the standard embedding followed by the group homomorphism γ.*

Moreover, the extension of the standard embedding $\varphi\colon \mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Cr}_n$ as well as the extension of the embedding $\gamma \circ \varphi$, to the group H_n is unique, where γ is any field homomorphism of $\mathbb{C}$. The embedding $\varphi \circ \alpha$ does not extend to a homomorphism from H_n to Cr_n.

By the theorem of Noether and Castelnuovo, Corollary A.3 implies in particular the theorem of Déserti about automorphisms of Cr_2.

We often use the following relations between elements of the Cremona group:

Lemma A.4. — *In Cr_2 the following relations hold:*

(1) $\sigma\tau(\tau\sigma)^{-1} = \mathrm{id}$ *for all* $\tau \in \mathcal{S}_3$,
(2) $\sigma d = d^{-1}\sigma$ *for all diagonal maps* $d \in D_2$ *and*
(3) $(\sigma h)^3 = \mathrm{id}$ *for* $h = [x_2 - x_0 : x_2 - x_1 : x_2]$.

Proof. — One checks the identities by a direct calculation. □

Denote by $\mathrm{Cr}_n^0 \subset \mathrm{Cr}_n$ the subgroup consisting of elements that contract only rational hypersurfaces. We have $\mathrm{H}_n \subset \mathrm{Cr}_n^0$. On the other hand, it seems to be an interesting open question, whether there exist elements in Cr_n^0 that are not contained in H_n for any $n \geqslant 3$ (cf. [35]).

Lemma A.5. — *The group* H_n *is generated by the group* $\mathrm{PGL}_{n+1}(\mathbb{C})$ *and the two birational transformations* $\sigma_n := (x_1^{-1}, x_2^{-1}, \dots, x_n^{-1})$ *and* $f_B := (x_1x_2, x_2, x_3, \dots, x_n)$.

Proof. — It is known that $\mathrm{GL}_n(\mathbb{Z})$ is generated by the subgroup of permutation matrices in $\mathrm{GL}_n(\mathbb{Z})$ and the two elements

$$A := \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & \dots & & \\ & & & & \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \text{ and } B := \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & \dots & & \\ & & & & \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

(see for example [31, III.A.2]). Notice that f_B is the birational transformation in W_n corresponding to B. Let f_A be the birational transformation corresponding to A. In [11] it is shown that f_A is contained in $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. □

Proof of Corollary A.2. — By Lemma A.5 it is enough to show that σ_n and f_B are contained in the normal subgroup containing $\mathrm{PGL}_{n+1}(\mathbb{C})$. Let

$$g_n := [x_n - x_0 : x_n - x_1 : \cdots : x_n - x_{n-1} : x_n] \in \mathrm{PGL}_{n+1}(\mathbb{C}).$$

Then $\sigma_n g_n \sigma_n g_n \sigma_n g_n = \mathrm{id}$. In particular, $\sigma_n g_n$ conjugates σ_n to g_n.

Let

$$C := \begin{pmatrix} -1 & 2 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & \dots & & \\ & & & & \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \; D := \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ & & \dots & & \\ & & & & \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

$$E := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & & 0 & \dots & 0 \\ & & \dots & & \\ & & & & \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z})$$

and let f_C, f_D and f_E be the corresponding elements in W_n. It is shown in [11] that f_C is contained in $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. Moreover, one calculates that

$$f_B = f_D f_C f_E f_D^{-1},$$

which implies that f_B is conjugate to an element in $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n \rangle$. □

Proof of Corollary A.3. — By Theorem A.1 we may assume that there exists a field homomorphism $\gamma\colon \mathbb{C} \to \mathbb{C}$, such that, up to conjugation, the restriction of Ψ to $\mathrm{PGL}_{n+1}(\mathbb{C})$ is the standard embedding composed with γ or the standard embedding composed with γ and the automorphism α of $\mathrm{PGL}_{n+1}(\mathbb{C})$ given by $\alpha(g) = {}^t g^{-1}$.

Therefore, after conjugation, the restriction of Ψ to D_2 is the standard embedding composed with γ. In particular, $\Psi(D_2) = \gamma(D_2)$ is dense in D_2 and therefore $\Psi(\mathrm{W}_n)$ is contained in $D_n \rtimes \mathrm{W}_n$. Assume that $\Psi(\sigma_n) = d\tau$ for some $d \in D_n$ and $\tau \in \mathrm{W}_n$. The relation $\Psi(\sigma_n)\Psi(e)\Psi(\sigma_n) = \Psi(e)^{-1}$ for all $e \in D_n$ implies $\Psi(\sigma_n)e\Psi(\sigma_n) = e^{-1}$ for all $e \in D_2$ and hence $\tau = \sigma_n$. Note that the restriction of Ψ to $\mathcal{S}_{n+1}$ is the standard embedding. So for all $\tau \in \mathcal{S}_{n+1}$ we obtain

$$\tau d\sigma_n = d\sigma_n\tau = d\tau\sigma_n\,.$$

The only element in D_n that commutes with $\mathcal{S}_{n+1}$ is the identity. Hence $\Psi(\sigma_n) = \sigma_n$.

Let g_n be as in the proof of Corollary A.2. The relation $\sigma_n g_n \sigma_n g_n \sigma_n g_n =$ id implies that $\Psi|_{\mathrm{PGL}_{n+1}(\mathbb{C})}$ is the standard embedding composed with γ, since

$$\sigma_n\alpha(g_n)\sigma_n\alpha(g_n)\sigma_n\alpha(g_n) \neq \mathrm{id}\,.$$

It remains to show that $\Psi(f_B) = f_B$. Let $d \in D_n$, and $\rho \in \mathrm{W}_n$ such that $\Psi(f_B) = d\rho$. The image $\Psi(f_B)$ acts on $\Psi(D_n)$ by conjugation. The action of $\Psi(f_B)$ on $\Psi(D_n)$ is determined by ρ. Since $\Psi|_{D_n}$ is the standard embedding composed with γ, we obtain $\rho = f_B$. Let $d = (d_1x_1, \dots, d_nx_n)$. The image $\Psi(f_B)$ commutes with σ_n. We obtain

$$d^{-1}\sigma_n f_B = \sigma_n d f_B = d f_B \sigma_n = d\sigma_n f_B$$

and hence $d_i = \pm 1$ for all i.

The image $\Psi(f_B)$ commutes with all elements of $\mathcal{S}_{n+1}$ that fix the coordinates x_1 and x_2. Similarly as above, this yields that d commutes with all elements of $\mathcal{S}_{n+1}$ that fix the coordinates x_1 and x_2 and we get $d_i = 1$ for $i \neq 1$ and $i \neq 2$.

In [11] it is shown that f_B^2 is contained in $\langle \mathrm{PGL}_{n+1}(\mathbb{C}), \sigma_n\rangle$. By what we proved above, this gives

$$\Psi(f_B^2) = f_B^2 = d f_B d f_B = dd' f_B^2\,,$$

where $d' = (d_1d_2x_1, d_2x_2, \dots, d_nx_n)$. So $dd' = \mathrm{id}$, which yields $d_1^2d_2 = 1$ and therefore $d_2 = 1$. This means that we have either $\Psi(f_B) = f_B$ or $\Psi(f_B) = df_B$ with $d = (-x_1, x_2, \dots, x_n)$.

Let

$$r_1 := [x_0 : x_1 : \cdots : x_{n-1} : x_n + x_1], \quad r_2 := [x_n : x_1 : \cdots : x_{n-1} : x_0],$$
$$r_3 := [x_n : x_0 : x_2 : \cdots : x_{n-1} : x_1], \qquad t := [x_n : x_0 : \cdots : x_{n-1}].$$

We have the relation

$$(r_2 t f_B t^{-1} r_3) r_1 (r_2 t f_B t^{-1} r_3) = r_1$$

and therefore

$$(r_2 t \Psi(f_B) t^{-1} r_3) r_1 (r_2 t \Psi(f_B) t^{-1} r_3) = r_1 .$$

One calculates that, if $\Psi(f_B) = (-x_1, x_2, \ldots, x_n) f_B$, then this relation is not satisfied. Hence $\Psi(f_B) = f_B$. □

BIBLIOGRAPHY

[1] D. N. Akhiezer, *Lie group actions in complex analysis*, Aspects of Mathematics, vol. 27, Friedr. Vieweg & Sohn, 1995, viii+201 pages.

[2] M. Alberich-Carramiñana, *Geometry of the plane Cremona maps*, Lecture Notes in Mathematics, vol. 1769, Springer, 2002, xvi+257 pages.

[3] I. V. Artamkin, "Stable bundles with $c_1 = 0$ on rational surfaces", *Izv. Akad. Nauk SSSR Ser. Mat.* **54** (1990), no. 2, p. 227-241.

[4] A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc & P. Swinnerton-Dyer, "Variétés stablement rationnelles non rationnelles", *Ann. Math.* **121** (1985), no. 2, p. 283-318.

[5] B. A. Białynicki-Birula, "Remarks on the action of an algebraic torus on k^n I", *Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys.* **14** (1966), p. 177-181.

[6] J. Blanc, "Conjugacy classes of affine automorphisms of $\mathbb{K}^n$ and linear automorphisms of $\mathbb{P}^n$ in the Cremona groups", *Manuscr. Math.* **119** (2006), no. 2, p. 225-241.

[7] ———, "Sous-groupes algébriques du groupe de Cremona", *Transform. Groups* **14** (2009), no. 2, p. 249-285.

[8] ———, "Symplectic birational transformations of the plane", *Osaka J. Math.* **50** (2013), no. 2, p. 573-590.

[9] J. Blanc & J. Déserti, "Degree growth of birational maps of the plane", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **14** (2015), no. 2, p. 507-533.

[10] J. Blanc & J.-P. Furter, "Topologies and structures of the Cremona groups", *Ann. Math.* **178** (2013), no. 3, p. 1173-1198.

[11] J. Blanc & I. Hedén, "The group of Cremona transformations generated by linear maps and the standard involution", *Ann. Inst. Fourier* **65** (2015), no. 6, p. 2641-2680.

[12] A. Borel & J. Tits, "Homomorphismes "abstraits" de groupes algébriques simples", *Ann. Math.* **97** (1973), p. 499-571.

[13] M. Brion, "Spherical Varieties: An Introduction", in *Topological methods in algebraic transformation groups*, Progress in Mathematics, vol. 80, Springer, 1989, p. 11-26.

[14] M. Brunella, "Feuilletages holomorphes sur les surfaces complexes compactes", *Ann. Sci. Éc. Norm. Supér.* **30** (1997), no. 5, p. 569-594.

[15] S. Cantat, “Endomorphismes des variétés homogènes”, *Enseign. Math.* **49** (2003), no. 3-4, p. 237-262.

[16] ———, “Morphisms between Cremona groups, and characterization of rational varieties”, *Compos. Math.* **150** (2014), no. 7, p. 1107-1124.

[17] S. Cantat & A. Zeghib, “Holomorphic actions, Kummer examples, and Zimmer program”, *Ann. Sci. Éc. Norm. Supér.* **45** (2012), no. 3, p. 447-489.

[18] D. Cerveau & J. Déserti, “Birational maps preserving the contact structure on $\mathbb{P}^3_{\mathbb{C}}$”, `http://arxiv.org/abs/1602.08866v1`, to appear in *J. Math. Soc. Japan*, 2016.

[19] A. Corti & A.-S. Kaloghiros, “The Sarkisov program for Mori fibred Calabi-Yau pairs”, *Algebr. Geom.* **3** (2016), no. 3, p. 370-384.

[20] M. Demazure, “Sous-groupes algébriques de rang maximum du groupe de Cremona”, *Ann. Sci. Éc. Norm. Supér.* **3** (1970), p. 507-588.

[21] J. Déserti, “On the Cremona group: some algebraic and dynamical properties”, Theses, Université Rennes 1 (France), 2006, `https://tel.archives-ouvertes.fr/tel-00125492/file/these.pdf`.

[22] ———, “Sur les automorphismes du groupe de Cremona”, *Compos. Math.* **142** (2006), no. 6, p. 1459-1478.

[23] ———, “Some properties of the group of birational maps generated by the automorphisms of $\mathbb{P}^n_{\mathbb{C}}$ and the standard involution”, `http://arxiv.org/abs/1403.0346v1`, 2015.

[24] J. A. Dieudonné, *La géométrie des groupes classiques*, 3e ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 5, Springer, 1971, viii+129 pages.

[25] J. Diller & J.-L. Lin, “Rational surface maps with invariant meromorphic two-forms”, *Math. Ann.* **364** (2016), no. 1-2, p. 313-352.

[26] I. V. Dolgachev, *Classical algebraic geometry. A modern view*, Cambridge Univ. Press, 2012, A modern view, xii+639 pages.

[27] F. Enriques, “Sui gruppi continui di trasformazioni cremoniane nel piano”, *Rend. Accad. Lincei, 1er sem* **5** (1893), no. 1, p. 468-473.

[28] W. Fulton & J. Harris, *Representation theory. A first course*, Graduate Texts in Mathematics, vol. 129, Springer, 1991, xvi+551 pages.

[29] M. Gizatullin, “On some tensor representations of the Cremona group of the projective plane”, in *New trends in algebraic geometry (Warwick, 1996)*, London Math. Soc. Lecture Note Ser., vol. 264, Cambridge University Press, 1999, p. 111-150.

[30] ———, “Klein’s conjecture for contact automorphisms of the three-dimensional affine space”, *Mich. Math. J.* **56** (2008), no. 1, p. 89-98.

[31] P. de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, 2000, vi+310 pages.

[32] H. P. Hudson, *Cremona transformation in Plane and Space*, Cambridge University Press, 1927.

[33] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 21, Springer, 1975, xiv+247 pages.

[34] S. Lamy, “Une preuve géométrique du théorème de Jung”, *Enseign. Math.* **48** (2002), no. 3-4, p. 291-315.

[35] ———, “On the genus of birational maps between threefolds”, in *Automorphisms in birational and affine geometry*, Springer Proceedings in Mathematics & Statistics, vol. 79, Springer, 2014, p. 141-147.

[36] I. Pan, “Une remarque sur la génération du groupe de Cremona”, *Bol. Soc. Bras. Mat., Nova Sér.* **30** (1999), no. 1, p. 95-98.

[37] V. L. Popov, “Tori in the Cremona groups”, *Izv. Ross. Akad. Nauk Ser. Mat.* **77** (2013), no. 4, p. 103-134.

[38] C. Procesi, *Lie groups. An approach through invariants and representations*, Universitext, Springer, 2007, xxiv+596 pages.

[39] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer, 1979, Translated from the French by Marvin Jay Greenberg, viii+241 pages.

[40] ———, "Le groupe de Cremona et ses sous-groupes finis", in *Séminaire Bourbaki 2008/2009*, Astérisque, vol. 332, Société Mathématique de France, 2010, p. 75-100.

[41] I. Stampfli, "A note on automorphisms of the affine Cremona group", *Math. Res. Lett.* **20** (2013), no. 6, p. 1177-1181.

[42] H. Sumihiro, "Equivariant completion", *J. Math. Kyoto Univ.* **14** (1974), p. 1-28.

[43] H. Umemura, "Maximal algebraic subgroups of the Cremona group of three variables. Imprimitive algebraic subgroups of exceptional type", *Nagoya Math. J.* **87** (1982), p. 59-78.

[44] ———, "On the maximal connected algebraic subgroups of the Cremona group. I", *Nagoya Math. J.* **88** (1982), p. 213-246.

[45] ———, "On the maximal connected algebraic subgroups of the Cremona group. II", in *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, Adv. Stud. Pure Math., vol. 6, North-Holland, Amsterdam, 1985, p. 349-436.

[46] C. Urech, "Subgroups of Cremona groups", PhD Thesis, University of Basel (Switzerland) / University of Rennes 1 (France), 2017.

[47] A. Weil, "On algebraic groups of transformations", *Am. J. Math.* **77** (1955), p. 355-391.

[48] A. R. Williams, "Birational transformations in 4-space and 5-space", *Bull. Am. Math. Soc.* **44** (1938), no. 4, p. 272-278.

[49] D. Zaitsev, "Regularization of birational group operations in the sense of Weil", *J. Lie Theory* **5** (1995), no. 2, p. 207-224.

[50] D.-Q. Zhang, "The g-periodic subvarieties for an automorphism g of positive entropy on a compact Kähler manifold", *Adv. Math.* **223** (2010), no. 2, p. 405-415.

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