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COVARIANT BI-DIFFERENTIAL OPERATORS ON MATRIX SPACE

by Jean-Louis CLERC

Abstract. — A family of bi-differential operators from $C^\infty(\mathrm{Mat}(m,\mathbb{R})\times\mathrm{Mat}(m,\mathbb{R}))$ into $C^\infty(\mathrm{Mat}(m,\mathbb{R}))$ which are covariant for the projective action of the group $SL(2m,\mathbb{R})$ on $\mathrm{Mat}(m,\mathbb{R})$ is constructed, generalizing both the *transvectants* and the *Rankin–Cohen brackets* (case $m=1$).

Résumé. — On construit une famille d'opérateurs bi-différentiels de $C^\infty(\mathrm{Mat}(m,\mathbb{R})\times\mathrm{Mat}(m,\mathbb{R}))$ dans $C^\infty(\mathrm{Mat}(m,\mathbb{R}))$ qui sont covariants pour l'action projective du groupe $SL(2m,\mathbb{R})$ sur $\mathrm{Mat}(m,\mathbb{R})$. Dans le cas $m=1$, cette construction fournit une nouvelle approche des *transvectants* et des *crochets de Rankin–Cohen*.

Introduction

Let $X = Gr(m,2m,\mathbb{R})$ the Grassmannian of m-planes in $\mathbb{R}^{2m}$, and consider the projective action of the group $G=SL(2m,\mathbb{R})$ on X, given for $g\in G$ and $p\in X$ by $g.p=\{gv, v\in p\}$. Choose an origin o and let P be the stabilizer of o in G. The group P is a maximal parabolic subgroup and $X\sim G/P$. The characters $\chi_{\lambda,\epsilon}$ of P are indexed by $(\lambda,\epsilon)\in\mathbb{C}\times\{\pm\}$. For $(\lambda,\epsilon)\in\mathbb{C}\times\{\pm\}$, let $\pi_{\lambda,\epsilon}$, be the corresponding representation induced from P, realized on the space $\mathcal{E}_{\lambda,\epsilon}$ of smooth sections of the line bundle $E_{\lambda,\epsilon}=X\times_{P,\,\chi_{\lambda,\epsilon}}\mathbb{C}$ (degenerate principal series). For the purpose of this paper, it is more convenient to work with the *noncompact realization* of $\pi_{\lambda,\epsilon}$ on a space $\mathcal{H}_{\lambda,\epsilon}$ of smooth functions on $V=\mathrm{Mat}(m,\mathbb{R})$.

The *Knapp–Stein intertwining operators* form a meromorphic family (in λ) of operators which intertwines $\pi_{\lambda,\epsilon}$ and $\pi_{2m-\lambda,\epsilon}$ (in our notation). In the non compact picture, for generic λ, the corresponding operators,

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denoted by $J_{\lambda,\epsilon}$ are convolution operators on V by certain tempered distributions. The properties of this family of operators are presented in Section 3 and are mostly consequences of the theory of *local zeta functions* and their functional equation on (the simple real Jordan algebra) V. Incidentally, the results for $\epsilon = -1$ seem to be new, at least in the present form.

Let $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$ and consider the tensor product $\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}$, realized (after completion) on a space $\mathcal{H}_{(\lambda,\epsilon),(\mu,\eta)}$ of smooth functions on $V \times V$. Because of the *covariance property* (see (1.9)) of the kernel $k(x, y) = \det(x-y)$ under the diagonal action of G on $V \times V$, the multiplication M by $\det(x - y)$ intertwines $\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}$ and $\pi_{\lambda-1,-\epsilon} \otimes \pi_{\mu-1,-\eta}$ (Proposition 4.2).

Let $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$ and consider the following diagram

$$\begin{array}{ccc}
\mathcal{H}_{(\lambda,\epsilon),(\mu,\eta)} & \xrightarrow{\ ?\ } & \mathcal{H}_{(\lambda+1,-\epsilon),(\mu+1,-\eta)} \\
\Big\downarrow {\scriptstyle J_{\lambda,\epsilon}\otimes J_{\mu,\eta}} & & \Big\downarrow {\scriptstyle J_{\lambda+1,-\epsilon}\otimes J_{\mu+1,-\eta}} \\
\mathcal{H}_{(2m-\lambda,\epsilon),(2m-\mu,\eta)} & \xrightarrow{\ M\ } & \mathcal{H}_{(2m-\lambda-1,-\epsilon),(2m-\mu-1,-\eta)}
\end{array}$$

The main result of the paper is a (rather explicit) construction of a *differential operator* on $V \times V$ which completes the diagram (Theorem 4.1). The proof uses the Fourier transform on V and some delicate calculation specific to the matrix space V, based in particular on *Bernstein–Sato's identities* for $(\det x)^s$ (Section 2). Up to some normalization factors, this yields a family of differential operators $D_{\lambda,\mu}$ with polynomial coefficients on $V \times V$, covariant w.r.t. $\big(\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}, \pi_{\lambda+1,-\epsilon} \otimes \pi_{\mu+1,-\eta})\big)$. Their expression does not depend on ϵ and η, and the family depends holomorphically on (λ, μ). See also Theorem 4.4 for a formulation of the same result in the compact picture.

From this result, it is then easy to construct families of projectively covariant bi-differential operators from $C^\infty(V \times V)$ into $C^\infty(V)$. For any integer k, define

$$B_{\lambda,\mu\,;\,k} = \operatorname{res} \circ D_{\lambda+k,\mu+k} \circ \cdots \circ D_{\lambda+1,\mu+1} \circ D_{\lambda,\mu}$$

where res is the restriction map from $V \times V$ to the diagonal $\operatorname{diag}(V \times V) \sim V$. Clearly, $B_{\lambda,\mu;k}$ is G-covariant w.r.t. $(\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}, \pi_{\lambda+\mu+2k,\epsilon\eta})$. For k fixed, the family depends holomorphically on λ, μ and is generically non trivial.

For $m = 1$, there is another classical construction of such projectively covariant bi-differential operators. The Ω-*process*, a cornerstone in classical invariant theory leads to the construction of the *transvectants*, which are covariant bi-differential operators for special values of the parameters λ and μ connected to the *finite-dimensional representations* of $G = SL(2, \mathbb{R})$.

The *Rankin–Cohen brackets*, much used in the theory of modular forms, are other examples of such covariant bi-differential operators, for special values of (λ,μ) connected to the *holomorphic discrete series* of $SL(2,\mathbb{R})$. There is a vast literature about Rankin–Cohen brackets, see e.g. [6, 7, 21, 22, 23].

In case $m=1$, it has been observed later (see e.g. [16]) that the Ω-process can be extended to general (λ,μ), yielding both the transvectants and the Rankin–Cohen brackets as special cases. As computations are easy when $m=1$, the present construction can be shown to coincide with the approach through the Ω-process, and the operators $B_{\lambda,\mu;k}$ for special of values of (λ,μ), essentially coincide with the transvectants or the Rankin–Cohen brackets. For another related but different point of view see [13] (specially Section 9) or [12]. The situation where $m\geqslant 2$ is further commented in Section 6. Although not directly related to the present approach, it might be worth to mention the papers [17] and [10], for other approaches to multivariable analogues of Rankin–Cohen brackets.

The striking fact that the operator $D_{\lambda,\mu}$, although obtained by composing non-local operators, is a *differential* operator (hence local) was already observed in another geometric context, namely for conformal geometry on the sphere $S^d, d\geqslant 3$ (see [2, 5]). It seems reasonable to conjecture that similar results are valid for any (real or complex) simple Jordan algebra and its conformal group (see [1] for analysis on these spaces).

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1. The degenerate principal series for $Gr(m,2m,\mathbb{R})$

Let $X=Gr(m,2m;\mathbb{R})$ be the Grassmannian of m-dimensional vector subspaces of $\mathbb{R}^{2m}$. The group $G=SL(2m,\mathbb{R})$ acts transitively on X.

Let $(\epsilon_1,\epsilon_2,\dots,\epsilon_{2m})$ be the standard basis of $\mathbb{R}^{2m}$ and let

$$p_0=\bigoplus_{j=m+1}^{2m}\mathbb{R}\epsilon_j\,,\qquad p_\infty=\bigoplus_{j=1}^{m}\mathbb{R}\epsilon_j\,.$$

The stabilizer of p_0 in G is the parabolic subgroup P given by

$$P=\left\{\begin{pmatrix}a&0\\c&d\end{pmatrix},\ a,d\in GL(m,\mathbb{R}),\ \det a\det d=1\right\},$$

and $X\simeq G/P$.

Two subspaces p and q in X are said to be *transverse* if $p\cap q=\{0\}$, and this relation is denoted by $p\pitchfork q$. Let $\mathcal{O}=\Big\{p\in X, p\pitchfork p_\infty\Big\}$. Then

$\mathcal{O}$ is a dense open subset of X. Any subspace p transverse to p_∞ can be realized as the graph of some linear map $x : p_0 \longrightarrow p_\infty$, and vice versa. More explicitly, any $p \in \mathcal{O}$ can be realized as

$$p = p_x = \left\{ \begin{pmatrix} x\xi \\ \xi \end{pmatrix},\ \xi \in \mathbb{R}^m \right\},$$

where ξ is interpreted as a column vector in $\mathbb{R}^m$ and x is viewed as an element of $V = \mathrm{Mat}(m, \mathbb{R})$.

Let $g \in G$ and $x \in V$. The element $g \in G$ is said to be *defined at x* if $g.p_x \in \mathcal{O}$ and then $g(x)$ is defined by $p_{g(x)} = g.p_x$. More explicitly, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$g.p_x = \left\{ \begin{pmatrix} (ax+b)\,\xi \\ (cx+d)\,\xi \end{pmatrix}, \xi \in \mathbb{R}^m \right\},$$

so that g is defined at x iff $(cx+d)$ is invertible, and then

$$g(x) = (ax+b)(cx+d)^{-1}\,.$$

Define $\alpha : G \times V \longrightarrow \mathbb{R}$ by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \alpha(g, x) = \det(cx+d)\,. \tag{1.1}$$

The following elementary calculation is left to the reader.

Lemma 1.1. — *Let $g, g' \in G$ and $x \in V$, and assume that g' is defined at x and g is defined at $g'(x)$. Then gg' is defined at x and*

$$\alpha(gg', x) = \alpha\big(g, g'(x)\big)\alpha(g', x)\,. \tag{1.2}$$

The map $x \longmapsto p_x$ is a homeomorphism of V onto $\mathcal{O}$. The reciprocal of this map $\kappa : \mathcal{O} \to V$ is a local chart, thereafter called the *principal chart*. For any $g \in G$, let $\mathcal{O}_g = g^{-1}(\mathcal{O})$ and $\kappa_g : \mathcal{O}_g \longrightarrow V$ defined by $\kappa_g = \kappa \circ g$. Then $\big(\mathcal{O}_g, \kappa_g\big)_{g\in G}$ is an atlas for X.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Then

$$V_g := \kappa(\mathcal{O}_g \cap \mathcal{O}) = \{x \in V, \det(cx+d) \neq 0\}\,,$$

and the change of coordinates between the charts $\mathcal{O}$ and $\mathcal{O}_g$ is given by

$$V_g \ni x \quad \longmapsto g(x) = (ax+b)(cx+d)^{-1}\ .$$

The group P admits the Langlands decomposition $P = L \ltimes N$, where

$$L = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},\ \det a \det d = 1 \right\}, \qquad N = \left\{ t_v = \begin{pmatrix} \mathbf{1}_m & 0 \\ v & \mathbf{1}_m \end{pmatrix}, v \in V \right\}.$$

The group L acts on V by

$$l = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \qquad l(x) = axd^{-1} \ .$$

Let

$$\overline{N} = \left\{ \overline{n}_y = \begin{pmatrix} \mathbf{1}_m & y \\ 0 & \mathbf{1}_m \end{pmatrix},\ y \in V \right\} \sim V$$

be the opposite unipotent subgroup. The subgroup $\overline{N}$ acts on V by translations, i.e. $\overline{n}_y(x) = x + y$ for $y \in V$.

Let $\iota = \begin{pmatrix} 0 & \mathbf{1}_m \\ -\mathbf{1}_m & 0 \end{pmatrix}$ be the *inversion*. It is defined on the open set $V^\times$ of invertible matrices and acts by $\iota(x) = -x^{-1}$. Its differential $D\iota(x)$ is given by $V \ni u \longmapsto D\iota(x)u = x^{-1}ux^{-1}$.

It is a well-known result that G is generated by $L, \overline{N}$ and ι (a special case of a theorem valid for the *conformal group* of a simple (real or complex) Jordan algebra).

An element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ belongs to $\overline{N}P$ iff $\det d \neq 0$ and then the following *Bruhat decomposition* holds

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbf{1}_m & bd^{-1} \\ 0 & \mathbf{1}_m \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ c & d \end{pmatrix}. \tag{1.3}$$

Let χ be the character of P defined by

$$P \ni p = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \qquad \chi(p) = \det a = (\det d)^{-1}. \tag{1.4}$$

Lemma 1.2. — *Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, x \in V$ and assume that g is defined at x.*

(1) *the differential $Dg(x)$ belongs to L*

(2) $\chi(Dg(x)) = \alpha(g,x)^{-1}$

(3) *the Jacobian of g at x is equal to*

$$j(g,x) = \chi\big(Dg(x)\big)^{2m} = \alpha(g,x)^{-2m} \ . \tag{1.5}$$

Proof. — By elementary calculation, the statements are verified for elements of N, L and for ι. As these elements generate G, the conclusion follows by using the cocycle relations satisfied by $\alpha(g,x)$ (see (1.2)) and by $\chi(Dg(x)$ or $j(g,x)$ as consequences of the chain rule. □

Let $\lambda \in \mathbb{C}$ and $\epsilon \in \{\pm\}$. For $t \in \mathbb{R}^*$ let $t^{\lambda,\epsilon}$ be defined by

$$t \longmapsto \begin{cases} |t|^\lambda & \text{if } \epsilon = + \\ \operatorname{sgn}(t)|t|^\lambda & \text{if } \epsilon = - \end{cases} .$$

The map $t \longmapsto t^{\lambda,\epsilon}$ is a smooth character of $\mathbb{R}^*$, and any smooth character is of this form.

Let $\chi^{\lambda,\epsilon}$ be the character of P defined by

$$\chi^{\lambda,\epsilon}(p) = \chi(p)^{\lambda,\epsilon} .$$

Let $E_{\lambda,\epsilon}$ be the line bundle over $X = G/P$ associated to the character $\chi^{\lambda,\epsilon}$ of P. Let $\mathcal{E}_{\lambda,\epsilon}$ be the space of smooth sections of $E_{\lambda,\epsilon}$. Then G acts on $\mathcal{E}_{\lambda,\epsilon}$ by the natural action on the sections of $E_{\lambda,\epsilon}$ and gives raise to a representation $\pi_{\lambda,\epsilon}$ of G on $\mathcal{E}_{\lambda,\epsilon}$.

A smooth section of $E_{\lambda,\epsilon}$ can be realized as a smooth function F on G which satisfies

$$F(gp) = \chi(p^{-1})^{\lambda,\epsilon} F(g) .$$

To each such function F, associate its restriction to $\overline{N}$, which can be viewed as a function f on V defined for $y \in V$ by

$$f(y) = F(\overline{n}_y) = F\left(\begin{pmatrix} \mathbf{1}_m & y \\ 0 & \mathbf{1}_m \end{pmatrix}\right) .$$

Using the Bruhat decomposition (1.3), the function F can be recovered from f as

$$F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (\det d)^{\lambda,\epsilon} f(bd^{-1}) .$$

The formula is valid for $g \in \overline{N}P$ and extends by continuity to all of G.

This yields the realization of $\pi_{\lambda,\epsilon}$ in the *noncompact picture*, namely for $g \in G$, such that $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{aligned} \pi_{\lambda,\epsilon}(g) f(y) &= \big(\det(cy+d)^{-1}\big)^{\lambda,\epsilon} f\big((ay+b)(cy+d)^{-1}\big) \\ &= \alpha(g^{-1}, y)^{-\lambda,\epsilon} f(g^{-1}(y)) . \end{aligned}$$

In the noncompact picture, the representation $\pi_{\lambda,\epsilon}$ is defined on the image $\mathcal{H}_{\lambda,\epsilon}$ of $\mathcal{E}_{\lambda,\epsilon}$ by the principal chart. The local expression of an element of $\mathcal{H}_{\lambda,\epsilon}$ is a function $f \in C^\infty(V)$. For $g \in G$, the function $x \mapsto \big(\alpha(g,x)^{-1}\big)^{-\lambda,\epsilon} f\big(g(x)\big)$ is a priori defined on the (dense open) subset $\mathcal{O}_g$

of V. Hence a (rather nasty) characterization of the space is as follows : a smooth function f on V belongs to $\mathcal{H}_{\lambda,\epsilon}$ if and only if,

$$(1.6)\quad \forall\, g\in G,\quad x\mapsto \big(\alpha(g,x)^{-1}\big)^{-\lambda,\epsilon} f\big(g(x)\big) \quad \text{extends as a } C^\infty \text{ function on } V\,.$$

Let $(\lambda,\epsilon),(\mu,\eta)\in\mathbb{C}\times\{\pm\}$, and let $\pi_{\lambda,\epsilon}\boxtimes\pi_{\mu,\eta}$ be the corresponding product representation of $G\times G$. The space of the representation $\mathcal{E}_{(\lambda,\epsilon),(\mu,\eta)}$ (after completion) is the space of smooth sections of the fiber bundle $E_{\lambda,\epsilon}\boxtimes E_{\mu,\eta}$ over $X\times X$. For the non-compact realization, observe that $\mathcal{O}^2=\mathcal{O}\times\mathcal{O}$ is an open dense set in $X\times X$. For any $g\in G$, let $\mathcal{O}^2_g$ be the image of $\mathcal{O}^2$ under the *diagonal* action of g^{-1}, i.e. $\mathcal{O}^2_g=\{g(x),g(y),x\in\mathcal{O},y\in\mathcal{O}\}$. Then the family $\big(\mathcal{O}^2_g, g\in G\big)$ is a covering of $X\times X$. Using the corresponding atlas, the local expressions in the principal chart $\kappa\otimes\kappa:\mathcal{O}^2\to V\times V$ of $\mathcal{E}_{(\lambda,\epsilon),(\mu,\eta)}$ is the space $\mathcal{H}_{(\lambda,\epsilon),(\mu,\eta)}$ of C^∞ functions f on $V\times V$ such that, for any $g\in G$

$$(1.7)\quad \alpha(g,x)^{-\lambda,\epsilon} f\big(g(x),g(y)\big)\alpha(g,y)^{-\mu,\eta} \quad \text{extends as a } C^\infty \text{function on } V\times V\,.$$

The group $G\times G$ acts on $\mathcal{H}_{(\lambda,\epsilon),(\mu,\eta)}$ by

$$(1.8)\quad (\pi_\lambda\boxtimes\pi_\mu)(g_1,g_2)f(x,y) = \alpha(g_1^{-1},x)^{-\lambda,\epsilon}\alpha(g_2^{-1},y)^{-\mu,\eta} f\big(g_1^{-1}(x),g_2^{-1}(y)\big)\,.$$

LEMMA 1.3. — *Let $g\in G$, $x,y\in V$ such that g is defined at x and at y. Then*

$$(1.9)\qquad \det\big(g(x)-g(y)\big)=\alpha(g,x)^{-1}\det(x-y)\,\alpha(g,y)^{-1}\,.$$

Proof. — If $g\in\overline{N}$, g acts by translations on V and hence (1.9) is trivial. If $g=\begin{pmatrix}a&0\\0&d\end{pmatrix}$, then $g(x)-g(y)=a(x-y)d^{-1}$, $\alpha(g,x)=\alpha(g,y)=\det a^{-1}\det d$ and (1.9) is easily verified. When $g=\iota$, then

$$\det(-x^{-1}+y^{-1})=\det(x^{-1}(x-y)y^{-1})=\det x^{-1}\det(x-y)\det y^{-1}$$
$$\forall\, v\in V,\quad D\iota(x)v=x^{-1}vx^{-1},\qquad \alpha(\iota,x)=\det x$$

and (1.9) follows easily. The cocycle property (1.2) satisfied by α and the fact that G is generated by $\overline{N},L$ and ι imply (1.9) in full generality. □

PROPOSITION 1.4. — *The function $k(x,y)=\det(x-y)$ belongs to $\mathcal{H}_{(-1,-),(-1,-)}$ and is invariant under the diagonal action of G.*

Proof. — Let $x, y \in V$ and $g \in G$ defined at x and y. (1.9) implies

$$\alpha(g,x)k(g(x),g(y))\alpha(g,y) = k(x,y)$$

which shows that k belongs to $\mathcal{H}_{(-1,-),(1,-)}$ by the criterion (1.7). Further apply (1.8) for $g_1 = g_2 = g$ to get the invariance of k under the diagonal action of G. □

2. Some functional identities in $\mathrm{Mat}(m,\mathbb{C})$ and $\mathrm{Mat}(m,\mathbb{R})$

Let $\big(\mathbb{E}, (\,.\,,.\,)\big)$ be a complex finite dimensional Hilbert space. To any holomorphic polynomial p on $\mathbb{E}$, associate the holomorphic differential operator $p\left(\frac{\partial}{\partial z}\right)$ defined by

$$p\left(\frac{\partial}{\partial z}\right) e^{(z,\xi)} = p\left(\overline{\xi}\right) e^{(z,\xi)}\ .$$

Let $e_1, e_2, \dots, e_n$ is an orthonormal basis, with corresponding coordinates $z_1, z_2, \dots, z_n$. For $I = (i_1, i_2, \dots, i_n)$ a n-tuple of integers, set

$$z^I = z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}, \qquad \left(\frac{\partial}{\partial z}\right)^I = \left(\frac{\partial}{\partial z_1}\right)^{i_1} \left(\frac{\partial}{\partial z_2}\right)^{i_2} \dots \left(\frac{\partial}{\partial z_n}\right)^{i_n}.$$

Let $p(z) = \sum_{|I|\leqslant N} a_I z^I$ be a holomorphic polynomial on $\mathbb{E}$. Then

$$p\left(\frac{\partial}{\partial z}\right) = \sum_{|I|\leqslant N} a_I \left(\frac{\partial}{\partial z}\right)^I.$$

Let $\big(E, \langle\,.\,,.\,\rangle\big)$ be a finite dimensional Euclidean vector space. To any polynomial p on E associate the differential operator $p\big(\frac{\partial}{\partial x}\big)$ such that

$$p\left(\frac{\partial}{\partial x}\right) e^{\langle x,\xi\rangle} = p(\xi) e^{\langle x,\xi\rangle}.$$

Lemma 2.1. — *Let $\big(\mathbb{E}, (\,.\,,.\,)\big)$ be a complex finite dimensional Hilbert space, and let $\big(E, \langle\,.\,,.\,\rangle\big)$ be a real form of $\mathbb{E}$ such that*

$$\forall\, x, y \in E, \quad (x,y) = \langle x, y\rangle\ .$$

Let p be a holomorphic polynomial on $\mathbb{E}$. Let $\mathcal{O}$ be an open subset of $\mathbb{E}$ such that $\omega = \mathcal{O} \cap E \neq \emptyset$. Let f be a holomorphic function f on $\mathcal{O}$. Then for $x \in \omega$

$$p\left(\frac{\partial}{\partial z}\right) f(x) = p\left(\frac{\partial}{\partial x}\right) f_{|\omega}(x)\ . \tag{2.1}$$

Now let $\mathbb{E} = \mathrm{Mat}(m, \mathbb{C}) = \mathbb{V}$ with the inner product $(z, w) = \operatorname{tr} zw^*$. The restriction of this inner product to the real form $E = \mathrm{Herm}(m, \mathbb{C})$ is equal to

$$\langle x, y\rangle = \operatorname{tr} xy^* = \operatorname{tr} xy = \operatorname{tr} y^t x^t = \operatorname{tr} \overline{yx} = \overline{\operatorname{tr} xy} = \overline{\operatorname{tr} xy^*} = \overline{\langle x, y\rangle}$$

and conditions of Lemma 2.1 are satisfied. Denote by $\Omega_m \subset E$ the open cone of positive-definite Hermitian matrices.

Let $k \in \{1, 2, \dots m\}$. For $z \in \mathbb{V}$, let $\Delta_k(z)$ be the principal minor of order k of the matrix z. Let $\Delta_k^c(z)$ be the $(m-k)$ anti-principal minor of z. Both Δ_k and Δ_k^c are holomorphic polynomials on $\mathbb{V}$.

Let $\mathbb{V}^\times$ be the set of invertible matrices in $\mathbb{V}$. Let $z_0 \in \mathbb{V}^\times$. Choose a local determination of $\ln \det z$ on a neighborhood of z_0, and, for $s \in \mathbb{C}$ define $(\det z)^s = e^{s \ln \det z}$ accordingly. Any other local determination of $\ln \det z$ is of the form $\ln \det z + 2ik\pi$ for some $k \in \mathbb{Z}$, and the associated local determination of $(\det z)^s$ is given by $e^{2ik\pi s}(\det z)^s$.

Recall the *Pochhammer's symbol*, for $s \in \mathbb{C}, n \in \mathbb{N}$

$$(s)_0 = 1\,, \quad (s)_1 = s\,, \quad \dots \quad (s)_n = s(s+1)\dots(s+n-1)\,.$$

Proposition 2.2. — *For any $z \in \mathbb{V}^\times$ and for any local determination of* $\ln \det$ *in a neighborhood of z*

$$\Delta_k\left(\frac{\partial}{\partial z}\right)(\det z)^s = (s)_k\, \Delta_k^c(z)\, (\det z)^{s-1}. \tag{2.2}$$

Proof. — Let $z_0 \in \mathbb{V}^\times$. Choose an open neighborhood $\mathcal{V}$ of z contained in $\mathbb{V}^\times$ which is simply connected and such that $\mathcal{V} \cap \Omega_m \neq \emptyset$. On Ω_m, $\det x > 0$ so that $\operatorname{Ln} \det z$ (where Ln is the principal determination of the logarithm on $\mathbb{C} \setminus (-\infty, 0])$ is an appropriate determination of $\ln \det z$ in a neighborhood of Ω_m, which can be analytically continued to $\mathcal{V}$ and used for defining $(\det z)^s$ on $\mathcal{V}$. For $x \in \Omega_m$, the identity

$$\Delta_k\left(\frac{\partial}{\partial x}\right)(\det x)^s = (s)_k\, \Delta_k^c(x)\, (\det x)^{s-1}$$

holds. It is a special case of [8, Proposition VII.1.6] for the simple Euclidean Jordan algebra $Herm(m, \mathbb{C})$. By Lemma 2.1, (2.2) is satisfied for $z \in \mathcal{V} \cap Herm(m, \mathbb{C})$. As both sides of (2.2) are holomorphic functions, (2.2) yields everywhere on $\mathcal{V}$. But if (2.2) is valid for *some* local determination of $\ln \det z$ it is valid for *any* local determination. □

There is a real version of these identities.

Proposition 2.3. — *The following identity holds for $x \in V^\times$*

$$\Delta_k\left(\frac{\partial}{\partial x}\right)(\det x)^{s,\epsilon} = (s)_k\, \Delta_k^c(x)\, (\det x)^{s-1,-\epsilon}\,. \tag{2.3}$$

Proof. — Let $x \in V^{\times}$ and assume first that $\det x > 0$. In a neighbourhood of x in $\mathbb{V}^{\times}$ choose $\mathrm{Ln}(\det z)$ as a local determination of $\ln(\det z)$. Then $(\det x)^s = |\det x|^s$ and hence, using Lemma 2.1 and (2.2)

$$\Delta_k\left(\frac{\partial}{\partial x}\right)\left|\det x\right|^s = (s)_k\, \Delta_k^c(x)\left|\det x\right|^{s-1}.$$

Next assume that $\det x < 0$. In a neighborhood of x in $\mathbb{V}^{\times}$ choose $\mathrm{Ln}(-\det z) + i\pi$ as a local determination of $\ln(\det z)$. Then $(\det x)^s = e^{is\pi}|\det x|^s$, so that, using again Lemma 2.1 and (2.2)

$$e^{is\pi}\Delta_k\left(\frac{\partial}{\partial x}\right)\left|\det x\right|^s = e^{i(s-1)\pi}(s)_k\Delta_k^c(x)|\det x|^{s-1}.$$

The identity (2.3) follows. □

Let $a = (a_{ij})$ be a $m \times m$ matrix with real or complex entries a_{ij}. Let I and J be two subsets of $\{1, 2, \ldots, m\}$ both of cardinality $k, 0 \leqslant k \leqslant m$. After deleting the $m - k$ rows (resp. the $m - k$ columns) corresponding to the indices not in I (resp. not in J), the determinant of the $k \times k$ remaining matrix is the *minor* associated to (I, J) and will be denoted by $\Delta_{I,J}(a)$. For $k = 0$, i.e. $I = J = \emptyset$, by convention $\Delta_{\emptyset,\emptyset}(a) = 1$. For $k = m$, $I = J = \{1, 2, \ldots, m\}$, $\Delta_{I,J}(a) = \det a$.

For $I = \{i_1 < i_2 < \cdots < i_k\}$, let $|I| = i_1 + i_2 + \cdots + i_k$. Also denote by I^c the complement of I in $\{1, 2, \ldots, m\}$, which is a subset of cardinality $m - k$. Recall the following elementary result.

Lemma 2.4. — *Let $I = \{i_1 < i_2 < \cdots < i_k\}$ be a subset of $\{1, 2, \ldots, m\}$ of cardinality k. Let $I^c = \{i'_1 < i'_2 < \cdots < i'_{m-k}\}$. The permutation σ_I defined by*

$$\sigma_I(1) = i_1, \ldots, \sigma_I(k) = i_k, \quad \sigma_I(k+1) = i'_1, \ldots, \sigma_I(m) = i'_{m-k}$$

has signature equal to $\epsilon(\sigma_I) = (-1)^{|I|}$.

The next lemma is a variation on (and a consequence of) the previous lemma.

Lemma 2.5. — *Let $I = \{i_1 < i_2 < \cdots < i_k\}$, $J = \{j_1 < j_2 < \cdots < j_k\}$ be two subsets of $\{1, 2, \ldots, m\}$ both of cardinality k. Let*

$$I^c = \{i'_1 < i'_2 < \cdots < i'_{m-k}\}, \quad J^c = \{j'_1 < j'_2 < \cdots < j'_{m-k}\}.$$

The permutation $\sigma = \sigma_{I,J}$ given by

$$\sigma(i_1) = j_1, \ldots, \sigma(i_k) = j_k, \quad \sigma(i'_1) = j'_1, \ldots, \sigma(i'_{m-k}) = j'_{m-k}$$

has signature $\epsilon(I, J) := \epsilon(\sigma_{I,J}) = (-1)^{|I|+|J|}$.

A permutation σ such that $\sigma(I) = J$ can be written in a unique way as $\sigma = (\tau \vee \tau_c) \circ \sigma_{I,J}$, where τ is a permutation of J and τ_c is a permutation of J^c, and $\tau \vee \tau_c$ is the permutation of $\{1, 2, \dots, m\}$ which coincides on J with τ and on J^c with τ_c.

Proposition 2.6. — *Let $I, J \subset \{1, 2, \dots, n\}$ of equal cardinality k. Then, for $x \in \mathbb{V}^\times$*

$$\partial(\Delta_{I,J})\big(\Delta^s\big)(x) = \epsilon(I, J)(s)_k \, \Delta_{I^c,J^c}(x)\, \Delta(x)^{s-1} \ . \tag{2.4}$$

Proof. — By permuting raws and columns properly, the minor $\Delta_{I,J}$ becomes the k-th principal minor and Δ^{I^c,J^c} becomes the $m-k$ anti-principal minor, up to a sign. Hence (2.4) is a consequence of (2.2) and Lemma 2.4. □

Proposition 2.7. — *Let f, g be two smooth functions defined on $\mathbb{V}$. Then*

$$\det\left(\frac{\partial}{\partial x}\right)(fg) = \sum_{\substack{I,J\subset\{1,2,\dots,m\}\\ \#I=\#J}} \epsilon(I,J)\Delta_{I,J}\left(\frac{\partial}{\partial x}\right) f \ \Delta_{I^c,J^c}\left(\frac{\partial}{\partial x}\right) g \tag{2.5}$$

Proof. — For $\sigma \in \mathfrak{S}_m$

$$\frac{\partial^m}{\partial a_{1\sigma(1)}\partial a_{2\sigma(2)}\dots\partial a_{m\sigma(m)}}(fg) = \sum_{I\subset\{1,2,\dots,m\}} \left(\prod_{i\in I}\frac{\partial}{\partial a_{i\sigma(i)}}\right) f \left(\prod_{i\in I^c}\frac{\partial}{\partial a_{i\sigma(i)}}\right) g \ .$$

Now, given $I \subset \{1, 2, \dots, m\}$,

$$\sum_{\sigma\in\mathfrak{S}_m} = \sum_{\substack{J\subset\{1,2,\dots,m\}\\ \#J=\#I}} \ \sum_{\substack{\sigma\in\mathfrak{S}_m\\ \sigma(I)=J}}$$

so that

$$\begin{aligned}
&\partial(\Delta)(fg)\\
&\quad= \sum_{\sigma\in\mathfrak{S}_m}\epsilon(\sigma) \sum_{I\subset\{1,2,\dots,m\}} \left(\prod_{i\in I}\frac{\partial}{\partial a_{i\sigma(i)}}\right) f \left(\prod_{i\in I^c}\frac{\partial}{\partial a_{i\sigma(i)}}\right) g\\
&\quad= \sum_{I\subset\{1,2,\dots,m\}} \ \sum_{\substack{J\subset\{1,2,\dots,m\}\\ \#I=\#J}} \ \sum_{\substack{\sigma\in\mathfrak{S}_m\\ \sigma(I)=J}} \epsilon(\sigma) \left(\prod_{i\in I}\frac{\partial}{\partial a_{i\sigma(i)}}\right) f \left(\prod_{i\in I^c}\frac{\partial}{\partial a_{i\sigma(i)}}\right) g \ .
\end{aligned}$$

Let

$$I = \{i_1 < i_2 < \cdots < i_k\}, \quad J = \{j_1 < j_2 < \cdots < j_k\}$$
$$I^c = \{i'_1 < i'_2, \cdots < i'_{m-k}\}, \qquad J^c = \{j'_1 < j'_2, \cdots < j'_{m-k}\}.$$

As noted after the proof of Lemma 2.5, a permutation σ such that $\sigma(I) = J$ can be written in a unique way as

$$\sigma = (\tau \vee \tau_c) \circ \sigma_{I,J}$$

where $\tau \in \mathfrak{S}(J), \tau_c \in \mathfrak{S}(J^c)$. Hence

$$\sum_{\substack{\sigma\in\mathfrak{S}_m\\ \sigma(I)=J}} \epsilon(\sigma) \left(\prod_{i\in I} \frac{\partial}{\partial a_{i\sigma(i)}}\right) f \left(\prod_{i\in I^c} \frac{\partial}{\partial a_{i\sigma(i)}}\right) g$$
$$= \epsilon(I,J) \sum_{\tau\in\mathfrak{S}(J)} \sum_{\tau_c\in\mathfrak{S}(J^c)} \epsilon(\tau)\epsilon(\tau_c) \frac{\partial^k f}{\partial a_{i_1\tau(j_1)} \dots \partial a_{i_k\tau(j_k)}} \times \frac{\partial^{m-k} g}{\partial a_{i'_1\tau_c(j'_1)} \dots \partial a_{i'_{m-k}\tau_c(j'_{m-k})}}$$
$$= \epsilon(I,J)\Delta_{I,J}\left(\frac{\partial}{\partial x}\right) f\, \Delta_{I^c,J^c}\left(\frac{\partial}{\partial x}\right) g\,.$$

Formula (2.5) follows by summing over I and J. □

There is a similar *relative* result, allowing to compute $\Delta_{I,J}(fg)$ for I, J two subsets of $\{1,2,\dots,m\}$, both of cardinality $k \leqslant m$. Let

$$I = \{i_1 < i_2 < \cdots < i_k\}, \qquad J = \{j_1 < j_2 < \cdots < j_k\}\ .$$

A subset $P \subset I$ (resp. $Q \subset J$) of cardinality $l \leqslant k$ can be uniquely written as

$$P = \{i_{p_1} < i_{p_2}, \cdots < i_{p_l}\}, \quad \text{resp. } Q = \{j_{q_1}, j_{q_2}, \dots, j_{q_l}\}.$$

Set

$$\epsilon(P:I, Q:J) = (-1)^{p_1+p_2+\cdots+p_l}(-1)^{q_1+q_2+\cdots+q_l}\ .$$

Proposition 2.8. — *Let I, J be two subsets of $\{1,2,\dots,m\}$, both of cardinality $k \leqslant m$. Let f, g be two smooth functions defined on $\mathbb{V}$. Then*

$$\Delta_{I,J}\left(\frac{\partial}{\partial x}\right)(fg) = \sum_{\substack{P\subset I\\ Q\subset J\\ \#P=\#Q}} \epsilon(P:I,Q:J)\, \Delta_{P,Q}\left(\frac{\partial}{\partial x}\right) f\, \Delta_{I\smallsetminus P, J\smallsetminus Q}\left(\frac{\partial}{\partial x}\right) g\,. \tag{2.6}$$

Proof. — In order to calculate the left hand side of (2.6), it is possible to "freeze" all variables x_{ij} for $(i,j)\notin I\times J$. For $x\in\mathbb{V}$, let

$$\mathbb{V}^x_{I,J}=\left\{z=\begin{pmatrix} & & \\ & z_{ij} & \\ & & \end{pmatrix}\in \operatorname{Mat}(m,\mathbb{C}), z_{ij}=x_{ij} \text{ for } (i,j)\notin I\times J\right\}.$$

Then $\mathbb{V}^x_{I,J}\sim \operatorname{Mat}(k,\mathbb{C})$. Now to compute the left hand side of (2.6) at x, apply (2.5) to the restrictions of f and g to $\mathbb{V}^x_{I,J}$. □

Proposition 2.9. — *Let $s,t\in\mathbb{C}$. Then, for $f\in C^\infty(\mathbb{V}\times\mathbb{V})$ and $x,y\in\mathbb{V}$, such that $x,y-x\in\mathbb{V}^\times$*

$$\det\left(\frac{\partial}{\partial x}\right)\Big(\det(x)^s\det(y-x)^t f(x,y)\Big) = \det(x)^{s-1}\det(y-x)^{t-1}\big(E_{s,t}f\big)(x,y) \tag{2.7}$$

where $E_{s,t}$ is the differential operator on $\mathbb{V}\times\mathbb{V}$ given by

$$E_{s,t}f(x,y)=\sum_{k=0}^m\sum_{\substack{I,J\subset\{1,2,\dots,m\}\\ \#I=\#J=k}} p_{I,J}(x,y;s,t)\,\Delta_{I^c,J^c}\left(\frac{\partial}{\partial x}\right)f(x,y)$$

where, for I,J of cardinality k

$$p_{I,J}(x,y;s,t)=\sum_{0\leqslant l\leqslant k}(-1)^l(s)_{(k-l)}(t)_l \times\sum_{\substack{P\subset I,Q\subset J\\ \#P=\#Q=l}}\epsilon(P:I,Q:J)\,\Delta_{I^c\cup P,J^c\cup Q}(x)\,\Delta_{P^c,Q^c}(y-x)\,.$$

Proof. — Using (2.5), the statement is equivalent to, for any $I,J\subset\{1,2,\dots,n\},\#I=\#J=k$,

$$\epsilon(I,J)\det(x)^{-s+1}\det(y-x)^{-t+1}\,\Delta_{I,J}\left(\frac{\partial}{\partial x}\right)\Big(\det(x)^s\det(y-x)^t\Big)$$

a priori defined for $x\in\mathbb{V}^\times, y-x\in\mathbb{V}^\times$ extends as a polynomial in (x,y) equal to $p_{I,J}(x,y;s,t)$.

Use (2.6) to obtain

$$\Delta_{I,J}\left(\frac{\partial}{\partial x}\right)(\det x)^s\big(\det(y-x)\big)^t = \sum_{l=0}^{k}\sum_{\substack{P\subset I,Q\subset J\\ \#P=\#Q=l}} \epsilon(P:I,Q:J)\,\Delta_{I\smallsetminus P,J\smallsetminus Q}\left(\frac{\partial}{\partial x}\right)(\det x)^s \times \Delta_{P,Q}\left(\frac{\partial}{\partial x}\right)\big(\det(y-x)\big)^t.$$

By (2.4),

$$\det(x)^{-s+1}\Delta_{I\smallsetminus P,J\smallsetminus Q}\left(\frac{\partial}{\partial x}\right)(\det x)^s = \epsilon(I\smallsetminus P,J\smallsetminus Q)\,(s)_{k-l}\,\Delta_{I^c\cup P,J^c\cup Q}(x)\ .$$

Moreover, as any constant coefficients differential operator, $\Delta_{K,L}\big(\frac{\partial}{\partial x}\big)$ commutes to translations, so that again by (2.4)

$$\det(y-x)^{-t+1}\Delta_{P,Q}\left(\frac{\partial}{\partial x}\right)(\det(y-x))^t = \epsilon(P,Q)(-1)^l\,(t)_l\,\Delta_{P^c,Q^c}(y-x)\,.$$

Next, as $|I\smallsetminus P|+|P|=|I|$ and $|J\smallsetminus Q|+|Q|=|J|$

$$\epsilon(P,Q)\epsilon(I\smallsetminus P,J\smallsetminus Q)=\epsilon(I,J)\,.$$

It remains to gather all formulæ to finish the proof of Proposition 2.9. □

Let p be a polynomial on $\mathbb{V}$, and let q be the polynomial on $\mathbb{V}\times\mathbb{V}$ given by $q(x,y)=p(x-y)$. Let f be a function on $\mathbb{V}\times\mathbb{V}$. Let g be the function on $\mathbb{V}\times\mathbb{V}$ defined by $g(u,v)=f(u,v-u)$ or equivalently $g(x,x+y)=f(x,y)$. Then

$$\left(q\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)f\right)(x,y)=\left(p\left(\frac{\partial}{\partial u}\right)g\right)(x,x+y)\,. \tag{2.8}$$

In the sequel, for commodity reason, the operator $q\big(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\big)$ will be denoted by $p\big(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\big)$

Proposition 2.10. — *Let $s,t\in\mathbb{C}$. For any smooth function on $\mathbb{V}\times\mathbb{V}$ and for $x,y\in\mathbb{V}^\times$*

$$\det\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)\big((\det x)^s(\det y)^t f\big)(x,y) = (\det x)^{s-1}(\det y)^{t-1}F_{s,t}f(x,y) \tag{2.9}$$

where $F_{s,t}$ is the differential operator on $\mathbb{V}\times\mathbb{V}$ given by

$$F_{s,t}f(x,y) = \sum_{k=0}^{m} \sum_{\substack{I,J\subset\{1,2,\dots,m\}\\ \#I=\#J=k}} q_{I,J}(x,y;s,t)\Delta_{I^c,J^c}\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right) f(x,y)$$

where, for I, J of cardinality k

$$q_{I,J}(x,y;s,t) = \sum_{0\leqslant l\leqslant k} (-1)^l (s)_{(k-l)}\,(t)_l \times \sum_{\substack{P\subset I, Q\subset J\\ \#P=\#Q=l}} \epsilon(P:I,Q:J)\Delta_{I^c\cup P,J^c\cup Q}(x)\,\Delta_{P^c,Q^c}(y)\,.$$

Proof. — Apply the change of variable formula (2.8) to $p = \det$. □

There is a real version of these identities and they are obtained by the same method used to prove the *real* Bernstein–Sato identities (see the proof of (2.3)).

Proposition 2.11. — *Let $s,t\in\mathbb{C}$. For any $f\in C^\infty(V\times V)$ and $x,y\in V^\times$*

$$\left[\det\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)\right](\det x)^{s,\epsilon}(\det y)^{t,\eta}f(x,y) = (\det x)^{s-1,-\epsilon}(\det y)^{t-1,-\eta}F_{s,t}f(x,y)\,. \tag{2.10}$$

3. Knapp–Stein intertwining operators

The definition and properties of the *Knapp–Stein intertwining operators* to be introduced later in this section are based on the study of the two (families of) distributions $(\det x)^{s,\epsilon}$. In a different terminology, there are the *local Zeta functions* on $\mathrm{Mat}(n,\mathbb{R})$. Many authors contributed to the study of these distributions, more generally in the context of simple Jordan algebras or in the context of prehomogeneous vector spaces (see [3, 4, 9, 14, 18, 19, 20]). For the present situation [1] turned out to be the most complete and most useful reference.

Let first consider the case where $\epsilon = +1$, and write $|\det x|^s$ instead of $(\det x)^{s,+}$. Use the notation $\mathcal{S}(V)$ (resp. $\mathcal{S}'(V)$) for the Schwartz space of smooth rapidly decreasing functions (resp. of tempered distributions) on V. Also define, for $s\in\mathbb{C}$

$$\Gamma_V(s) = \Gamma\left(\frac{s+1}{2}\right)\dots\Gamma\left(\frac{s+m}{2}\right). \tag{3.1}$$

Proposition 3.1.

(1) *For any* $\varphi \in \mathcal{S}(V)$ *the integral* $\int_V \varphi(x)|\det(x)|^s\,dx$ *converges for* $\Re(s) > -1$ *and defines a tempered distribution* $T_{s,+}$ *on* $\mathcal{S}(V)$*.*

(2) *The* $\mathcal{S}'(V)$*-valued function* $s \mapsto T_{s,+}$ *defined for* $\Re(s) > -1$ *can be analytically continued as a meromorphic function on* $\mathbb{C}$*.*

(3) *The function* $s \longmapsto \frac{1}{\Gamma_V(s)} T_{s,+}$ *extends as an entire function of* s *(denoted by* $\widetilde{T}_{s,+}$*) with values in the space of tempered distributions.*

Proof. — See [1], especially Theorem 5.12. A careful examination of the Γ factors in the normalizing factor $\Gamma_V(s)$ shows that the poles are at $s = -1, -2, \dots$ if $m > 1$ and at $s = -1, -3, \dots$ if $m = 1$. □

For $f \in \mathcal{S}(V)$, define the *Euclidean Fourier transform* $\mathcal{F}f$ by

$$\mathcal{F}f(x) = \int_V e^{-2i\pi\langle x,y\rangle} f(y)\,\mathrm{d}y\,.$$

The Fourier transform is extended to various functional spaces, and in particular to the space of tempered distributions $\mathcal{S}'(V)$. Recall the elementary formulæ, for $p \in \mathcal{P}(V)$

$$\mathcal{F}\left(p\left(\frac{\partial}{\partial x}\right) f\right) = p(2i\pi\,.\,)\mathcal{F}f, \qquad \mathcal{F}(p\,f) = p\left(-\frac{1}{2i\pi}\frac{\partial}{\partial x}\right)(\mathcal{F}f)\,. \tag{3.2}$$

Proposition 3.2. — *The Fourier transform of the tempered distribution* $\widetilde{T}_{s,+}$ *is given by*

$$\mathcal{F}(\widetilde{T}_{s,+}) = \pi^{-\frac{m^2}{2}-ms}\,\widetilde{T}_{-m-s,+} \tag{3.3}$$

or equivalently

$$\mathcal{F}\Big(\frac{1}{\Gamma_V(s)}|\det(\,.\,)|^s\Big) = \frac{\pi^{-\frac{m^2}{2}-ms}}{\Gamma_V(-s-m)}|\det(\,.\,)|^{-m-s}\,. \tag{3.4}$$

Proof. — See [1, Theorem 4.4 and Theorem 5.12]. □

Now let $\epsilon = -1$. The corresponding results do not seem to have been written, although they could be deduced from [4]. In our approach, the results for $(\det x)^{s,+}$ are used to prove those for $(\det x)^{s,-}$.

Proposition 3.3.

(1) *For any* $\varphi \in \mathcal{S}(V)$ *the integral* $\int_V \varphi(x)(\det x)^{s,-}\,dx$ *converges for* $\Re(s) > -1$ *and defines a tempered distribution* $T_{s,-}$ *on* $\mathcal{S}(V)$*.*

(2) *The* $\mathcal{S}'(V)$*-valued function* $s \mapsto T_{s,-}$ *defined for* $\Re(s) > -1$ *can be analytically continued as a meromorphic function on* $\mathbb{C}$*.*

(3) *The function* $s \mapsto \frac{1}{s\,\Gamma_V(s-1)}\,T_{s,-}$ *extends as an entire function of* s *(denoted by* $\widetilde{T}_{s,-}$*) with values in* $\mathcal{S}'(V)$*.*

Proof. — As a special case of (2.3), the following identity holds on $V^\times$

$$\det\left(\frac{\partial}{\partial x}\right)(\det x)^{s+1,+} = (s+1)_m\,(\det x)^{s,-}\ . \tag{3.5}$$

Next

$$\begin{aligned}\frac{\Gamma_V(s+1)}{\Gamma_V(s-1)} &= \frac{\Gamma(\frac{s}{2}+1)\dots\Gamma(\frac{s+m-1}{2}+1)}{\Gamma(\frac{s}{2})\dots\Gamma(\frac{s+m-1}{2})}\\ &= 2^{-m}\,(s)_m = 2^{-m}\,\frac{s}{s+m}\,(s+1)_m\,.\end{aligned}$$

Rewrite (3.5) as

$$\frac{1}{s\,\Gamma_V(s-1)}(\det x)^{s,-} = 2^{-m}\frac{1}{s+m}\det\left(\frac{\partial}{\partial x}\right)\Big(\frac{1}{\Gamma_V(s+1)}(\det x)^{s+1,+}\Big).$$

For $\Re s$ large enough, both sides extend as continuous functions on V and hence coincide as distributions. Viewed now as a distribution-valued function of s, the right hand side extends holomorphically to all of $\mathbb{C}$ except perhaps at $s=-m$. To get the statements of Proposition 3.3, it suffices to prove that at $s=-m$ the right hand side can be continued as a holomorphic function. In turn this is a consequence of the following lemma.

Lemma 3.4.

$$\det\left(\frac{\partial}{\partial x}\right)\big(\widetilde{T}_{-m+1,+}\big) = 0\,. \tag{3.6}$$

Proof. — The Fourier transform of the distribution $\widetilde{T}_{-m+1,+}$ is equal (up to a non vanishing constant) to $\widetilde{T}_{-1,+}$ (see (3.3)). Hence the statement of the lemma is equivalent to

$$(\det x)\,\widetilde{T}_{-1,+} = 0\,. \tag{3.7}$$

But $\widetilde{T}_{-1,+}$ (the "first" residue of the meromorphic function $s\mapsto T_{s,+}$) is equal (up to a non vanishing constant) to the quasi-invariant measure on the L-orbit $\mathcal{O}_1 = \{x\in V, \operatorname{rank}(x) = m-1\}$ (see [1, Theorem 5.12]). As $\mathcal{O}_1\subset\{x\in V, \det x = 0\}$, (3.7) follows. □

This finishes the proof of Proposition 3.3. A careful analysis of the normalization factor $s\Gamma_V(s-1)$ shows that $T_{s,-}$ has poles at $s=-1,-2,-3,\dots$ if $m>1$, and at $s=-2,-4,\dots$ if $m=1$. □

Proposition 3.5.

$$\mathcal{F}(\widetilde{T}_{s,-}) = -i^m \pi^{-\frac{m^2}{2}-ms}\ \widetilde{T}_{-m-s,-}\,. \tag{3.8}$$

Proof. — During the proof of Proposition 3.3, it was established that

$$\widetilde{T}_{s,-} = 2^{-m}\frac{1}{s+m}\det\left(\frac{\partial}{\partial x}\right)T_{s+1,+}\,.$$

Hence, using (3.4)

$$\mathcal{F}(\widetilde{T}_{s,-}) = 2^{-m}\frac{1}{s+m}\pi^{-\frac{m^2}{2}-m(s+1)}(2i\pi)^m(\det x)\,\widetilde{T}_{-s-m-1,+}$$

which, for generic s can be rewritten as

$$i^m\pi^{-\frac{m^2}{2}-ms}\ \frac{1}{s+m}\ \frac{1}{\Gamma_V(-s-m-1)}\,(\det x)\,T_{-s-m-1,+}\,.$$

Next, for $\Re(s)$ large enough, $(\det x)\,T_{s,+} = T_{s+1,-}$, and by analytic continuation this holds for any s where both sides are defined. Use this result to obtain (3.8) for generic s, and by continuity for all s. □

For $(s,\epsilon)\in\mathbb{C}\times\{\pm\}$, let

$$\gamma(s,\epsilon) = \begin{cases}\frac{1}{\Gamma_V(s)} & \text{if } \epsilon = 1\\ \frac{1}{s\Gamma_V(s-1)} & \text{if } \epsilon=-1\end{cases}$$

so that

$$\widetilde{T}_{s,\epsilon} = \gamma(s,\epsilon)T_{s,\epsilon}\,. \tag{3.9}$$

Let

$$\rho(s,\epsilon) = \begin{cases}\pi^{-\frac{m^2}{2}-ms} & \text{if } \epsilon=+1\\ -i^m\pi^{-\frac{m^2}{2}-ms} & \text{if } \epsilon=-1\end{cases}$$

so that

$$\mathcal{F}(\widetilde{T}_{s,\epsilon}) = \rho(s,\epsilon)\,\widetilde{T}_{-s-m,\epsilon}\,. \tag{3.10}$$

The Knapp–Stein intertwining operators play a central role in semi-simple harmonic analysis (see [11] for general results). The present approach takes advantage of the specific situation to give more explicit results.

For $(\lambda,\epsilon)\in\mathbb{C}\times\{\pm\}$ consider the following operator (*Knapp–Stein intertwining operator*) (formally) defined by

$$J_{\lambda,\epsilon}f(x) = \int_V \det(x-y)^{-2m+\lambda,\epsilon}f(y)\,\mathrm{d}y\,. \tag{3.11}$$

The operator $J_{\lambda,\epsilon}$ verifies the following (formal) intertwining property.

Proposition 3.6. — *For any $g\in G$,*

$$J_{\lambda,\epsilon}\circ\pi_{\lambda,\epsilon}(g) = \pi_{2m-\lambda,\epsilon}(g)\circ J_{\lambda,\epsilon}\,.$$

Proof.

$$J_{\lambda,\epsilon}\big(\pi_{\lambda,\epsilon}(g)f\big)(x) = \int_V (\det(x-y))^{-2m+\lambda,\epsilon}\ \alpha(g^{-1},y)^{-\lambda,\epsilon} f\big(g^{-1}(y)\big)\,\mathrm{d}y$$

which, by using (1.9) and the cocycle property of α can be rewritten as

$$\alpha(g^{-1},x)^{-2m+\lambda,\epsilon}\int_V \det\big(g^{-1}(x)-g^{-1}(y)\big)^{-2m+\lambda,\epsilon}\alpha(g^{-1},y)^{-2m-\lambda+\lambda,\epsilon^2}\,\mathrm{d}y$$

and use the change of variable $z=g^{-1}(y),\ dz=|\alpha(g^{-1},y)|^{-2m}\,\mathrm{d}y$ to get

$$\begin{aligned}J_{\lambda,\epsilon}\big(\pi_{\lambda,\epsilon}(g)f\big)(x) &= \alpha(g^{-1},x)^{-(2m-\lambda),\epsilon}\int_V \det\big(g^{-1}(x)-z\big)^{-2m+\lambda,\epsilon}f(z)\,\mathrm{d}z\\ &= \pi_{2m-\lambda,\epsilon}(g)\big(J_{\lambda,\epsilon}f\big)(x)\,.\end{aligned}$$

□

To pass from a formal operator to an actual operator, notice that the Knapp–Stein operator is a convolution operator and hence (3.11) can be rewritten as

$$J_{\lambda,\epsilon}f = T_{-2m+\lambda,\epsilon}\star f\,.$$

The study of the distributions $T_{s,\pm}$ strongly suggests to define the *normalized* intertwining operator $\widetilde{J}_{\lambda,\epsilon}$ by

$$\widetilde{J}_{\lambda,\epsilon}f = \widetilde{T}_{-2m+\lambda,\epsilon}\star f \tag{3.12}$$

for $f\in\mathcal{S}(V)$, or more explicitly

$$\widetilde{J}_{\lambda,+}f(x) = \frac{1}{\Gamma_V(-2m+\lambda)}\int_V |\det(x-y)|^{-2m+\lambda}f(y)\,\mathrm{d}y\,,$$

$$\widetilde{J}_{\lambda,-}f(x) = \frac{1}{(-2m+\lambda)\Gamma_V(-2m+\lambda-1)}\int_V (\det(x-y))^{-2m+\lambda,-}f(y)\,\mathrm{d}y\,.$$

The representation $\pi_{\lambda,\epsilon}$ is not properly defined on $\mathcal{S}(V)$, but its infinitesimal version is. In fact, let $\varphi\in C_c^\infty(V)$. For $g\in G$ sufficiently close to the identity, g is defined on the compact $Supp(\varphi)$, so that the following definition makes sense : for $X\in\mathfrak{g}$ let

$$d\pi_{\lambda,\epsilon}(X)\varphi = \left(\frac{d}{dt}\right)_{t=0}\pi_{\lambda,\epsilon}(\exp tX)\varphi\,.$$

Moreover, it is well known that the resulting operator $d\pi_{\lambda,\epsilon}(X)$ is a differential operator of order 1 on V with polynomial coefficients, hence can be extended as a continuous operator on the Schwartz space $\mathcal{S}(V)$, and by duality as an operator on $\mathcal{S}'(V)$. An operator $J:\mathcal{S}(V)\to\mathcal{S}'(V)$ is said to be an intertwining operator w.r.t. $(\pi_{\lambda,\epsilon},\pi_{2m-\lambda,\epsilon})$ if for any $X\in\mathfrak{g}$,

$$J\circ d\pi_{\lambda,\epsilon}(X) = d\pi_{2m-\lambda,\epsilon}(X)\circ J\,.$$

The next statement is easily obtained by combining the results on the family of distributions $\widetilde{T}_{s,\epsilon}, (s,\epsilon) \in \mathbb{C}\times\{\pm\}$ (see Propositions 3.1, 3.3), and the formal intertwining property.

Proposition 3.7.

(1) *the operator $\widetilde{J}_{\lambda,\epsilon}$ is a continuous operator form $\mathcal{S}(V)$ into $\mathcal{S}'(V)$.*

(2) *the operator $\widetilde{J}_{\lambda,\epsilon}$ intertwines the representations $\pi_{\lambda,\epsilon}$ and $\pi_{2m-\lambda,\epsilon}$*

(3) *the (operator-valued) function $\lambda \longmapsto \widetilde{J}_{\lambda,\epsilon}$ is holomorphic.*

4. Construction of the families $D_{\lambda,\mu}$ and $B_{\lambda,\mu;k}$

Recall the differential operator $F_{s,t}$ on $V\times V$, constructed in Section 2 (Proposition 2.10). Define for $s,t\in\mathbb{C}$

$$H_{s,t} = \mathcal{F}^{-1}\circ F_{s,t}\circ \mathcal{F} \tag{4.1}$$

As $F_{s,t}$ is a differential operator with polynomial coefficients, $H_{s,t}$ is also a differential operator with polynomial coefficients. To be more explicit, according to (3.2), the passage from $F_{s,t}$ to $H_{s,t}$ consists in changing $p(\frac{\partial}{\partial x},\frac{\partial}{\partial y})$ to multiplication by $p(-2i\pi x,-2i\pi y))$, and multiplication by $p(x,y)$ to the differential operator $p\big(\frac{1}{2i\pi}\frac{\partial}{\partial x},\frac{1}{2i\pi}\frac{\partial}{\partial y}\big)$. Observe that $q_{I,J}$ is homogeneous of degree $2m-k$ and Δ_{I^c,J^c} is homogeneous of degree $m-k$, where $k=\#I=\#J$. This leads to

$$H_{s,t} = \left(\frac{i}{2\pi}\right)^m\sum_{k=0}^m(-1)^k\sum_{\substack{I,J\subset\{1,2,\dots,m\}\\ \#I=\#J=k}} h_{I,J}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y};s,t\right) \times\Big(\Delta_{I^c,J^c}(x-y)f(x,y)\Big) \tag{4.2}$$

where the polynomial $h_{I,J}(\xi,\eta;s,t)$ is given by

$$h_{I,J}(\xi,\eta;s,t) = \sum_{0\leqslant l\leqslant k}(s)_{(k-l)}\,(t)_l\sum_{\substack{P\subset I,Q\subset J\\ \#P=\#Q=l}}\epsilon(P:I,Q:J) \times\Delta_{I^c\cup P,J^c\cup Q}(\xi)\,\Delta_{P^c,Q^c}(\eta)\,.$$

Theorem 4.1. — *The operator $H_{m-\lambda,m-\mu}$ is G-covariant with respect to $(\pi_{\lambda,\epsilon}\otimes\pi_{\mu,\eta},\pi_{\lambda+1,-\epsilon}\otimes\pi_{\mu+1,-\eta})$.*

The (rather long) proof will be given at the end of this section. The next results are preparations for the proof.

Let M be the continuous operator on $\mathcal{S}(V\times V)$ given by

$$M\varphi(x,y) = \det(x-y)\varphi(x,y)\,.$$

Proposition 4.2. — *The operator M intertwines $\pi_{\lambda,\epsilon}\otimes\pi_{\mu,\eta}$ and $\pi_{\lambda-1,-\epsilon}\otimes\pi_{\mu-1,-\eta}$.*

Proof. — Let $\varphi\in C_c^\infty(V\times V)$. Let $g\in G$, and assume that g is defined on $Supp(\varphi)$.

$$\Big(M\circ\big(\pi_{\lambda,\epsilon}(g)\otimes\pi_{\mu,\eta}(g)\big)\varphi\Big)(x,y)$$
$$=\det(x-y)\,\alpha(g^{-1},x)^{-\lambda,\epsilon}\,\alpha(g^{-1},y)^{-\mu,\eta}\,\varphi\big(g^{-1}(x),g^{-1}(y)\big)$$

whereas

$$\Big(\big(\pi_{\lambda-1,-\epsilon}(g)\otimes\pi_{\mu-1,-\eta}(g)\big)\circ M\Big)\varphi(x,y)$$
$$=\det\big(g^{-1}(x)-g^{-1}(y)\big)\alpha(g^{-1},x)^{-\lambda+1,-\epsilon}\alpha(g^{-1},y)^{-\mu+1,-\eta}$$
$$\times\,\varphi\big(g^{-1}(x)-g^{-1}(y)\big)\,.$$

Use (1.9) to conclude that

$$\Big(M\circ\big(\pi_{\lambda,\epsilon}(g)\otimes\pi_{\mu,\eta}(g)\Big)\varphi=\Big(\big(\pi_{\lambda-1,-\epsilon}(g)\otimes\pi_{\mu-1,-\eta}(g)\big)\circ M\Big)\varphi\,.$$

For $X\in\mathfrak{g}$, and for t small enough, $g_t=\exp tX$ is defined on $Supp(\varphi)$. Apply the previous result to g_t, differentiate w.r.t. t at $t=0$ to get

$$M\circ\big(\mathrm{d}(\pi_{\lambda,\epsilon}\otimes\pi_{\mu,\eta})(X)\big)\varphi=\big(\mathrm{d}(\pi_{\lambda-1,-\epsilon}\otimes\pi_{\mu-1,-\eta})(X)\big)\circ M\varphi$$

for any $\varphi\in C_c^\infty(V\times V)$, and extend this equality to any φ in $\mathcal{S}(V\times V)$ by continuity. □

The next proposition is the key result towards the proof.

Proposition 4.3. — *For $f\in\mathcal{S}(V\times V)$*

$$M\circ(\widetilde{J}_{\lambda,\epsilon}\otimes\widetilde{J}_{\mu,\eta})f =\mathrm{d}\big((\lambda,\epsilon),(\mu,\eta)\big)\,\big((\widetilde{J}_{\lambda+1,-\epsilon}\otimes\widetilde{J}_{\mu+1,-\eta})\circ H_{-m+2\lambda,-m+2\mu}\big)f\,, \tag{4.3}$$

where $\mathrm{d}\big((\lambda,\epsilon),(\mu,\eta)\big)$ is equal to

$$\begin{cases}\dfrac{\pi^{4m^2}}{(\lambda-m)\dots(\lambda-2m+2)(\mu-m)\dots(\mu-2m+2)} & \epsilon=+1,\eta=+1\\[2mm] \dfrac{2^{-m}\pi^{4m^2}}{(\lambda-m)\dots(\lambda-2m+2)(\mu-m)} & \epsilon=+1,\eta=-1\\[2mm] \dfrac{2^{-m}\pi^{4m^2}}{(\lambda-m)(\mu-m)\dots(\mu-2m+2)} & \epsilon=-1,\eta=+1\\[2mm] \dfrac{2^{-2m}\pi^{4m^2}}{(\lambda-m)(\mu-m)} & \epsilon=-1,\eta=-1\,.\end{cases}$$

Proof. — As the operators $\widetilde{J}_{\lambda,\epsilon}$ and $\widetilde{J}_{\mu,\eta}$ are convolution operators by a tempered distribution, the left hand side is well defined as a tempered distribution on $V \times V$, and so is its Fourier transform.

In order to alleviate the proof, $c_1, \dots, c_4$ are used during the proof to mean complex numbers depending on $\lambda, \epsilon, \mu, \eta$ but neither on f nor on $(x, y) \in V \times V$. Their actual values are listed at the end of the computation. By (3.4),

$$\begin{aligned}\mathcal{F}\big((\widetilde{J}_{\lambda,\epsilon} \otimes \widetilde{J}_{\mu,\eta})f\big)(x,y) &= \mathcal{F}(\widetilde{T}_{-2m+\lambda,\epsilon})(x)\mathcal{F}(\widetilde{T}_{-2m+\mu,\eta})(y)\mathcal{F}f(x,y)\\ &= c_1\widetilde{T}_{m-\lambda,\epsilon}(x)\widetilde{T}_{m-\mu,\eta}(x)\mathcal{F}f(x,y)\,.\end{aligned} \tag{4.4}$$

Next, for p a polynomial on $V \times V$, and $\Phi \in \mathcal{S}'(V)$,

$$\mathcal{F}(p\Phi)(x,y) = p\left((-2i\pi)^{-1}\frac{\partial}{\partial x}, (-2i\pi)^{-1}\frac{\partial}{\partial y}\right)(\mathcal{F}\Phi)(x,y)\,.$$

Hence

$$\begin{aligned}&\mathcal{F}\big(M \circ (\widetilde{J}_{\lambda,\epsilon} \otimes \widetilde{J}_{\mu,\eta})f\big)(x,y)\\ &\qquad = c_1c_2 \det\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\big((\det x)^{m-\lambda,\epsilon}(\det y)^{m-\mu,\eta}\mathcal{F}f(x,y)\big)\,.\end{aligned} \tag{4.5}$$

Assume temporarily that $\Re\lambda, \Re\mu \ll 0$ so that $(\det x)^{m-\lambda,\epsilon}(\det y)^{m-\mu,\eta}$ is a sufficiently many times differentiable function on $V \times V$. Then, use Proposition 2.11 to get

$$\begin{aligned}&\mathcal{F}\big(M \circ (\widetilde{J}_{\lambda,\epsilon} \otimes \widetilde{J}_{\mu,\eta})f\big)(x,y)\\ &\quad = c_1c_2(\det x)^{m-(\lambda+1),-\epsilon)}(\det y)^{m-(\mu+1),-\eta}F_{m-\lambda,m-\mu}\left(\mathcal{F}f\right)(x,y)\,,\end{aligned} \tag{4.6}$$

the equality being valid *a priori* on $V^\times \times V^\times$, but thanks to the assumption on λ and μ it extends to all of $V \times V$. Next, by the definition of the operator $H_{s,t}$,

$$\begin{aligned}&\mathcal{F}(M \circ (\widetilde{J}_{\lambda,\epsilon} \otimes \widetilde{J}_{\mu,\eta})f)(x,y)\\ &\quad = c_1c_2(\det x)^{m-\lambda-1,-\epsilon)}(\det y)^{m-\mu-1,-\eta}\mathcal{F}\big(H_{-m+2\lambda,-m+2\mu}f\big)(x,y)\\ &\qquad\qquad c_1c_2c_3\,\widetilde{T}_{m-\lambda-1,-\epsilon}(x)\,\widetilde{T}_{m-\mu-1,-\eta}(y)\,\mathcal{F}\big(H_{m-\lambda,m-\mu}f\big)(x,y)\,.\end{aligned} \tag{4.7}$$

Use inverse Fourier transform and (3.10) to conclude that

$$M \circ (\widetilde{J}_\lambda \otimes \widetilde{J}_\mu)f \;= c_1c_2c_3c_4\left((\widetilde{J}_{\lambda+1,-\epsilon} \otimes \widetilde{J}_{\mu+1,-\eta}) \circ H_{m-\lambda,m-\mu}\right)f\,. \tag{4.8}$$

The values of the constants c_1, c_2, c_3 and c_4 are given by

$$c_1 = \rho(-2m+\lambda,\epsilon)\,\rho(-2m+\mu,\eta)$$

$$c_2 = (-1)^m(2\pi)^{-2m}\gamma(m-\lambda,\epsilon)\,\gamma(m-\mu,\eta)$$

$$c_3 = \frac{1}{\gamma(m-\lambda-1,-\epsilon)\,\gamma(m-\mu-1,-\eta)}$$

$$c_4 = \frac{1}{\gamma(\lambda+1,-\epsilon)\,\gamma(\mu+1,-\eta)}$$

so that $c_1c_2c_3c_4$ is equal to

$$\begin{cases} \dfrac{\pi^{4m^2}}{(\lambda-m)\dots(\lambda-2m+2)(\mu-m)\dots(\mu-2m+2)} & \epsilon=+1, \eta=+1 \\ \dfrac{2^{-m}\pi^{4m^2}}{(\lambda-m)\dots(\lambda-2m+2)(\mu-m)} & \epsilon=+1, \eta=-1 \\ \dfrac{2^{-m}\pi^{4m^2}}{(\lambda-m)(\mu-m)\dots(\mu-2m+2)} & \epsilon=-1, \eta=+1 \\ \dfrac{2^{-2m}\pi^{4m^2}}{(\lambda-m)(\mu-m)} & \epsilon=-1, \eta=-1\,. \end{cases}$$

By analytic continuation, (4.3) holds for all λ, μ, thus proving Proposition 4.3. Incidentally, notice that the last step implies the vanishing of $\big((\widetilde{J}_{\lambda+1,-\epsilon}\otimes\widetilde{J}_{\mu+1,-\eta})\circ H_{-m+2\lambda,-m+2\mu}\big)$ at the poles of $d\big((\lambda,\epsilon),(\mu,\eta)\big)$. $\square$

To finish the proof of Theorem 4.1, note that, by Lemma 4.2 and Proposition 3.7 the operator $M\circ(\widetilde{J}_{\lambda,\epsilon}\otimes\widetilde{J}_{\mu,\eta})$ is covariant with respect to $(\pi_{\lambda,\epsilon}\otimes\pi_{\mu,\eta}), (\pi_{2m-\lambda-1,-\epsilon}\otimes\pi_{2m-\mu-1,-\eta})$. Using Proposition 4.3, this implies, generically in (λ,μ) that for any $f\in C_c^\infty(V\times V)$ and any $g\in G$ which is defined on $Supp(f)$,

$$\begin{aligned}\big((\widetilde{J}_{\lambda+1,-\epsilon}\otimes\widetilde{J}_{\mu+1,-\eta})\circ(\pi_{\lambda+1,-\epsilon}(g)\otimes\pi_{\mu+1,-\eta}(g))\circ H_{-m+2\lambda,-m+2\mu}\big)f \\ = \big((\widetilde{J}_{\lambda+1,-\epsilon}\otimes\widetilde{J}_{\mu+1,-\epsilon})\circ H_{m-\lambda,m-\mu}\circ(\pi_{\lambda,\epsilon}(g)\otimes\pi_{\mu,\eta}(g)\big)f\,.\end{aligned}$$

Generically in (λ,μ), the convolution operator $\widetilde{J}_{\lambda+1,-\epsilon}\otimes\widetilde{J}_{\mu+1,-\eta}$ is injective on $C_c^\infty(V)$ as can be seen after performing a Fourier transform, so that

$$\begin{aligned}\big(\big(\pi_{\lambda+1,-\epsilon}(g)\otimes\pi_{\mu+1,-\eta}(g)\big)\circ H_{m-\lambda,m-\mu}\big)f \\ = \big(H_{m-\lambda,m-\mu}\circ\big(\pi_{\lambda,\epsilon}(g)\otimes\pi_{\mu,\eta}(g)\big)\big)f\,.\end{aligned}$$

The covariance of $H_{m-\lambda,m-\mu}$ follows, at least generically in λ,μ and hence everywhere by analytic continuation. This completes the proof of Theorem 4.1.

For convenience in the sequel, let shift the parameters in the notation by setting

$$D_{\lambda,\mu} = H_{m-\lambda,m-\mu}\,.$$

Perhaps is it enlightening to state a version of Theorem 4.1 in the compact picture. Going back to the notation of the Introduction, the (outer) tensor product $\mathcal{E}_{\lambda,\epsilon}\boxtimes\mathcal{E}_{\mu,\eta}$ can be completed to a space $\mathcal{E}_{(\lambda,\epsilon),(\mu,\eta)}$ of smooth sections of the line bundle $E_{\lambda,\mu}\boxtimes E_{\mu,\eta}$ over $X\times X$. The operator M can also be transferred as a continuous operator from $\mathcal{E}_{(\lambda,\epsilon),(\mu,\eta)}$ into $\mathcal{E}_{(\lambda-1,-\epsilon),(\mu-1,-\eta)}$. Denote by $\widetilde{I}_{\lambda,\epsilon}:\mathcal{E}_{\lambda,\epsilon}$ into $\mathcal{E}_{2m-\lambda,\epsilon}$ the normalized Knapp–Stein operator, which corresponds to $\widetilde{J}_{\lambda,\epsilon}$ in the principal chart. The formulation to be given below is a consequence of Theorem 4.1, using the well-known fact that the Knapp–Stein intertwining operators are invertible, at least generically in λ, the inverse of $\widetilde{I}_{\lambda,\epsilon}$ being equal (up to a scalar) to $\widetilde{I}_{2m-\lambda,\epsilon}$.

Theorem 4.4. — *The operator $D_{(\lambda,\epsilon),(\mu,\eta)}$ defined as*

$$D_{(\lambda,\epsilon),(\mu,\eta)} = \left(\widetilde{I}_{2m-\lambda-1,-\epsilon}\otimes\widetilde{I}_{2m-\mu-1,-\eta}\right)\circ M\circ\left(\widetilde{I}_{\lambda,\epsilon}\otimes\widetilde{I}_{\mu,\eta}\right)$$

which, by construction intertwines $\pi_{\lambda,\epsilon}\otimes\pi_{\mu,\eta}$ and $\pi_{\lambda+1,-\epsilon}\otimes\pi_{\mu+1,-\eta}$ (as representations of G) is a differential operator *on $X\times X$.*

Let $\operatorname{res}: C^\infty(V\times V)\longrightarrow C^\infty(V)$ be the *restriction map* defined by

$$\operatorname{res}(\varphi)(x) = \varphi(x,x)\,.$$

For any λ,ϵ and μ,η in $\mathbb{C}\times\{\pm\}$, the restriction map intertwines the representations $\pi_{\lambda,\epsilon}\otimes\pi_{\mu,\eta}$ and $\pi_{\lambda+\mu,\epsilon\eta}$.

Let $\lambda,\mu\in\mathbb{C}$, and $k\in\mathbb{N}$. Let $B_{\lambda,\mu,k}: C^\infty(V\times V)\longrightarrow C^\infty(V)$ be the bi-differential operator defined by

$$B_{\lambda,\mu;k} = \operatorname{res}\circ D_{\lambda+k-1,\mu+k-1}\circ\dots\circ D_{\lambda,\mu}\,.$$

The covariance property of the operators $D_{\lambda,\mu}$ and of res imply the following result.

Theorem 4.5. — *Let $(\lambda,\epsilon),(\mu,\eta)$ be in $\mathbb{C}\times\{\pm\}$. The operator $B_{\lambda,\mu;k}$ is covariant w.r.t. $(\pi_{\lambda,\epsilon}\otimes\pi_{\mu,\eta},\pi_{\lambda+\mu+2k,\epsilon\eta})$.*

A remarkable fact is that whereas the operator $H_{\lambda,\mu}$ has polynomial functions as coefficients, the operator $B_{\lambda,\mu;k}$ has *constant* coefficients, i.e. is of the form

$$\varphi\longmapsto\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} a_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}}{\partial y^{\boldsymbol{\alpha}}\partial z^{\boldsymbol{\beta}}}\varphi\right)(x,x)$$

where $a_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ are complex numbers. In fact, this is merely a consequence of the invariance of the $B_{\lambda,\mu;k}$ under the action of the translations (action of $\overline{N}$). More concretely,this is due to the vanishing on the diagonal diag(V) of many of the coefficients of the operators $H_{\lambda,\mu}$. It seems however difficult to find a closed formula for the coefficients of $B_{\lambda,\mu;k}$ except if $m = 1$.

5. The case $m = 1$ and the Ω-process

For $m = 1$, a simple calculation yields

$$F_{s,t}f = (-tx + sy)f + xy\left(\frac{\partial^2}{\partial x\partial y}\right)f \tag{5.1}$$

$$H_{s,t}f = \frac{1}{2i\pi}\left(-(t-1)\frac{\partial}{\partial x}f + (s-1)\frac{\partial}{\partial y}f - (x-y)\frac{\partial^2 f}{\partial x\partial y}\right) \tag{5.2}$$

$$D_{\lambda,\mu} = \frac{1}{2i\pi}\left(\mu\frac{\partial}{\partial x} - \lambda\frac{\partial}{\partial y} - (x-y)\frac{\partial^2}{\partial x\partial y}\right). \tag{5.3}$$

There is a relation with the Ω-*process*, which we now recall following the classical spirit (see e.g. [15]), but in terms adapted to our situation.

Let $(\lambda,\epsilon) \in \mathbb{C}\times\{\pm\}$ and let $\mathcal{F}_{\lambda,\epsilon}$ be the space of smooth functions defined on $\mathbb{R}^2 \smallsetminus \{0\}$ which satisfy

$$\forall\, t \in \mathbb{R}^* \qquad F(tx_1, tx_2) = t^{-\lambda,\epsilon}F(x_1,x_2)\,.$$

To $F \in \mathcal{F}_{\lambda,\epsilon}$ associate the function f given by $f(x) = F(x,1)$. Then f is a smooth function on $\mathbb{R}$, and F can be recovered from f by

$$F(x_1,x_2) = x_2^{-\lambda,\epsilon}f(\frac{x_1}{x_2})\,,$$

at least for $x_2 \neq 0$ and then extended by continuity.

Let $g \in SL_2(\mathbb{R})$ and let $g^{-1} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$. The function $F\circ g^{-1}$ also belongs to $\mathcal{F}_{\lambda,\epsilon}$, and is explicitly given by

$$F\circ g^{-1}(x_1,x_2) = F(ax_1 + bx_2, cx_1 + dx_2)\,.$$

Its associated function on $\mathbb{R}$ is given by

$$(F\circ g^{-1})(x,1) = F(ax+b, cx+d) = (cx+d)^{-\lambda,\epsilon}f\left(\frac{ax+b}{cx+d}\right),$$

so that the natural action of $G = SL(2,\mathbb{R})$ on $\mathcal{F}_{\lambda,\epsilon}$ is but another realization of the representation $\pi_{\lambda,\epsilon}$.

Now let $(\lambda,\epsilon),(\mu,\eta) \in \mathbb{C}\times\{\pm\}$ and consider the space $\mathcal{F}_{(\lambda,\epsilon),(\mu,\eta)}$ of smooth functions F on $\mathbb{R}^2\smallsetminus\{0\}\times\mathbb{R}^2\smallsetminus\{0\}$ which satisfy

$$\forall\, t,s \in \mathbb{R}^*, \qquad F(t(x_1,x_2), s(y_1,y_2)) = t^{-\lambda,\epsilon}s^{-\mu,\eta}F\big((x_1,x_2),(y_1,y_2)\big)\,.$$

The group $SL_2(\mathbb{R})$ acts naturally (diagonally) on $\mathcal{F}_{(\lambda,\epsilon),(\mu,\eta)}$, and this action yields a realization of $\pi_{\lambda,\epsilon}\otimes\pi_{\mu,\eta}$. More explicitly, let

$$f(x,y)=F((x,1),(y,1))\,.$$

Then for $g\in SL_2(\mathbb{R})$ such that $g^{-1}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

$$F\circ g^{-1}((x,1),(y,1))=(cx+d)^{-\lambda,\epsilon}(cy+d)^{-\mu,\eta}f\left(\frac{ax+b}{cx+d},\frac{ay+b}{cy+d}\right).$$

The polynomial $\det\begin{pmatrix}x_1 & y_1\\ x_2 & y_2\end{pmatrix}$ is invariant by the action of $SL_2(\mathbb{R})$ and so is the differential operator

$$\Omega=\frac{\partial^2}{\partial x_1\partial y_2}-\frac{\partial^2}{\partial x_2\partial y_1}\,.$$

The operator Ω maps $\mathcal{F}_{(\lambda,\epsilon),(\mu,\eta)}$ to $\mathcal{F}_{(\lambda+1,-\epsilon),(\mu+1,-\eta)}$ and yields a covariant differential w.r.t. $(\pi_{\lambda,\epsilon}\otimes\pi_{\mu,\eta},\pi_{\lambda+1,-\epsilon}\otimes\pi_{\mu+1,-\eta})$.

Let $F\in\mathcal{F}_{(\lambda,\epsilon),(\mu,\eta)}$. As above, let f be the function on $\mathbb{R}\times\mathbb{R}$ obtained by deshomogenization of F i.e. $f(x,y)=F\big((x,1),(y,1)\big)$. The corresponding differential operator on $\mathbb{R}\times\mathbb{R}$ is given by

$$\omega_{\lambda,\mu}f(x,y))=\big(\Omega F\big)\big((x,1),(y,1)\big)=-\mu\frac{\partial f}{\partial x}+\lambda\frac{\partial f}{\partial y}+(x-y)\frac{\partial^2 f}{\partial x\partial y}\,,$$

independently of ϵ and η, so that $D_{\lambda,\mu}=-2i\pi\omega_{\lambda,\mu}$.

For $k\in\mathbb{N}$, let $R_k : C^\infty(\mathbb{R}^2\times\mathbb{R}^2)\longmapsto C^\infty(\mathbb{R}^2)$ be the bi-differential operator given by $R_k=\operatorname{res}\circ\Omega^k$ or more explicitely

$$x\in V,\qquad R_kF(x)=\Omega^kF(x,x) \tag{5.4}$$

The operator R_k commutes to the action of $SL(2,\mathbb{R})$. If F belongs to $\mathcal{F}_{(\lambda,\epsilon),(\mu,\eta)}$, the function R_kF is homogeneous of degree $(\lambda+\mu+2k,\epsilon\eta)$. By deshomogenization, the corresponding operator is

$$r_{\lambda,\mu;k}=\operatorname{res}\circ\,\omega_{\lambda+k-1,\mu+k-1}\circ\cdots\circ\omega_{\lambda,\mu}$$

so that $B_{\lambda,\mu;k}=(-2i\pi)^k r_{\lambda,\mu:k}$.

A classical computation in the theory of the Ω-process yields an explicit expression for $r_{\lambda,\mu,k}$

$$r_{\lambda,\mu;k}=\operatorname{res}\circ\left(k!\sum_{i+j=k}(-1)^j\binom{-\lambda-i}{j}\binom{-\mu-j}{i}\frac{\partial^k}{\partial x^i\partial y^j}\right). \tag{5.5}$$

The computation can be found in [16], where the indices λ and μ are supposed to be negative integers, but the computation goes through without this assumption.

Two special cases are worth being reported, both corresponding to cases where the representations $\pi_{\lambda,\epsilon}, \pi_{\mu,\eta}$ are *reducible*.

Suppose that $\lambda = k \in \mathbb{Z}$. Choose $\epsilon = (-1)^k$, so that for any $t \in \mathbb{R}^*, t^{\lambda,\epsilon} = t^k$. Then for $g \in G$ such that $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\pi_{k,(-1)^k}(g)f(x) = (cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right).$$

Let first consider the case where $\lambda \in -\mathbb{N}$, say $\lambda = -l, l \in \mathbb{N}$. Then the space $\mathcal{P}_l$ of polynomials of degree less than l is preserved by the representation $\pi_{-l,(-1)^l}$ Similarly, let $\mu = -m$ for some $m \in \mathbb{N}$. Let $p \in \mathcal{P}_l, q \in \mathcal{P}_m$. Let P (resp. Q) be the homogeneous polynomial on $\mathbb{R}^2$ obtained by homogenization of p (resp.q). For $k \leqslant \inf(l,m)$, the function $R_k(P\otimes Q)$ is a polynomial which is homogeneous of degree $l+m-2k$ and which in the classical theory of invariants is called the k^{th} *transvectant* of P and Q usually denoted by $[P,Q]_k$. So $B_{-l,-m;k}$ just expresses the k-th transvectant at the level of inhomogeneous polynomials.

Now suppose that $\lambda = l, l \in \mathbb{N}$. Then restrictions of holomorphic functions to $\mathbb{R}$ are preserved by the representation $\pi_{l,(-1)^l}$. Suppose also $\mu = m \in \mathbb{N}$. Then the operators $D_{l,m}$ and $B_{l,m,k}$, extended as holomorphic differential operators are still covariant under the action of G. If f is an automorphic form of degree l and g of degree m, then the covariance property of $B_{l,m;k}$ implies that $B_{l,m,k}(f\otimes g)$ is an automorphic form of degree $l+m+2k$. The operators $B_{l,m;k}$ essentially coincide with the *Rankin–Cohen brackets*, as easily deduced from formula (5.5).

6. The general case and some open problems

When $m \geqslant 2$, the Ω-process can be extended along the same lines (see [16]). Let $\mathcal{F}_{\lambda,\epsilon}$ be the space of functions $F : V\times V$ which are *determinantially homogeneous of weight* (λ,ϵ), i.e. satisfying

$$\forall\, \gamma \in GL(V) \qquad F(x\gamma, y\gamma) = (\det \gamma)^{-\lambda,\epsilon} F(x,y)\,.$$

To such a function F, associate the function f on V defined by $f(x) = F(x, \mathbf{1}_m)$. Then F can be recovered from f by

$$F(x,y) = (\det y)^{-\lambda,\epsilon} f(xy^{-1})\,, \tag{6.1}$$

at least when $y \in V^\times$ and everywhere by continuity.

The group $G = SL(2m, \mathbb{R})$ acts on $V \times V$ by left multiplication, i.e. if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\big(g, (x, y)\big) \longmapsto g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

The determinantial homogeneity of functions is preserved by this action, and hence the representation of G on $\mathcal{F}_{\lambda,\epsilon}$ is but another realization of $\pi_{\lambda,\epsilon}$ as can be seen by transferring the action through the correspondance $F \mapsto f$ given by (6.1). Using this time the polynomial $\det_{2m} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$, an operator Ω can be defined along the same line as in the case $m = 1$. As the action of G commutes to the action (on the right) of $GL(V)$, Ω maps $\mathcal{F}_{\lambda,\epsilon} \otimes F_{\mu,\eta}$ into $\mathcal{F}_{\lambda+1,-\epsilon} \otimes F_{\mu+1,-\eta}$ and is covariant for the action of G. Again, using the correspondence $F \mapsto f$, Ω lifts to a differential operator on $V \times V$ which is covariant w.r.t. $(\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}, \pi_{\lambda+1,-\epsilon} \otimes \pi_{\mu+1,-\eta})$ and which can be used for defining the covariant bi-differential operators. It is not clear wether the two approaches coincide, as computations get very complicated.

BIBLIOGRAPHY

[1] L. Barchini, M. R. Sepanski & R. Zierau, "Positivity of zeta distributions and small unitary representations", in *The ubiquitous heat kernel*, Contemporary Mathematics, vol. 398, American Mathematical Society, 2006, p. 1-46.

[2] R. Beckmann & J.-L. Clerc, "Singular invariant trilinear forms and covariant (bi)-differential operators under the conformal group", *J. Funct. Anal.* **262** (2012), no. 10, p. 4341-4376.

[3] S. Ben Saïd, "The functional equation of zeta distributions associated with non-Euclidean Jordan algebras.", *Can. J. Math.* **58** (2006), no. 1, p. 3-22.

[4] N. Bopp & H. Rubenthaler, *Local zeta functions attached to the minimal spherical series for a class of symmetric spaces*, Mem. Am. Math. Soc., vol. 821, American Mathematical Society, 2005, 233 pages.

[5] J.-L. Clerc, "Singular conformally invariant trilinear forms. II: The higher multiplicity cases", to appear in *Transform. Groups*.

[6] G. van Dijk & M. Pevzner, "Ring structures for holomorphic discrete series and Rankin-Cohen brackets", *J. Lie Theory* **17** (2007), no. 2, p. 283-305.

[7] A. M. El Gradechi, "The Lie theory of the Rankin-Cohen brackets and allied bi-differential operators", *Adv. Math.* **207** (2006), no. 2, p. 484-531.

[8] J. Faraut & A. Korányi, *Analysis on symmetric cones*, Oxford Mathematical Publications, Clarendon Press, 1994, xii+382 pages.

[9] S. S. Gelbart, *Fourier analysis on matrix space*, Mem. Am. Math. Soc., vol. 108, American Mathematical Society, 1971, 77 pages.

[10] T. Ibukiyama, T. Kuzumaki & H. Ochiai, "Holonomic systems of Gegenbauer polynomials of matrix arguments related with Siegel modular forms", *J. Math. Soc. Japan* **64** (2012), no. 1, p. 273-316.

[11] A. W. Knapp, *Representation theory of semisimple groups, an overview based on examples*, Princeton Mathematical Series, vol. 36, Princeton University Press, 1986, xvii+773 pages.

[12] T. Kobayashi, T. Kubo & M. Pevzner, "Vector-valued covariant differential operators for the Möbius transformation", in *Lie theory and its applications in physics*, Springer Proceedings in Mathematics & Statistics, vol. 111, Springer, 2014, p. 67-85.

[13] T. Kobayashi & M. Pevzner, "Differential symmetry breaking operators. II: Rankin-Cohen operators for symmetric pairs", *Sel. Math.* **22** (2016), no. 2, p. 847-911.

[14] I. Muller, "Décomposition orbitale des espaces préhomogènes réguliers de type parabolique commutatif et application", *C. R. Acad. Sci., Paris* **303** (1986), p. 495-498.

[15] P. J. Olver, *Classical Invariant Theory*, London Mathematical Society Student Texts, vol. 44, Cambridge University Press, 1999, xxi+280 pages.

[16] P. J. Olver, M. Petitot & P. Solé, "Generalized Transvectants and Siegel modular forms", *Adv. Appl. Math.* **38** (2007), no. 3, p. 404-418.

[17] L. Peng & G. Zhang, "Tensor products of holomorphic representations and bilinear differential operators", *J. Funct. Anal.* **210** (2004), no. 1, p. 171-192.

[18] M. Sato & T. Shintani, "On zeta functions associated with prehomogeneous vector spaces", *Ann. Math.* **100** (1974), p. 131-170.

[19] E. M. Stein, "Analysis in matrix spaces and some new representations of $SL(N,\mathbb{C})$", *Ann. Math.* **86** (1967), p. 461-490.

[20] J. T. Tate, "Fourier analysis in number fields and Hecke's zeta-functions", PhD Thesis, Princeton University, USA, 1950.

[21] A. Unterberger & J. Unterberger, "Algebras of symbols and modular forms", *J. Anal. Math.* **68** (1996), p. 121-143.

[22] D. Zagier, "Modular forms and differential operators", *Proc. Indian Acad. Sci.* **104** (1994), no. 1, p. 57-75.

[23] G. Zhang, "Rankin-Cohen brackets, transvectants and covariant differential operators", *Math. Z.* **264** (2010), no. 3, p. 513-519.

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