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PARTIAL PERIODIC QUOTIENTS OF GROUPS ACTING ON A HYPERBOLIC SPACE

by **Rémi B. COULON (*)**

ABSTRACT. — In this article, we construct partial periodic quotients of groups which have a non-elementary acylindrical action on a hyperbolic space. In particular, we provide infinite quotients of mapping class groups where a fixed power of every homeomorphism is identified with a periodic or reducible element.

RÉSUMÉ. — Dans cet article, nous construisons des quotients partiellement périodiques de groupes admettant une action acylindrique sur un espace hyperbolique. En particulier, nous produisons des quotients infinis de groupes modulaires de surfaces, dans lesquelles une puissance fixée de tout homéomorphisme s'identifie avec un élément réductible ou un élément d'ordre fini.

1. Introduction

A group G is *periodic* with *exponent* n if for every $g \in G$, $g^n = 1$. In 1902, W. Burnside asked whether or not a finitely generated periodic group was necessarily finite. Despite the simplicity of the statement, this question remained open for a long time and motivated many developments in group theory. In 1968, P.S. Novikov and S.I. Adian achieved a breakthrough by providing the first examples of infinite finitely generated periodic groups [31]. See also [32] and [13]. We now know that if G is a hyperbolic group which is not virtually cyclic then there exists an integer n such that G has an infinite quotient of exponent n [27]. As opposed to this situation any finitely generated periodic linear group is finite [36].

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The original motivation for our work was the following question. What are the finitely generated groups which admit an infinite periodic quotient? With this level of generality, it is very difficult to understand what could be the periodic quotients of an arbitrary non-hyperbolic group G. In this article we are interested in partial periodic quotients of the form G/S^n where S^n stands for the normal subgroup generated by the n-th power of every element in a large subset S of G. Our construction provides various examples of quotients with exotic properties. Let us mention two applications.

Quotient of amalgamated products. A subgroup H of a group G is *malnormal* if for every $g \in G$, $gHg^{-1} \cap H$ is trivial provided g does not belong to H.

Theorem 1.1. — *Let A and B be two groups without involution. Let C be a subgroup of A and B malnormal in A or B. There is an integer n_1 such that for every odd exponent $n \geqslant n_1$ there exists a quotient Q of $A *_C B$ with the following properties.*

- *(i) The natural projection $A *_C B \twoheadrightarrow Q$ induces an embedding of A and B into Q.*
- *(ii) For every $g \in Q$, if g is not a conjugate of an element of A or B then $g^n = 1$.*
- *(iii) There are infinitely many elements in Q which are not conjugates of elements of A or B.*

A similar statement has been obtained by K. Lossov in his Ph.D. dissertation using a diagrammatical version of small cancellation theory. To the best of our knowledge the proof has never been published though.

Mapping class group. Our next example is new and comes from the geometry of surfaces. Let Σ be a compact surface of genus g with p boundary components. The *mapping class group* $\mathrm{MCG}(\Sigma)$ of Σ is the group of orientation preserving self homeomorphisms of Σ defined up to homotopy. A mapping class $f \in \mathrm{MCG}(\Sigma)$ is *periodic* if it has finite order; *reducible* if it permutes a collection of essential non-peripheral curves (up to isotopy); *pseudo-Anosov* if there exists an homotopy in the class of f that preserves a pair of transverse foliations and rescale these foliations in an appropriate way. It follows from Thurston's work that any element of $\mathrm{MCG}(\Sigma)$ falls into one these three categories [42]. We produce an infinite quotient of $\mathrm{MCG}(\Sigma)$ where a fixed power of every element "becomes" periodic or reducible.

Theorem 1.2. — *Let Σ be a compact surface of genus g with p boundary components such that $3g + p - 3 > 1$. There exist integers κ and n_0*

such that for every odd exponent $n \geqslant n_0$ *there is a quotient* Q *of* $\mathrm{MCG}(\Sigma)$ *with the following properties.*

(i) *If* E *is a subgroup of* $\mathrm{MCG}(\Sigma)$ *that does not contain a pseudo-Anosov element, then the projection* $\mathrm{MCG}(\Sigma) \twoheadrightarrow Q$ *induces an isomorphism from* E *onto its image.*

(ii) *Let* f *be a pseudo-Anosov element of* $\mathrm{MCG}(\Sigma)$*. Either* $f^{\kappa n} = 1$ *in* Q *or there exists a periodic or reducible element* $u \in \mathrm{MCG}(\Sigma)$ *such that* $f^{\kappa} = u$ *in* Q*. In particular, for every* $f \in \mathrm{MCG}(\Sigma)$*, there exists a periodic or reducible element* $u \in \mathrm{MCG}(\Sigma)$ *such that* $f^{\kappa n} = u$ *in* Q*.*

(iii) *There are infinitely many elements in* Q *which are not the image of a periodic or reducible element of* $\mathrm{MCG}(\Sigma)$*.*

Remarks. — The question of infinite periodic quotients of mapping class groups was raised by N.V. Ivanov [26, §13]. Let Σ be a compact of surface of genus g. Let n be an integer. Using quantum representations, L. Funar and T. Kohno established that if n is large enough, the quotient of $\mathrm{MCG}(\Sigma)$ by the subgroup generated by the n-th power of every Dehn-twist is infinite [18, Corollary 1.2] and contains non-abelian free subgroups [20, Remark 3.13]. In Section 6 we provide an alternative proof of this fact (Theorem 6.17). To the best of our knowledge, Theorem 1.2 is the first construction of infinite quotient of higher genus mapping class groups which “neutralizes” a fixed power of all pseudo-Anosov homeomorphisms. L. Funar provided also some answers regarding *full* periodic quotients of mapping class group. If $g \geqslant 3$, then there exist arbitrarily large integers n such that $\mathrm{MCG}(\Sigma)$ has non-trivial periodic quotients of exponent n [19, Theorem 1.2]. If $g = 2$, these quotients are actually infinite [19, Theorem 1.3].

A ping-pong argument shows that $\mathrm{MCG}(\Sigma)$ contains many free purely pseudo-Anosov subgroups. By *purely pseudo-Anosov subgroup* we mean that any non-trivial element of this subgroup is pseudo-Anosov. Until recently it was an open whether $\mathrm{MCG}(\Sigma)$ had purely pseudo-Anosov *normal* subgroups. This question was for instance listed in Kirby’s book as Problem 2.12(A) [28]. See also [26, Problem 3] and [16, Paragraph 2.4]. In [11], F. Dahmani, V. Guirardel and D. Osin provide many examples of such groups. More precisely they prove the following. There exists an integer n (that only depends on the surface Σ) such that if $f \in \mathrm{MCG}(\Sigma)$ is pseudo-Anosov, then the normal closure of f^n is free and purely pseudo-Anosov [11, Theorem 8.1]. One could ask whether or not there is an integer n such that the normal subgroup N of $\mathrm{MCG}(\Sigma)$ generated by the n-th power of *every* pseudo-Anosov element is purely pseudo-Anosov. However such an integer cannot exist. Indeed one can find a pseudo-Anosov element f and an infinite order reducible element u such that $f^n u$ is pseudo-Anosov. If both f^n and $(f^n u)^n$ belong to N, then the reducible element

$$u^n = \left(u^{n-1} f^{-n} u^{-(n-1)}\right) \cdots \left(u^2 f^{-n} u^{-2}\right) \left(u f^{-n} u^{-1}\right) f^{-n} \left(f^n u\right)^n,$$

would also belong to N. Nevertheless, if Q stands for the quotient given by Theorem 1.2, then the kernel K of the projection $\mathrm{MCG}(\Sigma) \twoheadrightarrow Q$ provides a purely pseudo-Anosov normal subgroup that contains a fixed power of most of the pseudo-Anosov elements of $\mathrm{MCG}(\Sigma)$. Following [26], we wonder whether this kernel is a free group.

Corollary 1.3. — *Let Σ be a compact surface of genus g with p boundary components such that $3g + p - 3 > 1$. There exist integers κ and n_0 such that for every odd exponent $n \geqslant n_0$ there is a subgroup K of* $\mathrm{MCG}(\Sigma)$ *with the following properties.*

(i) K is normal and purely pseudo-Anosov.
(ii) As a normal subgroup, K is not finitely generated.
(iii) For every $f \in \mathrm{MCG}(\Sigma)$ either $f^{\kappa n}$ belongs to K or there exists a periodic or reducible element $u \in \mathrm{MCG}(\Sigma)$ such that $f^{\kappa} u$ belongs to K.

In his seminal paper M. Gromov introduced the concept of δ-hyperbolic spaces [23]. Using a simple four point inequality, he captured most of the large scale features of negative curvature. For a group G, being hyperbolic means that its Cayley is hyperbolic as a metric space. Generalizing this idea, M. Gromov also defined the notion of relatively hyperbolic groups. For many purposes the Cayley graph is not the most appropriate space to work with. To take advantage of the hyperbolic geometry what really matters though is to have G acting "nicely" on a hyperbolic space. However not all actions will do the job. Indeed every group admits a proper action on a hyperbolic space. To make this idea works the action needs to satisfy some finiteness condition. For instance a group G is

(i) *hyperbolic* if and only if it acts properly co-compactly on a hyperbolic length space X.
(ii) *relatively hyperbolic* if and only if it acts properly on a hyperbolic length space X with some finiteness condition for the induced action of G on the boundary at infinity ∂X of X.

These two classes already cover numerous examples of groups: geometrically finite Kleinian groups, fundamental groups of finite volume manifolds with pinched sectional curvature, small cancellation groups, amalgamated products over finite groups, etc. In this article we focus on a weaker condition: *acylindricity*. It was first used by Z. Sela for actions on a tree [37]. The following formulation is due to B. Bowditch [3].

DEFINITION 1.4. — *The action of a group G on a metric space X is* acylindrical *if for every $l \geqslant 0$, there exist $d \geqslant 0$ and $N \geqslant 0$ with the following property. For every $x, x' \in X$ with $|x-x'| \geqslant d$, the set of elements $u \in G$ satisfying $|ux-x| \leqslant l$ and $|ux'-x'| \leqslant l$ contains at most N elements.*

Roughly speaking, it means that the stabilizers of long paths are finite with some uniform bound on their cardinality.

Example 1.5. — Let A and B be two groups. Let C be a subgroup of A and B which is malnormal in A or B. The action of the amalgamated product $A *_C B$ on the corresponding Bass-Serre tree is acylindrincal [37].

Example 1.6. — Let Σ be a compact surface of genus g with p boundary components. The *complex of curves* X is a simplicial complex associated to Σ introduced by W. Harvey [24]. The simplices of X are collections of homotopy classes of curves of Σ that can be disjointly realized. H. Masur and Y. Minsky proved that this space is hyperbolic [29]. By construction, X is endowed with an action by isometries of $\mathrm{MCG}(\Sigma)$. Moreover B. Bowditch showed that this action is acylindrical [3].

An acylindrical group action on a metric space is *non-elementary* if its orbits are neither bounded or quasi-isometric to a line. D. Osin studied the class of groups that admit a non-elementary acylindrical action on a hyperbolic space X. It turns out that this class is very large [34]. Besides the two examples previously mentioned it also contains hyperbolic groups, relatively hyperbolic groups, outer automorphism groups of free groups, right angled Artin groups which are not cyclic or split as a direct product, the Cremona group, etc. More examples are given in the work of A. Minasyan and D. Osin [30].

Let G be a group acting acylindrically on hyperbolic space X. Just as with hyperbolic groups, an element $g \in G$ is either elliptic (its orbits are bounded) or loxodromic (given $x \in X$, the map $\mathbb{Z} \to X$ that sends m to $g^m x$ is a quasi-isometric embedding). Every elementary subgroup E of G either has bounded orbits or is virtually $\mathbb{Z}$. The number $e(G, X)$ is the least common multiple of the exponents of the holomorph $\mathrm{Hol}(F) = F \rtimes \mathrm{Aut}\,(F)$, where F runs over the maximal finite normal subgroup of all maximal non-elliptic elementary subgroups of G. Provided this number is odd, our main result explains how to build a quotient Q of G with the following properties. Any elliptic element is not affected; a fixed power of every loxodromic element is identified with an elliptic one. More precisely we prove the following statement.

Theorem 1.7. — *Let X be a hyperbolic length space. Let G be a group without involution acting by isometries on X. We assume that the action of G is acylindrical and non-elementary. Suppose that $e(G, X)$ is odd. There is a critical exponent n_1 such that for every odd integer $n \geqslant n_1$ which is a multiple of $e(G, X)$, there exists a quotient Q of G with the following properties.*

- *If E is an elliptic subgroup of G, then the projection $G \twoheadrightarrow Q$ induces an isomorphism from E onto its image.*
- *For every element $g \in Q$, either $g^n = 1$ or g is the image an elliptic element of G.*
- *There are infinitely many elements in Q which do not belong to the image of an elliptic subgroup of G.*

Theorem 1.7 applied with the amalgamated product $A *_C B$ of Example 1.5 gives Theorem 1.1. The mapping class group $\mathrm{MCG}(\Sigma)$ of a surface Σ does contain elements of order 2. Therefore we cannot directly apply Theorem 1.7. However it has a finite index torsion-free normal subgroup N. A variation on Theorem 1.7 (see Theorem 6.9) leads to Theorem 1.2. The constant κ in Theorem 1.2 is exactly the least common multiple of $e(N, X)$ and the index of N in $\mathrm{MCG}(\Sigma)$.

Our theorem actually holds for some group action which are not acylindrical (see Theorem 6.9). However the statement requires additional invariants associated to G and X (see Section 3.5). This larger framework allows in particular the group G to contain parabolic isometries/subgroups which is never the case for an acylindrical action.

The proof of Theorem 1.7 relies on techniques introduced by T. Delzant and M. Gromov to study free Burnside groups of odd exponents. Recall that the *free Burnside group* $\mathbf{B}_r(n)$ of rank r and exponent n is the quotient of the free group $\mathbf{F}_r$ of rank r by the normal subgroup $\mathbf{F}_r^n$ generated the n-th power of every element. It is the largest group of rank r and exponent n. In [13], T. Delzant and M. Gromov provide an alternative proof of the infiniteness of $\mathbf{B}_r(n)$ for sufficiently large odd integers n. To that end they construct a sequence of non-elementary hyperbolic groups

$$\mathbf{F}_r = G_0 \twoheadrightarrow G_1 \twoheadrightarrow G_2 \twoheadrightarrow \cdots \twoheadrightarrow G_k \twoheadrightarrow \dots$$

whose direct limit is $\mathbf{B}_r(n)$. Each group G_k is obtained by adjoining to the previous group new relations of the form g^n, using a geometric form of small cancellation theory. The infiniteness of $\mathbf{B}_r(n)$ follows from the hyperbolicity of the approximation groups G_k. For a detailed presentation of this approach we refer the reader to the notes written by the author [10].

It appears that small cancellation theory can be extended to a larger class of groups. In the previous process if G_k is a group acting "nicely" on a hyperbolic space X_k one can construct a hyperbolic space X_{k+1} on which G_{k+1} acts with similar properties [8, 11]. The main difficulty is to make sure that one can indefinitely iterate this construction. In the case of free Burnside groups of odd exponents T. Delzant and M. Gromov used two invariants (the injectivity radius and the invariant A, see Definition 3.36 and Definition 3.43) to control the small cancellation parameters during the process. The other key ingredient involved in their proof is the algebraic structure of the approximation groups G_k: every elementary subgroup of G_k is cyclic. This remarkable property explains why the case of odd exponents is much easier than the even one. If, instead of a free group, we initiate the construction with a group G acylindrically acting on a hyperbolic space, then the algebraic structure of G will never be as simple. Indeed the elliptic subgroups of G can be anything. To handle this difficulty we use a new invariant $\nu(G, X)$. Formally, it is the smallest integer m with the following property. Given any two elements $g, h \in G$ with h loxodromic, if g, $h^{-1}gh, \dots, h^{-m}gh^m$ generate an elliptic subgroup, then g and h generate an elementary subgroup of G (see Definition 3.40). This new parameter will allow us to control the structure of elementary subgroups which are not elliptic.

Outline of the paper. In Section 2 and Section 3 we review some of the standard facts on hyperbolic spaces and groups acting on a hyperbolic space. In particular, we define in Section 3.5 all the invariants that are needed to iterate later the small cancellation process. In Section 4 we recall the cone-off construction which is one of the key tool in the geometric approach of small cancellation. Section 5 is dedicated to small cancellation theory. If G is a group acting on a hyperbolic space X we explain how to use small cancellation theory to produce a quotient $\bar{G}$ with an action on a hyperbolic space $\bar{X}$. Moreover we show that the invariants associated to the action of $\bar{G}$ on $\bar{X}$ can be controlled using the ones describing the action of G on X. In the beginning of Section 6, we prove a statement (Proposition 6.1) that will be used as the induction step in the proof of the main theorem (Theorem 6.9). Finally we discuss some applications of our results.

2. Hyperbolic geometry

In this section we review some of the basic ideas about hyperbolic spaces in the sense of M. Gromov. For more details we refer the reader to Gromov's original paper [23] or [7, 22].

2.1. Definitions

Notations and vocabulary. Let X be a metric length space. Unless otherwise stated a path is a rectifiable path parametrized by arc length. Given two points x and x' of X, we denote by $|x-x'|_X$ (or simply $|x-x'|$) the distance between them. We write $B(x,r)$ for the open ball of center x and radius r. The space is said to be *proper* if every closed bounded subset is compact. Let Y be a subset of X. We write $d(x,Y)$ for the distance between a point $x \in X$ and Y. We denote by $Y^{+\alpha}$, the *α-neighborhood* of Y, i.e. the set of points $x \in X$ such that $d(x,Y) \leqslant \alpha$. The *open α-neighborhood* of Y is the set of points $x \in X$ such that $d(x,Y) < \alpha$. Let $\eta \geqslant 0$. A point p of Y is an *η-projection* of $x \in X$ on Y if $|x-p| \leqslant d(x,Y)+\eta$. A 0-projection is simply called a *projection*.

The four point inequality. The Gromov product of three points $x,y,z \in X$ is defined by

$$\langle x,y\rangle_z = \frac{1}{2}\left\{|x-z|+|y-z|-|x-y|\right\}.$$

The space X is *δ-hyperbolic* if for every $x,y,z,t \in X$

$$\langle x,z\rangle_t \geqslant \min\left\{\langle x,y\rangle_t, \langle y,z\rangle_t\right\} - \delta, \tag{2.1}$$

or equivalently

$$|x-z|+|y-t| \leqslant \max\left\{|x-y|+|z-t|, |x-t|+|y-z|\right\}+2\delta. \tag{2.2}$$

Remarks. — Note that in the definition of hyperbolicity we do not assume that X is geodesic or proper. If X is 0-hyperbolic, then it can be isometrically embedded in an $\mathbb{R}$-tree, [22, Chapitre 2, Proposition 6]. For our purpose though, we will always assume that the hyperbolicity constant δ is positive. It is indeed more convenient to define particular subsets without introducing other auxiliary positive parameters (see Definition 2.17 of a hull or Definition 3.9 of an axis). The hyperbolicity constant of the hyperbolic plane $\mathbf{H}$ will play a particular role. We denote it by $\boldsymbol{\delta}$ (bold delta).

From now on we assume that X is δ-hyperbolic. It is known that triangles in a geodesic hyperbolic space are thin (every side lies in a uniform neighborhood of the union of the two other ones). Since our space is not geodesic, we use instead the following metric inequalities. In this lemma the Gromov products $\langle x,z\rangle_t$, $\langle x,y\rangle_s$ and $\langle x,y\rangle_t$ should be thought as very small quantities.

LEMMA 2.1 ([10, Lemma 2.2]). — *Let x, y, z, s and t be five points of X.*

(i) $\langle x,y\rangle_t \leqslant \max\left\{|x-t| - \langle y,z\rangle_x, \langle x,z\rangle_t\right\} + \delta$,
(ii) $|s-t| \leqslant ||x-s|-|x-t|| + 2\max\left\{\langle x,y\rangle_s, \langle x,y\rangle_t\right\} + 2\delta$,
(iii) The distance $|s-t|$ is bounded above by

$$\max\Big\{ ||x-s|-|x-t|| + 2\max\left\{\langle x,y\rangle_s, \langle x,z\rangle_t\right\}, \\ |x-s|+|x-t|-2\langle y,z\rangle_x \Big\} + 4\delta.$$

The boundary at infinity. Let x be a base point of X. A sequence (y_n) of points of X *converges to infinity* if $\langle y_n,y_m\rangle_x$ tends to infinity as n and m approach to infinity. The set $\mathcal{S}$ of such sequences is endowed with a binary relation defined as follows. Two sequences (y_n) and (z_n) are related if

$$\lim_{n\to+\infty}\langle y_n,z_n\rangle_x = +\infty.$$

If follows from (2.1) that this relation is actually an equivalence relation. The *boundary at infinity* of X denoted by ∂X is the quotient of $\mathcal{S}$ by this relation. If the sequence (y_n) is an element in the class of $\xi\in\partial X$ we say that (y_n) *converges* to ξ and write

$$\lim_{n\to+\infty} y_n = \xi.$$

Note that the definition of ∂X does not depend on the base point x. If Y is a subset of X, we denote by ∂Y the set of elements of ∂X which are limits of sequences of points of Y. Since X is not proper, ∂Y might be empty even though Y is unbounded.

The Gromov product of three points can be extended to the boundary. Let $x\in X$ and $y,z\in X\cup\partial X$. We define $\langle y,z\rangle_x$ as the infimum of

$$\liminf_{n\to+\infty}\langle y_n,z_n\rangle_x$$

where (y_n) and (z_n) run over all sequences which respectively converge to y and z. This definition coincides with the original one when $y,z\in X$. Two points ξ and η of ∂X are equal if and only if $\langle \xi,\eta\rangle_x = +\infty$. Let $x\in X$. Let

(y_n) and (z_n) be two sequences of points of X respectively converging to y and z in $X \cup \partial X$. It follows from (2.1) that

$$\langle y, z\rangle_x \leqslant \liminf_{n\to+\infty} \langle y_n, z_n\rangle_x \leqslant \limsup_{n\to+\infty} \langle y_n, z_n\rangle_x \leqslant \langle y, z\rangle_x + k\delta, \tag{2.3}$$

where k is the number of points of $\{y, z\}$ that belong to ∂X. Moreover, for every $t \in X$, for every $x, y, z \in X \cup \partial X$, the four point inequality (2.1) leads to

$$\langle x, z\rangle_t \geqslant \min\left\{\langle x, y\rangle_t, \langle y, z\rangle_t\right\} - \delta. \tag{2.4}$$

The next lemma is an analog of Lemma 2.1 with one point in the boundary of X. It will be used in situations where the Gromov products $\langle x, \xi\rangle_s$, $\langle x, \xi\rangle_t$ and $\langle y, \xi\rangle_t$ are very small.

Lemma 2.2. — *Let $\xi \in \partial X$. Let x, y, s and t be four points of X. We have the following inequalities*

(i) $\langle x, \xi\rangle_t \leqslant \max\left\{|x-t| - \langle y, \xi\rangle_x, \langle x, y\rangle_t\right\} + \delta$,

(ii) $|s-t| \leqslant ||x-s| - |x-t|| + 2\max\left\{\langle x, \xi\rangle_s, \langle x, \xi\rangle_t\right\} + 3\delta$,

(iii) The distance $|s-t|$ is bounded above by

$$\max\Big\{ \langle x, \xi\rangle_s + \langle y, \xi\rangle_t + 2\delta, |x-y| + ||x-s| - |y-t|| + 2\max\{\langle x, \xi\rangle_s, \langle y, \xi\rangle_t\}\Big\} + 2\delta.$$

Proof. — Points (i) and (ii) follow directly from Lemma 2.1 (i) and (ii) combined with (2.3). Let us focus on Point (iii). By hyperbolicity we have

$$\min\left\{\langle x, t\rangle_s, \langle t, \xi\rangle_s\right\} \leqslant \langle x, \xi\rangle_s + \delta, \tag{2.5}$$

$$\min\left\{\langle y, s\rangle_t, \langle s, \xi\rangle_t\right\} \leqslant \langle y, \xi\rangle_t + \delta. \tag{2.6}$$

Assume that the minimum in (2.5) is achieved by $\langle x, t\rangle_s$. It follows that

$$|s-t| \leqslant |x-t| - |x-s| + 2\langle x, \xi\rangle_s + 2\delta.$$

Combined with the triangle inequality we obtain

$$|s-t| \leqslant |x-y| + ||x-s| - |y-t|| + 2\langle x, \xi\rangle_s + 2\delta.$$

The same kind of arguments hold if the minimum in (2.6) is achieved by $\langle y, s\rangle_t$. Therefore we can now assume that $\langle t, \xi\rangle_s \leqslant \langle x, \xi\rangle_s + \delta$ and $\langle s, \xi\rangle_t \leqslant \langle y, \xi\rangle_t + \delta$. For every $z \in X$, we have $|s-t| = \langle s, z\rangle_t + \langle t, z\rangle_s$. If follows from (2.3) that $|s-t| \leqslant \langle s, \xi\rangle_t + \langle t, \xi\rangle_s + 2\delta$. Consequently $|s-t| \leqslant \langle x, \xi\rangle_s + \langle y, \xi\rangle_t + 4\delta$. ☐

Lemma 2.3. — *Let $x \in X$ and $\xi \in \partial X$. For every $l \geqslant 0$, for every $\eta > 0$, there exists a point $y \in X$ such that $|x-y| = l$ and $\langle x, \xi\rangle_y \leqslant \delta + \eta$.*

Proof. — Let $l \geqslant 0$ and $\eta > 0$. Let (z_n) be a sequence of points of X which converges to ξ. In particular, there exists $N \in \mathbb{N}$ such that for all $n, m \geqslant N$, $\langle z_n, z_m \rangle_x \geqslant l$. We choose for y a point of X such that $|x-y| = l$ and $\langle x, z_N \rangle_y \leqslant \eta$. By Lemma 2.1 (i), we get for every $n \geqslant N$,

$$\langle x, z_n \rangle_y \leqslant \max\left\{ |x-y| - \langle z_N, z_n \rangle_x , \langle x, z_N \rangle_y \right\} + \delta \leqslant \langle x, z_N \rangle_y + \delta \leqslant \delta + \eta.$$

Consequently $\langle x, \xi \rangle_y \leqslant \delta + \eta$. □

2.2. Quasi-geodesics

Definition 2.4. — *Let $l \geqslant 0$, $k \geqslant 1$ and $L \geqslant 0$. Let $f : X_1 \to X_2$ be a map between two metric spaces X_1 and X_2. We say that f is a* (k, l)-quasi-isometric embedding *if for every $x, x' \in X_1$,*

$$k^{-1} |f(x) - f(x')| - l \leqslant |x - x'| \leqslant k |f(x) - f(x')| + l.$$

We say that f is an L-local (k, l)-quasi-isometric embedding *if its restriction to any subset of diameter at most L is a (k, l)-quasi-isometric embedding. Let I be an interval of $\mathbb{R}$. A path $\gamma : I \to X$ that is a (k, l)-quasi-isometric embedding is called a* (k, l)-quasi-geodesic. *Similarly, we define* L-local (k, l)-quasi-geodesics.

Remarks. — We assumed that our paths are rectifiable and parametrized by arc length. Thus a (k, l)-quasi-geodesic $\gamma : I \to X$ satisfies a more accurate property: for every $t, t' \in I$,

$$|\gamma(t) - \gamma(t')| \leqslant |t - t'| \leqslant k |\gamma(t) - \gamma(t')| + l.$$

In particular, if γ is a $(1, l)$-quasi-geodesic, then for every $t, t', s \in I$ with $t \leqslant s \leqslant t'$, we have $\langle \gamma(t), \gamma(t') \rangle_{\gamma(s)} \leqslant l/2$. Since X is a length space for every $x, x' \in X$, for every $l > 0$, there exists a $(1, l)$-quasi-geodesic joining x and x'.

Proposition 2.5 ([10, Proposition 2.4]). — *Let $\gamma : I \to X$ be a $(1, l)$-quasi-geodesic of X.*

(i) Let x be a point of X and p an η-projection of x on $\gamma(I)$. For all $y \in \gamma$, $\langle x, y \rangle_p \leqslant l + \eta + 2\delta$.

(ii) For every $x \in X$, for every $y, y' \in \gamma$, we have $d(x, \gamma) \leqslant \langle y, y' \rangle_x + l + 3\delta$.

Let $\gamma : \mathbb{R}_+ \to X$ be a (k, l)-quasi-geodesic. There exists a point $\xi \in \partial X$ such that for every sequence (t_n) diverging to infinity, $\lim_{n\to+\infty} \gamma(t_n) = \xi$. In this situation we consider ξ as an endpoint (at infinity) of γ and write $\lim_{t\to+\infty} \gamma(t) = \xi$.

Stability of quasi-geodesics. One important feature of hyperbolic spaces is the stability of quasi-geodesic paths recalled below.

Proposition 2.6 (Stability of quasi-geodesics [7, Chapitre 3, Th. 1.2, 1.4 et 3.1]). — *Let $k \geqslant 1$, $k' > k$ and $l \geqslant 0$. There exist L and D which only depend on δ, k, k' and l with the following properties*

(i) Every L-local (k, l)-quasi-geodesic is a (global) (k', l)-quasi-geodesic.

(ii) The Hausdorff distance between two L-local (k, l)-quasi-geodesics joining the same endpoints (possibly in ∂X) is at most D.

In this article we are mostly using L-local $(1, l)$-quasi-geodesics. For these paths one can provide a precise value for D (see next corollary). This is not really necessary but will decrease the number of parameters that we have to deal with in all the proofs.

Corollary 2.7 ([10, Corollaries 2.6 and 2.7]). — *Let $l_0 \geqslant 0$. There exists $L = L(l_0, \delta)$ which only depends on δ and l_0 with the following properties. Let $l \in [0, l_0]$. Let $\gamma\colon I \to X$ be an L-local $(1, l)$-quasi-geodesic.*

(i) The path γ is a (global) $(2, l)$-quasi-geodesic.

(ii) For every $t, t', s \in I$ with $t \leqslant s \leqslant t'$, we have $\langle \gamma(t), \gamma(t') \rangle_{\gamma(s)} \leqslant l/2 + 5\delta$.

(iii) For every $x \in X$, for every y, y' lying on γ, we have $d(x, \gamma) \leqslant \langle y, y' \rangle_x + l + 8\delta$.

(iv) The Hausdorff distance between γ and any other L-local $(1, l)$-quasi-geodesic joining the same endpoints (possibly in ∂X) is at most $2l + 5\delta$.

Remark. — Using a rescaling argument, one can see that the best value for the parameter $L = L(l, \delta)$ satisfies the following property: for all $l, \delta \geqslant 0$ and $\lambda > 0$, $L(\lambda l, \lambda\delta) = \lambda L(l, \delta)$. This allows us to define a parameter L_S that will be use all the way through.

Definition 2.8. — *Let $L(l, \delta)$ be the best value for the parameter $L = L(l, \delta)$ given in Corollary 2.7. We denote by L_S a number larger than 500 such that $L(10^5\delta, \delta) \leqslant L_S\delta$.*

Quasi-rays. If X is a proper geodesic space, thanks to the Azerlà-Ascoli Theorem, any two distinct points in $X \cup \partial X$ can be joined by a geodesic. Here, X is not necessarily proper or geodesic. Therefore we substitute this property for the following lemma.

Lemma 2.9. — *Let $x \in X$ and $\xi \in \partial X$. For every $L > 0$, for every $l > 0$, there exists an L-local $(1, l + 10\delta)$-quasi-geodesic joining x to ξ.*

Proof. — Let $L \geqslant L_S\delta$ and $\eta \in (0,\delta)$. According to Lemma 2.3 for every $n \in \mathbb{N}$, there exists a point $x_n \in X$ such that $|x - x_n| = nL$ and $\langle x, \xi \rangle_{x_n} \leqslant \eta + \delta$. By construction (x_n) converges to ξ. We claim that for every $n \in \mathbb{N}^*$,

$$|x_n - x_{n-1}| \geqslant L \quad \text{and} \quad \langle x_{n+1}, x_{n-1} \rangle_{x_n} \leqslant 2\eta + 5\delta.$$

Let $n \in \mathbb{N}^*$. First, the triangle inequality gives $|x_n - x_{n-1}| \geqslant L$ and $|x_{n+1} - x_{n-1}| \geqslant 2L$. On the other hand, applying Lemma 2.2 (ii) we get $|x_n - x_{n-1}| \leqslant L + 2\eta + 5\delta$. The claim is a consequence of these inequalities. For every $n \in \mathbb{N}$, we choose a $(1,\eta)$-quasi-geodesic γ_n joining x_n to x_{n+1}. We define $\gamma : \mathbb{R}_+ \to X$ as the concatenation of these paths. It follows from the previous inequalities that γ is an L-local $(1, 8\eta + 10\delta)$-quasi-geodesic. By choice of L, γ is also a $(2, 8\eta + 10\delta)$-quasi-geodesic (Corollary 2.7), thus it has an endpoint at infinity. Since (x_n) lies on γ, this endpoint is ξ. If η is chosen sufficiently small, γ is the desired path. ☐

2.3. Quasi-convex and strongly quasi-convex subsets

Definition 2.10. — *Let $\alpha \geqslant 0$. A subset Y of X is* α-quasi-convex *if for every $x \in X$, for every $y, y' \in Y$, $d(x, Y) \leqslant \langle y, y' \rangle_x + \alpha$.*

Since X is not a geodesic space our definition of quasi-convex slightly differs from the usual one (every geodesic joining two points of Y remains in the α-neighborhood of Y). However if X is geodesic, an α-quasi-convex subset in the usual sense is $(\alpha + 3\delta)$-quasi-convex in our sense and conversely. For instance it follows from the four point inequality (2.2) that any ball is 2δ-quasi-convex. According to Proposition 2.5 every $(1,l)$-quasi-geodesic is $(l + 3\delta)$-quasi-convex. If L is sufficiently large then every L-local $(1,l)$-quasi-geodesic is $(l + 8\delta)$-quasi-convex (Corollary 2.7). For our purpose we will also need a slightly stronger version of quasi-convexity.

Definition 2.11. — *Let $\alpha \geqslant 0$. Let Y be a subset of X connected by rectifiable paths. The length metric on Y induced by the restriction of $|\,.\,|_X$ to Y is denoted by $|\,.\,|_Y$. We say that Y is* strongly quasi-convex *if it is 2δ-quasi-convex and for every $y, y' \in Y$,*

$$|y - y'|_X \leqslant |y - y'|_Y \leqslant |y - y'|_X + 8\delta.$$

Remark. — The first inequality is just a consequence of the definition of $|\,.\,|_Y$. The second one gives a way to compare Y, seen as a length space, with X.

Lemma 2.12 ([7, Chapitre 10, Prop. 1.2], [10, Prop. 2.13]). — *Let Y be an α-quasi-convex subset of X. For every $A \geqslant \alpha$, the A-neighborhood of Y is 2δ-quasi-convex.*

Lemma 2.13. — *Let Y be an α-quasi-convex subset of X. Let $A > \alpha + 2\delta$. The* open *A-neighborhood of Y is strongly quasi-convex.*

Proof. — Let us denote by Z the *open* A-neighborhood of Y. Let $x \in X$. Let z_1 and z_2 be two points of Z. By definition there exist $y_1, y_2 \in Y$ such that $|y_1 - z_1|, |y_2 - z_2| < A$. It follows from the four point inequality (2.1) that

$$\min\left\{\langle z_1, y_1\rangle_x, \langle y_1, y_2\rangle_x, \langle y_2, z_2\rangle_x\right\} \leqslant \langle z_1, z_2\rangle_x + 2\delta.$$

Since Y is α-quasi-convex, $d(x, Y) \leqslant \langle y_1, y_2\rangle_x + \alpha < \langle y_1, y_2\rangle_x + A$. On the other hand, the triangle inequality gives

$$\langle z_1, y_1\rangle_x \geqslant |x - y_1| - |y_1 - z_1| > d(x, Y) - A.$$

In the same way $\langle z_2, y_2\rangle_x > d(x, Y) - A$. Hence $d(x, Y) < \langle z_1, z_2\rangle_x + A + 2\delta$. However X is a length-space. Thus $d(x, Z) \leqslant \langle z_1, z_2\rangle_x + 2\delta$. Consequently Z is 2δ-quasi-convex.

Let $\eta > 0$ such that $|y_1 - z_1| + \eta < A$, $|y_2 - z_2| + \eta < A$ and $A \geqslant \alpha + 2\delta + \eta$. We denote by γ_1 a $(1, \eta)$-quasi-geodesic joining y_1 to z_1. By choice of η, this path is contained in Z. We denote by x_1 a point of γ_1 such that $|x_1 - y_1| = \min\{A - 2\delta - \eta, |z_1 - y_1|\}$. In particular $|z_1 - x_1| \leqslant 2\delta + \eta$. We construct in the same way a $(1, \eta)$-quasi-geodesic γ_2 joining y_2 to z_2 and a point x_2 lying on γ_2. Let γ be a $(1, \eta)$-quasi-geodesic joining x_1 to x_2. Let p be a point lying on γ. By hyperbolicity we get

$$\min\{\langle x_1, y_1\rangle_p, \langle y_1, y_2\rangle_p, \langle y_2, x_2\rangle_p\} \leqslant \langle x_1, x_2\rangle_p + 2\delta \leqslant \eta/2 + 2\delta. \tag{2.7}$$

Since Y is α-quasi-convex, we have

$$d(p, Y) \leqslant \langle y_1, y_2\rangle_p + \alpha \leqslant \langle y_1, y_2\rangle_p + A - 2\delta - \eta \tag{2.8}$$

On the other hand, the triangle inequality yields

$$d(p, Y) \leqslant |p - y_1| \leqslant |x_1 - y_1| + \langle x_1, y_1\rangle_p \leqslant \langle x_1, y_1\rangle_p + A - 2\delta - \eta. \tag{2.9}$$

The same inequality holds with $\langle x_2, y_2\rangle_p$. Combining (2.7)-(2.9) we get $d(p, Y) < A$. In particular, γ is contained in Z. So are γ_1 and γ_2. Recall that $|z_1 - x_1| \leqslant 2\delta + \eta$ and $|z_2 - x_2| \leqslant 2\delta + \eta$. Hence there is a path of length at most $L(\gamma) + 4\delta + 3\eta$ joining z_1 to z_2 and contained in Z. By the triangle inequality $L(\gamma) \leqslant |z_1 - z_2| + 4\delta + 3\eta$. It follows that $|z_1 - z_2|_Z \leqslant |z_1 - z_2|_X + 8\delta + 6\eta$. This inequality holds for every sufficiently small η, hence Z is strongly quasi-convex. ☐

Lemma 2.14 (Projection on a quasi-convex [7, Chapitre 10, Prop. 2.1], [10, Lemma 2.12]). — *Let Y be an α-quasi-convex subset of X.*

(i) *If p is an η-projection of $x \in X$ on Y, then for all $y \in Y$, $\langle x, y \rangle_p \leqslant \alpha + \eta$.*
(ii) *Let $x, x' \in X$. If p and p' are respective η- and η'-projections of x and x' on Y, then*

$$|p - p'| \leqslant \max\left\{|x - x'| - |x - p| - |x' - p'| + 2\epsilon, \epsilon\right\},$$

where $\epsilon = 2\alpha + \eta + \eta' + \delta$.

The next two lemmas respectively generalize Lemma 2.12 and Lemma 2.13 of [10] where they are stated for the intersection of two quasi-convex subsets. However the proofs work exactly in the same way and are left to the reader.

Lemma 2.15 ([10, Lemma 2.12]). — *Let $Y_1, \dots, Y_m$ be a collection of subsets of X such that for every $j \in \{1, \dots, m\}$, Y_j is α_j-quasi-convex. We denote by Z the intersection*

$$Z = Y_1^{+\alpha_1 + 3\delta} \cap \ldots \cap Y_m^{+\alpha_m + 3\delta}$$

It is a 7δ-quasi-convex subset of X.

Lemma 2.16 ([10, Lemma 2.13]). — *Let $Y_1, \dots, Y_m$ be a collection of subsets of X such that for every $j \in \{1, \dots, m\}$, Y_j is α_j-quasi-convex. For all $A \geqslant 0$, we have*

$$\operatorname{diam}\left(Y_1^{+A} \cap \ldots \cap Y_m^{+A}\right) \leqslant \operatorname{diam}\left(Y_1^{+\alpha_1 + 3\delta} \cap \ldots \cap Y_m^{+\alpha_m + 3\delta}\right) + 2A + 4\delta.$$

Definition 2.17. — *Let Y be a subset of X. The* hull *of Y, denoted by* hull (Y), *is the union of all $(1, \delta)$-quasi-geodesics joining two points of Y.*

Lemma 2.18 ([10, Lemma 2.15]). — *Let Y be a subset of X. The hull of Y is 6δ-quasi-convex.*

3. Group acting on a hyperbolic space

3.1. Classification of the isometries

Let x be a point of X. An isometry g of X is either

- *elliptic*, i.e. the orbit $\langle g\rangle x$ is bounded,
- *loxodromic*, i.e. the map from $\mathbb{Z}$ to X that sends m to $g^m x$ is a quasi-isometry,
- or *parabolic*, i.e. it is neither loxodromic or elliptic.

Note that these definitions do not depend on the point x. In order to measure the action of g on X, we use two translation lengths. By the *translation length* $[g]_X$ (or simply $[g]$) we mean

$$[g]_X = \inf_{x\in X} |gx - x| \, .$$

The *asymptotic translation length* $[g]^\infty_X$ (or simply $[g]^\infty$) is

$$[g]^\infty_X = \lim_{n\to+\infty} \frac{1}{n} |g^n x - x| \, .$$

The isometry g is loxodromic if and only if its asymptotic translation length is positive [7, Chapitre 10, Proposition 6.3]. These two lengths are related as follows.

Proposition 3.1 ([7, Chapitre 10, Proposition 6.4]). — *Let g be an isometry of X. Its translation lengths satisfy*

$$[g]^\infty \leqslant [g] \leqslant [g]^\infty + 16\delta$$

Lemma 3.2 ([10, Lemma 2.22]). — *Let x, x' and y be three points of X. Let g be an isometry of X. Then $|gy - y| \leqslant \max\{|gx-x|, |gx'-x'|\} + 2\langle x, x'\rangle_y + 6\delta$.*

By construction, the group of isometries of X acts on the boundary at infinity ∂X of X. The different types of isometries of X can be characterized in terms of accumulation points in ∂X. Given a group G acting by isometries on X, we denote by ∂G the set of accumulation points of Gx in ∂X. Note that it does not depend on $x \in X$. It is also G-invariant. If g is a loxodromic isometry of X then $\partial\langle g\rangle$ contains exactly two points:

$$g^- = \lim_{n\to-\infty} g^n x \text{ and } g^+ = \lim_{n\to+\infty} g^n x$$

They are the only points of ∂X fixed by g [7, Chapitre 10, Proposition 6.6].

Definition 3.3. — *Let g be an isometry of X. Let $l \geqslant 0$. A path $\gamma\colon \mathbb{R} \to X$ is called an* l-nerve *of g if there exists $T \in \mathbb{R}$ with $[g] \leqslant T \leqslant [g] + l$ such that γ is a T-local $(1, l)$-quasi-geodesic and for every $t \in \mathbb{R}$, $\gamma(t + T) = g\gamma(t)$. The parameter T is called the* fundamental length *of γ.*

Remark. — For every $l > 0$, one can construct an l-nerve of g as follows. Let $\eta > 0$. There exists $x \in X$ such that $|gx - x| < [g] + \eta$. Let $\gamma : [0, T] \to X$ be a $(1, \eta)$-quasi-geodesic joining x to gx. In particular $[g] \leqslant T < [g] + 2\eta$. We extend γ into a path $\gamma : \mathbb{R} \to X$ in the following way: for every $t \in [0, T)$, for every $m \in \mathbb{Z}$, $\gamma(t + mT) = g^m\gamma(t)$. It turns out that γ is a T-local $(1, 2\eta)$-quasi-geodesic. Thus if η is chosen sufficiently small then γ is an l-nerve.

This kind of path will be used to simplify some proofs. Recall that L_S is the parameter given by the stability of quasi-geodesics (see Definition 2.8). If $[g] > L_S\delta$ (in particular g is hyperbolic) and $l \leqslant 10^5\delta$, by stability of quasi-geodesics γ is actually $(l + 8\delta)$-quasi-convex. Moreover it joins g^- to g^+. Thus it provides a g-invariant line than can advantageously be used as a substitution for an axis or a cylinder (see Definition 3.9 and Definition 3.12).

Recall that we did not assume that X was proper. Therefore there might exist unbounded subsets of Y of X such that ∂Y is empty. However this pathology does not happen if Y is the orbit of a group G. To prove this fact we need the following lemma.

Lemma 3.4 ([7, Chapitre 9, Lemme 2.3]). — *Let g and h be two isometries of X which are not loxodromic. If there exists a point $x \in X$ such that $|gx - x| \geqslant 2\langle gx, hx\rangle_x + 6\delta$ and $|hx - x| \geqslant 2\langle gx, hx\rangle_x + 6\delta$ then $g^{-1}h$ is loxodromic.*

Proposition 3.5. — *Let G be a group acting by isometries on X. Either one (and thus every) orbit of G is bounded or ∂G is non-empty.*

Proof. — Let x be point of X. Assume that, contrary to our claim, G is unbounded and ∂G is empty. In particular, G cannot contain a loxodromic element. On the other hand, there exists a sequence (g_n) of elements of G such that $\lim_{n\to+\infty} |g_nx - x| = +\infty$ and $\langle g_nx, g_mx\rangle_x$, $n \neq m$ is bounded. It follows from Lemma 3.4 that if n and m are sufficiently large distinct integers, then $g_n^{-1}g_m$ is a loxodromic element of G. Contradiction. □

Proposition 3.6. — *Let G be a group acting by isometries on X. If ∂G has at least two points then G contains a loxodromic isometry.*

Proof. — Let us denote by ξ and η two distinct points of ∂G. They are respectively limits of two sequences $(g_n x)$ and $(h_n x)$ where g_n and h_n belong to G. Thus the following holds.

- $\lim\limits_{n\to+\infty} |g_n x - x| = +\infty$ and $\lim\limits_{n\to+\infty} |h_n x - x| = +\infty$,
- $\limsup\limits_{n\to+\infty} \langle g_n x, h_n x\rangle_x \leqslant \langle \xi, \eta\rangle_x + 2\delta < +\infty$.

In particular, there exists $n \in \mathbb{N}$ such that $|g_n x - x| \geqslant 2\langle g_n x, h_n x\rangle_x + 6\delta$ and $|h_n x - x| \geqslant 2\langle g_n x, h_n x\rangle_x + 6\delta$. If g_n and h_n are not already loxodromic, then by Lemma 3.4 $g_n^{-1}h_n$ is. ☐

Corollary 3.7. — *An isometry g of X is parabolic if and only if $\partial\langle g\rangle$ has exactly one point.*

Lemma 3.8. — *Let G be a group acting by isometries on X. If ∂G has at least three points then G contains two loxodromic isometries g and h such that $\{g^-, g^+\} \neq \{h^-, h^+\}$.*

Proof. — By Proposition 3.6 G contains a loxodromic isometry g. We denote by g^- and g^+ the points of ∂X fixed by g. They belong to ∂G. According to the stability of quasi-geodesics (Corollary 2.7) the Hausdorff distance between two $L_S\delta$-local $(1,\delta)$-quasi-geodesics with the same end-points is at most 7δ. We denote by Y the union of all $L_S\delta$-local $(1,\delta)$-quasi-geodesics joining g^- and g^+. This set is non-empty (it contains the nerve of a sufficiently large power of g). Moreover $\partial Y = \{g^-, g^+\}$. We assume now that for every $u \in G$, we have $u\{g^-, g^+\} = \{g^-, g^+\}$. It follows that Y is G-invariant. Thus every point of ∂G is the limit of a sequence of points of Y. In other words ∂G is contained in $\{g^-, g^+\}$. Contradiction. Hence there exists $u \in G$ such that $u\{g^-, g^+\} \neq \{g^-, g^+\}$. The isometries g and $h = ugu^{-1}$ satisfy the conclusion of the lemma. ☐

3.2. Axis of an isometry

Definition 3.9. — *Let g be an isometry of X. The* axis *of g denoted by A_g is the set of points $x \in X$ such that $|gx - x| < [g] + 8\delta$.*

Remarks. — Note that we do not require g to be loxodromic. This definition works also for parabolic or elliptic isometries. For every $l \in (0, 4\delta)$, every l-nerve of g is contained in A_g. On the other hand, for every $x \in A_g$ there is a 8δ-nerve of g going through x.

Proposition 3.10 ([10, Proposition 2.24]). — *Let g be an isometry of X. Let x be a point of X.*

(i) $|gx - x| \geqslant 2d(x, A_g) + [g] - 6\delta$,
(ii) *if* $|gx - x| \leqslant [g] + A$, *then* $d(x, A_g) \leqslant A/2 + 3\delta$,
(iii) A_g *is* 10δ*-quasi-convex.*

Proposition 3.11. — *Let g be an isometry of X. Let ξ be a point of ∂X fixed by g. Let $x \in X$. Let $l \in [0\,, 10^5\delta]$ and $\gamma\colon \mathbb{R}_+ \to X$ be an $L_S\delta$-local $(1,l)$-quasi-geodesic joining x to ξ. There exists $t_0 \in \mathbb{R}_+$ such that for every $t \geqslant t_0$, $\gamma(t)$ lies in the $(l/2 + 31\delta)$-neighborhood of A_g. In particular ξ belongs to ∂A_g.*

Proof. — Assume that the statement is false. There exists $t > 2d(x, A_g) + 2l + 62\delta$ such that the distance between $y = \gamma(t)$ and A_g is larger than $l/2 + 31\delta$. In particular $|x - y| > d(x, A_g) + l/2 + 31\delta$ (Corollary 2.7). We are going to prove that under these assumption ξ cannot be a fixed point of g. Let p and q be respective δ-projections of x and y on A_g. We first claim that $\langle p, x\rangle_y > l/2 + 7\delta$. By projection on a quasi-convex (Lemma 2.14) we have

$$|p - q| \leqslant \max\left\{|x - y| - |x - p| - |y - q| + 46\delta, 23\delta\right\}.$$

Consequently, either $\langle x, y\rangle_p \leqslant 23\delta$ or $|p - q| \leqslant 23\delta$. Assume that $|p - q| \leqslant 23\delta$. Then the triangle inequality leads to

$$\begin{aligned}\langle p, x\rangle_y &\geqslant |x - y| - |x - p| \geqslant |x - y| - |x - q| - 23\delta \\ &\geqslant |x - y| - d(x, A_g) - 24\delta > l/2 + 7\delta.\end{aligned}$$

On the other hand if $\langle x, y\rangle_p \leqslant 23\delta$, then we get

$$\langle p, x\rangle_y = |y - p| - \langle x, y\rangle_p \geqslant d(y, A_g) - \langle x, y\rangle_p - \delta > l/2 + 7\delta,$$

which proves our claim.

Let z be a point of A_g. Recall that γ is an $L_S\delta$-local $(1,l)$-quasi-geodesic hence $\langle \xi, x\rangle_y \leqslant l/2 + 5\delta$ (Corollary 2.7). Applying the four point inequality (2.4) twice we get

$$\min\left\{\langle \xi, z\rangle_y, \langle z, p\rangle_y, \langle p, x\rangle_y\right\} \leqslant \langle \xi, x\rangle_y + 2\delta \leqslant l/2 + 7\delta.$$

Thanks to our previous claim the minimum cannot be achieved by $\langle p, x\rangle_y$. It cannot be achieved by $\langle z, p\rangle_y$ either. Indeed, A_g being 10δ-quasi-convex this would lead to $d(y, A_g) \leqslant \langle z, p\rangle_y + 10\delta \leqslant l/2 + 17\delta$, which contradicts our assumption. Thus $\langle \xi, z\rangle_y \leqslant l/2 + 7\delta$. It follows from (2.3) that

$$\langle y, \xi\rangle_z \geqslant |y - z| - \langle \xi, z\rangle_y - 2\delta \geqslant d(y, A_g) - \langle \xi, z\rangle_y - 2\delta > 10\delta.$$

Hence we proved that for every $z \in A_g$, $\langle y, \xi \rangle_z > 10\delta$. In particular, $\langle y, \xi \rangle_p > 10\delta$ and $\langle gy, g\xi \rangle_p > 10\delta$. By Proposition 3.10 we get

$$|gy - y| \geqslant 2d(y, A_g) + [g] - 6\delta \geqslant 2\,|y - p| + |gp - p| - 16\delta.$$

Thus $\langle gy, y \rangle_p \leqslant 8\delta$. Applying again the four point inequality (2.4) we obtain

$$\min\left\{ \langle gy, g\xi \rangle_p, \langle g\xi, \xi \rangle_p, \langle \xi, y \rangle_p \right\} \leqslant \langle gy, y \rangle_p + 2\delta \leqslant 10\delta.$$

However we proved before that the minimum cannot be achieved by $\langle gy, g\xi \rangle_p$ or $\langle \xi, y \rangle_p$. Hence $\langle g\xi, \xi \rangle_p \leqslant 10\delta$. Consequently g does not fix ξ, which contradicts our original assumption. □

Definition 3.12. — *Let g be a loxodromic isometry of X. We denote by Γ_g the union of all $L_S\delta$-local $(1, \delta)$-quasi-geodesics joining g^- to g^+. The* cylinder *of g, denoted by Y_g, is the* open 20δ-neighborhood *of Γ_g.*

Lemma 3.13. — *Let g be a loxodromic isometry of X. The cylinder of g is strongly quasi-convex.*

Proof. — According to Lemma 2.13, it is sufficient to prove that the union Γ_g of all $L_S\delta$-local $(1, \delta)$-quasi-geodesics joining g^- to g^+ is 16δ-quasi-convex. Let $y, y' \in \Gamma_g$ and $x \in X$. By definition there exist γ and γ' two $L_S\delta$-local $(1, \delta)$-quasi-geodesics joining g^- to g^+ such that y and y' respectively lie on γ and γ'. We denote by p a projection of y' on γ. By stability of quasi-geodesic, the Hausdorff distance between γ and γ' is at most 7δ (Corollary 2.7). Thus $|y' - p| \leqslant 7\delta$. As an $L_S\delta$-local $(1, \delta)$-quasi-geodesic γ is 9δ-quasi-convex hence

$$d(x, \Gamma_g) \leqslant d(x, \gamma) \leqslant \langle y, p \rangle_x + 9\delta \leqslant \langle y, y' \rangle_x + 16\delta.$$

Consequently, Γ_g is 16δ-quasi-convex. □

Lemma 3.14 ([10, Lemma 2.32]). — *Let g be a loxodromic isometry of X. Let Y be a g-invariant α-quasi-convex subset of X. Then the cylinder Y_g is contained in the $(\alpha + 42\delta)$-neighborhood of Y. In particular Y_g is contained in the 52δ-neighborhood of A_g.*

Lemma 3.15 ([10, Lemma 2.33]). — *Let g be an isometry of X such that $[g] > L_S\delta$. Let $l \in [0, \delta]$. Let γ be an $L_S\delta$-local $(1, l)$-quasi-geodesic of X joining g^- to g^+. Then A_g is contained in the $(l + 9\delta)$-neighborhood of γ. In particular A_g is contained in Y_g.*

The next lemma explains the following fact. Let g be a loxodromic isometry of X. A quasi-geodesic contained in the neighborhood of the axis of g almost behaves like a nerve of g.

Lemma 3.16 ([10, Lemma 2.34]). — *Let g be an isometry of X such that $[g] > L_S\delta$. Let $l \in [0, \delta]$ and $\gamma : [a, b] \to X$ be a $[g]$-local $(1, l)$-quasi-geodesic contained in the C-neighborhood of A_g. Then there exists $\epsilon \in \{\pm 1\}$ such that for every $s \in [a, b]$ if $s \leqslant b - [g]$ then*

$$|g^{\epsilon}\gamma(s) - \gamma(s + [g])| \leqslant 4C + 4l + 88\delta.$$

3.3. Weakly properly discontinuous action

From now on we fix a group G acting by isometries of X. Recall that we do not require X to be proper. Similarly we do not make for the moment any assumption on the action of G on X. In particular, it is not necessarily proper. Instead we use a weak notion of properness introduced by M. Bestvina and K. Fujiwara in [2].

Definition 3.17. — *A loxodromic element g of G satisfies the* weak proper discontinuity property *(WPD property) if for every $x \in X$, for every $l \geqslant 0$, there exists $n \in \mathbb{N}$ such that the set of elements $u \in G$ satisfying $|ux - x| \leqslant l$ and $|ug^n x - g^n x| \leqslant l$ is finite. The action of G on X is said to be* weakly properly discontinuous *(WPD) if every loxodromic element of G satisfies the WPD property.*

We are interested in situations where X is hyperbolic. In this context the WPD property follows from a local condition (see Proposition 3.19). Before proving this statement we start with the following lemma.

Lemma 3.18. — *Let g be a loxodromic element of G. Let $l \geqslant 0$. Assume that there exist $y, y' \in Y_g$ such that the set of elements $u \in G$ satisfying $|uy - y| \leqslant l + 110\delta$ and $|uy' - y'| \leqslant l + 110\delta$ is finite. Then there exists n_0 such that for every $x \in X$, for every $n \geqslant n_0$, the set of elements $u \in G$ satisfying $|ux - x| \leqslant l$ and $|ug^n x - g^n x| \leqslant l$ is finite.*

Proof. — We write S for the set of elements $u \in G$ satisfying $|uy - y| \leqslant l + 110\delta$ and $|uy' - y'| \leqslant l + 110\delta$. Since g is loxodromic, there exists $k \in \mathbb{N}$ such that $k[g]^{\infty} > L_S\delta$. We denote by $\gamma : \mathbb{R} \to X$ a δ-nerve of g^k and T its fundamental length. By stability of quasi-geodesics Y_g is contained in the 27δ-neighborhood of γ (Corollary 2.7). Therefore there exist $q = \gamma(s)$ and $q' = \gamma(s')$ such that $|y - q| \leqslant 27\delta$ and $|y' - q'| \leqslant 27\delta$. By swaping if necessary y and y', we can assume that $s \leqslant s'$. We choose for n_0 an integer such that $n_0[g]^{\infty} \geqslant |s' - s| + T + 73\delta$.

Let x be a point of X and $n \geqslant n_0$ an integer. We denote by p and r respective projections of x and $g^n x$ on γ. Without loss of generality we can assume that $p = \gamma(0)$. We write $r = \gamma(t)$. Let r' be a projection of $g^n p$ on γ (see Figure 3.1). By stability of quasi-geodesics, the Hausdorff distance

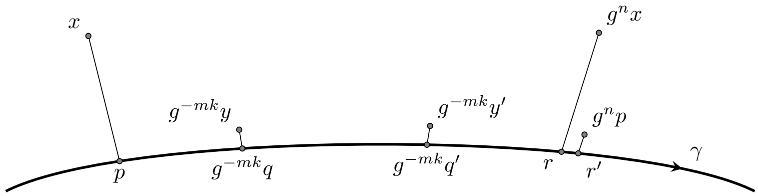

Figure 3.1. Projections on the δ-nerve γ

between γ and $g^n\gamma$ is at most 7δ, thus $|g^n p - r'| \leqslant 7\delta$ (Corollary 2.7). Moreover r' is a 14δ-projection of $g^n x$ on γ. It follows from the projection on quasi-convex subsets that $|r - r'| \leqslant 66\delta$. Consequently,

$$|t-0| \geqslant |r-p| \geqslant |g^n p - p| - 73\delta \geqslant n[g]^\infty - 73\delta \geqslant |s'-s| + T.$$

In particular there exists $m \in \mathbb{Z}$ such that $s - mT$ and $s' - mT$ are between 0 and t. Recall that γ is a δ-nerve of g^k, hence $g^{-mk}q = \gamma(s - mT)$ and $g^{-mk}q' = \gamma(s' - mT)$ are two points lying on γ between p and r. Using the projection on a quasi-convex combined with Corollary 2.7 we get

$$\langle x, g^n x\rangle_{g^{-mk}q} \leqslant 25\delta \quad \text{and} \quad \langle x, g^n x\rangle_{g^{-mk}q'} \leqslant 25\delta.$$

Let $u \in G$ such that $|ux - x| \leqslant l$ and $|ug^n x - g^n x| \leqslant l$. Lemma 3.2 yields $|ug^{-mk}q - g^{-mk}q| \leqslant l + 56\delta$. Consequently $|ug^{-mk}y - g^{-mk}y| \leqslant l + 110\delta$. Similarly we get $|ug^{-mk}y' - g^{-mk}y'| \leqslant l + 110\delta$. In other words ug^{-mk} belongs to S. Thus there are only finitely many $u \in G$ such that $|ux - x| \leqslant l$ and $|ug^n x - g^n x| \leqslant l$. □

Proposition 3.19. — *Let g be a loxodromic element of G. The isometry g satisfies the WPD property if and only if there exist $y, y' \in Y_g$ such that the set of elements $u \in G$ satisfying $|uy - y| \leqslant 486\delta$ and $|uy' - y'| \leqslant 486\delta$ is finite.*

Remark. — It follows in particular that a loxodromic element g satisfies the WPD property if and only if for every $n \in \mathbb{N}^*$, so does g^n. The proof follows the idea provided by F. Dahmani, V. Guirardel and D. Osin in [11] for the case of an acylindrical action.

Proof. — Assume first that g satisfies the WPD property. Fix a point y in Y_g. By assumption there exists $n \in \mathbb{N}$ such that the set of elements

$u \in G$ satisfying $|uy - y| \leqslant 486\delta$ and $|ug^ny - g^ny| \leqslant 486\delta$ is finite. Since Y_g is g-invariant $y' = g^ny$ is a point of Y_g. Consequently y and y' satisfy the the statement of the proposition.

Assume now that there exist $y, y' \in Y_g$ such that the set of elements $u \in G$ satisfying $|uy - y| \leqslant 486\delta$ and $|uy' - y'| \leqslant 486\delta$ is finite. Let $x \in X$ and $l \geqslant 0$. The element g being loxodromic there exists $k \in \mathbb{N}$ such that $k[g]^\infty > \max\{L_S\delta, l + 37\delta\}$. Let γ be a δ-nerve of g^k and T its fundamental length. We denote by p a projection of x on γ. For simplicity, we let $q = g^kp$ (which also lies on γ). According to Lemma 3.18, there exists $n_0 \in \mathbb{N}^*$ such that for every integer $n \geqslant n_0$ the set of elements $u \in G$ satisfying $|uq - q| \leqslant 376\delta$ and $|ug^{nk}q - g^{nk}q| \leqslant 376\delta$ is finite. We put $m = n_0k$ and $n = (n_0 + 2)k$.

We denote by S the set of elements $u \in G$ such that $|ux - x| \leqslant l$ and $|ug^nx - g^nx| \leqslant l$. We want to prove that S is finite. Let $N = \lceil (|x - q| + l)/\delta \rceil$. For every integer $i \in \{0, \dots, N\}$, we denote by x_i a point of X such that $|x - x_i| = i\delta$ and $\langle x, g^nx \rangle_{x_i} \leqslant \delta$ (see Figure 3.2). Such points exist because

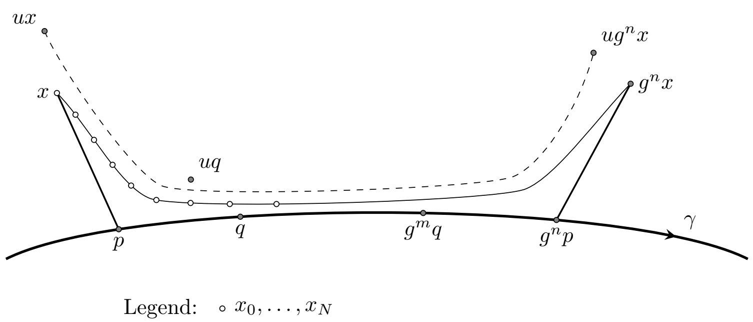

Figure 3.2. The points $x_0, \dots, x_N$

$|g^nx - x| \geqslant |x - q| + l$. Let $u \in S$. It follows from the projection on quasi-convex that $\langle x, g^nx \rangle_q \leqslant 25\delta$ and $\langle q, g^nx \rangle_{g^mq} \leqslant 15\delta$ whereas $|x - q|$ and $|g^nx - q|$ are at least $l + 28\delta$. By hyperbolicity, we have

$$\min\left\{\langle ux, x \rangle_{uq}, \langle x, g^nx \rangle_{uq}, \langle g^nx, ug^nx \rangle_{uq}\right\} \leqslant \langle ux, ug^nx \rangle_{uq} + 2\delta \leqslant 27\delta.$$

However $\langle ux, x \rangle_{uq} \geqslant |x - q| - |x - ux| > 27\delta$. Hence the minimum cannot be achieved by $\langle ux, x \rangle_{uq}$. With a similar argument we see that it cannot be achieved by $\langle g^nx, ug^nx \rangle_{uq}$ either. Therefore $\langle x, g^nx \rangle_{uq} \leqslant 27\delta$. By Lemma 2.1 (ii), $|uq - x_i| \leqslant ||x - uq| - i\delta| + 56\delta$. However, by triangle inequality $|x - uq| \leqslant |x - q| + l \leqslant N\delta$. Thus there exists $i \in \{0, \dots, N\}$ such that $|uq - x_i| \leqslant 57\delta$. Consequently there is $i \in \{0, \dots, N\}$ and a subset S_i of S such that for every $u \in S_i$, $|uq - x_i| \leqslant 57\delta$ and $\#S \leqslant (N+1)\#S_i$ (where $\#S$ denotes the cardinality, possibly infinite, of S).

Fix now $u_0 \in S_i$. Let $v \in u_0^{-1}S_i$. By construction $|vq - q| \leqslant 114\delta$ and $|vg^n x - g^n x| \leqslant 2l$. It follows from the triangle inequality that

$$\langle q, vg^n x\rangle_{vg^m q} \leqslant \langle vq, vg^n x\rangle_{vg^m q} + |vq - q| = \langle q, g^n x\rangle_{g^m q} + |vq - q| \leqslant 129\delta.$$

Applying Lemma 2.1 (iii) in the "triangle" $[q, g^n x, vg^n x]$ we obtain $|vg^m q - g^m q| \leqslant 376\delta$. Consequently for every $v \in u_0^{-1}S_i$, $|vq - q| \leqslant 114\delta$ and $|vg^m q - g^m q| \leqslant 376\delta$. It follows from the definition of m that S_i is finite. However S_i has been built in such a way that $\#S \leqslant (N+1)\#S_i$, therefore S is finite as well, which completes the proof. □

From now on we assume that the action of G on X is WPD.

Lemma 3.20. — *Let g be a loxodromic element of G. Let $x \in X$ and $l \geqslant 0$. The set of elements $u \in G$ satisfying $|ux - x| \leqslant l$ and $ug^+ = g^+$ is finite.*

Proof. — Without loss of generality we can assume that $[g] > L_S\delta$. We denote by γ a δ-nerve of g. Let p be a projection of x on γ. By definition of WPD property, there exists $n \in \mathbb{N}$ such that the set S of elements $u \in G$ satisfying $|up - p| \leqslant l + 34\delta$ and $|ug^n p - g^n p| \leqslant l + 34\delta$ is finite. By projection on a quasi-convex (Lemma 2.14) we have $\langle x, g^+\rangle_p \leqslant 9\delta$. Since γ is a δ-nerve of g, $g^n p$ lies on γ between p and g^+. It follows that $\langle x, g^+\rangle_{g^n p} \leqslant 15\delta$.

Let u be an element of G such that $|ux - x| \leqslant l$ and $ug^+ = g^+$. The estimates of the previous Gromov products give $\langle ux, g^+\rangle_{up} \leqslant 9\delta$ and $\langle ux, g^+\rangle_{ug^n p} \leqslant 15\delta$. Applying Lemma 2.2 (iii) we obtain

$$|up - p| \leqslant |ux - x| + 22\delta \leqslant l + 22\delta$$
$$\text{and } |ug^n p - g^n p| \leqslant |ux - x| + 34\delta \leqslant l + 34\delta.$$

Consequently u belongs to the finite set S. □

Definition 3.21. — *A subgroup H of G is called* elementary *if ∂H contains at most two points. Otherwise it is said* non-elementary.

Remark. — Note that this notion implicitly depends on the action of G on X. For instance a free group acting trivially on a hyperbolic space is not considered in this sense as a non-elementary groups. In the next lemmas we briefly recall how a free group quasi-isometrically embeds into any non-elementary subgroup of G.

Proposition 3.22. — *Let g and h be two loxodromic elements of G. Then $\{g^-, g^+\}$ and $\{h^-, h^+\}$ are either disjoint or equal.*

Proof. — By replacing if necessary g and h by some powers we can assume that $[g] > L_S\delta$ and $[h] > L_S\delta$. We suppose that $\{g^-, g^+\}$ and $\{h^-, h^+\}$ have one common point that we denote ξ. Let γ_g (respectively γ_h) be a δ-nerve of g (respectively h). We denote by T the fundamental length of γ_h. We fix a point x of γ_h and y a projection of x on γ_g. Since γ_g is 9δ-quasi-convex we have $\langle \xi, x\rangle_y \leqslant 9\delta$. In particular there exists a point z on γ_h such that $|y - z| \leqslant 19\delta$. Up to reparametrizing γ_h we can assume that $z = \gamma_h(0)$.

Let $p \in \mathbb{N}$. By replacing if necessary g by its inverse we can assume that $g^p y$ is a point of γ_g between y and ξ. In particular, $\langle \xi, z\rangle_{g^p y} \leqslant \langle \xi, y\rangle_{g^p y} + |y - z| \leqslant 25\delta$. The path γ_h being 9δ-quasi-convex, there exists a point s on γ_h such that $|g^p y - s| \leqslant 35\delta$. We can write $s = \gamma_h(r - qT)$ where $q \in \mathbb{Z}$ and $r \in [-T/2, T/2]$. It follows from the triangle inequality that

$$|h^q g^p y - y| \leqslant |g^p y - s| + |\gamma_h(r) - \gamma_h(0)| + |z - y| \leqslant T/2 + 54\delta.$$

The isometries g and h also fix the point ξ. According to Lemma 3.20 there exists a finite subset S of G with the following property. For every $p \in \mathbb{N}$, there is $q \in \mathbb{Z}$ such that $h^q g^p$ belongs S. Consequently there exist $p, q \in \mathbb{Z}^*$ such that $g^p = h^q$. It implies that $\{g^-, g^+\} = \{h^-, h^+\}$. $\square$

Lemma 3.23 ([12, Lemmes 1.1 and 1.2] or [22, Chapitre 5, Thm. 16]). *Let $k > 0$. Let $g_1, \dots, g_r$ be a collection of isometries of X. Let $x \in X$. We assume that for every $i, j \in \{1, \dots, r\}$, for every $\epsilon \in \{\pm 1\}$, if $g_i^{-\epsilon} g_j$ is not trivial then*

$$2\langle g_i^\epsilon x, g_j x\rangle_x < \min\{|g_i x - x|, |g_j - x|\} - 2\delta.$$

Then $g_1, \dots, g_r$ generate a free group $\mathbf{F}_r$ of rank r. Moreover the map $\mathbf{F}_r \to X$ which send $g \in \mathbf{F}_r$ to gx is a quasi-isometric embedding.

Remark. — Combined with Proposition 3.22 one consequence of this lemma is the following. A subgroup H of G is non-elementary if and only if it contains a copy of $\mathbf{F}_2$ such that for some (and thus every) $x \in X$, the map $\mathbf{F}_2 \to X$ that sends g to gx is a quasi-isometric embedding. Given two elements u and v of G we now state a sufficient condition under which they generate a non-elementary subgroup. Note that the assumptions allow u and v to be elliptic.

Lemma 3.24. — *Let $A \geqslant 0$. Let $u, v \in G$ and $x \in X$. We assume that*

(i) $2\left\langle u^{\pm 1}x, v^{\pm 1}x\right\rangle_x < \min\{|ux - x|, |vx - x|\} - A - 8\delta,$
(ii) $2\left\langle ux, u^{-1}x\right\rangle_x < |ux - x| + A,$
(iii) $2\left\langle vx, v^{-1}x\right\rangle_x < |vx - x| + A.$

Then the subgroup of G generated by u and v is non-elementary.

Proof. — Put $g_1 = uv$ and $g_2 = vu$. We are going to prove that g_1 and g_2 satisfy the assumptions of Lemma 3.23. First note that $|g_1x - x| = |ux - x| + |vx - x| - 2\left\langle u^{-1}x, vx\right\rangle_x$. In particular

$$|g_1x - x| > \max\left\{|ux - x|, |vx - x|\right\} + A + 8\delta.$$

The same inequality holds for g_2. On the other hand, the four point inequality (2.1) gives

$$\min\left\{\langle vx, g_2x\rangle_x, \left\langle g_2x, g_1^{-1}x\right\rangle_x, \left\langle g_1^{-1}x, v^{-1}x\right\rangle_x\right\} \leqslant \left\langle vx, v^{-1}x\right\rangle_x + 2\delta,$$

which leads to

$$\min\left\{\langle x, ux\rangle_{v^{-1}x}, \left\langle g_2x, g_1^{-1}x\right\rangle_x, \left\langle u^{-1}x, x\right\rangle_{vx}\right\} < \frac{1}{2}|vx - x| + \frac{1}{2}A + 2\delta.$$

Note that the minimum on the left hand side cannot be achieved by $\langle x, ux\rangle_{v^{-1}x}$. If it was the case we would have indeed

$$\begin{aligned}\frac{1}{2}|vx - x| + \frac{1}{2}A + 4\delta < |vx - x| - \left\langle v^{-1}x, ux\right\rangle_x &= \langle x, ux\rangle_{v^{-1}x} \\ &< \frac{1}{2}|vx - x| + \frac{1}{2}A + 2\delta.\end{aligned}$$

Similarly it cannot be achieved by $\left\langle u^{-1}x, x\right\rangle_{vx}$. Thus we get

$$\left\langle g_2x, g_1^{-1}x\right\rangle_x < \frac{1}{2}|vx - x| + \frac{1}{2}A + 2\delta < \frac{1}{2}\min\left\{|g_1x - x|, |g_2x - x|\right\} - 2\delta$$

With similar arguments we obtain the upper bound for the other Gromov products which are required to apply Lemma 3.23. Thus the subgroup of $\langle u, v\rangle$ generated by g_1 and g_2 is a free group of rank 2 which quasi-isometrically embeds into X. Therefore $\langle u, v\rangle$ is not elementary. □

3.4. Elementary subgroups

For some of the results in this section, the cited reference only provides a proof for the case of geodesic metric spaces. However, by relaxing if necessary some constants, which we do here, the same proof works in the more general context of length spaces.

Following the classification of isometries, we sort the elementary subgroups of G into three categories. A subgroup H of G is

(i) *elliptic* if its orbits are bounded;
(ii) *parabolic* if ∂H contains exactly one point;
(iii) *loxodromic* if ∂H contains exactly two points.

In this section we give a brief exposition of the properties of these subgroups. We still assume that the action of G on X is WPD.

Lemma 3.25. — *Let E be a subgroup of G and g an element of E. Assume that $\langle g\rangle$ is a finite index subgroup of E. Then E is elementary. Moreover E is elliptic (respectively parabolic, loxodromic) if and only if g is elliptic (respectively parabolic, loxodromic).*

Proof. — Let x be a point of X. Since $\langle g\rangle$ is a finite index subgroup of E, the Hausdorff distance between the orbits $\langle g\rangle x$ and Ex is finite. Therefore $\partial E = \partial\langle g\rangle$. The lemma follows from this equality. □

Elliptic subgroups.

Definition 3.26. — *Let H be an elliptic subgroup of G. The* characteristic set *associated to H is the following subset of X.*

$$C_H = \{x \in X \mid \forall h \in H,\ |hx - x| \leqslant 11\delta\},$$

Proposition 3.27 ([10, Corollaries 2.37 and 2.38]). — *The subset C_H is 9δ-quasi-convex. Let Y be a non-empty H-invariant α-quasi-convex subset of X. For every $A > \alpha$, the A-neighborhood of Y contains a point of C_H.*

Loxodromic subgroups. Let H be a loxodromic subgroup of G. According to Proposition 3.6, H contains a loxodromic isometry g. In particular, g^- and g^+ are exactly the two points of ∂H. Moreover H stabilizes ∂H. There exists a subgroup H^+ of H of index at most 2 which pointwise fixes ∂H. If $H^+ \neq H$ the subgroup H is said to be of *dihedral type*.

Lemma 3.28. — *Let g be a loxodromic element of G. Let E be the subgroup of G stabilizing $\{g^-, g^+\}$. Then E is a loxodromic subgroup of G. Moreover every elementary subgroup of G containing g lies in E.*

Proof. — By definition g belongs to E therefore ∂E contains $\{g^-, g^+\}$. If ∂E has an other point, then by Lemma 3.8 it contains an other loxodromic isometry h such that $\{h^-, h^+\} \neq \{g^-, g^+\}$. As an element of E, h^2 fixes g^- and g^+. On the other hand, since h is loxodromic the only points of ∂X fixed by h^2 are h^- and h^+. Contradiction. Therefore E is a loxodromic subgroup.

Let H be an elementary subgroup of G containing g. In particular g^- and g^+ belong to ∂H. Since H is elementary, there is no other point in ∂H. As we noticed H stabilizes ∂H, thus H is contained in E. □

Proposition 3.29. — *Let $g \in G$ be a loxodromic isometry and E the subgroup of G which stabilizes $\{g^-, g^+\}$. Then $\langle g\rangle$ is a finite index subgroup of E.*

Proof. — Note that it is sufficient to prove that $\langle g\rangle$ has finite index in E^+ – the subgroup of E fixing pointwise $\{g^+, g^-\}$. The isometry g is loxodromic. Thus, by replacing if necessary g by a power of g, we can assume that $[g] > L_S\delta$. Let $\gamma : \mathbb{R} \to X$ be a δ-nerve of g and T its fundamental length. The point x stands for $\gamma(0)$.

Let u be an element of E^+. By definition of E^+, $u\gamma$ is a T-local $(1,\delta)$-quasi-geodesic joining g^- to g^+. According to the stability of quasi-geodesics (Corollary 2.7), there exists a point p on γ such that $|ux - p| \leqslant 7\delta$. We can write $p = \gamma(r - mT)$ where $m \in \mathbb{Z}$ and $r \in [-T/2, T/2]$. It follows from the triangle inequality that

$$|g^m ux - x| \leqslant |ux - p| + |\gamma(r) - \gamma(0)| \leqslant T/2 + 7\delta.$$

The isometries u and g also fix the point g^+. By Lemma 3.20 there exists a finite subset S of G with the following property. For every $u \in E^+$, there is $m \in \mathbb{Z}$ such that $g^m u$ belongs to S. Thus $\langle g\rangle$ is a finite index subgroup of E^+. □

The next corollary is a well-known consequence of the previous proposition and a Schur Theorem [35, Theorem 5.32].

Corollary 3.30. — *Let H be a loxodromic subgroup of G. The set F of all elements of finite order of H^+ is a finite normal subgroup of H. Moreover there exists a loxodromic element $g \in H^+$ such that H^+ is isomorphic to $F \rtimes \mathbb{Z}$ where $\mathbb{Z}$ is the subgroup generated by g acting by conjugacy on F.*

Remark. — The subgroup F is the unique maximal finite subgroup of H^+. In addition, if H is of dihedral type then H is isomorphic to $F \rtimes \mathbf{D}_\infty$ where $\mathbf{D}_\infty$ stands for the infinite dihedral group $\mathbf{D}_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. In particular F is the unique maximal *normal* finite subgroup of H.

Definition 3.31. — *Let g be a loxodromic element of G. Let E be the subgroup of G stabilizing $\{g^-, g^+\}$ and F its maximal normal finite subgroup. We say that g is* primitive *if its image in $E^+/F \equiv \mathbb{Z}$ is -1 or 1.*

Corollary 3.32. — *Let A and B be two elementary subgroups of G which are not loxodromic. If A and B generate a loxodromic subgroup then it is necessarily of dihedral type.*

Proof. — Assume that the subgroup H generated by A and B is not of dihedral type. It follows from our previous discussion that H is isomorphic to the semi-direct product $F \rtimes \mathbb{Z}$ where F is a finite group and $\mathbb{Z}$ is generated by a loxodromic element g acting by conjugacy on F. Every element h of H can be written $h = g^m u$ with $m \in \mathbb{Z}$ and $u \in F$. Moreover h is loxodromic if and only if $m \neq 0$. Consequently every elliptic or parabolic element of H belongs to F (and thus has finite order). In particular, A and B are both contained in F. Therefore they cannot generate a loxodromic subgroup. Contradiction. □

Lemma 3.33 ([10, Lemma 2.39]). — *Let g be a hyperbolic element of G and H a subgroup of G fixing pointwise $\{g^-, g^+\}$. Let F be the maximal finite subgroup of H. The cylinder Y_g of g is contained in the 51δ-neighborhood of C_F.*

Parabolic subgroups.

Lemma 3.34. — *Let H be a parabolic subgroup of G. Let E be the subgroup of G fixing ∂H. Then $\partial E = \partial H$. In particular E is parabolic.*

Proof. — By construction E contains H. Therefore ∂H is a subset of ∂E. Assume now that ∂E has at least two points. By Proposition 3.6, E contains a loxodromic element g. This element fixes exactly two points of ∂X, g^- and g^+, one of them being the unique point of ∂H. Without loss of generality we can assume that $\partial H = \{g^+\}$. Let u be an element of H. The conjugate ugu^{-1} is a loxodromic element of E such that $(ugu^{-1})^+ = g^+$. According to Proposition 3.22, $(ugu^{-1})^- = g^-$. Hence u fixes pointwise $\{g^-, g^+\}$. By Proposition 3.29 the stabilizer of $\{g^-, g^+\}$ contains a finite subgroup F such that every non-loxodromic element fixing pointwise $\{g^-, g^+\}$ belongs to F. In particular H lies in F, which contradicts the fact that H is parabolic. □

To every elliptic subgroup F of G we associated a characteristic subset C_F. We would like to have an analogue of such a set for a parabolic group H. By definition, there is no point $x \in X$ which is moved by a small distance by all the elements of H. However this fact remains true for any finite subset of H. Let ξ be the unique point of ∂H. Given any finite subset S of H one can find indeed a sufficiently small horoball around ξ whose points are hardly moved by the elements of S. For our purpose we do not need to use horoball. The following statement will be sufficient.

Lemma 3.35. — *Let H be a parabolic subgroup of G and ξ the unique point of ∂H. Let $l \in [0, 10^5\delta]$. Let $\gamma : \mathbb{R}_+ \to X$ be an $L_S\delta$-local $(1, l)$-quasi-geodesic such that $\lim_{t\to+\infty} \gamma(t) = \xi$. Let S be a finite subset of $\operatorname{Stab}(\xi)$. There exists $t_0 \geqslant 0$ such that for every $t \geqslant t_0$, $|g\gamma(t) - \gamma(t)| \leqslant l + 86\delta$.*

Proof. — Note that it is sufficient to prove the lemma for a set S with a single element. Let us call it g. According to Proposition 3.11 there exists $t_0 \in \mathbb{R}_+$ such that for every $t \geqslant t_0$, $\gamma(t)$ lies in the $(l/2 + 31\delta)$-neighborhood of A_g. Let $t \geqslant t_0$ and $y = \gamma(t)$. Since g belongs to a parabolic subgroup it cannot be loxodromic. Hence $[g] \leqslant 16\delta$ (Proposition 3.1). It follows from the triangle inequality that $|gy - y| < 2d(y, A_g) + [g] + 8\delta \leqslant l + 86\delta$. □

3.5. Group invariants

We now introduce several invariants associated to the action of G on X. During the final induction they will be useful to ensure that the set of relations we are looking at satisfies a small cancellation assumption. In all this section we assume that the action of G on X is WPD.

Definition 3.36. — *The* injectivity radius *of G on X, denoted by $r_{inj}(G, X)$ is*

$$r_{inj}(G, X) = \inf \left\{ [g]^\infty \mid g \in G,\ g \text{ loxodromic} \right\}.$$

Let F be a finite group. Its *holomorph*, denoted by $\mathrm{Hol}(F)$, is the semi-direct product $F \rtimes \mathrm{Aut}(F)$, where $\mathrm{Aut}(F)$ stands for the automorphism group of F. The *exponent* of $\mathrm{Hol}(F)$ is the smallest integer n such that for every $g \in \mathrm{Hol}(F)$, $g^n = 1$.

Definition 3.37. — *The integer $e(G, X)$ is the least common multiple of the exponents of* $\mathrm{Hol}(F)$*, where F runs over the maximal finite normal subgroups of all maximal loxodromic subgroups of G.*

Remark. — If the loxodromic subgroups of G are all cyclic (for instance if G is torsion-free) then $e(G, X) = 1$.

Lemma 3.38 (Compare [27, Lemma 19]). — *Let n be an integer multiple of $e(G, X)$. Let E be a loxodromic subgroup of G and F its maximal finite normal subgroup. For every loxodromic element $g \in E$, for every $u \in F$, we have the following identities*

$$(ug)^n = g^n \quad \text{and} \quad ug^nu^{-1} = g^n.$$

Proof. — Without loss of generality we can assume that E is a maximal loxodromic subgroup of G. Let g be a loxodromic element of E and u an element of F. Recall that g acts by conjugacy on F. We denote by ψ the corresponding automorphism of F. The first identity is a consequence of the following observations.

$$\begin{aligned}(ug)^n &= u\left(gug^{-1}\right)\left(g^2ug^{-2}\right)\dots\left(g^{n-1}ug^{-(n-1)}\right)g^n \\ &= u\psi(u)\psi^2(u)\dots\psi^{n-1}(u)g^n.\end{aligned}$$

However in $\mathrm{Hol}(F)$ we have

$$\big(u\psi(u)\psi^2(u)\dots\psi^{n-1}(u),1\big) = (u,\psi)^n(1,\psi)^{-n} = 1.$$

Thus $(ug)^n = g^n$. Since F is a normal subgroup of F, $gu^{-1}g^{-1}$ also belongs to F. The previous identity yields

$$ug^nu^{-1} = \big(ugu^{-1}\big)^n = \big[(ugu^{-1}g^{-1})g\big]^n = g^n. \qquad \square$$

Proposition 3.39. — *Let n be an integer multiple of $e(G,X)$. Let g and h be two loxodromic elements of G which are primitive. Either g and h generate a non-elementary subgroup or $\langle g^n\rangle = \langle h^n\rangle$.*

Proof. — Let E be the subgroup of G stabilizing $\{g^-, g^+\}$. We write F for its maximal finite normal subgroup. Since g is primitive (see Definition 3.31), E^+ is isomorphic to the semi-direct product $F \rtimes \mathbb{Z}$ where $\mathbb{Z}$ is the subgroup generated by g acting by conjugacy on F. Assume that g and h generate an elementary subgroup. In particular h belongs to E and $\{h^-, h^+\} = \{g^-, g^+\}$. However being loxodromic, h fixes pointwise $\{g^-, g^+\}$ thus h belongs to E^+. The element h is also primitive, thus there exists $u \in F$ such that $g = uh^{\pm 1}$. It follows from Lemma 3.38 that $g^n = h^{\pm n}$, hence $\langle g^n\rangle = \langle h^n\rangle$. $\square$

Definition 3.40. — *The invariant $\nu(G,X)$ (or simply ν) is the smallest positive integer m satisfying the following property. Let g and h be two isometries of G with h loxodromic. If g, $h^{-1}gh$,..., $h^{-m}gh^m$ generate an elementary subgroup which is not loxodromic then g and h generate an elementary subgroup of G.*

Example. — If G acts properly co-compactly on a hyperbolic space X, then $\nu(G,X)$ is finite. Moreover if every elementary subgroup of G is cyclic then $\nu(G,X) = 1$ [13, Lemme 2.4.2]. Other examples are given in Section 6.3.

Proposition 3.41. — *Let g and h be two elements of G with h loxodromic. Let m be an integer such that g, $h^{-1}gh$,..., $h^{-m}gh^m$ generate an elementary (possibly loxodromic) subgroup of G. We assume that $m \geqslant \nu(G,X)$ and G has no involution. Then g and h generate an elementary subgroup of G.*

Proof. — We write H for the subgroup of G generated by g, $h^{-1}gh$,..., $h^{-m}gh^m$. We first assume that g is not loxodromic. We denote by p the largest integer such that g, $h^{-1}gh$,..., $h^{-p}gh^p$ generate an elementary subgroup which is not loxodromic, that we denote E. If $p \geqslant \nu(G,X)$, then by definition g and h generate an elementary subgroup. Therefore we can assume that $p \leqslant \nu(G,X) - 1 \leqslant m - 1$. Since p is maximal E and hEh^{-1} generate a loxodromic subgroup of H. According to Corollary 3.32, this loxodromic subgroup is of dihedral type. This is not possible since G has no involution. Consequently, we can assume that g is loxodromic. In particular, ∂H contains exactly two points g^- and g^+ which are also the accumulation points of $h^{-1}gh$. It follows that h stabilizes $\{g^-, g^+\}$. Consequently, g and h are contained in the elementary subgroup of G which stabilizes $\{g^-, g^+\}$ (Lemma 3.28). □

Notation 3.42. — If $g_0, \dots, g_m$ are m elements of G we denote by $A(g_0, \dots, g_m)$ the quantity

$$A(g_0, \dots, g_m) = \operatorname{diam}\left(A_{g_0}^{+13\delta} \cap \dots \cap A_{g_m}^{+13\delta}\right).$$

Recall that the parameter L_S is the constant given by the stability of quasi-geodesics (Definition 2.8).

Definition 3.43. — *Assume that $\nu = \nu(G,X)$ is finite. We denote by $\mathcal{A}$ the set of $(\nu+1)$-uples $(g_0, \dots, g_\nu)$ such that $g_0, \dots, g_\nu$ generate a non-elementary subgroup of G and for all $j \in \{0, \dots, \nu\}$, $[g_j] \leqslant L_S\delta$. The parameter $A(G,X)$ is given by*

$$A(G,X) = \sup_{(g_0,\dots,g_\nu)\in\mathcal{A}} A\left(g_0, \dots, g_\nu\right).$$

Proposition 3.44. — *Let g and h be two elements of G which generate a non-elementary subgroup.*

(i) If $[g] \leqslant L_S\delta$, then $A(g,h) \leqslant \nu[h] + A(G,X) + 154\delta$.

(ii) Without assumption on g we have,

$$A(g,h) \leqslant [g] + [h] + \nu \max\{[g],[h]\} + A(G,X) + 680\delta.$$

Remark. — If g is a loxodromic element such that $[g] \leqslant L_S\delta$, the same proof shows that

$$A(g,h) \leqslant [h] + A(G,X) + 154\delta.$$

Proof. — We prove Point (i) by contradiction. Assume that

$$A(g,h) > \nu[h] + A(G,X) + 154\delta.$$

Let $\eta \in (0,\delta)$ such that

$$A(g,h) > \nu([h] + \eta) + A(G,X) + 4\eta + 154\delta.$$

By definition of $A(G,X)$ we have $[h] > L_S\delta$, otherwise g and h would generate an elementary subgroup. We denote by $\gamma : \mathbb{R} \to X$ an η-nerve of h and by T its fundamental length. In particular, $T \leqslant [h]+\eta$. By Lemma 3.15, its $(\eta + 9\delta)$-neighborhood contains A_g, therefore applying Lemma 2.16, we get

$$\operatorname{diam}\left(A_g^{+13\delta} \cap \gamma^{+12\delta}\right) > \nu([h]+\eta) + A(G,X) + 2\eta + 106\delta.$$

In particular, there exist $x = \gamma(s)$ and $x' = \gamma(s')$ two points of γ which also belong to the 25δ-neighborhood of A_g and such that

$$|x - x'| > \nu([h]+\eta)+A(G,X)+2\eta+82\delta \geqslant \nu T+A(G,X)+2\eta+82\delta. \tag{3.1}$$

By replacing if necessary h by h^{-1} we can assume that $s \leqslant s'$. By stability of quasi-geodesics, for all $t \in [s\,,s']$, $\langle x, x'\rangle_{\gamma(t)} \leqslant \eta/2 + 5\delta$ (Corollary 2.7). The 25δ-neighborhood of A_g is 2δ-quasi-convex (Lemma 2.12). It follows that $\gamma(t)$ lies in the $(\eta/2 + 32\delta)$-neighborhood of A_g. Thus $|g\gamma(t) - \gamma(t)| \leqslant [g] + \eta + 72\delta$.

According to (3.1) there exists $t \in [s\,,s']$ such that $|x-\gamma(t)| = A(G,X)+ 2\eta + 82\delta$. We let $y = \gamma(t)$. Note that

$$|s' - t| \geqslant |x' - y| \geqslant |x - x'| - |x - y| \geqslant \nu T.$$

Let $m \in \{0,\dots,\nu\}$. By construction $h^m x = \gamma(s + mT)$ and $h^m y = \gamma(t + mT)$. Using our previous remark $s+mT$ and $t+mT$ belong to $[s\,,s']$. Hence

$$\max\left\{|gh^m x - h^m x|\,, |gh^m y - h^m y|\right\} \leqslant \left[h^m g h^{-m}\right] + \eta + 72\delta.$$

It follows from Proposition 3.10, that x and y belong to the $(\eta/2 + 39\delta)$-neighborhood of $h^m A_g$. This holds for every $m \in \{0,\dots,\nu\}$. Consequently x and y are two points of

$$A_g^{+\eta/2+39\delta} \cap \ldots \cap h^\nu A_g^{+\eta/2+39\delta}.$$

Applying Lemma 2.16, we obtain

$$A\left(g, hgh^{-1}, \dots, h^\nu g h^{-\nu}\right) \geqslant |x - y| - \eta - 82\delta > A(G,X).$$

Moreover, for every $m \in \{0,\dots,\nu\}$, we have $[h^m g h^{-m}] = [g] \leqslant L_S\delta$. By definition of $A(G,X)$ the isometries g, hgh^{-1}, $\dots$, $h^\nu g h^{-\nu}$ generate an elementary subgroup. It follows from Proposition 3.41 that g and h also generate an elementary subgroup. Contradiction.

We now prove Point (ii). According to the previous point we can assume that $[g] > L_S\delta$ and $[h] > L_S\delta$. Without loss of generality we can suppose $[h] \geqslant [g]$. Assume that contrary to our claim

$$A(g,h) > [g] + (\nu+1)[h] + A(G,X) + 680\delta.$$

Let $\eta \in (0,\delta)$ such that

$$A(g,h) > [g] + (\nu+1)[h] + A(G,X) + 15\eta + 680\delta.$$

We denote by γ an η-nerve of h and by T its fundamental length. Its $(\eta + 9\delta)$-neighborhood contains A_h thus

$$\operatorname{diam}\left(\gamma^{+12\delta} \cap A_g^{+13\delta}\right) > [g] + (\nu+1)[h] + A(G,X) + 13\eta + 632\delta.$$

In particular there exist $x = \gamma(s)$, $x' = \gamma(s')$ lying in the 25δ-neighborhood of A_g such that

$$|x - x'| > [g] + (\nu+1)[h] + A(G,X) + 13\eta + 608\delta.$$

Without loss of generality we can assume that $s \leqslant s'$. As previously, the restriction of γ to $[s\,, s']$ is contained in the $(\eta/2+32\delta)$-neighborhood of A_g. According to Lemma 3.16 by replacing if necessary g by g^{-1} the following holds. For every $t \in [s\,, s']$ if $t \leqslant s' - [g]$ then

$$|g\gamma(t) - \gamma(t + [g])| \leqslant 6\eta + 222\delta.$$

Consequently, for every $t \in [s\,, s']$ such that $t \leqslant s' - [g] - T$, we have

$$|gh\gamma(t) - hg\gamma(t)| \leqslant |g\gamma(t+T) - h\gamma(t+[g])| + 6\eta + 222\delta \leqslant 12\eta + 444\delta.$$

It follows that the translation length of the isometry $u = h^{-1}g^{-1}hg$ is at most $L_S\delta$ and for all $t \in [s\,, s']$, if $t \leqslant s' - [g] - T$, then $\gamma(t)$ is in the $(6\eta+225\delta)$-neighborhood of A_u. Let $y = \gamma(t)$ be a point such that $t \in [s\,, s']$ and $|x' - y| = [g] + T$. In particular,

$$|x - y| \geqslant |x - x'| - |x' - y| > \nu[h] + A(G,X) + 12\eta + 608\delta.$$

Moreover x and y belong to the $(6\eta + 225\delta)$-neighborhood of A_u and A_h. Therefore

$$A(g,u) \geqslant |x - y| - 12\eta - 454\delta > \nu[h] + A(G,X) + 154\delta.$$

It follows from the previous point that h and u generate an elementary subgroup. Hence so do h and $g^{-1}hg$. However h is a loxodromic isometry. Consequently g and h generate an elementary subgroup. Contradiction. $\square$

Corollary 3.45. — *Let m be an integer such that $m \leqslant \nu(G,X)$. Let $g_0, \dots, g_m$ be $m+1$ elements of G. If they do not generate an elementary subgroup, then*

$$A\left(g_0, \dots, g_m\right) \leqslant (\nu+2) \sup_{0 \leqslant i \leqslant m} [g_i] + A(G,X) + 680\delta.$$

Proof. — We distinguish two cases. If for every $i \in \{0, \dots, m\}$ we have $[g_i] \leqslant L_S\delta$, then it follows from the definition of $A(G,X)$ that $A\left(g_0, \dots, g_m\right) \leqslant A(G,X)$. Assume now that there exists $i \in \{0, \dots, m\}$ such that $[g_i] > L_S\delta$. In particular g_i is loxodromic. Suppose that the corollary is false. Then by Proposition 3.44, for every $j \in \{0, \dots, m\}$ the elements g_i and g_j generate an elementary subgroup. Therefore g_j belongs to the maximal elementary subgroup containing g_i. Consequently $g_0, \dots, g_m$ cannot generate a non-elementary subgroup. Contradiction. $\square$

4. Cone-off over a metric space

In this section we recall the so called *cone-off* construction. The goal is to build a metric space $\dot{X}$ obtained by attaching a family of cones on a base space X. In particular, we need to understand its curvature. Most of the results of this section follow from the general exposition given by the author in [10].

Let ρ be a positive number. Its value will be made precise later. It should be thought as a very large parameter.

4.1. Cone over a metric space

Definition 4.1. — *Let Y be a metric space. The* cone of radius ρ over Y*, denoted by $Z_\rho(Y)$ or simply $Z(Y)$, is the quotient of $Y \times [0,\rho]$ by the equivalence relation that identifies all the points of the form $(y,0)$.*

The equivalence class of $(y,0)$, denoted by v, is called the *apex* of the cone. By abuse of notation we still write (y,r) for the equivalence class of (y,r). The cone over Y is endowed with a metric characterized as follows [5, Chapter I.5, Proposition 5.9]. Let $x=(y,r)$ and $x'=(y',r')$ be two points of $Z(Y)$ then

$$\cosh|x-x'| = \cosh r \cosh r' - \sinh r \sinh r' \cos\theta\left(y,y'\right),$$

where $\theta(y,y')$ is the *angle at the apex* defined by $\theta(y,y') = \min\{\pi, |y-y'|/\sinh\rho\}$. If Y is a length space, then so is $Z(Y)$. This metric is modeled on the one of the hyperbolic place $\mathbf{H}$ (see [10] for the geometric interpretation). In particular the cone $Z(Y)$ is $2\boldsymbol{\delta}$-hyperbolic, where $\boldsymbol{\delta}$ is the hyperbolicity constant of $\mathbf{H}$ [10, Proposition 4.6].

In order to compare the cone $Z(Y)$ and its base Y we introduce a map $\iota : Y \to Z(Y)$ which sends y to (y,ρ). It follows from the definition of the metric on $Z(Y)$ that for all $y,y' \in Y$,

$$|\iota(y)-\iota(y')|_{Z(Y)} = \mu\left(|y-y'|_Y\right),$$

where $\mu\colon \mathbb{R}_+ \to \mathbb{R}_+$ is the map characterized as follows. For every $t \geqslant 0$,

$$\cosh\mu(t) = \cosh^2\rho - \sinh^2\rho\cos\left(\min\left\{\pi, \frac{t}{\sinh\rho}\right\}\right).$$

In addition, the map μ satisfies the following proposition whose proof is Calculus exercise.

Proposition 4.2. — *The map μ is continuous, concave, non-decreasing. Moreover, we have the followings.*

(i) For all $t \geqslant 0$, $t - \frac{1}{24}\left(1 + \frac{1}{\sinh^2 \rho}\right) t^3 \leqslant \mu(t) \leqslant t$.
(ii) For all $t \in [0, \pi \sinh \rho]$, $t \leqslant \pi \sinh(\mu(t)/2)$.

Lemma 4.3. — *Let $r \in [0, \rho]$. The map from Y to $Z(Y)$ which sends y to (y, r) is κ-Lipschtiz, with $\kappa = \sinh r / \sinh \rho$. In particular, if $\gamma : I \to Y$ is a rectifiable path then the path $\tilde{\gamma} : I \to Z(Y)$ defined by $\tilde{\gamma}(t) = (\gamma(t), r)$ is rectifiable and $L(\tilde{\gamma}) \leqslant \kappa L(\gamma)$.*

Proof. — Let y and y' be two points of Y. We let $x = (y, r)$ and $x' = (y', r)$. Assume first that that $|y - y'| \leqslant \pi \sinh \rho$. By definition of the metric on $Z(Y)$ we have

$$\cosh\left(|x - x'|\right) = 1 + \sinh^2 r \left[1 - \cos\left(\frac{|y - y'|}{\sinh \rho}\right)\right] \leqslant 1 + \frac{1}{2} \cdot \frac{\sinh^2 r}{\sinh^2 \rho} |y - y'|^2.$$

It follows that $|x - x'| \leqslant \kappa |y - y'|$, where $\kappa = \sinh r / \sinh \rho$. The same inequality holds if $|y - y'| > \pi \sinh \rho$. Thus the map $Y \to Z(Y)$ which sends y to (y, r) is κ-Lipschitz. The property about the path $\tilde{\gamma}$ follows from this fact. □

Group action on a cone. Let Y be a metric space endowed with an action by isometries of a group H. This action naturally extends to an action by isometries on $Z(Y)$ in the following way. For every point $x = (y, r)$ of $Z(Y)$, for every $h \in H$, we let $h \cdot x = (hy, r)$.

Lemma 4.4 ([10, Lemma 4.7]). — *Let Y be a metric space and H a group acting by isometries on Y. Assume that for every $h \in H \setminus \{1\}$, $[h] \geqslant \pi \sinh \rho$. Then for every point $x \in Z(Y)$, for every $h \in H \setminus \{1\}$, $|hx - x| = 2|x - v|$.*

Note that H fixes the apex v of the cone. Therefore this action is not necessarily proper (even if the one of H on Y is). One should think of H as a rotation group with center v. Nevertheless if H acts properly on Y, then the metric on $Z(Y)$ induces a distance on $Z(Y)/H$. Moreover the spaces $Z(Y)/H$ and $Z(Y/H)$ are isometric. For every point x in $Z(Y)$, we denote by $\bar{x}$ its image in $Z(Y)/H$.

Lemma 4.5 ([10, Lemma 4.8]). — *Let $l \geqslant 2\pi \sinh \rho$. We assume that for every $h \in H \setminus \{1\}$, $[h] \geqslant l$. Let $x = (y, r)$ and $x' = (y', r')$ be two points of $Z(Y)$. If $|y - y'|_Y \leqslant l - \pi \sinh \rho$ then $|\bar{x} - \bar{x}'| = |x - x'|$.*

4.2. The cone-off construction. Definition and curvature

We now explain how the cones introduced in the previous section can be attached on a metric space. Let X be a δ-hyperbolic length space. We consider a collection $\mathcal{Y}$ of strongly quasi-convex subsets of X. Let $Y \in \mathcal{Y}$. We denote by $|\,.\,|_Y$ the length metric on Y induced by the restriction of $|\,.\,|_X$ to Y. We write $Z(Y)$ for the cone of radius ρ over $(Y, |\,.\,|_Y)$. Its comes with a natural map $\iota : Y \hookrightarrow Z(Y)$ as defined in Section 4.1.

Definition 4.6. — *The* cone-off of radius ρ over X relative to $\mathcal{Y}$ *denoted by $\dot{X}_\rho(\mathcal{Y})$ (or simply $\dot{X}$) is obtained by attaching for every $Y \in \mathcal{Y}$, the cone $Z(Y)$ on X along Y according to ι.*

In other words the space $\dot{X}$ is the quotient of the disjoint union of X and all the $Z(Y)$ by the equivalence relation which identifies every point $y \in Y$ with its image $\iota(y) \in Z(Y)$. By abuse of notation, we use the same letter to designate a point of this disjoint union and its image in $\dot{X}$.

Metric on the cone-off. For the moment $\dot{X}$ is just a set of points. We now define a metric on $\dot{X}$ and recall its main properties. Note that we did not require the attachment maps ι to be isometries. We endow the disjoint union of X and all the $Z(Y)$ (where $Y \in \mathcal{Y}$) with the distance induced by $|\,.\,|_X$ and $|\,.\,|_{Z(Y)}$. This metric is not finite: the distance between two points in distinct components is infinite. Let x and x' be two points of $\dot{X}$. We define $\|x - x'\|$ to be the infimum over the distances between two points in the previous disjoint union whose images in $\dot{X}$ are respectively x and x'.

(i) Let $Y \in \mathcal{Y}$. If $x \in Z(Y) \setminus \iota(Y)$ and $x' \notin Z(Y)$, then $\|x - x'\| = +\infty$. In particular, $\|\,.\,\|$ is not a distance on $\dot{X}$ (it does not satisfy the triangle inequality).

(ii) Let x and x' be two points of X. Using the properties of μ (Proposition 4.2) we get
$$\mu\left(|x - x'|_X\right) \leqslant \|x - x'\| \leqslant |x - x'|_X .$$

Let x and x' be two points of $\dot{X}$. A *chain* between x and x' is a finite sequence $C = (z_1, \ldots, z_m)$ of points of $\dot{X}$ such that $z_1 = x$ and $z_m = x'$.

Its length, denoted by $l(C)$, is

$$l(C) = \sum_{j=1}^{m-1} \|z_{j+1} - z_j\| .$$

The following map endows $\dot{X}$ with a length metric [10, Proposition 5.10].

$$\begin{array}{rclcl} \dot{X} \times \dot{X} & \to & \mathbb{R}_+ & & \\ (x,x') & \to & |x-x'|_{\dot{X}} & = & \inf\left\{ l(C) | C \text{ chain between } x \text{ and } x' \right\} . \end{array}$$

For every $Y \in \mathcal{Y}$, the natural map $Z(Y) \to \dot{X}$ is a 1-Lipschitz embedding. The same holds for the map $X \to \dot{X}$. The next lemmas detail the relationship between the metric of these spaces.

Lemma 4.7 ([10, Lemma 5.8]). — *For every $x, x' \in X$, $\mu\left(|x-x'|_X\right) \leqslant |x-x'|_{\dot{X}} \leqslant |x-x'|_X$.*

Lemma 4.8 ([10, Lemma 5.7]). — *Let $Y \in \mathcal{Y}$. Let $x \in Z(Y) \setminus \iota(Y)$. Let $d(x,Y)$ be the distance between x and $\iota(Y)$ computed with $|\ .\ |_{Z(Y)}$. For all $x' \in \dot{X}$, if $|x-x'|_{\dot{X}} < d(x,Y)$ then x' belongs to $Z(Y)$. Moreover $|x-x'|_{\dot{X}} = |x-x'|_{Z(Y)}$.*

Remark. — If v stands for the apex of the cone $Z(Y)$, then the previous lemma implies that $Z(Y) \setminus \iota(Y)$ is exactly the ball of $\dot{X}$ of center v and radius ρ.

Large scale geometry of the cone-off. In [14] C. Drutu and M. Sapir introduced the notion of tree-graded spaces. If X is tree-graded with respect to $\mathcal{Y}$, then $\dot{X}$ has a very precise geometry: it is tree-graded with respect to $\{Z(Y)\ |Y \in \mathcal{Y}\}$ and $2\boldsymbol{\delta}$-hyperbolic. From a qualitative point of view some of the metric features of $\dot{X}(\mathcal{Y})$ still hold after a small "perturbation" of the geometry of X. To make this statement precise we need to introduce a parameter that controls the overlap between two elements of $\mathcal{Y}$. We let

$$\Delta(\mathcal{Y}) = \sup_{Y_1 \neq Y_2 \in \mathcal{Y}} \operatorname{diam}\left(Y_1^{+5\delta} \cap Y_2^{+5\delta}\right) .$$

Theorem 4.9 ([10, Proposition 6.4]). — *There exist positive numbers δ_0, Δ_0 and $\rho_0 > 10^{20}\boldsymbol{\delta}$ with the following property. Let X be a δ-hyperbolic length space with $\delta \leqslant \delta_0$. Let $\mathcal{Y}$ be a family of strongly quasi-convex subsets of X with $\Delta(\mathcal{Y}) \leqslant \Delta_0$. Let $\rho \geqslant \rho_0$. Then the cone-off $\dot{X}_\rho(\mathcal{Y})$ of radius ρ over X relative to $\mathcal{Y}$ is $\dot{\delta}$-hyperbolic with $\dot{\delta} \leqslant 900\boldsymbol{\delta}$.*

Remark. — It is important to note that in this statement the constants δ_0, Δ_0 and ρ_0 do not depend on X or $\mathcal{Y}$. Moreover δ_0 and Δ_0 (respectively ρ_0) can be chosen arbitrarily small (respectively large).

4.3. Group action on the cone-off

In this section ρ is a real number, X a δ-hyperbolic length space and $\mathcal{Y}$ a collection of strongly quasi-convex subsets of X. We assume that $\delta \leqslant \delta_0$, $\Delta(\mathcal{Y}) \leqslant \Delta_0$ and $\rho \geqslant \rho_0$ where δ_0, Δ_0 and ρ_0 are the constants given by Theorem 4.9. In particular, $\dot{X}$ is $\dot{\delta}$-hyperbolic with $\dot{\delta} \leqslant 900\boldsymbol{\delta}$.

Let G be a group acting by isometries on X. We assume that G acts by left translation on $\mathcal{Y}$. The action of G on X can be extended by homogeneity into an action on $\dot{X}$ as follows. Let $Y \in \mathcal{Y}$ and $x = (y, r)$ be a point of the cone $Z(Y)$. Let g be an element of G. Then gx is the point of the cone $Z(gY)$ defined by $gx = (gy, r)$. It follows from the definition of the metric on $\dot{X}$ that G acts by isometries on $\dot{X}$.

Recall that the map $X \to \dot{X}$ is 1-Lipschitz. Therefore if an element of G is elliptic (respectively parabolic) for the action of G on X, then it is elliptic (respectively parabolic or elliptic) for its action on $\dot{X}$.

Proposition 4.10. — *If the action of G on X is WPD, so is the one on $\dot{X}$.*

Proof. — We apply here the criterion provided by Proposition 3.19. Let g be an element of G which is loxodromic for its action on $\dot{X}$. Its cylinder Y_g in the cone-off $\dot{X}$ is unbounded, therefore it contains a point y in X. Being loxodromic as an isometry of $\dot{X}$, g is also loxodromic as an isometry of X. In particular, it satisfies the WPD property. Consequently there exists $n \in \mathbb{N}$ such that the set S of elements $u \in G$ satisfying $|uy - y|_X \leqslant \pi \sinh(243\dot{\delta})$ and $|ug^n y - g^n y|_X \leqslant \pi \sinh(243\dot{\delta})$ is finite. Note that the point $y' = g^n y$ also belongs to $Y_g \subset \dot{X}$. Let $u \in G$ such that $|uy - y|_{\dot{X}} \leqslant 486\dot{\delta}$ and $|uy' - y'|_{\dot{X}} \leqslant 486\dot{\delta}$. It follows from Lemma 4.7 that

$$\mu\left(|uy - y|_X\right) \leqslant |uy - y|_{\dot{X}} \leqslant 486\dot{\delta} < 2\rho.$$

By Proposition 4.2, $|uy - y|_X \leqslant \pi \sinh(243\dot{\delta})$. Similarly we get $|uy' - y'|_X \leqslant \pi \sinh(243\dot{\delta})$. Thus u belongs to the finite set S. By Proposition 3.19, g is WPD for the action of G on $\dot{X}$. □

For the rest of this section, we assume that the action of G on X (and thus on $\dot{X}$) is WPD. We now study how the type of an elementary subgroup of G for its action on X is related to the the type of the same subgroup for the action of G on $\dot{X}$

Lemma 4.11. — *Let H be a subgroup of G. If H is elliptic (respectively parabolic, loxodromic) for the action on X, then H is elliptic (respectively parabolic or elliptic, elementary) for the action on $\dot{X}$.*

Proof. — We use one more time the fact that the map $X \to \dot{X}$ is 1-Lipschitz. In particular, it directly gives that if H is elliptic for the action on X so is it for the action on $\dot{X}$. Assume now that H is parabolic for the action on X. Since $\partial H \subset \partial X$ has only one point, H does not contain a loxodromic element for the action on X, and thus for the action on $\dot{X}$. According to Proposition 3.6 (applied in $\dot{X}$) H is either parabolic or elliptic. Assume now that H is loxodromic for the action on X. By Proposition 3.29, H contains a loxodromic element g such that $\langle g \rangle$ has finite index in H. It follows from Lemma 3.25 that H is elementary for the action of G on $\dot{X}$. ☐

Proposition 4.12. — *Let H be a subgroup of G. If H is parabolic for its action on $\dot{X}$, then so is it for its action on X.*

Proof. — We denote by ξ the unique point of $\partial H \subset \partial \dot{X}$. According to Lemma 2.9 there exists an $L_S\dot{\delta}$-local $(1, 11\dot{\delta})$-quasi-geodesic $\gamma : \mathbb{R}_+ \to \dot{X}$ such that $\lim_{t\to+\infty} \gamma(t) = \xi$. Let $g \in H$. By Lemma 3.35 there is t_0 such that for every $t \geqslant t_0$, $|g\gamma(t) - \gamma(t)|_{\dot{X}} \leqslant 97\dot{\delta}$. Since the path γ is infinite there exists $t \geqslant t_0$ such that $x = \gamma(t)$ lies in X. We obtain

$$\mu\left(|gx - x|_X\right) \leqslant |gx - x|_{\dot{X}} \leqslant 97\dot{\delta} < 2\rho.$$

Hence $|gx - x|_X \leqslant \pi \sinh(49\dot{\delta})$ (Proposition 4.2). Consequently, for every $g \in H$, $[g]_X \leqslant \pi \sinh(49\dot{\delta})$. Therefore H cannot contain a loxodromic element for its action on X. By Proposition 3.6, H is either elliptic or parabolic for this action. It follows from Lemma 4.11 that H is parabolic. ☐

5. Small cancellation theory

5.1. Small cancellation theorem

In this section X is a δ-hyperbolic length space, endowed with an action by isometries of a group G. We assume that the action of G on X is WPD and that G is non-elementary. We consider a family $\mathcal{Q}$ of pairs (H, Y) such that Y is a strongly quasi-convex subset of X and H a subgroup of the stabilizer $\mathrm{Stab}(Y)$ of Y. We suppose that G acts on $\mathcal{Q}$ and $\mathcal{Q}/G$ is finite. The action of G on $\mathcal{Q}$ is defined as follows: for every $g \in G$, for every $(H, Y) \in \mathcal{Q}$, $g(H, Y) = (gHg^{-1}, gY)$. We denote by K the (normal) subgroup of G generated by the subgroups H with $(H, Y) \in \mathcal{Q}$. The goal is to understand the action of the quotient $\bar{G} = G/K$ on an appropriate space. We use here the small cancellation theory.

In order to control the small cancellation parameters at each step during the final induction (see Proposition 6.1 and Theorem 6.9) we will not use the properties of the whole group G but only of a normal subgroup . To that end, we need additional assumptions on the subgroups H that can be stated as follows. Let N be a normal subgroup of G *without involution*. As a subgroup of G, the action of N on X is WPD. Note that the definition of a primitive element (Definition 3.31) depends on the ambient group. Let g be a loxodromic element of N. The maximal elementary subgroup of N containing g is a priori smaller than the one of G with the same property. Consequently g might be primitive viewed as an element of N but a proper power as an element of G. Keeping this subtlety in mind we can now state our last assumptions. Let $n \geqslant 100$ be an odd integer. We suppose that for every $(H, Y) \in \mathcal{Q}$, there exists a loxodromic element $h \in G$ which is the n-th power of *a primitive element of* N such that

(i) H is the cyclic subgroup generated by h,
(ii) Y is the cylinder Y_h of h.

By assumption K is contained in N. We denote by $\bar{N}$ the image N/K of N in $\bar{G}$.

Remark 5.1. — Let $(H, Y) \in \mathcal{Q}$. By construction $\operatorname{Stab}(Y)$ is a loxodromic subgroup of G. In particular it admits a maximal normal finite subgroup F (see Corollary 3.30). Every element $u \in F$ fixes pointwise ∂Y. Since N has no involution, every element of $\operatorname{Stab}(Y) \cap N$ also fixes pointwise ∂Y. In particular it is either elliptic and thus belongs to F or loxodromic. Said differently, the set of elliptic elements of $\operatorname{Stab}(Y) \cap N$ is a subgroup of F. We will very often use this property later. According to Proposition 3.29, H has finite index in $\operatorname{Stab}(Y)$. Thus $\operatorname{Stab}(Y)/H$ is finite.

Note also that for every (H, Y), Y is strongly quasi-convex (Lemma 3.13). Therefore we can apply the cone-off construction described in Section 4 with the space X and the collection $\mathcal{Y} = \{Y \mid (H, Y) \in \mathcal{Q}\}$. Let $\rho > 0$. We denote by $\dot{X}$ the cone-off of radius ρ over X relative to the collection $\mathcal{Y}$. As we explained previously, G acts by isometries on $\dot{X}$. The space $\bar{X}$ is defined to be the quotient of $\dot{X}$ by K. It is endowed with an action on $\bar{G}$. We denote by $\zeta : \dot{X} \to \bar{X}$ the canonical map from $\dot{X}$ to $\bar{X}$. We write $v(\mathcal{Q})$ for the subset of $\dot{X}$ consisting in all apices of the cones $Z(Y)$ where $(H, Y) \in \mathcal{Q}$. Its image in $\bar{X}$ is denoted by $\bar{v}(\mathcal{Q})$.

To study the action of $\bar{G}$ on $\bar{X}$ we consider two parameters which respectively play the role of the length of the largest piece and the length of the smallest relation in the usual small cancellation theory. Both quantities are measured with the metric of X.

$$\Delta(\mathcal{Q}) = \sup\left\{ \operatorname{diam}\left(Y_1^{+5\delta} \cap Y_2^{+5\delta}\right) \;\middle|\; (H_1, Y_1) \neq (H_2, Y_2) \in \mathcal{Q}\right\},$$
$$T(\mathcal{Q}) = \inf\left\{[h] \mid h \in H, (H, Y) \in \mathcal{Q}\right\}.$$

Theorem 5.2 (Small cancellation theorem [10, Proposition 6.7]). — *There exist positive constants δ_0, Δ_0 and ρ_0 which do not depend on X, G or $\mathcal{Q}$ and satisfying the following property. Assume that $\delta \leqslant \delta_0$, $\rho \geqslant \rho_0$. If in addition $\Delta(\mathcal{Q}) \leqslant \Delta_0$ and $T(\mathcal{Q}) \geqslant 8\pi \sinh \rho$ then the following holds.*

(i) The cone-off $\dot{X}$ is a $\dot{\delta}$-hyperbolic length space with $\dot{\delta} \leqslant 900\boldsymbol{\delta}$.
(ii) The space $\bar{X}$ is a $\bar{\delta}$-hyperbolic length space with $\bar{\delta} \leqslant 54.10^4\boldsymbol{\delta}$.
(iii) The group $\bar{G}$ acts by isometries on $\bar{X}$.
(iv) For every $(H, Y) \in \mathcal{Q}$, the projection $G \twoheadrightarrow \bar{G}$ induces an isomorphism from $\operatorname{Stab}(Y)/H$ onto its image.

Remarks. — By increasing if necessary $\bar{\delta}$ we can assume that $\dot{\delta} \leqslant \bar{\delta}$, thus $\dot{X}$ is also $\bar{\delta}$-hyperbolic. This is not really accurate, however it will allow us to decrease the number of parameters we have to deal with. As in Theorem 4.9, the constants δ_0 and Δ_0 (respectively ρ_0) can be chosen arbitrarily small (respectively large). From now on, we will always assume that $\rho_0 > 10^{20} L_S\boldsymbol{\delta}$ whereas $\delta_0, \Delta_0 < 10^{-10}\boldsymbol{\delta}$. These estimates are absolutely not optimal. We chose them very generously to be sure that all the inequalities that we might need later will be satisfied. What really matters is their orders of magnitude recalled below.

$$\max\left\{\delta_0, \Delta_0\right\} \ll \boldsymbol{\delta} \ll \rho_0 \ll \pi \sinh \rho_0.$$

An other important point to remember is the following. The constants δ_0, Δ_0 and $\pi \sinh \rho_0$ are used to describe the geometry of X whereas $\boldsymbol{\delta}$ and ρ_0 refers to the one of $\dot{X}$ or $\bar{X}$. From now on and until the end of Section 5 we assume that X, G and $\mathcal{Q}$ are as in Theorem 5.2. In particular $\dot{X}$ and $\bar{X}$ are $\bar{\delta}$-hyperbolic.

Notation. — In this section we work with three metric spaces namely X, its cone-off $\dot{X}$ and the quotient $\bar{X}$. Since the map $X \hookrightarrow \dot{X}$ is an embedding we use the same letter x to designate a point of X and its image in $\dot{X}$. We write $\bar{x}$ for its image in $\bar{X}$. Unless stated otherwise, we keep the notation $|\,.\,|$ (without mentioning the space) for the distances in X or $\bar{X}$. The metric on $\dot{X}$ will be denoted by $|\,.\,|_{\dot{X}}$.

5.2. The geometry of $\bar{X}$

In this section we look more closely at the geometric features of the space $\bar{X}$.

Quasi-geodesics in $\bar{X}$. We study here the quasi-geodesics of $\bar{X}$. We explain how to build quasi-geodesic paths of $\bar{X}$ that avoid the set of apices $\bar{v}(\mathcal{Q})$. In addition, we prove that the set $\bar{v}(\mathcal{Q})$ of apices of $\bar{X}$ contains at least 2 elements.

Proposition 5.3 ([10, Corollary 3.12]). — *The space $\dot{X} \setminus v(\mathcal{Q})$ is a covering space of $\bar{X} \setminus \bar{v}(\mathcal{Q})$. Let $r \geqslant 0$ and $x \in \dot{X}$. If for every $v \in v(\mathcal{Q})$, $|v - x|_{\dot{X}} \geqslant r$, then for every $g \in K \setminus \{1\}$, $|gx - x|_{\dot{X}} \geqslant \min\{2r, \rho/5\}$.*

Proposition 5.4 ([10, Proposition 3.15]). — *Let $r \in (0, \rho/20]$ and $x \in \dot{X}$. If for every $v \in v(\mathcal{Q})$, $|v - x|_{\dot{X}} \geqslant 2r$, then the map $\zeta : \dot{X} \to \bar{X}$ induces an isometry from $B(x, r)$ onto $B(\bar{x}, r)$.*

Remark. — On important consequence of this proposition is the following. If $\gamma \colon I \to \dot{X}$ is a $(1,l)$-quasi-geodesic of $\dot{X}$ that stays in the d-neighborhood of X, then for every $L < \min\{\rho - d, \rho/10\}$, the path $\bar{\gamma} \colon I \to \bar{X}$ induced by γ is an L-local $(1,l)$-quasi-geodesic of $\bar{X}$. In particular, if d and l are sufficiently small, we can apply the stability of quasi-geodesics (Corollary 2.7) to the path $\bar{\gamma}$.

Lemma 5.5. — *Let $(H, Y) \in \mathcal{Q}$ and $r \in (0, \rho)$. We denote by v the apex of the cone $Z(Y)$ and by h a generator of H. Let $\bar{x}, \bar{x}' \in \bar{X} \setminus \{\bar{v}\}$ such that $|\bar{x} - \bar{v}| \leqslant r$ and $|\bar{x}' - \bar{v}| \leqslant r$. There exists a path $\bar{\gamma} : I \to \bar{X}$ joining $\bar{x}$ and $\bar{x}'$ such that*

(i) for every $t \in I$, $0 < |\bar{\gamma}(t) - \bar{v}| \leqslant r$,
(ii) $\bar{\gamma}$ is rectifiable and its length is at most $(\sinh r / \sinh \rho)[h]$.

Proof. — By construction, the ball $B(\bar{v}, \rho)$ is the image of $Z(Y) \setminus \iota(Y)$ in $\bar{X}$. In particular $\bar{x}$ and $\bar{x}'$ are the respective images of some points $x = (y, s)$ and $x' = (y', s')$ of $Z(Y)$. Without loss of generality we can assume that $0 < s \leqslant s' \leqslant r$. The subset Y (which is the cylinder of h) is 27δ-close to any δ-nerve of h (Corollary 2.7). Therefore, by translating if necessary x' by h we can always assume that $|y - y'| \leqslant [h]/2 + 55\delta$. Since Y is strongly quasi-convex there exists a path $\gamma : I \to Y$ joining y to y' whose length (as a path of X) is at most

$$L(\gamma) \leqslant |y - y'| + 9\delta \leqslant \frac{1}{2}[h] + 64\delta.$$

We define the path $\gamma_1 : I \to Z(Y)$ by $\gamma_1(t) = (\gamma(t), s)$. It joins x to (y', s). By Lemma 4.3, the length of γ_1 (as a path of $\dot{X}$) is at most $(\sinh s/\sinh\rho)([h]/2 + 64\delta)$. Moreover for every $t \in I$, $|\gamma_1(t) - v|_{\dot{X}} = s$. We write $\gamma_2 : [s\,, s'] \to \dot{X}$ for the radial path defined by $\gamma_2(t) = (y', s)$. It joins (y', s) to x'. Its length (as a path of $\dot{X}$) is at most $s' - s \leqslant r$ and for every $t \in [s\,, s']$, $|\gamma_2(t) - v|_{\dot{X}} = t$. We choose for $\bar{\gamma}$ the path of $\bar{X}$ induced by the concatenation of γ_1 and γ_2. Recall that $[h] \geqslant 8\pi \sinh\rho$ (Theorem 5.2). Hence the length of $\bar{\gamma}$ is bounded above as follows.

$$\begin{aligned} L(\bar{\gamma}) \leqslant L(\gamma_1) + L(\gamma_2) &\leqslant \frac{\sinh s}{\sinh\rho}\left(\frac{1}{2}[h] + 64\delta\right) + s' - s \\ &\leqslant \frac{\sinh r}{\sinh\rho}\left(\frac{1}{2}[h] + 64\delta\right) + r \\ &\leqslant \frac{\sinh r}{\sinh\rho}[h]. \end{aligned}$$

The other properties of $\bar{\gamma}$ follow from the construction of γ_1 and γ_2. □

Lemma 5.6. — *For every $\bar{x}, \bar{x}' \in \bar{X}\setminus\bar{v}(\mathcal{Q})$, for every $l > 0$, there exists a $(1, l)$-quasi-geodesic of $\bar{\gamma} : I \to \bar{X}$ joining $\bar{x}$ to $\bar{x}'$ such that for every $t \in I$, $\bar{\gamma}(t)$ does not belong to $\bar{v}(\mathcal{Q})$.*

Proof. — By assumption $\mathcal{Q}/G$ is finite. Therefore there exists $D \geqslant 0$ such that for every $(H, Y) \in \mathcal{Q}$, if h is a generator of H then $[h] \leqslant D$. Let $\bar{x}$ and $\bar{x}'$ be two points of $\bar{X}$. Two apices of $\bar{v}(\mathcal{Q})$ are at least 2ρ far apart from each other. Therefore there is only a finite number, say M, of points $\bar{v} \in \bar{v}(\mathcal{Q})$ such that $\langle \bar{x}, \bar{x}'\rangle_{\bar{v}} \leqslant \bar{\delta}$.

Fix $\eta \in (0, 2\bar{\delta})$ such that $M\sinh(2\eta)D/\sinh\rho + \eta \leqslant l$. Let $\bar{\gamma} : [a\,, b] \to \bar{X}$ be a $(1, \eta)$-quasi-geodesic joining $\bar{x}$ to $\bar{x}'$. For every $t \in [a\,, b]$, $\langle \bar{x}, \bar{x}'\rangle_{\bar{\gamma}(t)} \leqslant \eta/2$. Hence by choice of η, there are at most M distinct points of $\bar{v}(\mathcal{Q})$ lying on $\bar{\gamma}$. We denote them $\bar{v}_1 = \bar{\gamma}(t_1), \dots, \bar{v}_m = \bar{\gamma}(t_m)$ (with $m \leqslant M$). Without loss of generality we can assume that $t_1 < t_2 < \cdots < t_m$. Note that for every $j \in \{1, \dots, m-1\}$, $|t_{j+1} - t_j| \geqslant 2\rho$. Let $j \in \{1, \dots, m\}$. The path $\bar{\gamma}$ is not a geodesic, thus it can go through the same apex several times. However if we let $s_j = \max\{t_j - 2\eta, a\}$ and $s'_j = \min\{t_j + 2\eta, b\}$, then $\bar{\gamma}$ restricted to $[a\,, s_j]$ or $[s'_j\,, b]$ does not contain $\bar{v}_j$. Moreover, by Lemma 5.5 there exists a path $\bar{\gamma}_j$ joining $\bar{\gamma}(s_j)$ to $\bar{\gamma}(s'_j)$ whose length is at most $\sinh(2\eta)D/\sinh\rho$ that does not contain any apex. We now define a new path $\bar{\gamma}'$ joining $\bar{x}$ to $\bar{x}'$ as follows. For every $j \in \{1, \dots, m\}$, we replace the subpath of $\bar{\gamma}$ between times s_j and s'_j by the path $\bar{\gamma}_j$. By construction, $\bar{\gamma}'$ does not contain any apex. Moreover its length is at most

$$L(\bar{\gamma}') \leqslant L(\bar{\gamma}) + M\sinh(2\eta)D/\sinh\rho \leqslant L(\bar{\gamma}) + l - \eta.$$

Since $\bar{\gamma}$ is a $(1,\eta)$-quasi-geodesic, $\bar{\gamma}'$ is a $(1,l)$-quasi-geodesic. $\square$

Lemma 5.7. — *Let $\bar{x} \in \bar{X} \setminus \bar{v}(\mathcal{Q})$ and $\bar{\xi} \in \partial\bar{X}$. For every $L > 0$, for every $l > 0$, there exists an L-local $(1, l + 10\bar{\delta})$-quasi-geodesic $\bar{\gamma} : \mathbb{R}_+ \to \bar{X}$ joining $\bar{x}$ to $\bar{\xi}$ such that for every $t \in \mathbb{R}_+$, $\bar{\gamma}(t)$ does not belong to $\bar{v}(\mathcal{Q})$.*

Proof. — The proof works just as the one of Lemma 2.9, using Lemma 5.6 to avoid the apices of $\bar{X}$. $\square$

Proposition 5.8. — *The set $\bar{v}(\mathcal{Q})$ contains at least two distinct apices.*

Proof. — Let $(H,Y) \in \mathcal{Q}$. By assumption the action of G on X is non-elementary. Therefore there exists $g \in G$ such that $\operatorname{Stab}(Y) \neq g\operatorname{Stab}(Y)g^{-1}$. In particular $(H,Y) \neq g(H,Y)$. In other words $\mathcal{Q}$ contains at least two elements. Let $\eta \in (0, \dot{\delta})$. We now fix two distinct apices v and v' in $v(\mathcal{Q})$ such that for every $w, w' \in v(\mathcal{Q})$, $|v - v'|_{\dot{X}} \leqslant |w - w'|_{\dot{X}} + \eta$. Let $\gamma : [a\,, b] \to \dot{X}$ be a $(1,\eta)$-quasi-geodesic joining v to v'. Recall that two distinct points of $v(\mathcal{Q})$ are at least 2ρ far apart from each other. Hence $|b - a| \geqslant 2\rho$. We let $t = a + \rho/4 + \eta$ and $t' = b - \rho/4 - \eta$. The points x and x' respectively stand for $x = \gamma(t)$ and $x' = \gamma(t')$. It follows from the triangle inequality that $|x - x'|_{\dot{X}} \geqslant 3\rho/2 - 2\eta$. We claim that γ restricted to $[t\,, t']$ lies in the $3\rho/4$-neighborhood of X. First, γ being a $(1,\eta)$-quasi-geodesic, for every $s \in [t\,, t']$, $|\gamma(s) - v|_{\dot{X}} \geqslant \rho/4$ and $|\gamma(s) - v'|_{\dot{X}} \geqslant \rho/4$. We now focus on the other apices of $\dot{X}$. Let $w \in v(\mathcal{Q}) \setminus \{v, v'\}$. Assume that w lies in the $\rho/4$-neighborhood of γ. It follows that

$$\min\left\{|v - w|_{\dot{X}}\,, |v' - w|_{\dot{X}}\right\} \leqslant \frac{1}{2}\,|v - v'|_{\dot{X}} + \rho/4 + \eta.$$

However two distinct apices of $v(\mathcal{Q})$ are at a distance at least 2ρ apart, hence

$$\min\left\{|v - w|_{\dot{X}}\,, |v' - w|_{\dot{X}}\right\} < |v - v'|_{\dot{X}} - \eta,$$

which contradicts our choice of v and v'. Consequently γ restricted to $[t\,, t']$ lies in the $3\rho/4$-neighborhood of X. Let $\bar{\gamma} : [t\,, t'] \to \bar{X}$ be the path of $\bar{X}$ induced by the restriction of γ to $[t\,, t']$. According to Proposition 5.4 $\bar{\gamma}$ is a $\rho/20$-local $(1,\eta)$-quasi-geodesic (Corollary 2.7). By stability of quasi-geodesics it is a (global) $(2,\eta)$-quasi-geodesic. Consequently,

$$|\bar{x} - \bar{x}'| \geqslant \frac{1}{2}\,|t - t'| - \frac{\eta}{2} \geqslant \frac{1}{2}\,|x - x'|_{\dot{X}} - \frac{\eta}{2} \geqslant \frac{3\rho}{4} - \frac{3\eta}{2} > \frac{\rho}{2} + 2\eta.$$

It implies that $\bar{v} \neq \bar{v}'$. Indeed by construction $|\bar{x} - \bar{v}| \leqslant \rho/4 + \eta$ and $|\bar{x}' - \bar{v}'| \leqslant \rho/4 + \eta$. Thus if $\bar{v}$ and $\bar{v}'$ were the same apex we would have $|\bar{x} - \bar{x}'| \leqslant \rho/2 + 2\eta$. $\square$

Stabilizers of apices. The next results deals with the stabilizers of the apices in $\bar{X}$. In particular given an apex $\bar{v} \in \bar{v}(\mathcal{Q})$, we are interested in how an element $\bar{g} \in \operatorname{Stab}(\bar{v})$ acts on the ball $B(\bar{v}, \rho)$. For the remainder of this paragraph we fix a pair $(H, Y) \in \mathcal{Q}$. We denote by v the apex of the cone $Z(Y)$. We write F for the maximal finite normal subgroup of $\operatorname{Stab}(Y)$. Let h be a generator of the cyclic subgroup H. By assumption, there exist an odd integer $n \geqslant 100$ such that h is the n-th power of an element of N.

Proposition 5.9. — *Let $u \in F$. Let $g \in \operatorname{Stab}(Y)$ such that $[h]^\infty/4 \leqslant [g]^\infty \leqslant 3[h]^\infty/4$.*

(i) *For every $x \in B(v, \rho)$, $|ux - x|_{\dot{X}} \leqslant \bar{\delta}$; in particular, $|\bar{u}\bar{x} - \bar{x}| \leqslant \bar{\delta}$.*
(ii) *For every $\bar{x} \in \bar{X}$, $\langle \bar{x}, \bar{u}\bar{g}\bar{x} \rangle_{\bar{v}} \leqslant 2\bar{\delta}$.*

Proof. — Let C_F be the characteristic subgroup of F (Definition 3.26). According to Lemma 3.33, Y is contained in the 51δ-neighborhood of C_F. Hence u moves the points of Y by a distance at most 113δ. Let x be a point of $B(v, \rho)$ and $\bar{x}$ its image in $\bar{X}$. It can be seen as a point $x = (y, r)$ of the cone $Z(Y)$. The map $\zeta : \dot{X} \to \bar{X}$ shortens the distances. Moreover Y is strongly quasi-convex. Hence we get

$$|\bar{u}\bar{x} - \bar{x}| \leqslant |ux - x|_{\dot{X}} \leqslant |uy - y|_Y \leqslant |uy - y| + 8\delta \leqslant 121\delta \leqslant \bar{\delta},$$

which proves the first point.

Let $\bar{x}$ be a point of $B(\bar{v}, \rho/3)$ and $x = (y, r)$ a preimage of $\bar{x}$ in $B(v, \rho/3)$. Reasoning as previously we see that

$$\begin{aligned} |ugy - y| \geqslant |gy - y| - 113\delta \geqslant [g] - 113\delta \geqslant [h]^\infty/4 - 113\delta \\ \geqslant T(\mathcal{Q})/4 - 113\delta \geqslant \pi \sinh \rho. \end{aligned}$$

It follows from Lemma 4.4 that $|ugx - x|_{Z(Y)} = 2r$. Recall that the metric on $Z(Y)$ and $\dot{X}$ coincide on the ball $B(v, \rho/3)$, thus $|ugx - x|_{\dot{X}} = 2r$ (Lemma 4.8). Let $k \in \mathbb{Z}^*$. The point y belongs to the cylinder of h and therefore is contained in the 52δ-neighborhood the axis of g (Lemma 3.14). Combined with Proposition 3.1 it leads to

$$\begin{aligned} |ugy - y| \leqslant |gy - y| + 113\delta \leqslant [g] + 225\delta \\ \leqslant 3[h]^\infty/4 + 241\delta \leqslant \left[h^k\right] - [h]^\infty/4 + 241\delta. \end{aligned}$$

Consequently for every $k \in \mathbb{Z}^*$, $|ugy - y|_Y \leqslant \left[h^k\right] - \pi \sinh \rho$. According to Lemma 4.5, $|\bar{u}\bar{g}\bar{x} - \bar{x}| = |ugx - x|_{\dot{X}} = 2r$. By construction $|\bar{x} - \bar{v}| = |\bar{u}\bar{g}\bar{x} - \bar{v}| = r$, thus $\langle \bar{x}, \bar{u}\bar{g}\bar{x} \rangle_{\bar{v}} = 0$. Assume now that $\bar{x}$ is a point of $\bar{X} \setminus B(\bar{v}, \rho/3)$. Let $\bar{z}$ be an $\bar{\delta}$-projection of $\bar{x}$ on $B(\bar{v}, \rho/3)$. It follows from the four point inequality (2.1) combined with the previous observation that

$$\min\left\{\langle \bar z, \bar x\rangle_{\bar v}, \langle \bar x, \bar u\bar g\bar x\rangle_{\bar v}, \langle \bar u\bar g\bar x, \bar u\bar g\bar z\rangle_{\bar v}\right\} \leqslant \langle \bar z, \bar u\bar g\bar z\rangle_{\bar v} + 2\bar\delta = 2\bar\delta.$$

Since $\bar x$ does not belong to $B(v,\rho/3)$, $|\bar v - \bar z| \geqslant \rho/3 - \bar\delta$. By projection on a quasi-convex we have $\langle \bar x, \bar v\rangle_{\bar z} \leqslant 2\bar\delta$. Hence the minimum in the previous inequality cannot be achieved by $\langle \bar z, \bar x\rangle_{\bar v} = |\bar v - \bar z| - \langle \bar x, \bar v\rangle_{\bar z}$. Similarly it cannot be achieved by $\langle \bar u\bar g\bar x, \bar u\bar g\bar z\rangle_{\bar v} = \langle \bar x, \bar z\rangle_{\bar v}$. Consequently $\langle \bar x, \bar u\bar g\bar x\rangle_{\bar v} \leqslant 2\bar\delta$. □

Corollary 5.10. — *Let $\bar g \in \operatorname{Stab}(\bar v)$. If $\bar g$ is not the image of an elliptic element of* $\operatorname{Stab}(Y)$ *then there exists $k \in \mathbb{Z}$ such that the axis of $\bar g^k$ is contained in the $6\bar\delta$-neighborhood of $\bar v$. In particular, $\bar v$ is the unique apex of $\bar X$ fixed by $\bar g$.*

Proof. — Let g be a preimage of $\bar g$ in $\operatorname{Stab}(Y)$. By assumption h and g are loxodromic elements, thus they fix pointwise ∂Y. Let c be a primitive element of $\operatorname{Stab}(Y)$. Recall that h is the n-th power of an element of G (possibly not primitive as an element of G). Consequently there is $u, u' \in F$ and $p, q \in \mathbb{Z}$ such that $h = c^{np}u$ and $g = c^q u'$. Since g is not the image of an elliptic element of $\operatorname{Stab}(Y)$, $q \neq 0 \mod np$. Thus there exist integers $k, l \in \mathbb{Z}$ such that $m = kq + lnp$ is between $np/4$ and $4np/4$. Since F is a normal subgroup of $\operatorname{Stab}(Y)$, there exists $f \in F$ such that $h^l g^k = c^m f$. In particular $\bar g^k = \bar c^m \bar f$. It follows from Proposition 3.29 that $[c]^\infty = [h]^\infty/np$. Hence $[h]^\infty/4 \leqslant [c^m]^\infty \leqslant 3[h]^\infty/4$. Let $\bar x$ be a point of $\bar X$. According to Proposition 5.9, $\left\langle \bar x, \bar g^k\bar x\right\rangle_{\bar v} \leqslant 2\bar\delta$. Thus $|\bar g^k\bar x - \bar x| \geqslant 2|\bar v - \bar x| - 4\delta$. However $\bar g$ fixes $\bar v$, thus $\left[\bar g^k\right] = 0$. Consequently the points of $\bar X$ which belong to the axis of $\bar g^k$ are $6\bar\delta$-close to $\bar v$. □

Corollary 5.11. — *There exists $\bar g \in \operatorname{Stab}(\bar v)$ such that for every $\bar x \in \bar X$, $\langle \bar x, \bar g\bar x\rangle_{\bar v} \leqslant 2\bar\delta$ and $\left\langle \bar g^{-1}\bar x, \bar g\bar x\right\rangle_{\bar x} \leqslant |\bar g\bar x - \bar x|/2 + 4\bar\delta$.*

Proof. — By assumption there exists $b \in N$ and an odd integer $n \geqslant 100$ such that H is generated by $h = b^n$. Thus there exists an integer m such that $n/4 \leqslant m \leqslant 3m/8$. We let $\bar g = \bar b^m$. Let $\bar x \in \bar X$. By Proposition 5.9 we get that $\langle \bar g\bar x, \bar x\rangle_{\bar v} \leqslant 2\bar\delta$ and $\left\langle \bar g^{-1}\bar x, \bar g\bar x\right\rangle_{\bar v} \leqslant 2\bar\delta$. It follows from the triangle inequality that

$$\left\langle \bar g^{-1}\bar x, \bar g\bar x\right\rangle_{\bar x} \leqslant |\bar x - \bar v| + 2\bar\delta \leqslant |\bar g\bar x - \bar x|/2 + 4\bar\delta. \qquad \square$$

Lifting figures. The next propositions are two key ingredients for the coming study of $\bar G$. We explain how some figure in $\bar X$ can be lift to a picture of $\dot X$.

Proposition 5.12 ([10, Proposition. 3.21]). — *Let $\alpha \geqslant 0$ and $d \geqslant \alpha$. Let $\bar Z$ be an α-quasi-convex subset of $\bar X$. Let $\bar z_0$ be a point of $\bar Z$ and z_0*

a preimage of $\bar{z}_0$ in $\dot{X}$. We assume that for every $\bar{v} \in \bar{v}(\mathcal{Q})$, $\bar{Z}$ does not intersect $B(\bar{v}, \rho/20 + d + 10\bar{\delta})$. Then there exists a subset Z of $\dot{X}$ satisfying the following properties.

(i) The map $\zeta : \dot{X} \to \bar{X}$ induces an isometry from Z onto $\bar{Z}$.

(ii) For every $\bar{g} \in \bar{G}$, if $\bar{g}\bar{Z}$ lies in the d-neighborhood of $\bar{Z}$ then there exists a preimage $g \in G$ of $\bar{g}$ such that for every $z, z' \in Z$, $|gz' - z|_{\dot{X}} = |\bar{g}\bar{z}' - \bar{z}|$.

(iii) The projection $\pi : G \to \bar{G}$ induces an isomorphism from $\operatorname{Stab}(Z)$ *onto* $\operatorname{Stab}(\bar{Z})$

Let $\bar{\gamma} : I \to \bar{X}$ be a quasi-geodesic $\bar{X}$. If $\bar{\gamma}$ stays far away from the apices (e.g. if it is a small path with endpoints in $\zeta(X)$) Proposition 5.12 provides a tool to lift it in an appropriate manner as a path γ of $\dot{X}$ with the same length. In particular, if an isometry $\bar{g} \in \bar{G}$ moves the endpoints of $\bar{\gamma}$ by a small distance, one can find a preimage $g \in G$ of $\bar{g}$ that moves the endpoints of γ by a small distance. This property might fail if $\bar{\gamma}$ is an arbitrary path (take a long path with loops around apices). The next proposition explain how to handle that case.

Proposition 5.13. — *Let $l \in [0, 10^5\bar{\delta}]$. Let x and y be two points of X. Let $\gamma : [a, b] \to \dot{X}$ be a path joining x to y such that the path $\bar{\gamma} : [a, b] \to \bar{X}$ that it induces is an $L_S\bar{\delta}$-local $(1, l)$-quasi-geodesic. Let S be a subset of G such that for every $g \in S$, $|gx - x|_{\dot{X}} \leqslant \rho/50$ and $|\bar{g}\bar{y} - \bar{y}| \leqslant \rho/50$. In addition, we suppose that S satisfies the following property. Let $(H, Y) \in \mathcal{Q}$. Let v be the apex of $Z(Y)$ and F the maximal finite normal subgroup of* $\operatorname{Stab}(Y)$*. If $\bar{\gamma}$ intersects $B(\bar{v}, 9\rho/10)$, then for every $g \in S$, $\bar{g}$ is the image of an element of F. Under these assumptions, for every $g \in S$, $|gy - y|_{\dot{X}} = |\bar{g}\bar{y} - \bar{y}|$.*

Remark. — Let $g \in S$. By assumption $\bar{\gamma}$ is a local quasi-geodesic It follows from Corollary 2.7 and Lemma 3.2 that for every $t \in [a, b]$, $|\bar{g}\bar{\gamma}(t) - \bar{\gamma}(t)| \leqslant \rho/50 + l + 16\bar{\delta}$. If this path was entirely contained in the neighborhood of $\zeta(X)$ we could apply Proposition 5.12 to lift to path $\dot{X}$. However $\bar{\gamma}$ might go through the cones. In this case g will fix the apex of the cone. The strategy is to subdivide $\bar{\gamma}$ into subpaths of two types: the ones which stay far away from the apices and the ones contained in a cone. Once this is done, we lift them one after the other.

Proof. — Let $v_1, \dots, v_m$ be the apices of $v(\mathcal{Q})$ which are $9\rho/10$-close to γ. For every $j \in \{1, \dots, m\}$, we denote by $\gamma(c_j)$ a projection of v_j on γ. By reordering the apices we can always assume that $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_m$. For simplicity of notation we put $c_0 = a$ and $c_{m+1} = b$. Let $j \in \{1, \dots, m\}$. Since $\bar{\gamma}$ is an $L_S\bar{\delta}$-local $(1, l)$-quasi-geodesic of $\bar{X}$ so is γ. In particular, it is a (global) $(2,l)$-quasi-geodesic. Hence we can find $b_{j-1} \in (c_{j-1}, c_j]$ and $a_j \in [c_j, c_{j+1})$ with the following properties.

(i) $|v_j - \gamma(b_{j-1})| = 9\rho/10$ and $|v_j - \gamma(a_j)| = 9\rho/10$,

(ii) $\gamma \cap B(v_j, 2\rho/5)$ is contained in $\gamma((b_{j-1}, a_j))$

In addition, we let $a_0 = a$, $b_m = a_{m+1} = b$ (see Figure 5.1). We claim that

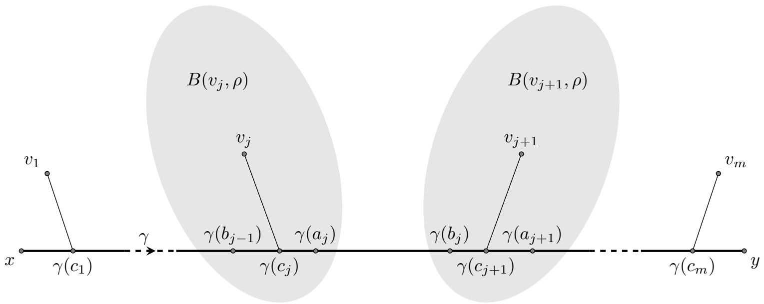

Figure 5.1. The cones intersecting γ.

for every $j \in \{0, \dots, m+1\}$, for every $g \in S$, we have

$$|\bar{g}\bar{\gamma}(a_j) - \bar{\gamma}(a_j)| = |g\gamma(a_j) - \gamma(a_j)|_{\dot{X}}\,.$$

The proof is by induction on j. If $j = 0$ then $\gamma(a_j) = x$. The claim follows from the fact that the map $\zeta : \dot{X} \to \bar{X}$ induces an isometry from $B(x, \rho/20)$ onto $B(\bar{x}, \rho/20)$ (Proposition 5.4). Assume now that our claim is true for $j \in \{0, \dots, m\}$. Since γ is a local quasi-geodesic, $a_j \leqslant b_j$. We denote by $\bar{\gamma}_j$ the restriction of $\bar{\gamma}$ to $[a_j\,, b_j]$. By construction $\bar{\gamma}_j$ is $(l+8\bar{\delta})$-quasi-convex and contained in the $3\rho/5$-neighborhood of $\zeta(X)$. Applying Proposition 5.12 there exists a continuous path $\gamma_j : [a_j\,, b_j] \to \dot{X}$ starting at $\gamma(a_j)$ and lifting $\bar{\gamma}_j$ with the following property. Given $\bar{g} \in \bar{G}$, if $\bar{g}\bar{\gamma}_j$ lies in the $\rho/10$-neighborhood of $\bar{\gamma}_j$ then there exists $g \in G$ such that for every $t \in [a_j\,, b_j]$, $|\bar{g}\bar{\gamma}_j(t) - \bar{\gamma}_j(t)| = |g\gamma_j(t) - \gamma_j(t)|_{\dot{X}}$. According to Proposition 5.3, $\dot{X} \setminus v(\mathcal{Q})$ is a covering space of $\bar{X} \setminus \bar{v}(\mathcal{Q})$. Thus γ_j is exactly the restriction of γ to $[a_j\,, b_j]$.

Take now an element g in S and write $\bar{g}$ for its image in $\bar{G}$. By assumption $|\bar{g}\bar{x} - \bar{x}| \leqslant \rho/50$ and $|\bar{g}\bar{y} - \bar{y}| \leqslant \rho/50$. It follows from Lemma 3.2 that for every $t \in [a\,, b]$, $|\bar{g}\bar{\gamma}(t) - \bar{\gamma}(t)| \leqslant \rho/50 + l + 16\bar{\delta}$. In particular $\bar{g}$ moves the points of $\bar{\gamma}_j$ by a distance at most $\rho/10$. Using the properties of the lift γ_j, there exists $u \in K$ such that for every $t \in [a_j\,, b_j]$, $|\bar{g}\bar{\gamma}(t) - \bar{\gamma}(t)| = |gu\gamma(t) - \gamma(t)|_{\dot{X}}$. Thus $|\bar{g}\bar{\gamma}(a_j) - \bar{\gamma}(a_j)| = |gu\gamma(a_j) - \gamma(a_j)|_{\dot{X}}$. On the other hand, using the induction assumption $|\bar g\bar\gamma(a_j)-\bar\gamma(a_j)| = |g\gamma(a_j)-\gamma(a_j)|_{\dot X}$. It follows from the triangle inequality that

$$\begin{aligned}|u\gamma(a_j)-\gamma(a_j)|_{\dot X} &\leqslant |gu\gamma(a_j)-\gamma(a_j)|_{\dot X} + |g\gamma(a_j)-\gamma(a_j)|_{\dot X}\\ &\leqslant 2\,|\bar g\bar\gamma(a_j)-\bar\gamma(a_j)|\\ &\leqslant \rho/25+2l+32\bar\delta.\end{aligned}$$

However, $K\setminus\{1\}$ moves the points of the $\rho/10$-neighborhood of $X\subset\dot X$ by a distance at least $\rho/5$ (Proposition 5.3). Consequently $u=1$. In particular $|g\gamma(b_j)-\gamma(b_j)|_{\dot X} = |\bar g\bar\gamma(b_j)-\bar\gamma(b_j)|$ is at most $\rho/50+l+16\bar\delta$. If $j=m$, then $a_{m+1}=b_m$, thus the claim holds for $j+1$. Otherwise, $|v_{j+1}-\gamma(b_j)| = 9\rho/10$, thus g necessarily belongs to $\mathrm{Stab}(v_{j+1})$. Moreover by assumption, $\bar g$ is the image of an element in the maximal normal finite subgroup F_{j+1} of $\mathrm{Stab}(v_{j+1})$. Since g moves the point $\gamma(b_j)\in B(v_{j+1},\rho)$ by a small distance, g is the elliptic preimage of $\bar g$. Therefore it moves all the points of $B(v_{j+1},\rho)$ by a distance at most $\bar\delta$ (Proposition 5.9). In particular, $|g\gamma(a_{j+1})-\gamma(a_{j+1})|_{\dot X}\leqslant\bar\delta$. However, the map $\zeta:\dot X\to\bar X$ induces an isometry from the ball $B(\gamma(a_{j+1}),\rho/20)$ onto its image, hence $|g\gamma(a_{j+1})-\gamma(a_{j+1})|_{\dot X} = |\bar g\bar\gamma(a_{j+1})-\bar\gamma(a_{j+1})|$. This proves our claim for $j+1$. The statement of the lemma follows from our claim for $j=m+1$. □

5.3. Elementary subgroups

Proposition 5.14. — *The action of $\bar G$ on $\bar X$ is WPD.*

Proof. — Let $\bar g$ be a loxodromic element of $\bar G$. We claim that there exist $\bar y$ and $\bar y'$ in $Y_{\bar g}$ such that the set of elements $\bar u\in\bar G$ satisfying $|\bar u\bar y-\bar y|\leqslant 486\bar\delta$ and $|\bar u\bar y'-\bar y'|\leqslant 486\bar\delta$ is finite. According to Proposition 3.19 it is sufficient to show that $\bar g$ satisfies the WPD property. By replacing if necessary $\bar g$ by a power of $\bar g$ we can assume that $[\bar g]>L_S\bar\delta$. Let $\bar\gamma:\mathbb R\to\bar X$ be a $\bar\delta$-nerve of $\bar g$ and T its fundamental length. By definition $\bar\gamma$ is contained in the cylinder $Y_{\bar g}$ of $\bar g$. We now distinguish two cases.

Assume first that there exists $\bar v\in\bar v(\mathcal Q)$ lying in the $\rho/10$-neighborhood of $\bar\gamma([0,T])$. There is $(H,Y)\in\mathcal Q$ such that $\bar v$ is the image in $\bar X$ of the apex of $Z(Y)$. Let $\bar y=\bar y'=\bar\gamma(s)$ be a projection of $\bar v$ on $\bar\gamma([0,T])$. Let $\bar u$ be an element of $\bar G$ such that $|\bar u\bar y-\bar y|\leqslant 486\bar\delta$. It follows from the triangle inequality that

$$|\bar u\bar v-\bar v|\leqslant 2\,|\bar v-\bar y|+|\bar u\bar y-\bar y|<2\rho.$$

However two distinct apices of $\bar{X}$ are at a distance at least 2ρ apart. Thus $\bar{u}\bar{v} = \bar{v}$. Hence $\bar{u}$ belongs to the finite group $\operatorname{Stab}(\bar{v}) = \operatorname{Stab}(Y)/H$, which proves our claim.

Assume now that for every $\bar{v} \in \bar{v}(\mathcal{Q})$, $\bar{\gamma}([0,T])$ does not intersect $B(\bar{v}, \rho/10)$. The set $\bar{v}(\mathcal{Q})$ being $\bar{G}$-invariant, for every $\bar{v} \in \bar{v}(\mathcal{Q})$, $\bar{\gamma}$ does not intersect $B(\bar{v}, \rho/10)$. Recall that $[\bar{g}] > L_S\bar{\delta}$ thus $\bar{\gamma}$ is a $9\bar{\delta}$-quasi-convex subset of $\bar{X}$. We let $\bar{y} = \bar{\gamma}(0)$ and denote by y a preimage of $\bar{y}$ in $\dot{X}$. According to Proposition 5.12, there exists a map $\gamma : \mathbb{R} \to \dot{X}$ and a preimage g of $\bar{g}$ with the following properties.

(i) $y = \gamma(0)$.

(ii) For every $t \in \mathbb{R}$, $\gamma(t)$ is a preimage in $\dot{X}$ of $\bar{\gamma}(t)$.

(iii) For every $t \in \mathbb{R}$, $\gamma(t+T) = g\gamma(t)$.

Recall that the map $\zeta : \dot{X} \to \bar{X}$ is 1-Lipschitz. Thus $\bar{g}$ being a loxodromic isometry of $\bar{X}$, g is a loxodromic isometry of $\dot{X}$. According to Proposition 4.10, the action of G on $\dot{X}$ is WPD. Hence there exists $n \in \mathbb{N}$ such that the set S of elements $u \in G$ satisfying $|uy - y|_{\dot{X}} \leqslant 486\bar{\delta}$ and $|ug^n y - g^n y| \leqslant 486\bar{\delta}$ is finite. We put $y' = g^n y = \gamma(nT)$. By construction $\bar{y}'$ is a point on $\bar{\gamma} \subset Y_{\bar{g}}$. Let $\bar{u}$ be an element of $\bar{G}$ such that $|\bar{u}\bar{y} - \bar{y}| \leqslant 486\bar{\delta}$ and $|\bar{u}\bar{y}' - \bar{y}'| \leqslant 486\bar{\delta}$. Recall that K acts properly on $\dot{X} \setminus v(\mathcal{Q})$ hence there exists $u \in G$ such that $|uy - y|_{\dot{X}} = |\bar{u}\bar{y} - \bar{y}| \leqslant 486\bar{\delta}$. We are going to apply Proposition 5.13 with the set $\{u\}$ and the path γ joining y to y'. By construction $|uy - y|_{\dot{X}} \leqslant 486\bar{\delta}$ and $|\bar{u}\bar{y}' - \bar{y}'| \leqslant 486\bar{\delta}$. Being a $\bar{\delta}$-nerve of $\bar{g}$, $\bar{\gamma}$ is an $L_S\bar{\delta}$-local $(1,\bar{\delta})$-quasi-geodesic. Moreover for every $v \in v(\mathcal{Q})$, $\bar{\gamma}$ does not intersect $B(\bar{v}, 9\rho/10)$, hence the last assumption of Proposition 5.13 is vacuous. It follows that $|uy' - y'|_{\dot{X}} = |\bar{u}\bar{y}' - \bar{y}'| \leqslant 486\bar{\delta}$. Hence u belongs to the finite set S, which proves our claim for the second case. □

Proposition 5.15. — *The group $\bar{G}$ is non-elementary (for its action on $\bar{X}$).*

Proof. — The idea of the proof is to exhibit two elements of $\bar{G}$ satisfying the criterion provided by Lemma 3.24. According to Proposition 5.8, $\bar{v}(\mathcal{Q})$ contains two distinct apices $\bar{v}_1$ and $\bar{v}_2$. By Corollary 5.11, for each $j \in \{1,2\}$, there exists $\bar{g}_j \in \operatorname{Stab}(\bar{v}_j)$ such that for every $\bar{x} \in \bar{X}$,

$$\langle \bar{g}_j\bar{x}, \bar{x}\rangle_{\bar{v}} \leqslant 2\bar{\delta} \quad \text{and} \quad 2\left\langle \bar{g}_j^{-1}\bar{x}, \bar{g}_j\bar{x}\right\rangle_{\bar{x}} \leqslant |\bar{g}_j\bar{x} - \bar{x}| + 8\bar{\delta}. \tag{5.1}$$

Let $\bar{x}$ be a $\bar{\delta}$-projection of $\bar{v}_2$ on $B(\bar{v}_1, \rho)$. Recall that $B(\bar{v}_1, \rho)$ is $2\bar{\delta}$-quasi-convex. Thus $\langle \bar{v}_1, \bar{v}_2\rangle_{\bar{x}} \leqslant 3\bar{\delta}$. Applying the four point inequality (2.1) we get

$$\min\left\{\langle \bar{v}_1, \bar{g}_1\bar{x}\rangle_{\bar{x}}, \langle \bar{g}_1\bar{x}, \bar{g}_2\bar{x}\rangle_{\bar{x}}, \langle \bar{g}_2\bar{x}, \bar{v}_2\rangle_{\bar{x}}\right\} \leqslant \langle \bar{v}_1, \bar{v}_2\rangle_{\bar{x}} + 2\bar{\delta} \leqslant 5\bar{\delta}. \tag{5.2}$$

According to the triangle inequality $\langle \bar v_1, \bar g_1\bar x\rangle_{\bar x} \geqslant |\bar v_1 - \bar x| - \langle \bar x, \bar g_1 \bar x\rangle_{\bar v_1}$ and $\langle \bar g_2\bar x, \bar v_2\rangle_{\bar x} \geqslant |\bar v_2 - \bar x| - \langle \bar g_2 \bar x, \bar x\rangle_{\bar v_2}$. By construction, $\rho - \bar\delta \leqslant |\bar v_1 - \bar x| \leqslant \rho$. Since $\bar v_1$ and $\bar v_2$ are 2ρ far apart we get $|\bar v_2 - \bar x| \geqslant \rho$. Consequently the minimum in (5.2) can only be achieved by $\langle \bar g_1 \bar x, \bar g_2 \bar x\rangle_{\bar x}$. Thus $\langle \bar g_1 \bar x, \bar g_2 \bar x\rangle_{\bar x} \leqslant 5\bar\delta$. Similarly we prove that $\left\langle \bar g_1^{\pm 1}\bar x, \bar g_2^{\pm 1}\bar x\right\rangle_{\bar x} \leqslant 5\bar\delta$. However by construction $\langle \bar g_1 \bar x, \bar x\rangle_{\bar v_1} \leqslant 2\bar\delta$ and $\langle \bar g_2 \bar x, \bar x\rangle_{\bar v_2} \leqslant 2\bar\delta$. Thus $|\bar g_1 \bar x - \bar x| \geqslant 2|\bar x - \bar v_1| - 4\bar\delta \geqslant 2\rho - 6\bar\delta$ and $|\bar g_2 \bar x - \bar x| \geqslant 2\rho - 4\bar\delta$. Consequently,

$$2\left\langle \bar g_1^{\pm 1}\bar x, \bar g_2^{\pm 1}\bar x\right\rangle_{\bar x} < \min\left\{|\bar g_1\bar x - \bar x|, |\bar g_2 \bar x - \bar x|\right\} - 16\bar\delta.$$

The other inequalities needed to apply Lemma 3.24 are given by (5.1). It follows that $\bar g_1$ and $\bar g_2$ generate an non-elementary subgroup of $\bar G$. □

Proposition 5.16. — *The image in $\bar G$ of an elliptic (respectively parabolic, loxodromic) subgroup of G is elliptic (respectively parabolic or elliptic, elementary).*

Proof. — The map $X \to \bar X$ shortens the distance. Hence the proof works exactly as the one of Lemma 4.11. □

Proposition 5.17. — *Let E be a non-loxodromic elementary subgroup of G. Then the projection $\pi : G \twoheadrightarrow \bar G$ induces an isomorphism from E onto its image.*

Proof. — Let g be a non-trivial element of E. Since E is not loxodromic, g cannot be loxodromic, (Corollary 3.7). In particular $[g]^\infty = 0$, thus $[g] \leqslant 16\delta$ (Propositon 3.1). We distinguish two cases. Assume first that g does not act trivially on X. In particular, there exists a point $x \in X$ such that $|gx - x| > 0$. Without loss of generality we can assume that $|gx - x| \leqslant 17\delta$. It follows that

$$0 < \mu\left(|gx - x|\right) \leqslant |gx - x|_{\dot X} \leqslant |gx - x| \leqslant 17\delta.$$

However the map $\zeta : \dot X \to \bar X$ induces an equivariant isometry from $B(x, \rho/20)$ onto its image. Therefore $|\bar g \bar x - \bar x| \neq 0$, hence $\bar g \neq 1$. Assume now that g acts trivially on X. Let $(H, Y) \in \mathcal Q$. Then g belongs to the stabilizer of Y. Moreover, being non-loxodromic g does not belong to H, thus it induces a non-trivial element of $\operatorname{Stab}(Y)/H$. However we know that $\operatorname{Stab}(Y)/H$ embeds into $\bar G$. Therefore $\bar g \neq 1$. □

From now on we are interested in the elementary subgroups of $\bar N$. Our goal is to find a way, to lift any elementary subgroup of $\bar N$ to an elementary subgroup of N. Recall that we assumed that N is a normal subgroup *without involution*. Hence for every $(H, Y) \in \mathcal Q$, the elements of $\operatorname{Stab}(Y) \cap N$ are either loxodromic or in the maximal normal finite subgroup of $\operatorname{Stab}(Y)$.

On the other hand, the kernel K of the projection $G \twoheadrightarrow \bar{G}$ is contained in N. Thus for every $\bar{g} \in \bar{N}$, any preimage $g \in G$ of $\bar{g}$ belongs to N.

Elliptic subgroups. The following result follows the ideas of T. Delzant and M. Gromov in [13].

Proposition 5.18. — *Let $\bar{E}$ be an elliptic subgroup of $\bar{G}$. One of the following holds.*

(i) There is an elliptic subgroup E of G (for its action on X) such that the projection $\pi : G \to \bar{G}$ induces an isomorphsim from E onto $\bar{E}$.

(ii) There exists $\bar{v} \in \bar{v}(\mathcal{Q})$ such that $\bar{E}$ is contained in $\mathrm{Stab}(\bar{v})$. *Moreover there exists $\bar{g} \in \bar{E}$ such that $A_{\bar{g}}$ lies in the $6\bar{\delta}$-neighborhood of $\{\bar{v}\}$.*

Proof. — Recall that $C_{\bar{E}}$ is the set of points $\bar{x} \in \bar{X}$ such that for every $\bar{g} \in \bar{E}$, $|\bar{g}\bar{x} - \bar{x}| \leqslant 11\bar{\delta}$. It is an $\bar{E}$-invariant $9\bar{\delta}$-quasi-convex (Proposition 3.27). We distinguish two cases. Assume first that $C_{\bar{E}}$ contains a point $\bar{x}$ in the $50\bar{\delta}$-neighborhood of $\zeta(X)$. We write $\bar{Z}$ for the hull of the $\bar{E}$-orbit of $\bar{x}$ (Definition 2.17). It is an $\bar{E}$-invariant $6\bar{\delta}$-quasi-convex contained in the $56\bar{\delta}$-neighborhood of $\zeta(X)$. By Proposition 5.12, there exists a subset Z of $\dot{X}$ such that the map $\zeta : \dot{X} \to \bar{X}$ induces an isometry from Z onto $\bar{Z}$ and the projection $G \twoheadrightarrow \bar{G}$ induces an isomorphism from $\mathrm{Stab}(Z)$ onto $\mathrm{Stab}(\bar{Z})$. In particular $\bar{E}$ is isomorphic to a subgroup E of $\mathrm{Stab}(Z)$. Let x be the preimage of $\bar{x}$ in Z and y a projection of x on X. Thus $|x-y|_{\dot{X}} \leqslant 50\bar{\delta}$. Let $g \in E$ we have then

$$\begin{aligned}\mu\left(|gy-y|\right) \leqslant |gy-y|_{\dot{X}} \leqslant |gx-x|_{\dot{X}} + 100\bar{\delta} = |\bar{g}\bar{x}-\bar{x}|_{\bar{X}} + 100\bar{\delta} \\ \leqslant 111\bar{\delta} < 2\rho.\end{aligned}$$

It follows that $|gx-x|_X \leqslant \pi\sinh(56\bar{\delta})$ (Proposition 4.2). In particular E has a bounded orbit in X, thus it is an elliptic subgroup of G.

Assume now that $C_{\bar{E}}$ does not contain any point $\bar{x}$ in the $50\bar{\delta}$-neighborhood of $\zeta(X)$. Since $C_{\bar{E}}$ is $9\bar{\delta}$-quasi-convex, there is $\bar{v} \in \bar{v}(\mathcal{Q})$ such that $C_{\bar{E}}$ lies in the ball $B(\bar{v}, \rho - 50\bar{\delta})$. Let $\bar{x}$ be a point of $C_{\bar{E}}$. Any element $\bar{g}$ of $\bar{E}$ moves $\bar{x}$ by a distance at most $11\bar{\delta}$. The triangle inequality yields $|\bar{g}\bar{v} - \bar{v}| < 2\rho$, hence $\bar{g}$ fixes $\bar{v}$. Consequently $\bar{E}$ is a subgroup of $\mathrm{Stab}(\bar{v})$. There exists $(H, Y) \in \mathcal{Q}$ such that $\bar{v}$ is the image of the apex v of the cone $Z(Y)$ over Y. We claim that there exists an element $\bar{g} \in \bar{E}$ which is not the image of an elliptic element of $\mathrm{Stab}(Y)$. Assume on the contrary that our claim is false. In particular, $E = \pi^{-1}(\bar{E}) \cap \mathrm{Stab}(Y)$ is a subgroup of G that only contains elliptic elements. Since $\mathrm{Stab}(Y)$ is a loxodromic subgroup, E is elliptic (Corollary 3.30). Thus there exists $x \in X$ such that for every $g \in E$, $|gx-x| \leqslant 11\delta$ (Proposition 3.27). Since the map $\zeta : X \to \bar{X}$ reduces the distances, it follows that the image $\bar{x}$ of x in $\bar{X}$ is moved by a distance at most $11\bar{\delta}$ by any element of $\bar{E}$. However $\bar{x}$ lies in $\zeta(X)$. It contradicts the fact that $C_{\bar{E}}$ does not contain any point $\bar{x}$ in the $50\bar{\delta}$-neighborhood of $\zeta(X)$. According to Corollary 5.10 there exists $k \in \mathbb{Z}$ such that the axis of $\bar{g}^k$ is contained in the $6\bar{\delta}$-neighborhood of $\bar{v}$. □

Corollary 5.19. — *The subgroup $\bar{N}$ has no involution.*

Proof. — Let $\bar{g}$ be an element of $\bar{N}$ and assume that $\bar{g}$ has order 2. Recall the kernel K of the projection $\pi : G \to \bar{G}$ is contained in N. Hence every preimage of $\bar{g}$ is contained in N. According to Proposition 5.18 there are two cases.

(i) *There exists a preimage $g \in N$ of $\bar{g}$ with order* 2, which contradicts the fact that N has no involution.

(ii) *There exists $\bar{v} \in \bar{v}(\mathcal{Q})$ such that $\bar{g}$ belongs to* $\operatorname{Stab}(\bar{v})$. There is $(H, Y) \in \mathcal{Q}$ such that $\bar{v}$ is the image of the apex of the cone $Z(Y)$. We write F for the maximal finite normal subgroup of $\operatorname{Stab}(Y)$. By assumption, there exist an element $c \in N$, which is primitive (as an element of N) and an odd integer n such that H is generated by $h = c^n$. It follows that $\operatorname{Stab}(Y) \cap N$ is isomorphic to the semi-direct product $(F \cap N) \rtimes \mathbb{Z}$ where $\mathbb{Z}$ the subgroup generated by c acting by conjugacy on $F \cap N$. Let $g \in N$ be a preimage of $\bar{g}$ in $\operatorname{Stab}(Y)$. There exists $u, u' \in F \cap N$ and $m \in \mathbb{Z}$ such that $g = c^m u$ and $g^2 = c^{2m} u'$. We noticed that $\operatorname{Stab}(\bar{v})$ is isomorphic to $\operatorname{Stab}(Y)/H$ (Theorem 5.2). Consequently there exists $p \in \mathbb{Z}$ such that $c^{pn} = g^2$. Thus $pn = 2m$ (and $u' = 1$). However n is odd, thus n divides m. It follows $\bar{g}$ is the image of u. Restricted to F the projection $G \twoheadrightarrow \bar{G}$ is one-to-one, hence u has order 2. It contradicts again the fact that N has no involution.

Thus $\bar{N}$ cannot contain an involution. □

Proposition 5.20. — *Let E be an elliptic subgroup of N (for its action on X). Let S be a subset of G and y a point of X such that for every $u \in S$, $|uy - y|_{\dot{X}} < \rho/100$. If the image $\bar{S}$ of S in $\bar{G}$ is contained in $\bar{E}$, then there exists $g \in K$ such that gSg^{-1} lies in E.*

Proof. — We fix a point x in $C_E \subset X$. There exists $g \in K$ such that $|gy - x|_{\dot{X}} \leqslant |\bar{y} - \bar{x}| + \bar{\delta}$. By Proposition 5.17, the map $G \to \bar{G}$ induces an isomorphism from E onto its image. We denote by S' the preimage of $\bar{S}$ in E. We claim that $z = gy$ is hardly moved by the elements of S'. Let $\gamma : I \to \dot{X}$ be a $(1, \bar{\delta})$-quasi-geodesic joining x to z. Let $\bar{\gamma} : I \to \bar{X}$ the path of $\bar{X}$ induced by γ. By choice of g the length of $\bar{\gamma}$ satisfies

$$L(\bar{\gamma}) \leqslant L(\gamma) \leqslant |z-x|_{\dot{X}} + \bar{\delta} \leqslant |\bar{z}-\bar{x}| + 2\bar{\delta}.$$

Hence $\bar{\gamma}$ is a $(1, 2\bar{\delta})$-quasi-geodesic of $\bar{X}$. Let u be an element of S and u' the preimage of $\bar{u}$ in S'. We are going to apply Proposition 5.13 with the path γ and the set $\{u'\}$. Since u' belongs to E we have $|u'x-x|_{\dot{X}} \leqslant |u'x-x| \leqslant 11\delta \leqslant \bar{\delta}$. On the other hand g lies in K and $\bar{u} = \bar{u}'$ in $\bar{S}$, thus

$$|\bar{u}'\bar{z}-\bar{z}| = |\bar{u}\bar{y}-\bar{y}| \leqslant |uy-y|_{\dot{X}} < \rho/100.$$

Let $(H,Y) \in \mathcal{Q}$. Let v be the apex of $Z(Y)$. Assume that $\bar{u}'$ belongs to $\operatorname{Stab}(\bar{v})$. Recall that u' belongs to $E \subset N$. If $\bar{u}'$ is not the image of an element in the maximal normal finite subgroup of $\operatorname{Stab}(Y)$, then by Corollary 5.10, the characteristic subset $C_{\bar{E}}$ lies in the $15\bar{\delta}$-neighborhood of $\bar{v}$. However $\bar{x}$ is by construction a point of this characteristic subset. Contradiction. It follows then from Proposition 5.13 that $|u'z-z|_{\dot{X}} = |\bar{u}'\bar{z}-\bar{z}| \leqslant \rho/100$, which proves our claim. Applying the triangle inequality we get

$$\begin{aligned}\left|gug^{-1}z-u'z\right|_{\dot{X}} &\leqslant \left|gug^{-1}z-z\right|_{\dot{X}} + |u'z-z|_{\dot{X}}\\ &\leqslant |uy-y|_{\dot{X}} + |u'z-z|_{\dot{X}}\\ &\leqslant \rho/50.\end{aligned}$$

However $\bar{u} = \bar{u}'$, thus $u'gu^{-1}g^{-1}$ belongs to K. Applying Proposition 5.3, we get $u' = gug^{-1}$. In particular gug^{-1} belongs to E. □

Corollary 5.21. — *Let u and u' be two elements of N. We assume that $[u] < \rho/100$ and u' is elliptic (for the action on X). If $\bar{u} = \bar{u}'$ then u and u' are conjugated in G.*

Proof. — We apply Proposition 5.20 with the elliptic subgroup $E = \langle u' \rangle$ and the set $S = \{u\}$. In particular there exists $g \in K$ such that gug^{-1} belongs to E. However by Proposition 5.17, the map $G \twoheadrightarrow \bar{G}$ induces an isomorphism from E onto its image. It follows that $gug^{-1} = u'$. □

Corollary 5.22. — *Let u and u' be two elements of N. We assume that $[u] < \rho/100$ and u' is elliptic (for the action on X). If $\bar{u}$ and $\bar{u}'$ are conjugated in $\bar{G}$ then u and u' are conjugated in G.*

Proof. — Assume that $\bar{u}$ and $\bar{u}'$ are conjugated in $\bar{G}$. In particular there exists $g \in G$ such that $\bar{u} = \bar{g}\bar{u}'\bar{g}^{-1}$. However $gu'g^{-1}$ is also an elliptic element of N. The corollary follows from Corollary 5.21 applied to u and $gu'g^{-1}$. □

Parabolic subgroups. Proposition 5.18 explains how we can lift an elliptic subgroup of $\bar{N}$ to a particular subgroup of N. We need a similar procedure for the parabolic subgroups of $\bar{N}$. This is the purpose of Proposition 5.23 to Proposition 5.25. Let $\bar{E}$ be a parabolic subgroup of $\bar{N}$ (for its action on $\bar{X}$). We denote by $\bar{\xi}$ the unique point of $\partial\bar{E} \subset \partial\bar{X}$. By Lemma 3.34, $\operatorname{Stab}(\bar{\xi})$ is a parabolic subgroup of $\bar{G}$. We also fix a point x_0 in X and write $\bar{x}_0$ for its image in $\bar{X}$. According to Lemma 5.7, there exits an $L_S\bar{\delta}$-local $(1, 11\bar{\delta})$-quasi-geodesic $\bar{\gamma} : \mathbb{R}_+ \to \bar{X}$ joining $\bar{x}_0$ to $\bar{\xi}$ and avoiding the apices of $\bar{v}(\mathcal{Q})$. Recall that $\dot{X} \setminus v(\mathcal{Q})$ is a covering space of $\bar{X} \setminus \bar{v}(\mathcal{Q})$ (Proposition 5.3). Therefore there exists a continuous path $\gamma : \mathbb{R}_+ \to \dot{X}$ starting at x_0 such that for every $t \in \mathbb{R}_+$, $\gamma(t)$ is a preimage of $\bar{\gamma}(t)$. Since the map $\dot{X} \setminus v(\mathcal{Q}) \to \bar{X} \setminus \bar{v}(\mathcal{Q})$ is a local isometry (Proposition 5.3), γ is an $L_S\bar{\delta}$-local $(1, 11\bar{\delta})$-quasi-geodesic of $\dot{X}$. In particular it defines a point $\xi = \lim_{t\to+\infty} \gamma(t)$ in the boundary at infinity of $\dot{X}$. Our goal is to prove that $\operatorname{Stab}(\xi)$ is a parabolic subgroup of G (for its action on $\dot{X}$ and thus on X) and that the map $G \twoheadrightarrow \bar{G}$ induces an isomorphism from $\operatorname{Stab}(\xi) \cap N$ onto $\operatorname{Stab}(\bar{\xi}) \cap \bar{N}$. The next proposition is the key result for our proof.

Proposition 5.23. — *Let $\bar{g} \in \operatorname{Stab}(\bar{\xi}) \cap \bar{N}$. There exists a preimage $g \in N$ of $\bar{g}$ and $t_0 \in \mathbb{R}_+$ such that for every $t \geqslant t_0$, $|g\gamma(t) - \gamma(t)|_{\dot{X}} \leqslant 114\bar{\delta}$. In particular g belongs to* $\operatorname{Stab}(\xi)$.

Proof. — By Lemma 3.35, there exists $t_0 \in \mathbb{R}_+$ such that for every $t \geqslant t_0$, $|\bar{g}\bar{\gamma}(t) - \bar{\gamma}(t)| \leqslant 97\bar{\delta}$. Without loss of generality, we can assume that $\gamma(t_0)$ lies in X. However the map $\zeta : \dot{X} \to \bar{X}$ induces an isometry from $B(\gamma(t_0), \rho/20)$ onto $B(\bar{\gamma}(t_0), \rho/20)$ (Proposition 5.4). Therefore there exists a preimage $g \in N$ of $\bar{g}$ such that $|g\gamma(t_0) - \gamma(t_0)|_{\dot{X}} = |\bar{g}\bar{\gamma}(t_0) - \bar{\gamma}(t_0)|$. Let $t \geqslant t_0$. Since γ is an infinite continuous path, there exists $t_1 \geqslant t$ such that $\gamma(t_1)$ belongs to X. In addition, $|\bar{g}\bar{\gamma}(t_1) - \bar{\gamma}(t_1)| \leqslant 97\bar{\delta}$. Let $(H, Y) \in \mathcal{Q}$. We denote by v the apex of the cone $Z(Y)$ and F the maximal normal finite subgroup of $\operatorname{Stab}(Y)$. Assume $\bar{\gamma}$ restricted to $[t_0\,, t_1]$ intersects $B(\bar{v}, 9\rho/10)$. It follows from the triangle inequality that $|\bar{g}\bar{v} - \bar{v}| < 2\rho$, thus $\bar{g}$ belongs to $\operatorname{Stab}(\bar{v})$. We claim that $\bar{g}$ is the image of an element of F. Assume on the contrary that this is false. Since $\bar{g}$ belongs to $\bar{N}$, $\bar{g}$ is not the image of an elliptic element of $\operatorname{Stab}(Y)$. According to Corollary 5.10, there exists $k \in \mathbb{Z}$ such that the axis of $\bar{g}^k$ is contained in the $6\bar{\delta}$-neighborhood of $\bar{v}$. However $\bar{g}^k$ is also an element of $\operatorname{Stab}(\bar{\xi})$. Thus by Lemma 3.35, there exists $t_2 \in \mathbb{R}_+$ such that for every $t \geqslant t_2$, $|\bar{g}^k\bar{\gamma}(t) - \bar{\gamma}(t)| \leqslant 97\bar{\delta}$, which leads to a contradiction. It follows then from Proposition 5.13 that $|g\gamma(t_1) - \gamma(t_1)|_{\dot{X}} = |\bar{g}\bar{\gamma}(t_1) - \bar{\gamma}(t_1)|$.

Applying Lemma 3.2, we get that

$$\begin{aligned}&|g\gamma(t)-\gamma(t)|_{\dot X}\\&\leqslant \max\left\{|g\gamma(t_0)-\gamma(t_0)|_{\dot X},|g\gamma(t_1)-\gamma(t_1)|_{\dot X}\right\}+2\left\langle\gamma(t_0),\gamma(t_1)\right\rangle_{\gamma(t)}+6\bar\delta\\&\leqslant 114\bar\delta.\end{aligned}$$

□

Proposition 5.24. — *The subgroup* $\mathrm{Stab}(\xi)$ *is parabolic for the action of* G *on* X*.*

Proof. — According to Proposition 4.12 it is sufficient to prove that $\mathrm{Stab}(\xi)$ is parabolic for the action of G on $\dot X$. Let $\bar g$ be an element of the parabolic subgroup $\bar E$. In particular $\bar g$ belongs to $\mathrm{Stab}(\bar\xi)\cap\bar N$. We denote by $g\in\mathrm{Stab}(\xi)$ the preimage of $\bar g$ given by Proposition 5.23. We write E for the set of all preimages of elements of $\bar E$ obtained in this way. It is a subset of $\mathrm{Stab}(\xi)\cap N$. Since $\bar E$ is parabolic the orbit $\bar E\bar x_0$ is not bounded. The map $\zeta:\dot X\to\bar X$ being 1-Lipschitz Ex_0 is unbounded as well (in $\dot X$). Consequently, $\mathrm{Stab}(\xi)$ cannot be an elliptic subgroup of G. Therefore it is sufficient to show that $\mathrm{Stab}(\xi)$ does not contain a loxodromic element. Assume on the contrary that there exists $g\in\mathrm{Stab}(\xi)$ which is a loxodromic isometry of $\dot X$. By replacing if necessary g by a power of g we can assume that $[g]_{\dot X}>L_S\bar\delta$. As a loxodromic isometry g fixes exactly two points of $\partial\dot X$, namely g^- and g^+. Being an element of $\mathrm{Stab}(\xi)$, g also fixes ξ, thus $\xi\in\{g^-,g^+\}$. We denote by $\sigma:\mathbb R\to\dot X$ a $\bar\delta$-nerve of g. According to Proposition 3.11, there exists $t_0\in\mathbb R_+$ such that for every $t\geqslant t_0$, $\gamma(t)$ is in the $37\bar\delta$-neighborhood of the axis A_g of g in $\dot X$ (Lemma 3.14). In particular, for every $t\geqslant t_0$, $|g^\epsilon\gamma(t)-\gamma(t)|_{\dot X}\leqslant[g]_{\dot X}+82\bar\delta$. Since the map $\zeta:\dot X\to\bar X$ reduces the distances, for every $t\geqslant t_0$, $|\bar g\bar\gamma(t)-\bar\gamma(t)|\leqslant[g]_{\dot X}+82\bar\delta$. Hence $\bar g$ belongs to $\mathrm{Stab}(\bar\xi)$. On the other hand $\bar\gamma$ is an $L_S\bar\delta$-local $(1,11\bar\delta)$-quasi-isometry, hence a $(2,11\bar\delta)$-quasi-geodesic (Corollary 2.7). Thus for every $t\geqslant t_0$,

$$|\bar g\bar\gamma(t)-\bar\gamma(t)|\geqslant|\bar\gamma(t+[g]_{\dot X})-\bar\gamma(t)|-500\bar\delta\geqslant\frac12[g]_{\dot X}-506\bar\delta>97\bar\delta.$$

This last point contradicts Lemma 3.35 applied with the path $\bar\gamma$ and the parabolic subgroup $\mathrm{Stab}(\bar\xi)$. □

Proposition 5.25. — *The projection* $G\twoheadrightarrow\bar G$ *induces a one-to-one map from* $\mathrm{Stab}(\xi)$ *into* $\mathrm{Stab}(\bar\xi)$*. It sends* $\mathrm{Stab}(\xi)\cap N$ *onto* $\mathrm{Stab}(\bar\xi)\cap\bar N$*. The preimage* E *of* $\bar E$ *in* $\mathrm{Stab}(\xi)\cap N$ *is a parabolic subgroup of* G *for its action on* X*.*

Proof. — Let g be an element of $\mathrm{Stab}(\xi)$. According to Proposition 5.24 $\mathrm{Stab}(\xi)$ is parabolic for the action of G on $\dot X$. By Lemma 3.35, there exits

$t_0 \in \mathbb{R}_+$ such that for every $t \geqslant t_0$, $|g\gamma(t) - \gamma(t)|_{\dot{X}} \leqslant 97\bar{\delta}$. It follows that for every $t \geqslant t_0$, $|\bar{g}\bar{\gamma}(t) - \bar{\gamma}(t)| \leqslant 97\bar{\delta}$. In particular $\bar{g}$ belongs to $\mathrm{Stab}(\bar{\xi})$. The subgroup $\mathrm{Stab}(\xi)$ is elementary and not loxodromic, thus Proposition 5.17 says that the map $G \twoheadrightarrow \bar{G}$ restricted to $\mathrm{Stab}(\xi)$ is one-to-one. The surjectivity follows from Proposition 5.23. According to Proposition 5.24, E is elementary either elliptic or parabolic. However it cannot be elliptic otherwise its image $\bar{E}$ in $\bar{G}$ would be elliptic too. □

Loxodromic subgroups. We finish this study with the case of loxodromic subgroups.

Proposition 5.26. — *Let $\bar{E}$ be a loxodromic subgroup of $\bar{N}$ (for its action on $\bar{X}$). Then $\bar{E}$ is isomorphic to a loxodromic subgroup E of N (for its action on X). Moreover if $\bar{E}$ is a* maximal *loxodromic subgroup of $\bar{N}$, then E is a also a* maximal *loxodromic subgroup of N.*

Proof. — By Corollary 5.19, $\bar{N}$ has no involution, thus $\bar{E}$ is not of dihedral type. We denote by $\bar{F}$ its maximal normal finite subgroup. There exists a loxodromic element $\bar{g} \in \bar{E}$ such that $\bar{E}$ is isomorphic to the semi-direct product $\bar{F} \rtimes \mathbb{Z}$, where $\mathbb{Z}$ is the cyclic group generated by $\bar{g}$ acting by conjugacy on $\bar{F}$. According to Lemma 3.33, the cylinder $Y_{\bar{g}}$ of $\bar{g}$ is contained in the $51\bar{\delta}$-neighborhood of $C_{\bar{F}}$. Since $Y_{\bar{g}}$ contains bi-infinite local quasi-geodesics it cannot be a subset of a ball $B(\bar{v}, \rho)$ with $v \in v(\mathcal{Q})$. Therefore we can find a point $\bar{x}$ in $C_{\bar{F}}$ which is at the same time in the $51\bar{\delta}$-neighborhood of $\zeta(X)$. Let $\bar{Z}$ be the hull of $\bar{F}\bar{x}$. It is an $\bar{F}$-invariant $6\bar{\delta}$-quasi-convex subset of $\bar{X}$ contained in the $57\bar{\delta}$-neighborhood of $\zeta(X)$. It follows from Proposition 5.12 that there exits a subset Z of $\dot{X}$ with the following properties.

(i) The map $\zeta : \dot{X} \to \bar{X}$ induces an isometry from Z onto $\bar{Z}$.
(ii) The projection $G \twoheadrightarrow \bar{G}$ induces an isomorphism from $\mathrm{Stab}(Z)$ onto $\mathrm{Stab}(\bar{Z})$.

We denote by x the preimage of $\bar{x}$ in Z and by F the preimage of $\bar{F}$ in $\mathrm{Stab}(Z)$. In particular, for every $u \in F$, $|ux - x|_{\dot{X}} \leqslant 11\bar{\delta}$. There exists a preimage $g \in N$ of $\bar{g}$ such that $|gx - x|_{\dot{X}} \leqslant |\bar{g}\bar{x} - \bar{x}| + \bar{\delta}$. As a preimage of $\bar{g}$, g is loxodromic (for its action on $\dot{X}$ and thus on X). Let $\gamma : I \to \dot{X}$ be a $(1, \bar{\delta})$-quasi-geodesic between x and gx. We denote by $\bar{\gamma}$ the path of $\bar{X}$ induced by γ. Its length satisfies the following

$$L(\bar{\gamma}) \leqslant L(\gamma) \leqslant |gx - x|_{\dot{X}} + \bar{\delta} \leqslant |\bar{g}\bar{x} - \bar{x}| + 2\bar{\delta}.$$

Thus $\bar{\gamma}$ is a $(1, 2\bar{\delta})$-quasi-geodesic. Recall that $\bar{F}$ is a normal subgroup of $\bar{E}$, consequently $C_{\bar{F}}$ is $\bar{g}$-invariant. In particular, F is finite and for every

$\bar{u} \in \bar{F}$, $|\bar{u}\bar{g}\bar{x} - \bar{g}\bar{x}| \leqslant 11\bar{\delta}$. We want to apply Proposition 5.13, with the path γ and the whole group F as the subset S. Let $(H, Y) \in \mathcal{Q}$. Let v be the apex of the cone $Z(Y)$ and $\bar{v}$ its image in $\bar{X}$. Assume that $\bar{g}$ intersects $B(\bar{v}, 9\rho/10)$. We denote by $\bar{p}$ a projection of $\bar{v}$ on $\bar{\gamma}$. Let $u \in F$ and $\bar{u}$ its image in $\bar{G}$. Recall that the endpoints $\bar{x}$ and $\bar{g}\bar{x}$ of $\bar{\gamma}$ are moved by $\bar{u}$ by at distance at most $11\bar{\delta}$. It follows from Lemma 3.2 that $|\bar{u}\bar{p} - \bar{p}| \leqslant 19\bar{\delta}$. Combined with the triangle inequality, we get $|\bar{u}\bar{v}-\bar{v}| < 2\rho$, hence $\bar{u}$ belongs to $\mathrm{Stab}(\bar{v})$. If $\bar{u}$ is not the image of an elliptic element of $\mathrm{Stab}(Y)$, then by Corollary 5.10, then there exists a power of $\bar{u}$ whose axis is contained in the $6\bar{\delta}$-neighborhood of $\bar{v}$. In particular, the characteristic subset $C_{\bar{F}}$ is contained in the $15\bar{\delta}$-neighborhood of $\bar{v}$. This contradicts the fact that $\bar{x}$ belongs to this characteristic subset. Consequently, by Proposition 5.13 for every $u \in F$, $|ugx - gx|_{\dot{X}} = |\bar{u}\bar{g}\bar{x} - \bar{g}\bar{x}|$. Let u be an element of F. Since $\bar{g}$ normalizes $\bar{F}$, the image of $g^{-1}ug$ in $\bar{N}$ is an element of $\bar{F}$. We denote by u' its preimage in F. We claim that $g^{-1}ug = u'$. Using the conclusions of Proposition 5.12 and Proposition 5.13 we have the following equalities

$$\begin{aligned} |u'x - x|_{\dot{X}} &= \left|\bar{g}^{-1}\bar{u}\bar{g}\bar{x} - \bar{x}\right| = |\bar{u}\bar{g}\bar{x} - \bar{g}\bar{x}| \\ \left|g^{-1}ugx - x\right|_{\dot{X}} &= |ugx - gx|_{\dot{X}} = |\bar{u}\bar{g}\bar{x} - \bar{g}\bar{x}|\,. \end{aligned}$$

However $\bar{g}\bar{x}$ belongs to $C_{\bar{F}}$. We get from the triangle inequality that

$$\left|g^{-1}u^{-1}gu'x - x\right| \leqslant |u'x - x| + \left|x - g^{-1}ugx\right| = 2\,|\bar{u}\bar{g}\bar{x} - \bar{g}\bar{x}| \leqslant 22\bar{\delta}.$$

Recall that u' and $g^{-1}ug$ are two preimages of the same element of $\bar{N}$. Hence $g^{-1}u^{-1}gu'$ belongs to K. By Proposition 5.3, we have $g^{-1}ug = u'$, which completes the proof of our claim. Not only g normalizes F but the projection $G \twoheadrightarrow \bar{G}$ identifies the action by conjugacy of g on F and the one of $\bar{g}$ on $\bar{F}$. Consequently the subgroup E of N generated by g and F is a loxodromic subgroup isomorphic to $\bar{E}$.

Assume now that $\bar{E}$ is a maximal loxodromic subgroup of $\bar{N}$. Let us denote by E' the maximal loxodromic subgroup of N containing E. According to Proposition 5.16, the image $\bar{E}'$ of E' in $\bar{G}$ is an elementary subgroup of $\bar{N}$. By maximality $\bar{E}' = \bar{E}$. Let g' be an element of E' whose image in $\bar{G}$ is trivial. According to Corollary 3.30 g' is either elliptic or loxodromic. If it is loxodromic, then $\langle g' \rangle$ has finite index in E', thus $\bar{E}'$ is finite, which is impossible. Hence g' is elliptic. Applying Propositon 5.17 we get $g' = 1$. In other words, the projection $G \twoheadrightarrow \bar{G}$ restricted to E' is also one-to-one, which completes the proof of the last assertion. □

5.4. Invariants of the action on $\bar{X}$.

In Section 3.5 we associated several invariants to the action of a group on a hyperbolic space. In this section we explain how the invariants for the action of $\bar{N}$ on $\bar{X}$ are related to the ones for the action of N on X.

Proposition 5.27. — *The number $e(\bar{N}, \bar{X})$ divides $e(N, X)$.*

Proof. — It follows directly from Proposition 5.26 and the definition of $e(\bar{N}, \bar{X})$ (Definition 3.37). □

Proposition 5.28. — *The invariant $\nu(\bar{N}, \bar{X})$ is at most $\nu(N, X)$.*

Proof. — Let $m \geqslant \nu(N, X)$ be an integer. Let $\bar{g}$ and $\bar{h}$ be two elements of $\bar{N}$ with $\bar{h}$ loxodromic such that $\bar{g}$, $\bar{h}^{-1}\bar{g}\bar{h}$,…, $\bar{h}^{-m}\bar{g}\bar{h}^m$ generate an elementary subgroup $\bar{E}$ of $\bar{N}$ which is not loxodromic. For every $j \in \{0, \dots, m\}$, we let $\bar{g}_j = \bar{h}^{-j}\bar{g}\bar{h}^j$.

Claim 1. — *If $\bar{E}$ is elliptic, then for every $\bar{v} \in \bar{v}(\mathcal{Q})$, $C_{\bar{E}}$ is not contained in $B(\bar{v}, \rho - 50\bar{\delta})$.*

Assume on the contrary that Claim 1 is false. Hence $\bar{E}$ is elliptic and there exists $\bar{v} \in \bar{v}(\mathcal{Q})$ such that $C_{\bar{E}}$ is contained in $B(\bar{v}, \rho - 50\bar{\delta})$. Let (H, Y) be a pair of $\mathcal{Q}$ such that $\bar{v}$ is the image in $\bar{X}$ of the apex v of the cone $Z(Y)$. The elements of $\bar{E}$ move the points of $C_{\bar{E}}$ by a distance at most $11\bar{\delta}$. Thus $\bar{E}$ is contained in $\operatorname{Stab}(\bar{v})$. Since N has no involution, the set of elliptic elements of $\operatorname{Stab}(Y) \cap N$ forms a subgroup F of $\operatorname{Stab}(Y)$ whose image in $\bar{N}$ will be denoted by $\bar{F}$ (see Remark 5.1). Note that at least one of the elements $\bar{g}_0, \dots, \bar{g}_m$ does not belong to $\bar{F}$. Indeed, if it was the case, $\bar{E}$ would be a subgroup of $\bar{F}$ and thus by Propsition 5.9, $B(\bar{v}, \rho)$ should lie in $C_{\bar{E}}$, which contradicts the assumption of Case 1. Assume that $\bar{g}_0$ does not belong to $\bar{F}$ (the proof works similarly for the other elements). According to Corollary 5.10, $\bar{v}$ is the only apex fixed by $\bar{g}_0$. However $\bar{g}_1 = \bar{h}^{-1}\bar{g}_0\bar{h}$ also belongs to $\bar{E}$ and thus $\operatorname{Stab}(\bar{v})$. It follows that $\bar{h}\bar{v}$ is also an apex fixed by $\bar{g}_0$. Hence $\bar{h}\bar{v} = \bar{v}$. Consequently $\bar{g}$ and $\bar{h}$ belong to $\operatorname{Stab}(\bar{v})$. It contradicts the fact that $\bar{h}$ is loxodromic, and completes the proof of Claim 1.

Claim 2. — *There exists an elementary subgroup E of N which is not loxodromic and a point $x \in X$ with the following properties.*

- *The map $G \twoheadrightarrow \bar{G}$ induces an isomorphism from E onto $\bar{E}$.*
- *For every $j \in \{0, \dots, m\}$, the preimage g_j of $\bar{g}_j$ in E satisfies $|g_jx - x|_{\dot{X}} = |\bar{g}_j\bar{x} - \bar{x}| \leqslant 111\bar{\delta}$.*

Moreover, for every $\bar{u} \in \bar{E}$ there exists $\bar{y}$ in $\zeta(X)$ such that $|\bar{u}\bar{y} - \bar{y}| \leqslant 111\bar{\delta}$.

The proof of Claim 2 requires to distinguish two cases.

Case 1: $\bar{E}$ is elliptic. — We claim that there exists a point of $\zeta(X)$ lying in the $50\bar{\delta}$-neighborhood of $C_{\bar{E}}$. Assume that it is not the case. According to Claim 1, $C_{\bar{E}}$ is not contained in a single ball $B(\bar{v}, \rho - 50\bar{\delta})$ for some $\bar{v} \in \bar{v}(\mathcal{Q})$. Consequently there exists two distinct apices $\bar{v}$ and $\bar{v}'$ and two points $\bar{z}, \bar{z}' \in C_{\bar{E}}$ which respectively belong to $B(\bar{v}, \rho - 50\bar{\delta})$ and $B(\bar{v}', \rho - 50\bar{\delta})$. Let $\bar{c}$ be a $(1, \bar{\delta})$-quasi-geodesic joining $\bar{z}$ to $\bar{z}'$. The points $\bar{z}$ and $\bar{z}'$ being in distinct cones, $\bar{c}$ passes through $\zeta(X)$. However $C_{\bar{E}}$ is $9\bar{\delta}$-quasi-convex. It follows that any point of $\bar{\gamma}$ is $10\bar{\delta}$-close to $C_{\bar{E}}$. In particular there exists a point of $\zeta(X)$ that is $10\bar{\delta}$-close to $C_{\bar{E}}$, which contradicts our assumption. Hence there exists a point $\bar{x} \in \zeta(X)$ that belongs to the $50\bar{\delta}$-neighborhood of $C_{\bar{E}}$. Let $x \in X$ be a preimage of $\bar{x}$ in $\dot{X}$. Applying Proposition 5.12 with the hull of $\bar{E}\bar{x}$ we get that there exists an elliptic subgroup E of N such that the map $G \twoheadrightarrow \bar{G}$ induces an isomorphism from E onto $\bar{E}$ and for every $g \in E$, $|gx - x|_{\dot{X}} = |\bar{g}\bar{x} - \bar{x}| \leqslant 111\bar{\delta}$. In this case $\bar{y} = \bar{x}$ works for all the elements of $\bar{E}$.

Case 2: $\bar{E}$ is parabolic. — We denote by $\bar{\xi}$ the unique point of $\partial\bar{E} \subset \partial\bar{X}$. Let x_0 be a point of X. According to Lemma 5.7 there exits an $L_S\bar{\delta}$-local $(1, 11\bar{\delta})$-quasi-geodesic $\bar{c} : \mathbb{R}_+ \to \bar{X}$ joining $\bar{x}_0$ to $\bar{\xi}$ and avoiding the points of $\bar{v}(\mathcal{Q})$. Recall that $\dot{X} \setminus v(\mathcal{Q})$ is a covering space of $\bar{X} \setminus \bar{v}(\mathcal{Q})$ (Proposition 5.3). Therefore the path $\bar{c}$ lifts to a continuous path $c : \mathbb{R}_+ \to \dot{X}$ starting at x_0. Since the map $\dot{X} \setminus v(\mathcal{Q}) \to \bar{X} \setminus \bar{v}(\mathcal{Q})$ is a local isometry (Proposition 5.4), c is an $L_S\bar{\delta}$-local $(1, 11\bar{\delta})$-quasi-geodesic of $\dot{X}$. In particular it defines a point $\xi = \lim_{t \to +\infty} c(t)$ in the boundary at infinity of $\dot{X}$. It follows from Proposition 5.25 that the map $G \twoheadrightarrow \bar{G}$ induces an isomorphism from $\operatorname{Stab}(\xi) \cap N$ onto $\operatorname{Stab}(\bar{\xi}) \cap \bar{N}$. We denote by E the preimage in $\operatorname{Stab}(\xi) \cap N$ of $\bar{E}$. Applying Lemma 3.35, for every $u \in E$, there exists $t_0 \in \mathbb{R}_+$ such that for every $t \geqslant t_0$, $|uc(t) - c(t)|_{\dot{X}} \leqslant 97\bar{\delta}$, which completes the proof of Claim 2.

We are going to apply Proposition 5.13 with the set $S = \{g_0, \dots g_{m-1}\}$ and a path γ that will be define later. We first check the assumption relative to the stabilizers of vertices. Let $(H, Y) \in \mathcal{Q}$. We denote by v the apex of the cone $Z(Y)$ and F the maximal finite normal subgroup of $\operatorname{Stab}(Y)$. Let $j \in \{0, \dots, m-1\}$. We claim that if $\bar{g}_j$ belongs to $\operatorname{Stab}(\bar{v})$ then $\bar{g}_j$ is the image of an element of F. Assume this is false. The element $\bar{g}_j$ belongs to $\bar{N}$, thus it is not the image of an elliptic element of $\operatorname{Stab}(Y)$. By Corollary 5.10, there exists $k \in \mathbb{Z}$ such that the axis of $\bar{u} = \bar{g}_j^k$ is contained in the $6\bar{\delta}$-neighborhood of $\bar{v}$. On the other hand, Claim 2 tells us that there exists $\bar{y}$ in $\zeta(X)$ such that $|\bar{u}\bar{y} - \bar{y}| \leqslant 111\bar{\delta}$. It contradicts Proposition 3.10.

We now fix a preimage $h \in N$ of $\bar{h}$ such that $|hx - x|_{\dot{X}} \leqslant |\bar{h}\bar{x} - \bar{x}| + \bar{\delta}$. Let $\gamma : I \to \dot{X}$ be an $L_S\bar{\delta}$-local $(1, \bar{\delta})$-quasi-geodesic joining x to hx. The path $\bar{\gamma} : I \to \bar{X}$ induced by γ is an $L_S\bar{\delta}$-local $(1, 2\bar{\delta})$-quasi-geodesic joining $\bar{x}$ to $\bar{h}\bar{x}$. We can now apply Proposition 5.13 with the path γ and the set $S = \{g_0, \dots g_{m-1}\}$. Thus for every $j \in \{0, \dots, m-1\}$, $|g_jhx - hx|_{\dot{X}} = |\bar{g}_j\bar{h}\bar{x} - \bar{h}\bar{x}|$. We denote by g the preimage of $\bar{g}$ in E ($g = g_0$). Let $j \in \{0, \dots, m-1\}$. We claim that $h^{-1}g_jh = g_{j+1}$. The proof is very similar to the one of Proposition 5.26. It follows from Claim 2 that

$$\left|\bar{g}_j\bar{h}\bar{x} - \bar{h}\bar{x}\right| = |\bar{g}_{j+1}\bar{x} - \bar{x}| = |g_{j+1}x - x|_{\dot{X}} \leqslant 111\bar{\delta}$$

By choice of h we have the following equations

$$\begin{aligned} |g_{j+1}x - x|_{\dot{X}} &= |\bar{g}_{j+1}\bar{x} - \bar{x}| = \left|\bar{g}_j\bar{h}\bar{x} - \bar{h}\bar{x}\right| \\ \left|h^{-1}g_jhx - x\right|_{\dot{X}} &= |g_jhx - hx|_{\dot{X}} = \left|\bar{g}_j\bar{h}\bar{x} - \bar{h}\bar{x}\right|. \end{aligned}$$

Since $\bar{x}$ is moved by a small distance by $\bar{g}_{j+1}$ we get

$$\begin{aligned} \left|h^{-1}g_j^{-1}hg_{j+1}x - x\right|_{\dot{X}} &\leqslant |g_{j+1}x - x|_{\dot{X}} + \left|x - h^{-1}g_jhx\right|_{\dot{X}} \\ &\leqslant 2\left|\bar{g}_{j+1}\bar{x} - \bar{x}\right| \leqslant 222\bar{\delta}. \end{aligned}$$

However g_{j+1} and $h^{-1}g_jh$ are two preimages of the same element of $\bar{G}$. Hence $h^{-1}g_j^{-1}hg_{j+1}$ belongs to K. By Proposition 5.3 we get $h^{-1}g_jh = g_{j+1}$, which completes the proof of our claim. In particular, for every $j \in \{0, \dots, m\}$, $h^{-j}gh^j = g_j$ belongs to E. Thus g, $h^{-1}gh$, …, $h^{-m}gh^m$ generate an elementary subgroup of N which is not loxodromic. However we assumed that $m \geqslant \nu(N, X)$. Consequently g and h generate an elementary subgroup of N. By Proposition 5.16, $\bar{g}$ and $\bar{h}$ generate an elementary subgroup of $\bar{N}$.

It follows from our discussion that $\nu(\bar{N}, \bar{X}) \leqslant m$. ☐

Proposition 5.29. — *Let m be an integer. Let $\bar{g}_0, \dots, \bar{g}_m$ be a collection elements of $\bar{G}$ such that for every $j \in \{0, \dots, m\}$, $[\bar{g}_j] \leqslant L_S\bar{\delta}$. One of the following holds.*

(i) *There exists $\bar{v} \in \bar{v}(\mathcal{Q})$ such that for every $j \in \{0, \dots, m\}$, $\bar{g}_j$ belongs to* $\operatorname{Stab}(\bar{v})$.
(ii) *There exist preimages $g_0, \dots, g_m$ in G of $\bar{g}_0, \dots, \bar{g}_m$ such that for every $j \in \{0, \dots, m\}$, $[g_j] \leqslant \pi \sinh[(L_S + 34)\bar{\delta}]$ and*

$$A(\bar{g}_0, \dots, \bar{g}_m) \leqslant A(g_0, \dots, g_m) + \pi \sinh\left[(L_S + 34)\,\bar{\delta}\right] + (L_S + 45)\bar{\delta}.$$

Remark. — Recall that $A(g_0, \dots, g_m)$ stands for

$$A(g_0, \dots, g_m) = \operatorname{diam}\left(A_{g_0}^{+13\delta} \cap \dots \cap A_{g_m}^{+13\delta}\right)$$

In the statement of the proposition all the metric objects are measured either with the distance of X or $\bar{X}$, but not with the one of $\dot{X}$.

Proof. — Without loss of generality we can assume that the intersection of the $13\bar{\delta}$-neighborhoods of $A_{\bar{g}_0}, \dots, A_{\bar{g}_m}$ is not empty. Let us call $\bar{Z}$ this intersection. Assume that there exists $\bar{v} \in \bar{v}(\mathcal{Q})$ and a point $\bar{z} \in \bar{Z}$ such that $|\bar{v} - \bar{z}| \leqslant \rho - (L_S/2 + 17)\bar{\delta}$. By definition any $\bar{g}_j$ moves $\bar{z}$ by a distance smaller than $[\bar{g}_j] + 34\bar{\delta} \leqslant (L_S + 34)\bar{\delta}$. It follows from the triangle inequality that every $\bar{g}_j$ belongs to $\operatorname{Stab}(\bar{v})$, which provides the first case.

We now assume that for every $\bar{v} \in \bar{v}(\mathcal{Q})$, $\bar{Z}$ does not intersect the ball of center $\bar{v}$ and radius $\rho - (L_S/2+17)\bar{\delta}$. By Lemma 2.15, $\bar{Z}$ is $7\bar{\delta}$-quasi-convex. Moreover, for every $j \in \{0, \dots, m\}$, $\bar{g}_j$ moves any point of $\bar{Z}$ by at most $(L_S + 34)\bar{\delta}$. According to Proposition 5.12, there exists a subset Z of $\dot{X}$ and a collection $g_0, \dots, g_m$ of preimages of $\bar{g}_0, \dots, \bar{g}_m$ satisfying the following properties.

(i) The map $\zeta : \dot{X} \to \bar{X}$ induces an isometry from Z onto $\bar{Z}$.
(ii) For every $z \in Z$, for every $j \in \{0, \dots, m\}$, we have $|g_j z - z|_{\dot{X}} = |\bar{g}_j \bar{z} - \bar{z}|$.

We now denote by $\bar{z}$ and $\bar{z}'$ two points of $\bar{Z}$ such that

$$|\bar{z} - \bar{z}'| \geqslant A(\bar{g}_0, \dots, \bar{g}_m) - \bar{\delta}.$$

The points z and z' stand for their preimages in Z. We write x and x' for respective projections of z and z' on X. By assumption, $\bar{Z}$ lies in the $(L_S/2+17)\bar{\delta}$-neighborhood of $\zeta(X)$. Thus $|x-z|_{\dot{X}}, |x'-z'|_{\dot{X}} \leqslant (L_S/2+17)\bar{\delta}$. In particular for every $j \in \{0, \dots, m\}$,

$$\begin{aligned}
\mu\left(|g_j x - x|\right) \leqslant |g_j x - x|_{\dot{X}} &\leqslant |g_j z - z|_{\dot{X}} + (L_S + 34)\bar{\delta} \\
&\leqslant |\bar{g}_j \bar{z} - \bar{z}| + (L_S + 34)\bar{\delta} \\
&\leqslant 2(L_S + 34)\bar{\delta} < 2\rho.
\end{aligned}$$

It follows that $|g_j x - x| \leqslant \pi \sinh[(L_S + 34)\bar{\delta}]$ (Proposition 4.2). The same holds for x'. In particular,

$$[g_j] \leqslant \pi \sinh\left[(L_S + 34)\,\bar{\delta}\right].$$

Moreover x and x' belong to the C-neighborhood of A_{g_j} where $C = \pi \sinh[(L_S + 34)\bar{\delta}]/2 + 3\bar{\delta}$ (Proposition 3.10). By Lemma 2.16,

$$|x - x'| \leqslant A(g_0, \dots, g_m) + \pi \sinh\left[(L_S + 34)\,\bar{\delta}\right] + 10\bar{\delta}$$

On the other hand, the map $X \to \dot{X}$ shorten the distances. Therefore

$$|x - x'| \geqslant |x - x'|_{\dot{X}} \geqslant |z - z'|_{\dot{X}} - (L_S + 34)\bar{\delta} = |\bar{z} - \bar{z}'| - (L_S + 34)\bar{\delta}.$$

However by construction $|\bar{z} - \bar{z}'| \geqslant A(\bar{g}_0, \dots, \bar{g}_m) - \bar{\delta}$. The conclusion of the second case follows from the last two inequalities. □

Corollary 5.30. — *The invariant $A(\bar{N}, \bar{X})$ satisfies the following inequality*

$$A(\bar{N}, \bar{X}) \leqslant A(N, X) + (\nu + 4)\pi \sinh\left(2L_S\bar{\delta}\right),$$

where ν stands for $\nu = \nu(N, X)$.

Proof. — Let $\bar{\nu}$ be the invariant $\bar{\nu} = \nu(\bar{N}, \bar{X})$. We denote by $\mathcal{A}$ the set of $(\bar{\nu}+1)$-uples $(\bar{g}_0, \dots, \bar{g}_{\bar{\nu}})$ of $\bar{N}$ such that $\bar{g}_0, \dots, \bar{g}_{\bar{\nu}}$ generate a non-elementary subgroup of $\bar{N}$ and for every $j \in \{0, \dots, \bar{\nu}\}$, $[\bar{g}_j] \leqslant L_S\bar{\delta}$. Let $(\bar{g}_0, \dots, \bar{g}_{\bar{\nu}}) \in \mathcal{A}$. Since $\bar{g}_0, \dots, \bar{g}_{\bar{\nu}}$ do not generate an elementary subgroup of $\bar{G}$, there is no apex $\bar{v} \in \bar{v}(\mathcal{Q})$ such that they all belong to $\operatorname{Stab}(\bar{v})$. According to Proposition 5.29 there exist respective preimages $g_0, \dots, g_{\bar{\nu}}$ of $\bar{g}_0, \dots, \bar{g}_{\bar{\nu}}$ in N such that

(i) for every $j \in \{0, \dots, \bar{\nu}\}$, $[g_j] \leqslant \pi \sinh[(L_S + 34)\bar{\delta}]$,
(ii) $A(\bar{g}_0, \dots, \bar{g}_{\bar{\nu}}) \leqslant A(g_0, \dots, g_{\bar{\nu}}) + \pi \sinh[(L_S + 34)\bar{\delta}] + (L_S + 45)\bar{\delta}$.

By Proposition 5.16 the subgroup of N generated by $g_0, \dots, g_{\bar{\nu}}$ is not elementary. In addition $\bar{\nu} \leqslant \nu(N, X)$ (Proposition 5.28). It follows from Corollary 3.45 that

$$\begin{aligned} A(\bar{g}_0, \dots, \bar{g}_{\bar{\nu}}) &\leqslant A(N, X) + (\nu + 3)\pi \sinh\left[(L_S + 34)\,\bar{\delta}\right] + (L_S + 725)\bar{\delta} \\ &\leqslant A(N, X) + (\nu + 4)\pi \sinh\left(2L_S\bar{\delta}\right). \end{aligned}$$

This inequality holds for every $(\bar{\nu}+1)$-uple in $\mathcal{A}$, which provides the required conclusion. □

Proposition 5.31. — *We denote by l the infimum over the stable translation length (in X) of loxodromic elements of N which do not belong to some $\operatorname{Stab}(Y)$ for $(H, Y) \in \mathcal{Q}$. Let $\bar{g}$ be an isometry of $\bar{N}$ which is not elliptic. If every preimage of $\bar{g}$ in N is loxodromic then $[\bar{g}]^\infty \geqslant \min\left\{\kappa l, \bar{\delta}\right\}$, where $\kappa = \bar{\delta}/\pi \sinh(26\bar{\delta})$.*

Remark. — If every loxodromic element of N is contained in some $\operatorname{Stab}(Y)$, $(H, Y) \in \mathcal{Q}$ then we use the convention $l = -\infty$. However this situation will never happen. According to Proposition 5.15, the quotient $\bar{G}$ is non-elementary. For our application, $\bar{N}$ will be a finite index subgroup of $\bar{G}$. In particular it will be non-elementary. Thus it will be possible a find a loxodromic element in N which does not belong to some $\operatorname{Stab}(Y)$ for $(H, Y) \in \mathcal{Q}$.

Proof. — Recall that for every $m \in \mathbb{N}$, we have $m[\bar g]^\infty \geqslant [\bar g^m] - 16\bar\delta$ (Proposition 3.1). Therefore it suffices to find an integer m such that $[\bar g^m] \geqslant m \min\left\{\kappa l, \bar\delta\right\} + 16\bar\delta$. We denote by m the largest integer satisfying $m \min\left\{\kappa l, \bar\delta\right\} \leqslant 2\bar\delta$. Assume that $[\bar g^m]$ is smaller than $m\min\left\{\kappa l, \bar\delta\right\} + 16\bar\delta$. In particular, $[\bar g^m] \leqslant 18\bar\delta$, thus every point in $A_{\bar g^m}$ is moved by $\bar g^m$ by a distance at most $26\bar\delta$. In follows that for every $\bar v \in \bar v(\mathcal{Q})$, the set $A_{\bar g^m}$ does not intersect $B(\bar v, \rho - 13\bar\delta)$. Indeed if it was the case, $\bar g^m$ would fix $\bar v$ which contradicts the fact that $\bar g$ is not elliptic. By Proposition 5.12, there exists a subset A of $\dot X$ such that the map $\zeta : \dot X \to \bar X$ induces an isometry from A onto $A_{\bar g^m}$ and the projection $\pi : G \twoheadrightarrow \bar G$ induces an isomorphism from $\operatorname{Stab}(A)$ onto $\operatorname{Stab}(A_{\bar g^m})$. We denote by g the preimage of $\bar g$ in $\operatorname{Stab}(A)$. By assumption g is a loxodromic element of N, therefore $[g]^\infty \geqslant l$. Let $\bar x$ be a point of $A_{\bar g^m}$, x the preimage of $\bar x$ in A and y a projection of x on X. Recall that $\bar x$ lies in the 13δ-neighborhood of $\zeta(X)$, thus $|x - y|_{\dot X} \leqslant 13\bar\delta$. Using the triangle inequality we get

$$\begin{aligned}\mu\left(|g^m y - y|_X\right) \leqslant |g^m y - y|_{\dot X} \leqslant |g^m x - x|_{\dot X} + 26\bar\delta = |\bar g^m \bar x - \bar x|_{\bar X} + 26\bar\delta\\ \leqslant 52\bar\delta < 2\rho.\end{aligned}$$

By Proposition 4.2,

$$ml \leqslant m[g]^\infty \leqslant |g^m y - y|_X \leqslant \pi \sinh(26\bar\delta) = \kappa^{-1}\bar\delta,$$

which contradicts the maximality of m. □

Corollary 5.32. — *We denote by l the infimum over the stable translation length (in X) of loxodromic elements of N which do not belong to some* $\operatorname{Stab}(Y)$ *for $(H, Y) \in \mathcal{Q}$. Then $r_{inj}\left(\bar N, \bar X\right) \geqslant \min\left\{\kappa l, \bar\delta\right\}$, where $\kappa = \bar\delta/\pi\sinh(26\bar\delta)$.*

6. Applications

6.1. Partial periodic quotients

The next proposition will play the role of the induction step in the proof of the main theorem.

Proposition 6.1. — *There exist positive constants δ_1, L_S, A_0, r_0, α such that for every positive integer ν_0 there is an integer n_0 with the following properties. Let G be a group acting by isometries on a δ_1-hyperbolic length space X. We assume that this action is WPD and non-elementary. Let N be a normal subgroup of G without involutions. Let $n_1 \geqslant n_0$ and*

$n \geqslant n_1$ be an odd integer. We denote by P the set of loxodromic elements h of N which are primitive as elements of N such that $[h] \leqslant L_S\delta_1$. Let K be the (normal) subgroup of G generated by $\{h^n, h \in P\}$ and $\bar{G}$ the quotient of G by K. We make the following assumptions.

(i) *$e(N, X)$ divides n.*
(ii) *$\nu(N, X) \leqslant \nu_0$.*
(iii) *$A(N, X) \leqslant \nu_0 A_0$.*
(iv) *$r_{inj}(N, X) \geqslant r_0/\sqrt{n_1}$.*

Then there exists a δ_1-hyperbolic length space $\bar{X}$ on which $\bar{G}$ acts by isometries. This action is WPD and non-elementary. The image $\bar{N}$ of N in $\bar{G}$ has no involution. Moreover it satisfies Assumptions (i)-(iv). In addition, the map $G \to \bar{G}$ has the following properties.

- *For every $g \in G$, if $\bar{g}$ stands for its image in $\bar{G}$, we have*
$$[\bar{g}]^\infty_{\bar{X}} \leqslant \frac{\alpha}{\sqrt{n_1}}[g]^\infty_X.$$
- *For every non-loxodromic elementary subgroup E of G, the map $G \to \bar{G}$ induces an isomorphism from E onto its image $\bar{E}$ which is elementary and non-loxodromic.*
- *Let $\bar{g}$ be an elliptic (respectively parabolic) element of $\bar{N}$. Either $\bar{g}^n = 1$ or $\bar{g}$ is the image of an elliptic (respectively parabolic) element of N.*
- *Let $u, u' \in N$ such that $[u] < L_S\delta_1$ and u' is elliptic. If the respective images of u and u' are conjugated in $\bar{G}$ then so are u and u' in G.*

Vocabulary. — Let G be a group acting by isometries on a space X and N a normal subgroup of G. Once ν_0, n_1 and n have been fixed, if G, N and X satisfy the assumption of the proposition including Points (i)-(iv), we will write that (G, N, X) *satisfies the induction hypotheses for exponent n*. The proposition says in particular that if (G, N, X) satisfies the induction hypotheses for exponent n then so does $(\bar{G}, \bar{N}, \bar{X})$.

Proof. — We start by defining all the constants that appear at the beginning of the proposition. The parameter L_S is still the one that comes from the stability of quasi-geodesics (see Definition 2.8). The parameters ρ_0, δ_0 and Δ_0 are the given by the small cancellation theorem (Theorem 5.2). We let $\delta_1 = 54.10^4\boldsymbol{\delta}$. The constant $\kappa = \delta_1/\pi\sinh(26\delta_1)$ is chosen to apply later Corollary 5.32. We let

$$A_0 = 6\pi\sinh(2L_S\delta_1), \quad r_0 = 2\sqrt{\pi\sinh\rho_0\kappa L_S\delta_1}, \quad \text{and} \quad \alpha = 4\sqrt{\frac{\pi\sinh\rho_0}{\kappa L_S\delta_1}}.$$

Note that

$$\alpha r_0 = 8\pi \sinh \rho_0, \quad \text{while} \quad \alpha\kappa L_S \delta_1/2 = r_0. \tag{6.1}$$

Let $\nu_0 \geqslant 1$ be an integer. We now define the critical exponent n_0. To that end we consider a rescaling parameter λ_n depending on an integer n

$$\lambda_n = \frac{\alpha}{\sqrt{n}} = 4\sqrt{\frac{\pi \sinh \rho_0}{n\kappa L_S \delta_1}}.$$

The sequence (λ_n) converges to 0 as n approaches infinity. Therefore there exists an integer $n_0 \geqslant 100$ such that for every $n \geqslant n_0$

$$\lambda_n \delta_1 \leqslant \delta_0 \tag{6.2}$$

$$\lambda_n \left(\nu_0 A_0 + 118\delta_1\right) \leqslant \min\left\{\Delta_0, \pi \sinh(2L_S\delta_1)\right\} \tag{6.3}$$

$$\lambda_n \kappa L_S \delta_1/2 \leqslant \delta_1 \tag{6.4}$$

$$\lambda_n \rho_0 \leqslant \rho_0 \tag{6.5}$$

Let $n_1 \geqslant n_0$ and $n \geqslant n_1$ be an odd integer. For simplicity we denote by λ the rescaling parameter $\lambda = \lambda_{n_1}$. Let G be a group acting by isometries on a metric space X and N a normal subgroup of G such that (G, N, X) satisfies the induction hypotheses for exponent n. We denote by P the set of loxodromic elements h of N which are primitive as elements of N such that $[h] \leqslant L_S\delta_1$. Let K be the normal subgroup of G generated by $\{h^n, h \in P\}$. Note that P in invariant under conjugacy, thus K is contained in N. We write $\bar{G} = G/K$ for the quotient of G by K and $\bar{N} = N/K$ for the image of N in $\bar{G}$. We are going to prove that $\bar{G}$ is a small cancellation quotient of G. To that end we consider the action of G on the rescaled space λX. In particular it is a δ-hyperbolic space, with $\delta = \lambda\delta_1 \leqslant \delta_0$. Unless stated otherwise, we will always work with the rescaled space λX. We define the family $\mathcal{Q}$ by

$$\mathcal{Q} = \left\{ \left(\langle h^n \rangle, Y_h\right) \mid h \in P \right\}.$$

Lemma 6.2. — *The family $\mathcal{Q}$ satisfies the following assumptions:*

- $\Delta\left(\mathcal{Q}\right) \leqslant \Delta_0$
- $T\left(\mathcal{Q}\right) \geqslant 8\pi \sinh \rho_0$.

Proof. — We start with the upper bound of $\Delta(\mathcal{Q})$. Let h_1 and h_2 be two elements of P such that $(\langle h_1^n \rangle, Y_{h_1}) \neq (\langle h_2^n \rangle, Y_{h_2})$. According to Lemma 3.14, Y_{h_1} and Y_{h_2} are respectively contained in the 52δ-neighborhood of A_{h_1} and A_{h_2}, thus by Lemma 2.16

$$\operatorname{diam}\left(Y_{h_1}^{+5\delta} \cap Y_{h_2}^{+5\delta}\right) \leqslant \operatorname{diam}\left(A_{h_1}^{+13\delta} \cap A_{h_2}^{+13\delta}\right) + 118\delta.$$

According to Proposition 3.39, h_1 and h_2 generate a non-elementary subgroup of N. On the other hand, their translation lengths in λX are at most $L_S\delta$, thus

$$\begin{aligned}\operatorname{diam}\left(Y_{h_1}^{+5\delta}\cap Y_{h_2}^{+5\delta}\right) \leqslant A(N,\lambda X)+118\delta &\leqslant \lambda A(N,X)+118\lambda\delta_1\\ &\leqslant \lambda(\nu_0 A_0+118\delta_1).\end{aligned}$$

Thus by (6.3), $\Delta(\mathcal{Q}) \leqslant \Delta_0$. Let us focus now on $T(\mathcal{Q})$. Equation (6.1) allows to bound from below the injectivity radius of N on λX as follows

$$r_{inj}\left(N,\lambda X\right) \geqslant \frac{\lambda r_0}{\sqrt{n_1}} = \frac{\alpha r_0}{n_1} = \frac{8\pi\sinh\rho_0}{n_1} \geqslant \frac{8\pi\sinh\rho_0}{n}$$

In particular, for every $h\in P$ we have $[h^n]^\infty = n[h]^\infty \geqslant 8\pi\sinh\rho_0$. Hence $T(\mathcal{Q}) \geqslant 8\pi\sinh\rho_0$. □

On account of the previous lemma, we can now apply the small cancellation theorem (Theorem 5.2) to the action of G on the rescaled space λX and the family $\mathcal{Q}$. We denote by $\dot{X}$ the space obtained by attaching on λX for every $(H,Y)\in\mathcal{Q}$, a cone of radius ρ_0 over the set Y. The quotient of $\dot{X}$ by K is the space $\bar{X}$. According to Theorem 5.2, $\bar{X}$ is a δ_1-hyperbolic length space and $\bar{G}$ acts by isometries on it. By Proposition 5.14 and Proposition 5.15 this action is WPD and non-elementary. It follows from Corollary 5.19 that $\bar{N}$ has no involution. We now prove that the action of $\bar{N}$ on $\bar{X}$ also satisfies Assumptions (i)-(iv).

Lemma 6.3. — *The invariant $e(\bar{N},\bar{X})$ and $\nu(\bar{N},\bar{X})$ satisfy the following:*

- *$e(\bar{N},\bar{X})$ divides n*
- *$\nu(\bar{N},\bar{X}) \leqslant \nu_0$*

Proof. — By Proposition 5.27, $e(\bar{N},\bar{X})$ divides $e(N,X)$. Thus the first point follows from Assumption (i) of the proposition. The second one is a consequence of Proposition 5.28 and Assumption (ii) □

Lemma 6.4. — *The constant $A(\bar{N},\bar{X})$ is bounded above by ν_0A_0 whereas $r_{inj}\left(\bar{N},\bar{X}\right)$ is bounded below by $r_0/\sqrt{n_1}$.*

Proof. — We start with the upper bound of $A(\bar{N},\bar{X})$. According to Corollary 5.30,

$$\begin{aligned}A(\bar{N},\bar{X}) &\leqslant A(N,\lambda X)+(\nu(N,X)+4)\pi\sinh\left(2L_S\delta_1\right)\\ &\leqslant A(N,\lambda X)+(\nu_0+4)\pi\sinh\left(2L_S\delta_1\right).\end{aligned}$$

However the inequality (6.3) gives

$$A(N,\lambda X)=\lambda A(N,X) \leqslant \lambda\nu_0 A_0 \leqslant \pi\sinh\left(2L_S\delta_1\right).$$

Thus $A(\bar{N},\bar{X})$ is bounded above by $(\nu_0+5)\pi \sinh(2L_S\delta_1)$. However $\nu_0 \geqslant 1$, hence $A(\bar{N},\bar{X}) \leqslant \nu_0 A_0$. We now focus on the injectivity radius of $\bar{N}$. Let g be a loxodromic isometry of N which does not belong to the stabilizer of Y_h for any $h \in P$. Its asymptotic translation length in λX is larger than $\lambda L_S\delta_1/2$ (Proposition 3.1). Corollary 5.32 combined with (6.4) and (6.1) gives

$$r_{inj}\left(\bar{N},\bar{X}\right) \geqslant \min\left\{\frac{\lambda\kappa L_S\delta_1}{2},\delta_1\right\} = \frac{\lambda\kappa L_S\delta_1}{2} = \frac{\alpha\kappa L_S\delta_1}{2\sqrt{n_1}} = \frac{r_0}{\sqrt{n_1}}. \qquad \square$$

Lemma 6.3 and Lemma 6.4 show that $(\bar{G},\bar{N},\bar{X})$ satisfies the induction hypotheses for exponent n. To finish the proof we focus on the properties on the map $G \to \bar{G}$.

Lemma 6.5. — *For every $g \in G$, we have*

$$[\bar{g}]^{\infty}_{\bar{X}} \leqslant \frac{\alpha}{\sqrt{n_1}}[g]^{\infty}_{X}.$$

Proof. — Let $g \in G$. The asymptotic translation length of g in the rescaled space λX is $[g]^{\infty}_{\lambda X} = \lambda[g]^{\infty}_{X}$. On the other hand the map $\lambda X \to \bar{X}$ shortens the distances, thus $[\bar{g}]^{\infty}_{\bar{X}} \leqslant \lambda[g]^{\infty}_{X}$. □

Lemma 6.6. — *Let E be a non-loxodromic elementary subgroup of G. The map $G \to \bar{G}$ induces an isomorphism from E onto its image $\bar{E}$ which is elementary and non-loxodromic.*

Proof. — This lemma follows from Propositions 5.16 and 5.17. □

Lemma 6.7. — *Let $\bar{g}$ be an elliptic (respectively parabolic) element of $\bar{N}$. Either $\bar{g}^n = 1$ or $\bar{g}$ is the image of an elliptic (respectively parabolic) element of N.*

Proof. — If $\bar{g}$ is parabolic, it follows from Proposition 5.25. Assume now that $\bar{g}$ is elliptic. We denote by $\bar{E}$ the subgroup of $\bar{N}$ generated by $\bar{g}$. According to Proposition 5.18, there are two cases.

- (i) In the first case, there exists $h \in P$ such that $\bar{E}$ is embedded in $\operatorname{Stab}(Y_h)/\langle h^n\rangle$. However $e(N,X)$ divides n. Therefore the order of any element of $\bar{N}$ in this group divides n (see Definition 3.37).
- (ii) In the second case $\bar{E}$ is isomorphic to an elliptic subgroup E of G. Hence $\bar{g}$ has an elliptic preimage in G. □

Lemma 6.8. — *Let $u,u' \in N$ such that $[u] < L_S\delta_1$ and u' is elliptic. If the respective images of u and u' are conjugated (in $\bar{G}$) so are u and u' in G.*

Proof. — This lemma follows directly from Corollary 5.22. □

These last lemmas complete the proof of Proposition 6.1. □

Theorem 6.9. — *Let X be a hyperbolic length space. Let G be a group acting by isometries on X. We suppose that this action is WPD and non-elementary. Let N be a normal subgroup of G without involution. In addition we assume that $e(N,X)$ is odd, $\nu(N,X)$ and $A(N,X)$ are finite and $r_{inj}(N,X)$ is positive. There is a critical exponent n_1 such that every odd integer $n \geqslant n_1$ which is a multiple of $e(N,X)$ has the following property. There exists a normal subgroup K of G contained in N such that*

- *if E is an elementary subgroup of G which is not loxodromic, then the projection $G \twoheadrightarrow G/K$ induces an isomorphism from E onto its image;*
- *for every element $g \in N/K$, either $g^n = 1$ or g is the image an elliptic or a parabolic element of N;*
- *there are infinitely many elements in N/K which are not the image of an elliptic or a parabolic element of G;*
- *every non-trivial element of K is loxodromic;*
- *As a normal subgroup, K is not finitely generated.*

Remark. — For most of our examples we will simply take $N = G$. However this more general statement is useful to avoid some problems coming from the 2-torsion.

Proof. — The main ideas of the proof are the following. Using Proposition 6.1 we construct by induction a sequence of groups $G_0 \to G_1 \to G_2 \to \dots$ where G_{k+1} is obtained from G_k by adding new relations of the form h^n with $h \in N$. Then we chose for the quotient G/K the direct limit of these groups. Let us put $\nu_0 = \nu(N,X)$ (which is finite by assumption). The parameters L_S, δ_1, A_0, r_0, α and n_0 are the one given by Proposition 6.1.

Critical exponent. The invariant $A(N,X)$ is finite. By rescaling if necessary the space X we can assume that $\delta \leqslant \delta_1$ and $A(N,X) \leqslant \nu_0 A_0$. By assumption $r_{inj}(N,X) > 0$. Therefore, there exists an integer $n_1 \geqslant n_0$ such that $r_{inj}(N,X) \geqslant r_0/\sqrt{n_1}$. Without loss of generality we can also assume that $\alpha/\sqrt{n_1} < 1$. From now on, we fix an odd integer $n \geqslant n_1$ which is a multiple of $e(N,X)$.

Initialization. We let $G_0 = G$, $N_0 = N$ and $X_0 = X$. In particular, (G_0, N_0, X_0) satisfies the induction hypotheses for exponent n.

Induction. We assume that we already constructed the groups G_k, N_k and the space X_k such that (G_k, N_k, X_k) satisfies the induction hypotheses for exponent n. We denote by P_k the set of loxodromic elements $h \in N_k$ such that $[h]_{X_k} \leqslant L_S\delta_1$ which are primitive *as elements of* N_k. Let K_k be the normal subgroup of G_k generated by $\{h^n, h \in P_k\}$. We write G_{k+1} for the quotient of G_k by K_k and N_{k+1} for the image of N_k in G_{k+1}. In particular N_{k+1} is a normal subgroup of G_{k+1}. By Proposition 6.1, there exists a metric space X_{k+1} such that $(G_{k+1}, N_{k+1}, X_{k+1})$ satisfies the induction hypotheses for exponent n. Moreover the projection $G_k \twoheadrightarrow G_{k+1}$ fulfills the following properties.

(i) For every $g \in G_k$, if we still denote by g its image in G_{k+1} we have $[g]^\infty_{X_{k+1}} \leqslant (\alpha/\sqrt{n_1})[g]^\infty_{X_k}$.
(ii) For every non-loxodromic elementary subgroup E of G_k, the map $G_k \twoheadrightarrow G_{k+1}$ induces an isomorphism from E onto its image which is elementary and non-loxodromic.
(iii) For every elliptic or parabolic element $g \in N_{k+1}$, either $g^n = 1$ or g is the image of an elliptic or a parabolic element of N_k.
(iv) Let $u, u' \in N_k$ such that $[u]_{X_k} < L_S\delta_1$ and u' is elliptic. If the respective images of u and u' are conjugated in G_{k+1} so are u and u' in G_k.

Direct limit. The direct limit of the sequence (G_k) is a quotient G/K of G. We claim that this group satisfies the announced properties. Let g be an element of G. To shorten the notation we will still denote by g its images in G, G_k or G/K.

Properties of G/K. Let E be an elementary subgroup of G which is not loxodromic. A proof by induction on k shows that for every $k \in \mathbb{N}$, the map $G \twoheadrightarrow G_k$ induces an isomorphism from E onto its image which is an elementary subgroup of G_k either elliptic or parabolic. It follows that $G \twoheadrightarrow G/K$ induces an isomorphism from E onto its image. This proves the first point of the theorem.

Let g be a non-trivial element of K. Assume that contrary to our claim g is not loxodromic. Then $\langle g \rangle$ is an elementary subgroup of G either elliptic or parabolic. Therefore the map $G \twoheadrightarrow G/K$ induces an isomorphism from $\langle g \rangle$ onto its image. In particular, g is not trivial in G/K, and thus cannot belong to K. Contradiction.

A proof by induction on k shows that if g is a element of N_k which is not loxodromic, then either $g^n = 1$ or g is the image of an elliptic or a parabolic element of N. Let g be an element of N/K which is not the image of an elliptic or a parabolic element of N. We still denote by g a preimage of g in N. In particular, g is loxodromic. It follows from the construction of the sequence (G_k) that for every $k \in \mathbb{N}$, we have $[g]^\infty_{X_k} \leqslant (\alpha/\sqrt{n_1})^k [g]^\infty_X$. Recall that $\alpha/\sqrt{n_1} < 1$. Hence, there exists an integer k such that $[g]^\infty_{X_k} < r_0/\sqrt{n_1} \leqslant r_{inj}(N_k, X_k)$. As an element of G_k the isometry g is not loxodromic. Consequently, as an element of N_k, $g^n = 1$. The same holds in G/K.

Denote by $\mathcal{P}$ for the set of all loxodromic elements of N which are not identified in G/K with an elliptic or a parabolic element of G. Assume that the image of $\mathcal{P}$ in N/K is finite. In particular there exists a finite subset $\mathcal{S}$ of $\mathcal{P}$ such that $\mathcal{P}$ lies in $\mathcal{S} \cdot K$. Using a similar argument as previously we see that there exists $s \in \mathbb{N}$ such that every element of $\mathcal{S}$ is elliptic or parabolic in N_s. Fix $g \in \mathcal{P}$ a preimage in N of an element of P_s (recall that P_s is the set of elements of N_s whose n-th power is "killed" in G_{s+1}). By construction g is loxodromic in N_s with $[g]_{X_s} \leqslant L_S \delta_1$ and elliptic in N_{s+1}. However $\mathcal{P}$ is a subset of $\mathcal{S} \cdot K$. Therefore there exits $t > s$ such that g belongs to $\mathcal{S}$ as an element of N_t. An induction using the Property (iv) about conjugates shows that g is actually conjugated to an element of $\mathcal{S}$ in N_s. However in N_s, g is loxodromic whereas all elements of S are elliptic. Contradiction.

For every $k \in \mathbb{N}$, the action of G_k on X_k is non-elementary. It follows that the sequence (G_k) does not ultimately stabilize. Thus K is infinitely generated as a normal subgroup. □

6.2. Acylindrical action on a hyperbolic space

Our main source of examples comes from groups acting acylindrically on a hyperbolic space. We recall and prove here a few properties of this kind of actions. They will be useful to satisfy the assumptions of Theorem 6.9. In this section, X is a δ-hyperbolic length space endowed with an action by isometries of a group G.

Definition 6.10. — *The action of G on X is* acylindrical *if for every $l \geqslant 0$, there exist $d \geqslant 0$ and $N > 0$ such that for all $x, x' \in X$ with $|x - x'| \geqslant d$, there are at most N elements $u \in G$ satisfying $|ux - x| \leqslant l$ and $|ux' - x'| \leqslant l$.*

Note that if a group G acts acylindrically on a hyperbolic space, this action is also WPD (see Definition 3.17). However the acylindricity condition is much stronger. In particular the parameters d and N are uniform:

they only depend on l and not on the points x and x'. As a consequence, of G acts acylindrically on X then G has no parabolic element (for this action) [3, Lemma 2.2]. A proper and co-compact action on a hyperbolic space is acylindrical. An other example is the action of the mapping class group of a surface on the complex of curves. More examples are detailed in Section 6.3. From now on, we will assume that G acts acylindrically on X.

Lemma 6.11 (Bowditch [3, Lemma 2.2]). — *The injectivity radius $r_{inj}(G, X)$ is positive.*

Lemma 6.12. — *The invariant $\nu(G, X)$ is finite.*

Proof. — By acylindricity, there exist positive constants d and N with the following property. For every $x, x' \in X$ with $|x - x'| \geqslant d$, there are at most N elements $u \in G$ satisfying $|ux - x| \leqslant 97\delta$ and $|ux' - x'| \leqslant 97\delta$. Moreover there exists $M > 0$ such that $M r_{inj}(G, X) \geqslant d$ (Lemma 6.11). Let m be an integer such that $m \geqslant N + M$. Let $g, h \in G$ with h loxodromic. Assume that $g, h^{-1}gh, \dots, h^{-m}gh^m$ generate an elementary subgroup of G which is not loxodromic. According to Proposition 3.27 and Lemma 3.35 there exists a point $x \in X$ such that for every $j \in \{0, \dots, m\}$, $|h^{-j}gh^jx - x| \leqslant 97\delta$. In particular for every $j \in \{0, \dots, N\}$, we have

$$\left|h^{-j}gh^jx - x\right| \leqslant 97\delta \quad \text{and} \quad \left|h^{-j}gh^j(h^Mx) - h^Mx\right| \leqslant 97\delta.$$

However by choice of M, $|h^Mx - x| \geqslant d$. It follows from acylindricity that the set

$$\left\{h^{-j}gh^j \;\middle|\; 0 \leqslant j \leqslant N\right\}$$

contains at most N elements. Therefore there exists $j \in \{1, \dots, N\}$ such that $h^{-j}gh^j = g$. Hence g stabilizes $\{h^-, h^+\}$ where h^- and h^+ are the points of the boundary ∂X fixed by h. In particular, g and h generate an elementary subgroup of G. Consequently, $\nu(G, X)$ is bounded above by $N + M$. □

We now focus on the invariant $A(G, X)$. Recall first that given $(m + 1)$ elements $g_0, \dots, g_m$ of G the quantity $A(g_0, \dots, g_m)$ is defined by

$$A(g_0, \dots, g_m) = \operatorname{diam}\left(A_{g_0}^{+13\delta} \cap \dots \cap A_{g_m}^{+13\delta}\right).$$

Lemma 6.13. — *There exist $\ell \in \mathbb{N}$ and $C > 0$ with the following property. Let $m \in \mathbb{N}$. Let $g_0, \dots, g_m$ be $(m + 1)$ elements of G which generate a non-elementary subgroup. If $A(g_0, \dots, g_m) > C$ then there exists a loxodromic element which is the product of at most ℓ elements of $\{g_0, \dots, g_m\}$ and their inverses.*

Proof. — Since G acts acylindrically on X there exist $N \in \mathbb{N}$ and $d > 0$ with the following property. For every $x, x' \in X$ with $|x - x'| \geqslant d$, there are at most N elements $u \in G$ such that $|ux - x| \leqslant 50\delta$ and $|ux' - x'| \leqslant 50\delta$. We let $\ell = N$ and $C = d + (50\ell + 10)\delta$. Let $g_0, \dots, g_m$ be $(m+1)$ elements of G which generate a non-elementary subgroup H of G. Suppose that $A(g_0, \dots, g_m) > C$. We denote by S the set of elements of G that can be written as a product of at most ℓ elements of $\{g_0, \dots, g_m\}$ and their inverses. Assume that, contrary to our claim, no element of S is loxodromic. In particular, $g_0, \dots, g_m$ are not loxodromic, thus their translation length it at most 16δ (Proposition 3.1).

Let $h \in S$. Let x be a point in the intersection of the respective 13δ-neighborhoods of the axis $A_{g_1}, \dots, A_{g_m}$. For every $j \in \{0, \dots, m\}$, $|g_j x - x| \leqslant 50\delta$. It follows from the triangle inequality that $|hx - x| \leqslant 50\ell\delta$. According to Proposition 3.10 (ii), x lies in the $(25\ell + 3)\delta$-neighborhood of A_h. It follows that

$$\operatorname{diam}\left(\bigcap_{h \in S} A_h^{+(25\ell+3)\delta}\right) \geqslant A(g_0, \dots, g_m) > C.$$

Applying Lemma 2.16, we get that

$$\operatorname{diam}\left(\bigcap_{h \in S} A_h^{+13\delta}\right) > C - (50\ell + 10)\delta \geqslant d.$$

In particular, there exist two points $x, x' \in X$ with $|x - x'| \geqslant d$ such that for every $h \in S$, x and x' belong to the 13δ-neighborhood of A_h. By assumption the elements of S are not loxodromic, thus for every $h \in S$, $|hx - x| \leqslant 50\delta$ and $|hx' - x'| \leqslant 50\delta$. By choice of N and d, the set S contains at most N elements. However $\ell = N$. It follows that every element of S which is exactly the product of ℓ elements of $\{g_0, \dots, g_m\}$ and their inverses can be written as a shorter product. In particular, any element of the subgroup H generated by $\{g_0, \dots, g_m\}$ can be written as a product of at most $N - 1$ elements of $\{g_1, \dots, g_m\}$ and their inverses. Thus H is finite. It contradicts the fact that H is non-elementary. □

Lemma 6.14. — *The invariant $A(G, X)$ is finite.*

Proof. — We first need to define a few parameters. For simplicity we let $\nu = \nu(G, X)$ which is finite according to Lemma 6.12. As in Section 3.5, we denote by $\mathcal{A}$ the set of all $(\nu+1)$-uples $(g_0, \dots, g_\nu)$ such that $g_0, \dots, g_\nu$ generate a non-elementary subgroup of G and for all $j \in \{0, \dots, \nu\}$, $[g_j] \leqslant L_S\delta$. According to Lemma 6.13, there exist $\ell \in \mathbb{N}$ and $C > 0$ with the following property. For every $(g_0, \dots, g_\nu) \in \mathcal{A}$, if $A(g_0, \dots, g_\nu) > C$ then there exists a loxodromic element which is the product of at most ℓ elements of $\{g_0, \dots, g_\nu\}$ and their inverses. By Lemma 6.11, $r_{inj}(G, X)$ is positive, thus there is an integer m such that $mr_{inj}(G, X) > L_S\delta$. Finally, by acylindricity, there exist $N \in \mathbb{N}$ and $d > 0$ such that for every $x, y \in X$ with $|x - y| \geqslant d$, there are at most N elements $u \in G$ satisfying $|ux - x| \leqslant (L_S + 74)\delta$ and $|uy - y| \leqslant (L_S + 74)\delta$. We claim that

$$A(G, X) \leqslant \max\{C, d + (N + 1)m\ell(L_S + 34)\delta + (N + 54)\delta\}.$$

Assume that our assertion is false. There exists $(g_0, \dots, g_\nu) \in \mathcal{A}$ such that

$$A(g_0, \dots, g_\nu) > \max\{C, d + (N + 1)m\ell(L_S + 34)\delta + (N + 54)\delta\}.$$

In particular, $A(g_0, \dots, g_\nu) > C$. By choice of C and ℓ there exists a loxodromic element which is the product of at most ℓ elements of $\{g_1, \dots, g_\nu\}$ and their inverses. Taking its m-th power we obtain an element $h \in G$ with the following properties.

(i) h is the product of at most $m\ell$ elements of $\{g_1, \dots, g_\nu\}$ and their inverses.

(ii) $[h] \geqslant mr_{inj}(G, X) > L_S\delta$.

Let $\gamma\colon \mathbb{R} \to X$ be a δ-nerve of h and T its fundamental length. Let x be a point in the intersection of the respective 13δ-neighborhoods of the axis $A_{g_0}, \dots, A_{g_\nu}$. By definition for every $j \in \{0, \dots, \nu\}$, $|g_jx - x| \leqslant (L_S + 34)\delta$. It follows from the triangle inequality that $|hx - x| \leqslant m\ell(L_S + 34)\delta$. Hence

$$T \leqslant [h] + \delta \leqslant m\ell(L_S + 34)\delta + \delta.$$

Moreover, according to Proposition 3.10 (ii), the distance between x and A_h is at most $m\ell(L_S/2 + 17)\delta + 3\delta$. Since $[h] > L_S\delta$, the axis A_h lies in the 10δ-neighborhood of γ (Lemma 3.15). Thus x belongs to the D-neighborhood of γ where $D = m\ell(L_S/2 + 17)\delta + 13\delta$. In particular

$$\operatorname{diam}\left(\gamma^{+D} \cap A_{g_0}^{+13\delta} \cap \dots \cap A_{g_\nu}^{+13\delta}\right) = A(g_0, \dots, g_\nu).$$

By Lemma 2.16, we get that for every $j \in \{0, \dots, \nu\}$,

$$\begin{aligned}\operatorname{diam}\left(\gamma^{+12\delta} \cap A_{g_j}^{+13\delta}\right) &\geqslant A(g_0, \dots, g_\nu) - 2D - 4\delta \\ &> d + Nm\ell(L_S + 34)\delta + (N + 24)\delta\end{aligned}$$

Let $j \in \{0, \dots, \nu\}$. According to the previous inequality there exists points $x = \gamma(s)$ and $x' = \gamma(s')$ in the 25δ-neighborhood of the axis of g_j such that

$$|x - x'| \geqslant d + Nm\ell(L_S + 34)\delta + N\delta \geqslant d + NT. \tag{6.6}$$

By replacing if necessary h by h^{-1} we can assume that $s \leqslant s'$. By stability of quasi-geodesics, for all $t \in [s, s']$, $\langle x, x' \rangle_{\gamma(t)} \leqslant 6\delta$ (Corollary 2.7). Since the

25δ-neighborhood of A_{g_j} is 2δ-quasi-convex (Lemma 2.12), it follows that $\gamma(t)$ lies in the 33δ-neighborhood of A_{g_j}. Thus $|g_j\gamma(t) - \gamma(t)| \leqslant (L_S + 74)\delta$. According to (6.6), there exists $t \in [s, s']$ such that $|\gamma(t) - x| = d$. We let $y = \gamma(t)$. Note that

$$|s' - t| \geqslant |y - x'| \geqslant |x - x'| - |x - y| \geqslant NT.$$

Let $k \in \{0, \dots, N\}$. By construction $h^k x = \gamma(s + kT)$ and $h^k y = \gamma(t + kT)$. Using our previous remark, we see that $s + kT$ and $t + kT$ belongs to $[s, s']$. Thus

$$\max\left\{\left|g_j h^k x - h^k x\right|, \left|g_j h^k y - h^k y\right|\right\} \leqslant (L_S + 74)\delta.$$

In other words, for every $k \in \{0, \dots, N\}$, $|h^{-k} g_j h^k x - x| \leqslant (L_S + 74)\delta$ and $|h^{-k} g_j h^k y - y| \leqslant (L_S + 74)\delta$. However $|x - y| \geqslant d$. By choice of d and N, there exists $k \in \{1, \dots, N\}$ such that g_j and h^k commute. Since h is loxodromic, g_j fixes pointwise $\{h^-, h^+\} \subset \partial X$. Hence g_j belongs to the maximal elementary subgroup containing h. This statement holds for every $j \in \{0, \dots, \nu\}$. Consequently $g_0, \dots, g_\nu$ do not generate a non-elementary subgroup. Contradiction. □

In view of Lemma 6.11, Lemma 6.12 and Lemma 6.14, Theorem 6.9 leads to the following result.

Theorem 6.15. — *Let X be a hyperbolic length space. Let G be a group acting by isometries on X. We assume that the action of G is acylindrical and non-elementary. Let N be a normal subgroup of G without involution. Assume that $e(N, X)$ is odd. There exists a critical exponent n_1 such that every odd integer $n \geqslant n_1$ which is a multiple of $e(N, X)$ has the following property. There exists a normal subgroup K of G contained in N such that*

- *if E is an elementary subgroup of G which is not loxodromic, then the projection $G \twoheadrightarrow G/K$ induces an isomorphism from E onto its image;*
- *for every element $g \in N/K$, either $g^n = 1$ or g is the image an elliptic element of N;*
- *every non-trivial element of K is loxodromic;*
- *there are infinitely many elements in N/K which are not the image of an elliptic element of G.*
- *As a normal subgroup, K is not finitely generated.*

6.3. Examples

Mapping class groups. Let Σ be a compact surface of genus g with p boundary components. In the rest of this paragraph we assume that its complexity $3g+p-3$ is larger than 1. The *mapping class group* $\mathrm{MCG}(\Sigma)$ of Σ is the group of orientation preserving self homeomorphisms of Σ defined up to homotopy. A mapping class $f \in \mathrm{MCG}(\Sigma)$ is

(i) *periodic*, if it has finite order;
(ii) *reducible*, if it permutes a collection of essential non-peripheral curves (up to isotopy);
(iii) *pseudo-Anosov*, if there exists an homotopy in the class of f that preserves a pair of transverse foliations and rescale these foliations in an appropriate way.

It follows from Thurston's work that any element of $\mathrm{MCG}(\Sigma)$ falls into one these three categories [42, Theorem 4]. The *complex of curves* X is a simplicial complex associated to Σ. It has been first introduced by W. Harvey [24]. A k-simplex of X is a collection of $k+1$ homotopy classes of curves of Σ that can be disjointly realized. H. Masur and Y. Minsky proved that this new space is hyperbolic [29]. By construction, X is endowed with an action by isometries of $\mathrm{MCG}(\Sigma)$. Moreover B. Bowditch showed that this action is acylindrical [3, Theorem 1.3]. This is an example of a group acting acylindrically but not properly on a hyperbolic space. Indeed the stabilizer of a point, i.e. the set of mapping classes preserving a curve, is far from being finite. This action provides an other characterization of the elements of $\mathrm{MCG}(\Sigma)$. An element of $\mathrm{MCG}(\Sigma)$ is periodic or reducible (respectively pseudo-Anosov) if and only it is elliptic (respectively loxodromic) for the action on the complex of curves [29].

THEOREM 6.16. — *Let Σ be a compact surface of genus g with p boundary components such that $3g+p-3>1$. There exist integers κ and n_0 such that for every odd exponent $n \geqslant n_0$ there is a quotient Q of $\mathrm{MCG}(\Sigma)$ with the following properties.*

(i) If E is a subgroup of $\mathrm{MCG}(\Sigma)$ that does not contain a pseudo-Anosov element, then the projection $\mathrm{MCG}(\Sigma) \twoheadrightarrow Q$ induces an isomorphism from E onto its image.
(ii) Let f be a pseudo-Anosov element of $\mathrm{MCG}(\Sigma)$. Either $f^{\kappa n}=1$ in Q or there exists a periodic or reducible element $u \in \mathrm{MCG}(\Sigma)$ such that $f^{\kappa}=u$ in Q. In particular, for every $f \in \mathrm{MCG}(\Sigma)$, there exists a periodic or reducible element $u \in \mathrm{MCG}(\Sigma)$ such that $f^{\kappa n}=u$ in Q.

(iii) There are infinitely many elements in Q which are not the image of a periodic or reducible element of $\mathrm{MCG}(\Sigma)$.

Proof. — We would like to apply Theorem 6.15 with the mapping class group $\mathrm{MCG}(\Sigma)$ acting on the complex of curve X of Σ. However $\mathrm{MCG}(\Sigma)$ does contains elements of order 2. To avoid this difficulty we consider a torsion-free finite-index normal subgroup N of $\mathrm{MCG}(\Sigma)$ [17, 38]. We write κ for the index of N in $\mathrm{MCG}(\Sigma)$. This groups acts acylindrically on X thus $r_{inj}(N, X)$ is positive, $\nu(N, X)$ and $A(G, X)$ are finite. Since N has no torsion, $e(N, X) = 1$. Note also that for every $f \in \mathrm{MCG}(\Sigma)$, f^κ belongs to N. Thus the theorem follows from Theorem 6.15. □

Remark. — Corollary 1.3 is a similar consequence of Theorem 6.15.

Theorem 6.16 studies quotients of mapping class groups of the form $\mathrm{MCG}(\Sigma)/S^n$ where S consists in a large set of pseudo-Anosov homotopies. As opposed to this situation we also examples where S only contains Dehn twists. Let α be a simple close curve of Σ and T a tubular neighborhood of α homeomorphic to $S^1 \times [0, 1]$. A *Dehn twist around* α is a homeomorphism f whose restriction to T (identified with $S^1 \times [0, 1]$) is given by $f(\theta, t) = (\theta + 2\pi t, t)$ and whose is the identity outside of T. The set of all Dehn-twist generate $\mathrm{MCG}(\Sigma)$.

THEOREM 6.17. — *Let Σ a surface of genus g with $g > 1$. Let S be the set of all Dehn twists of Σ. There exists an integer n_0 such that for every odd exponent $n \geqslant n_0$, the free group $\mathbf{F}_2$ embeds into* $\mathrm{MCG}(\Sigma)/S^n$.

Proof. — Every homeomorphism of Σ induces an automorphism of the fundamental group of Σ which is well defined up to a conjugation. More precisely $\mathrm{MCG}(\Sigma)$ embeds into $\mathrm{Out}(\pi_1(\Sigma))$ (as a subgroup of index 2). For simplicity let us denote by $G = \pi_1(\Sigma)$ the fundamental group of Σ. Let $n \in \mathbb{N}$. The subgroup G^n is characteristic. It provides a canonical map $\mathrm{Out}(G) \to \mathrm{Out}(G/G^n)$. Let α be a simple close curve of Σ and f the Dehn twist around α. We claim that the image of f^n in $\mathrm{Out}(G/G^n)$ is trivial. To that end we distinguish two cases.

- Assume first that α splits the surface into two connected components Σ_1 and Σ_2. We fix a base point $x_0 \in \Sigma_1$ for Σ. According to the van Kampen theorem G is isomorphic to $A_1 *_{\mathbb{Z}} A_2$ where A_i is the fundamental group of Σ and $\mathbb{Z}$ the subgroup generated by α. The automorphism ϕ of $G \equiv \pi(\Sigma, x_0)$ induced by f preserves the decomposition: for every $g \in A_1$, $\phi(g) = g$ and for every $g \in A_2$, $\phi(g) = hgh^{-1}$ where h stands for the generator t of $\mathbb{Z}$ induced by the curve α. In particular for every $g \in A_2$, $\phi^n(g) = h^n g h^{-n}$ which equals g in G/G^n. Consequently ϕ^n induces the identity of G/G^n.

- Assume now that α does not split Σ. Let us fix a base point x_0 in $\Sigma \setminus \alpha$. Then G can be seen as an HNN extension $A*_{\mathbb{Z}}$ where A is the fundamental group of $\Sigma \setminus \alpha$. Moreover the automorphism ϕ induced by f acts as follows. For every $g \in A$, $\phi(g) = g$. If t stands for the generator of $\mathbb{Z}$ then $\phi(t) = th$ where h correspond to the class of α. In particular, $\phi^n(t) = th^n$ which equals t in G/G^n. As previously ϕ^n induces the identity of G/G^n which proves our claim.

It follows from our claim that $\mathrm{MCG}(\Sigma) \to \mathrm{Out}\,(G)$ induces a homomorphism from $\mathrm{MCG}(\Sigma)/S^n$ into $\mathrm{Out}\,(G/G^n)$. However if n is a sufficiently large integer, then the image of $\mathrm{MCG}(\Sigma)$ in $\mathrm{Out}\,(G/G^n)$ contains $\mathbf{F}_2$ [9, Corollaire IV.3.4]. Hence so does $\mathrm{MCG}(\Sigma)/S^n$. □

Remark. — We keep the notations of the previous proof. In particular G stands for the fundamental group of the surface Σ. In [9, Corollaire IV.3.2] we also proved the following fact. Given a pseudo-Anosov homeomorphism f of Σ, there exists an integer n_0, such that for every odd exponent n, f induces an infinite order automorphism of G/G^n. In particular f has infinite order as an element of $\mathrm{MCG}(\Sigma)/S^n$. From this point of view this second type of quotient is diametrically opposed to the one given by Theorem 6.16: it "kills" reducible elements and preserves numerous pseudo-Anosov homeomorphisms. These results can also be obtained through quantum representations of mapping class groups [20, 21].

Amalgamated product. Let G be a group. A subgroup H of G is *malnormal* if for every $g \in G$, $gHg^1 \cap H = \{1\}$ unless g belongs to H. The following theorem is known from specialists in the field. However it has not be published so far.

THEOREM 6.18. — *Let A and B be two groups without involution. Let C be a subgroup of A and B malnormal in A or B. There is an integer n_1 such that for every odd exponent $n \geqslant n_1$ there exists a quotient Q of $A *_C B$ with the following properties.*

(i) The natural projection $A *_C B \twoheadrightarrow Q$ induces an embedding of A and B into Q.
(ii) For every $g \in Q$, if g is not a conjugate of an element of A or B then $g^n = 1$.
(iii) There are infinitely many elements in Q which are not conjugates of elements of A or B.

Proof. — We denote by X the Bass-Serre tree associated to the amalgamated product $G = A *_C B$ (see for instance [39]). By construction $A *_C B$ acts by isometries on X. An element $g \in A *_C B$ is elliptic for this action if and only if it is a conjugate of an element of A or B. It is loxodromic otherwise. In particular $A *_C B$ does not contain any element of order 2. Moreover A and B are elliptic subgroups. Since C is malnormal in A or B the stabilizer of any path of length at least 3 is trivial. It follows that the action of G on X is acylindrical. On the other hand, any elementary loxodromic subgroup is cyclic infinite, hence $e(A *_C B, X) = 1$. The theorem follows from Theorem 6.15. □

Hyperbolic groups. Let G be a group acting properly co-compactly on a hyperbolic length space. In particular G is a hyperbolic group. Moreover this action is acylindrical. In this particular case, the invariant $e(G, X)$ can be characterized algebraically. Indeed the elementary loxodromic subgroups of G are exactly the ones containing $\mathbb{Z}$ as a finite index subgroup. Therefore we simply write $e(G)$ for $e(G, X)$.

If G is torsion-free, there exists an integer n_0 such that for every odd exponent $n \geqslant n_0$ the quotient G/G^n is infinite. This result was first proved by A.Y. Ol'shanskii [33]. The work of T. Delzant and M. Gromov provides an alternative prove of the same result [13] (see also [10]). Our study allow us to add some harmless torsion in the original group G, We recover here a particular case of a theorem proved by A.Y. Ol'shanskii and S.V. Ivanov in [27] (their result works for also for hyperbolic groups with 2-torsion).

Theorem 6.19. — *Let G be a non-elementary hyperbolic group without involution such that $e(G)$ is odd. There exist integers κ and n_1 such that for every odd integer $n \geqslant n_1$, the quotient $G/G^{\kappa n}$ is infinite.*

Proof. — Since G is hyperbolic, its action on its Cayley graph X is proper and co-compact. In particular it is acylindrical. Moreover it contains only a finite number of conjugacy classes of elliptic elements (see [7, Lemme 3.5]). Since G has no involution, there exists an odd integer κ, multiple of $e(G)$ such that for every elliptic element u of G, the order of u divides κ. Hence we can apply Theorem 6.15 with $G = N$. There exists an integer n_1 such that for every odd exponent $n \geqslant n_1$ there exists an infinite quotient Q of G with the following property. For every element $g \in Q$ either $g^{\kappa n} = 1$ or there exists an elliptic element $u \in G$ such that $g = u$ in Q. However for every elliptic element $u \in G$, we have $u^\kappa = 1$. It follows that Q is an infinite quotient of $G/G^{\kappa n}$, hence $G/G^{\kappa n}$ is infinite. □

Remark. — One can actually prove that the quotient Q that appears in the proof is exactly $G/G^{\kappa n}$. However this is not needed here.

Relatively hyperbolic groups. The notion of a group being hyperbolic relative to a collection of subgroups was introduced by Gromov in [23]. This class extends the one of hyperbolic groups and covers various examples like fundamental groups a negatively curved manifold with finite volume, HNN extensions over finite groups, geometrically finite Kleinian groups, etc. Since Gromov's original paper, several different definitions have emerged, see for instance [4, 15]. These definitions have been shown to be almost equivalent [4, 41, 25]. For our purpose we will use the following one.

Definition 6.20 ([25, Definition 3.3]). — *Let G be a group and $\{H_1, \dots, H_m\}$ be a collection of subgroups of G. We say that G is* hyperbolic relative to $\{H_1, \dots, H_m\}$ *if there exists a proper geodesic hyperbolic space X and a collection $\mathcal{Y}$ of disjoint open horoballs satisfying the following properties.*

(i) G acts properly by isometries on X and $\mathcal{Y}$ is G-invariant.
(ii) If U stands for the union of the horoballs of $\mathcal{Y}$ then G acts co-compactly on $X \setminus U$.
(iii) $\{H_1, \dots, H_m\}$ is a set of representatives of the G-orbits of $\{\mathrm{Stab}(Y) \mid Y \in \mathcal{Y}\}$.

The action of G on the space X given by Definition 6.20 is not acylindrical. Indeed the subgroups H_j can be parabolic. This cannot happen with an acylindrical action [3, Lemma 2.2]. More generally, the non-loxodromic elementary subgroups of G are exactly the finite subgroups of G and the ones which are conjugated to a subgroup of some H_j. As in the case of hyperbolic groups, the invariant $e(G, X)$ can be characterized algebraically. Indeed a subgroup E of G is loxodromic if and only if $\mathbb{Z}$ is a finite-index subgroup of E and E is not conjugated to a subgroup of some H_j. Therefore we simply write $e(G)$ for $e(G, X)$. Note that this notation implicitly depends on the collection $\{H_1, \dots, H_m\}$ though.

As in the case of groups with an acylindrical action, one can prove that $r_{inj}(G, X)$ is positive whereas $\nu(G, X)$ and $A(G, X)$ are finite. Theorem 6.9 gives the following result.

Theorem 6.21. — *Let G be a group without involution and $\{H_1, \dots, H_m\}$ be a collection of subgroups of G. Assume that G is hyperbolic relatively to $\{H_1, \dots, H_m\}$ and $e(G)$ is odd. There is a critical exponent n_1 such that every odd integer $n \geqslant n_1$ which is a multiple of*

$e(G)$ has the following property. There exists a quotient G/K of G such that

- *if E is a finite subgroup of G or conjugated to some H_j, then the projection $G \twoheadrightarrow G/K$ induces an isomorphism from E onto its image;*
- *for every element $g \in G/K$, either $g^n = 1$ or g is the image a non-loxodromic element of G;*
- *there are infinitely many elements in G/K which do not belong to the image of an elementary non-loxodromic subgroup of G.*

Other examples. In [34], D. Osin investigates the class of groups that admit a non-elementary acylindrical action on a hyperbolic space. He called them *acylindrically hyperbolic groups.* It turns out that this class is very large. Here are a few examples in addition to the one we already studied.

(i) If a group G is not virtually cyclic and admits an action on a hyperbolic space with at least one loxodromic element satisfying the WPD property, then G is acylindrically hyperbolic. In particular for every $r \geqslant 2$, the outer automorphism group $\operatorname{Out}(\mathbf{F}_r)$ of the free group $\mathbf{F}_r$ of rank r is acylindrically hyperbolic. Indeed given any automorphism $\phi \in \operatorname{Out}(\mathbf{F}_r)$ which is irreducible with irreducible powers (iwip), M. Bestvina and M. Feighn constructed a hyperbolic $\operatorname{Out}(\mathbf{F}_r)$-complex where ϕ satisfies the WPD property [1].

(ii) If G contains a proper infinite hyperbolically embedded subgroup (see [11] for a precise definition) G is acylindrically hyperbolic. One example is the *Cremona group* $\mathbf{Bir}(P^2_{\mathbb{C}})$. It is the group of birationnal transformations of the projective planes. It has been shown by S. Cantat and S. Lamy that $\mathbf{Bir}(P^2_{\mathbb{C}})$ admits an action on a hyperbolic space with many loxodromic elements [6]. F. Dahmani, V. Guirardel and D. Osin used then these data to prove that $\mathbf{Bir}(P^2_{\mathbb{C}})$ contains virtually cyclic hyperbolically embedded subgroups [11].

(iii) In [40] A. Sisto proved that if G is a group acting properly on a proper CAT(0) space, then every rank 1 element of G is contained in a hyperbolically embedded virtually cyclic subgroup. which provides other examples of acylindrically hyperbolic groups. In particular every right-angled Artin Group which is not cyclic, or directly decomposable is acylindrically hyperbolic.

(iv) In [30], A. Minasyan and D. Osin used actions on trees to provide other examples of acylindrically hyperbolic group. Among others, they gave the following results. For every field k, the group $\mathrm{Aut}\,(k[x,y])$ of automorphisms of the polynomial algebra $k[x,y]$ is acylindrically hyperbolic. Any one relator group with at least three generators is acylindrically hyperbolic.

For all these examples we can apply Theorem 6.15 provided we can deal with the even torsion. However we do not necessarily have an intrinsic characterization for the type (elliptic or loxodromic) of the elements of G for the corresponding action on X. For instance, it is not known if there exists an acylindrical action of $\mathrm{Out}\,(\mathbf{F}_r)$ on a hyperbolic space such that the loxodromic elements are exactly the iwip automorphisms of $\mathbf{F}_r$.

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