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ON THE CONTINUITY OF FOURIER MULTIPLIERS ON THE HOMOGENEOUS SOBOLEV SPACES $\dot{W}_1^1(R^d)$

by Krystian KAZANIECKI & Michał WOJCIECHOWSKI (*)

Abstract. — In this paper we prove that every Fourier multiplier on the homogeneous Sobolev space $\dot{W}_1^1(\mathbb{R}^d)$ is a continuous function. This theorem is a generalization of the result of A. Bonami and S. Poornima for Fourier multipliers, which are homogeneous functions of degree zero.

Résumé. — Dans cet article, nous prouvons que chaque multiplicateur de Fourier sur l'espace homogène $\dot{W}_1^1(\mathbb{R}^d)$ de Sobolev est une fonction continue. Notre théorème est une généralisation du résultat de A. Bonami et S. Poornima sur les multiplicateurs de Fourier, qui sont des fonctions homogènes de degré zéro.

1. Introduction

We consider the invariant operators on the homogeneous Sobolev spaces on $\mathbb{R}^d$ given by Fourier multipliers. The homogeneous Sobolev space $\dot{W}_1^1(\mathbb{R}^d)$ consists of functions on $\mathbb{R}^d$ whose distributional gradient is integrable. A measurable function $m : \mathbb{R}^d \to \mathbb{R}$ is called a (Fourier) multiplier if the operator given by the formula $T_m f = \mathscr{F}^{-1}(m \cdot \mathscr{F}(f))$ is bounded. Fourier transforms of bounded measures are examples of multipliers. Indeed, the convolution with a bounded measure is a bounded operator on every translation invariant space with continuous shifts operators, in particular on the homogeneous Sobolev space.

However, the class of Fourier multipliers on $\dot{W}_1^1(\mathbb{R}^d)$ is wider than the class of Fourier transforms of measures (Proposition 2.2 in [10]). One of the

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important questions about the invariant subspaces of L^1 is a description of bounded singular operators acting on it e.g. the Calderon-Zygmund operators are given by multiplier with singularity at zero. Therefore, the question of the continuity of a multiplier arises quite naturally in the theory.

The simplest case of noncontinuous multipliers was settled by A. Bonami and S. Poornima who proved that the only homogeneous multipliers of degree zero are the constants. In their beautiful proof they use very delicate result by Ornstein (cf. [9]) on the non-majorization of a partial derivative by other derivatives of the same order. While the class of homogeneous multipliers, containing e.g. Riesz transforms, is the most important one, the question of the continuity of general multipliers remained open. The aim of this paper is to fill the gap. We prove that any multiplier acting on the homogeneous Sobolev space with integral norm is a continuous function.

Our proof uses three main ingredients. The first one is the Bonami - Poornima result. The second is the Riesz product technique which allows us to make the crucial estimates on the torus group. This would be sufficient for our purpose, provided we are able to transfer the problem from $\mathbb{R}^d$ to $\mathbb{T}^d$. Such transference in the case of multipliers on L^p spaces is the subject of the theorem of deLeeuw (cf. [7]). However, in the case of multipliers on the homogeneous Sobolev space no version of the deLeeuw transference theorem is known. We are able to overcome this difficulty due to the special form of functions on which the multiplier reaches its norm. The question of general deLeeuw type theorem for the homogeneous Sobolev spaces remains open. We believe that this paper will provide a motivation for further research in this direction.

One can ask whether a similar approach could be used to prove the Ornstein's non-inequality. Indeed in some special cases this technique works, for more details one can check [5].

For a formal statement of the main theorem we use standard definitions and notations.

- $L^p(\mathbb{R}^d)$ - space of Lebesgue p-integrable functions on $\mathbb{R}^d$
- $\mathscr{D}(\mathbb{R}^d)$ - space of $C^\infty(\mathbb{R}^d)$ functions with compact support on $\mathbb{R}^d$.
- $\mathscr{D}'(\mathbb{R}^d)$ - space of distributions on $\mathbb{R}^d$.
- $\mathscr{S}(\mathbb{R}^d)$ - Schwartz function space on $\mathbb{R}^d$.
- $\mathscr{S}'(\mathbb{R}^d)$ - space of tempered distributions on $\mathbb{R}^d$.
- $C_b(\mathbb{R}^d)$ - space of bounded continuous functions on $\mathbb{R}^d$.
- $\mathscr{F}(\cdot)$ - Fourier transform of tempered distributions.
- $\mathscr{F}^{-1}(\cdot)$ - inverse Fourier transform of tempered distributions.

One can find more details on the function spaces mentioned above in [11]. For the definition of the Fourier transform we follow [12]. As usual, C will denote a generic constant, whose value can change from line to line. We write $W_k^p(\mathbb{R}^d)$ for the Sobolev space, given by

$$W_k^p(\mathbb{R}^d) := \left\{ f \in L^p(\mathbb{R}^d) \,:\, D^{\boldsymbol{\alpha}} f \in L^p(\mathbb{R}^d) \text{ for } |\boldsymbol{\alpha}| \leqslant k \right\}$$

with the norm

$$\|f\|_{W_k^p(\mathbb{R}^d)} := \sum_{0 \leqslant |\boldsymbol{\alpha}| \leqslant k} \|D^{\boldsymbol{\alpha}} f\|_{L^p(\mathbb{R}^d)}$$

where $\boldsymbol{\alpha}$ is a multi-index and $D^{\boldsymbol{\alpha}}$ is the corresponding distributional derivative and $k \in \mathbb{N}^+$. Analogously we write $\dot{W}_p^k(\mathbb{R}^d)$ for the homogeneous Sobolev space, given by

$$\dot{W}_k^p(\mathbb{R}^d) := \left\{ f \in \mathscr{D}'(\mathbb{R}^d) \,:\, D^{\boldsymbol{\alpha}} f \in L^p(\mathbb{R}^d) \text{ for } |\boldsymbol{\alpha}| = k \right\}$$

with the seminorm

$$\|f\|_{\dot{W}_k^p(\mathbb{R}^d)} := \sum_{|\boldsymbol{\alpha}| = k} \|D^{\boldsymbol{\alpha}} f\|_{L^p(\mathbb{R}^d)}$$

where $\boldsymbol{\alpha}$, $D^{\boldsymbol{\alpha}}$ and k are the same as above. The homogeneous Sobolev spaces are special cases of Beppo-Levi spaces which are discussed in [3]. Later we will use the symbol $\dot{W}_k^p(\mathbb{R}^d)$ to denote the quotient space $\dot{W}_k^p(\mathbb{R}^d)/\mathscr{P}^k$, where $\mathscr{P}^k$ stands for the space of polynomials of degree strictly less than k. The space $\dot{W}_k^p(\mathbb{R}^d)/\mathscr{P}^k$ with the quotient norm is a Banach space.

We say that the function $m \in L^\infty(\mathbb{R}^d)$ is a Fourier multiplier on X, where X is either the Lebesgue space, the Sobolev space or the homogeneous Sobolev space $\dot{W}_1^1(\mathbb{R}^d)$, if there exists a bounded operator $T_m : X \to X$ such that

$$\mathscr{F}(T_m f) = m \mathscr{F}(f) \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^d).$$

We use the symbol $\mathscr{M}(X, X)$ to denote the space of the Fourier multipliers on X with the norm

$$\|m\|_{\mathscr{M}(X,X)} := \|T_m\| \qquad \forall m \in \mathscr{M}(X, X).$$

Now we can state the main result of this paper

Theorem 1.1. — *If $d \geqslant 2$ and $m \in \mathscr{M}(\dot{W}_1^1(\mathbb{R}^d), \dot{W}_1^1(\mathbb{R}^d))$ then $m \in C_b(\mathbb{R}^d)$.*

In the proof we will use the theorem of A. Bonami and S. Poornima on the homogeneous Fourier multipliers on $\dot{W}_1^1(\mathbb{R}^d)$.

Theorem 1.2 (A. Bonami, S. Poornima). — *Let Ω be a continuous function on $\mathbb{R}^d\backslash\{0\}$, homogeneous of degree zero i.e.*

$$\Omega(\varepsilon\boldsymbol{x}) = \Omega(\boldsymbol{x}) \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^d.$$

Then

$$\Omega \in \mathscr{M}(\dot{W}_1^1(\mathbb{R}^d), \dot{W}_1^1(\mathbb{R}^d)) \Leftrightarrow \Omega \equiv K \in \mathbb{C}.$$

For the proof see [1].

In the next section we prove Theorem 1.1. To focus the attention on the main line of the proof, some technical lemmas are formulated there without proofs. For the reader's convenience proofs of the technical lemmas are given in the last section.

2. Proof of the main theorem

Let the function $m \in \mathscr{M}(\dot{W}_1^1(\mathbb{R}^d), \dot{W}_1^1(\mathbb{R}^d))$. Hence $\xi_i m(\boldsymbol{\xi})\mathscr{F}(f)(\boldsymbol{\xi})$ is a Fourier transform of an integrable function for every $f \in \mathscr{S}(\mathbb{R}^d)$. Therefore m is a continuous function on $\mathbb{R}^d\backslash\{0\}$. Thus it is enough to show the existence of the limit $\lim_{\boldsymbol{x}\to 0} m(\boldsymbol{x})$.

Prior to the proof we need one more definition. Let $f : \mathbb{R}^d \to \mathbb{R}$.

(*): We say that the function f has *almost radial limits* at 0 iff for every vector $\boldsymbol{w} \in \mathbb{S}^{d-1}$ there exists a scalar $g(\boldsymbol{w}) \in \mathbb{R}$ such that for every sequences $t_k \to 0$ and $\boldsymbol{w}^k \to \boldsymbol{w}$ ($t_k \in \mathbb{R}$; $\boldsymbol{w}^k \in \mathbb{S}^{d-1}$) we have

$$\lim_{k\to\infty} f(t_k\boldsymbol{w}^k) = g(\boldsymbol{w}).$$

Proof of Theorem 1.1. — Since m is bounded, there are three possibilities:

Case I The multiplier m has almost radial limits at 0 (*).

Case II The multiplier m does not satisfy condition (*). Then there exists a sequence $\{\boldsymbol{a}^n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$, $\boldsymbol{a}^n \to 0$, a vector $v \in \mathbb{S}^1$ and two different scalars a and b such that

$$\lim_{n\to\infty} \frac{\boldsymbol{a}^n}{|\boldsymbol{a}^n|} = \boldsymbol{v}$$

and one of the following is satisfied

(a) Symmetric case.

$$\begin{aligned} \lim_{n\to\infty} m(\boldsymbol{a}^{2n}) &= \lim_{n\to\infty} m(-\boldsymbol{a}^{2n}) &&= a, \\ \lim_{n\to\infty} m(\boldsymbol{a}^{2n+1}) &= \lim_{n\to\infty} m(-\boldsymbol{a}^{2n+1}) &&= b. \end{aligned} \tag{2.1}$$

(b) Asymmetric case.

$$\lim_{n\to\infty} m(\boldsymbol{a}^n) = a,$$
$$\lim_{n\to\infty} m(-\boldsymbol{a}^n) = b.$$

2.1. Proof in the Case I

We will use the following lemma, stated in [1], on the pointwise convergence of multipliers.

LEMMA 2.1. — *Let $\{m_k\}$ be a sequence of Fourier multipliers on $\dot{W}_1^1(\mathbb{R}^d)$ and assume that the corresponding operators have commonly bounded norms. If m_k converge pointwise to a function $m(\cdot)$ then $m(\cdot)$ is a Fourier multiplier on $\dot{W}_1^1(\mathbb{R}^d)$.*

In the next lemma we use Theorem 1.2 to show that the multipliers satisfying condition (*) are continuous.

LEMMA 2.2. — *If $d \geqslant 2$ and $m \in \mathscr{M}(\dot{W}_1^1(\mathbb{R}^d), \dot{W}_1^1(\mathbb{R}^d))$ satisfies condition (*), then $\lim_{\boldsymbol{\xi}\to 0} m(\boldsymbol{\xi})$ exists and is finite.*

Proof. — Note first that m has the radial limit at 0 (we apply (*) to fixed $\boldsymbol{v} = \boldsymbol{v}^k = \boldsymbol{w}^k$). Hence the formula

$$\Omega(\boldsymbol{\xi}) := \lim_{n\to\infty} m(\frac{1}{n}\boldsymbol{\xi}).$$

defines a homogeneous function on $\mathbb{R}^d\backslash\{0\}$. The condition (*) implies continuity of Ω on $\mathbb{R}^d\backslash\{0\}$. Indeed

$$\lim_{\boldsymbol{\xi}_k\to\boldsymbol{\xi}} \Omega(\boldsymbol{\xi}_k) = \lim_{\boldsymbol{\xi}_k\to\boldsymbol{\xi}} \lim_{n\to\infty} m(\frac{1}{n}\boldsymbol{\xi}_k) \overset{(*)}{=} \lim_{n\to\infty} m(\frac{1}{n}\boldsymbol{\xi}) = \Omega(\boldsymbol{\xi}) \tag{2.2}$$

Since the norm of multipliers from $\mathscr{M}(\dot{W}_1^1(\mathbb{R}^d), \dot{W}_1^1(\mathbb{R}^d))$ is invariant under rescaling, the functions $m(\frac{1}{n}\cdot)$ are Fourier multipliers with equal norms. By Lemma 2.1 their pointwise limit, being bounded and continuous on $\mathbb{R}^d\backslash\{0\}$, is a Fourier multiplier on $\dot{W}_1^1(\mathbb{R}^d)$. Then Theorem 1.2 implies that Ω is a constant function which in turn means that all radial limits of m are equal. In similar way as in (2.2) we check that a function which has all radial limits equal and satisfies condition (*) is continuous at zero. Hence multiplier m is a continuous function. $\square$

2.2. Proof in the Case IIa

From now on we assume that $d = 2$. This allows us to simplify the notation yet not loosing the generality. We can also assume, transforming linearly if necessary, that $a = 1$, $b = -1$ and $\boldsymbol{v} = (1, 0)$. We will estimate the norm of the multiplier m from the following lemma:

Lemma 2.3 (cf. [13]). — *There exists constant $C > 0$ such that for every $s \in \mathbb{N}^+$, there exists M_s such that*

$$\left\| \sum_{j=1}^{s} (-1)^j \cos\left(2\pi\langle \boldsymbol{c}^j, \boldsymbol{\xi}\rangle\right) \prod_{1\leqslant k<j} \left(1 + \cos\left(2\pi\langle \boldsymbol{c}^k, \boldsymbol{\xi}\rangle\right)\right) \right\|_{L^1(\mathbb{T}^d)} \geqslant Cs \tag{2.3}$$

whenever $\{\boldsymbol{c}^k\}_{k=1}^s \subset \mathbb{Z}^d$ satisfies

$$|\boldsymbol{c}^{k+1}| > M_s|\boldsymbol{c}^k|.$$

Remark 2.4. — The value of M_s could be derived from [8], where it is proved that whenever $\sum_{k=1}^s \left(\frac{|\boldsymbol{c}^k|}{|\boldsymbol{c}^{k+1}|}\right) < \infty$ then the expression appearing in the inequality (2.3) is equivalent to the similar one with functions $\boldsymbol{\xi} \mapsto \cos(2\pi\langle \boldsymbol{c}^j, \boldsymbol{\xi}\rangle)$ replaced by cosines of certain independent random variables, for which it follows by the theorem by R. Latała (Theorem 1 in [6]). In [2] the weaker condition $\sum_{k=1}^s \left(\frac{|\boldsymbol{c}^k|}{|\boldsymbol{c}^{k+1}|}\right)^2 < \infty$ is claimed to be sufficient (see [4] for more detailed discusion). Similar inequality was obtained and used by M. Wojciechowski in [13].

In the rest of the paper we put $N := \left(\left|\frac{\log(M_s)}{\log(2)}\right| + 2\right)$.

Let us assume that the operator T_m corresponding to the multiplier m is bounded. For every $s \in \mathbb{N}$ we will construct a function h_s with norm bounded by a constant independent of s, such that

$$\|T_m h_s\|_{\dot{W}_1^1(\mathbb{R}^d)} \geqslant Cs.$$

Let $\varepsilon > 0$, be fixed later. We construct the sequence of balls $\mathbf{B}(\boldsymbol{c}^k, r_k)$ and $\mathbf{B}(-\boldsymbol{c}^k, r_k)$ for $k \in \{1, 2, \ldots, s\}$, such that the following conditions hold:

- II-A. $|m(\boldsymbol{\xi}) - (-1)^k| < \varepsilon$ for $\boldsymbol{\xi} \in \boldsymbol{B}(\boldsymbol{c}^k, r_k) \cup \boldsymbol{B}(-\boldsymbol{c}^k, r_k)$ for $k = 1, 2, \ldots, s$,
- II-B. $r_n \leqslant 2^{-N} r_{n+1}$ for $n = 1, 2, \ldots, s-1$,
- II-C. $\boldsymbol{c}^n \in \mathbb{Q} \times \mathbb{Q}$ for $n = 1, 2, \ldots, s$,
- II-D. $|\boldsymbol{c}^{n+1}| < 2^{-N} r_n$ for $n = 1, 2, \ldots, s-1$,
- II-E. $|c_2^n|/|c_1^n| \leqslant \frac{1}{3^{s+2}s}$ for $n = 1, 2, \ldots, s$,
- II-F. $|\boldsymbol{c}^n| < 2^{-N}|\boldsymbol{c}^{n+1}|$ for $n = 1, 2, \ldots, s-1$,

II-G. $|c_i^n| > r_n$ for $n = 1, 2, \dots, s$ and $i = 1, 2$,

II-H. $|c_i^n| < 2^{-N}|c_i^{n+1}|$ for $n = 1, 2, \dots, s$ and $i \in \{1, 2\}$.

II-I. $\boldsymbol{B}(\sum_{j=1}^n \zeta_j \boldsymbol{c}^j, r_1) \subset \boldsymbol{B}(\zeta_n \boldsymbol{c}^k, r_k)$ for $\zeta_k \in \{-1, 1\}$, $\zeta_j \in \{-1, 0, 1\}$. and $n = 1, 2, \dots, s$.

We define sequences $\{\boldsymbol{c}^k\}$ and $\{r_k\}$ by backward induction. There is no problem with r_n because it is chosen always after $\boldsymbol{c}^n$ and for II-B and II-G we take it sufficiently small. For $\boldsymbol{c}^n$ note that the conditions II-D and II-F require only that $\boldsymbol{c}^n$ is small enough. Conditions II-A, II-E, II-H will be satisfied if we take as $\boldsymbol{c}^n$ a vector $\boldsymbol{a}^k$ with sufficiently large index k s.t. $k \equiv n \mod 2$. At the end we adjust our choice to the condition II-C: since the rationals are dense in $\mathbb{R}$ and all other inequalities are strict, we can do this in such a way that inequalities remain valid.

The condition II-I follows from II-B, II-D and II-F. Indeed for $k \in \{1, \dots, s-1\}$, $\zeta_j \in \{-1, 0, 1\}$, $j \in \{1, ..., k-1\}$ and $\zeta_k \in \{-1, 1\}$ we have

$$\sum_{j=1}^{k-1} r_j < 2^{-N} \sum_{j=2}^{k-1} r_j + 2^{-N} r_k < \dots < \left(\sum_{j=1}^{k} 2^{-Nj} \right) r_k < \frac{1}{2} r_k.$$

Hence

$$\left| \zeta_k \boldsymbol{c}^k - \sum_{j=1}^{k} \zeta_j \boldsymbol{c}^j \right| = \left| \sum_{j=1}^{k-1} \zeta_j \boldsymbol{c}^j \right| < \sum_{j=1}^{k-1} |\zeta_j \boldsymbol{c}^j| < \sum_{j=1}^{k-1} 2^{-N} r_j < \frac{1}{2} r_k. \tag{2.4}$$

By condition II-B we have $r_l < \frac{r_k}{4}$ for $k > l$. Therefore by (2.4)

$$\boldsymbol{B}(\sum_{j=1}^{k} \zeta_j \boldsymbol{c}^j, r_1) \subset \boldsymbol{B}(\zeta_k \boldsymbol{c}^k, r_k) \qquad \forall k \in \{1, 2, \dots, s\}.$$

The norm of T_m is invariant under rescaling. Then by condition II-C for fixed s multiplying $\boldsymbol{c}^j$'s by suitable scalar and rescaling multiplier m by the same scalar, we may assume that $\boldsymbol{c}^1, \dots, \boldsymbol{c}^s \in \mathbb{Z}^2$ and the conditions II-A – II-I are still satisfied. Note that if $\boldsymbol{q} \in \mathbb{Z}^2$ has the representation

$$\boldsymbol{q} = \sum_{j=1}^{s} \zeta_j(\boldsymbol{q}) \boldsymbol{c}^j \quad \text{ where } \zeta_j(\boldsymbol{q}) \in \{-1, 0, 1\}, \tag{2.5}$$

it is unique. For $\boldsymbol{q} \in \mathbb{Q}^2$ we denote by $\chi(\boldsymbol{q})$ the number of non zero terms in the representation (2.5). We define the set

$$\Lambda_s := \{\boldsymbol{q} : \boldsymbol{q} = \sum_{j=1}^{s} \zeta_j(\boldsymbol{q}) \boldsymbol{c}^j ; \boldsymbol{q} \neq 0 \text{ where } \zeta_j(\boldsymbol{q}) \in \{-1, 0, 1\}\}. \tag{2.6}$$

If $\boldsymbol{q}$, $\tilde{\boldsymbol{q}} \in \Lambda_s$ are two different vectors then

$$|\boldsymbol{q} - \tilde{\boldsymbol{q}}| \geqslant \inf |\boldsymbol{c}^j| \geqslant 1. \tag{2.7}$$

We will construct a function h_s in such a way that one of its derivatives behaves like a Riesz product. Let

$$g(t) := \max\{1 - |t|, 0\}^2$$

and

$$G(\boldsymbol{\xi}) := g(\xi_1)g(\xi_2).$$

We denote by R_s the modified Riesz product:

$$R_s(\boldsymbol{t}) := -1 + \Pi_{k=1}^{s}(1 + \cos(2\pi\langle \boldsymbol{t}\,, \boldsymbol{c}^k\rangle))$$

For fixed $\theta \in \mathbb{N}^+$ we define a function $H^\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$H^\theta(\boldsymbol{\xi}) := \sum_{\boldsymbol{q}\in\Lambda_s} \frac{1}{2^{\chi(\boldsymbol{q})}} G\left(2^\theta(\boldsymbol{\xi} - \boldsymbol{q})\right) = \sum_{\boldsymbol{q}\in\mathbb{Z}^2} \widehat{R}_s(\boldsymbol{q}) G\left(2^\theta(\boldsymbol{\xi} - \boldsymbol{q})\right). \tag{2.8}$$

Since R_s are densities of periodic measures with uniformly bounded norms and the inverse Fourier transform of the function G decays sufficiently fast at infinity we get:

Corollary 2.5. — *For every $\theta \in \mathbb{N}^+$ the following inequality is satisfied*

$$\|\mathscr{F}^{-1}(H^\theta)\|_{L^1(\mathbb{R}^2)} \leqslant C\|R_s\|_{L^1(\mathbb{T}^2)} \leqslant C,$$

where the constant C is independent of s.

In the next lemma we state another property of H^θ.

Lemma 2.6. — *There exists $\theta = \theta(s) \in \mathbb{N}^+$ such that*

$$\left\|\mathscr{F}^{-1}\left(\frac{\xi_2}{\xi_1}H^\theta\right)\right\|_{L^1(\mathbb{R}^2)} \leqslant C$$

where the constant C is independent of s.

The proof of this fact one can find in the Appendix. From now on we put $H := H^{\theta(s)}$.

Remark 2.7. — Note that homogeneous, non-constant functions are never multipliers on $L^1(\mathbb{R}^d)$. The above lemma holds true only due to the special form of H^θ, mainly the strong concentration of its support near x_1-axis and because of small size of its support.

Since H is bounded, continuous and has compact support separated from the axis $\{\xi_1 = 0\}$, the function $\frac{H}{\xi_1}$ is a tempered distribution. We define a tempered distribution h by the formula

$$h(\psi) := \frac{H}{x_1}(\mathscr{F}^{-1}\psi) \qquad \forall \psi \in \mathscr{S}.$$

By standard properties of the Fourier transform on the space of tempered distributions, we get

$$\begin{aligned} \mathscr{F}\left(\frac{\partial}{\partial x_1} h\right) &= H \\ \mathscr{F}\left(\frac{\partial}{\partial x_2} h\right) &= \frac{\xi_2}{\xi_1} H. \end{aligned} \tag{2.9}$$

We proved that both H and $\frac{\xi_2}{\xi_1}H$ are the Fourier transforms of L^1 functions. Hence equalities (2.9) mean that $h \in \dot{W}_1^1(\mathbb{R}^d)$ with the norm bounded by a constant independent of s.

Now we estimate the norm of $T_m h$ from below.

Since $T_m : \dot{W}_1^1(\mathbb{R}^2) \to \dot{W}_1^1(\mathbb{R}^2)$, obviously $\frac{\partial}{\partial x_1} T_m h \in L^1(\mathbb{R}^2)$. We denote by P the periodization of the function $\frac{\partial}{\partial x_1} T_m h$. It is only the fact that, when the function is in $L^1(\mathbb{R}^d)$, then its periodization is in $L^1(\mathbb{T}^d)$. We have

$$\|T_m h\|_{\dot{W}_1^1(\mathbb{R}^2)} \geqslant \left\| \frac{\partial}{\partial x_1} T_m h \right\|_{L^1(\mathbb{R}^2)} \geqslant \|P\|_{L_1(\mathbb{T}^2)}. \tag{2.10}$$

One can check that the function P is a polynomial given by the formula

$$P(\boldsymbol{\xi}) = \sum_{\boldsymbol{p} \in \Lambda_s} m(\boldsymbol{p}) H(\boldsymbol{p}) e^{2\pi i \langle \boldsymbol{p}, \boldsymbol{\xi} \rangle}. \tag{2.11}$$

We put

$$a(\boldsymbol{p}) := \begin{cases} (-1)^k H(\boldsymbol{p}) & \text{when } \boldsymbol{p} \in \Lambda_s \text{ and } \boldsymbol{p} \in \boldsymbol{B}(\boldsymbol{c}^k, r_k) \cup \boldsymbol{B}(-\boldsymbol{c}^k, r_k), \\ 0 & \text{otherwise.} \end{cases}$$

Since Λ_s is a finite set, the function

$$Z(\boldsymbol{\xi}) := \sum_{\boldsymbol{p} \in \mathbb{Z}^2} a(\boldsymbol{p}) e^{2\pi i \langle \boldsymbol{p}, \boldsymbol{\xi} \rangle}$$

is a polynomial. By the triangle inequality,

$$\|P\|_{L_1(\mathbb{T}^2)} \geqslant \|Z\|_{L^1(\mathbb{T}^2)} - \|P - Z\|_{L^1(\mathbb{T}^2)}. \tag{2.12}$$

By the conditions II-I and II-A, all coefficients of Z differ by at most ε from the corresponding coefficients of P. Since both polynomials have no more then 3^s non-zero coefficients, we get

$$\|Z - P\|_{L^1(\mathbb{T}^2)} \leqslant \varepsilon 3^s. \tag{2.13}$$

It is easy to verify that

$$Z(\boldsymbol{\xi}) = \sum_{j=1}^{s} (-1)^j \cos\left(2\pi\langle \boldsymbol{c}^j, \boldsymbol{\xi}\rangle\right) \prod_{1\leqslant k<j} \left(1 + \cos\left(2\pi\langle \boldsymbol{c}^k, \boldsymbol{\xi}\rangle\right)\right).$$

By the condition II-F and Lemma 2.3,

$$\|Z\|_{L^1(\mathbb{T}^2)} \geqslant Cs.$$

Combining now successively (2.10), (2.12) and (2.13), we get

$$\|T_m h\|_{\dot{W}_1^1(\mathbb{R}^2)} \geqslant Cs - \varepsilon 3^s.$$

Setting $\varepsilon = C3^{-s-1}s$

$$\|T_m h\|_{\dot{W}_1^1(\mathbb{R}^2)} \geqslant Cs$$

which by the uniform boundedness of $\|h\|_{\dot{W}_1^1(\mathbb{R}^2)}$ proves that T is unbounded.

2.3. Proof in Case IIb

The proof in this case is very similar to Case IIa. The only difference is that, due to lack of symmetry, we have to replace Lemma 2.3 by its asymmetric counterpart. We will use the following result from [13].

Lemma 2.8. — *There exist $C > 0$ such that for every $n \in \mathbb{N}^+$ there exists $M = M(n)$ such that for any sequence $\{\boldsymbol{c}^k\}_{k=1}^n \subset \mathbb{Z}^d$, which satisfies*

$$|\boldsymbol{c}^{k+1}| > M|\boldsymbol{c}^k|,$$

following inequality holds

$$\left\| \sum_{j=1}^{n} e^{2\pi i\langle \boldsymbol{c}^j, \boldsymbol{\xi}\rangle} \prod_{1\leqslant k<j} \left(1 + \cos\left(\langle 2\pi \boldsymbol{c}^k, \boldsymbol{\xi}\rangle\right)\right) \right\|_{L_1(\mathbb{T}^r)} \geqslant Cn.$$

For fixed $\varepsilon > 0$ we construct the sequence of balls $\boldsymbol{B}(\boldsymbol{c}^n, r_n)$ and $\boldsymbol{B}(-\boldsymbol{c}^n, r_n)$ satisfying conditions II-B – II-I and

II-A'. $|m(\boldsymbol{\xi}) - 1| < \varepsilon$ for $\boldsymbol{B}(\boldsymbol{c}^n, r_n)$ and $|m(\boldsymbol{\xi})| < \varepsilon$ for $\boldsymbol{\xi} \in \boldsymbol{B}(-\boldsymbol{c}^n, r_n)$ and $n = 1, 2, \ldots, s$.

The inductive construction is similar as in the Case IIa. Then, similarly as in the Case IIa, we define $\theta(s)$ and h, and we get

$$\|h\|_{\dot{W}_1^1(\mathbb{R}^2)} \leqslant C,$$

where the constant $C > 0$ is independent of s. Analogously as in the Case IIa we define polynomial P by (2.11) and by similar reasons

$$\|T_m h\|_{\dot{W}_1^1(\mathbb{R}^2)} \geqslant \|P\|_{L^1(\mathbb{T}^2)}.$$

Then we put

$$a(\boldsymbol{p}) := \begin{cases} H(\boldsymbol{p}) & \text{when } \boldsymbol{p} \in \Lambda_s \text{ and } \boldsymbol{p} \in \boldsymbol{B}(\boldsymbol{c}^k, r_k), \\ 0 & \text{otherwise }, \end{cases}$$

where $k \in \{1, 2, \ldots, s\}$. The function $a(\cdot)$ differs from its analog from Case IIa. We define a polynomial Z by

$$Z(\boldsymbol{\xi}) := \sum_{\boldsymbol{p} \in \mathbb{Z}^2} a(\boldsymbol{p}) e^{2\pi i \langle \boldsymbol{p}, \boldsymbol{\xi} \rangle}.$$

It is easy to check that

$$Z(\xi) = \sum_{j=1}^{2n} e^{2\pi i \langle \boldsymbol{c}^j, \boldsymbol{\xi} \rangle} \prod_{1 \leqslant k < j} \left(1 + \cos\left(2\pi \langle \boldsymbol{c}^k, \boldsymbol{\xi} \rangle\right)\right),$$

and similar reasoning as in the Case IIa (2.13) gives

$$\|P\|_{L^1(\mathbb{T}^2)} \geqslant \|Z\|_{L^1(\mathbb{T}^2)} - \varepsilon 3^s.$$

By Lemma 2.8,

$$\|Z\|_{L_1(\mathbb{T}^2)} \geqslant Cs.$$

Hence

$$\|T_m h\|_{\dot{W}_1^1(\mathbb{R}^2)} \geqslant Cs - \varepsilon 3^s,$$

and setting $\varepsilon = C 3^{-s-1} s$ we get

$$\|T_m h\|_{\dot{W}_1^1(\mathbb{R}^2)} \geqslant Cs$$

which by uniform boundedness of $\|h\|_{\dot{W}_1^1(\mathbb{R}^2)}$ proves that T is unbounded. □

3. Appendix

3.1. Proof of lemma 2.6

We begin with two lemmas. We study the operator given by sufficiently smooth multiplier acting on a subspace of L^1 functions with compactly supported Fourier transform. Let k be the smallest even number greater then $\lceil \frac{d}{2} \rceil$, $d \geqslant 2$. We fix function $\eta \in C_0^\infty$ supported in ball of radius 1.

Lemma 3.1. — *Let $0 < \varepsilon \leqslant r < 1$ and $f \in C^{k+1}(\boldsymbol{B}(\mathbf{0}, r))$ with all derivatives of order less than or equal to k vanishing at $\mathbf{0}$. Then the following inequality holds*

$$\|\mathscr{F}^{-1}(\eta_\varepsilon f)\|_{L^1(\mathbb{R}^d)} \leqslant C(\eta, d)\, \varepsilon \left(\sum_{|\boldsymbol{\alpha}|=k+1} |D^{\boldsymbol{\alpha}} f(\mathbf{0})| + o(\varepsilon) \right), \tag{3.1}$$

where $\eta_\varepsilon(\boldsymbol{x}) := \eta(\varepsilon \boldsymbol{x})$.

Proof. — We recall that for such k the left hand side is bounded up to a constant by $\|\eta f\|_{W_1^k}$ (cf. [12]). By the Leibnitz Formula, it is sufficient to prove that all derivatives $D^{\boldsymbol{\beta}} f$ are dominated by $\sum_{|\boldsymbol{\alpha}|=k+1} |D^{\boldsymbol{\alpha}} f(0)| + o(\varepsilon)$ for $|\boldsymbol{\beta}| \leqslant k$ on $\boldsymbol{B}(\mathbf{0}, \varepsilon)$. This is a consequence of Taylor's Formula.

$$\begin{aligned} D^{\boldsymbol{\beta}} f(\boldsymbol{x}) &= \sum_{|\boldsymbol{\alpha}| \leqslant k+1-|\boldsymbol{\beta}|} D^{\boldsymbol{\alpha}+\boldsymbol{\beta}} f(\mathbf{0}) \boldsymbol{x}^{\boldsymbol{\alpha}} + o(|\boldsymbol{x}|^{k+1-|\boldsymbol{\beta}|}) \\ &= \sum_{|\boldsymbol{\alpha}|=k+1} D^{\boldsymbol{\alpha}} f(\mathbf{0}) + o(\varepsilon). \end{aligned} \tag{3.2}$$

□

Lemma 3.2. — *Let $0 < \varepsilon \leqslant r \leqslant 1$ and $f \in C^{k+1}(\boldsymbol{B}(0, r))$ then the following inequality holds*

$$\|\mathscr{F}^{-1}(\eta_\varepsilon f)\|_{L^1(\mathbb{R}^d)} \leqslant C(\eta, d) \left(|f(\mathbf{0})| + \varepsilon \left(\sum_{|\boldsymbol{\alpha}| \leqslant k+1} |D^{\boldsymbol{\alpha}} f(\mathbf{0})| \right) + o(\varepsilon) \right). \tag{3.3}$$

Proof. — Writing f as the sum of a polynomial of degree k and a function satisfying the assumptions of the previous lemma, we see that it is sufficient to consider only polynomials and by linearity monomials. For $f(\boldsymbol{\xi}) = (2i\pi\boldsymbol{\xi})^{\boldsymbol{\alpha}}$, we have

$$\|\mathscr{F}^{-1}(\eta_\varepsilon f)(\boldsymbol{x})\|_{L_1} = \|\varepsilon^{d+|\boldsymbol{\alpha}|} D^{\boldsymbol{\alpha}} \eta(\frac{\boldsymbol{x}}{\varepsilon})\|_{L_1} \leqslant C(\eta)\varepsilon^{\boldsymbol{\alpha}}. \tag{3.4}$$

Hence inequality (3.3) follows. □

Now we can prove the Lemma 2.6.

Proof of Lemma 2.6. — By the definition of H^θ we see that its support is contained in the union of disjoint balls of radius r with centered in points of Λ_s. Radius r depends only on the parameter θ, so we can choose it as small as we wish. Let $\eta_{\boldsymbol{q}} \in C^\infty$ be rescaled and translated copies of the same function η with $\operatorname{supp} \eta_{\boldsymbol{q}} \subset B(\boldsymbol{q}, 2r)$ and $\eta_{\boldsymbol{q}}(\boldsymbol{\xi}) = 1$, $\boldsymbol{\xi} \in B(\boldsymbol{q}, r)$ for every $\boldsymbol{q} \in \Lambda_s$. The following identity holds

$$\frac{\xi_2}{\xi_1} H^\theta(\boldsymbol{\xi}) = \sum_{\boldsymbol{\phi} \in \Lambda_s} \eta_{\boldsymbol{q}}(\boldsymbol{\xi}) \frac{\xi_2}{\xi_1} H^\theta(\boldsymbol{\xi}). \tag{3.5}$$

By the condition II-G (page 1252) the function $f = \frac{\xi_2}{\xi_1}$ satisfies conditions of Lemma 3.2 on these balls. Hence for r small enough by the triangle inequality, (3.3), and (3.5)

$$\|\mathscr{F}^{-1}(\eta_{\boldsymbol{q}} f H^\theta)\|_{L^1(\mathbb{R}^2)} \leqslant C(\eta) \sum_{\boldsymbol{q} \in \Lambda_s} \left(|f(\boldsymbol{q})| + \varepsilon \left(\sum_{|\alpha| \leqslant k+1} |D^\alpha f(\boldsymbol{q})| \right) + o(\varepsilon) \right) \cdot \|\mathscr{F}^{-1}(H^\theta)\|_{L^1(\mathbb{R}^2)}.$$

By conditions II-E and II-H,

$$\left|\frac{q_2}{q_1}\right| = \left| \frac{c_2^k + \sum_{j=1}^{k-1} \zeta_j c_2^j}{c_1^k + \sum_{j=1}^{k-1} \zeta_j c_1^j} \right| \leqslant \frac{k|c^k|}{|c_1^k| - \sum_{j=1}^{k-1} |c_1^j|} \leqslant \frac{s}{3} \left| \frac{c_1^k}{c_2^k} \right| \leqslant \frac{1}{2 \cdot 3^s}.$$

Since $|\Lambda_s| \leqslant 3^s$ we can choose sufficiently small $\varepsilon > 0$ such that

$$\|\mathscr{F}^{-1}(\frac{\xi_2}{\xi_1} H^\theta)\|_{L_1(\mathbb{R}^2)} \leqslant C \|\mathscr{F}^{-1}(H^\theta)\|_{L_1(\mathbb{R}^2)},$$

where the constant C does not depend on s. □

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