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ON THE HILBERT GEOMETRY OF SIMPLICIAL TITS SETS

by Xin NIE (*)

ABSTRACT. — The moduli space of convex projective structures on a simplicial hyperbolic Coxeter orbifold is either a point or the real line. Answering a question of M. Crampon, we prove that in the latter case, when one goes to infinity in the moduli space, the entropy of the Hilbert metric tends to 0.

RÉSUMÉ. — L'espace des modules de structures projectives convexes sur un orbifold simplicial hyperbolique est soit un point soit la droite réelle. En répondant à une question de M. Crampon, nous prouvons que dans ce dernier cas, quand on tend vers l'infini dans l'espace des modules, l'entropie de la métrique de Hilbert tend vers 0.

1. Statements of results

A (real) projective structure on an orbifold X is a maximal atlas with charts taking values in the real projective space $\mathbb{P}^n$ and with transition functions taking values in the group of projective transformations.

An extensively studied class of projective structures is given by the following construction. An open subset $\Omega \subset \mathbb{P}^n$ is said to be *properly convex* if it is a bounded convex subset of an affine chart $\mathbb{R}^n \subset \mathbb{P}^n$. Let $X = \widetilde{X}/\Pi$ be an orbifold, where $\widetilde{X}$ is a manifold homeomorphic to $\mathbb{R}^n$ and Π is a group acting properly discontinuously on $\widetilde{X}$. A *convex projective structure* on X consists of a faithful representation $\rho : \Pi \to \mathrm{PGL}(n+1, \mathbb{R})$ and a ρ-equivariant homeomorphism from $\widetilde{X}$ to a properly convex open set $\Omega \subset \mathbb{P}^n$.

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The convex set Ω is determined by ρ because Ω is the convex hull of any $\rho(\Pi)$-orbit in Ω [13], so the *moduli space of convex projective structures* is defined as the subset in the moduli space of representations

$$\mathfrak{P}(X) \subset \mathrm{Hom}\left(\Pi, \mathrm{PGL}(n+1,\mathbb{R})\right) / \mathrm{PGL}(n+1,\mathbb{R})$$

consisting of those $\rho \in \mathrm{Hom}(\Pi, \mathrm{PGL}(n+1,\mathbb{R}))$ which arise from convex projective structures. It is known that $\mathfrak{P}(X)$ is an open and closed subset in the moduli space of representations [2] and is homeomorphic to $\mathbb{R}^{16g-16}$ when X is a closed oriented surface of genus g [7].

In this article we investigate the case where X is a hyperbolic simplicial Coxeter orbifold, *i.e.* $X = \mathbb{H}^n/\Gamma$, where $\mathbb{H}^n$ is the real hyperbolic n-space and $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is generated by orthogonal reflections with respect to the faces of a bounded n-simplex $P \subset \mathbb{H}^n$ with totally geodesic faces, such that P is a fundamental domain of Γ.

The angle between the i^{th} and j^{th} face of such a simplex P must be π/m_{ij} for some integer $m_{ij} \geqslant 2$ (where $0 \leqslant i < j \leqslant n$). The *Coxeter diagram* J associated to P is a graph with nodes labelled by $0, 1, \cdots, n$, such that i and j are joint by an edge with weight m_{ij} if $m_{ij} \geqslant 3$ and are not joint by any edge if $m_{ij} = 2$. J determines P up to isometry, hence determines the orbfold X, which we denote from now on by X_J.

The J's occurring in this way are the *hyperbolic Coxeter diagrams*, which are classified by F. Lannér [10] as in Figure 1.1. In particular, they exist only when $n \leqslant 4$. We divide them into two classes: circular and non-circular ones.

The following result should be well known to specialists and is stated in [6] in the two-dimensional case.

Proposition 1.1. — *Let J be a hyperbolic Coxeter diagram, then*

$$\mathfrak{P}(X_J) \cong \begin{cases} \mathbb{R}_+ & \text{if } J \text{ is circular,} \\ \text{a point} & \text{otherwise.} \end{cases}$$

A proof of Proposition 1.1 is given in Section 3 below.

For a circular hyperbolic Coxeter diagram J, we prove a result on how the convex set Ω deforms as ρ goes to 0 or $+\infty$ in $\mathfrak{P}(X_J) \cong \mathbb{R}_+$.

Proposition 1.2. — *Let P be a simplex in $\mathbb{P}^n$. Let $X_J = \mathbb{H}^n/\Gamma$ be a hyperbolic simplicial Coxeter orbifold given by a circular diagram J. Then there exists a one-parameter family of representations $\{\rho_t\}_{t\in\mathbb{R}_+}$ of Γ into $\mathrm{PGL}(n+1,\mathbb{R})$ such that*

(1) *ρ_t sends generators of Γ to projective reflections with respect to face of P.*

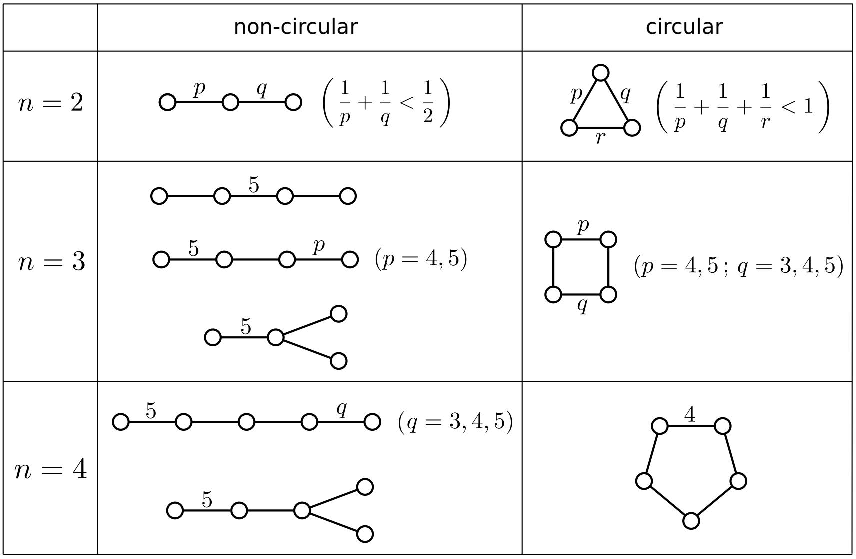

Figure 1.1. All hyperbolic Coxeter diagrams. Here each edge without specified weight has weight 3.

(2) *ρ_t yields a convex projective structure on X_J, where the properly convex set is $\Omega_t := \bigcup_{\gamma \in \Gamma} \rho_t(P)$. The map $\mathbb{R}_+ \to \mathfrak{P}(X_J)$, $t \mapsto [\rho_t]$ is bijective.*

(3) *The convex set Ω_t converges to the simplex P with respect to the Hausdorff topology as t tends to* 0 *or to* $+\infty$.

See Figure 1.2 for a 2-dimensional example of Proposition 1.2.

Our main result is concerned with metric properties of the above family of convex sets. Any properly convex open set $\Omega \subset \mathbb{P}^n$ carries a canonical Finsler metric d_Ω, called the *Hilbert metric*, which is invariant under projective transformations preserving Ω. In particular, if Ω is an ellipsoid, then (Ω, d_Ω) is isometric to the real hyperbolic n-space $\mathbb{H}^n$.

Consider the convex set Ω_t from Proposition 1.2. We denote the Hilbert metric on it by d_t. By projective invariance, d_t induces a metric on X_J. From Proposition 1.2 we can readily deduce some simple geometric properties of the family of metrics $\{d_t\}$. For example, the diameter of (X_J, d_t) tends to infinity as $t \to 0$ or $t \to +\infty$. The purpose of this paper is to study a more subtle quantity, the entropy, as defined below.

Definition 1.3. — *Let $(\widetilde{X}, d)$ be a metric space and Γ be a group acting properly discontinuously on $\widetilde{X}$ by isometries. Given a base point $x_0 \in \widetilde{X}$,*

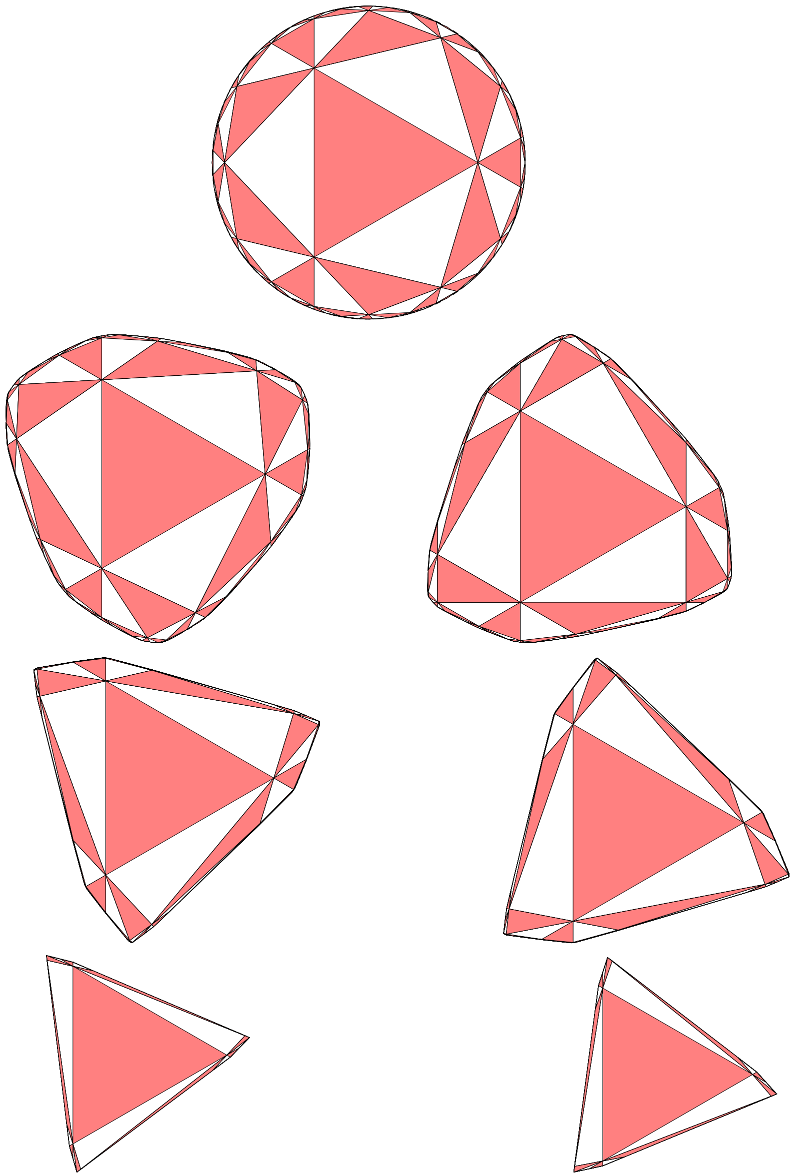

Figure 1.2. Deformation of Ω_t when t tends to 0 and $+\infty$.

the (exponential) growth rate of the orbit $\Gamma.x_0$ is defined as

$$\delta(\widetilde{X}, d, \Gamma, x_0) = \varlimsup_{R\to+\infty} \frac{1}{R} \log \#\big(\Gamma.x_0 \cap B(x_0, R)\big),$$

where $B(x_0, R)$ is the ball of radius R centered at x_0.

This notion originally arose from the case where $(\widetilde{X}, d)$ is the universal cover of a compact non-positively curved Riemaniann manifold X and $\Gamma = \pi_1(X)$. In this case the above growth rate is independent of the choice of x_0 and equals the topological entropy of the geodesic flow on the unit tangent bundle of X [11]. This result easily generalizes to geodesic flows of compact convex projective manifolds endowed with the Hilbert metric, see [4]. For this reason, we refer to the orbit growth rate as the *entropy* and omit x_0 in the notation.

For any properly convex open set $\Omega \subset \mathbb{P}^n$ acted upon by a discrete group $\Gamma \subset \mathrm{PGL}(n+1, \mathbb{R})$ with compact fundamental domain, M. Crampon [4] proved that the entropy is bounded from above

$$\delta(\Omega, d_\Omega, \Gamma) \leqslant n - 1$$

and the equality is achieved if and only if Ω is an ellipsoid. He then asked whether $\delta(\Omega, d_\Omega, \Gamma)$ has a lower bound.

Our main result gives a negative answer:

Theorem 1.4. — *Let $X_J = \mathbb{H}^n/\Gamma$ be a hyperbolic simplicial Coxeter orbifold where J is a circular diagram. Let ρ_t and Ω_t be given by Proposition 1.2 and d_t be the Hilbert metric on Ω_t. Then*

$$\delta(\Omega_t, d_t, \Gamma) \to 0 \textit{ as } t \to 0 \textit{ or } t \to +\infty.$$

The main ingredient in the proof of Theorem 1.4 is the following result, which is a manifestation of the reflectional symmetries of the metric spaces (Ω_t, d_t).

Lemma 1.5. — *There exists a constant C depending only on the Coxeter diagram J, such that if A and B are two k-dimensional cells of the simplex P and $E = A \cap B$ is a $(k-1)$-dimensional cell, where $1 \leqslant k \leqslant n-1$, then we have*

$$C d_t(x, y) \geqslant d_t(x, E) + d_t(y, E)$$

for any $x \in A$, $y \in B$ and $t \in \mathbb{R}_+$.

As another consequence of Lemma 1.5, we construct families of convex projective structures on surfaces which answer Crampon's question and have some other curious properties.

Corollary 1.6. — *On an oriented closed surface of genus $g \geqslant 2$, there exists an one-parameter family of convex projective structures such that when the parameter goes to infinity, the entropy of Hilbert metric tends to 0, whereas the systole and constant of Gromov hyperbolicity tends to $+\infty$.*

Recall that for a metrized manifold (X,d), the *systole* is defined as the infimum of lengths of homotopically non-trivial closed curves on X. Let $\widetilde{X}$ be the universal covering of X, then the *constant of Gromov hyperbolicity* is defined to be the supremum of sizes of geodesic triangles in $\widetilde{X}$. Here the “size" of a geodesic triangle Δ is the minimal perimeter of all geodesic triangles inscribed in Δ (*c.f.* [5]).

Corollary 1.6 raises the problem of studying the entropy as a function on the moduli space of convex projective structures $\mathfrak{P}(\Sigma)$ for a closed hyperbolic surface Σ. The Teichmüller space $\mathcal{T}(\Sigma)$ is naturally contained in $\mathfrak{P}(\Sigma)$ and the entropy is identically 1 on $\mathcal{T}(\Sigma)$, while Corollary 1.6 provides a curve transverse to $\mathcal{T}(\Sigma)$ along which the entropy tends to 0. Recently, using a different method, T. Zhang [16] constructed higher-dimensional submanifold of $\mathfrak{P}(\Sigma)$ transverse to $\mathcal{T}(\Sigma)$ along which the entropy tends to 0 as well.

The paper is organized as follows. After recalling some backgrounds about reflection groups in Section 2, we prove Proposition 1.1 and 1.2 in Section 3. In Section 4 we prove Theorem 1.4 and Corollary 1.6 assuming Lemma 1.5. Finally we prove Lemma 1.5 in Section 5.

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2. Preliminaries

In this section we recall some well known facts about reflection groups and Tits set. See [3, 14] for details.

A projective transformation $s \in \mathrm{PGL}(n+1,\mathbb{R})$ is called a *reflection* if it is conjugate to $\pm\,\mathrm{diag}(-1,1,\cdots,1)$. The fixed point set of a reflection s is $\mathrm{Fix}(s) = F \sqcup f$ for some hyperplan F in $\mathbb{P}^n$ and some point $f \notin F$. Reflections are in one-to-one correspondence with pairs (f,F) with $f \notin F$.

Let P be a n-dimesnional simplex in $\mathbb{P}^n$ with faces P_i, where $i = 0,\cdots,n$. We choose a reflection s_i with respect to each P_i. We are interested in the group $\Gamma \subset \mathrm{PGL}(n+1,\mathbb{R})$ generated by the s_i's and call it a *simplicial reflection group*, and call P the *fundamental simplex*. Since simplices in $\mathbb{P}^n$

are conjugate to each other by projective transformations, when studying Γ up to conjugacy, we can assume

$$P = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_i \geqslant 0, \forall i\},$$

whose faces are

$$P_i = \{[x_0 : \cdots : x_n] \mid x_i = 0, x_k \geqslant 0, \forall k \neq i\}.$$

Γ is determined by $f_1, \cdots, f_n \in \mathbb{P}^n$, where $f_i \notin P_i$ is a fixed point of s_i. Suppose $f_i = [a_{0i} : \cdots : a_{ni}]$. We can assume $a_{ii} = 1$ since $f_i \notin P_i$. We record these f_i's by the matrix $\mathbf{A} = (a_{ij})$ with 1 on diagonals and denote the resulting reflection group by $\Gamma_{\mathbf{A}}$.

Let $H^+ \subset \mathrm{PGL}(n+1, \mathbb{R})$ be the subgroup consisting of positive diagonal matrices $\lambda = \mathrm{diag}(\lambda_0, \cdots, \lambda_n)$, $\lambda_i > 0$. So H^+ is the identity component of the stabilizer of P. Given $\lambda \in H^+$ and a reflection group $\Gamma_{\mathbf{A}}$ as above, the conjugate $\lambda\Gamma_{\mathbf{A}}\lambda^{-1}$ is just $\Gamma_{\lambda\mathbf{A}\lambda^{-1}}$. Put

$$\mathcal{M}_{n+1} = \{\mathbf{A} = (a_{ij}) \in \mathbb{R}^{(n+1)\times(n+1)} \mid a_{ii} = 1 \text{ for any } i\}.$$

The quotient $\mathcal{M}_{n+1}/H^+$ by conjugation action is the moduli space of simplicial reflection groups. We now proceed to discuss discreteness of such groups.

By a *Coxeter diagram with n nodes* we mean a collection of integers $J = (m_{ij})$, where $i, j \in \{0, \cdots, n\}$ are distinct, such that $2 \leqslant m_{ij} \leqslant \infty$. Note that $m_{ij} = \infty$ is allowed. J is a “diagram" because we view it as a graph with

- n nodes labelled by $0, 1, \cdots, n$,
- at most one weigted edge joining any two nodes i and j: no edge if $m_{ij} = 2$ and an edge of weight m_{ij} if $m_{ij} \geqslant 3$.

The Coxeter diagram $J = (m_{ij})$ determines an abstract Coxeter group W_J through the presentation

$$W_J := \left\langle \tau_0, \cdots, \tau_n \;\middle|\; (\tau_i\tau_j)^{m_{ij}} = \tau_i^2 = 1, \forall\, i \neq j \right\rangle.$$

By convention, $(\tau_i\tau_j)^\infty = 1$ means $\tau_i\tau_j$ has infinite order.

The *Cartan matrix* of J, denoted by $\mathbf{C}_J$, is defined as the symmetric matrix whose diagonal entries are 1 and the (i,j)-entry is $-\cos(\pi/m_{ij})$ (where $i \neq j$).

We have the following sufficient condition on $\mathbf{A} \in \mathcal{M}_{n+1}$ which ensures that $\Gamma_{\mathbf{A}}$ is discrete. This is a special case of Theorem 1.5 in [3].

Theorem (Tits, Vinberg). — *Let $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{n+1}$. Let P° be the interior of the fundamental simplex P. The translates $\gamma(P^\circ)$, $\gamma \in \Gamma_{\mathbf{A}}$ are* *pairwise disjoint if and only if there is a Coxeter diagram $J = (m_{ij})$ such that $\mathbf{A}$ satisfies the following condition, referred to as Condition (J):*

$$\text{(J)} \qquad \begin{cases} a_{ij} = a_{ji} = 0 & \textit{if } m_{ij} = 2, \\ a_{ij} < 0 \textit{ and } a_{ij}a_{ji} = \cos^2(\pi/m_{ij}) & \textit{if } 3 \leqslant m_{ij} < \infty, \\ a_{ij} < 0 \textit{ and } a_{ij}a_{ji} \geqslant 1 & \textit{if } m_{ij} = \infty. \end{cases}$$

Furthermore, when Condition (J) is satisfied, the following assertions hold.

(1) *Let f_i be the point in $\mathbb{P}^n$ whose homogeneous coordinates are given by the i^{th} column of $\mathbf{A}$. Let $s_i \in \mathrm{PGL}(n+1, \mathbb{R})$ be the reflection fixing P_i and f_i. Then there is an isomorphism $\rho_{\mathbf{A}} : W_J \to \Gamma_{\mathbf{A}}$ defined on generators by*

$$\rho_{\mathbf{A}}(\tau_i) = s_i.$$

(2) *The* Tits set *$\Omega_{\mathbf{A}}$ of the reflection group $\Gamma_{\mathbf{A}}$, defined by*

$$\Omega_{\mathbf{A}} = \bigcup_{\gamma \in \Gamma_{\mathbf{A}}} \gamma(P),$$

is either the whole $\mathbb{P}^n$ (this occurs if and only if W_J is finite) or a convex subset in some affine chart of $\mathbb{P}^n$.

(3) *$\Omega_{\mathbf{A}}$ is open if and only if the stabilizer in $\Gamma_{\mathbf{A}}$ of each vertex of P is finite.*

We let $\mathcal{M}_J \subset \mathcal{M}_{n+1}$ denote the set of matrices satisfying Condition (J). Note that the H^+-action on $\mathcal{M}_{n+1}$ preserves $\mathcal{M}_J$.

There are only a few choices of J for which $\Omega_{\mathbf{A}}$ (where $\mathbf{A} \in \mathcal{M}_J$) is open and is not the whole $\mathbb{P}^n$, so that $\Omega_{\mathbf{A}}/\Gamma_{\mathbf{A}}$ is a convex projective orbifold. Indeed, $\Omega_{\mathbf{A}} \neq \mathbb{P}^n$ and $\Omega_{\mathbf{A}}$ being open are respectively equivalent to following two constraints on the Cartan matrix $\mathbf{C}_J$.

(a) $\mathbf{C}_J$ is not positively definite.

(b) Every proper principle submatrix of $\mathbf{C}_J$ is positively definite.

Such $\mathbf{C}_J$'s are completely classified. The corresponding J's are divided into classes (*c.f.* [14]):

- **Euclidean Coxeter diagrams**, *i.e.* the J's satisfying conditions (a) and (b) with $\mathbf{C}_J$ degenerate. In this case $\mathbf{C}_J$ has corank 1 and there is a faithful representation $W_J \to \mathrm{Isom}(\mathbb{E}^n)$ realizing W_J as an Euclidean reflection group. All such J's are enumerated by H.S.M.Coxeter himself. It follows from a result of Margulis and

Vinberg ([12] Lemma 8) that the Tits set $\Omega_{\mathbf{A}}$ ($\mathbf{A} \in \mathcal{M}_J$) in this case is either an open simplex containing P or an affine chart.

- **Hyperbolic Coxeter diagrams**, *i.e.* the J's satisfying conditions (a) and (b) with $\mathbf{C}_J$ non-degenerate. In this case $\mathbf{C}_J$ has signature $(1, n)$ and there is a faithful representation $\rho_0 : W_J \to \mathrm{Isom}(\mathbb{H}^n)$ which realize W_J as a hyperbolic simplicial reflection group. F. Lannér [10] enumerated all hyperbolic Coxeter diagrams as in Figure 1.1 above (*c.f.* [14]). Note that they exist only for dimension $n \leqslant 4$. Since W_J is a word-hyperbolic group, a theorem of Benoist [1] says that $\Omega_{\mathbf{A}}$ ($\mathbf{A} \in \mathcal{M}_J$) is strictly convex, *i.e.* $\partial\Omega_{\mathbf{A}}$ does not contain any line segment. These $\Omega_{\mathbf{A}}$'s are our main concern in the following sections. Historically, they provide the first "non-trivial" examples of convex projective structures [15]. See Figure 1.2 for some 2-dimensional examples.

3. Deformation of simplicial Tits sets

We fix a hyperbolic Coxeter diagram J and consider W_J as a hyperbolic reflection group with fundamental simplex $P \subset \mathbb{H}^n$. Put $X_J = \mathbb{H}^n/W_J$. The goal of this section is to prove Proposition 1.1 and Proposition 1.2.

The hyperbolic n-space $\mathbb{H}^n$ is a ball in $\mathbb{P}^n$ (the Klein-Beltrami model). Let $P_0, \cdots, P_n$ be the faces of P and L_i be the hyperplane in $\mathbb{P}^n$ containing P_i. Consider a faithful representation $\rho : W_J \to \mathrm{PGL}(n+1, \mathbb{R})$ which defines a convex projective structure, *i.e.* there is some convex open set Ω_ρ and a ρ-equivariant homeomorphism $\Phi : \mathbb{H}^n \to \Omega_\rho$. Since $\rho(\tau_i)$ has order 2, its fixed point set in $\mathbb{P}^n$ is the disjoint union of a k-dimensional subspace and a $(n-k)$-dimensional subspace. On the other hand, $\rho(\tau_i)$ fixes pointwisely $\Phi(L_i)$, a $(n-1)$-dimensional submanifold of Ω_ρ, so we conclude that $\Phi(L_i)$ is a hyperplan and $\rho(\tau_i)$ is a reflection. $\rho(W_J)$ is thus a simplicial projective reflection group. But the discussions in the previous section implies that such groups, up to conjugacy, correspond to matrices $\mathbf{A} \in \mathcal{M}_J$ up to conjugation by H^+. Therefore we get an identification

$$\mathfrak{P}(X_J) = \mathcal{M}_J/H^+.$$

We now need to determined the latter quotient. To this end, for a $(n+1) \times (n+1)$ matrix $\mathbf{A} = (a_{ij})$ and a cyclically ordered multi-index $I = (i_1, \cdots, i_k)$ (where $i_1, \cdots, i_k \in \{0, \cdots, n\}$), we put

$$\mathbf{A}(I) = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}.$$

In particular, $\mathbf{A}(i) = a_{ii}$, $\mathbf{A}(i,j) = a_{ij}a_{ji}$. Let $|I| = k$ denote the length of I.

Let $\mathcal{M}_{n+1}$ denote the set of $(n+1) \times (n+1)$ real matrices with 1 on diagonals. Consider the subset

$$\mathcal{M}^{\circ}_{n+1} = \{\mathbf{A} = (a_{ij}) \in \mathcal{M}_{n+1} \mid \text{ for any } i \neq j,\ a_{ij} = 0 \text{ if and only if } a_{ji} = 0\}.$$

Lemma 3.1. — *Suppose $\mathbf{A}, \mathbf{B} \in \mathcal{M}^{\circ}_{n+1}$. Then $\mathbf{A} = \lambda \mathbf{B} \lambda^{-1}$ for some positive diagonal matrix $\lambda = \mathrm{diag}(\lambda_0, \cdots, \lambda_n)$, $\lambda_i > 0$ if and only if $\mathbf{A}(I) = \mathbf{B}(I)$ for any multi-index I with $|I| \geqslant 2$.*

Proof. — The "only if" part is immediate and we only treat the "if" part.

We say that $\mathbf{A}$ is *irreducible* if it cannot be brought into block-diagonal form by a permutation of basis. For a multi-index $I = (ij)$ of length 2, the equality $\mathbf{A}(ij) = \mathbf{B}(ij)$ implies that $a_{ij} = 0$ if only if $b_{ij} = 0$. Therefore, after a permutation of basis if necessary, we can assume that $\mathbf{A}$ and $\mathbf{B}$ are both block-diagonal with irreducible blocks and that the r^{th} block of $\mathbf{A}$ has the same size as the r^{th} block of $\mathbf{B}$. Such $\mathbf{A}$ and $\mathbf{B}$ are conjugate through a diagonal matrix if and only if their blocks are, so we can assume that $\mathbf{A}$ and $\mathbf{B}$ are irreducible.

We look for $\lambda_0, \cdots, \lambda_n > 0$ such that $\lambda_i a_{ij} \lambda_j^{-1} = b_{ij}$, or equivalently,

$$\frac{\lambda_i}{\lambda_j} = \frac{b_{ij}}{a_{ij}} \text{ for all } i \neq j \text{ such that } a_{ij} \neq 0 \tag{3.1}$$

Put $\lambda_0 = 1$ to begin. Irreducibility implies that for each $i \in \{0, 1, 2, \cdots, n\}$ there is sequence of distinct indices $0, i_1, i_2, \cdots, i_k, i$, such that $a_{0i_1}, a_{i_1 i_2}, \cdots, a_{i_{k-1} i_k}, a_{i_k i}$ are all non-zero. Since

$$\lambda_i = \frac{\lambda_i}{\lambda_{i_k}} \frac{\lambda_{i_k}}{\lambda_{i_{k-1}}} \cdots \frac{\lambda_{i_1}}{\lambda_0},$$

in view of of Eq.(3.1) which we need to fulfill, we set

$$\lambda_i = \frac{b_{ii_k}}{a_{ii_k}} \frac{b_{i_k i_{k-1}}}{a_{i_k i_{k-1}}} \cdots \frac{b_{i_1 1}}{a_{i_1 1}}. \tag{3.2}$$

Let us check the so-defined λ_i does not depend on the choice of the sequence of indices, namely,

$$\frac{b_{ii_k}}{a_{ii_k}} \frac{b_{i_k i_{k-1}}}{a_{i_k i_{k-1}}} \cdots \frac{b_{i_1 0}}{a_{i_1 0}} = \frac{b_{ij_m}}{a_{ij_m}} \frac{b_{j_m j_{m-1}}}{a_{j_m j_{m-1}}} \cdots \frac{b_{j_1 0}}{a_{j_1 0}} \tag{3.3}$$

for another sequence $0, j_1, j_2, \cdots, j_m, i$. Using the hypothesis

$$\mathbf{A}(i,j) = a_{ij}a_{ji} = b_{ij}b_{ji} = \mathbf{B}(i,j),$$

we can write the right-hand side of Eq.(3.3) as

$$\frac{a_{j_m i}}{b_{j_m i}}\frac{a_{j_{m-1}j_m}}{b_{j_{m-1}j_m}}\cdots\frac{a_{1j_1}}{b_{1j_1}}.$$

This coincides the left-hand side because of the equality $\mathbf{A}(I)=\mathbf{B}(I)$ for $I=(1,j_1,\cdots,j_m,i,i_k,i_{k-1},\cdots,i_1)$. A similar equality shows that the λ_i's defined by Eq.(3.2) satisfy (3.1). □

Proof of Proposition 1.1. — The hyperbolic Coxeter diagram $J=(m_{ij})$ is not circular if and only if $\mathbf{A}(I)=0$ for any $\mathbf{A}\in\mathcal{M}_J$ and any multi-index I with $|I|\geqslant 3$. But we also have $\mathbf{A}(i,j)=a_{ij}a_{ji}=\cos^2(\pi/m_{ij})$ by definition of $\mathcal{M}_J$. Therefore, for any given I with $|I|\geqslant 2$, $\mathbf{A}(I)$ has the same value for any $\mathbf{A}\in\mathcal{M}_J$. Lemma 3.1 implies that elements in $\mathcal{M}_J$ are conjugate to each other through H^+.

If J is circular, then any $\mathbf{A}\in\mathcal{M}_J$ looks like the following one (where $n=4$):

$$\mathbf{A}=\begin{pmatrix}1 & a_{01} & 0 & 0 & a_{04}\\ a_{10} & 1 & a_{12} & 0 & 0\\ 0 & a_{21} & 1 & a_{23} & 0\\ 0 & 0 & a_{32} & 1 & a_{34}\\ a_{40} & 0 & 0 & a_{43} & 1\end{pmatrix}.$$

Again, given i and j, the value of $\mathbf{A}(i,j)$ is the same for any $\mathbf{A}\in\mathcal{M}_J$. The only two non-zero $\mathbf{A}(I)$'s for $|I|\geqslant 3$ are

$$\begin{aligned}\mathbf{A}(0,1,\cdots,n) &= a_{01}\cdots a_{n-1,n}a_{n0}\,,\\ \mathbf{A}(n,n-1,\cdots,0) &= a_{n,n-1}\cdots a_{10}a_{0n}.\end{aligned}$$

They determine each other because the product is a constant

$$\mathbf{A}(0,1,\cdots,n)\cdot\mathbf{A}(n,n-1,\cdots,0)$$
$$=\cos^2\left(\frac{\pi}{m_{01}}\right)\cos^2\left(\frac{\pi}{m_{12}}\right)\cdot\cos^2\left(\frac{\pi}{m_{n0}}\right).$$

Therefore, Lemma 3.1 implies that $\mathbf{A}\in\mathcal{M}_J$ is determined up to H^+-conjugacy by $\mathbf{A}(0,1,\cdots,n)$, which is always positive (resp. negative) if n is odd (resp. even). Thus we get a homeomorphism

$$\begin{aligned}\mathfrak{P}(X_J)=\mathcal{M}_J/H^+ &\to \mathbb{R}_+\\ [\mathbf{A}] &\mapsto |\mathbf{A}(0,1,\cdots,n)|\end{aligned}$$

□

In order to study how the Tits set $\Omega_{\mathbf{A}}$ deforms when $[\mathbf{A}]$ goes to 0 or $+\infty$ in $\mathfrak{P}(X_J)\cong\mathbb{R}_+$, we need the following lemma, which bounds $\Omega_{\mathbf{A}}$ by a simplex.

Lemma 3.2. — *Let J be a circular hyperbolic Coxeter diagram and let $\mathbf{A} \in \mathcal{M}_J$. Let $f_i \in \mathbb{P}^n$ be the point with coordinates given by the i^{th} column of $\mathbf{A}$. Then there is a simplex with vertices $f_0, \cdots, f_n$ which contains the Tits set $\Omega_{\mathbf{A}}$.*

Proof. — The arguments given here are illustrated in the two-dimensional case by Figure 3.1. Let L_i be the hyperplane in $\mathbb{P}^n$ spanned by $f_0, \cdots, f_{i-1}, f_{i+1}, \cdots, f_n$ and let $H_i = \{[x_0, \cdots, x_n] \mid x_i = 0\}$ be the hyperplane containing the face P_i. Assume by contradiction that $\Omega := \Omega_{\mathbf{A}}$ is not contained in any simplex with vertices $f_0, \cdots, f_n$, or equivalently, Ω meets some L_i, say, L_0.

Recall that Ω is preserved by the group $\Gamma_{\mathbf{A}}$, which is in turn generated by $s_0, \cdots, s_n$, where s_i is the reflection with fixed points $\mathrm{Fix}(s_i) = f_i \sqcup H_i$. In general, a reflection s with $\mathrm{Fix}(s) = f \sqcup H$ stabilizes any projective subspace containing f. It follows that L_0 is stabilized by the subgroup $\Gamma_0 \subset \Gamma_{\mathbf{A}}$ generated by $s_1, \cdots, s_n$. Γ_0 is a finite group because J is hyperbolic.

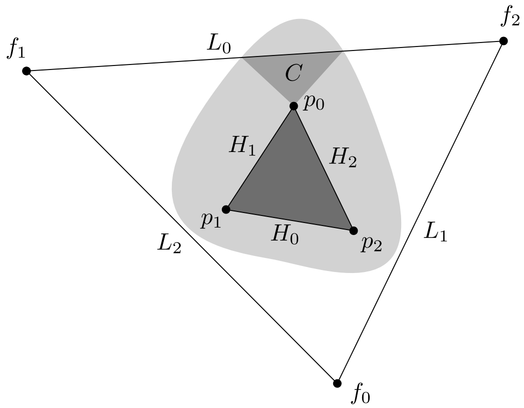


Figure 3.1. Proof of Lemma 3.2.

We claim that the vertex $p_0 = [1 : 0, \cdots, 0]$ of P is not in L_0. Indeed, on one hand, since J is circular, each column of $\mathbf{A} \in \mathcal{M}_J$ has at least two non-zero off-diagonal entries. In other words, each f_i lies outside at least two H_j's, thus

$$\mathrm{Fix}(\Gamma_0) = (f_1 \cup H_1) \cap \cdots \cap (f_n \cup H_n) = H_1 \cap \cdots \cap H_n = \{p_0\}.$$

i.e. p_0 is the only fixed point of Γ_0. On the other hand, Γ_0 preserves the affine chart $\mathcal{A}_0 = \mathbb{P}^n \backslash L_0$ and hence has fixed points in $\mathcal{A}_0$, namely, barycenters of orbits. Thus $p_0 \in \mathcal{A}_0$ and the claim is proved.

To finish the contradiction argument, we put

$$C = \bigcup_{x \in \Omega \cap L_0} [p_0, x],$$

where $[p_0, x]$ is the segment joining p_0 and x within Ω. In other words, C is the cone over p_0 generated by $\Omega \cap L_0$. We consider $\mathcal{A}_0$ as a vector space with origin p_0. Properly convexity of Ω implies that C is a properly convex cone in $\mathcal{A}_0$ (*i.e.* a cone whose projectivization is a properly convex subset of $\mathbb{P}(\mathcal{A}_0)$). Since Γ_0 preserves C, taking the barycenter of a non-zero Γ_0-orbit in C gives a fixed point of Γ_0 different from p_0, contradicting the fact $\mathrm{Fix}(\Gamma_0) = \{p_0\}$ which we have established above. □

Proof of Proposition 1.2. — We only consider the $n = 3$ case to simplify the notations. Thus we fix a circular hyperbolic Coxeter diagram $J = (m_{ij})$ with nodes $\{0, 1, 2, 3\}$. Any $\mathbf{A} \in \mathcal{M}_J$ has the form

$$\mathbf{A} = \begin{pmatrix} 1 & a_{01} & 0 & a_{03} \\ a_{10} & 1 & a_{12} & 0 \\ 0 & a_{21} & 1 & a_{23} \\ a_{30} & 0 & a_{32} & 1 \end{pmatrix}$$

with $a_{ij} < 0$ and $a_{ij}a_{ji} = \cos^2(\pi/m_{ij})$.

We define a one-parameter family of matrices $\{\mathbf{A}_t\}_{t\in\mathbb{R}} \subset \mathcal{M}_J$ by

$$\mathbf{A}_t = \begin{pmatrix} 1 & -t\cos^2(\frac{\pi}{m_{01}}) & 0 & -t^{-1} \\ -t^{-1} & 1 & -t\cos^2(\frac{\pi}{m_{12}}) & 0 \\ 0 & -t^{-1} & 1 & -t\cos^2(\frac{\pi}{m_{23}}) \\ -t\cos^2(\frac{\pi}{m_{30}}) & 0 & -t^{-1} & 1 \end{pmatrix}.$$

Since $|\mathbf{A}_t(0,1,2,3)| = t^4$, by the proof of Proposition 1.1, every $\mathbf{A} \in \mathcal{M}_J$ is H^+-conjugate to a unique $\mathbf{A}_t$. We claim that the family of representations

$$\rho_t := \rho_{\mathbf{A}_t} : W_J \to \mathrm{PGL}(n+1, \mathbb{R}), \quad t \in \mathbb{R}_+$$

given by the Tits-Vinberg theorem is the required one.

To see this, let $f_i(t)$ be the point in $\mathbb{P}^n$ with coordinates given by the i^{th} column of $\mathbf{A}_t$ and let $p_0 = [1:0:0:0], \cdots, p_3 = [0:0:0:1]$ be the vertices of P. We see from the above expression of $\mathbf{A}_t$ that

$$\lim_{t\to 0} f_i(t) = p_{i+1}, \quad \lim_{t\to+\infty} f_i(t) = p_{i-1},$$

where the indices are counted mod 4. Therefore the simplex bounding $\Omega_t = \Omega_{\mathbf{A}_t}$ given by Lemma 3.2 converges to P in the Hausdorff topology, hence so does Ω_t. □

4. Metric geometry of simplicial Tits sets

The *Hilbert metric* d_Ω on a properly convex open set $\Omega \subset \mathbb{P}^n$ is defined as follows. Take any affine chart $\mathbb{R}^n$ containing the closure $\overline{\Omega}$. For $x, y \in \Omega$, let x', y' be the points on the boundary $\partial\Omega$ such that x', x, y, y' lie consecutively on the segment $[x', y']$. We then put

$$d_\Omega(x,y) = \frac{1}{2}\log\,[x',x,y,y'], \tag{4.1}$$

where $[x',x,y,y'] = \frac{(x'-y)(y'-x)}{(x'-x)(y'-y)}$ is the cross-ratio.

See [8] for basic properties of d_Ω. A crucial property which we will use implicitly several times below is that geodesics in (Ω, d_Ω) are straight lines.

We proceed to study the geometry of the Hilbert metric $d_t := d_{\Omega_t}$ on the Tits set Ω_t produced by Proposition 1.2 and studied in the previous section. The goal of this section is to prove Theorem 1.4 and Corollary 1.6, admitting Lemma 1.5.

Observe that Ω_t is a simplicial complex whose k-cells are translates of the k-cells of P by the W_J-action. We denote the k-skeleton of Ω_t by $\Omega_t^{(k)}$ and let $d_t^{(k)}$ be the geodesic metric on $\Omega_t^{(k)}$ induced by d_t, *i.e.*

$$d_t^{(k)}(x,y) = \min\{l_t(\gamma) \mid \gamma \subset \Omega_t^{(k)} \text{ is a piecewise geodesic joining } x \text{ and } y\}.$$

Here $l_t(\gamma)$ is the length of γ measured under d_t. In particular, $(\Omega_t^{(1)}, d_t^{(1)})$ is a metric graph, whereas $d_t^{(n)}$ is just d_t itself.

Lemma 1.5 implies that these metrics are uniformly equivalent to each other:

Lemma 4.1. — *Suppose $2 \leqslant k \leqslant n$. There is a constant C depending only on J such that for any $t \in \mathbb{R}_+$ and $x, y \in \Omega_t^{(k-1)}$, we have*

$$d_t^{(k)}(x,y) \leqslant d_t^{(k-1)}(x,y) \leqslant C d_t^{(k)}(x,y) \tag{4.2}$$

As a result, iterating the above inequality for $k = 2, \cdots, n$, we get

$$d_t(x,y) \leqslant d_t^{(1)}(x,y) \leqslant C' d_t(x,y)$$

for a constant C' depending only on J.

Proof. — The first "$\leqslant$" in (4.2) follows immediately from the definition of the $d_t^{(k)}$'s.

We prove the second "$\leqslant$" in (4.2). Let $c : [0,1] \to \Omega_t^{(k)}$ be a piecewise geodesic joining $x, y \in \Omega_t^{(k-1)}$ such that the length of c equals $d_t^{(k)}(x,y)$.

Let $t_0, t_1, t_2, \cdots, t_r \in [0,1]$ with $t_0 = 0$ and $t_r = 1$ be such that each $c([t_{i-1}, t_i])$ lies in a single k-cell and that the $c(t_i)$'s are in $\Omega_t^{(k-1)}$. Since

c is length-minimizing, each $c[t_{i-1}, t_i]$ must be a segment, whose length equals the distance between the two end points. Thus if we could prove

$$d_t^{(k-1)}(c(t_{i-1}), c(t_i)) \leqslant C d_t^{(k)}(c(t_{i-1}), c(t_i))$$

then we can take the sum over $1 \leqslant i \leqslant r$ and use the triangle inequality to obtain

$$d_t^{(k-1)}(x, y) \leqslant C d_t^{(k)}(x, y).$$

Therefore, we can assume that both x and y lie on the boundary of a k-cell. Since each k-cell is isometric to some sub-cell of P, it is sufficient to prove that, for any k-dimensional sub-cell F of P we have

$$d_t^{(k-1)}(x, y) \leqslant C d_t^{(k)}(x, y) = C d_t(x, y).$$

for $t \in \mathbb{R}_+$ and $x, y \in F$.

If x, y both lie on the same $(k-1)$-dimensional sub-cell of F, then we have $d_t^{(k)}(x, y) = d_t^{(k-1)}(x, y)$ and there is nothing to prove. So we assume that x and y belong to $(k-1)$-dimensional sub-cells A and B, respectively. $E = A \cap B$ is a $(k-2)$-dimensional sub-cell. Let x_0, y_0 be a point in E nearest to x, y, respectively, *i.e.* $d_t(x, E) = d_t(x, x_0)$ and $d_t(y, E) = d_t(y, y_0)$.

The three segments $[x, x_0]$, $[x_0, y_0]$ and $[y_0, y]$ lie in $\Omega_t^{(k-1)}$ and form a piecewise segment joining x, y, so the definition of $d_t^{(k-1)}$ implies

$$d_t^{(k-1)}(x, y) \leqslant d_t(x, x_0) + d_t(x_0, y_0) + d_t(y_0, y). \tag{4.3}$$

By the triangle inequality, we have

$$d_t(x_0, y_0) \leqslant d_t(x_0, x) + d_t(x, y) + d_t(y, y_0). \tag{4.4}$$

(4.3) and (4.4) gives

$$\begin{aligned} d_t^{(k-1)}(x, y) &\leqslant 2(d_t(x, x_0) + d_t(y, y_0)) + d_t(x, y) \\ &= 2(d_t(x, E) + d_t(y, E)) + d_t(x, y) \end{aligned}$$

Now we apply Lemma 1.5, and conclude that

$$d_t^{(k-1)}(x, y) \leqslant (2C + 1) d_t(x, y)$$

this is the required inequality. □

Proof of Theorem 1. — Note that each vertex of the simplex P lies on different orbits of W_J, so the vertex set $\Omega_t^{(0)}$ is the union of $n + 1$ orbits. Hence, fixing any vertex v_0, we have the follow expression for the entropy $\delta_t = \delta(\Omega_t, d_t, W_J)$:

$$\delta_t = \overline{\lim_{R \to \infty}} \frac{1}{R} \log \#\{v \in \Omega_t^{(0)} | d_t(v, v_0) \leqslant R\}.$$

We shall compare δ_t with the entropy $\delta_t^{(1)} = \delta(\Omega_t^{(1)}, d_t^{(1)}, W_J)$ of the metric graph $(\Omega_t^{(1)}, d_t^{(1)})$, which is defined by

$$\delta_t^{(1)} = \varlimsup_{R\to\infty} \frac{1}{R} \log \#\{v \in \Omega_t^{(0)} | d_t^{(1)}(v, v_0) \leqslant R\}$$

The comparison of $d_t^{(1)}$ and d_t given by Lemma 4.1 implies there is a constant C' depending only on J such that

$$\delta_t^{(1)} \leqslant \delta_t \leqslant C'\delta_t^{(1)}.$$

So it is sufficient to prove

$$\delta_t^{(1)} \to 0 \quad \text{ as } t \to 0 \text{ or } t \to +\infty.$$

To this end, let $m(t)$ be the minimal length of edges of P under d_t. We have seen in Proposition 1.2 that Ω_t approaches the simplex P when $t \to 0$ or $+\infty$. Using the expression of Hilbert metric (4.1) one can see the length of each edge of P tends to $+\infty$, thus $m(t) \to +\infty$.

On the other hand, a W_J-invariant geodesic metric on the graph $\Omega_t^{(1)}$ is uniquely determined by lengths of the edges of P and is monotone with respect to each of these lengths. Therefore, if we let d' be the metric on the graph defined by setting all edge lengths to be 1, then we have $d_t^{(1)} \geqslant m(t)d'$. This allows us to compare the entropy $\delta_1^{(1)}$ defined by $d_t^{(1)}$ with the one defined by d':

$$\delta_t^{(1)} \leqslant \frac{1}{m(t)} \delta(\Omega_t^{(1)}, d', W_J).$$

But the right-hand side tends to 0 because $\delta(\Omega_t^{(1)}, d', W_J)$ is a constant. $\square$

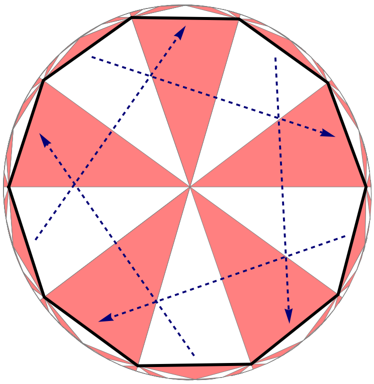
Proof of Corollary 1.6. — Let Σ be a surface with genus $\geqslant 2$. We claim that there are integers $p, q, r \geqslant 3$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ and a subgroup Π of finite index in the (p, q, r)-triangle group $\Delta = \Delta_{p,q,r}$ such that Π acts freely on the hyperbolic plan $\mathbb{H}^2$ with quotient $\mathbb{H}^2/\Pi \cong \Sigma$. Restricting the one-parameter family of representations $\rho_t : \Delta \to \mathrm{PGL}(n+1, \mathbb{R})$ given by Proposition 1.1 and 1.2 to Π, we obtain an one-parameter family of convex projective structures on Σ. We shall show that this family fulfils the requirements.

Since $\Pi \subset \Delta$ has finite index, the entropy $\delta(\Omega_t, d_t, \Pi)$ of the convex projective surface equals the entropy $\delta(\Omega_t, d_t, \Delta)$ which tends to 0 by Theorem 1.4.

Lemma 1.5 implies that for any t, every triangle inscribed in the fundamental triangle P has perimeter greater than $\frac{1}{C}$ times the perimeter of P

(measure by d_t). But the latter perimeter tends to $+\infty$ because of convergence of Ω_t to P. Thus the constant of Gromov hyperbolicity of Ω_t tends to $+\infty$.

To show that the systole tends to $+\infty$, we take a homotopically nontrivial closed curve c which is the shortest under d_t. The image of c under the orbifold covering map $\Sigma \cong \mathbb{H}^2/\Pi \to \mathbb{H}^2/\Delta \cong P$ is a closed billiard trajectory in the triangle P which hits each of the three sides. The same argument as in the previous paragraph shows that the length of c goes to $+\infty$.



Finally, we prove the claim using an explicit constructions as indicated by the above picture. The boldfaced 10-gon consists of ten fundamental domains of the triangle group $\Delta = \Delta_{5,5,5}$. We take the five elements in Δ indicated by the arrows, each of them pushing the 10-gon to an adjacent one. One checks that the group Π generated by them has the 10-gon as a fundamental domain. The quotient $\mathbb{H}^2/\Pi$ is a surface obtained by pairwise gluing edges of the 10-gon. A calculation of Euler characteristic shows $\mathbb{H}^2/\Pi$ have genus 2. Since closed surfaces of higher genus covers the surface of genus 2, by taking subgroups of Π, we conclude that all surfaces of genus $\geqslant 2$ is the quotient of $\mathbb{H}^2$ by some subgroup of Δ, and the claim is proved. $\square$

5. Proof of Lemma 1.5

To begin with, we need the following fact concerning the cellular structure of Ω_t. Looking at Figure 1.2, one observes that the 1-skeleton of Ω_t consists of straight lines. More generally, in higher dimensions, the k-skeleton

$\Omega_t^{(k)}$ is also a union of k-dimensional subspaces[(1)], or equivalently, the k-dimensional subspace L containing some k-cell must be an union of k-cells. This can be proved using the fact that the tangent space of a vertex in Ω_t has the structure of a finite Coxeter complex, and it is well known that the above assertion holds for finite Coxeter complexes (see *e.g.* [9]). We omit the details.

We first present a proof of Lemma 1.5 for the 2-dimensional case, where the main idea is most transparent.

Proof of Lemma 1.5 for $n = 2$. — We may assume

$$W_J = \langle\, \tau_1, \tau_2, \tau_3 \mid (\tau_1\tau_2)^p = (\tau_2\tau_3)^q = (\tau_3\tau_1)^r = \tau_1^2 = \tau_2^2 = \tau_3^2 = 1 \,\rangle.$$

Suppose x and y lie on the sides A and B of a triangle P in $\mathbb{P}^2$, respectively. Denote the common vertex of A and B by E. We need to prove that

$$C d_t(x, y) \geqslant d_t(x, E) + d_t(y, E), \quad \forall t$$

Put $s_1 = \rho_t(\tau_1)$ and $s_2 = \rho_t(\tau_2)$. So s_1 and s_2 are reflections with respect to A and B, respectively, whereas $s_1 s_2$ is a rotation of order p.

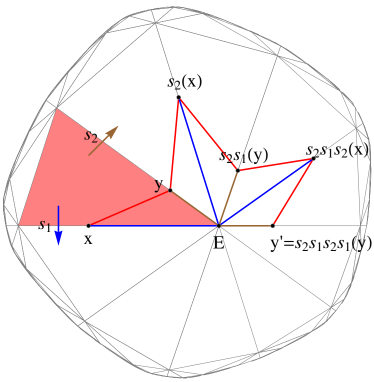


Figure 5.1. $p = 5$

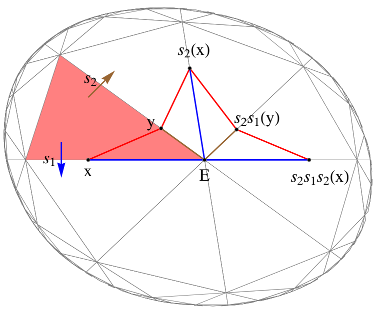


Figure 5.2. $p = 4$

If p is odd, we put

$$y' = \underbrace{s_2 s_1 s_2 \cdots s_1}_{p-1 \text{ reflections}} (y).$$

(1) By a "subspace" of Ω_t, we mean the intersection of a projective subspace of $\mathbb{P}^n$ with Ω_t.

Then y' lies on the opposite half ray of the geodesic ray $\vec{Ex}$ (see Figure 5.1). On the other hand, the successive images of $[x, y]$ by the sequence of projective transforms

$$s_2,\ s_2 s_1,\ s_2 s_1 s_2,\ \cdots,\ \underbrace{s_2 s_1 s_2 \cdots s_1}_{p-1}$$

consititue a piecewise geodesic γ joining x and y'. γ consists of p pieces, each one with the same length $d_t(x, y)$. Thus we have

$$p\, d_t(x, y) \geqslant d_t(x, y') \geqslant d_t(x, E).$$

When p is even, we obtain the above inequality with $x' = s_2 s_1 \cdots s_2(x)$ replacing y' in the same way (see Figure 5.2).

Interchanging the roles of x and y, we get

$$p\, d_t(x, y) \geqslant d_t(y, E)$$

and conclude that

$$2p\, d_t(x, y) \geqslant d_t(x, E) + d_t(y, E).$$

□

In order to tackle higher-dimensional cases, we first introduce a terminology. Let E be a $(k-1)$-cell of Ω_t. Two k-cells are said to be *E-colinear* if they lie on the same k-dimensional subspace and their intersection is E. As explained at the beginning of this section, the k-dimensional subspace of Ω_t containing a k-cell A is a union of k-cells, so for any $(k-1)$-sub-cell E of A, there is a unique k-cell in Ω_t which is E-colinear to A.

The crucial point of the above proof in dimension 2 is the following fact: let V be the k-cell E-colinear to A. Then we can connect $x \in A$ and some point in V by a curve piecewise isometric to the segment $[x, y]$, where the number of pieces is bounded by a combinatorial constant.

In higher dimensions, if this fact still holds, we would immediately get the required inequality as in the two-dimensional case, because the distance from x to any point of V is greater than the distance from x to E.

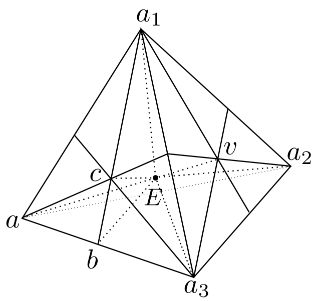
However, the situation is more delicate: the cell V which is E-colinear to A may not be a translate of A or B and this prevents us from constructing a curve going from x to V which is piecewise isometric to $[x, y]$.

In this situation, our alternative strategy is to take a cell $A' = \rho_t(\gamma)A$, the translate of A by some $\gamma \in W_J$, such that A' and V are contained in the same top-dimensional cell. Now we can go from x to A' along a curve piecewise isometric to $[x, y]$. To prove Lemma 1.5, we then need to show that the distance from x to A' is greater than the distance from x to E.

We will develop some lemmas concerning distance comparison in Hilbert geometry in order to carry out this strategy. But before going into that, the reader might find it useful to keep in mind the following typical example of the above situation. Take $n = 3$ and $k = 1$, so that E is a vertex while A and V are edges. Assume that the sub-diagram of J corresponding to E [(2)] is

○———○———○

where the first two nodes form the sub-diagram corresponding to A. The 3-cells containing E then form the configuration of the barycentric subdivision of a tetrahedron, as partly shown in the picture below.



Here P is the tetrahedron $abcE$, while A and V are the segments $[a, E]$ and $[v, E]$, respectively. We can take A' to be either $[a_1, E]$, $[a_2, E]$ or $[a_3, E]$.

We now discuss distance comparison in Hilbert geometry. Using the definition of Hilbert metrics Eq.(4.1), it is easy to check that if a properly convex set $\Omega \subset \mathbb{P}^n$ is strictly convex (*c.f.* the last paragraph of Section 2), then the Hilbert metric d_Ω has the following property. Let $L \subset \Omega$ be a subspace and $x \in \Omega \setminus L$. Among all points of L, there is a unique $x_0 \in L$ whose distance to x is minimal. We call x_0 the *projection* of x on L, and denote it by $x_0 = \mathrm{Pr}(x, L)$.

Let $L \subset \Omega$ be a hyperplane, *i.e.* a subspace of codimension 1. We say that Ω *have reflectional symmetry s with respect to L* if $s \in \mathrm{PGL}(n+1, \mathbb{R})$ is a reflection preserving Ω and fixing each point of L. In this case, the triangle inequality and the fact that geodesics with respect to d_Ω are straight lines yield a simple characterization of projections,

$$\mathrm{Pr}(x, L) = [x, s(x)] \cap L. \tag{5.1}$$

[(2)] Given a sub-cell E of the simplex P, the sub-diagram of J corresponding to E is the Coxeter diagram consisting of nodes i satisfying $E \subset P_i$ and all the weighted edges joining these nodes.

LEMMA 5.1. — *Let $\Omega \subset \mathbb{P}^n$ be a properly strictly convex open set with reflectional symmetry s with respect to a hyperplane L. Then for any $x, y \in \Omega$, we have*

$$d_\Omega(\mathrm{Pr}(x, L), \mathrm{Pr}(y, L)) \leqslant d_\Omega(x, y).$$

In particular, if $x \in L$, then for any $y \in \Omega$ we have

$$d_\Omega(x, \mathrm{Pr}(y, L)) \leqslant d_\Omega(x, y).$$

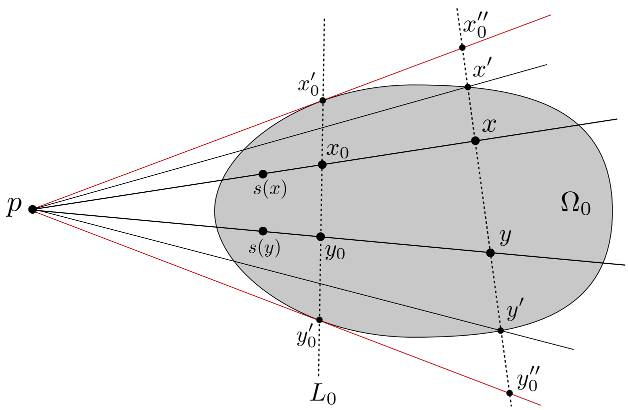


Figure 5.3. $d_\Omega(x_0, y_0) \leqslant d_\Omega(x, y)$

Proof. — Denote $x_0 = \mathrm{Pr}(x, L)$ and $y_0 = \mathrm{Pr}(y, L)$. The reflection s has another fixed point $p \in \mathbb{P}^n$ outside L. We have $x_0 = \overline{xp} \cap L$ and $y_0 = \overline{yp} \cap L$. Therefore the four points x, y, x_0, y_0 lie on the plane $\overline{pxy}$, which is preserved by s. So we are reduced to the 2-dimensional case by restricting the consideration to $\Omega_0 = \overline{pxy} \cap \Omega$ and $L_0 = \overline{pxy} \cap L$.

Suppose that L_0 intersects $\partial\Omega_0$ at x_0' and y_0', see Figure 5.3. Since Ω_0 has reflectional symmetry with respect to L_0, the lines $\overline{px_0'}$ and $\overline{py_0'}$ are tangent to Ω. Let x_0'' (resp. y_0'') be the intersection of $\overline{px_0'}$ (resp. $\overline{py_0'}$) with $\overline{xy}$. It is a basis fact from projective geometry that we have an equality of cross-ratios

$$[x_0', x_0, y_0, y_0'] = [x_0'', x, y, y_0''].$$

Since x' and y' lie strictly inside the segment $[x_0'', y_0'']$, we have

$$[x_0', x_0, y_0, y_0'] = [x_0'', x, y, y_0''] \leqslant [x', x, y, y'].$$

It follows that $d_\Omega(x_0, y_0) \leqslant d_\Omega(x, y)$. $\square$

The next lemma, which strengthens the above one in the case where there are several reflectional symmetries, is the main nontrivial ingredient that we need to add to above two-dimensional proof of Lemma 1.5 in order to generalize it to higher dimensions.

Lemma 5.2. — *Let $\Omega \subset \mathbb{P}^n$ be a properly strictly convex open set with reflectional symmetries $s_1, \cdots, s_m$ $(m \leqslant n-1)$ with respect to hyperplanes $L_1, L_2, \cdots, L_m$, such that the s_i's generate a finite group $\Gamma \subset \mathrm{PGL}(n+1, \mathbb{R})$. Assume that $W = L_1 \cap \cdots \cap L_m$ has dimension $n-m$ (*i.e. *the L_i's are in general position) and $W \cap \Omega \neq \emptyset$. Let D be a Γ-invariant convex subset of Ω. Then for any $x \in W \cap \Omega$ and any $x' \in D$, there is some point x_0 in $W \cap D$ such that*

$$d_\Omega(x, x_0) \leqslant d_\Omega(x, x')$$

Proof. — Fix $x \in W \cap \Omega$ and $x' \in W \cap D$. We can choose an affine chart $\mathcal{A} \subset \mathbb{P}^n$, an origin $x_0 \in \mathcal{A}$ (so as to consider $\mathcal{A}$ as a vector space) and a Euclidean scalar product on $\mathcal{A}$ such that

(1) $\mathcal{A}$ contains the closure of Ω;
(2) $L_1, \cdots, L_m$ are linear subspaces of $\mathcal{A}$, or equivalently, $x_0 \in W$;
(3) Γ preserves the Euclidean scalar product;
(4) $x' \in W^\perp$, where $W^\perp$ is the orthogonal complement of W.

Our aim is to show that the origin x_0 is contained in D and satisfies the required inequality. Let us denote x_0 simply by 0.

We will mainly work on the subspace $W^\perp$ of $\mathcal{A}$. Each $L_i' = L_i \cap W^\perp$ is a subspace of $W^\perp$ of codimension 1, and the intersection $L_1' \cap \cdots \cap L_m'$ is trivial. Since $D \cap W^\perp$ is invariant by Γ and is convex, the barycenter of the Γ-orbit of x lies in $D \cap W^\perp$ and is fixed by Γ. But $L_1' \cap \cdots \cap L_m' = \{0\}$ implies that the only fixed point of Γ in $W^\perp$ is 0, thus $0 \in D$ as required.

We proceed to prove $d_\Omega(x, 0) \leqslant d_\Omega(x, x')$. Put

$$C_i(\theta) = \{y \in W^\perp | \angle(y, L_i') \geqslant \theta \text{ or } y = 0\}$$

where $\angle(y, L_i')$ is the Euclidean angle between the 1-dimensional subspace spanned by y and the subspace L_i'. Namely, $C_i(\theta)$ consists of points in $W^\perp$ which are away from L_i' by an angle θ. We take θ small enough so that

$$C_1(\theta) \cup \cdots \cup C_m(\theta) = W^\perp. \tag{5.2}$$

Condition (3) above implies that the restriction of the reflection s_i to $W^\perp$ is just the Euclidean reflection with respect to L_i'. So it follows from Eq.(5.1) that for any $y \in W^\perp$, the projection $\mathrm{Pr}(y, L_i)$ coincides with the

Euclidean projection (within $W^\perp$) of y on L_i'. Therefore, by definition of $C_i(\theta)$, any $y \in C_i(\theta)$ verifies

$$|\operatorname{Pr}(y, L_i)| \leqslant |y| \cos\theta \tag{5.3}$$

We construct a sequence of points y_0, y_1, y_2, $\cdots \in W^\perp \cap D$ converging to 0 as follows. Put $x' = y_0$. Assume by recurrence that we already have y_k. By Eq.(5.2), there is some $i_k \in \{1, \cdots, m\}$ such that $y_k \in C_{i_k}$. We then put

$$y_{k+1} = \operatorname{Pr}(y_k, L_{i_k}).$$

The inequality (5.3) yields

$$|y_k| \leqslant |y_{k-1}| \cos\theta \leqslant \cdots \leqslant |y_0| (\cos\theta)^k.$$

Hence $\lim_{k\to\infty} y_k = 0$.

On the other hand, Lemma 5.1 yields

$$d_\Omega(x, y_k) \leqslant d_\Omega(x, y_{k-1}) \leqslant \cdots \leqslant d_\Omega(x, y_0) = d_\Omega(x, x').$$

Therefore, by continuity of the metric d_Ω, we conclude that

$$d_\Omega(x, 0) = \lim_{k\to\infty} d_\Omega(x, y_k) \leqslant d_\Omega(x, x').$$

□

Return to the specific convex set Ω_t. For any cell V of Ω_t, we let $\operatorname{St}(V)$ denote the union of all closed n-cells containing V (“St" for “star-like" or “saturated").

LEMMA 5.3. — $\operatorname{St}(V)$ *is a convex subset of* Ω_t.

Proof. — Let F be a $(n-1)$-cell on the boundary of $\operatorname{St}(V)$ and let L be the hyperplane containing F. L does not contain V, so V is contained in one of the two “half-spaces" in Ω_t bounded by L. Using the fact that L is an union of $(n-1)$-cells, we conclude the whole $\operatorname{St}(V)$ lie in the same half-space as V. Therefore, $\operatorname{St}(V)$ is an intersection of half-spaces, hence is convex. □

With the above preparations, we can finally give the proof of Lemma 1.5 in arbitrary dimensions.

Proof of Lemma 1.5. — Fix a hyperbolic Coxeter diagram J. The Coxeter group is

$$W_J = \langle\, \tau_0, \cdots, \tau_n \mid (\tau_i\tau_j)^{m_{ij}} = \tau_i^2 = 1, \forall i \neq j \,\rangle.$$

By definition of hyperbolic Coxeter diagrams, for any $I \subsetneqq \{0, \cdots, n\}$, the subgroup W_I generated by $\{\tau_i\}_{i\in I}$ is finite. The *word-length* of $\sigma \in W_I$, denoted by $l(\sigma)$, is defined as the minimum of the integer k such that

$\sigma = \tau_{i_1} \cdots \tau_{i_k}$ for some $i_1, \cdots, i_k \in I$. We define the *word-length-diameter* of W_I as

$$\mathrm{diam}(W_I) = \max_{\sigma \in W_I} l(\sigma)$$

and put

$$C = \max_{I \subsetneq J} \mathrm{diam}(W_I)\,.$$

We only need to show that

$$C\, d_t(x,y) \geqslant d_t(x,E)$$

because if this is true, exchanging the roles of x and y gives $Cd_t(x,y) \geqslant d_t(y,E)$ and the required inequality follows.

Let V be the k-cell which is E-colinear to A. Let $\gamma \in \mathrm{Stab}_{W_J}(E)$ be such that $P' = \rho_t(\gamma)P$ contains V. We denote $A' = \rho_t(\gamma)A$ and $x' = \rho_t(\gamma)x$.

We first show that there is a curve joining x and x' which is piecewise isometric to $[x,y]$ with at most C pieces.

Denote $s_i = \rho_t(\tau_i)$. Let $J_E \subset \{0, 1, \cdots, n\}$ be the set of indices of those P_i such that $E \subset P_i$. Then J_E has $n-k+1$ elements and $\mathrm{Stab}_{W_J}(E)$ is generated by $\{s_i\}_{i \in J_E}$.

We can write $\gamma = \tau_{i_1} \cdots \tau_{i_m}$, with $i_1, \cdots, i_m \in J_E$ and $m \leqslant \mathrm{diam}(W_{J_E}) \leqslant C$. Then $\rho_t(\gamma) = s_{i_1}s_{i_2} \cdots s_{i_m}$. Consider the sequence of segments

$$s_{i_1}([x,y]),\ s_{i_1}s_{i_2}([x,y]),\ \cdots,\ s_{i_1}s_{i_2}\cdots s_{i_m}([x,y]).$$

The k-cell A contains the $(k-1)$-cell E, so A has only one vertex a lying outside E. Similarly B has only one vertex b outside E. Each face P_i of P must contain at least one of the two points a and b. Hence a face containing E also contains either A or B. It follows that if $i \in J_E$ then s_i fixes either x or y. Therefore, each segment in the above sequence shares at least one end point with the next one. So the union of these segments is connected, and we can extract a subset of these segments to form a curve joining x and x' which is piecewise isometric to $[x,y]$. The number of pieces is bounded by C.

Next, using the triangle inequality, we conclude that

$$Cd_t(x,y) \geqslant d_t(x,x').$$

So it remains to be shown that

$$d_t(x,x') \geqslant d_t(x,E). \tag{5.4}$$

To this end, we use Lemma 5.2. Let $\mathrm{St}(V)$ be the convex set D in Lemma 5.2, which contains x' (in the above example $\mathrm{St}(V)$ is the tetrahedron $a_1a_2a_3E$). Let $J_A \subset \{0, 1, \cdots, n\}$ be the set of indices of those faces P_i which contain A and let L_i be the hyperplane containing P_i. $W = \cap_{i \in J_A} L_i$

is the k-dimensional subspace containing A and V (in the above example W is the line passing through a, E and v). For each $i \in J_A$, the reflection s_i preserves $\mathrm{St}(V)$ since $V \subset L_i$. Thus the hypothesis of Lemma 5.2 is verified and we conclude that there is $x_0 \in V = \mathrm{St}(V) \cap W$ such that

$$d_t(x, x') \geqslant d_t(x, x_0).$$

The union $A \cup V$ is convex because it is the intersection of $\mathrm{St}(E)$ and a k-dimensional subspace. So $[x, x_0]$ intersects E at some point x_1 and we have

$$d_t(x, x_0) \geqslant d_t(x, x_1) \geqslant d_t(x, E)$$

Hence we have obtained (5.4), and the proof is complete. □

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