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ON THE GRIFFITHS NUMBERS FOR HIGHER DIMENSIONAL SINGULARITIES

by Rong DU & Yun GAO (*)

ABSTRACT. — We show that Yau's conjecture on the inequalities for $(n-1)$-th Griffiths number and $(n-1)$-th Hironaka number does not hold for isolated rigid Gorenstein singularities of dimension greater than 2. But his conjecture on the inequality for $(n-1)$-th Griffiths number is true for irregular singularities.

RÉSUMÉ. — Nous montrons que la conjecture de Yau sur les inégalités concernant le $(n-1)$-ième nombre de Griffiths et le $(n-1)$-ième nombre de Hironaka n'est pas vraie en général pour les singularités de Gorenstein isolées rigides de dimension supérieure à 2. Cependant, la première conjecture sur les inégalités concernant le $(n-1)$-ième nombre de Griffiths est vraie pour les singularités irrégulières.

1. Introduction

In singularity theory, one always wants to find invariants associated to singularities. Let (V, o) be a Stein analytic space with o as its only singularity of dimension $n \geqslant 2$. In [8], Yau introduced a bunch of invariants which are naturally attached to isolated singularities. These invariants are used to characterize the different notions of sheaves of germs of holomorphic differential forms on analytic spaces. Various formulas which relate to all these invariants were proved in [8]. Among these invariants the Griffiths number $g^{(p)}$, the Hironaka number $h^{(p)}$ and $\delta^{(p)}$ are the most interesting invariants. In 1981, Yau conjectured that the following two inequalities of

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these invariants should be true for general isolated normal singularities in [7].

Conjecture 1.1. — *Let (V,o) be a Stein analytic space with o as its only normal singularity of dimension $n \geqslant 2$. Then*

(i) $g^{(n-1)} \geqslant n-1$;
(ii) $\delta^{(n-1)} \geqslant h^{(n-1)} + n - 1$.

In ([7]), he confirmed his conjecture for surface singularities and non-rational singularities with "good" $\mathbb{C}^*$-action of dimension greater than 2. As an application, he showed that any Gorenstein surface singularities with "good" $\mathbb{C}^*$-action are not rigid. Although this conjecture inspired researches in singularity theory for long time, we will show that this conjecture is not true in general. But the first part of his conjecture holds for irregular singularities.

2. Preliminaries

Let (V,o) be a normal isolated singularity of dimension $n \geqslant 2$. It is well known that holomorphic functions defined on $V - \{o\}$ can be extended across o. However for holomorphic forms, the situation is completely different. Even if we assume that the holomorphic forms defined on $V - \{o\}$ are L^2-integrable in a neighborhood of o in the sense of Griffiths ([2]), it is not clear whether holomorphic forms can be extended across o. In [8], the Griffiths number $g^{(p)}$ was introduced to measure how many L^2-integrable holomorphic p-forms on $V - \{o\}$ cannot be extended across o. Similarly, Yau defined another class of invariants $\delta^{(p)}$ which measures how many holomorphic p-forms on $V - \{o\}$ cannot be extended across o in [7].

In [8], Yau studied the relations among all kinds of sheaves of germs of holomorphic forms which were also considered by Grauert-Grothendieck, Noether, Ferrari and Siu.

(1) Noether: $\bar{\Omega}_V^p := \pi_*\Omega_M^p$, where $\pi : M \longrightarrow V$ is a resolution of singularities of V.
(2) Grauert-Grothendieck: $\Omega_V^p := \Omega_{\mathbb{C}^N}^p/\mathscr{K}^p$, where $\mathscr{K}^p = \{f\alpha + dg\wedge\beta : \alpha \in \Omega_{\mathbb{C}^N}^p; \beta \in \Omega_{\mathbb{C}^N}^{p-1}; f, g \in \mathscr{I}\}$ and $\mathscr{I}$ is the ideal sheaf of V in $\mathbb{C}^N$.
(3) Ferrari: $\widetilde{\Omega}_V^p := \Omega_{\mathbb{C}^N}^p/\mathscr{H}^p$, where $\mathscr{H}^p = \{\omega \in \Omega_{\mathbb{C}^N}^p : \omega|_{V\backslash V_{\text{sing}}} = 0\}$.
(4) Siu: $\bar{\bar{\Omega}}_V^p := \theta_*\Omega_{V\backslash V_{\text{sing}}}^p$ where $\theta : V\backslash V_{\text{sing}} \longrightarrow V$ is the inclusion map and V_{sing} is the singular set of V.

If V is a normal variety, then dualizing sheaf ω_V of Grothendieck is actually the sheaf $\bar{\bar{\Omega}}^n_V$. Clearly Ω^p_V, $\tilde{\Omega}^p_V$ are both coherent. $\bar{\Omega}^p_V$ is a coherent sheaf because π is a proper map. $\bar{\bar{\Omega}}^p_V$ is also a coherent sheaf by the theorem of Siu (see Theorem A of [5]).

Lemma 2.1. — *There are several short exact sequences:*

$$0 \longrightarrow K^p \longrightarrow \Omega^p_V \longrightarrow \tilde{\Omega}^p_V \longrightarrow 0, \tag{2.1}$$

$$0 \longrightarrow \tilde{\Omega}^p_V \longrightarrow \bar{\Omega}^p_V \longrightarrow H^p \longrightarrow 0, \tag{2.2}$$

and

$$0 \longrightarrow \bar{\Omega}^p_V \longrightarrow \bar{\bar{\Omega}}^p_V \longrightarrow J^p \longrightarrow 0, \tag{2.3}$$

where K^p, H^p and J^p are all supported on the singular set.

Definition 2.2. — *Let (V, o) be a Stein analytic space with o as its only singularity. Let K^p, H^p and J^p be defined as in (2.1), (2.2), and (2.3). Then the invariants $m^{(p)}$, $g^{(p)}$ and $s^{(p)}$ at o are defined to be $\dim K^p_o$, $\dim H^p_o$ and $\dim J^p_o$ respectively. $\delta^{(p)}$ is defined to be the dimension of the co-kernel of the natural map $\Omega^P_V \to \bar{\bar{\Omega}}^p_V$. We also define the geometric genus of the singularity p_g to be $s^{(n)}$ and irregularity of the singularity q to be $s^{(n-1)}$.*

The following lemma can be found as Lemma 2.7 in [8]. We will provide a short proof here.

Lemma 2.3. — *Let (V, o) be a Stein analytic space with o as its only isolated singularity. Let $\pi : M \to V$ be a resolution of the singularity. Then*

$$g^{(p)} = \dim \Gamma(M, \Omega^p_M)/\pi^*\Gamma(V, \Omega^p_V),$$

and

$$\delta^{(p)} = \dim \Gamma(V - \{o\}, \Omega^p_V)/\Gamma(V, \Omega^p_V),$$

where we identify $\Gamma(V, \Omega^p_V)$ with its image in $\Gamma(V - \{o\}, \Omega^p_V)$.

Proof. — Because K^p is supported on the isolated singularity o and $\tilde{\Omega}^p_V$ is coherent on the Stein space V, they both have vanishing cohomology groups of degree greater than 0. So, from (2.1) and (2.2), we can have an exact sequence as follows

$$0 \longrightarrow K^p_o \longrightarrow \Gamma(V, \Omega^p_V) \longrightarrow \Gamma(V, \bar{\Omega}^p_V) \longrightarrow H^p_o \longrightarrow 0.$$

Therefore we get the first equality by identifying $\Gamma(V, \bar{\Omega}^p_V)$ with $\Gamma(M, \Omega^p_M)$. The second equality is obvious by the definition. □

3. Inequalities for invariants of singularities

We will show that Yau's conjecture mentioned in the first section is not true in general.

Theorem 3.1. — *Let (V, o) be an isolated rigid Gorenstein singularity of dimension $n \geqslant 3$. Then (V, o) does not satisfy the above two inequalities.*

Proof. — From the exact sequences in Lemma 2.1 one immediately concludes the inclusions of finite dimensional vector spaces

$$\tilde{\Omega}_V^p \subset \bar{\Omega}_V^p \subset \bar{\bar{\Omega}}_V^p$$

from which the equality $\delta^{(p)} = g^{(p)} + s^{(p)}$ immediately follows (see Lemma 1.3 in [7]).

A rigid Gorenstein singularity has $\delta^{(n-1)} = 0$ (*cf.* Theorem 4.6 in [7]) from which it follows that $g^{(n-1)} = 0$. Hence a rigid Gorenstein singularity provides a counterexample to the conjectures that $g^{(n-1)} \geqslant n - 1$ and $\delta^{(n-1)} \geqslant p_g + n - 1$. □

Remark 3.2. — The isolated rigid Gorenstein singularities of dimension $n \geqslant 3$ do exist. For example, the existence of an isolated Gorenstein finite quotient singularity of dimension $n \geqslant 3$ is given by [3], [9] or [1]. From Theorem 3 in [4], such singularity is rigid.

Next, we are going to show that the inequality (3.4) holds for irregular singularities.

Definition 3.3. — *A normal isolated singularity is called regular if the irregularity $q = 0$. Otherwise, it is called irregular.*

Theorem 3.4. — *Let (V, o) be a normal isolated irregular singularity of dimension $n \geqslant 2$. Then*

$$g^{(n-1)} \geqslant n - 1.$$

Proof. — We will improve the methods used in Yau's paper ([7] Theorem 3.1) to estimate the lower bound of $g^{(n-1)}$. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with $A = \cup_{i=1}^s A_i$ as the exceptional set, where each A_i is an irreducible component. We can assume without loss of generality that the exceptional set A is a divisor in M with normal crossings. Embed (V, o) into $(\mathbb{C}^m, 0)$ whose coordinate functions are supposed to be $z_1, \cdots, z_m$. Since the singularity is irregular, we know that

$$q = \dim \Gamma(M \backslash A, \Omega_M^{n-1}) / \Gamma(M, \Omega_M^{n-1}) > 0.$$

So there exists a holomorphic $(n-1)$-form ω_0 on $M\backslash A$ that can not be extended across A, *i.e.*, $\omega_0 \in \Gamma(M\backslash A, \Omega_M^{n-1})\backslash\Gamma(M, \Omega_M^{n-1})$, which means that ω_0 must have poles along some irreducible component A_1 of A. Suppose ω_1 has the lowest order of poles along A_1 among all $\omega \in \Gamma(M\backslash A, \Omega_M^{n-1})\backslash\Gamma(M, \Omega_M^{n-1})$. Then $\pi^*(z_j)\omega_1$ is holomorphic along A_1 for all j, $1 \leqslant j \leqslant m$. Otherwise, it contracts to the assumption. If $\pi^*(z_j)\omega_1 \notin \Gamma(M, \Omega_M^{n-1})$ for some j, it must have poles along another irreducible component A_2 of A. Suppose $\omega_2 \in \Gamma(M\backslash A, \Omega_M^{n-1})$ is holomorphic along A_1 and has the lowest order of poles along A_2. Then $\pi^*(z_j)\omega_2$ must be holomorphic along A_1 and A_2, for all j, $1 \leqslant j \leqslant m$. Similarly, we can continue such kind of steps to generate ω_3, ω_4, $\cdots$ if it is available. Since the number of irreducible components of A is finite, by induction, there exists a non-empty set $W_k = \{\omega \in \Gamma(M\backslash A, \Omega_M^{n-1})\backslash\Gamma(M, \Omega_M^{n-1}) : \omega$ has poles along some irreducible component A_k and holomorphic along $A_1, \ldots, A_{k-1}$ such that $\pi^*(z_j)\omega \in \Gamma(M, \Omega_M^{n-1})$ for all j, $1 \leqslant j \leqslant m\}$.

Suppose $\omega_k \in W_k$. We will separate our argument into two parts according to the order of poles of ω_k.

- The order of poles of ω_k is greater than 1 along some exceptional component A_k:

  Choose a point b in A_k which is a smooth point of A. Let $(x_1, x_2, \cdots, x_n)$ be a coordinate system centered at b such that A_k is given locally by $x_1 = 0$ at b. Take the power series expansion of $\pi^*(z_j)$ around b:

$$\pi^*(z_j) \mathrel{\overset{\circ}{=}} x_1^{r_j} f_j, 1 \leqslant j \leqslant m, \tag{3.1}$$

  where f_j is a holomorphic function such that $f_j(0, x_2, \cdots, x_n) \neq 0$ and "$\overset{\circ}{=}$" means local equality around b. Without loss of generality, we may assume $r_1 = min\{r_1, \ldots, r_m\}$. It is easy to see that the holomorphic $(n-1)$-form

$$d\pi^*(z_{i_1}) \wedge d\pi^*(z_{i_2}) \wedge \cdots \wedge d\pi^*(z_{i_{n-1}}) \tag{3.2}$$

  has vanishing order at least $(n-1)r_1 - 1$ along A_k. So the vanishing orders of the $(n-1)$-form $(\pi^*(z_1))^j\omega_k$, $1 \leqslant j \leqslant n-1$, along A_k are at most $(n-1)r_1 - 2$. These $(n-1)$-forms cannot be the linear combinations of (3.2). Therefore we have produced at least $n-1$ holomorphic $(n-1)$-forms on M which are not obtained by pulling back of holomorphic $(n-1)$-forms on V.
- The maximal order of poles of ω_k is equal to 1:

  Because

$$H^0(M, \Omega_M^{n-1}) = H^0(M, \Omega_M^{n-1}(\log A))$$

from Theorem 1.3 in [6], there exists an exceptional component A_k such that the local expression of ω_k around some smooth point b' on A contains a summand of the form

$$x_1^{-1} g dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n,$$

where g is a holomorphic function such that $g(0, x_2, \cdots, x_n) \neq 0$ and $(x_1, x_2, \cdots, x_n)$ is a coordinate system centered at b' such that A_k is given locally by $x_1 = 0$ at b'. Similarly, if we express $\pi^*(z_j)$, where $1 \leqslant j \leqslant m$, locally as (3.1) and still assume $r_1 = min\{r_1, \ldots, r_m\}$, then the coefficient before $dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n$ in the expression of (3.2) has vanishing order of x_1 at least $(n-1)r_1$. However the coefficient before $dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n$ in the expression of the $(n-1)$-form $(\pi^*(z_1))^j \omega_k$, $1 \leqslant j \leqslant n-1$, has vanishing order of x_1 at most $(n-1)r_1 - 1$. These $(n-1)$-forms cannot be the linear combinations of (3.2). Therefore we also have produced at least $n-1$ holomorphic $(n-1)$-forms on M which are not obtained by pulling back of holomorphic $(n-1)$-forms on V.

□

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