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LOCAL-GLOBAL PRINCIPLE FOR QUADRATIC FORMS OVER FRACTION FIELDS OF TWO-DIMENSIONAL HENSELIAN DOMAINS

by Yong HU

Abstract. — Let R be a 2-dimensional normal excellent henselian local domain in which 2 is invertible and let L and k be its fraction field and residue field respectively. Let Ω_R be the set of rank 1 discrete valuations of L corresponding to codimension 1 points of regular proper models of Spec R. We prove that a quadratic form q over L satisfies the local-global principle with respect to Ω_R in the following two cases: (1) q has rank 3 or 4; (2) q has rank $\geqslant 5$ and $R = A[[y]]$, where A is a complete discrete valuation ring with a not too restrictive condition on the residue field k, which is satisfied when k is C_1.

Résumé. — Soit R un anneau local intègre de dimension 2, normal, excellent et hensélien dans lequel 2 est inversible. Soient L son corps de fractions et k son corps résiduel. Soit Ω_R l'ensemble des valuations discrètes de rang 1 de L correspondant aux points de codimension 1 des modèles propres réguliers de Spec R. On démontre qu'une forme quadratique q sur L satisfait le principe local-global par rapport à Ω_R dans les deux cas suivants : (1) q est de rang 3 ou 4 ; (2) q est de rang $\geqslant 5$ et $R = A[[y]]$, où A est un anneau de valuation discrète complet, avec une condition sur le corps résiduel k qui est satisfaite lorsque k est C_1.

1. Statements of results

Let R be a 2-dimensional excellent henselian local domain and let L and k be respectively its fraction field and residue field. Assume that the characteristic of k is not 2.

Colliot-Thélène, Ojanguren and Parimala [2] proved that any quadratic form of rank at least 5 over L is isotropic when k is separably closed, and that the local-global principle with respect to all discrete valuations (of

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rank 1) on L holds for quadratic forms of rank 3 or 4 when k is separably closed or finite. For the first result, the special case where $R = \mathbb{C}[[x, y]]$ was proven earlier in [1] using the Weierstraß preparation theorem. On the other hand, Jaworski [6] proved that if k is an algebraically closed field, then quadratic forms of any rank over $L = k((x, y))$ satisfy the local-global principle with respect to all discrete valuations on L.

In the case where k is finite, however, whether the local-global principle holds for quadratic forms of rank $\geqslant 5$ is left an open question. In this paper, we give an affirmative answer to this question in the case where $R = A[[y]]$ is the ring of formal power series in one variable over a complete discrete valuation ring A. Also, we prove that the result of Colliot-Thélène, Ojanguren and Parimala about the local-global principle for quadratic forms of rank 3 or 4 is still valid without the assumption that k is separably closed or finite.

The more precise statements are the following.

Theorem 1.1. — *Let R be a 2-dimensional normal excellent henselian local domain in which 2 is invertible. Let L and k be respectively the fraction field and the residue field of R. For any regular integral scheme $\mathcal{M}$ equipped with a proper birational morphism $\mathcal{M} \to \operatorname{Spec} R$, let $\Omega_{\mathcal{M}}$ denote the set of rank 1 discrete valuations of L that correspond to codimension 1 points of $\mathcal{M}$. Let Ω_R be the union of all $\Omega_{\mathcal{M}}$.*

Then the local-global principle with respect to Ω_R holds for quadratic forms of rank 3 or 4 over L. Namely, if a quadratic form of rank 3 or 4 over L has a nontrivial zero over the w-adic completion L_w for every $w \in \Omega_R$, then it has a nontrivial zero over L.

Theorem 1.2. — *Let A be a complete discrete valuation ring in which 2 is invertible, and let K and k be respectively its fraction field and residue field. Let $R = A[[y]]$ and $L = \operatorname{Frac}(R)$ the fraction field of R. Define Ω_R as in Theorem 1.1.*

Assume that the residue field k has the following property:

- (∗) *for every finite field extension k'/k, every quadratic form of rank $\geqslant 3$ over k' is isotropic.*

Then the local-global principle with respect to discrete valuations in Ω_R holds for quadratic forms of rank $\geqslant 5$ over L.

Recall that a field k is called a C_i field if every homogeneous polynomial of degree d in $n > d^i$ variables has a nontrivial zero over k. A finite field extension of a C_i field is again a C_i field. Clearly, a C_1 field k has property (∗). So as typical examples to which Theorem 1.2 applies, we may take

$R = \mathbb{F}[[x, y]]$ where $\mathbb{F}$ is a finite field of characteristic > 2, or $R = \mathcal{O}_K[[y]]$ where $\mathcal{O}_K$ is the ring of integers of a p-adic number field K (p is an odd prime).

Remark 1.3. — Note that property $(*)$ implies the following:

$(**)$ for every finite field extension K'/K, every quadratic form of rank $\geqslant 5$ over K' is isotropic.

Indeed, the integral closure A' of A in K' is a complete discrete valuation ring and is finite over A (*cf.*, [12, p. 28, § II.2, Prop. 3]). The residue field k' of A' is a finite extension of k. Any quadratic form q over K' is isometric to a form $q_1 \perp t.q_2$, where t is a uniformizer of A' and the coefficients of q_1, q_2 are all units in A'. When q has rank $\geqslant 5$ and k has property $(*)$, a standard argument using Springer's lemma (*cf.*, Lemma 4.1) shows that q is isotropic over K'.

Let A, k, K and so on be as in Theorem 1.2. Let $x \in A$ be a uniformizer of A and $F = K(y)$ the function field of $\mathbb{P}^1_K$. For any regular integral scheme $\mathcal{P}$ equipped with a proper flat morphism $\mathcal{P} \to \operatorname{Spec} A$ with generic fiber $\mathcal{P} \times_A K \cong \mathbb{P}^1_K$, let $\Omega_{\mathcal{P}}$ denote the set of rank 1 discrete valuations of F that correspond to codimension 1 points of $\mathcal{P}$. Let Ω_A be the union of all $\Omega_{\mathcal{P}}$. Then we have the following proposition.

Proposition 1.4. — *With notation as above, let $q/F = \langle a_1, \dots, a_r\rangle$ be a nonsingular diagonal quadratic form of rank $r \geqslant 5$ with $a_i \in A[y]$. Let $\Sigma \subseteq A$ be a fixed set of representatives of k^* in A. Assume that*

$$a_i = \lambda_i . x^{n_i} . P_i, \tag{1.1}$$

where $\lambda_i \in \Sigma, n_i \in \{0, 1\}$ and P_i is a distinguished polynomial of degree m_i in $A[y]$ (meaning that P_i is a monic polynomial in $A[y]$ whose reduction mod x is $y^{m_i} \in k[y]$).

If for every $w \in \Omega_R$, q is isotropic over the completion L_w of L with respect to w, then for every $v \in \Omega_A$, q is isotropic over the completed field F_v.

As we shall see at the end of the paper, Theorem 1.2 follows by combining the above proposition with a theorem of Colliot-Thélène, Parimala and Suresh [3] on quadratic forms over $F = K(y)$, whose proof builds upon earlier work of Harbater, Hartmann and Krashen [4].

2. Valuations coming from blow-ups

Lemma 2.1. — *Let A be an excellent local domain with residue field k and $\mathcal{X}$ an integral A-scheme of finite type. Let F be the function field of $\mathcal{X}$ and v a rank 1 discrete valuation of F with valuation ring $\mathcal{O}_v$. Assume that v is centered on $\mathcal{X}$ at a point x in the closed fiber $X := \mathcal{X} \times_A k$ and that the residue field $\kappa(v)$ of $\mathcal{O}_v$ has transcendence degree $\mathrm{trdeg}_k\, \kappa(v) = \dim \mathcal{X} - 1$ over k. Let $\mathcal{Y} = \mathrm{Spec}\, \mathcal{O}_v$ and $y \in \mathcal{Y}$ the closed point of $\mathcal{Y}$. Let $f : \mathcal{Y} \to \mathcal{X}$ be the natural morphism induced by the inclusion $\mathscr{O}_{\mathcal{X},x} \subseteq \mathcal{O}_v$. Define schemes $\mathcal{X}_n, n \in \mathbb{N}$ and morphisms $f_n : \mathcal{Y} \to \mathcal{X}_n, n \in \mathbb{N}$ as follows:*

Set $\mathcal{X}_0 = \mathcal{X}$ and $f_0 = f$. When $f_i : \mathcal{Y} \to \mathcal{X}_i$ is already defined, let $\mathcal{X}_{i+1} \to \mathcal{X}_i$ be the blow-up of $\mathcal{X}_i$ along the closure of $x_i := f_i(y)$ and let $f_{i+1} : \mathcal{Y} \to \mathcal{X}_{i+1}$ be the induced morphism.

Then for some large enough n, the morphism $f_n : \mathcal{Y} \to \mathcal{X}_n$ induces an isomorphism $\mathscr{O}_{\mathcal{X}_n,x_n} \cong \mathcal{O}_v$.

Proof. — The following proof is an easy adaptation of the proof of the geometric case, as given in [7, p. 61, Lemma 2.45].

Let $\mathscr{O}_n := \mathscr{O}_{\mathcal{X}_n,x_n}$. The ring theoretic construction of $\mathscr{O}_n$ is as follows. Assume that $\mathscr{O}_n$ (with maximal ideal $\mathfrak{m}_n$) is already defined. Pick a system of generators $z_1, \dots, z_r$ of $\mathfrak{m}_n$ such that $v(z_1) \leqslant \cdots \leqslant v(z_r)$. Let $\mathscr{O}'_n = \mathscr{O}_n[z_2/z_1, \dots, z_r/z_1]$. Then $\mathscr{O}_{n+1}$ is the localization of $\mathscr{O}'_n$ at $\mathscr{O}'_n \cap \mathfrak{m}_v$, where $\mathfrak{m}_v$ denotes the maximal ideal of $\mathcal{O}_v$.

The same argument as in the proof of [7, p. 61, Lemma 2.45] applies here and shows that $\mathcal{O}_v = \bigcup_{n \geqslant 0} \mathscr{O}_n$. Pick elements $u_1, \dots, u_t \in \mathcal{O}_v \subseteq F$ such that the reductions $\overline{u}_i$ form a transcendence basis of $\kappa(v) = \mathcal{O}_v/\mathfrak{m}_v$ over k. Choose n big enough so that $u_1, \dots, u_t \in \mathscr{O}_n$. Then $\kappa(v) = \mathcal{O}_v/\mathfrak{m}_v$ is an algebraic extension of $\kappa(x_n) = \mathscr{O}_n/\mathfrak{m}_n$ and

$$\mathrm{trdeg}_k\, \kappa(x_n) = \mathrm{trdeg}_k\, \kappa(v) = \dim \mathcal{X} - 1.$$

The closure $Z_n := \overline{\{ x_n \}}$ of x_n in $\mathcal{X}_n$ is an algebraic scheme over k. So we have

$$\dim Z_n = \mathrm{trdeg}_k\, \kappa(x_n) = \dim \mathcal{X} - 1.$$

By [10, p. 334, Coro. 8.2.7], we have $\dim \mathcal{X}_n = \dim \mathcal{X}$. Hence,

$$\dim \mathscr{O}_n = \mathrm{codim}(Z_n, \mathcal{X}_n) \leqslant \dim \mathcal{X}_n - \dim Z_n = 1.$$

But $\mathscr{O}_n \subseteq \mathcal{O}_v$ and the discrete valuation ring $\mathcal{O}_v$ is unequal to its fraction field $F = \mathrm{Frac}(\mathcal{O}_v) = \mathrm{Frac}(\mathscr{O}_n)$, so $\dim \mathscr{O}_n = 1$. Let $R' \subseteq F$ be the normalization of $\mathscr{O}_n$ and let $\mathfrak{m}' = \mathfrak{m}_v \cap R$. Then R' is a Dedekind domain and $R'_{\mathfrak{m}'}$ is a discrete valuation ring contained in $\mathcal{O}_v$ with fraction field F. Therefore, $R'_{\mathfrak{m}'} = \mathcal{O}_v$. The ring $\mathscr{O}_n$ is a Nagata ring (see *e.g.*, [10, p. 340, Prop. 8.2.29

and p. 343, Thm 8.2.39]). So R' is a finitely generated $\mathscr{O}_n$-module. Thus we have $R' \subseteq \mathscr{O}_N$ for some large $N \in \mathbb{N}$. Then it follows that $\mathcal{O}_v = \mathscr{O}_{N+1}$. The lemma is thus proved. □

3. Proof of Theorem 1.1

Theorem 1.1 is a statement generalizing [2, Thm 3.1], where the result is only established under the hypothesis that k is separably closed or finite. In our proof the observation that [2, Prop. 1.14] holds over an arbitrary field k is the key point which makes it possible to get rid of this restriction on k. In addition, Lemma 2.1 will be used in order to obtain the local-global principle for valuations in the subset Ω_R instead of the set of all discrete valuations.

Lemma 3.1. — *Let R be a two-dimensional normal excellent henselian local domain with fraction field L, L'/L a finite field extension and R' the integral closure of R in L'. Let w' be a discrete valuation of L' lying over a discrete valuation w of L.*

If w' corresponds to a codimension 1 point on a regular proper model $\mathcal{X}'$ of R' (i.e., $\mathcal{X}'$ is a regular integral scheme equipped with a proper birational morphism $\mathcal{X}' \to \operatorname{Spec} R'$), then w corresponds to a codimension 1 point on a regular proper model $\mathcal{X}$ of R.

Proof. — Let k (resp. k') be the residue field of R (resp. R'). Since R is excellent, R' is finite over R and hence k'/k is a finite extension. Let $x' \in \mathcal{X}'$ be the center of w' on $\mathcal{X}'$, p' the canonical image of x' in $\operatorname{Spec} R'$ and p the canonical image of p' in $\operatorname{Spec} R$.

If p is not the closed point of $\operatorname{Spec} R$, then it has codimension 1 in $\operatorname{Spec} R$ and the valuation ring $\mathcal{O}_w$ of w is equal to the local ring of p in $\operatorname{Spec} R$, since R is a 2-dimensional normal local domain. Let V be the complement of the closed point in $\operatorname{Spec} R$. For any regular proper model $\pi : \mathcal{X} \to \operatorname{Spec} R$, which exists by resolution of singularities, $\pi^{-1}(V) \to V$ is an isomorphism since R is normal (*cf.*, [10, p. 150, Coro. 4.4.3]). Hence, the point $x = \pi^{-1}(p)$ has codimension 1 in $\mathcal{X}$ and is the center of w on $\mathcal{X}$.

Now assume that p is the closed point of $\operatorname{Spec} R$. Then $x' \in \mathcal{X}'$ lies in the closed fiber of $\mathcal{X}'/R'$ and is the generic point of an integral curve over $k' = \kappa(p')$. Hence, the residue field $\kappa(w')$ of w' has transcendence degree 1 over k'. Since k'/k and $\kappa(w')/\kappa(w)$ are finite extensions, this implies that the residue field $\kappa(w)$ has transcendence degree 1 over k. By taking any regular proper model $\mathcal{X} \to \operatorname{Spec} R$ and applying Lemma 2.1 to the ring

R and the R-scheme $\mathcal{X}$, we conclude that there is a morphism $\mathcal{X}_n \to \mathcal{X}$ obtained by a sequence of blow-ups such that the center of w on $\mathcal{X}_n$ is a point of codimension 1, which completes the proof. $\square$

Given a scheme Y, we will denote by $\mathrm{Br}(Y) = H^2_{\text{ét}}(Y, \mathbb{G}_m)$ its cohomological Brauer group.

Proof of Theorem 1.1. — For any $a, b \in L^*$, the isotropy of the rank 3 form $\langle 1, a, b\rangle$ is equivalent to the isotropy of the rank 4 form $\langle 1, a, b, ab\rangle$. So we may restrict to the case of rank 4 forms. Let q be a rank 4 quadratic form over L which is isotropic over L_w for every $w \in \Omega_R$. After scaling we may assume without loss of generality that $q = \langle 1, a, b, abd\rangle$ with $a, b, d \in L^*$.

First assume that d is a square in L. Then the quadratic form q is isomorphic to the norm form of a quaternion algebra, whose class in the Brauer group $\mathrm{Br}(L)$ will be denoted by α. The form q is isotropic if and only if $\alpha = 0$ in the Brauer group.

Take a proper birational morphism $\mathcal{X} \to \operatorname{Spec} R$ with $\mathcal{X}$ a regular integral scheme such that the closed fiber X of $\mathcal{X}/R$ is a curve over k. For each $w \in \Omega_R$ corresponding to a codimension 1 point of $\mathcal{X}$, the canonical image α_w of α in $\mathrm{Br}(L_w)$ is trivial since q is isotropic over L_w by assumption. In particular, the residue of α at every codimension 1 point of $\mathcal{X}$ is trivial. Since $\mathcal{X}$ is a regular integral scheme, it follows that $\alpha \in \mathrm{Br}(L)$ lies in the subgroup $\mathrm{Br}(\mathcal{X})$. By [2, Thm 1.8 (c) and Lemma 1.6], we have canonical isomorphisms $\mathrm{Br}(\mathcal{X}) \cong \mathrm{Br}(X) \cong \mathrm{Br}(X_{\mathrm{red}})$. Identify $\alpha \in \mathrm{Br}(\mathcal{X})$ with its canonical image in $\mathrm{Br}(X_{\mathrm{red}})$. We will apply [2, Prop. 1.14] to show that $\alpha = 0$.

Let $f : Z \to X_{\mathrm{red}}$ be the normalization of the reduced curve X_{red}/k and let $D \subseteq X_{\mathrm{red}}$ be the closed subscheme defined by the conductor of f. Then [2, Prop. 1.14] says that the natural map $\mathrm{Br}(X_{\mathrm{red}}) \to \mathrm{Br}(Z) \times \mathrm{Br}(D)$ is injective. Let $(\alpha_1, \alpha_2) \in \mathrm{Br}(Z) \times \mathrm{Br}(D)$ be the image of $\alpha \in \mathrm{Br}(X_{\mathrm{red}})$. Each reduced irreducible component T of Z is a regular integral curve whose function field $k(T)$ is the residue field $\kappa(w)$ of a codimension 1 point w of the 2-dimensional regular scheme $\mathcal{X}$. Since α vanishes in $\mathrm{Br}(L_w)$ by hypothesis, the specialisation of α in $\mathrm{Br}(\kappa(w)) = \mathrm{Br}(k(T))$ is zero. The natural map $\mathrm{Br}(T) \to \mathrm{Br}(k(T))$ is an injection for the regular scheme T, so the canonical image of α in $\mathrm{Br}(T)$ is zero. Since this holds for every irreducible component T of Z, we have $\alpha_1 = 0$ in $\mathrm{Br}(Z)$.

To show that $\alpha_2 = 0$ in $\mathrm{Br}(D)$, it suffices to prove that α_2 vanishes at each closed point x of X_{red}, by a 0-dimensional variant of [2, Lemma 1.6]. The point x is also a closed point of $\mathcal{X}$. We may choose a 1-dimensional closed integral subscheme C of $\mathcal{X}$ which contains x as a regular point and let $\omega \in \mathcal{X}$ be the generic point of C. Our hypothesis implies that $\alpha \in \mathrm{Br}(\mathcal{X})$ vanishes at ω, and it follows that there is a regular open subscheme U of C, containing x, such that $\alpha|_U = 0$ in $\mathrm{Br}(U) \subseteq \mathrm{Br}(\kappa(\omega))$. Hence, $\alpha_2(x) = \alpha(x) = 0$. We have thus proved that $\alpha = 0$ in $\mathrm{Br}(L)$, whence the isotropy of the rank 4 quadratic form $q = \langle 1, a, b, abd\rangle$.

Now suppose that d is not a square in L. Let $L' = L(\sqrt{d})$ and R' the integral closure of R in L'. Then R' and L' satisfy the same assumptions as R and L. Let w' be a discrete valuation on L' corresponding to a codimension 1 point of a regular proper model $\mathcal{X}'/R'$. By Lemma 3.1, w' lies over a discrete valuation w in Ω_R. The isotropy of q over L_w implies the isotropy of $q_{L'}$ over $L'_{w'}$.

Thus the quadratic form $q_{L'}$ over L' has trivial determinant and is isotropic over $L'_{w'}$ for every $w' \in \Omega_{R'}$, where the set $\Omega_{R'}$ of discrete valuations of L' is defined in the same way as Ω_R. By the previous case, $q_{L'}$ is isotropic over L'. By [8, p. 197, Chap. VII, Thm 3.1], either q is isotropic over L or q contains a multiple of $\langle 1, -d\rangle$. In the latter case, since $\det(q) = d$ mod $(L^*)^2$, q also contains a rank 2 form of determinant -1. Hence q is isotropic over L, which completes the proof. $\square$

4. Valuations centered on the special fiber

Most of the present section and the next will be devoted to the proof of Proposition 1.4. The lemma below will be used frequently and referred to as Springer's lemma in what follows.

Lemma 4.1 (Springer's lemma, [8, p. 148, Prop. VI.1.9]). — *Let A be a complete discrete valuation ring in which 2 is invertible. Let K and k be respectively its fraction field and residue field. Let $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_s$ be units of A and let $\overline{\alpha}_i \in k$ and $\overline{\beta}_j \in k$ be their residue classes. Let π be a uniformizer of A.*

Then the quadratic form $\langle \alpha_1, \dots, \alpha_r\rangle \perp \pi.\langle \beta_1, \dots, \beta_s\rangle$ over K is anisotropic if and only if the two residue forms

$$\langle \overline{\alpha}_1, \dots, \overline{\alpha}_r\rangle \quad \text{and} \quad \langle \overline{\beta}_1, \dots, \overline{\beta}_s\rangle$$

are both anisotropic over k.

We shall now start the proof of Proposition 1.4. Recall that Ω_A is the union of all $\Omega_{\mathcal{P}}$, where $\mathcal{P}$ is a regular integral proper flat A-scheme with generic fiber $\mathcal{P} \times_A K \cong \mathbb{P}^1_K$ and $\Omega_{\mathcal{P}}$ is the set of rank 1 discrete valuations on $F = K(y)$ that correspond to codimension 1 points of $\mathcal{P}$. We will fix a discrete valuation $v \in \Omega_A$ and let $\mathcal{O}_v \subseteq F$ denote the valuation ring of v, $\pi_v \in \mathcal{O}_v$ a uniformizer of v, $\mathfrak{m}_v = \pi_v\mathcal{O}_v$ and $\kappa(v)$ the residue field of $\mathcal{O}_v$. The v-adic completion of $\mathcal{O}_v \subseteq F$ will be written as $\widehat{\mathcal{O}}_v \subseteq F_v$. If w is a discrete valuation of L, similar notations like $\mathcal{O}_w$, $\mathfrak{m}_w$, $\kappa(w)$, $\widehat{\mathcal{O}}_w \subseteq L_w$ and so on will be used.

Put $\mathcal{X} = \mathbb{P}^1_A$. Let $\mathcal{X}_K = \mathbb{P}^1_K$ and $\mathcal{X}_s = \mathbb{P}^1_k$ be respectively the generic and special fiber of $\mathcal{X}$ over A. Let $\eta \in \mathcal{X}_s = \mathbb{P}^1_k$ denote the generic point of $\mathcal{X}_s$. The valuation $v \in \Omega_A$ has a unique center on the model $\mathcal{X} = \mathbb{P}^1_A$, which will be denoted $P \in \mathcal{X}$. We have the following cases:

(1) $P \in \mathcal{X}_s = \mathbb{P}^1_k$, $P \neq 0, \infty, \eta$;
(2) $P = \eta \in \mathcal{X}_s = \mathbb{P}^1_k$;
(3) $P = \infty \in \mathcal{X}_s = \mathbb{P}^1_k$ or $P = \infty \in \mathcal{X}_K = \mathbb{P}^1_K$;
(4) $P = 0 \in \mathcal{X}_s = \mathbb{P}^1_k$;
(5) P is a closed point of $\mathbb{A}^1_K \subseteq \mathcal{X}_K = \mathbb{P}^1_K$.

Our proof of Proposition 1.4 will be a case-by-case argument, which is divided into two parts with details in what follows.

Proof of Proposition 1.4 (Part I). — In the first part of the proof, we treat cases (1)–(4).

Case (1). The valuation v is centered at $P \in \mathcal{X}_s \setminus \{0, \infty, \eta\}$.

In this case, we have $v(x) > 0$ and $v(y) = 0$. We may assume without loss of generality that for some $0 \leqslant r_1 \leqslant r$, the numbers n_i in (1.1) satisfy:

$$n_1 = \cdots = n_{r_1} = 0 \quad \text{and} \quad n_{r_1+1} = \cdots = n_r = 1. \tag{4.1}$$

Then $a_1, \dots, a_{r_1}$ and $a'_{r_1+1} = a_{r_1+1}/x, \dots, a'_r = a_r/x$ are units for v. Let

$$q_1 = \langle a_1, \dots, a_{r_1}\rangle \quad \text{and} \quad q_2 = \langle a'_{r_1+1}, \dots, a'_r\rangle. \tag{4.2}$$

Then $q = \langle a_1, \dots, a_r\rangle = q_1 \perp x.q_2$ is anisotropic only if q_1 and q_2 are both anisotropic. By Springer's lemma (or Hensel's lemma), q_i is anisotropic over F_v if and only if its residue form $\overline{q}_i := q_i \pmod{\mathfrak{m}_v}$ is anisotropic over $\kappa(v)$. In the present situation, the two residue forms $\overline{q}_i, i = 1, 2$ have coefficients in the subfield $\kappa(P) \subseteq \kappa(v)$. Since $r \geqslant 5$, either q_1 or q_2 has rank $\geqslant 3$. Assume for example q_1 has rank $\geqslant 3$. The residue field $\kappa(P)$ is a finite extension of k, so property $(*)$ implies that $\overline{q}_1$ is isotropic over $\kappa(P)$ and a fortiori over $\kappa(v)$. It follows that q is isotropic over F_v as desired.

Case (2). The valuation v is centered at the generic point η of the special fiber $\mathcal{X}_s = \mathbb{P}^1_k$.

In this case, v is the x-adic valuation on $A[y]$ and $\kappa(v) = k(y)$. Let w be the x-adic valuation on $A[[y]]$, so that $w|_{A[y]} = v|_{A[y]}$ and $\kappa(w) = k((y))$.

Define q_1 and q_2 as in (4.2). We have

$$\begin{aligned} \overline{q}_1 &:= q_1 \pmod{\mathfrak{m}_w} = \langle \lambda_1 y^{m_1}, \dots, \lambda_{r_1} y^{m_{r_1}} \rangle, \\ \overline{q}_2 &:= q_2 \pmod{\mathfrak{m}_w} = \langle \lambda_{r_1+1} y^{m_{r_1+1}}, \dots, \lambda_r y^{m_r} \rangle. \end{aligned} \tag{4.3}$$

Here we have identified each $\lambda_i \in \Sigma \subseteq A$ with its canonical image in k. By hypothesis and Springer's lemma, we may assume one of the two residue forms, say $\overline{q}_1$, is isotropic over $k((y))$. By (4.3), $\overline{q}_1$ has coefficients in $k(y)$ and is isometric to $\mu_1 \perp y.\mu_2$ over $k(y)$ for some nonsingular quadratic forms μ_i over k. Indeed, if I (resp. J) denotes the subset of $\{ 1, \dots, r_1 \}$ consisting of indices i such that m_i is even (resp. odd), then we may take $\mu_1 = \langle \lambda_i \rangle_{i \in I}$ (resp. $\mu_2 = \langle \lambda_i \rangle_{i \in J}$). Applying Springer's lemma to the form $\overline{q}_1 / k((y))$ with respect to the discrete valuation ring $k[[y]]$, we conclude that either μ_1 or μ_2 is isotropic over k. Then it is clear that $\overline{q}_1 \cong \mu_1 \perp y.\mu_2$ is isotropic over $k(y) = \kappa(v)$. Since the residue forms of q mod v coincide with those mod w, it follows from Springer's lemma that q is isotropic over F_v.

Case (3). The valuation v is centered at $P = \infty \in \mathcal{X}_s = \mathbb{P}^1_k$ or $P = \infty \in \mathcal{X}_K = \mathbb{P}^1_K$.

In this case, we have $v(y) < 0$ and $v(x) \geqslant 0$. Put $z = y^{-1} \in F = K(y)$. We want to prove that q is isotropic over F_v.

Recall that the coefficients of the diagonal form q have the form $a_i = \lambda_i . x^{n_i} . P_i$, where $\lambda_i \in \Sigma$, $n_i \in \{ 0, 1 \}$ and P_i is a distinguished polynomial in $A[y]$ for each i. Let $m_i = \deg P_i$ be the degree of P_i with respect to the variable y. Then in $F = K(y)$ we have

$$P_i(y) = y^{m_i}(1 + z.\rho_i) \quad \text{for some } \rho_i \in A[z].$$

Set $b_i = \lambda_i . x^{n_i} . y^{m_i} \in F$ and let q'/F be the diagonal quadratic form $\langle b_1, \dots, b_r \rangle$. The two forms $q = \langle a_i \rangle$ and $q' = \langle b_i \rangle$ are isometric over F_v since $1 + z.\rho_i$ is a square in F_v for each i. So it suffices to prove the isotropy over F_v of the form $q' = \langle b_i \rangle$.

We may assume the numbers n_i are given as in (4.1), so that $q' = q'_1 \perp x.q'_2$ with

$$q'_1 = \langle \lambda_1 y^{m_1}, \dots, \lambda_{r_1} y^{m_{r_1}} \rangle \,, \quad q'_2 = \langle \lambda_{r_1+1} y^{m_{r_1+1}}, \dots, \lambda_r y^{m_r} \rangle.$$

There are diagonal quadratic forms $\mu_j, j = 1, \dots, 4$, where μ_1, μ_2 have coefficients in $\{ \lambda_1, \dots, \lambda_{r_1} \} \subseteq \Sigma$ and μ_3, μ_4 have coefficients in $\{ \lambda_{r_1+1}, \dots, \lambda_r \} \subseteq \Sigma$, such that $q'_1 \cong \mu_1 \perp y.\mu_2$ and $q'_2 \cong \mu_3 \perp y.\mu_4$ over $F = K(y)$. Observe that the two residue forms of q with respect to the x-adic valuation on F are isometric to the forms $\mu_1 \perp y.\mu_2$ and $\mu_3 \perp y.\mu_4$. A close inspection of the above proof for case (2) shows that not all of the four forms μ_j are anisotropic over k. Since

$$q' \cong \mu_1 \perp y.\mu_2 \perp x.(\mu_3 \perp y.\mu_4) \quad \text{over} \quad F = K(y),$$

it follows easily that q' is isotropic over F_v, whence the isotropy of q over F_v.

Case (4). The valuation v is centered at the origin $P = 0 \in \mathbb{P}^1_k$ of the special fiber.

By the definition of the set Ω_A, the valuation $v \in \Omega_A$ corresponds to a codimension 1 point p of a regular proper model $\mathcal{P}/A$ of $\mathbb{P}^1_K$. Since the center of v on $\mathcal{X}$ lies in the special fiber, $v(x) > 0$. The point $p \in \mathcal{P}$ lies in the special fiber of $\mathcal{P}/A$ since otherwise the valuation v must be trivial on $K = \mathrm{Frac}(A)$. The residue field $\kappa(v)$ is then the function field of a curve over k. So we have

$$\mathrm{trdeg}_k\, \kappa(v) = 1 = \dim \mathbb{P}^1_A - 1.$$

By Lemma 2.1, there is a scheme $\mathcal{X}_n \to \mathcal{X} = \mathbb{P}^1_A$ obtained by a sequence of blow-ups at closed points lying over $0 \in \mathcal{X}_s = \mathbb{P}^1_k$ such that $\mathcal{O}_v = \mathscr{O}_{\mathcal{X}_n, x_n} \subseteq F$ for some codimension 1 point $x_n \in \mathcal{X}_n$. If we consider the same sequence of blow-ups which is carried out on $\mathrm{Spec}\, A[[y]]$ this time, then we get a discrete valuation $w \in \Omega_R$ of L which extends v. Now we have inclusions $A[y] \subseteq \mathcal{O}_v \subseteq \mathcal{O}_w$ and $\kappa(v) = \kappa(w)$. Let q_1, q_2 be diagonal quadratic forms with coefficients in $\widehat{\mathcal{O}}_v^*$ such that

$$q \cong q_1 \perp \pi_v.q_2 \quad \text{over} \quad F_v.$$

Since q is isotropic over L_w by assumption, applying Springer's lemma to w shows that $\overline{q}_1 = q_1 \pmod{\mathfrak{m}_v}$ or $\overline{q}_2 = q_2 \pmod{\mathfrak{m}_v}$ has a nontrivial zero in $\kappa(w) = \kappa(v)$. One more application of Springer's lemma, with respect to v this time, proves that q is isotropic over F_v. □

5. End of the proof

To prove Proposition 1.4 in case (5), we need the following form of the Weierstraß preparation theorem.

LEMMA 5.1 (Weierstraß). — *Let A be a complete discrete valuation ring and $A[[y]]$ the ring of formal power series in one variable over A. Let $P \in A[y]$ be a distinguished polynomial and $f \in A[y]$.*

(i) *For any $g \in A[[y]]$, there is a unique expression*

$$g = Q.P + R$$

where $Q \in A[[y]]$ *and* $R \in A[y]$ *is a polynomial of degree* $\leqslant \deg P - 1$. *In particular,*

$$A[y]/(P) \cong A[[y]]/(P).$$

(ii) *If* f *divides* P *in* $A[y]$, *then there is a unit* u *in* A *such that* uf *is a distinguished polynomial.*

Proof. — (i) See *e.g.*, [13, p. 114, Prop. 7.4]. Note that the isomorphism $A[y]/(P) \cong A[[y]]/(P)$ implies that P is irreducible in $A[y]$ if and only if P is irreducible in $A[[y]]$ and that P divides a polynomial f in $A[y]$ if and only if P divides f in $A[[y]]$.

(ii) Assume $P = fg$ with $g \in A[y]$. The hypothesis implies that the coefficient a_0 of $y^{\deg f}$ in f is a unit in A since P is a monic polynomial. Let k be the residue field of A and let $A[y] \to k[y]$, $F \mapsto \overline{F}$ denote the canonical reduction map. By considering the factorization $y^{\deg P} = \overline{P} = \overline{f} \cdot \overline{g}$ in $k[y]$, we see that $u := a_0^{-1} \in A^*$ has the required property. □

Proof of Proposition 1.4 (Part II). — We now consider the only remaining case, case (5). This is the case where the center P of the valuation v lies in $\mathbb{A}^1_K \subseteq \mathcal{X}_K = \mathbb{P}^1_K$.

We have $\mathscr{O}_{\mathcal{X},P} = \mathcal{O}_v$ since the two rings are both discrete valuation rings with fraction field F. So v is defined by an irreducible polynomial $f \in A[y]$ with $x \nmid f$.

If none of the polynomials $P_i, i = 1, \dots, r$ is divisible by f, then q has coefficients in $\mathcal{O}_v^* = \mathscr{O}_{\mathcal{X},P}^*$. Now the residue field $\kappa(v) = \kappa(P)$ is a finite extension of K and the residue form $\overline{q} = q \pmod{\mathfrak{m}_v}$ has rank $r \geqslant 5$. By property $(**)$ (*cf.*, Remark 1.3), $\overline{q}$ is isotropic over $\kappa(v)$. It follows from Springer's lemma (or Hensel's lemma) that q is isotropic over F_v.

Assume next f divides some P_i, say $f | P_1$. By Lemma 5.1, multiplying f by a unit in A if necessary, we may assume that f is an irreducible distinguished polynomial. In $A[[y]]$, f is still an irreducible element. The f-adic valuation on $R = A[[y]]$ determines a discrete valuation $w \in \Omega_R$ which extends $v \in \Omega_A$. We have

$$\kappa(v) = \mathrm{Frac}(A[y]/(f)) = \mathrm{Frac}(A[[y]]/(f)) = \kappa(w)$$

and $F_v \subseteq L_w$. Using the argument with the first and second residue forms and Springer's lemma, we conclude as in case (4) that q is isotropic over F_v. □

We are now ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. — Let q be any quadratic form of rank $r \geqslant 5$ over $L = \mathrm{Frac}(R)$ and assume that q is isotropic over L_w for every $w \in \Omega_R$. Without loss of generality, we may assume $q = \langle a_1, \dots, a_r \rangle$ for some nonzero elements $a_i \in R = A[[y]]$. By the usual form of the Weierstraß preparation theorem (see *e.g.*, [13, p. 115, Thm 7.3]), each a_i may be written as

$$a_i = x^{n_i}.P_i.U_i \quad \text{with } n_i \in \mathbb{N}, U_i \in R^* \text{ and } P_i \text{ a distinguished polynomial in } A[y].$$

For any power series $f = \sum_{i=0}^{\infty} a_i y^i \in R = A[[y]]$ which is invertible in R, letting $\lambda \in \Sigma$ be the unique element such that $\lambda^{-1}a_0 \equiv 1 \pmod{xA}$, we have

$$\lambda^{-1} f \equiv 1 \pmod{\mathfrak{m}_R}.$$

Since R is complete, it follows that $\lambda^{-1}f$ is a square in R. So after scaling out squares we may assume that the coefficients a_i have the form described in Proposition 1.4. Now the quadratic form q is defined over $F = K(y)$ and by Proposition 1.4, it is isotropic over F_v for every $v \in \Omega_A$. The local-global principle with respect to discrete valuations in Ω_A is proved for quadratic forms of rank $\geqslant 3$ in [3, Thm 3.1 and Remark 3.2]. Hence, q is isotropic over F and a fortiori over L. □

Remark 5.2. — In Theorem 1.2, assume that $A = k[[x]]$ with k a C_1 field of characteristic $\neq 2$ or $A = \mathcal{O}_K$ with K a p-adic number field (p an odd prime). Then every quadratic form of rank $\geqslant 9$ is isotropic over $F = K(y)$. In the former case, it is well-known that $F = k((x))(y)$ is a C_3 field. For the case $A = \mathcal{O}_K$, this statement is firstly proved by Parimala and Suresh [11], and then two more recent proofs using different methods are given in [4, Coro. 4.15] and [3, Coro. 3.4] as consequences of their main theorems. Still another proof (including the case $p = 2$), which builds upon the work of Heath-Brown [5], has been announced by Leep [9].

An easy argument using the Weierstraß preparation theorem shows that every quadratic form of rank $\geqslant 9$ is isotropic over $L = \mathrm{Frac}(A[[y]])$. So in these cases, the local-global principle in Theorem 1.2 is only interesting for quadratic forms of rank $5 \leqslant r \leqslant 8$.

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