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UNCERTAINTY PRINCIPLES FOR THE SCHRÖDINGER EQUATION ON RIEMANNIAN SYMMETRIC SPACES OF THE NONCOMPACT TYPE

by Angela PASQUALE & Maddala SUNDARI

ABSTRACT. — Let X be a Riemannian symmetric space of the noncompact type. We prove that the solution of the time-dependent Schrödinger equation on X with square integrable initial condition f is identically zero at all times t whenever f and the solution at a time $t_0 > 0$ are simultaneously very rapidly decreasing. The stated condition of rapid decrease is of Beurling type. Conditions respectively of Gelfand-Shilov, Cowling-Price and Hardy type are deduced.

RÉSUMÉ. — Soit X un espace riemannien symétrique de type non-compact. On montre que la solution de l'équation de Schrödinger dépendante du temps sur X, avec condition initiale de carré intégrable f, est nulle en tout temps t lorsque f et la solution à un temps $t_0 > 0$ donné sont simultanément très rapidement décroissantes. La condition de décroissance rapide considérée est de type Beurling. Des conditions respectivement de types Gelfand-Shilov, Cowling-Price et Hardy en sont déduites.

1. Introduction

Consider the initial value problem for the time-dependent Schrödinger equation on a Riemannian symmetric space of the noncompact type X:

(S)
$$i\partial_t u(t,x) + \Delta u(t,x) = 0$$
$$u(0,x) = f(x)$$

where Δ denotes the Laplace-Beltrami operator on X. In [5], S. Chanillo initiated the study of certain uniqueness properties for the solutions of (S) that are related with the uncertainty principle. According to this principle, the solution $u(t, \cdot)$ must be identically zero at all times t whenever the

initial condition f and the solution at a certain time $t_0 \neq 0$ are simultaneously very rapidly decreasing. The case considered by Chanillo corresponds to Riemannian symmetric spaces of the form $X = G/K$ where G is a noncompact connected semisimple Lie group possessing a complex structure and K is a maximal compact subgroup of G. Moreover, in [5] the initial condition f is assumed to be K-invariant, which ensures that the solution itself is K-invariant. Some related articles in the Euclidean setting are [6], [9], [10], [11] and [18]; for the Heisenberg group [3].

In this paper, we consider the initial value problem (S) on arbitrary Riemannian symmetric spaces X of the noncompact type and not-necessarily K-invariant initial data $f \in L^2(X)$. We show that if f and the solution $u(t_0, \cdot)$ at some time $t_0 > 0$ satisfy a suitable Beurling type condition, then $u(t, \cdot) = 0$ for all $t \in \mathbb{R}$. This describes an uncertainty principle because of the relation one gets between the Fourier transforms of $u_t = u(t, \cdot)$ and the initial condition f; see (4.1). Our result implies in particular the uniqueness property of solutions for the Schrödinger equation (S) proved by Chanillo using Hardy type conditions.

The rapid decrease of a function on X is measured by means of exponential powers of the distance function d induced by the Riemannian metric. If $o = eK$ denotes the base point in X, we set $\sigma(x) := d(o, x)$. The estimates are also in terms of the elementary spherical function of spectral parameter 0, denoted by Ξ. We denote by $C(\mathbb{R} : L^2(X))$ the space of continuous functions $t \mapsto u(t, \cdot)$ from $\mathbb{R}$ to $L^2(X)$. We refer to Section 2.4 for the precise definitions. Our main result is the following theorem.

THEOREM 1.1. — *Let X be a Riemannian symmetric space of the noncompact type and let $f \in L^2(X)$. Let $u(t, x) \in C(\mathbb{R} : L^2(X))$ denote the solution of (S) with initial condition f. If there is a time $t_0 > 0$ so that*

$$(1.1) \qquad \int_X \int_X |f(x)| \, |u(t_0, y)| \Xi(x) \Xi(y) e^{\frac{\sigma(x)\sigma(y)}{2t_0}} \, dx \, dy < \infty,$$

then $u(t, \cdot) = 0$ for all $t \in \mathbb{R}$.

The proof of Theorem 1.1 is given in Section 4.3. It is based on a reduction to a Euclidean situation by means of the Radon transform. This is a powerful technique that is commonly used to attack problems related to the uncertainty principle on symmetric spaces as well as on Euclidean spaces; see for instance [23, 24, 20, 21]. In our situation, the important feature is that for functions satisfying (1.1), the composition of the Radon transform and a Euclidean Fourier transform gives the Helgason-Fourier transform. A Beurling theorem for the Helgason-Fourier transform has been proved

by Sarkar and Sengupta (see [21], Theorem 4.1), under the assumption $f \in L^1(X) \cap L^2(X)$. However, unlike the Euclidean case treated in Section 3 below, we cannot deduce our uniqueness theorem from Beurling's theorem on Riemannian symmetric spaces. In fact, in the Euclidean situation, the moduli of the initial value function $f(x)$ and of the solution $u(t, x)$ are (up to constants) equal to those of a Fourier transform pair $(h, \widehat{h})$. So Beurling's theorem for $(h, \widehat{h})$ yields the required uniqueness for the Schrödinger solution. See Corollary 3.5. A similar property does not hold for general Riemannian symmetric spaces. This is a consequence of the lack of duality between the Helgason-Fourier transform and its inverse. One can in part recover this duality for K-invariant functions on Riemannian symmetric spaces G/K with G complex. In this case, one can work on a maximally flat geodesic submanifold, where the Helgason-Fourier transform reduced essentially to a Euclidean Fourier transform. This is the approach used in [5].

The classical version of Beurling's theorem on $\mathbb{R}^n$ is among the strongest qualitative uncertainty principles. By this, we mean theorems which allow to conclude that $f = 0$ by giving quantitative conditions on f and its Fourier transform $\widehat{f}$. For instance, Beurling's theorem implies the qualitative uncertainty principles of Gelfand-Shilov, Cowling-Price and Hardy. Following this line, in Section 5 we prove that the uniqueness property stated in Theorem 1.1 implies uniqueness properties respectively of Gelfand-Shilov, Cowling-Price and Hardy type, for the solution of the Schrödinger equation on a Riemannian symmetric space.

Finally, we study more closely the Hardy type conditions. Let α and β be the exponents measuring respectively the very rapid decay of the initial condition f and the solution u_{t_0} at a time $t_0 > 0$. Then the uniqueness property is proven to hold when the condition $16t_0^2 \alpha\beta > 1$ is satisfied. The value of t_0 is shown to be optimal. More precisely, in Theorem 5.9 we prove that, under the additional assumption that G admits a complex structure, the K-invariant initial conditions f for which the uniqueness property fails in the case $16t_0^2 \alpha\beta = 1$ are precisely the constant multiples of an explicitly given function.

2. Notation and preliminaries

2.1. Riemannian symmetric spaces of the noncompact type and their structure

Let X be a Riemannian symmetric space of the noncompact type. Then $X = G/K$ where G is a noncompact, connected, semisimple, real Lie group G with finite center and K is a maximal compact subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{k}$ ($\subset \mathfrak{g}$) be the Lie algebra of K. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the subspace of $\mathfrak{g}$ which is orthogonal to $\mathfrak{k}$ with respect to the Killing form B of $\mathfrak{g}$. Then $\mathfrak{p}$ can be identified with the tangent space to X at the base point $o = eK$. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. We denote by $\mathfrak{a}^*$ the (real) dual space of $\mathfrak{a}$ and by $\mathfrak{a}^*_{\mathbb{C}}$ its complexification. The Killing form B is a (positive definite) inner product on $\mathfrak{p}$ and hence on $\mathfrak{a}$. For $H_1, H_2 \in \mathfrak{a}$ we set $\langle H_1, H_2 \rangle := B(H_1, H_2)$ and $|H_1| := \langle H_1, H_1 \rangle^{1/2}$. We extend this inner product to $\mathfrak{a}^*$ by duality by setting $\langle \lambda, \mu \rangle := \langle H_\lambda, H_\mu \rangle$ if H_λ is the unique element of $\mathfrak{a}$ so that $\langle H, H_\lambda \rangle = \lambda(H)$ for all $H \in \mathfrak{a}$. The $\mathbb{C}$-bilinear extension of $\langle \cdot, \cdot \rangle$ to $\mathfrak{a}^*_{\mathbb{C}}$ will be indicated by the same symbol.

The set of (restricted) roots of the pair $(\mathfrak{g}, \mathfrak{a})$ is denoted by Σ. It consists of all $\alpha \in \mathfrak{a}^*$ for which the vector space

$$\mathfrak{g}_\alpha := \big\{ X \in \mathfrak{g} \colon [H, X] = \alpha(H)X \text{ for every } H \in \mathfrak{a} \big\}$$

contains nonzero elements. The dimension m_α of $\mathfrak{g}_\alpha$ is called the multiplicity of the root α. We shall adopt the convention that $m_{2\alpha} = 0$ if 2α is not a root. The subset of $\mathfrak{a}$ on which all roots vanish is a finite union of hyperplanes. We can therefore choose $Y \in \mathfrak{a}$ so that $\alpha(Y) \neq 0$ for all $\alpha \in \Sigma$. Put $\Sigma^+ := \{\alpha \in \Sigma \colon \alpha(Y) > 0\}$. Then Σ^+ is a system of positive

roots, and Σ is the disjoint union of Σ^+ and $-\Sigma^+$. A root $\alpha \in \Sigma$ is said to be indivisible if $\alpha/2 \notin \Sigma$. We denote by Σ_0 the set of indivisible roots and set $\Sigma_0^+ := \Sigma^+ \cap \Sigma_0$. The half-sum of the positive roots counted with multiplicities is denoted by ρ: hence

$$(2.1) \qquad \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$

The set $\mathfrak{a}^+ := \left\{ H \in \mathfrak{a} \colon \alpha(H) > 0 \text{ for all } \alpha \in \Sigma^+ \right\}$ is an open polyhedral cone called the positive Weyl chamber. The corresponding Weyl chamber in $\mathfrak{a}^*$ is $\mathfrak{a}_+^* := \left\{ \lambda \in \mathfrak{a}^* \colon \langle \lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Sigma^+ \right\}$.

Let $\exp \colon \mathfrak{g} \to G$ be the exponential map of G, and set $A := \exp \mathfrak{a}$ and $A^+ := \exp \mathfrak{a}^+$. Then A is an abelian subgroup of G that is diffeomorphic to $\mathfrak{a}$ under $\exp$. The inverse diffeomorphism is denoted by $\log$.

The Weyl group W of the pair $(\mathfrak{g}, \mathfrak{a})$ is the finite group of orthogonal transformations of $\mathfrak{a}$ generated by the reflections r_α with $\alpha \in \Sigma$, where

$$r_\alpha(H) := H - 2\frac{\alpha(H)}{\langle \alpha, \alpha \rangle} H_\alpha, \quad H \in \mathfrak{a}.$$

The Weyl group action extends to A via the exponential map, to $\mathfrak{a}^*$ by duality, and to $\mathfrak{a}_{\mathbb{C}}^*$ by complex linearity.

Set $\mathfrak{n} := \oplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Then $N := \exp \mathfrak{n}$ is a simply connected nilpotent subgroup of G. The subgroup A normalizes N. The map $(k, a, n) \mapsto kan$ is an analytic diffeomorphism of the product manifold $K \times A \times N$ onto G. The resulting decomposition $G = KAN = KNA$ is called the Iwasawa decomposition of G. Thus, for $g \in G$ we have $g = k(g) \exp H(g) n(g)$ for uniquely determined $k(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in G$. One can prove that

$$(2.2) \qquad |H(ak)| \leqslant |\log a|$$

for all $a \in A$ and $k \in K$.

Let M be the centralizer of A in K and set $B = K/M$. Then the map $A \colon X \times B \to \mathfrak{a}$ is defined by

$$(2.3) \qquad A(gK, kM) := -H(g^{-1}k).$$

Besides the Iwasawa decomposition, we will also need the polar decomposition $G = KAK$: every $g \in G$ can be written in the form $g = k_1 a k_2$ with $k_1, k_2 \in K$ and $a \in A$. The element a is unique up to W-invariance.

2.2. Normalization of measures and integral formulas

We shall adopt the normalization of measures as in [15], Ch. II, §3.1. In particular, the Haar measures dk and dm on the compact groups K and M are normalized to have total mass 1. The Haar measures da and $d\lambda$ on A and $\mathfrak{a}^*$, respectively, are normalized so that the Euclidean Fourier transform

$$(2.4) \qquad (\mathcal{F}_A f)(\lambda) := \int_A f(a) e^{-i\lambda(\log a)} \, da, \quad \lambda \in \mathfrak{a}^*,$$

of a sufficiently regular function $f \colon A \to \mathbb{C}$ is inverted by

$$(2.5) \qquad f(a) = \int_{\mathfrak{a}^*} (\mathcal{F}_A f)(\lambda) e^{i\lambda(\log a)} \, d\lambda, \quad a \in A.$$

The Haar measures dg and dn of G and N, respectively, are normalized so that $dg = e^{2\rho(\log a)} \, dk \, da \, dn$. Moreover, if U is a Lie group and P is a closed subgroup of U, with left Haar measures du and dp, respectively, then the U-invariant measure $d(uP)$ on the homogeneous space U/P (when it exists) is normalized by

$$(2.6) \qquad \int_U f(u) \, du = \int_{U/P} \left(\int_P f(up) \, dp \right) d(uP).$$

This condition normalizes the G-invariant measure $dx = d(gK)$ on $X = G/K$ and fixes the K-invariant measure $db = d(kM)$ on $B = K/M$ to have total mass 1.

Corresponding to the Iwasawa and the polar decompositions, we have the integral formulas

$$(2.7) \quad \int_X f(x) \, dx = \int_G f(g \cdot o) \, dg = \int_K \int_A \int_N f(kan \cdot o) e^{2\rho(\log a)} \, dk \, da \, dn$$

$$(2.8) \qquad\qquad = c \int_K \int_{A^+} f(ka \cdot o) \delta(a) \, dk \, da$$

where for $H \in \mathfrak{a}^+$

$$(2.9) \quad \delta(\exp H) = \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^{m_\alpha} = e^{2\rho(H)} \prod_{\alpha \in \Sigma^+} \left(\frac{1 - e^{-2\alpha(H)}}{2} \right)^{m_\alpha}$$

and c is a suitable positive constant.

2.3. Invariant differential operators on X and the Laplace-Beltrami operator

Let $\mathbb{D}(X)$ denote the commutative algebra of invariant differential operators on X which are invariant under the action of G on X by left translations. As a Riemannian manifold, a symmetric space of the noncompact type $X = G/K$ is endowed with a Laplace-Beltrami operator Δ defined by $\Delta f = \mathrm{div} \circ \mathrm{grad}\, f$ for $f \in C^\infty(X)$. It turns out that Δ is a self-adjoint differential operator on $L^2(X)$ belonging to $\mathbb{D}(X)$.

Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $b \in B$. Then the function $e_{\lambda,b} \colon X \to \mathbb{C}$ defined by

$$(2.10) \qquad e_{\lambda,b}(x) = e^{(i\lambda+\rho)(A(x,b))}, \quad x \in X,$$

is an eigenfuntion of Δ. In fact,

$$(2.11) \qquad \Delta e_{\lambda,b} = -(\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle) e_{\lambda,b}.$$

We refer to [14], Ch. II, for additional information.

2.4. K-invariant functions

Functions $f \colon X \to \mathbb{C}$ can be identified with right K-invariant functions on G, *i.e.* with functions $f \colon G \to \mathbb{C}$ so that $f(gk) = f(g)$ for all $g \in G$ and $k \in K$. Likewise, K-invariant functions on X can be identified with K-bi-invariant functions on G, *i.e.* with functions $f \colon G \to \mathbb{C}$ so that $f(k_1 g k_2) = f(g)$ for all $g \in G$ and $k_1, k_2 \in K$. In turn, because of the polar decomposition $G = KAK = K\overline{A^+}K$, a K-bi-invariant function on G can be identified with its W-invariant restriction to A or with its restriction to $\overline{A^+}$. The Riemannian distance from the origin o, defined by $\sigma(x) = d(x, o)$, is an example of K-invariant map on X. According to the above identifications, we shall sometimes write $\sigma(g)$ instead of $\sigma(g \cdot o)$ for $g \in G$. If $x = g \cdot o$ and $g = k_1 \exp H k_2$ with $g \in G$, $H \in \mathfrak{a}$ and $k_1, k_2 \in K$, then

$$(2.12) \qquad \sigma(x) = \sigma(g) = \sigma(\exp H) = |H|.$$

The function σ satisfies the following inequalities:

$$(2.13) \qquad \sigma(gh) \leqslant \sigma(g) + \sigma(h), \quad g, h \in G,$$

$$(2.14) \qquad \sigma(an) \geqslant \sigma(a), \quad a \in A, n \in N.$$

Another K-invariant function on X we shall employ is the spherical function Ξ of spectral parameter 0. It is defined by

$$(2.15) \qquad \Xi(x) = \int_B e_{0,b}(x)\, db = \int_K e^{-\rho(H(gk))}\, dk, \quad x = g \cdot o \in X.$$

It is a real analytic function and, writing $\Xi(g)$ instead of $\Xi(g \cdot o)$, it satisfies the following properties:

$$(2.16) \qquad 0 < \Xi(g) = \Xi(g^{-1}) \leqslant 1, \quad g \in G,$$

$$(2.17) \qquad \int_K \Xi(g_1 k g_2) \, dk = \Xi(g_1)\Xi(g_2), \quad g_1, g_2 \in G,$$

$$(2.18) \qquad e^{-\rho(H)} \leqslant \Xi(a) \leqslant e^{-\rho(H)}(1 + |H|)^d, \quad a = \exp H \in \overline{A^+},$$

where $d = |\Sigma_0^+|$ is the cardinality of the set of positive indivisible roots. The properties of the functions σ and Ξ can be found in [12], §4.6 and §6.2.

2.5. The Helgason-Fourier transform and the Radon transform

A reference for this section is [15], Ch. III. The Helgason-Fourier transform of a sufficiently regular function $f\colon X \to \mathbb{C}$ is the function $\mathcal{F}f$ defined by

$$(2.19)$$
$$\mathcal{F}f(\lambda, b) = \int_X f(x)e_{-\lambda, b}(x) \, dx = \int_{AN} f(kan \cdot o)e^{(-i\lambda+\rho)(\log a)} \, da \, dn$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $b = kM \in B$ for which this integral exists. The Plancherel theorem states that the Helgason-Fourier transform $\mathcal{F}$ extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}_+^* \times B, |c(\lambda)|^{-2}db \, d\lambda)$. The function $c(\lambda)$ occurring in the Plancherel density is Harish-Chandra's c-function. It is the meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ given explicitly by the Gindikin-Karpelevich product formula:

$$(2.20) \qquad c(\lambda) = c_0 \prod_{\alpha \in \Sigma_0^+} c_\alpha(\lambda)$$

where

$$(2.21) \qquad c_\alpha(\lambda) = \frac{2^{-i\lambda_\alpha} \, \Gamma(i\lambda_\alpha)}{\Gamma\left(\frac{i\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{1}{2}\right)\Gamma\left(\frac{i\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2}\right)}$$

and the constant c_0 is given by the condition $c(-i\rho) = 1$. In (2.21) we have employed the notation

$$(2.22) \qquad \lambda_\alpha := \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

for $\alpha \in \Sigma$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Observe that if $f \in L^2(X)$ then, by the Plancherel theorem, for almost all $\lambda \in \mathfrak{a}^*$ we have $\mathcal{F}f(\lambda, \cdot) \in L^2(B)$.

The Helgason-Fourier transform $\mathcal{F}f$ of a function $f \in L^1(X)$ is almost everywhere defined. More precisely, there exists a subset $B' \subset B$ (depending on f) with $B \setminus B'$ of zero measure, such that $\mathcal{F}f(\lambda, b)$ is defined for each $\lambda \in \mathfrak{a}^*$. In fact, $\mathcal{F}f(\lambda, b)$ is defined and holomorphic for λ in a small tube domain around $\mathfrak{a}^*$ in $\mathfrak{a}_{\mathbb{C}}^*$. It turns out that $\mathcal{F}$ is injective on $L^1(X)$. Furthermore, as in the Euclidean case, there is an inversion formula for functions $f \in L^1(X)$ with $\mathcal{F}f \in L^1(\mathfrak{a}^* \times B, |c(\lambda)|^{-2} db\, d\lambda)$: for almost all $x \in X$, we have

$$(2.23) \qquad f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^* \times B} \mathcal{F}f(\lambda, b) e_{\lambda, b}(x) \, \frac{db\, d\lambda}{|c(\lambda)|^2},$$

where $|W|$ denotes the cardinality of the Weyl group W.

By identifying the space G/MN of horocycles on X with $B \times A$, one can define the Radon transform of a sufficiently regular function $f \colon X \to \mathbb{C}$ as the function Rf defined by

$$(2.24) \qquad Rf(b, a) = e^{\rho(\log a)} \int_N f(kan \cdot o) \, dn$$

for all $b = kM \in B = K/M$ and $a \in A$ for which this integral exists. See [15], p. 220 or [8]. If $f \in L^1(X)$, then the integral defining $Rf(b, a)$ converges absolutely for almost all $(b, a) \in B \times A$ and $Rf \in L^1(B \times A)$. Indeed $\|Rf\|_{L^1(B \times A, db\, da)} \leqslant |W| \|f\|_1$; see [8], Lemma 3.2. By [15], Ch. II, Theorem 3.2, the Radon transform is injective on $L^1(X)$. Notice that if $Rf \in L^1(B \times A)$ then for almost all $a \in A$ we have $Rf(\cdot, a) \in L^1(B)$.

The Helgason-Fourier transform of a sufficiently regular function f is the Euclidean Fourier transform of the Radon transform. For instance, if $f \in L^1(X)$, then

$$(2.25) \qquad \mathcal{F}f(\lambda, b) = \mathcal{F}_A\big(Rf(b, \cdot)\big)(\lambda) = \int_A Rf(b, a) e^{-i\lambda(\log a)} \, da.$$

for almost all $b \in B$ and all $\lambda \in \mathfrak{a}^*$. See *e.g.* the proof of formula (43), Ch. III, §1 in [15].

2.6. Analysis on $B = K/M$

Let $\widehat{K}$ denote the set of (equivalence classes of) irreducible unitary representations of K. Fix $\delta \in \widehat{K}$ acting on the space V_δ of finite dimension $d(\delta)$. We consider V_δ endowed with an inner product $\langle \cdot, \cdot \rangle$ making δ unitary. Let $M = Z_K(A)$ be the centralizer of A in K. A vector $v \in V_\delta$ is said to be M-fixed if $\delta(m)v = v$ for all $m \in M$. Let V_δ^M denote the subspace of

M-fixed vectors of V_δ. We denote by $\widehat{K}_M$ the subset of $\widehat{K}$ consisting of (equivalence classes of) representations of K for which $\dim V_\delta^M > 0$.

As before, let $B = K/M$. For $f \colon B \to \mathbb{C}$ sufficiently regular, we define $f^\delta \colon B \to \operatorname{Hom}(V_\delta, V_\delta)$ by

$$(2.26) \qquad f^\delta(kM) = d(\delta) \int_K \delta(k_1^{-1}) f(k_1 kM)\, dk_1, \quad k \in K.$$

Then

$$(2.27) \qquad\qquad\qquad f^\delta(kM) = \delta(k) f^\delta(eM)$$

for all $k \in K$. In particular, $f^\delta = 0$ for $\delta \notin \widehat{K}_M$.

The functions f^δ determine $f \in L^2(B)$ by the Peter-Weyl theorem for vector-valued functions, stating that

$$(2.28) \qquad\qquad\qquad f = \sum_{\delta \in \widehat{K}_M} \operatorname{Trace}(f^\delta).$$

See [14], Ch. V, Corollary 3.4. More generally, (2.28) holds in the sense of distributions when $f \in L^1(B)$. See [14], p. 508. Let $v_1, \ldots, v_{d(\delta)}$ be an orthonormal basis of V_δ. For $i, j = 1, \ldots, d(\delta)$, define $f_{i,j}^\delta \in \mathbb{C}$ by

$$(2.29) \qquad f_{i,j}^\delta = d(\delta)^{-1} \left\langle f^\delta(eM) v_i, v_j \right\rangle = \int_K \left\langle \delta(k^{-1}) v_i, v_j \right\rangle f(kM)\, dk.$$

Observe that, by (2.27) and (2.29), we have $f^\delta = 0$ if and only if $f_{i,j}^\delta = 0$ for all $i, j = 1, \ldots, d(\delta)$. Moreover, by (2.28), $f = 0$ if and only if $f^\delta = 0$ for all $\delta \in \widehat{K}_M$.

3. The Euclidean case

3.1. The damped Schrödinger equation on $\mathbb{R}^n$

Let $\mathcal{F}_{\mathbb{R}^n}$ denote the Fourier transform on $\mathbb{R}^n$. For a (sufficiently regular) function f on $\mathbb{R}^n$, we also write $\widehat{f} = \mathcal{F}_{\mathbb{R}^n} f$. We fix as a measure on $\mathbb{R}^n$ the Lebesgue measure divided by the factor $(2\pi)^{n/2}$.

Consider the initial value problem for the time-dependent damped Schrödinger equation on $\mathbb{R}^n$:

$$(\mathrm{S}_c) \qquad \begin{aligned} & i\partial_t u(t, x) + (\Delta - c) u(t, x) = 0 \\ & u(0, x) = f(x) \end{aligned}$$

where Δ denotes the Laplace operator on $\mathbb{R}^n$ and $c \in \mathbb{R}$ is the damping parameter. Suppose $f \in L^2(\mathbb{R}^n)$. Then there is a unique $u \in C(\mathbb{R} \colon L^2(\mathbb{R}^n))$

satisfying (S_c) in the sense of distributions and such that $u(0, \cdot) = f$. In fact, let $|\lambda|$ denote the Euclidean norm of $\lambda \in \mathbb{R}^n$. The Fourier transform $\mathcal{F}_{\mathbb{R}^n}$ gives a unitary equivalence of Δ with the multiplication operator M on $L^2(\mathbb{R}^n)$ defined by $(Mf)(\lambda) = -|\lambda|^2 f(\lambda)$. It follows that the unique solution $u_t(x) = u(t, x)$ of (S_c) is characterized by the equation

$$(3.1) \qquad \widehat{u_t}(\lambda) = e^{-i(|\lambda|^2+c)t} \widehat{f}(\lambda).$$

Equivalently, $u_t = \mathcal{F}_{\mathbb{R}^n}^{-1}\big(e^{-i(|\lambda|^2+c)t} \mathcal{F}_{\mathbb{R}^n} f\big)$. See for instance [19], §3.5, Theorem 1 and Corollary 2.

We will need precise information on the solution of (3.1) in the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions on $\mathbb{R}^n$ for functions f which are not necessarily in $L^2(\mathbb{R}^n)$.

Let $O_M(\mathbb{R}^n)$ denote the space of functions φ on $\mathbb{R}^n$ which are C^∞ with slow growth (*i.e.* each derivative of φ is bounded by a polynomial), and let $O_C'(\mathbb{R}^n)$ be the space of rapidly decreasing distributions T on $\mathbb{R}^n$ (*i.e.* $T \in \mathcal{S}'(\mathbb{R}^n)$ and $T * \varphi \in \mathcal{S}(\mathbb{R}^n)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$). See [22], pp. 243–245. Then the Fourier transform is an isomorphism of $O_M(\mathbb{R}^n)$ onto $O_C'(\mathbb{R}^n)$. For every fixed $a \in \mathbb{R}$, we have $e^{-ia|x|^2} \in O_M(\mathbb{R}^n) \cap O_C'(\mathbb{R}^n)$. The following properties hold.

LEMMA 3.1. — *If* $S \in O_M(\mathbb{R}^n)$, $T \in O_C'(\mathbb{R}^n)$ *and* $F \in \mathcal{S}'(\mathbb{R}^n)$, *then*
(a) $SF \in \mathcal{S}'(\mathbb{R}^n)$ *and* $T * F \in \mathcal{S}'(\mathbb{R}^n)$.
(b) $\widehat{SF} = \widehat{S} * \widehat{F}$ *and* $\widehat{(T * F)} = \widehat{T}\widehat{F}$.

Proof. — See [22], Ch. VII, §8, théorème XV. □

Let $c \in \mathbb{R}$ and $t \neq 0$ be fixed. In $\mathcal{S}'(\mathbb{R}^n)$, we have, with our normalization of the Lebesgue measure:

$$\widehat{\gamma_{c,t}}(\lambda) = e^{-i(|\lambda|^2+c)t}$$

if

$$\gamma_{c,t}(x) = (2|t|)^{-n/2} e^{-ict} e^{-\pi i(\operatorname{sign} t)n/4} e^{i\frac{|x|^2}{4t}},$$

where $\operatorname{sign} t := t/|t|$. See for instance [16], Theorem 7.6.1. Thus part (b) in Lemma 3.1 gives for $F \in \mathcal{S}'(\mathbb{R}^n)$

$$e^{-i(|\lambda|^2+c)t} \widehat{F} = \widehat{\gamma_{c,t}}\widehat{F} = (\gamma_{c,t} * F)^\wedge.$$

Since the Fourier transform is injective on $\mathcal{S}'(\mathbb{R}^n)$, we obtain that the unique solution to the equation

$$\widehat{u_t} = e^{-i(|\lambda|^2+c)t} \widehat{f}$$

in $\mathcal{S}'(\mathbb{R}^n)$ is

$$(3.2) \qquad u_t = \gamma_{c,t} * f.$$

Applying this property to the damped Schrödinger equation, we obtain in particular that, if $f \in L^1(\mathbb{R}^n)$, then for $t \neq 0$ the solution to (S_c) can be written explicitly as

(3.3)
$$u(t,x) = (\gamma_{c,t} * f)(x) = (2\pi)^{n/2}(2|t|)^{-n/2}e^{-ict}e^{-\pi i(\operatorname{sign} t)n/4}e^{i\frac{|x|^2}{4t}} \, \widehat{h}_t\left(\frac{x}{2t}\right)$$

where

(3.4)
$$h_t(y) = e^{i\frac{|y|^2}{4t}} \, f(y).$$

If $f \in L^2(\mathbb{R}^n)$, then (3.3) and (3.4) hold as equalities in $L^2(\mathbb{R}^n)$.

3.2. Beurling-type uncertainty principles in $\mathbb{R}^n$

Beurling's theorem, proven by Hörmander in [17], provides one of the strongest quantitative versions of the uncertainty principles for the Fourier transform on $\mathbb{R}$.

THEOREM 3.2 (Beurling's theorem). — Let $f \in L^1(\mathbb{R})$. If

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)| \, |\widehat{f}(y)| e^{|xy|} \, dx \, dy < \infty$$

then $f = 0$ almost everywhere.

A simplified proof of Theorem 3.2 can be found in the appendix of [4]. Some higher dimensional versions of this theorem have been proven recently by Bagchi and Ray. One of these versions is the following theorem. See [2], Corollary 1.

THEOREM 3.3. — Let $f \in L^1(\mathbb{R}^n)$. Suppose that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)| \, |\widehat{f}(y)| e^{|x||y|} \, dx \, dy < \infty.$$

Then $f = 0$ almost everywhere.

COROLLARY 3.4. — Suppose f, u_t satisfy (3.1). If $f \in L^1(\mathbb{R}^n)$ and there is $t_0 > 0$ so that

(3.5)
$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)| \, |u(t_0, y)| e^{\frac{|x||y|}{2t_0}} \, dx \, dy < \infty$$

then $f = 0$. Hence $u(t, \cdot) = 0$ for all $t \in \mathbb{R}$.

Proof. — According to (3.5) and (3.3), we have

$$+\infty > \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)| \, |u(t_0, y)| e^{\frac{|x||y|}{2t_0}} \, dx \, dy$$

$$= \frac{1}{(2t_0)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| h_{t_0}(x) \widehat{h_{t_0}}\left(\frac{y}{2t_0}\right) \right| e^{\frac{|x||y|}{2t_0}} \, dx \, dy$$

$$= (2t_0)^{n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |h_{t_0}(x)| \, |\widehat{h_{t_0}}(y)| e^{|x||y|} \, dx \, dy$$

Theorem 3.3 gives then $h_{t_0} = 0$ and hence $f = 0$. Thus $u(t, \cdot) = 0$ for all $t \in \mathbb{R}$ by (3.3). $\qquad\square$

COROLLARY 3.5. — *Let $u(t, x) \in C(\mathbb{R} : L^2(\mathbb{R}^n))$ be the solution to the initial value problem (S_c) with initial condition $f \in L^2(\mathbb{R}^n)$. If there is $t_0 > 0$ so that (3.5) holds, then $f = 0$ and hence $u(t, \cdot) = 0$ for all $t \in \mathbb{R}$.*

Proof. — By Corollary 3.4, it suffices to prove that $f \in L^1(\mathbb{R}^n)$. If $u(t_0, \cdot) = 0$ a.e., then $\widehat{f} = e^{i(|\lambda|^2 + c)t_0} \widehat{u_{t_0}} = 0$. Hence $f = 0$. We can therefore suppose that $u(t_0, \cdot)$ is not a.e. zero. By (3.5), we have

$$(3.6) \qquad |u(t_0, y)| \int_{\mathbb{R}^n} |f(x)| e^{\frac{|x||y|}{2t_0}} \, dx < \infty$$

for almost all $y \in \mathbb{R}^n$. Since u_{t_0} does not vanish almost everywhere, (3.6) gives for some $y_0 \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} |f(x)| \, dx \leqslant \int_{\mathbb{R}^n} |f(x)| e^{\frac{|x||y_0|}{2t_0}} \, dx < \infty.$$

Thus $f \in L^1(\mathbb{R}^n)$. $\qquad\square$

4. The case of a Riemannian symmetric space of the noncompact type

4.1. The Schrödinger equation on X

Consider the initial value problem for the time-dependent Schrödinger equation on the Riemannian symmetric space $X = G/K$:

$$(S) \qquad \begin{aligned} i\partial_t u(t, x) + \Delta u(t, x) &= 0 \\ u(0, x) &= f(x) \end{aligned}$$

where Δ is the Laplace-Beltrami operator on X. Suppose $f \in L^2(X)$. Then the analysis of solutions of the Schrödinger equation on $\mathbb{R}^n$ carries out to X when the Fourier transform $\mathcal{F}_{\mathbb{R}^n}$ is replaced by the Helgason-Fourier transform $\mathcal{F}$. In fact, by (2.11), the function $e_{\lambda, b}$ appearing in

the definition of the Helgason-Fourier transform are eigenfunctions of the Laplace-Beltrami operator Δ. Hence, by the Plancherel theorem, $\mathcal{F}$ is a unitary equivalence of Δ with the multiplication operator M on $L^2(\mathfrak{a}^* \times B, \frac{d\lambda \, db}{|c(\lambda)|^2})$ defined by $(Mf)(\lambda, b) = -(|\lambda|^2 + |\rho|^2)f(\lambda, b)$. It follows that there is a unique $u_t(x) = u(t, x) \in C(\mathbb{R}: L^2(X))$ satisfying (S) in the sense of distributions on X and such that $u(0, \cdot) = f$. It is characterized by the equation

$$(4.1) \qquad\qquad (\mathcal{F}u_t)(\lambda, b) = e^{-i(|\lambda|^2 + |\rho|^2)t} \mathcal{F}f(\lambda, b).$$

Equivalently, $u_t = \mathcal{F}^{-1}\big(e^{-i(|\lambda|^2 + |\rho|^2)t} \mathcal{F}f\big)$. We observe, in particular, that $u \in C(\mathbb{R}: \mathcal{S}(X))$ if $f \in \mathcal{S}(X)$. Here $\mathcal{S}(X)$ is the Schwartz space of smooth rapidly decreasing functions on X; see *e.g.* [15], pp. 214–220.

Notice that, as $\mathcal{F}$ is a bijection on $L^2(X)$, the condition $f = 0$ implies $u_t = 0$ for all $t \in \mathbb{R}$. Conversely, suppose there is $t_0 \in \mathbb{R}$ so that $u_{t_0} = 0$. Then (4.1) gives $e^{-i(|\lambda|^2 + |\rho|^2)t_0} \mathcal{F}f(\lambda, b) = 0$ for all (λ, b). Hence $\mathcal{F}f = 0$. Thus $f = 0$ and $u_t = 0$ for all $t \in \mathbb{R}$. This remark applies more generally to solutions of (4.1) in $\mathcal{S}'(X)$, the space of tempered distributions on X.

4.2. The class of functions $L^1(X)_C$

Let Ξ and σ be the functions defined in (2.15) and (2.12), respectively. Let $C \geqslant 0$. For a measurable function $h: X \to \mathbb{C}$ we set

$$\|h\|_{1,C} := \int_X |h(x)| \Xi(x) e^{C\sigma(x)} \, dx.$$

We denote by $L^1(X)_C$ the $\mathbb{C}$-vector space of (equivalence classes of a.e. equal) functions on X for which $\|h\|_{1,C} < \infty$.

The motivation for introducing this class of functions is the following. Suppose f, u_{t_0} satisfy Beurling's condition (1.1). Then $f \in L^1(X)_C$ for some $C \geqslant 0$. Indeed if $u(t_0, \cdot)$ vanishes almost everywhere, then $f = 0$ by (4.1); if $u(t_0, \cdot)$ does not vanish almost everywhere, (1.1) implies

$$\big|u(t_0, y)\big| \Xi(y) \int_X \big|f(x)\big| \Xi(x) e^{\frac{\sigma(x)\sigma(y)}{2t_0}} \, dx \, dy < \infty$$

for almost all $y \in X$, whence the result with $C = \sigma(y)/2t_0$ for any $y \neq o$ such that $u(t_0, y) \neq 0$. Likewise (1.1) implies $u(t_0, \cdot) \in L^1(X)_{C'}$ for some $C' > 0$, and we may assume $C = C'$.

LEMMA 4.1. — *Let h be a measurable function on X and let $C \geqslant 0$ be a constant. Then*

$$(4.2) \qquad \int_X |h(x)| \Xi(x) e^{C\sigma(x)} \, dx = \int_G |h(g \cdot o)| e^{-\rho(H(g))} e^{C\sigma(g)} \, dg.$$

Moreover,

$$(4.3) \qquad \int_B \int_A \big(R|h|\big)(b,a) e^{C\sigma(a)} \, da \, db \leqslant \|h\|_{1,C}.$$

Consequently, the Radon transform Rh of a function $h \in L^1(X)_C$ is almost everywhere defined and $Rh \in L^1(B \times A, e^{C\sigma(a)} da \, db)$.

Proof. — Because of the definition of Ξ in (2.15), we have

$$\int_X |h(x)| \Xi(x) e^{C\sigma(x)} \, dx = \int_G \int_K |h(g \cdot o)| e^{-\rho(H(gk))} e^{C\sigma(g)} \, dg \, dk.$$

Then (4.2) follows by the right-K-invariance of σ and the map $g \mapsto h(g \cdot o)$.

Suppose now that $\int_X |h(x)| \Xi(x) e^{C\sigma(x)} \, dx < \infty$. By (4.2), the integral formula (2.7) for the Iwasawa decomposition and the inequality (2.14), we can write this integral as

$$\iiint_{K \times A \times N} |h(kan \cdot o)| e^{-\rho(\log a)} e^{C\sigma(an)} e^{2\rho(\log a)} \, dk \, da \, dn$$

$$\geqslant \int_K \int_A \left(e^{\rho(\log a)} \int_N |h(kan \cdot o)| \, dn \right) e^{C\sigma(a)} \, da \, dk$$

$$= \int_B \int_A \big(R|h|\big)(b,a) e^{C\sigma(a)} \, da \, db.$$

This proves (4.3). The final property then follows as

$$|Rh(a,b)| \leqslant e^{\rho(\log a)} \int_N |h(kan \cdot o)| \, dn = R|h|(a,b).$$

$\square$

The Radon transform Rh of a function $h \in L^1(X)_C$ is well defined and $Rh \in L^1(B \times A)$. In fact, we also have $h \in L^1(X)$ when C is sufficiently big.

LEMMA 4.2. — *For all $x \in X$, we have*

$$(4.4) \qquad\qquad \Xi(x) e^{|\rho|\sigma(x)} \geqslant 1.$$

Consequently, $L^1(X)_C \subset L^1(X)$ for $C \geqslant |\rho|$.

Proof. — By (2.18), for all $a = \exp H \in \overline{A^+}$ we have $\Xi(a \cdot o) e^{|\rho|\sigma(a \cdot o)} \geqslant \Xi(a \cdot o) e^{\rho(H)} \geqslant 1$. The inequality (4.4) then follows by K-invariance of Ξ and σ and by the decomposition $G = K\overline{A^+}K$. $\square$

Even if $h \in L^1(X)_C$ may not be in $L^1(X)$ unless C is sufficiently big, we have $Rh(b, \cdot) \in L^1(A)$ for almost all $b \in B$ by Lemma 4.1. So we can consider $\mathcal{F}_A\big(Rh(b, \cdot)\big)$. In fact, on $L^1(X)_C$ the equality $\mathcal{F} = \mathcal{F}_A \circ R$ holds. To prove this, we work on K-types. Let $\delta \in \widehat{K}_M$ and let $v_1, \ldots, v_{d(\delta)}$ be a fixed orthonormal basis of the space V_δ of δ. Recall from (2.29) the notation $f_{i,j}^\delta$ for $f \colon B = K/M \to \mathbb{C}$. Recall also that $f^\delta = 0$ if and only if $f_{i,j}^\delta = 0$ for all $i, j = 1, \ldots, d(\delta)$.

LEMMA 4.3. — *Let $C \geqslant 0$ be a constant and let $h \in L^1(X)_C$. For $a \in A$ and $\lambda \in \mathfrak{a}^*$ set*

$$(4.5) \qquad (Rh)_{i,j}^\delta(a) := \big((Rh)(\cdot, a)\big)_{i,j}^\delta = \int_K \big\langle \delta(k^{-1}) v_i, v_j \big\rangle Rh(kM, a) \, dk$$

$$(4.6) \qquad (\mathcal{F}h)_{i,j}^\delta(\lambda) := \big(\mathcal{F}h(\lambda, \cdot)\big)_{i,j}^\delta = \int_K \big\langle \delta(k^{-1}) v_i, v_j \big\rangle \mathcal{F}h(\lambda, kM) \, dk$$

For a function $g \colon \mathfrak{a}^* \to \mathbb{C}$, set $\|g\|_\infty := \sup_{\lambda \in \mathfrak{a}^*} |g(\lambda)|$. Then

$$(4.7) \qquad\qquad \|\mathcal{F}_A\big((Rh)_{i,j}^\delta\big)\|_\infty \leqslant \|h\|_{1,C}$$

$$(4.8) \qquad\qquad \|(\mathcal{F}h)_{i,j}^\delta\|_\infty \leqslant \|h\|_{1,C}$$

Suppose moreover that $Rh \in L^1(B \times A)$ and set

$$(4.9) \qquad \big[\mathcal{F}_A(Rh)\big]_{i,j}^\delta(\lambda) := \int_K \big\langle \delta(k^{-1}) v_i, v_j \big\rangle \mathcal{F}_A\big(Rh(kM, \cdot)\big)(\lambda) \, dk.$$

Then

$$(4.10) \qquad\qquad \big[\mathcal{F}_A(Rh)\big]_{i,j}^\delta = \mathcal{F}_A\big((Rh)_{i,j}^\delta\big)$$

Proof. — By (4.5) we have

$$(4.11) \quad \mathcal{F}_A\big((Rh)_{i,j}^\delta\big)(\lambda) = \int_A \int_K \big\langle \delta(k^{-1}) v_i, v_j \big\rangle Rh(kM, a) e^{i\lambda(\log a)} \, da \, dk.$$

Since δ is unitary and $v_1, \ldots, v_{d(\delta)}$ is an orthonormal basis, we obtain from (4.3):

$$\big|\mathcal{F}_A\big((Rh)_{i,j}^\delta\big)(\lambda)\big| \leqslant \int_A \int_B \big|Rh(b, a)\big| \, db \, da \leqslant \|h\|_{1,C}.$$

This proves (4.7). To prove (4.8), we have by (2.19):

$$\big|\mathcal{F}h(\lambda, b)\big| \leqslant \int_A \int_N \big|h(kan \cdot o)\big| e^{\rho(\log a)} \, da \, dn$$

$$\leqslant \int_A \left(e^{\rho(\log a)} \int_N |h(kan \cdot o)| \, dn \right) da$$

$$= \int_A \big(R|h|\big)(b, a) \, da.$$

Hence

$$|(\mathcal{F}h)_{i,j}^{\delta}(\lambda)| \leqslant \int_B |\mathcal{F}h(\lambda, b)| \, db$$
$$\leqslant \int_B \int_A (R|h|)(b, a) \, da \, db.$$

So (4.8) follows again from (4.3). Finally, (4.10) is a consequence of (4.11) and Fubini's theorem, which applies as $Rh \in L^1(B \times A)$. $\qquad\square$

COROLLARY 4.4. — *Suppose $h \in C_c^{\infty}(X)$. Then for all $i, j = 1, \ldots, d(\delta)$ and $\lambda \in \mathfrak{a}^*$*

$$(4.12) \qquad \left[\mathcal{F}_A(Rh)\right]_{i,j}^{\delta}(\lambda) = (\mathcal{F}h)_{i,j}^{\delta}(\lambda).$$

Proof. — For $h \in C_c^{\infty}(X)$ we have $\mathcal{F}_A \circ R = \mathcal{F}$. $\qquad\square$

PROPOSITION 4.5. — *Let $C \geqslant 0$. Let $\delta \in \widehat{K}_M$ and $v_1, \ldots, v_{d(\delta)}$ be as above. Then for all $h \in L^1(X)_C$*

$$(4.13) \qquad \mathcal{F}_A\big((Rh)_{i,j}^{\delta}\big) = \left[\mathcal{F}_A(Rh)\right]_{i,j}^{\delta} = (\mathcal{F}h)_{i,j}^{\delta}$$

as functions on $\mathfrak{a}^*$. Consequently, $\mathcal{F} = \mathcal{F}_A \circ R$ on $L^1(X)_C$.

Proof. — The first equality is a consequence of (4.10). For the second, let $h \in L^1(X)_C$, and let $h_n \in C_c^{\infty}(X)$ be a sequence converging to h in $L^1(X)_C$. By (4.7), (4.8) and (4.10), for all $i, j = 1, \ldots, d(\delta)$, the sequences $[\mathcal{F}_A(Rh_n)]_{i,j}^{\delta}$ and $(\mathcal{F}h_n)_{i,j}^{\delta}$ converge in $L^{\infty}(\mathfrak{a}^*)$ to $[\mathcal{F}_A(Rh)]_{i,j}^{\delta}$ and $(\mathcal{F}h)_{i,j}^{\delta}$, respectively. But $[\mathcal{F}_A(Rh_n)]_{i,j}^{\delta} = (\mathcal{F}h_n)_{i,j}^{\delta}$ by Corollary 4.4. Hence $[\mathcal{F}_A(Rh)]_{i,j}^{\delta} = (\mathcal{F}h)_{i,j}^{\delta}$ by uniqueness of the limit. Since this is true for all $\delta \in \widehat{K}_M$ and every orthonormal basis $v_1, \ldots, v_{d(\delta)}$ of the space of δ, we conclude that $\mathcal{F} = \mathcal{F}_A \circ R$ on $L^1(X)_C$. $\qquad\square$

We conclude this section by a remark on the convolution $h \times \psi$ of two elements in $L^1(X)_C$ under the additional assumption that ψ is K-invariant. Recall that the convolution of two sufficiently regular functions f_1 and f_2 on X is the function $f_1 \times f_2$ defined on X by $(f_1 \times f_2) \circ \pi = (f_1 \circ \pi) * (f_2 \circ \pi)$. Here $\pi \colon G \to X = G/K$ is the natural projection and $*$ denotes the convolution product of functions on G. This convolution is not commutative.

PROPOSITION 4.6. — *Let $C \geqslant 0$ and let $h, \psi \in L^1(X)_C$. Suppose that ψ is K-invariant. Then $h \times \psi \in L^1(X)_C$. More precisely, we have*

$$\int_X |(h \times \psi)(x)| \Xi(x) e^{C\sigma(x)} \, dx \leqslant \|h\|_{1,C} \|\psi\|_{1,C}.$$

Proof. — In the following we shall employ the same symbol to denote a function on X and the corresponding K-invariant function on G. By definition of convolution products on G,

$$\int_G (|h| * |\psi|)(g)\Xi(g)e^{C\sigma(g)}\,dg = \int_G\int_G |h(g)|\,|\psi(g')|\Xi(gg')e^{C\sigma(gg')}\,dg\,dg'$$

$$\leqslant \int_G\int_G |h(g)|\,|\psi(g')|\Xi(gg')e^{C\sigma(g)}e^{C\sigma(g')}\,dg\,dg'$$

since $\sigma(gg') \leqslant \sigma(g) + \sigma(g')$ and $C \geqslant 0$. Replacing g' by kg', $k \in K$, and using the K-invariance of ψ and σ, the latter integral becomes

$$\int_G\int_G |h(g)|\,|\psi(g')|\Xi(gkg')e^{C\sigma(g)}e^{C\sigma(g')}\,dg\,dg',$$

which does not depend on k. Integration over K leads to

$$\int_G\int_G |h(g)|\,|\psi(g')|\Xi(g)\Xi(g')e^{C\sigma(g)}e^{C\sigma(g')}\,dg\,dg' = \|h\|_{1,C}\|\psi\|_{1,C}$$

in view of the classical functional equation of spherical functions.   □

4.3. The Beurling-type condition for the Schrödinger equation on X

In this section, we shall prove Theorem 1.1. Recall that if $u_t = u(t, \cdot)$ is a solution of (S) with initial condition $f \in L^2(X)$ satisfying

$$(4.14) \qquad \int_X\int_X |f(x)|\,|u(t_0, y)|\Xi(x)\Xi(y)e^{\frac{\sigma(x)\sigma(y)}{2t_0}}\,dx\,dy < +\infty$$

for some $t_0 > 0$, then $f, u_{t_0} \in L^1(X)_C$ for some constant $C > 0$.

Proof of Theorem 1.1. — Observe that, by applying an argument similar to that in the proof of Lemma 4.1, we get

$$(4.15) \int_X\int_X |f(x)|\,|u(t_0, y)|\Xi(x)\Xi(y)e^{\frac{\sigma(x)\sigma(y)}{2t_0}}\,dx\,dy$$

$$= \int_G\int_G |f(g_1 \cdot o)|\,|u(t_0, g_2 \cdot o)|e^{-\rho(H(g_1)+H(g_2))}e^{\frac{\sigma(g_1)\sigma(g_2)}{2t_0}}\,dg_1\,dg_2.$$

By (4.1), to prove that $u(t, \cdot) = 0$ for all $t \in \mathbb{R}$, it is enough to prove that $f = 0$. But by Proposition 4.5, we have $\mathcal{F}f = \mathcal{F}_A(Rf)$ as $f \in L^1(X)_C$ for some $C > 0$. Since $\mathcal{F}$ is an isometry on $L^2(X)$, it therefore suffices to prove that $Rf = 0$. For this, we shall prove that $(Rf)^\delta_{i,j} = 0$ for all $\delta \in \widehat{K}_M$ and $i, j = 1, \ldots, d(\delta)$. This would complete the proof by the comments at the end of Section 2.6.

By proceeding as in the proof of (4.2), we have

$$\int_X \int_X |f(x)|\,|u(t_0,y)|\Xi(x)\Xi(y)e^{\frac{\sigma(x)\sigma(y)}{2t_0}}\,dx\,dy$$

$$\geqslant \int_A \int_B \int_A \int_B (R|f|)(b_1,a_1)(R|u_{t_0}|)(b_2,a_2)e^{\frac{|\log a_1||\log a_2|}{2t_0}}\,da_1\,da_2\,db_1\,db_2$$

$$\geqslant \int_A \int_A \Big(\int_B |Rf(b_1,a_1)|db_1\Big)\Big(\int_B |Ru_{t_0}(b_2,a_2)|db_2\Big)e^{\frac{|\log a_1||\log a_2|}{2t_0}}\,da_1\,da_2.$$

By (4.14) and (4.5), this implies that

$$(4.16) \qquad \int_A \int_A |(Rf)^\delta_{i,j}(a_1)|\,|(Ru_{t_0})^\delta_{i,j}(a_2)|e^{\frac{|\log a_1||\log a_2|}{2t_0}}\,da_1\,da_2 < \infty$$

for all $\delta \in \widehat{K}_M$ and $i,j = 1,\dots,d(\delta)$.

On the other hand, (4.1) together with (4.13) gives us

$$(4.17) \qquad \mathcal{F}_A\big((Ru_t)^\delta_{i,j}\big)(\lambda) = e^{-i(|\lambda|^2+|\rho|^2)t}\mathcal{F}_A\big((Rf)^\delta_{i,j}\big)(\lambda).$$

Now, by Corollary 3.4, we obtain

$$(Rf)^\delta_{i,j} = 0$$

for all $\delta \in \widehat{K}_M$ and $i,j = 1,\dots,d(\delta)$, concluding the result. $\square$

Remark 4.7. — Note that (4.15) give us the Beurling's condition in group terms.

As an immediate corollary of Theorem 1.1, we obtain the following result for compactly supported initial conditions.

COROLLARY 4.8. — *Let $u(t,x) \in C(\mathbb{R}\colon L^2(X))$ denote the solution of (S) with initial condition $f \in L^2(X)$. Suppose that f has compact support. If there is a time $t_0 > 0$ so that $u(t_0,\cdot)$ has compact support. Then $f = 0$ and hence $u(t,\cdot) = 0$ for all $t \in \mathbb{R}$.*

Proof. — It suffices to observe that (1.1) is always satisfied if f and $u(t_0,\cdot)$ are compactly supported. $\square$

Remark 4.9. — For $f \in L^2(X)$ the Radon transform $Rf(b,\cdot)$ as well as the K-types $(Rf)^\delta_{i,j}$ appearing in (4.17) need not be in $L^2(A)$. This can be easily seen for functions f satisfying $\mathcal{F}f = \mathcal{F}_A(Rf)$. Indeed the Euclidean Fourier transform is an isometric isomorphism of $L^2(A)$ onto $L^2(\mathfrak{a}^*)$. According to the Plancherel theorem, the image of $L^2(X)$ under the Helgason-Fourier transform is $L^2(\mathfrak{a}^*_+ \times B, |c(\lambda)|^{-2}\,d\lambda\,db)$. So $Rf(b,\cdot) \in L^2(A)$ for almost all $b \in B$ provided $L^2(\mathfrak{a}^*, |c(\lambda)|^{-2}\,d\lambda) \subset L^2(\mathfrak{a}^*,d\lambda)$. The latter condition depends on the Harish-Chandra's c-function appearing in

the Plancherel measure. Using the properties of the gamma function, one
can prove the asymptotic behavior

$$(4.18) \qquad \frac{1}{|c(\lambda)|^2} \asymp \prod_{\alpha \in \Sigma_0^+} |\langle \lambda, \alpha \rangle|^2 \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \lambda, \alpha \rangle|)^{m_\alpha + m_{2\alpha} - 2}.$$

See *e.g.* [1], Lemma 1. Here $f \asymp g$ means that there exists positive constants
C_1 and C_2 so that $C_1 g(\lambda) \leqslant f(\lambda) \leqslant C_2 g(\lambda)$ for all λ.

In the rank-one case, we have for instance the following result.

LEMMA 4.10. — *Suppose* $\dim \mathfrak{a}^* = 1$. *Let* $h \in L^2(\mathfrak{a}^*, |c(\lambda)|^{-2} \, d\lambda)$ *be
bounded on* $\{\lambda \in \mathfrak{a}^* : |\lambda| \leqslant r\}$ *for some* $r > 0$. *Then* $h \in L^2(\mathfrak{a}^*, d\lambda)$.

Proof. — The asymptotic formula (4.18) gives in this case: $|c(\lambda)|^{-2} \geqslant C$
for $|\lambda| \geqslant r$. If $h \in L^2(\mathfrak{a}^*, |c(\lambda)|^{-2} \, d\lambda)$, then h is square integrable with re-
spect to the Lebesgue measure on $\{\lambda \in \mathfrak{a}^* : |\lambda| \geqslant r\}$. Hence $h \in L^2(\mathfrak{a}^*, d\lambda)$,
as it is bounded on the compact set $\{\lambda \in \mathfrak{a}^* : |\lambda| \leqslant r\}$. $\square$

The boundedness of $\mathcal{F}f(b, \cdot)$ at $\lambda = 0$ for almost all $b \in B$ is obtained
under very weak assumptions on f. Recall for instance that for almost all
$b \in B$, the function $\mathcal{F}f(b, \cdot)$ is even holomorphic in a tube around $\mathfrak{a}^*$. See
Section 2.5. Moreover, in this case $\mathcal{F}f(b, \cdot)$ vanishes at infinity by Fatou's
lemma for $\mathcal{F}$. We can then prove that $Rf \in L^2(A)$ for $f \in L^1(X) \cap L^2(X)$
whenever the root system satisfies the following conditions:

(C)   Either there is no $\alpha \in \Sigma_0^+$ with multiplicity $m_\alpha = 1$, or $2\alpha \in \Sigma^+$.

When condition (C) is met, then we have

$$(4.19) \qquad \prod_{\alpha \in \Sigma_0^+} (1 + |\langle \lambda, \alpha \rangle|)^{m_\alpha + m_{2\alpha} - 2} \geqslant C$$

for all $\lambda \in \mathfrak{a}^*$. The result $Rf \in L^2(A)$ for $f \in L^2(X) \cap L^1(X)$ is then a
consequence of the above discussion and the following lemma.

LEMMA 4.11. — *Let* Σ *be a root system satisfying condition (C). Sup-
pose* $h \in L^2(\mathfrak{a}^*, |c(\lambda)|^{-2} \, d\lambda)$ *is continuous and vanishes at infinity. Then*
$h \in L^2(\mathfrak{a}^*, d\lambda)$.

Proof. — Set $\Pi(\lambda) = \prod_{\alpha \in \Sigma_0^+} \langle \lambda, \alpha \rangle$. By Lemma 5 in [1], the set $\Omega_1 = \{\lambda \in \mathfrak{a}^* : |\Pi(\lambda)| \leqslant 1\}$ has finite Lebesgue measure. Let $B_1 = \{\lambda \in \mathfrak{a}^* : \|\lambda\| \leqslant 1\}$ and $C_1 = \mathfrak{a}^* \setminus B_1$. Write

$$\int_{\mathfrak{a}^*} |h(\lambda)|^2 \, d\lambda = \int_{B_1} |h(\lambda)|^2 \, d\lambda + \int_{\Omega_1 \cap C_1} |h(\lambda)|^2 \, d\lambda + \int_{(\mathfrak{a}^* \setminus \Omega_1) \cap C_1} |h(\lambda)|^2 \, d\lambda.$$

The first integral is finite as h is continuous, hence bounded on the com-
pact B_1. The second integral is also finite as h is continuous and vanishes at

infinity, so it is bounded in C_1, and $\Omega_1 \cap C_1$ is a subset of Ω_1, hence of finite measure. For the convergence of the third integral, we use condition (C). In fact, condition (C) yields (4.19). Moreover, on $\mathfrak{a}^* \setminus \Omega_1$, we have $|\Pi(\lambda)| \geqslant 1$. Thus $|c(\lambda)|^{-2} \geqslant C$ for all $\lambda \in (\mathfrak{a}^* \setminus \Omega_1) \cap C_1$. Consequently,

$$\int_{(\mathfrak{a}^* \setminus \Omega_1) \cap C_1} |h(\lambda)|^2 \, d\lambda \leqslant C^{-1} \int_{(\mathfrak{a}^* \setminus \Omega_1) \cap C_1} |h(\lambda)|^2 \, \frac{d\lambda}{|c(\lambda)|^2} \; < +\infty.$$

$\square$

5. Applications

Let X be a Riemannian symmetric space of the noncompact type. In this section we collect some uniqueness conditions for the solution of the Schrödinger equation (S) on X which can be deduced from Theorem 1.1. They correspond to uncertainty principle conditions of Gelfand-Shilov type, Cowling-Price type and Hardy type. These results are parallel to the classical results of uncertainty principles for the Fourier transform on $\mathbb{R}^n$. Recall from (2.16) that $0 < \Xi(x) \leqslant 1$ for all $x \in X$.

THEOREM 5.1 (Gelfand-Shilov type). — Let $u(t, x) \in C(\mathbb{R} : L^2(X))$ be the solution of (S) with initial condition $f \in L^2(X)$. Suppose there exists positive constants α, β and a time $t_0 > 0$ so that

(5.1)
$$\int_X |f(x)| \Xi(x) e^{\frac{\alpha^p}{p} \sigma^p(x)} \, dx < \infty \quad \text{and} \quad \int_X |u(t_0, x)| \Xi(x) e^{\frac{\beta^q}{q} \sigma^q(x)} \, dx < \infty$$

where $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $2t_0 \alpha \beta \geqslant 1$, then $f = 0$ and hence $u(t, \cdot) = 0$ for all $t \in \mathbb{R}$.

Proof. — We have $\dfrac{\sigma(x)\sigma(y)}{2t_0} \leqslant \alpha\beta\sigma(x)\sigma(y) \leqslant \dfrac{\alpha^p}{p}\sigma^p(x) + \dfrac{\beta^q}{q}\sigma^q(y)$. The inequalities (5.1) imply then Beurling's condition (1.1). $\square$

THEOREM 5.2 (Cowling-Price type). — Let $u(t, x) \in C(\mathbb{R} : L^2(X))$ be the solution of (S) with initial condition $f \in L^2(X)$. Suppose there exists positive constants a, b and a time $t_0 > 0$ so that

(5.2)
$$\int_X \left(|f(x)| e^{a\sigma^2(x)} \right)^p \, dx < \infty \quad \text{and} \quad \int_X \left(|u(t_0, x)| e^{b\sigma^2(x)} \right)^q \, dx < \infty$$

where $1 \leqslant p, q \leqslant \infty$. If $16t_0^2 ab > 1$, then $f = 0$ and hence $u(t, \cdot) = 0$ for all $t \in \mathbb{R}$.

Proof. — Choose A, B so that $0 < A < a$, $0 < B < b$ and $16t_0^2 AB > 1$.
Let p' and q' so that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ and $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. Set $e_\eta(x) = e^{\eta\sigma^2(x)}$. Observe
that, by (2.8) and (2.9), the function $e_\eta \in L^p(X)$ for all $p \in [1, +\infty]$ if $\eta < 0$.
Indeed, by K-invariance of σ,

$$\int_X |e_\eta(x)|^p \, dx \leqslant c \int_{\mathfrak{a}^+} e^{p\eta|H|^2} \, \delta(H) \, dH \leqslant c \int_{\mathfrak{a}^+} e^{p\eta|H|^2 + 2\rho(H)} \, dH < \infty.$$

By (2.16), Hölder inequality and the assumption,

$$\|f \Xi e_A\|_1 \qquad \leqslant \|f e_A\|_1 \leqslant \|f e_a\|_p \|e_{(A-a)}\|_{p'} < \infty,$$
$$\|u(t_0, \cdot) \Xi e_B\|_1 \leqslant \|u(t_0, \cdot) e_B\|_1 \leqslant \|u(t_0, \cdot) e_b\|_q \|e_{(B-b)}\|_{q'} < \infty.$$

The stated result is then a consequence of Theorem 5.1 with $p = q = 2$,
$A = \alpha^2/2$ and $B = \beta^2/2$. $\qquad\square$

Remark 5.3. — Let ψ be a function on X satisfying $\psi(x)e^{\nu\sigma^2(x)} \in$
$L^\infty(X)$ and $\psi(x)^{-1} e^{\nu\sigma^2(x)} \in L^\infty(X)$ for all $\nu < 0$. Because of the strict
inequality $16t_0^2 ab > 1$ in Theorem 5.2, we can replace the functions $e^{\eta\sigma^2(x)}$,
with $\eta \in \{a, b\}$, measuring the growth of f and u_{t_0}, respectively, with
$\psi(x)e^{\eta\sigma^2(x)}$. For instance, we can choose $\psi(x) = \Xi(x)^M (1 + \sigma(x))^N$ for
some fixed integers M and N. Similar remarks apply to Corollary 5.4 and
Theorem 5.5 below.

COROLLARY 5.4. — *Let $u(t, x) \in C(\mathbb{R} : L^2(X))$ be the solution of (S)
with initial condition $f \in L^2(X)$. Suppose there exists positive constants
a, A, b, B and a time $t_0 > 0$ so that for all $x \in X$*

$$(5.3) \qquad |f(x)| \leqslant A e^{-\frac{a^p}{p} \sigma^p(x)} \quad \text{and} \quad |u(t_0, x)| \leqslant B e^{-\frac{b^q}{q} \sigma^q(x)}$$

*where $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $2t_0 ab > 1$, then $f = 0$ and hence
$u(t, \cdot) = 0$ for all $t \in \mathbb{R}$.*

Proof. — Choose α, β so that $0 < \alpha < a$, $0 < \beta < b$ and $2t_0\alpha\beta > 1$.
Then f and $u(t_0, \cdot)$ satisfy (5.1). $\qquad\square$

THEOREM 5.5 (Hardy type). — *Let f be a measurable function on X.
Suppose there exists positive constants A and α so that*

$$(5.4) \qquad\qquad\qquad |f(x)| \leqslant A e^{-\alpha\sigma^2(x)}$$

*for all $x \in X$. Then $f \in L^2(X)$. Let $u(t, x) \in C(\mathbb{R} : L^2(X))$ be the solution
of (S) with initial condition f. Suppose, moreover, that there is a time
$t_0 > 0$ and positive constants B and β so that*

$$(5.5) \qquad\qquad\qquad |u(t_0, x)| \leqslant B e^{-\beta\sigma^2(x)}$$

If $16\alpha\beta t_0^2 > 1$, then $u(t, \cdot) = 0$ for all $t \in \mathbb{R}$.

Proof. — By the K-bi-invariance of σ and the integral formula (2.8) and (2.9), the growth condition on f implies

$$\int_X |f(x)|^2 \, dx \leqslant A^2 \int_X e^{-2\alpha\sigma^2(x)} \, dx \leqslant cA^2 \int_{\mathfrak{a}^+} e^{-2\alpha|H|^2} \delta(H) \, dH$$

$$\leqslant cA^2 \int_{\mathfrak{a}^+} e^{-2\alpha|H|^2 + 2\rho(H)} \, dH < \infty.$$

Thus $f \in L^2(X)$. The uniqueness property is a consequence of Corollary 5.4 with $p = q = 2$ and $\alpha = a^2/2$, $\beta = b^2/2$. □

Remark 5.6. — Under the additional assumptions that f is K-invariant and G is endowed with a complex structure, Theorem 5.5 was proven in [5] using the explicit expression of the elementary spherical functions. In fact, the Euclidean reduction via Radon transform provides an elementary proof of Theorem 5.5 for arbitrary X and f. Observe first that if h is a measurable function on X and there is $C > 0$ for which $|h(x)| \leqslant A \, e^{-C\sigma^2(x)}$ for all $x \in X$, then $h \in L^1(X) \cap L^2(X)$. Suppose now that f and u_{t_0} satisfy (5.4) and (5.5), respectively. Choose a, b so that $0 < a < \alpha$, $0 < b < \beta$ and $16abt_0^2 > 1$. By Proposition 1 in [24], there are constants $A', B' > 0$ so that for all $(H, x) \in \mathfrak{a} \times B$ we have $|Rf(x, \exp H)| \leqslant A' \, e^{-a|H|^2}$ and $|Ru_{t_0}(x, \exp H)| \leqslant B' \, e^{-b|H|^2}$. By (4.1) and the fact that $\mathcal{F} = \mathcal{F}_A \circ R$ on $L^1(X)$, for all $x \in B$, we have that $Ru_t(x, \cdot)$ is the solution of the damped Schrödinger equation on $A \equiv \mathbb{R}^n$ with initial condition $Rf(x, \cdot) \in L^2(\mathbb{R}^n)$ and damping parameter $|\rho|^2$. The Hardy type uniqueness theorem for the Schrödinger equation on $\mathbb{R}^n$ (Theorem 2 in [5]) yields then $Rf(x, \cdot) = 0$ for all $x \in B$. Since R is injective on $L^1(X)$, we conclude that $f = 0$ and hence $u(t, \cdot) = 0$ for all $t \in \mathbb{R}$.

In [5], the value t_0 given in Theorem 5.5 by the inequality $16\alpha\beta t_0^2 > 1$ was proven to be optimal. Indeed, for the symmetric space $X = SL(2, \mathbb{C})/SU(2)$, Chanillo gave the following example. Let f be the function on X which agrees on the maximally flat geodesic submanifold $A \equiv \mathbb{R}$ with the function $e^{-x^2 - ix^2/4}$. Then f satisfies (5.4) with $\alpha = 1$. The solution $u(t, \cdot)$ of the Schrödinger equation (S) with initial condition f is not identically zero even if it satisfies (5.5) for $\beta = 1/16$ at $t_0 = (16\alpha\beta)^{-1/2} = 1$. Thus the uniqueness property fails in this case for some α, β and t_0 with $16\alpha\beta t_0^2 = 1$.

In the rest of this section we provide additional information on the optimality of t_0. We consider some Hardy type uniqueness conditions which are slightly more restrictive than those stated in Theorem 5.5. So they still imply that the solutions of (S) are identically zero provided the condition

$16\alpha\beta t_0^2 > 1$ holds. We then characterize the non-unique solutions in the case $t_0 = (16\alpha\beta)^{-1/2}$ under the additional assumption that G is endowed with a complex structure and the initial condition f is K-invariant.

Observe first that we can modify the right-hand side of the estimates in Theorem 5.5 by a suitable positive bounded factor ψ without loosing the uniqueness property when $16\alpha\beta t_0^2 > 1$. We choose here ψ to be the function used by Harish-Chandra to control the decay of the elements in the K-bi-invariant L^2-Schwartz space on G. See for instance [12], p. 256. Considered as a K-bi-invariant function on G, the function ψ is uniquely defined on G by the condition

$$(5.6) \qquad \psi(\exp H) = \prod_{\gamma\in\Sigma^+} \left(\frac{\gamma(H)}{\sinh\gamma(H)}\right)^{m_\gamma/2}, \qquad H\in\mathfrak{a}.$$

Then ψ is a positive K-invariant function on X which is bounded. More precisely, one can prove that for suitable positive constants c_1, c_2 and non-negative integers d_1, d_2 one has

$$c_1\Xi(x)(1+\sigma(x))^{-d_1} \leqslant \psi(x) \leqslant c_2\Xi(x)(1+\sigma(x))^{d_2}$$

for all $x\in X$. The following result is then a consequence of Theorem 5.5.

COROLLARY 5.7. — *Let f be a measurable function on X and assume there exist positive constants A and α so that*

$$(5.7) \qquad |f(x)| \leqslant A\,\psi(x)e^{-\alpha\sigma^2(x)}$$

for all $x\in X$. Then $f\in L^2(X)$. Let $u(t,x)\in C(\mathbb{R}\colon L^2(X))$ be the solution of (S) with initial condition f. Suppose, moreover, that there is a time $t_0 > 0$ and positive constants B and β so that

$$(5.8) \qquad |u(t_0,x)| \leqslant B\,\psi(x)e^{-\beta\sigma^2(x)}.$$

If $16\alpha\beta t_0^2 > 1$, then $u(t,\cdot) \equiv 0$ for all $t\in\mathbb{R}$.

Suppose now that G has a complex structure and f is a K-invariant function on X. A K-invariant function on X can be identified with a W-invariant function on $A = \exp\mathfrak{a} \equiv \mathfrak{a}$. Here, as before, W denotes the Weyl group. In this identification, the space of K-invariant functions in $L^2(X)$ corresponds to the space of W-invariants in $L^2(\mathfrak{a}, \eta^2(H)\,dH)$, where

$$(5.9) \qquad \eta(H) = \prod_{\gamma\in\Sigma^+} \sinh\gamma(H), \qquad H\in\mathfrak{a}.$$

Moreover, the radial component on $A^+ \equiv \mathfrak{a}^+$ of the Laplace-Beltrami operator Δ on X is

$$(5.10) \qquad \frac{1}{\eta(H)} \left(\Delta_{\mathfrak{a}} - |\rho|^2 \right) \circ \eta(H)$$

where $\Delta_{\mathfrak{a}}$ is the Laplace operator on $\mathfrak{a}$. If $n = \dim \mathfrak{a}$ and $\{H_j\}_{j=1}^n$ is an orthonormal basis of $\mathfrak{a}$ with respect to the inner product induced by the Killing form, then $\Delta_{\mathfrak{a}} = \sum_{j=1}^n \partial(H_j)^2$. Because of (5.10), $u_t = u(t, \cdot)$ is the solution to (S) with K-invariant initial condition $f \in L^2(X)$ if and only if $\eta(H)u_t(H)$ is the solution of the damped Schrödinger equation (S_c) on $\mathfrak{a} \equiv \mathbb{R}^n$ with damping parameter $c = |\rho|^2$ and W-skew-invariant initial condition $\eta(H)f(H)$.

When G admits a complex structure, all root multiplicities m_γ are equal to 2. Writing $\psi(H)$ instead of $\psi(\exp H)$, we therefore have

$$\psi(H) = \frac{\pi(H)}{\eta(H)}, \quad H \in \mathfrak{a},$$

where

$$\pi(H) = \prod_{\gamma \in \Sigma^+} \gamma(H).$$

Let α, β be two positive constants and let

$$f(x) = \psi(x)e^{-\alpha \sigma^2(x)}e^{-i\sqrt{\alpha\beta}\sigma^2(x)}, \quad x \in X.$$

Then f is a K-bi-invariant function in $L^2(X)$. Considered as a function on $A \equiv \mathfrak{a}$, we have

$$f(H) = \frac{\pi(H)}{\eta(H)} e^{-\alpha|H|^2}e^{-i\sqrt{\alpha\beta}|H|^2}, \quad H \in \mathfrak{a}.$$

Let $u(t, x)$ be the solution of (S) with initial condition f. Then $\eta(H)u(t, H)$, $H \in \mathfrak{a}$, is the solution of (S_c) with damping parameter $|\rho|^2$ and initial condition $\eta(H)f(H)$. Set $t_0 = (16\alpha\beta)^{-1/2}$. By (3.3) and (3.4) we have for a constant C_1 depending on t_0

$$\eta(H)u(t_0, H) = C_1 e^{i\frac{|H|^2}{4t_0}} \widehat{h_{t_0}} \left(\frac{H}{2t_0} \right)$$

where

$$h_{t_0}(Y) = \pi(Y)e^{-\alpha|Y|^2}.$$

Since

$$\widehat{h_{t_0}}(H) = \pi(i\partial)\left(e^{-\alpha|Y|^2} \right)^\wedge = C_2 \pi(i\partial)e^{-\frac{|H|^2}{4\alpha}}$$

and $16t_0^2\alpha = \beta^{-1}$, we conclude that

$$\eta(H)u(t_0, H) = C_3 e^{i\frac{|H|^2}{4t_0}} \pi(i\partial)e^{-\beta|H|^2}$$

where C_3 is a constant depending on t_0. Moreover, $\pi(i\partial) = \prod_{\gamma \in \Sigma^+} i\partial(H_\gamma)$, where H_γ is the unique element in $\mathfrak{a}$ so that $\gamma(H) = \langle H_\gamma, H \rangle$ for all $H \in \mathfrak{a}$. Since $\partial(H_\gamma)e^{-\beta|H|^2} = -2\beta\gamma(H)e^{-\beta|H|^2}$, we have $\pi(i\partial)e^{-\beta|H|^2} = P(H)e^{-\beta|H|^2}$ where $P(H)$ is a polynomial of degree $\deg P = \deg \pi$. Observe that $P(H)e^{-\beta|H|^2}$ is a constant multiple of $e^{-i\frac{|H|^2}{4t_0}}\eta(H)u(t_0, H)$. Since $\eta(H)$ is W-skew-invariant, so is $P(H)$. Hence $\pi(H)$ divides $P(H)$. Thus $P(H) = c\pi(H)$ for a constant c. Therefore

$$u(t_0, H) = c_{t_0} \frac{\pi(H)}{\eta(H)} e^{-\beta|H|^2} e^{i\frac{|H|^2}{4t_0}}$$

for all $H \in \mathfrak{a}$, i.e.

$$u(t_0, x) = c_{t_0} \psi(x) e^{-\beta\sigma^2(x)} e^{i\frac{\sigma^2(x)}{4t_0}}$$

for all $x \in X$. Since f and u_t respectively satisfy the estimates (5.7) and (5.8), the uniqueness property in Corollary 5.7 fails when $16t_0^2\alpha\beta = 1$.

In fact, we shall prove in Theorem 5.9 below that the function f we just considered is (up to constant multiples) the only function which can occur in the case $16t_0^2\alpha\beta = 1$. We shall derive this property from the following equality case of Hardy's theorem on $\mathbb{R}^n$; see [25], Theorem 1.4.4.

LEMMA 5.8. — *Suppose h is a measurable function on $\mathbb{R}^n$ that satisfies the estimates*

$$|h(x)| \leqslant C(1 + |x|^2)^m e^{-a|x|^2} \quad and \quad |\widehat{h}(\xi)| \leqslant C(1 + |\xi|^2)^m e^{-b|\xi|^2}$$

for some positive constants a and b. When $ab = \frac{1}{4}$, then $h(x) = P(x)e^{-a|x|^2}$ where P is a polynomial of degree $\leqslant 2m$.

THEOREM 5.9. — *Let $X = G/K$ be a Riemannian symmetric space with G complex. Suppose f is a K-invariant function on X satisfying (5.7). Let $u(t, \cdot)$ be the solution of (S) with initial condition f. Assume that $u(t_0, x)$ satisfies (5.8). If $16\alpha\beta t_0^2 = 1$, then there exists a constant C so that*

$$f(x) = C\,\psi(x)e^{-\alpha\sigma^2(x)}e^{-i\sqrt{\alpha\beta}\sigma^2(x)}$$

for all $x \in X$.

Proof. — It remains to prove that if f and u_{t_0} satisfy (5.7) and (5.8), respectively, then $f(x)$ is a constant multiple of $\psi(x)e^{-\alpha\sigma^2(x)}e^{-i\sqrt{\alpha\beta}\sigma^2(x)}$.

We know that $\eta(H)u_t(H)$ is the solution of the damped Schrödinger equation on $\mathfrak{a}$ with damping parameter $|\rho|^2$ and initial condition $\eta(H)f(H) \in L^2(\mathfrak{a}, dH)$. Hence, by (3.3),

$$\eta(H)u(t_0, H) = (\gamma_{|\rho|^2, t_0} * \eta \cdot f)(H) = C_{t_0} e^{i\frac{|H|^2}{4t_0}} \widehat{h_{t_0}}\left(\frac{H}{2t_0}\right)$$

where
$$h_{t_0}(H) = e^{i\frac{|H|^2}{4t_0}} \eta(H)f(H).$$

Hence
$$|h_{t_0}(H)| = |\eta(H)f(H)| = |\pi(H)\psi(H)^{-1}f(H)| \leqslant A|\pi(H)|e^{-\alpha|H|^2}$$

and
$$\left|\widehat{h_{t_0}}\left(\frac{H}{2t_0}\right)\right| = C_{t_0}^{-1}|\pi(H)\psi(H)^{-1}u(t_0, H)| \leqslant BC_{t_0}^{-1}|\pi(H)|e^{-\beta|H|^2}.$$

So
$$|\widehat{h_{t_0}}(\xi)| \leqslant C'_{t_0}|\pi(\xi)|e^{-4t_0^2\beta|\xi|^2}.$$

Let m be the smallest integer such that $2m \geqslant |\Sigma^+|$. Since $|\pi(H)| \leqslant C(1+|H|^2)^m$ and $\alpha(4t_0^2\beta) = 1/4$, we obtain from Lemma 5.8 that $h_{t_0}(H) = P(H)e^{-\alpha|H|^2}$ where $P(H)$ is a polynomial of degree $\leqslant 2m$. Therefore $e^{i\frac{|H|^2}{4t_0}}\eta(H)f(H) = P(H)e^{-\alpha|H|^2}$. This equality shows that $P(H)$ must be W-skew-invariant, so divisible by $\pi(H)$. Hence $P(H) = q(H)\pi(H)$ for a W-invariant polynomial $q(H)$ so that $\deg q = \deg P - \deg \pi \leqslant 2m - |\Sigma^+| \leqslant 1$. This is only possible when $q(H) = C$ is a constant. Thus $f(H) = C\frac{\pi(H)}{\eta(H)}e^{-i\frac{|H|^2}{4t_0}}e^{-\alpha|H|^2}$, which proves the claim. $\qquad\square$

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