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ANALYTIC TORSIONS ON CONTACT MANIFOLDS

by Michel RUMIN & Neil SESHADRI (*)

Abstract. — We propose a definition for analytic torsion of the contact complex on contact manifolds. We show it coincides with Ray–Singer torsion on any 3-dimensional CR Seifert manifold equipped with a unitary representation. In this particular case we compute it and relate it to dynamical properties of the Reeb flow. In fact the whole spectral torsion function we consider may be interpreted on CR Seifert manifolds as a purely dynamical function through Selberg-like trace formulae, that hold also in variable curvature.

Résumé. — Nous définissons et étudions la torsion analytique du complexe de contact sur les variétés de contact. Nous montrons qu'elle coïncide avec la torsion de Ray–Singer sur les variétés CR de Seifert munies d'une représentation unitaire. Nous la calculons dans ces cas et l'exprimons à l'aide de propriétés dynamiques du flot de Reeb. En fait, notre fonction spectrale de torsion analytique coïncide avec une fonction zêta dynamique naturelle. Ces formules de trace « à la Selberg » persistent ici pour des métriques de courbure non constante sur la base.

1. Introduction

As introduced by Ray and Singer in [33], the analytic torsion of a compact Riemannian manifold M may be seen as an infinite-dimensional analogue, on the de Rham complex $(\Omega^* M, d)$, of the Reidemeister–Franz torsion of a finite simplicial complex. More precisely, for $\lambda \geqslant 0$, let E^k_λ be the $]0,\lambda]$-spectral space of Hodge–de Rham Laplacian Δ_k on k-forms. Then the cut-off subcomplex (E^*_λ, d) is finite-dimensional and its Reidemeister–Franz torsion satisfies

(1.1)
$$2\ln \tau_R(E^*_\lambda, d) = \ln\Big(\prod_{k=0}^{n} \det(\Delta_{k\lceil E^k_\lambda})^{(-1)^{k+1}k}\Big) = \sum_{k=0}^{n} (-1)^k k \zeta'(\Delta_{k\lceil E^k_\lambda})(0)\,,$$

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where $\zeta(\Delta_{k\lceil E_\lambda^k})(s) = \mathrm{Tr}\big(\Delta_{k\lceil E_\lambda^k}^{-s}\big)$ is the truncated zeta function of Δ_k on E_λ^k. Taking these (E_λ^*, d) as successive 'approximations' to the full de Rham complex, Ray–Singer defined the analytic torsion T_{RS} as being

$$T_{RS} = \exp\Big(\frac{1}{2}\sum_{k=0}^{n}(-1)^k k\zeta'(\Delta_k)(0)\Big)\,, \tag{1.2}$$

while the Ray–Singer metric on $\mathcal{L} = \det H^*(\Omega^* M, d)$ is given by

$$\|\quad\|_{RS} = (T_{RS})^{-1}\,|\quad|_{L^2(\Omega^* M)}, \tag{1.3}$$

from the L^2 metric induced on $\mathcal{L}$ via identification of the cohomology by harmonic forms in $\Omega^* M$.

The first purpose of this work is to adapt this idea to the contact complex $(\mathcal{E}^*, d_H)$, a hypoelliptic differential form complex naturally defined ([36, 37]) on contact manifolds (M, H) of dimension $2n+1$. A specific feature of this complex is that the differential $D = d_H : \mathcal{E}^n \to \mathcal{E}^{n+1}$ in 'middle degree' is a second-order operator, which is due to a slower spectral sequence convergence at this degree; see [38, Proposition 3.3]. In order to find, as above, finite-dimensional cut-off subcomplexes (E_λ^k, d_H) approximating $(\mathcal{E}^*, d_H)$, we are led to consider *fourth*-order Laplacians Δ_k in *all* degrees k; see (2.8). The Reidemeister–Franz torsion of each cut-off subcomplex is then easily written (see Proposition 2.9) as

$$\begin{aligned} 4\ln \tau_R(E_\lambda^*, d_H) &= \ln\Big(\prod_{k=0}^{2n+1} \det(\Delta_{k\lceil E_\lambda^k})^{(-1)^{k+1}w(k)}\Big) \\ &= \sum_{k=0}^{2n+1}(-1)^k w(k)\zeta'(\Delta_{k\lceil E_\lambda^k})(0)\,, \end{aligned}$$

with $w(k) = k$ for $k \leqslant n$ and $w(k) = k+1$ for $k > n$ being the natural contact-weight of forms in $\mathcal{E}^k$; compare with (1.1). This leads us to define a candidate for the analytic torsion of the full contact complex by setting

$$T_C = \exp\Big(\frac{1}{4}\sum_{k=0}^{2n+1}(-1)^{k+1}w(k)\zeta'(\Delta_k)(0)\Big)\,.$$

We define also a torsion function

$$\kappa(s) = \frac{1}{2}\sum_{k=0}^{2n+1}(-1)^{k+1}w(k)\zeta(\Delta_k)(s)\,,$$

and a Ray–Singer metric $\|\quad\|_C$ on the determinant of the cohomology $\det H^*(\mathcal{E}, d_H)$,

$$\|\quad\|_C = T_C\,|\quad|_{L^2(\mathcal{E})}\,.$$

(Our convention for T_C is inverse to Ray–Singer's original definition (1.2), but is standard now since it is natural at the metric level, as compared to (1.3); see e.g. [13, 11].)

Having thus defined a torsion upon geometric and algebraic bases, we start then its analytical study. We first establish in Theorem 3.4 and Corollary 3.7 general variational formulae for $\kappa(0)$, $T_C = \exp(\kappa'(0)/2)$ and the Ray–Singer 'contact' metric $\| \quad \|_C$. It turns out that $\kappa(0)$ is a *contact* invariant given by the integration of an unknown universal polynomial in local curvature data. Its vanishing is necessary in order for the Ray–Singer metric to be even scale invariant under change of contact form $\theta \mapsto K\theta$ for K constant. We do not know whether $\kappa(0)$ vanishes in general except in dimension 3, as shown in Corollary 3.8. Therefore the rest of the paper deals with this lowest-dimensional case.

Corollary 3.8 also states that there exist 'universal' CR-invariant and contact-invariant 'corrections' to this Ray–Singer metric. Namely there exist universal constants $(C_i)_{1\leqslant i\leqslant 4}$ such that, on any contact manifold of dimension 3,

$$\| \quad \|_{\mathrm{CR}} = \exp\left(C_1 \int_M R^2\, \theta\wedge d\theta + C_2 \int_M |A|^2 \theta\wedge d\theta\right) \| \quad \|_C\,,$$

is a CR-invariant (i.e. independent of contact form) metric on $\det H^*(\mathcal{E}, d_H)$, where R and A are the Tanaka–Webster scalar curvature and torsion. Moreover

$$\| \quad \|_H^{\nu} = \exp(C_3\nu(M)) \| \quad \|_{\mathrm{CR}} \quad \text{and} \quad \| \quad \|_H^{D} = \exp(C_4\overline{\eta}(D*)) \| \quad \|_{\mathrm{CR}}$$

are contact-invariant metrics, where $\nu(M)$ is the ν-invariant of Biquard–Herzlich [6], and $\overline{\eta}(D*)$ is the CR-invariant correction to $\eta(D*)$; see [7, Theorem 9.4].

We do not know the values of the constants C_i. Though they are all related to the Heisenberg symbol of the Laplacians we consider, they might be difficult to compute: first since the Δ_k are fourth-order, but also because the Heisenberg symbolic calculus ([4, 21], §3.1) suitable for the hypoelliptic contact complex, is highly non-commutative.

Our next purpose is to compare the two analytic torsions and metrics coming from the de Rham and contact complexes. It is natural to expect they are related. Indeed, firstly these complexes have the same cohomology, being homotopy equivalent (Proposition 2.2), and in particular the determinants $\det H^*(\Omega^*M, d)$ and $\det H^*(\mathcal{E}, d_H)$ are canonically isomorphic. Moreover, from the point of view of spectral geometry, the non-exploding part of the Hodge–de Rham spectrum converges towards the contact complex spectrum when one takes the sub-Riemannian (or diabatic) limit $\varepsilon \searrow 0$ of calibrated metrics $g_\varepsilon = d\theta(\cdot, J\cdot) + \varepsilon^{-1}\theta^2$; see [38, 7]. Note that the classical Ray–Singer metric stays constant in this limit, being independent of the metric on M.

However, we cannot prove equality of metrics in general, but only on particular contact manifolds called *CR Seifert manifolds* in [7]. These are CR manifolds (M, H, J) of dimension 3 admitting a transverse locally free circle action preserving the CR structure (H, J); see Definition 4.1. The generator $T = d/dt$ of the circle action is the Reeb field of an invariant contact form θ. On such a manifold M, endowed with any unitary representation $\rho : \pi_1(M) \to U(N)$, Theorem 4.2 states that the two Ray–Singer metrics of the twisted de Rham and contact complexes coincide.

In the last part of this work we analyse in detail the torsion function $\kappa(s)$ for CR Seifert manifolds. It first turns out that $\kappa(s)$ is a dynamical function in this case, depending only on the topology of M, together with the holonomies of the representation along the various primitive closed orbits of the circle action, as stated in formula (5.7) and Theorem 5.4.

Specialising to $s = 0$ leads in Theorem 5.7 to an explicit formula for the Ray–Singer torsion and metric, twisted by any unitary representation. This Lefschetz-type formula extends a formula given by David Fried [17] in the *acyclic case*, i.e. $H^*(M, \rho) = \{0\}$, via topological methods and Reidemeister–Franz torsion. Fried interprets it as the identity

$$T_{RS}(M, \rho) = \left|\exp(Z_F(0))\right| , \tag{1.4}$$

where $Z_F(0)$ stands for the analytic continuation at $s = 0$ of the dynamical function

$$Z_F(s) = -\sum_C \operatorname{ind}(C) \operatorname{Tr}(\rho(C)) e^{-s\ell(C)} .$$

Here the sum describes all free homotopical classes of closed orbits of the Reeb flow T, $\operatorname{ind}(C)$ denotes its Fuller index (Proposition 5.8), $l(C)$ its length and $\rho(C)$ its holonomy.

Our approach leads to another viewpoint on (1.4). Namely, we show in Theorem 5.9 and (5.31) that our purely spectral torsion function $\kappa(s)$ may be seen as a dynamical zeta function, *in its whole*. Indeed it holds for any unitary representation that

$$\Gamma(s)\big(\kappa(s) - \kappa(M, \rho)\big) = \frac{2^{1-2s}}{\sqrt{\pi}} \Gamma(\frac{1}{2} - s) Z_\rho(2s) ,$$

for

$$Z_\rho(s) = \sum_C \operatorname{ind}(C)\, {}^{\mathrm{R}}\mathrm{Tr}(\rho(C))\, \ell(C)^s ,$$

where again the sum runs over all free homotopical classes of closed orbits of the Reeb flow, ${}^{\mathrm{R}}\mathrm{Tr}$ is the real part of the trace, and $\kappa(M,\rho) = 2\dim H^0(M,\rho) - \dim H^1(M,\rho)$ is a purely cohomological term.

This Selberg-type trace formula also has a counterpart at the level of heat kernels. Indeed let $\mathrm{Tr}_\kappa(e^{-t\Delta}) = 2\,\mathrm{Tr}(e^{-t\Delta_0}) - \mathrm{Tr}(e^{-t\Delta_1})$; then we show in Theorem 5.10 that

$$\mathrm{Tr}_\kappa(e^{-t\Delta}) = \dim V \frac{\sqrt{\pi}\chi(\Sigma)}{\sqrt{t}} + \frac{1}{\sqrt{\pi t}} \sum_C \ell(C)\,\mathrm{ind}(C)\,{}^{\mathrm{R}}\mathrm{Tr}(\rho(C)) e^{-\ell(C)^2/4t},$$

where $\chi(\Sigma)$ is the rational Euler class of the quotient surface orbifold $\Sigma = M/\mathbb{S}^1$. Hence our torsion heat trace (of fourth-order Laplacians) is closely related to a dynamical theta function. Such trace formulae are invariant under a contact form rescaling θ in $C\theta$ and don't hold using the usual Riemannian spectrum. They hold even if the curvature of Σ is not constant.

The second trace formula has some surprising consequences for the small time development of $\mathrm{Tr}_\kappa(e^{-t\Delta})$ on CR Seifert manifolds, but also on general 3-dimensional contact manifolds, as given in Corollaries 5.12 and 5.13.

The paper is organised as follows. In §2 we first review one construction of the contact complex $(\mathcal{E}^*, d_H)$ and recall the Ray–Singer argument, from the viewpoint of the Ray–Singer metric on the determinant of the cohomology. We then adapt this argument to the contact complex, which leads us to define the analytic torsion T_C, a torsion function κ and a Ray–Singer metric $\|\ \ \|_C$ on $\det H(\mathcal{E}^*, d_H)$.

In §3 we start the analytic study of this torsion. After reviewing relevant properties of hypoelliptic zeta functions and heat kernels, we establish variational formulae for $\kappa(0)$, the torsion T_C and the contact Ray–Singer metric $\|\ \ \|_C$. We then show that $\kappa(0) = 0$ in dimension 3, and introduce corrections of the metric $\|\ \ \|_C$ that give CR and contact invariants.

In §4 we compare Ray–Singer analytic torsion to ours and show that the two Ray–Singer metrics coincide on CR Seifert manifolds.

The final §5 is devoted to the study of the dynamical aspects of the torsion function of the contact complex, still on CR Seifert manifolds.

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2. Contact analytic torsion via a determinant bundle

Ray and Singer [33] defined analytic torsion of the de Rham complex as an infinite-dimensional analogue of the Reidemeister–Franz torsion of finite simplicial complexes. Our purpose in this section is to describe their argument and adapt it to a similar complex defined on contact manifolds, the construction of which we now review.

2.1. Contact complex

Let (M, H) be a smooth orientable contact manifold of dimension $2n+1$. This means that the smooth *contact distribution* $H \subset TM$ is given as the null space of a globally defined 1-form, called a *contact form*, satisfying the condition of maximal non-integrability $\theta \wedge (d\theta)^n \neq 0$. The contact forms comprise an equivalence class under multiplication by smooth non-vanishing functions.

The *contact complex* ([36, 37]) is a refinement of the de Rham complex on contact manifolds defined as follows. Let $\Omega^* M$ denote sections of the graded bundle of smooth differential forms on M, $\mathcal{I}$ the ideal in $\Omega^* M$ generated by θ and $d\theta$, and $\mathcal{J}$ the ideal in $\Omega^* M$ consisting of elements annihilated by θ and $d\theta$. One verifies that $\mathcal{I}^k = \Omega^k M$ for $k \geqslant n+1$, $\mathcal{J}^k = 0$ for $k \leqslant n$, and that the de Rham exterior derivative d naturally induces operators d_H to form two complexes

$$\Omega^0 M \xrightarrow{d_H} \Omega^1 M/\mathcal{I}^1 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^n M/\mathcal{I}^n$$

and

$$\mathcal{J}^{n+1} \xrightarrow{d_H} \mathcal{J}^{n+2} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{J}^{2n+1}.$$

It is clear that these two complexes are defined independently of the choice of θ. These two complexes are joined by a second-order differential operator $D : \Omega^n M/\mathcal{I}^n \to \mathcal{J}^{n+1}$ defined by setting $D[\alpha] = d\beta$, where $\beta \in \Omega^n M$ is defined by the following:

LEMMA 2.1 ([36, 37]). — *Let $\alpha \in \Omega^n M$. Then there exists a unique $\beta \in \Omega^n M$ such that $\beta \equiv \alpha$ (mod θ) and $\theta \wedge d\beta = 0$. Moreover $d\beta \in \mathcal{J}^{n+1}$, and if $\alpha \in \mathcal{I}^n$ then $d\beta = 0$.*

One can show that D may in fact be defined independently of the choice of θ. The contact complex is

$$\Omega^0 M \xrightarrow{d_H} \Omega^1 M/\mathcal{I}^1 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^n M/\mathcal{I}^n$$
$$\xrightarrow{D} \mathcal{J}^{n+1} \xrightarrow{d_H} \mathcal{J}^{n+2} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{J}^{2n+1}.$$

We also have:

PROPOSITION 2.2 ([37, p. 286]). — *The contact complex forms a resolution of the constant sheaf $\mathbb{R}$ and hence its cohomology coincides with the de Rham cohomology of M. Moreover the canonical projection $\pi : \Omega^k M \to \Omega^k M/\mathcal{I}^k$ for $k \leqslant n$ and injection $i : \mathcal{J}^k \to \Omega^k M$ for $k \geqslant n+1$ induce an isomorphism between the two cohomologies.*

The arguments being purely local, these results also apply to twisted versions of the complex with a flat bundle, as coming from a representation $\rho : \pi_1(M) \to U(N)$.

It is a basic fact that the symplectic bundle $(H, d\theta)$ admits a contractible homotopy class of *calibrated* almost complex structures, i.e. $J \in \mathrm{End}(H)$ is in this class if and only if $J^2 = -1$ and the *Levi metric* $d\theta(\cdot, J\cdot)$ is positive definite and Hermitian.

The *Reeb field* of θ is the unique vector field T satisfying $\theta(T) = 1$ and $T \lrcorner\, d\theta = 0$. Fixing a θ and a J, we may define a Riemannian metric g on M by using the Levi metric on vectors in H and declaring that the Reeb field T is of unit-length and orthogonal to H, i.e.

$$g = d\theta(\cdot, J\cdot) + \theta^2\,. \tag{2.1}$$

With these choices, one can identify the quotients of forms appearing in the lower-half of the contact complex with primitive horizontal forms:

$$\Omega^k M/\mathcal{I}^k \cong \{\alpha \in \Omega^k H \mid \Lambda\alpha \ = 0\} = \mathcal{E}^k, \tag{2.2}$$

where $\Omega^k H$ is the space of partially defined forms along H, and Λ is the adjoint of the operator $L : \Omega^k H \to \Omega^{k+2} H$, $L\alpha = d\theta \wedge \alpha$. As observed in [38, Remark 5.4] one has:

Proposition 2.3. — *The bundles $\mathcal{E}^k$ and the isomorphisms* (2.2) *only depend on H.*

Indeed, one has classically $\ker\Lambda = \ker L^{n-k+1}$ on $\Omega^k H$, which is independent from J and θ, since $L \mapsto fL$ when $\theta \mapsto f\theta$. Then the projections on $\ker\Lambda$ along $\operatorname{im} L$ are also contact-invariant. For $k \geqslant n+1$, we will write $\mathcal{E}^k = \mathcal{J}^k$ which are clearly contact-invariant sub-bundles of $\Omega^k M$.

We henceforth assume that M is compact. With the identification above we now have an L^2 inner product defined on the contact complex. Let δ_H, D^* denote the formal adjoint operators of d_H, D. It is straightforward to verify that

$$\delta_{H|\mathcal{E}^k} = (-1)^k * d_H *, \quad D^* = (-1)^{n+1} * D*, \tag{2.3}$$

where $* : \mathcal{E}^k \overset{\cong}{\to} \mathcal{E}^{2n+1-k}$ is induced by the usual Hodge $*$ operator.

As a last comment here, we mention there exist other approaches to this elementary construction of the contact complex. One possibility is via spectral sequence considerations, using a canonical filtration by Heisenberg weight of forms $\Omega^* M$; see [38, §3] and [7] where this approach is used in the study of the sub-Riemannian (diabatic) limit of the Hodge–de Rham spectrum. Another interesting viewpoint is to consider the contact complex as a curved version of a Bernstein–Gelfand–Gelfand complex in parabolic geometry; see [2, §8.1] for such a presentation on the 3-dimensional Heisenberg group.

2.2. Determinant bundles, metrics and Reidemeister–Franz torsion

We follow the presentation of Bismut and Zhang [13] to define the Reidemeister–Franz torsion of a finite-dimensional complex.

Let E be a finite-dimensional real vector space, and define the line

$$\det E = \wedge^{\max} E\,.$$

A useful convention here is to set $\det\{0\} = \mathbb{R}$ (compatible with $\det(E \oplus F) = \det E \otimes \det F$). If λ is a line, let $\lambda^{-1} = \lambda^*$ be its dual line. Then $\lambda \otimes \lambda^{-1} = \operatorname{End}(\lambda) = \mathbb{R}\operatorname{Id}$ is canonically isomorphic to $\mathbb{R}$.

One extends these notions to a finite-dimensional complex. Let

$$(E, d) \ : \ 0 \longrightarrow E_0 \xrightarrow{d} E_1 \xrightarrow{d} \cdots \xrightarrow{d} E_n \longrightarrow 0$$

be such a complex and $H^*(E,d)$ its cohomology. Define

$$\det E = \bigotimes_{k=0}^{n} (\det E_k)^{(-1)^k}$$

and

$$\det(H^*(E,d)) = \bigotimes_{k=0}^{n} (\det H^k(E,d))^{(-1)^k}.$$

Proposition 2.4 (Knudsen–Mumford [25]). — *The lines* $\det E$ *and* $\det(H^*(E,d))$ *are canonically isomorphic.*

Proof. — We include the proof as the explicit form of the isomorphism will be useful below. We follow [10]. Suppose first that $H^*(E,d) = \{0\}$ so that $\det H^*(E,d) = \mathbb{R}$. Then we need to find a canonical section of $\det E$. Let $N_k = \dim E_k$. Pick a non-vanishing element

$$s_0 = e_1 \wedge e_2 \wedge \cdots \wedge e_{N_0} \in \wedge^{N_0} E_0 = \det E_0\,; \tag{2.4}$$

then

$$ds_0 = de_1 \wedge de_2 \wedge \cdots \wedge de_{N_0} \in \wedge^{N_0} E_1 \tag{2.5}$$

is non-vanishing since $d : E_0 \to E_1$ is injective.

Next pick $s_1 \in \wedge^{N_1 - N_0} E_1$ such that $ds_0 \wedge s_1$ generates $\det E_1$, and so on, taking $s_k \in \wedge^{N_i - \cdots + (-1)^k N_0} E_i$ such that $ds_{k-1} \wedge s_k$ generates $\det E_k$. Consider now

$$S(E,d) = s_0 \otimes (ds_0 \wedge s_1)^{-1} \otimes (ds_1 \wedge s_2) \otimes \cdots \otimes (ds_{n-1})^{(-1)^n} \in \det E. \tag{2.6}$$

It is clear that the class $S(E,d)$ is non-zero and does not depend on the choices of s_k for $k = 0, \ldots, n$, completing the proof of the proposition in the acyclic case.

For the general case, observe that the determinants of the short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & dE_k & \to & \ker d_{|E_{k+1}} & \to & H^{k+1}(E,d) & \to & 0 \\ 0 & \to & \ker d_{|E_{k+1}} & \to & E_{k+1} & \to & dE_{k+1} & \to & 0 \end{array}$$

each have a canonical element, as was just shown above. So we have canonical isomorphisms

$$\begin{aligned} \det(\ker d_{|E_{k+1}}) &\cong \det(dE_k) \otimes \det(H^{k+1}(E,d)) \\ \det(E_{k+1}) &\cong \det(\ker d_{|E_{k+1}}) \otimes \det(dE_{k+1}), \end{aligned}$$

and then

$$\det(E_{k+1}) \cong \det(dE_k) \otimes \det(H^{k+1}(E,d)) \otimes \det(dE_{k+1}).$$

Finally taking tensor products over k gives

$$\det E_0 \otimes (\det E_1)^{-1} \otimes \det E_2 \otimes \cdots \cong \det H^0 \otimes (\det H^1)^{-1} \otimes \det H^2 \otimes \cdots$$

canonically. $\square$

Suppose now E is given a metric g. Hence $\det E$ has an induced metric. One can then define a metric on $\det H^*(E,d)$ by

$$\|\ \ \|_{\det H^*(E,d)} = \|\ \ \|_{\det E},$$

using the canonical isomorphism given by Proposition 2.4.

Let $d^* = \delta$ be the adjoint of d. By finite-dimensional Hodge theory, $H^*(E,d)$ identifies with the harmonic forms

$$\mathcal{H}^*(E,d) = \{s \in E \mid ds = d^*s = 0\}\,.$$

By their inclusion in E, the harmonic forms inherit a metric. We then have a a second metric $|\ \ |_{\det H^*(E,d)}$ on $\det H^*(E,d)$ via the above identification.

Definition 2.5. — *The* torsion *of the complex (E,d) with metric g is the ratio*

$$\tau(E,d,g) = \frac{\|\ \ \|_{\det H^*(E,d)}}{|\ \ |_{\det H^*(E,d)}}\,.$$

Remark 2.6. — Note that this definition of torsion, given in [13, §2] for instance, is quite natural at the metric level, but actually leads to the *inverse* of the original Reidemeister–Franz torsion (or R-torsion), i.e.

$$\tau_R(E,d,g) = 1/\tau(E,d,g)\,;$$

see [33, §1] and [17, §2].

One can be more explicit using the proof of Proposition 2.4. Consider

$$F = \mathcal{H}^*(E,d)^{\perp}\,.$$

The complex (F,d) is acyclic so we can construct the canonical class $S(F,d)$ as in (2.6).

Proposition 2.7. — *Let $P_k = \det\big(d^*d \mid E_k \cap (\ker d)^{\perp}\big)$. Then it holds*

$$\tau(E,d,g) = \|S(F,d)\|_{\det F} = \prod_{k=0}^{n-1} P_k^{(-1)^{k+1}/2}\,. \tag{2.7}$$

Proof. — The splitting $E = \mathcal{H}^*(E,d) \oplus F$ induces the canonical isomorphism

$$\begin{aligned} \det \mathcal{H}^*(E,d) &\xrightarrow{\cong} \det E = \det \mathcal{H}^*(E,d) \otimes \det F \\ s &\longmapsto s \otimes S(F,d)\,. \end{aligned}$$

Then

$$\|s \otimes S(F,d)\|_{\det E} = |s|_{\det H^*(E,d)} \|S(F,d)\|_{\det F}\,,$$

and by (2.6) and Definition 2.5

$$\begin{aligned}\tau(E,d,g) &= \|S(F,d)\|_{\det F} \\ &= \|s_0\| \times \|ds_0 \wedge s_1\|^{-1} \times \|ds_1 \wedge s_2\| \times \cdots \times \|ds_{n-1}\|^{(-1)^n} \\ &= \prod_{k=0}^{n-1} P_k^{(-1)^{k+1}/2}\,,\end{aligned}$$

since $\|ds_k \wedge s_{k+1}\| = \|ds_k\|\|s_{k+1}\| = P_k^{1/2}\|s_k\|\|s_{k+1}\|$ if choosing $s_k \in \det(\ker d)^{\perp}$. □

At this point in the Riemannian case ([33, 13]) one can guess the correct formula for analytic torsion by considering the Reidemeister–Franz torsion of finite-dimensional subcomplexes that approximate the infinite-dimensional de Rham complex (Ω^*, d). A natural choice of subcomplexes is obtained here by taking cut-off de Rham complexes using the spectrum of the Hodge–de Rham Laplacian $\Delta = d\delta + \delta d$; that is, one considers the energy levels $\Omega^*_{[0,\lambda]} = \{\Delta \leqslant \lambda\}$. One then expresses (2.7) using the determinants of Δ on $(\Omega^*_{[0,\lambda]}, d)$ and finally as combinations of differentiated zeta functions $\zeta'(\Delta)(0)$ in the limiting infinite-dimensional case. We carry out this procedure for the contact complex next.

2.3. Defining a contact analytic torsion

Consider now the contact complex

$$(\mathcal{E}, d_H) \;:\; \mathcal{E}^0 \xrightarrow{d_H} \mathcal{E}^1 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{E}^n \xrightarrow{D} \mathcal{E}^{n+1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{E}^{2n+1}.$$

We want to define finite-dimensional subcomplexes of the contact complex via finite energy cut-offs for a certain Laplacian Δ.

We shall use the following uniformly fourth-order Laplacian:

$$\Delta := \begin{cases} (d_H\delta_H + \delta_H d_H)^2 & \text{on } \mathcal{E}^k \text{ for } k \neq n, n+1 \\ (d_H\delta_H)^2 + D^*D & \text{on } \mathcal{E}^n \\ DD^* + (\delta_H d_H)^2 & \text{on } \mathcal{E}^{n+1}. \end{cases} \tag{2.8}$$

We denote by Δ_k the restriction of Δ to $\mathcal{E}^k$. The rationale behind our choice of Δ is as follows. In middle degrees, because D is second-order, one needs to square the terms involving d_H so that Δ has certain good analytical properties. In particular, Δ is maximally hypoelliptic and invertible in the Heisenberg symbolic calculus, while the standard combinations $d_H\delta_H + D^*D$ and $DD^* + d_H\delta_H$ are not; see §3.1 below. If we then consider the spectral spaces $E^n_{[0,\lambda]} = \{\Delta_n \leqslant \lambda\}$ as successive finite-dimensional approximations of $\mathcal{E}^n$, in order to include $E^n_{[0,\lambda]}$ in a finite-dimensional subcomplex of $(\mathcal{E}^*, d)$ we need the Laplacians outside middle degree to be fourth-order also.

Remark 2.8. — Note that Δ is different to the Laplacian Δ_Q defined in [37]. The latter was defined with nice algebraic properties, namely commutativity with J when $\mathcal{L}_T J = 0$. On the other hand, observe that Δ commutes with d_H, D and their adjoints, which, as we shall see in §3.2, is essential for analytic torsion of the contact complex having the correct variational behaviour. Moreover note that Δ is the Laplacian appearing in the sub-Riemannian limit ([38]).

Next let us set

$$E^*_{[0,\lambda]} = \bigoplus_{k=0}^{2n+1} \{\Delta_k \leqslant \lambda\}\,.$$

These are finite-dimensional subcomplexes of the contact complex.

PROPOSITION 2.9. — *The torsion of $(E^*_{[0,\lambda]}, d_H)$ is*

$$\tau(E^*_{[0,\lambda]}, d_H) = \prod_{k=0}^{2n+1} \det(\Delta_k \mid E^*_{]0,\lambda]})^{(-1)^k w(k)/4}\,, \tag{2.9}$$

where

$$w(k) = \begin{cases} k & \text{if} \quad k \leqslant n \\ k+1 & \text{if} \quad k > n\,. \end{cases} \tag{2.10}$$

Remark 2.10. — This $w(k)$ is the natural weight of $\mathcal{E}^k$ in the contact complex; see §3.1 and [38, §3].

Proof. — Let note also D by d_H in degree n, and recall that the spectrum of $d_H\delta_H$ on $E^k_{]0,\lambda]} \cap (\ker \delta_H)^\perp$ and $\delta_H d_H$ on $E^{k-1}_{]0,\lambda]} \cap (\ker d_H)^\perp$ coincide. Then by (2.8), one finds that

$$\det(\Delta_k \mid E^*_{]0,\lambda]}) = \begin{cases} P^2_{k-1}P^2_k & \text{if} \quad k \neq n, n+1 \\ P^2_{n-1}P_n & \text{if} \quad k = n \\ P_n P^2_{n+1} & \text{if} \quad k = n+1\,, \end{cases}$$

where $P_k = \det\big(\delta_H d_H \mid E^k_{]0,\lambda]} \cap (\ker d_H)^\perp\big)$. This leads directly to

$$\prod_{k=0}^{2n+1} \det(\Delta_k \mid E^*_{]0,\lambda]})^{(-1)^k w(k)} = \prod_{k=0}^{2n+1} P_k^{2(-1)^{(k+1)}} = \tau(E^*_{[0,\lambda]}, d_H)^4 \,,$$

by Proposition 2.7.

□

We finally introduce zeta functions of the contact Laplacian. If $\mathrm{spec}^*(\Delta_k)$ denotes the non-zero spectrum of Δ_k on $\mathcal{E}_k$, then we take

$$\zeta(\Delta_k)(s) = \dim H^k(\mathcal{E}, d_H) + \sum_{\lambda \in \mathrm{spec}^*(\Delta_k)} \lambda^{-s}.$$

Note that by hypoellipticity (or Proposition 2.2) $\dim H^k(\mathcal{E}, d_H)$ is finite. By the results in §3.1 below, $\zeta(\Delta_k)(s)$ admits a meromorphic extension to $\mathbb{C}$ that is regular at $s = 0$. On each subcomplex $(E^*_{[0,\lambda]}, d_H)$ we then have

$$\begin{aligned} \zeta'(\Delta_k \mid E_{]0,\lambda]})(0) &= - \sum_{\mu \in \mathrm{spec}^*(\Delta_k) \cap]0,\lambda]} \ln \mu \\ &= - \ln \det(\Delta_k \mid E_{]0,\lambda]}) \,. \end{aligned}$$

Thus formula (2.9) for the torsion of $(E^*_{[0,\lambda]}, d_H)$ can be written for $\lambda > 0$ as

$$\ln \tau(E^*_{[0,\lambda]}, d_H) = \frac{1}{4} \sum_{k=0}^{2n+1} (-1)^{k+1} w(k) \zeta'(\Delta_k \mid E^*_{]0,\lambda]})(0) \,. \tag{2.11}$$

We thus speculate in extending this formula to the whole contact complex by defining the *analytic torsion of the contact complex* as

$$\ln T_C = \frac{1}{4} \sum_{k=0}^{2n+1} (-1)^{k+1} w(k) \zeta'(\Delta_k)(0) \,. \tag{2.12}$$

This formula is very similar to that of Ray–Singer analytic torsion T_{RS} in the Riemannian setting. Namely, from [33, Definition 1.6], in dimension N

$$\ln T_{RS} = \frac{1}{2} \sum_{k=0}^{N} (-1)^k k \zeta'(\Delta_k)(0) \,, \tag{2.13}$$

for Hodge–de Rham Laplacians Δ_k. Note however the sign convention: T_{RS} coincides with Reidemeister–Franz torsion τ_R on finite-dimensional cut-off de Rham complexes, while our T_C leads to the inverse; see Remark 2.6.

By analogy with Definition 2.5 and [32, 13], we also define a *contact complex Ray–Singer metric* on $\det H^*(\mathcal{E}, d_H)$ by setting

$$\| \quad \|_C = T_C \, | \quad |_{L^2(\mathcal{E})} \,. \tag{2.14}$$

Here $|\ \ |_{L^2(\mathcal{E})}$ is the L^2 metric induced on $\det H^*(\mathcal{E}, d_H)$ by identification of $H^*(\mathcal{E}, d_H)$ with harmonic forms $\mathcal{H}^*(\mathcal{E}, d_H) \subset \mathcal{E}^*$.

Again, note that the Ray–Singer metric on the de Rham determinant $\det H^*(\Omega^* M, d)$ reads instead

$$\|\ \ \|_{RS} = (T_{RS})^{-1} |\ \ |_{L^2(\Omega^* M)} . \tag{2.15}$$

More generally, we can twist the contact complex with a flat bundle and then define the analytic torsion of this twisted contact complex $(\mathcal{E}^*_\rho, d_H)$. Indeed let $\rho : \pi_1(M) \to U(N)$ be a unitary representation on $\mathbb{C}^N$. Associated to ρ is an Hermitian complex rank N vector bundle V_ρ equipped with a canonical metric-preserving flat connection ∇_ρ. One sets $\mathcal{E}_\rho = \mathcal{E} \otimes V_\rho$ and $d_H(\alpha \otimes s) = d_H \alpha \otimes s$ for parallel s. From this we may define the contact analytic torsion $T_C(\rho)$ with associated contact complex Ray–Singer metric on $\det H^*(\mathcal{E}_\rho, d_H)$.

The conciseness of notations T_C and $T_C(\rho)$ should not be misleading. The (twisted) contact complex only depends on the contact structure H on M (and ρ), but the spectral invariants T_C and $T_C(\rho)$ also depend on the choices of a contact form θ and complex structure J, both being used in the metric g.

Since we have defined this analytic torsion through algebraic and formal considerations around Reidemeister–Franz torsion, we now need to study its analytical properties. That is the purpose of the next section.

3. Heat kernels and variational behaviour of the torsion

We first gather some properties of zeta functions and the heat development of hypoelliptic operators such as the Laplacian of the contact complex.

3.1. Heat kernels and zeta functions for hypoelliptic operators

The Laplacian Δ for the contact complex is not elliptic. However there is a (substantially more intricate) symbolic calculus that can be applied to it to obtain results on heat kernels qualitatively analogous to the elliptic case. This calculus is called the *(Volterra)-Heisenberg calculus* and was introduced by Beals–Greiner–Stanton [4, 3] and Taylor [44]. A short account of its properties may be found in [21], and its use for the contact complex has been presented by Julg and Kasparov in [23, §5]. This calculus has also been developed in a more general setting that includes the contact case by

Ponge [31]. Here we just briefly sketch the results that we shall need in the sequel.

THEOREM 3.1 ([31, Thm 5.4.10] and [4, Thm 5.6]). — *Let $\mathcal{V}$ be a vector bundle over a compact contact manifold (M, H) of dimension $2n+1$. Let $P : C^\infty(M, \mathcal{V}) \to C^\infty(M, \mathcal{V})$ be a differential operator of even Heisenberg order v that is self-adjoint and bounded from below. If P satisfies the Rockland condition at every point then the principal symbol of $\partial_t + P$ is an invertible Volterra–Heisenberg symbol and as $t \searrow 0$ the heat kernel $k_t(x, x)$ of P on the diagonal has the following asymptotics in $C^\infty(M, (\mathrm{End}\,\mathcal{V}) \otimes |\Lambda|(M))$:*

$$k_t(x, x) \sim \sum_{j=0}^{\infty} t^{\frac{2(j-n-1)}{v}} a_j(P)(x).$$

Some explanation about the proposition is in order. The *Heisenberg order* of P is defined by assuming that a derivative in the direction of the Reeb field T has weight 2, while derivatives in the direction of the contact distribution H have weight 1. The *Rockland condition* is a representation-theoretic condition defined in [31, Definition 3.3.8]. (The original formulation is due to Rockland [34].) An operator that satisfies this condition is hypoelliptic, in the sense of [31, Proposition 3.3.2]. Invertibility of an operator in the *Volterra–Heisenberg calculus* is explained in [4, §4], [31, Ch. 5] or [21, §4].

The next result describes the properties of the zeta function in this contact setting. For the non-negative operators P we are concerned with, the result follows from Theorem 3.1 by a classical argument using the Mellin transform of the heat kernel, see e.g. [22, §1.10].

THEOREM 3.2 ([30, §4]). — *Let P be as in Theorem 3.1. Then the zeta function*

$$\zeta(P)(s) = \dim\ker P + \mathrm{Tr}^*(P^{-s}) \ , \ s \in \mathbb{C},$$

is a well-defined holomorphic function for $\mathrm{Re}(s) \gg 1$ and admits a meromorphic extension to $\mathbb{C}$ with at worst simple poles occurring at $s \in \mathcal{S} = \{\frac{2(n+1-j)}{v} \mid j \in \mathbb{N}\} \setminus (-\mathbb{N})$. Moreover

$$\zeta(P)(0) = \int_M \mathrm{tr}(a_{n+1}(P))\, \theta \wedge (d\theta)^n$$

is the constant term in the development of $\mathrm{Tr}(e^{-tP})$ as $t \searrow 0$.

Now by [37, p. 300], the fourth-order Laplacian Δ_k on the contact complex (twisted with a flat bundle) satisfies the Rockland condition, hence Theorems 3.1 and 3.2 apply to it. Moreover one can be more precise in the nature of the coefficients $a_j(\Delta_k)$.

Proposition 3.3. — *The coefficients $a_j(\Delta_k)(x)$ in the development of the heat kernels $k_{e^{-t\Delta_k}}(x,x)$ are given by universal polynomials in the Tanaka–Tanno–Webster curvature, torsion and their covariant derivatives.*

Proof. — These coefficients can be computed algebraically (in theory) using the full symbol of Δ_k and the inverse of the leading symbol of $\partial_t + \Delta_k$. We refer to [21, §4] for a concise account of this parametrix technique and general formulae we rely on here.

By its construction and Proposition 2.3 the contact complex $(\mathcal{E}^k, d_H)$ is a contact-invariant differential complex. Hence the differentials, their adjoints and the Laplacians are given by universal tensorial expression in the Tanaka–Tanno–Webster connection and its curvature ([42, 43, 45]). Furthermore curvature terms only occur in lower order terms of these polynomial symbols. Then the leading fourth-order symbol of Δ_k at some point m does not contain curvature terms, and is thus an universal expression independent of m in normal coordinates. It is indeed the symbol of the model invariant operator Δ_{k,g_m} on the Heisenberg group endowed with the left invariant metric given by g_m. That means that the symbol of Δ_k is *uniform* in the sense of [4, Definition 4.12]. This implies the required property on the contact-heat coefficients by Theorem 4.14 in [4]; see also Proposition 7.19 and Theorem 7.30 in [4] or [21, Thm 4.1], where the arguments extend without changes to our operators. □

3.2. Variational behaviour of the analytic torsion

We consider the variation of contact analytic torsion and Ray–Singer metric in the direction of an arbitrary line of pairs $(\theta_\varepsilon, J_\varepsilon)$ of contact form and calibrated almost complex structure for (M, H). We shall see that the variation of the Ray–Singer metric is given entirely by local terms. Indeed this may be viewed as a necessary and sufficient condition for correctly defining an analytic torsion; see the approach of Branson [14].

First we define from (2.12) the *contact torsion function* by

$$\kappa(s) = \frac{1}{2} \sum_{k=0}^{2n+1} (-1)^{k+1} w(k) \zeta(\Delta_k)(s), \tag{3.1}$$

with $w(k)$ as in (2.10). Then the analytic torsion of the contact complex reads

$$T_C = \exp\Big(\frac{1}{2}\kappa'(0)\Big).$$

For simplicity, in this section we suppress the C from the notation, as well as the representation ρ, although all results stand for the twisted torsions and metrics as well.

THEOREM 3.4. — *Let a $\bullet$ superscript denote first variation $(d/d\varepsilon)|_{\varepsilon=0}$.*

(1) *One has $\kappa(0)^\bullet = 0$, so that $\kappa(0)$ is a contact invariant.*

(2) *The variation of the analytic torsion T is given by*

$$(\ln T)^\bullet = \sum_{k=0}^{n} (-1)^k \left(\int_M \operatorname{tr}(\alpha\, a_{t^0;\,k})\ \mathrm{dvol} - \operatorname{Tr}(\alpha \mathcal{P}_k) \right),$$

where $\alpha = *^{-1} *^\bullet$, $a_{t^0;\,k}$ is the t^0 coefficient in the diagonal small-time asymptotic expansion of the heat kernel of Δ_k, dvol is the volume form $\theta \wedge (d\theta)^n$, and $\mathcal{P}_k$ is orthogonal projection onto the null-space of Δ_k.

Proof. — By Hodge $*$ duality (for convenience we suppress the ε dependence), (3.1) reads

$$\kappa(s) = \sum_{k=0}^{n} c_k \zeta(\Delta_k)(s)$$

with

$$c_k = (-1)^k (n+1-k).$$

Let $f(s) = \Gamma(s)\kappa(s)$. By a Mellin transform and Theorem 3.1 one has for $\operatorname{Re} s$ large enough

(3.2)

$$f(s) = \sum_{k=0}^{n} c_k \int_0^{+\infty} t^{s-1} \operatorname{Tr}(e^{-t\Delta_k} - \mathcal{P}_k) dt + \Gamma(s) \sum_{k=0}^{n} c_k \dim H^k(\mathcal{E}, d_H),$$

where $\mathcal{P}_k$ denotes orthogonal projection onto the null-space of Δ_k. We need to take derivative in the metric of this formula. In the sequel we cover M with local orthonormal systems of horizontal vectors fields X_i, and fix the norm on the p-th horizontal Sobolev space W^p by

$$\|f\|_{W^p} = \sum_{|I| \leqslant p} \|X_I f\|_{L^2}\,.$$

LEMMA 3.5. — – *Given a calibrated metric g_0 and $p \in \mathbb{N}$, there exists a constant C_p such that for any calibrated metric g close enough to g_0, it holds that*

$$\|e^{-t\Delta_g}\|_{L^2, W^{4p}} \leqslant C_p t^{-p}\,.$$

– *Given a smooth variation of metric g_ε, it holds that*

$$(\operatorname{Tr}(e^{-t\Delta}))^\bullet = -t \operatorname{Tr}(\Delta^\bullet e^{-t\Delta}).$$

Proof. — – By maximal hypoellipticity of $\Delta_{g_0}^p$, one knows that there exists C such that

$$\|f\|_{W^{4p}} \leqslant C(\|\Delta_{g_0}^p f\|_2 + \|f\|_2)\,.$$

Since $\|(\Delta_{g_0}^p - \Delta_g^p)f\|_2 \leqslant (2C)^{-1}\|f\|_{W^{4p}}$ for g close enough to g_0, one obtains

$$\|\Delta_{g_0}^p f\|_2 \leqslant C'(\|\Delta_g^p f\|_2 + \|f\|_2)\,,$$

and the spectral calculus gives

$$\|e^{-t\Delta_g}\|_{L^2,W^{4p}} \leqslant C''\|(\Delta_g^p + 1)e^{-t\Delta_g}\|_{L^2,L^2} \leqslant C_p t^{-p}\,.$$

– Duhamel's formula (see e.g. [35, Proposition 3.15]) writes

$$e^{-t\Delta_\varepsilon} - e^{-t\Delta_0} = -\int_0^t e^{-(t-s)\Delta_\varepsilon}(\Delta_\varepsilon - \Delta_0)e^{-s\Delta_0}\,ds\,. \tag{3.3}$$

Let $\|P\|_{p,k}$ denotes $W^p \to W^k$ operator norm and $\|P\|_{\mathcal{L}^1} = \operatorname{Tr}|P|$. By hypoellipticity of Δ_0 one knows that $(1+\Delta_0)^{-N}$ is trace class for N large enough. Then one has for $0 \leqslant s \leqslant t/2$

$$\begin{aligned}
&\|e^{-(t-s)\Delta_\varepsilon}(\Delta_\varepsilon - \Delta_0)e^{-s\Delta_0}\|_{\mathcal{L}^1}\\
&\qquad \leqslant \|e^{-(t-s)\Delta_\varepsilon}(\Delta_\varepsilon - \Delta_0)(1+\Delta_0)^N e^{-s\Delta_0}\|_{0,0}\|(1+\Delta_0)^{-N}\|_{\mathcal{L}^1}\\
&\qquad \leqslant C\|e^{-(t-s)\Delta_\varepsilon}\|_{-4-4N,0}\|\Delta_\varepsilon - \Delta_0\|_{-4N,-4N-4}\\
&\qquad \leqslant C' t^{-4-4N}\|\Delta_\varepsilon - \Delta_0\|_{-4N,-4N-4}\,,
\end{aligned}$$

and a similar control for $t/2 \leqslant s \leqslant t$. Therefore one can take trace in (3.3). Moreover for a smooth family of metrics, $\varepsilon^{-1}(\Delta_\varepsilon - \Delta_0) \to \Delta^\bullet$ in any $(p, p-4)$-norm and one gets

$$(\operatorname{Tr}(e^{-t\Delta}))^\bullet = -\int_0^t \operatorname{Tr}(e^{-(t-s)\Delta}\Delta^\bullet e^{-s\Delta})\,ds.$$

Recalling that $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$, for smoothing operators A, B, we have

$$(\operatorname{Tr}(e^{-t\Delta}))^\bullet = -\int_0^t \operatorname{Tr}(\Delta^\bullet e^{-s\Delta}e^{-(t-s)\Delta})\,ds = -t\operatorname{Tr}(\Delta^\bullet e^{-t\Delta}),$$

as needed. □

By Lemma 3.5 and (3.2) one has for $\operatorname{Re} s$ large enough

$$f(s)^\bullet = -\sum_{k=0}^n c_k \int_0^{+\infty} t^s \operatorname{Tr}(\Delta_k^\bullet e^{-t\Delta_k})\,dt\,, \tag{3.4}$$

since $\operatorname{Tr}(\mathcal{P}_k) = \dim\ker\Delta_k = \dim H^k(\mathcal{E}, d_H)$ is certainly independent of θ and J.

Setting $\alpha = *^{-1}*^\bullet$, one computes using (2.3) the variation of the Laplacian as

$$\Delta^\bullet = \begin{cases} -d_H\alpha\delta_H d_H\delta_H + d_H\delta_H\alpha d_H\delta_H - d_H\delta_H d_H\alpha\delta_H + d_H\delta_H d_H\delta_H\alpha \\ \quad -\alpha\delta_H d_H\delta_H d_H + \delta_H\alpha d_H\delta_H d_H - \delta_H d_H\alpha\delta_H d_H + \delta_H d_H\delta_H\alpha d_H \\ \qquad \text{on } \mathcal{E}^k \text{ for } k=0,\dots,n-1\,, \\ -d_H\alpha\delta_H d_H\delta_H + d_H\delta_H\alpha d_H\delta_H - d_H\delta_H d_H\alpha\delta_H + d_H\delta_H d_H\delta_H\alpha \\ \quad -\alpha D^*D + D^*\alpha D \\ \qquad \text{on } \mathcal{E}^n\,. \end{cases}$$

A computation then shows that

$$\tag{3.5} \begin{aligned} &\sum_{k=0}^n c_k \operatorname{Tr}(\Delta_k^\bullet e^{-t\Delta_k}) \\ &\quad = 2\sum_{k=0}^{n-1} \operatorname{Tr}\left(\left(\alpha(c_k+c_{k-1})(d\delta)^2 - \alpha(c_k+c_{k+1})(\delta d)^2\right) e^{-t\Delta_k}\right) \\ &\qquad\qquad + 2\operatorname{Tr}\left(\left(\alpha(c_n+c_{n-1})(d\delta)^2 - \alpha c_n D^*D\right) e^{-t\Delta_n}\right). \end{aligned}$$

To move the α's to the front we have used the following facts: the heat kernel is a semigroup, implying $e^{-t\Delta} = e^{-(t/2)\Delta}e^{-(t/2)\Delta}$; if operators A, B are smoothing then $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$; and the Laplacian Δ commutes with d_H, D and their adjoints. We have also used that

$$\operatorname{Tr}(\alpha DD^* e^{-t\Delta_{n+1}}) = -\operatorname{Tr}(\alpha D^*D e^{-t\Delta_n}),$$

as $\alpha DD^* e^{-t\Delta_{n+1}} = -*^{-1}(\alpha D^*D e^{-t\Delta_n})*$, which follows from (2.3) and $(*^2)^\bullet = *\alpha + \alpha* = 0$.

Simplifying (3.5) yields

$$\begin{aligned} \sum_{k=0}^n c_k \operatorname{Tr}(\Delta_k^\bullet e^{-t\Delta_k}) &= 2\sum_{k=0}^n (-1)^{k+1}\operatorname{Tr}(\alpha\Delta_k e^{-t\Delta_k}) \\ &= -2\frac{d}{dt}\sum_{k=0}^n (-1)^{k+1}\operatorname{Tr}(\alpha e^{-t\Delta_k})\,. \end{aligned}$$

Hence after integrating by parts in (3.4), we obtain for $\operatorname{Re} s$ large enough

$$\tag{3.6} f(s)^\bullet = 2s\sum_{k=0}^n (-1)^k \int_0^{+\infty} t^{s-1}\operatorname{Tr}(\alpha e^{-t\Delta_k} - \alpha\mathcal{P}_k)\,dt = h(s)\,.$$

We have to extend analytically this identity near $s = 0$.

For the right side, one splits the integral in $\int_0^1 + \int_1^{+\infty}$ and uses the local heat development Theorem 3.1 at order $N = n+2$. This yields

$$\begin{aligned}\int_0^{+\infty} & t^{s-1}\operatorname{Tr}(\alpha e^{-t\Delta_k} - \alpha\mathcal{P}_k)dt \\ &= \sum_{j=0}^{n+2}\Big(s + \frac{j-n-1}{2}\Big)^{-1}\int_M \operatorname{tr}(\alpha(x)a_j(\Delta_k)(x))\,\mathrm{dvol} \\ &\qquad - s^{-1}\operatorname{Tr}(\alpha\mathcal{P}_k) + \text{holomorphic for } \operatorname{Re} s > -1/2\,.\end{aligned}$$

Hence h is regular at the origin with

$$h(0) = 2\sum_{k=0}^{n}(-1)^k\Big(\int_M \operatorname{tr}(\alpha a_{n+1}(\Delta_k))\,\mathrm{dvol} - \operatorname{Tr}(\alpha\mathcal{P}_k)\Big)\,. \tag{3.7}$$

For the left side of (3.6), we study the smoothness in the metric g of the analytic extension of $f = \Gamma\kappa$ near zero. Starting from (3.2), one has

$$f(s) = \sum_{k=0}^{n} c_k\Big(\int_0^1 + \int_1^{+\infty}\Big)t^{s-1}\operatorname{Tr}(e^{-t\Delta_k} - \mathcal{P}_k)dt + \Gamma(s)\mathrm{const}(g)\,. \tag{3.8}$$

By lemma 3.5, the $\int_1^{+\infty}$ part is clearly holomorphic on $\mathbb{C}$ as well as its derivative in g.

To study the $\int_0^1$ part we use a parametrix H_t at order N of $e^{-t\Delta_k}$. We briefly recall its construction as done in e.g. [21, p. 241]. Let p be the inverse of the leading (Volterra)-Heisenberg symbol of $\partial_t + \Delta_k$. Then if a denotes the full symbol of $\partial_t + \Delta_k$, the remainder $r = a\circ p - 1$ is of order -1, and one considers the Neumann series $P = \sum_{i=0}^{N}(-1)^i p\circ r^{\circ i}$. It holds that $a\circ P - 1$ and $P\circ a - 1$ are of order $\leqslant -N-1$, and one obtains the approximate parametrix H_t by quantizing these symbols P. By construction and the proof of Proposition 3.3, the symbol P depends smoothly in the metric. Moreover, following [21, p. 241], the family H_t is bounded in L^2 for small t, and the remainder

$$R_t = (\partial_t + \Delta_k)\circ H_t$$

satisfies $\|R_t\|_{p,p+M} \leqslant Ct^k$ for any p and $k \leqslant N - M - n$.

Then considering the $\int_0^1$ part in (3.8), one has

$$\int_0^1 t^{s-1}\operatorname{Tr}(e^{-t\Delta_k})dt = \int_0^1 t^{s-1}\operatorname{Tr} H_t dt + \int_0^1 t^{s-1}\operatorname{Tr}(e^{-t\Delta} - H_t)dt\,. \tag{3.9}$$

By its construction and the symbol calculus, see [21, §3-4], the trace of the quantized H_t is a rational expression $\sum_{i=0}^{N} t^{\frac{i-n-1}{2}}P_i(R)$, with $P_i(R)$ given by integral of universal polynomial expression in the curvature and their derivatives by Proposition 3.3. Hence the corresponding integral in (3.9) is a rational function $\sum_{i=0}^{N}(s+\frac{i-n-1}{2})^{-1}P_i(R)$ as above, with residues depending smoothly in the metric.

For the second integral in (3.9), we observe that

$$e^{-t\Delta} - H_t = -\int_0^t e^{-(t-u)\Delta} R_u du\,.$$

which is uniformly controlled in trace norm for $N \geqslant 3n+3$, since then

$$\|e^{-(t-u)\Delta} R_u\|_{\mathcal{L}^1} \leqslant \|R_u\|_{\mathcal{L}^1} \leqslant C\|R_u\|_{-2n-3,0}$$

is bounded for small u. The corresponding integral in (3.9) is holomorphic for $\operatorname{Re} s > -1$, with smooth control from the metric by construction of H_t and R_t.

Finally, one can apply (3.6) near zero, and writes there

$$f(s)^\bullet = \Gamma(s)\kappa^\bullet(s) = h(s).$$

Using (3.7) and $\Gamma(s) \sim s^{-1}$ leads to

$$\kappa(0)^\bullet = 0 \quad \text{and} \quad \kappa'(0)^\bullet = h(0)\,,$$

giving Theorem 3.4. □

Remark 3.6. — The previous proof actually shows that the torsion function

$$\kappa(s) = \sum_{k=0}^{n}(-1)^k(n+1-k)\zeta(\Delta_k)(s)$$

studied here is, up to a multiplicative factor, the *unique* combination of such zeta functions that leads to a variational formula like (3.6), i.e. local up to cohomological terms.

The variational formula for analytic torsion we obtained is more neatly expressed at the level of the Ray–Singer metric, since then the global term disappears. The next result is analogous to the variational formula for the Ray–Singer metric on Riemannian manifolds (see [13, Theorem 4.14] and [11, Theorem 1.18]).

Corollary 3.7. — *Let* $\|\ \ \|_C$ *denote the contact Ray–Singer metric on* $\det H^*(\mathcal{E}, d_H)$.

(1) *The following identity holds:*

$$(\ln\|\ \ \|_C)^\bullet = \sum_{k=0}^{n}(-1)^k \int_M \operatorname{tr}(\alpha a_{t^0;\,k})\,\theta\wedge(d\theta)^n\,. \tag{3.10}$$

(2) *Under conformal variations of the contact form* ($\theta_\varepsilon = e^{2\varepsilon\Upsilon}\theta, J_\varepsilon = J$), *for a function* Υ, *we have*

$$(\ln \|\quad \|_C)^\bullet = 2\sum_{k=0}^{n}(-1)^k(n+1-k)\int_M \Upsilon\,\mathrm{tr}(a_{t^0;\,k})\,\theta\wedge(d\theta)^n\,.$$

Proof. — Recalling from (2.14) and §2 the definition of the Ray–Singer metric, we have that

$$\ln\|\quad\|_C^2 = 2\ln T_C + \ln|\quad|^2_{L^2(\mathcal{E})}\,, \tag{3.11}$$

where the L^2 metric $|\quad|_{L^2(\mathcal{E})}$ is induced on $\det H^*(\mathcal{E}, d_H)$ from the inner product on $H^*(\mathcal{E}, d_H)$ defined by

$$\langle [u],[v]\rangle_{\theta,\,J} = \int_M \mathcal{P}u\wedge *\mathcal{P}v.$$

But for the orthogonal projection $\mathcal{P}$ onto harmonic forms $\mathcal{H}^*(\mathcal{E}, d_H)$, one checks that $\mathcal{P}^\bullet$ takes $\mathcal{H}^*(\mathcal{E}, d_H)$ to its orthogonal complement. Thus

$$\langle [u],[v]\rangle^\bullet = \int_M \mathcal{P}u\wedge(*^\bullet)\mathcal{P}v = \langle [u],\alpha[v]\rangle. \tag{3.12}$$

If we use Hodge $*$ duality in the definition of $\det H^*(\mathcal{E}, d_H)$, then take an orthonormal basis of each $\mathcal{H}^k(\mathcal{E}, d_H)$, $k = 0,\dots,n$, and finally use (3.12), it is easy to see that

$$(\ln|\quad|_C)^\bullet = \sum_{k=0}^{n}(-1)^k\,\mathrm{Tr}(\alpha\mathcal{P}_k). \tag{3.13}$$

This together with (3.11) and Theorem 3.4 completes the proof of (1).

Assertion (2) follows immediately from (1), since for conformal variations it is straightforward to check that on $\mathcal{E}^k$

$$\alpha = *^{-1}*^\bullet = 2(n+1-k)\Upsilon\,\mathrm{Id}\ .$$

□

Note that setting $\Upsilon \equiv 1$ in Corollary 3.7 (2), i.e. performing a constant rescaling $\theta \mapsto e^{2\varepsilon}\theta$, yields

$$\left(\|\quad\|_C\right)^\bullet = 2\kappa(0)\,. \tag{3.14}$$

In particular, if the contact invariant $\kappa(0) \neq 0$, then we could not hope for any invariance of Ray–Singer metric. Note that by definition

$$\begin{aligned}\kappa(0) &= \sum_{k=0}^{n} (-1)^k (n+1-k)\zeta(\Delta_k)(0) \\ &= \sum_{k=0}^{n} (-1)^k (n+1-k) \int_M \operatorname{tr}(a_{t^0;\,k})\, \theta \wedge (d\theta)^n\,,\end{aligned}$$

by Proposition 3.2, where again $a_{t^0;\,k}$ is the constant t^0 coefficient in the diagonal small-time asymptotic expansion of the heat kernel of Δ_k. Therefore by Proposition 3.3, $\kappa(0)$ is an integral over M of local curvature data, namely

$$\kappa(0) = \int_M P_n(R, A, T)\, \mathrm{dvol}\,, \tag{3.15}$$

for some universal polynomial in Tanaka–Tanno–Webster ([42, 43, 45]) curvature R, torsion A, Tanno's tensor $T = \nabla J$, and their covariant derivatives.

We show in Corollary 3.8 below that, in dimension 3 ($n = 1$), $\kappa(0)$ vanishes identically. Whether contact invariants such as $\kappa(0)$ vanish in all dimensions is an open problem. For further discussion in the contact case see [40, §7], and [7, Remark 9.3] for a similar problem arising for the eta function of the contact complex.

3.3. CR/contact invariants in dimension 3

In dimension 3, besides the vanishing of $\kappa(0)$ we mentioned, we can also obtain more explicit variational formulae, and get CR/contact-invariant corrections to the contact Ray–Singer metric.

COROLLARY 3.8. — *On 3-dimensional contact manifolds M:*

(1) *It holds that $\kappa(0) = 0$, and thus $\|\ \ \|_C$ is independent of a constant rescaling $\theta \mapsto K\theta$.*

(2) *There exist universal constants C_1, C_2 (i.e. independent of M) such that under a conformal variation ($\theta^\varepsilon = e^{2\varepsilon\Upsilon}\theta, J_\varepsilon = J$) we have*

$$(\ln \|\ \ \|_C)^\bullet = \int_M \Upsilon (C_1 \Delta_H R + C_2 \operatorname{Im} A_{11,}{}^{11})\, \theta \wedge d\theta\,,$$

where R, A, Δ_H and a comma subscript denote respectively the scalar curvature, torsion, sub-Laplacian and covariant differentiation with respect to the Tanaka–Webster connection ([42, 45]) of (θ, J).

(3) *Let $C_1' = -C_1/8$ and $C_2' = C_2/4$, with C_1, C_2 as above. Then*

$$\| \quad \|_{\mathrm{CR}} = \exp\left(C_1' \int_M R^2\, \theta \wedge d\theta + C_2' \int_M |A|^2 \theta \wedge d\theta \right) \| \quad \|_C$$

is a CR-invariant (i.e. independent of contact form) metric on $\det H^*(\mathcal{E}, d_H)$.

(4) *There exist universal constants C_3, C_4 such that both*

$$\| \quad \|_H^\nu = \exp(C_3 \nu(M)) \| \quad \|_{\mathrm{CR}} \quad \text{and} \quad \| \quad \|_H^D = \exp(C_4 \overline{\eta}(D*)) \| \quad \|_{\mathrm{CR}}$$

are contact-invariant metrics, with $\nu(M)$ the ν-invariant of Biquard–Herzlich [6]*, and $\overline{\eta}(D*)$ the CR–invariant correction to the pseudo-hermitian eta invariant $\eta(D*)$; see* [7, Theorem 9.4]*.*

Proof. — We complexify H and work in a local frame $\{Z_1, Z_{\overline{1}}\}$ and coframe $\{\theta^1, \theta^{\overline{1}}\}$, where $\theta^1(T) = 0 = \theta^{\overline{1}}(T)$ (T the Reeb field). Under a constant scaling of contact form $\widehat{\theta} = K\theta$, the relevant heat coefficient $a_{t^0;\,k}$, for $k = 0, 1$, scales as $\operatorname{tr}(\hat{a}_{t^0;\,k}) = K^{-2} \operatorname{tr}(a_{t^0;\,k})$. This is easily verified by an argument similar to that for [4, (6.48)]. Basic invariant theory (see e.g. [41, 7]) then tells us that $\operatorname{tr}(a_{t^0;\,k})$ must be a universal linear combination of

$$R^2,\ |A|^2,\ \Delta_H R,\ R_{,0} = 2 \operatorname{Re} A_{11,}{}^{11} \text{ and } \operatorname{Im} A_{11,}{}^{11}. \tag{3.16}$$

Now $\kappa(0)$ is the integral of a linear combination of these terms, which is moreover independent of the choice of θ. A familiar argument (see e.g. [7]) shows that the integral of a linear combination of R^2 and $|A|^2$ can never be contact-invariant. Thus $\kappa(0)$ is the integral of a divergence, and hence vanishes. This, together with (3.14), proves assertion (1).

Consider now the differential of $\ln \| \quad \|_C$ under a conformal change of θ. This may be seen as a real 1-form α on the space Θ of contact forms. By Corollary 3.7 (2) and (3.16) it can be written

$$\alpha_\theta(\Upsilon) = \int_M \Upsilon (c_1 R^2 + c_2 |A|^2 + c_3 \Delta_H R + c_4 R_{,0} + c_5 \operatorname{Im} A_{11,}{}^{11})\, \theta \wedge d\theta\,,$$

for some universal constants c_i. Here we identified the tangent space $T_\theta \Theta$ with functions Υ on M. By [7, Lemma 9.5], the general vanishing of such a 1-form on constant Υ implies that $c_1 = c_2 = 0$, while the fact that α is a *closed* form gives $c_4 = 0$; see [7] for details. This proves assertion (2).

Also by (83)–(84) in [7, §9], one has

$$\frac{d}{d\Upsilon} \int_M R^2 \theta \wedge d\theta = 8 \int_M \Upsilon\, (\Delta_H R)\, \theta \wedge d\theta\,,$$

and

$$\frac{d}{d\Upsilon}\int_M |A|^2\theta\wedge d\theta = -4\int_M \Upsilon\,(\mathrm{Im}\,A_{11,}{}^{11})\,\theta\wedge d\theta\,,$$

leading to assertion (3).

Assertion (4) is proved similarly as for the case of the contact eta invariant in [7, §9]. The CR deformations (i.e. of J) of the CR–invariants ν, $\overline{\eta}(D*)$ and $\ln\|\ \ \|_{\mathrm{CR}}$ are all given by multiples of

$$\int_M \langle Q, J^\bullet\,\rangle\,\theta\wedge d\theta\,,$$

where Q is Cartan's tensor; see [7, §9] for details. □

Remark 3.9. — As may be seen from (3.10), in order to determine the various universal constants in Corollary 3.8 and investigate whether $\|\ \ \|_C$ has any contact-invariant properties, one needs to calculate the local coefficients of t^0 in the diagonal small-time asymptotic expansion of the heat kernels of the fourth-order Laplacians we consider here. Formulae for calculating these coefficients are built into the pseudodifferential construction of the heat kernel, however implementing these in practice seems difficult. Another approach to fix the constants would be to compute $\|\ \ \|_C$ on a manifold with a family of contact forms and complex structures. The CR Seifert manifolds we will consider now don't help here, due to the rigidity of their contact form.

4. Contact and Ray–Singer analytic torsions of CR Seifert manifolds

We follow [7] to review the definition of a CR Seifert manifold and to fix notation. Note that in dimension 3 a calibrated almost complex structure J for the contact structure H is automatically integrable; the pair (H, J) is often called a *pseudoconvex CR structure*.

Definition 4.1. — *A* CR Seifert manifold *is a 3-dimensional compact manifold M endowed with a pseudoconvex CR structure (H, J) and a Seifert structure $\varphi : \mathbb{S}^1 \to \mathrm{Diff}(M)$ that are compatible in the following sense: the circle action φ preserves the CR structure and is generated by a Reeb field T.*

It is easily proved that existence of a Reeb field T satisfying $\varphi_*(d/dt) = T$ is equivalent to existence of a locally free action of $\mathbb{S}^1$ whose (never vanishing) infinitesimal generator preserves (H, J) and is transverse everywhere to H.

The quotient space $\Sigma = M/\mathbb{S}^1$ is an orbifold surface with conical singularities. Each CR Seifert manifold is then the $\mathbb{S}^1$-bundle inside a line orbifold bundle L over the compact Riemannian orbifold Σ. Singularities of L are located above the singularities of Σ in such a way that the total space M of the bundle is a smooth manifold: if the local fundamental group at $\sigma \in \Sigma$ is $\mathbb{Z}/\alpha\mathbb{Z}$ ($\alpha \in \mathbb{N}^*$), a generator acts on a local chart around σ as $e^{i\frac{2\pi}{\alpha}}$ and on the fibre above σ as $e^{i\frac{2\pi\beta}{\alpha}}$, where α and β are relatively prime integers with $1 \leqslant \beta < \alpha$.

Theorem 4.2. — *Let M be CR Seifert manifold and $\rho : \pi_1(M) \to U(N)$ a unitary representation. Then:*

- *The analytic torsion $T_C(\rho)$ of the twisted contact complex and Ray–Singer analytic torsion $T_{RS}(\rho)$ satisfy*

$$T_C(\rho) = \big(T_{RS}(\rho)\big)^{-1} .$$

- *The two Ray–Singer metrics on* $\det H^*(M,\rho)$, *corresponding to the de Rham and contact complexes (see* (2.14) *and* (2.15)*), coincide, i.e.*

$$\| \quad \|_C = \| \quad \|_{RS} ,$$

via the isomorphism $\det H^*(\mathcal{E}^*_\rho, d_H) \cong \det H^*(\Omega^*_\rho M, d)$ *coming from Proposition 2.2.*

The remainder of this section will be devoted to the proof of these results.

We first need to compare the two spectra coming from the de Rham and contact complexes. This has been done in [7, §§7, 8] in the untwisted case, i.e. for a trivial representation. We will rely on and refer to the spectral analysis done there and point out the few differences coming from the use of the representation ρ here.

4.1. Circle action and Fourier analysis

Let V be the flat complex vector bundle over M associated to ρ. It is the quotient of the trivial bundle $\widetilde{M} \times \mathbb{C}^N$ over the universal cover $\widetilde{M}$ of M by the deck transformations $\gamma.(m,v) = (\tau(\gamma)m, \rho(\gamma)v)$. In what follows the contact complex is twisted by ρ in order to take values in V.

Let φ_t be the circle action on M induced by the Reeb field T. It may be lifted on V, by parallel transport for the flat connection ∇_ρ, but no longer as a circle action. We have instead

$$\varphi_{2\pi} = \mathrm{holonomy}_\rho(f) = \rho(f)^{-1}, \tag{4.1}$$

where $f = \varphi_{[0,2\pi]}(m)$ is the generic closed orbit of the action, as seen in $\pi_1(M)$. This f is central, as comes from the presentation of the fundamental group of Seifert manifolds.

Proposition 4.3 (see e.g. [20, 29, 39]). — *Let M be the circle V-bundle L of rational degree $d = b + \sum_i \frac{\beta_i}{\alpha_i}$ over the orbifold surface Σ of integral genus g, with n conical points x_i of type $(\alpha_i,\ \beta_i)$. Then $\pi_1(M)$ admits the presentation*

$$\begin{aligned}\pi_1(M) = \langle f,\ a_j,\ b_j\ (1 \leqslant j \leqslant g),\ g_i\ (1 \leqslant i \leqslant n) \mid \\ [a_j, f] = [b_j, f] = [g_i, f] = g_i^{\alpha_i} f^{-\beta_i} = f^b \prod_j [a_j, b_j] \prod_i g_i = 1\rangle.\end{aligned}$$

We can split ρ into irreducible representations, on which $\rho(f)$ is scalar:

$$\rho(f) = e^{2i\pi x}$$

for some $x \in [0, 1[$. Let $V = \oplus V^x$ be the corresponding splitting of V into flat sub-bundles. By (4.1) we recover a circle action on each such component V^x by setting

$$\psi_t = e^{itx}\varphi_t\,. \tag{4.2}$$

Using the circle action ψ_t one can still perform a Fourier decomposition of sections of V^x as in [7, §7]. Namely for $s \in \mathbf{V}^x = \Gamma(M, V^x)$, let $\varphi_t(s)(m) = \varphi_t(s(\varphi_{-t}(m))$ and $\psi_t(s) = e^{itx}\varphi_t(s)$. The function $t \mapsto \psi_t(s)(m) \in V^x_m$ is 2π-periodic, hence one has

$$s = \sum_{n\in\mathbb{Z}} \pi_n s \quad \text{with} \quad \pi_n s = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\psi_t(s)dt\,.$$

Since $\psi_t(\pi_n s) = e^{int}\pi_n s$, it holds that $(\pi_n s)(\varphi_t(m)) = e^{i(x-n)t}(\pi_n s)(m)$ in a local flat trivialisation of V^x and $\nabla^\rho_T(\pi_n s) = i(x-n)\pi_n s$. Thus the spectrum of $iT = \nabla^\rho_{iT}$ becomes the shifted $\mathbb{Z} - x$ on $\mathbf{V}^x$. For $\lambda = n - x$ we shall note

$$\mathbf{V}_\lambda = \pi_n(\mathbf{V}^x) = \mathbf{V}^x \cap \{iT = \lambda\}\,. \tag{4.3}$$

As the circle action preserves the metric and the whole pseudohermitian structure (H, θ, J), we can split both the Hodge–de Rham and the contact complex spectra into their Fourier components. This is useful for comparing the spectra.

4.2. Comparing the Riemannian and sub-Riemannian spectra

We adapt Propositions 7.2 and 7.4 in [7] to our V-valued case, and consider the following spaces.

DEFINITION 4.4. — • *Let $\mathcal{H}^2_V$ be the space of vertical 2-forms $\alpha = \theta \wedge \beta$, with values in V, such that both α and $J\alpha$ are closed.*

• *Let also $\mathcal{H}^0_V$ be the space of pluri-CR functions in V, i.e.*

$$\mathcal{H}^0_V = \ker(\Delta_H^2 + T^2) = \ker \overline{\Box}_V \Box_V$$

with $\Delta_H = \delta_H d_H + d_H \delta_H$, $\Box_V = \partial_V^* \partial_V$ and $\overline{\Box}_V = \overline{\partial}_V^* \overline{\partial}_V$.

According to (63) in [7], and taking into account the tensorisation by the flat complex vector bundle V here, the non-zero spectrum of $D*$ splits as follows

$$\mathrm{spec}^*(D*) = \mathrm{spec}^*(-\Delta_H \mid (\mathcal{H}^0_V)^\perp) \cup \mathrm{spec}^*(-JT \mid \mathcal{H}^2_V)\,.$$

Therefore the non-zero spectrum of the non-positive second-order Laplacian $P = D * + \delta_H d_H$ on 2-forms splits into

$$\begin{aligned} \mathrm{spec}^*(P) &= \mathrm{spec}^*(D*) \ \cup\ \mathrm{spec}^*(\Delta_H) \\ &= \mathrm{spec}^*(\Delta_H) \ \cup\ \mathrm{spec}^*(-\Delta_H) \\ &\quad \bigcup \mathrm{spec}^*(-JT \mid \mathcal{H}^2_V) \setminus \mathrm{spec}^*(-\Delta_H \mid \mathcal{H}^0_V), \end{aligned} \tag{4.4}$$

where actually $\Delta_H = |T| = (-T^2)^{1/2}$ on $\mathcal{H}^0_V$ by Definition 4.4.

The torsion function κ_C of the contact complex is defined using the fourth-order Laplacians $\Delta_0 = \Delta_H^2$ on functions, and $\Delta_1 = D^*D + (d_H\delta_H)^2$ on 1-forms, with Δ_1 conjugated to P^2 by Hodge $*$ duality. Hence (4.4) yields

$$\mathrm{spec}^*(\Delta_1) = 2\times \mathrm{spec}^*(\Delta_H^2) \bigcup \mathrm{spec}^*(-T^2 \mid \mathcal{H}^2_V) \setminus \mathrm{spec}^*(-T^2 \mid \mathcal{H}^0_V)\,. \tag{4.5}$$

Finally by (3.1) the torsion function of the contact complex reads

$$\begin{aligned} \kappa_C(s) &= 2\zeta(\Delta_H^2)(s) - \zeta(\Delta_1)(s) \\ &= \zeta^*(-T^2 \mid \mathcal{H}^0_V)(s) - \zeta^*(-T^2 \mid \mathcal{H}^2_V)(s) + \kappa(M,\rho), \end{aligned} \tag{4.6}$$

where we have set

$$\begin{aligned} \kappa(M,\rho) &= 2\dim(\ker \Delta_H) - \dim(\ker \Delta_1) \\ &= 2\dim(H^0(M,\rho)) - \dim(H^1(M,\rho))\,, \end{aligned} \tag{4.7}$$

since the twisted contact complex is a resolution computing the cohomology of M with values in V.

We proceed similarly for the Hodge–de Rham spectrum, and work again with the calibrated metric $g = d\theta(\cdot, J\cdot) + \theta^2$. Set

$$Q^{\pm} = \pm\frac{1}{2} + \sqrt{\frac{1}{4} + \Delta_0^{dR}} \tag{4.8}$$

where Δ_0^{dR} is Hodge–de Rham (Riemannian) Laplacian acting on functions. According to [7, Corollary 7.6] the spectrum of $d*$ on 2-forms splits as

$$\mathrm{spec}^*(d*) = \mathrm{spec}^*(Q^+) \bigcup \mathrm{spec}^*(-Q^- \mid (\mathcal{H}_V^0)^{\perp}) \bigcup \mathrm{spec}^*(-JT \mid \mathcal{H}_V^2)\,. \tag{4.9}$$

By Definition 4.4, we have $\Delta_0^{dR} = \Delta_H - T^2 = |T| - T^2$ on $\mathcal{H}_V^0$ so that

$$Q^- = -1/2 + \sqrt{1/4 + \Delta_0^{dR}} = |T| \quad \text{on} \quad \mathcal{H}_V^0\,.$$

Then (4.9) also reads

$$\begin{aligned}\mathrm{spec}^*&(d*)\\ &= \mathrm{spec}^*(Q^+) \bigcup \mathrm{spec}^*(-Q^-) \setminus \mathrm{spec}^*(-|T| \mid \mathcal{H}_V^0) \bigcup \mathrm{spec}^*(-JT \mid \mathcal{H}_V^2),\end{aligned}$$

and since δd on 1-forms is $*$ conjugated to $(d*)^2$ on 2-forms, we get

$$\begin{aligned}&\mathrm{spec}^*(\delta d)\\ &= \mathrm{spec}^*(Q^+)^2 \bigcup \mathrm{spec}^*(Q^-)^2 \bigcup \mathrm{spec}^*(-T^2 \mid \mathcal{H}_V^2) \setminus \mathrm{spec}^*(-T^2 \mid \mathcal{H}_V^0).\end{aligned} \tag{4.10}$$

Now following our convention on analytic torsion, inverse to the original definition of Ray–Singer [33] (see Remark 2.6 and (2.13)), the torsion function of de Rham complex reads in dimension 3 as

$$\begin{aligned}\kappa_{RS}(s) &= \sum_{k=0}^{3} (-1)^{k+1} k \zeta(\Delta_k^{dR})(s)\\ &= 3\zeta(\Delta_0^{dR})(s) - \zeta(\Delta_1^{dR})(s)\,,\end{aligned}$$

with $\Delta_i^{dR} = d\delta + \delta d$ the Hodge–de Rham Laplacians on i-forms. Using

$$\zeta^*(\Delta_1^{dR})(s) = \zeta^*(\delta d)(s) + \zeta^*(\Delta_0^{dR})(s)$$

and (4.10) one finds that

$$\begin{aligned}\kappa_{RS}(s) &= 2\zeta^*(\Delta_0^{dR})(s) - \zeta^*(\delta d)(s) + 3\dim(\ker \Delta_0^{dR}) - \dim(\ker \Delta_1^{dR})\\ &= 2\zeta(\Delta_0^{dR})(s) - \zeta(Q^+)(2s) - \zeta(Q^-)(2s)\\ &\quad - \zeta^*(-T^2 \mid \mathcal{H}_V^2)(s) + \zeta^*(-T^2 \mid \mathcal{H}_V^0)(s) + \kappa(M, \rho)\,,\end{aligned}$$

since $\ker Q^+ = \{0\}$ and $\ker Q^- = \ker \Delta_0^{dR}$ by the definition (4.8) of $Q^{\pm}$. Comparing to the contact-complex torsion (4.6) we have shown the following result.

Proposition 4.5. — *On a CR Seifert manifold, the Ray–Singer and contact complex torsion functions twisted by a unitary representation satisfy*

$$\kappa_{RS}(s) - \kappa_C(s) = 2\zeta(\Delta_0^{dR})(s) - \zeta(Q^+)(2s) - \zeta(Q^-)(2s)\,, \tag{4.11}$$

where Δ_0^{dR} is the Hodge–de Rham Laplacian on functions and $Q^\pm = \pm 1/2 + \sqrt{1/4 + \Delta_0^{dR}}$.

Note at this stage that the right-hand side of (4.11) vanishes at $s = 0$, as needed by the vanishing of both torsion functions at $s = 0$; see [33] and Corollary 3.8. This also follows from $\zeta(\Delta_0^{dR})(0) = 0$, for the Hodge–de Rham Laplacian in odd dimension, and that $\zeta(Q^+)(0) = -\zeta(Q^-)(0)$, by [7, Lemma 8.5].

4.3. Proof of Theorem 4.2

In view of Proposition 4.5 we need to show that if we set

$$Q(s) = \zeta(Q^+)(s) + \zeta(Q^-)(s) - 2\zeta(\Delta)(s/2),$$

writing Δ instead of Δ_0^{dR} (the Hodge–de Rham Laplacian on functions) for brevity, then $Q'(0) = 0$. We have a hint that this is true by examining finite energy cut-offs: at any finite spectral level ($\Delta \leqslant \lambda$) it holds that

$$\begin{aligned}
\zeta(Q^+)'(0) + \zeta(Q^-)'(0) &= -\ln\det(Q^+) - \ln\det(Q^-) \\
&= -\ln\det(Q^+ \times Q^-) \\
&= -\ln\det\Delta \\
&= \zeta(\Delta)'(0)\,.
\end{aligned}$$

Hence $Q'(0)$ is insensitive to finite eigenvalues and behaves like a pseudo-differential invariant. It may indeed be seen as a multiplicative anomaly for the regularized determinant of the product of the two commuting operators $Q^\pm$. As thus $Q'(0)$ is related to a Wodzicky residue-type invariant; see [24, §6.5].

In fact the spectral function Q makes sense on any compact Riemannian manifold. The following result holds in a far more general setting than CR Seifert manifolds.

Lemma 4.6. — *On any odd-dimensional compact Riemannian manifold $Q(0) = Q'(0) = 0$.*

Proof. — Consider the one-parameter deformation

$$Q_\lambda^\pm = \pm\lambda + \sqrt{\lambda^2 + \Delta}\,,$$

so that, with $\lambda = 1/2$, $Q_{1/2}^\pm$ coincides with our original $Q^\pm$. Note that the product formula

$$Q_\lambda^+ \times Q_\lambda^- = \Delta$$

we already mentioned is preserved during the deformation. By ellipticity of $Q_\lambda^\pm$ and a Mellin transform

$$\zeta^*(Q_\lambda^\pm)(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1}(\mathrm{Tr}(e^{-tQ_\lambda^\pm}) - \dim\ker Q_\lambda^\pm)dt$$

is holomorphic for large s. Define a function, holomorphic for large s,

$$F^\pm(\lambda, s) = \int_0^{+\infty} t^{s-1}\,\mathrm{Tr}^*(e^{-tQ_\lambda^\pm})dt\,,$$

where here and in the sequel $\mathrm{Tr}^*(P) = \mathrm{Tr}(P) - P(\text{const. function} = 1)$. In particular

$$\begin{aligned}\zeta(Q_{1/2}^+)(s) = \zeta^*(Q_{1/2}^+)(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1}\,\mathrm{Tr}(e^{-tQ_{1/2}^+})dt \\ &= 1 + \Gamma(s)^{-1}F^+(1/2, s)\,,\end{aligned}$$

and

$$\begin{aligned}\zeta(Q_{1/2}^-)(s) = 1 + \zeta^*(Q_{1/2}^-)(s) &= 1 + \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1}\,\mathrm{Tr}(e^{-tQ_{1/2}^-} - 1)dt \\ &= 1 + \Gamma(s)^{-1}F^-(1/2, s)\,.\end{aligned}$$

Thus

$$\begin{aligned}Q(s) &= 1 + \Gamma(s)^{-1}F^+(1/2, s) + 1 + \Gamma(s)^{-1}F^-(1/2, s) - 2 - 2\Gamma(s)^{-1}F(0, s) \\ &\sim s(F^+(1/2, s) + F^-(1/2, s) - 2F(0, s))\end{aligned}$$

when $s \to 0$, with $F(0, s) = F^+(0, s) = F^-(0, s)$. Therefore we need to show that

$$F^+(1/2, 0) + F^-(1/2, 0) - 2F(0, 0) = 0,$$

for which it clearly suffices to show that

$$\partial_\lambda F^+(\lambda, 0) + \partial_\lambda F^-(\lambda, 0) = 0. \tag{4.12}$$

Now one has, for the smooth family of commuting elliptic first-order operators Q_λ,

$$\begin{aligned}\frac{d}{d\lambda}\big(e^{-tQ_\lambda^\pm}\big) &= -t\Big(\pm 1 + \frac{\lambda}{\sqrt{\lambda^2+\Delta}}\Big)e^{-tQ_\lambda^\pm}\\ &= \frac{\pm t}{\sqrt{\lambda^2+\Delta}}\frac{d}{dt}\big(e^{-tQ_\lambda^\pm}\big)\,,\end{aligned}$$

so that

$$\partial_\lambda F^+(\lambda,s) + \partial_\lambda F^-(\lambda,s) = \int_0^{+\infty} t^s \frac{d}{dt}\operatorname{Tr}^*\Big(\frac{e^{-tQ_\lambda^+} - e^{-tQ_\lambda^-}}{\sqrt{\lambda^2+\Delta}}\Big)dt$$

or after integrating by parts,

$$\begin{aligned}&= s\int_0^{+\infty} t^{s-1}\operatorname{Tr}^*\Big(\frac{e^{-tQ_\lambda^+} - e^{-tQ_\lambda^-}}{\sqrt{\lambda^2+\Delta}}\Big)dt\\ &= s\int_0^{+\infty} t^{s-1}2\sinh(t\lambda)\operatorname{Tr}^*\Big(\frac{e^{-t\sqrt{\lambda^2+\Delta}}}{\sqrt{\lambda^2+\Delta}}\Big)dt\,,\end{aligned}$$

or after again integrating by parts,

$$\partial_\lambda F^+(\lambda,s) + \partial_\lambda F^-(\lambda,s) = 2s\int_0^{+\infty} g(\lambda,t,s)\operatorname{Tr}^*(e^{-t\sqrt{\lambda^2+\Delta}})dt \tag{4.13}$$

with

$$g(\lambda,t,s) = \int_0^t u^{s-1}\sinh(u\lambda)du\,.$$

We therefore need to study the residue at $s=0$ of the integral expression in (4.13). First g is easily expanded as

$$\begin{aligned}g(\lambda,t,s) &= \int_0^t u^{s-1}\sum_{p\geqslant 0}\frac{\lambda^{2p+1}u^{2p+1}}{(2p+1)!}du\\ &= \sum_{p\geqslant 0}\frac{\lambda^{2p+1}t^{2p+1+s}}{(2p+1)!\,(2p+s+1)}\,. \qquad (4.14)\end{aligned}$$

Consider next the Poisson kernel $\operatorname{Tr}^*(e^{-t\sqrt{\lambda^2+\Delta}})$ in (4.13); the beginning of its asymptotic expansion as $t\searrow 0$ is related to that of the heat kernel $\operatorname{Tr}(e^{-t(\lambda^2+\Delta)})$ as follows. Recall from e.g. [22, Lemma 1.7.4] that the trace of the heat kernel of a second-order elliptic Laplacian such as $P=\lambda^2+\Delta$ develops when $t\searrow 0$ as

$$\operatorname{Tr}(e^{-tP}) \sim \sum_{k\geqslant 0} c_k t^{k-\frac{m}{2}}\,,$$

where m is the manifold dimension and the c_k are integrals of curvature terms.

Proposition 4.7. — *One has for P and c_k as above, for odd and even dimension m,*

$$\operatorname{Tr}^*(e^{-tP^{1/2}}) = \sum_{k=0}^{[m/2]} \frac{2^{m-2k}}{\sqrt{\pi}}\Gamma\Big(\frac{m-2k+1}{2}\Big)c_k t^{2k-m} - 1 + f(t) \tag{4.15}$$

where $f(t) \to 0$ when $t \searrow 0$.

Proof. — This is a particular case of [1, Theorem 3.1]. Indeed Bär and Moroianu gave there the full development of such kernels on the diagonal. Higher order terms in the development of the Poisson kernel are more involved in odd dimension since they contain log and even non-local coefficients.

Here is an alternative proof of the partial development we need. The classical Laplace transform $\mathcal{L}(t^{-1/2}e^{-1/t}) = \sqrt{\pi}p^{-1/2}e^{-2p^{1/2}}$ leads to the subordination formula

$$e^{-tP^{1/2}} = \pi^{-1/2}\int_0^{+\infty} e^{-u}u^{-1/2}e^{-t^2P/4u}du$$

between Poisson and heat kernels. Therefore summing at the trace level,

$$\begin{aligned}\operatorname{Tr}(e^{-tP^{1/2}}) &= \pi^{-1/2}\int_0^{+\infty} e^{-u}u^{-1/2}\operatorname{Tr}(e^{-t^2P/4u})du\\ &= \pi^{-1/2}\int_0^{+\infty} e^{-u}u^{-1/2}\Big(\sum_{k=0}^{[m/2]} c_k t^{2k-m}(4u)^{m/2-k} + B(t^2/4u)\Big)du\\ &= \sum_{k=0}^{[m/2]} \frac{2^{m-2k}}{\sqrt{\pi}}\Gamma\Big(\frac{m-2k+1}{2}\Big)c_k t^{2k-m}\\ &\qquad\qquad + \pi^{-1/2}\int_0^{+\infty} e^{-u}u^{-1/2}B\Big(\frac{t^2}{4u}\Big)du\end{aligned}$$

with $B(t^2/4u)$ bounded and $B(v) \to 0$ when $v \searrow 0$. This gives (4.15) by dominated convergence and the remark that

$$\operatorname{Tr}^*(e^{-tP^{1/2}}) = \operatorname{Tr}(e^{-tP^{1/2}}) - e^{-t\lambda} = \operatorname{Tr}(e^{-tP^{1/2}}) - 1 + o(1).$$

$\square$

We can now complete the proof of Lemma 4.6. We split (4.13) into

$$\partial_\lambda F^+(\lambda,s)+\partial_\lambda F^-(\lambda,s)=2s\Big(\int_0^1+\int_1^{+\infty}\Big)g(\lambda,t,s)\,\mathrm{Tr}^*(e^{-tP^{1/2}})dt\,.$$

By (4.14) the second integral here is meromorphic with simple poles at $s=-2n-1$ for $n\in\mathbb{N}$; in particular it is regular at $s=0$. Set

$$c'_k=\frac{2^{m-2k}}{\sqrt{\pi}}\Gamma\Big(\frac{m-2k+1}{2}\Big)c_k\,,$$

so that by (4.15)

$$\begin{aligned}
&\int_0^1 g(\lambda,u,s)\,\mathrm{Tr}^*(e^{-uP^{1/2}})du\\
&\qquad=\int_0^1\sum_{0\leqslant k\leqslant[m/2]}\sum_{p\geqslant 0}\frac{\lambda^{2p+1}u^{2p+1+s}}{(2p+1)!(2p+1+s)}c'_k u^{2k-m}du\\
&\qquad\quad+\int_0^1 g(\lambda,u,s)(f(u)-1)du\\
&\qquad=\sum_{0\leqslant k\leqslant[m/2]}\sum_{p\geqslant 0}\frac{c'_k\lambda^{2p+1}}{(2p+1)!(2p+1+s)(2p+2+2k-m+s)}\\
&\qquad\qquad+\text{holomorphic terms for }\mathrm{Re}(s)>-1\,.
\end{aligned}$$

This expression has no pole at $s=0$ if m is odd, giving (4.12) and hence Lemma 4.6. □

We now prove Theorem 4.2. First Proposition 4.5, Lemma 4.6 and (2.13) show that

$$T_C(\rho)=\exp(\kappa'_C(0)/2)=\exp(\kappa'_{RS}(0)/2)=(T_{RS}(\rho))^{-1}\,.$$

The equality of Ray–Singer metrics now comes from (2.14) and (2.15), using the equality of L^2 metrics on $H^*(M,\rho)$ when $H^*(M,\rho)$ is represented by harmonic forms in the de Rham and contact complexes. Indeed these latter two notions coincide on CR Seifert manifolds because of vanishing Tanaka–Webster torsion; see [37, Proposition 12].

Remark 4.8. — The equality of Ray–Singer metrics just proved on CR-Seifert manifolds doesn't help in computing the unknown universal constants in Corollary 3.8. Indeed, the contact form can't be deformed here, since we fix it by requiring that the Reeb flow is induced by the circle action in constant time 2π.

5. The torsion function of CR Seifert manifolds and its dynamical aspects

As an illustration of our viewpoint on analytic torsion, we first show how to compute it on any CR Seifert 3-manifold M equipped with a unitary representation $\rho : \pi_1(M) \to U(N)$.

As we will be only concerned with the contact torsion function κ_C in the sequel, we will denote it by κ for brevity.

Surprisingly, on CR Seifert manifolds, we will see that the *whole* contact torsion function κ, not only $\kappa'(0)$, is expressible using topological data and combinations of Riemann–Hurwitz zeta functions, parametrised by dynamical properties of ρ with respect to the circle action on M. This leads in particular to a Lefschetz-type formula for the torsion; see Theorem 5.7. This extends a result obtained by Fried [17] in the acyclic case using topological methods.

In fact it turns out that our spectral torsion function $\kappa(s)$ may also be seen as a purely dynamical zeta function, constructed from holonomies along all closed orbits of the Reeb field T and its length spectrum. This will be shown in §§5.4 and 5.5. We shall first interpret κ using holomorphic(CR) data on M.

5.1. The torsion function κ from the holomorphic viewpoint

Recall that V is the flat bundle associated to ρ. Let $\overline{V}$ be the conjugate complex vector space, i.e. the same underlying real space with the opposite complex structure, and set

$$W = V \oplus \overline{V}\,.$$

Using the complex structure J, one can split $\Omega^1 H \otimes \mathbb{C} = \Omega^{1,0} H + \Omega^{0,1} H$. We recall that $d_H^{0,1}$ is called the $\overline{\partial}_b$ operator and we consider the induced operator on the flat bundle W

$$\overline{\partial}_W : \Omega^0 W = \Gamma(M, W) \longrightarrow \Omega^{0,1} W = \Omega^{0,1} H \otimes W = \Gamma(M, \Lambda^{0,1} H^* \otimes W)\,. \tag{5.1}$$

Let

$$\mathcal{H}^0_W = \ker \overline{\partial}_W \quad \text{and} \quad \mathcal{H}^1_W = \ker \overline{\partial}^*_W \tag{5.2}$$

denote its cohomology. These spaces are related to $\mathcal{H}^0_V$ and $\mathcal{H}^2_V$ in Definition 4.4 as follows. First, one sees that

$$\begin{gathered}\Omega^{0,1}W = \Omega^{0,1}V \oplus \Omega^{0,1}\overline{V} \simeq \Omega^1 H \otimes V = *(\theta \wedge \Omega^1 H \otimes V)\\ (f,g) \mapsto f + \overline{g}\,,\end{gathered}$$

leading to the canonical isomorphism

$$\mathcal{H}^1_W \simeq *\mathcal{H}^2_V\,.$$

This also yields that

$$\operatorname{spec}(-JT \mid \mathcal{H}^2_V) = \operatorname{spec}(iT \mid \mathcal{H}^1_W). \tag{5.3}$$

Concerning the space $\mathcal{H}^0_W = \ker \overline{\partial}_V \oplus \ker \overline{\partial}_{\overline{V}}$, we have the isomorphism

$$\begin{gathered}\varphi : \mathcal{H}^0_W \cap (\ker T)^\perp \longrightarrow \mathcal{H}^0_V \cap (\ker T)^\perp\\ (f,g) \longmapsto f + \overline{g}\,.\end{gathered}$$

Proof. — We adapt the characterisation of pluri-CR functions given in [7, Prop.7.2].

Let $h \in \mathcal{H}^0_V \cap (\ker T)^\perp$. By Definition 4.4, one has $\Box_V h \in E = \ker \overline{\partial}_V \cap (\ker T)^\perp$. Now $\Box_V$ induces an isomorphism on E. Indeed from

$$\overline{\Box}_V - \Box_V = iT \quad \text{and} \quad \Delta_H = \Box_V + \overline{\Box}_V \tag{5.4}$$

(see e.g. [7, (57)]), one finds that $f = \Delta_H^{-1}\Box_V h \in E$ satisfies $\Box_V h = \Box_V f$, so that $h = \varphi(f, \overline{h - f})$. The injectivity of φ is due to $\ker \overline{\partial}_V \cap \ker \partial_V \subset \ker T$. $\Box$

It also comes from (5.4) that $iT = -\Delta_H = -|T|$ on $\mathcal{H}^0_V$, so that $\varphi(iT) = -|T|\varphi$ and

$$\operatorname{spec}^*(-|T| \mid \mathcal{H}^0_V) = \operatorname{spec}^*(iT \mid \mathcal{H}^0_W)\,.$$

This, together with (5.3), shows that the spectral decomposition (4.4) reads as follows.

Proposition 5.1. — *The spectrum of $P = D * + \delta_H d_H$ splits as*

$$\begin{aligned}\operatorname{spec}^*(P) = \operatorname{spec}^*(\Delta_H) \ &\cup\ \operatorname{spec}^*(-\Delta_H)\\ &\bigcup \operatorname{spec}^*(iT \mid \mathcal{H}^1_W) \setminus \operatorname{spec}^*(iT \mid \mathcal{H}^0_W)\,,\end{aligned}$$

where $W = V \oplus \overline{V}$ and $\mathcal{H}^*_W$ is the cohomology of $\overline{\partial}_W$ as in (5.1)–(5.2).

Remark 5.2. — Compared to the trivial representation case treated in [7, (68)], the only change here is the tensorisation by W.

Expressing (4.5) using the $\overline{\partial}_W$ complex also yields

(5.5)
$$\mathrm{spec}^*(\Delta_1) = 2 \times \mathrm{spec}^*(\Delta_H^2) \bigcup \mathrm{spec}^*(-T^2 \mid \mathcal{H}_W^1) \setminus \mathrm{spec}^*(-T^2 \mid \mathcal{H}_W^0)\,,$$

and (4.6) becomes

$$\begin{aligned}\kappa(s) &= 2\zeta(\Delta_H^2)(s) - \zeta(\Delta_1)(s) \\ &= \zeta^*(-T^2 \mid \mathcal{H}_W^1)(s) - \zeta^*(-T^2 \mid \mathcal{H}_W^0)(s) + \kappa(M,\rho)\,.\end{aligned} \tag{5.6}$$

This Lefschetz-type formula for κ can be seen more topologically. Indeed, Fourier decompose each $\mathbf{V}^x = \Gamma(M, V^x)$ into $\oplus \mathbf{V}_\lambda$ and let

$$\mathbf{W} = \Gamma(M, W) = \bigoplus_{\lambda \in \mathrm{spec}(iT)} \mathbf{W}_\lambda \quad \text{with} \quad \mathbf{W}_\lambda = \mathbf{V}_\lambda \oplus \overline{\mathbf{V}_\lambda}\,.$$

Then using the holomorphic genus

$$\chi_{\overline{\partial}}(\mathbf{W}_\lambda) = \dim \mathcal{H}^0_{\mathbf{W}_\lambda} - \dim \mathcal{H}^1_{\mathbf{W}_\lambda}\,,$$

the torsion function also reads as the Dirichlet series

$$\kappa(s) = \sum_{\lambda \in \mathrm{spec}^*(iT)} \frac{\chi_{\overline{\partial}}(\mathbf{W}_\lambda)}{\lambda^{2s}} + \kappa(M,\rho), \tag{5.7}$$

where, from (4.3), $\mathrm{spec}(iT)$ splits into copies of $(\mathbb{Z} - x)$ on each sub-representation V^x of V, on which $\rho(f) = e^{2i\pi x}$.

Remark 5.3. — For comparison, the eta function of $P = D * + \delta_H d_H$ twisted by ρ may be expressed using Proposition 5.1 in a similar manner. One gets

$$\eta_\rho(P)(s) = \sum_{\lambda \in \mathrm{spec}^*(iT)} \mathrm{sign}(\lambda) \frac{\chi_{\overline{\partial}}(\mathbf{V}_\lambda) - \chi_{\overline{\partial}}(\overline{\mathbf{V}_\lambda})}{|\lambda|^s}\,, \tag{5.8}$$

which is strikingly the 'odd version' of the formula (5.7) for the torsion function κ. Note that by [7, Theorem 8.8], $\eta_\rho(P)(0)$ identifies with $\eta_0(M,\rho)$, the diabatic limit of the Riemannian eta invariant with value in ρ, i.e., the constant term in the development of $\eta(M, g_\varepsilon, \rho)$ for the diabatic metrics (which we also consider in the present paper) $g_\varepsilon = \varepsilon^{-1} d\theta + \varepsilon^{-2}\theta^2$.

5.2. Torsion function κ and the Riemann–Roch–Kawasaki formula

In order to express the series (5.7) using the Riemann–Roch–Kawasaki formula, we need to see the spaces of sections $\mathbf{V}_\lambda$, a priori defined over M, as sections of some V(orbi)-bundles over the orbifold Σ, and compute their degrees and orbifold exponents.

Recall that by (4.3), sections $s \in \mathbf{V}_\lambda$ for $\lambda = n - x$ are sections of V^x over M such that $(iT)s = \lambda s$. Given $\sigma \in \Sigma$, let $S^1(\sigma)$ be the circle orbit in M over σ, and $V_\lambda(\sigma)$ be the vector space of sections of V^x along $S^1(\sigma)$ satisfying $(iT)s = \lambda s$, as above. Call V_λ this family of spaces over Σ. One has clearly that $\mathbf{V}_\lambda = \Gamma(\Sigma, V_\lambda)$, as wished. Moreover V_λ is a vector bundle of dimension $\dim V^x$ over the non singular points of Σ: since the circle action is free there.

To describe its orbifold structure near a singular point $\sigma_i \in \Sigma$, we recall that locally over σ_i, $M = S(L)$ is the quotient of the trivial bundle $\mathbb{C} \times \mathbb{R}$ by the group G_i generated by

$$g_i.(\sigma, t) = (e^{2i\pi/\alpha_i}\sigma, t + 2\pi\beta_i/\alpha_i) \quad \text{and} \quad f.(\sigma,t) = (\sigma, t + 2\pi)\,. \tag{5.9}$$

This G_i is the local fundamental group of M at σ_i, *i.e.* the fundamental group of a tubular neighborhood of the exceptional fiber over σ_i. It is indeed the subgroup of $\pi_1(M)$ generated by the elements also called g_i and f in the presentation given in Proposition 4.3. We note that G_i is generated by a single element $f_i \in \pi_1(M)$ corresponding to the closed primitive orbit over σ_i in M. It is induced by the path $\{0\} \times [0, 2\pi/\alpha_i] \subset \mathbb{C} \times \mathbb{R}$ and $f_i^{\alpha_i} = f$ holds in $\pi_1(M)$, where f is the generic circle orbit. While by shrinking a path linking (σ, t) to $g_i.(\sigma, t)$ in the σ factor, one sees that g_i is homotopic to $f_i^{\beta_i}$. To complete the picture, one observes that $f_i = g_i^{\gamma_i} f^{k_i}$ where $\gamma_i\beta_i + k_i\alpha_i = 1$ for the relatively prime numbers α_i and β_i.

As a consequence we see that, locally over σ_i, the flat bundle V^x over M is isomorphic to the quotient of the trivial bundle $\mathbb{C} \times \mathbb{R} \times \mathbb{C}^{\dim V^x}$ over $\mathbb{C} \times \mathbb{R}$ by the deck transforms

$$\begin{gathered} g_i.(\sigma, t, z_V) = (e^{2i\pi/\alpha_i}\sigma, t + 2\pi\beta_i/\alpha_i, \rho(g_i)z_V) \\ \text{and} \\ f.(\sigma, t, z_V) = (\sigma, t + 2\pi, \rho(f)z_V)\,, \end{gathered} \tag{5.10}$$

using the holonomies $\rho(g_i) = \rho(f_i)^{\beta_i}$ and $\rho(f) = e^{2i\pi x}$ here.

This leads to a description of the bundle V_λ near σ_i. Indeed let E_λ be the trivial bundle over $\mathbb{C}$, whose fiber $E_\lambda(\sigma)$ consists in functions f_{z_V} :

$\mathbb{R} \to \mathbb{C}^{\dim V^x}$ such that $f_{z_V}(t) = e^{-i\lambda t} z_V$. We embed $\Gamma(\mathbb{C}, E_\lambda)$ into $\Gamma(\mathbb{C} \times \mathbb{R}, \mathbb{C}^{\dim V^x})$ by sending

$$s : \sigma \mapsto f_{z_V(\sigma)} \quad \text{to} \quad S : (\sigma, t) \mapsto f_{z_V(\sigma)}(t) = e^{-i\lambda t} z_V(\sigma)\,.$$

Note that $(iT)S = \lambda S$. Therefore S goes down to a section of $\mathbf{V}_\lambda$ over M if it is invariant by g_i and f in (5.10). One has always

$$S(\sigma, t + 2\pi) = e^{-i(n-x)(t+2\pi)} z_V(\sigma) = e^{-i\lambda t} e^{2i\pi x} z_V(\sigma) = \rho(f) S(\sigma, t)$$

showing the f–invariance, while one finds that

$$S(e^{2i\pi/\alpha_i}\sigma, t + 2\pi\beta_i/\alpha_i) = \rho(g_i) S(\sigma, t)$$

if and only if

$$z_V(e^{2i\pi/\alpha_i}\sigma) = e^{2i\pi\lambda\beta_i/\alpha_i} \rho(g_i) z_V(\sigma)\,,$$

or equivalently, if the section s of $E_\lambda \simeq \mathbb{C} \times \mathbb{C}^{\dim V^x}$ is invariant under the transform

$$g_i.(\sigma, z_V) = (e^{2i\pi/\alpha_i}\sigma, e^{2i\pi\lambda\beta_i/\alpha_i} \rho(g_i) z_V)\,. \tag{5.11}$$

Note that $g_i^{\alpha_i} = \mathrm{Id}$ here since by Proposition 4.3, $\rho(g_i)^{\alpha_i} = \varphi(f)^{\beta_i} = e^{2i\pi\beta_i x}$.

We obtain that V_λ is a V-bundle over Σ, since locally over σ_i it is the quotient of E_λ by the finite group $\Gamma_i \simeq \mathbb{Z}/\alpha_i\mathbb{Z}$ generated by g_i in (5.11). We express its isotropy exponents using $\rho(f_i)$. As $f_i^{\alpha_i} = f$ in $\pi_1(M)$, $\rho(f_i)^{\alpha_i} = \rho(f) = e^{2i\pi x}$ on V^x and the spectra of $\rho(f_i)$ satisfy

$$\begin{cases} \operatorname{spec} \rho(f_i) = \{e^{2i\pi x_{i,j}} \mid 1 \leqslant j \leqslant \dim V^x\} \\ \text{with } x_{i,j} = \dfrac{x + k_{i,j}}{\alpha_i} \in [0,1) \text{ and } k_{i,j} \in \mathbb{Z}\,. \end{cases} \tag{5.12}$$

As recalled above $g_i = f_i^{\beta_i}$ in $\pi_1(M)$, so that $\rho(g_i) = \rho(f_i)^{\beta_i}$, hence

$$\operatorname{spec}(e^{2i\pi\lambda\beta_i/\alpha_i}\rho(g_i)) = \{e^{2i\pi(n+k_{i,j})\beta_i/\alpha_i}\}\,.$$

Then (5.11) shows that the isotopy exponents of the V-bundle V_λ are all the couples

$$(\alpha_i, (n + k_{i,j})\beta_i \mod \alpha_i)\,. \tag{5.13}$$

To compute the degree of the V-bundle V_λ, we consider the modified connection on V^x

$$\nabla^\lambda = \nabla^{\mathrm{flat}}_\rho + i\lambda\theta\,. \tag{5.14}$$

By definition, $\nabla^\lambda_T s = 0$ for any $s \in \mathbf{V}_\lambda$. Hence ∇^λ goes down on $T\Sigma$ as a unitary connection on V_λ. To compute its curvature, we lift vector fields X, $Y \in T\Sigma$ to horizontal ones $\widetilde{X}$, $\widetilde{Y}$ in $TM \cap \ker\theta$, so that

$$\begin{aligned}
R_{\nabla^\lambda}(X,Y) &= \nabla^\lambda_{\widetilde{X}}\nabla^\lambda_{\widetilde{Y}} - \nabla^\lambda_{\widetilde{Y}}\nabla^\lambda_{\widetilde{X}} - \nabla^\lambda_{[\widetilde{X},\widetilde{Y}]}\\
&= R_{\nabla^{\mathrm{flat}}_\rho}(\widetilde{X},\widetilde{Y}) - i\lambda\theta([\widetilde{X},\widetilde{Y}])\\
&= i\lambda d\theta(X,Y)\,.
\end{aligned}$$

Therefore the bundle $(V_\lambda, \nabla^\lambda)$ has curvature $\Omega = i\lambda d\theta$ and rational degree (see [7, 20, 29])

$$\begin{aligned}
\deg(V_\lambda) &= \frac{i}{2\pi}\int_\Sigma \mathrm{Tr}_{V_\lambda}(i\lambda d\theta)\\
&= \dim(V^x)\lambda\deg(L)\,,
\end{aligned}\tag{5.15}$$

because $\Omega_L = id\theta$ is the curvature of L. Recall that by Proposition 4.3, $\deg(L) = d = b + \sum_i \frac{\beta_i}{\alpha_i}$. By conjugation we have also

$$\deg(\overline{V_\lambda}) = -\dim(V^x)\lambda\deg(L) = -\deg(V_\lambda)\,,$$

so that finally

$$\deg(W_\lambda) = \deg(\overline{V_\lambda}) + \deg(V_\lambda) = 0\,,$$

as expected for this smooth part due to the real structure on $W_\lambda = V_\lambda \oplus \overline{V_\lambda}$; see [26, §14]. We did the above computation for completeness, as the degree of V_λ is needed to study the eta function given in Remark 5.3.

To complete our analysis of the series (5.7) as seen from Σ, we now interpret $\chi_{\overline{\partial}}(\mathbf{W}_\lambda)$ as the holomorphic Euler characteristic $\chi_{\overline{\partial}}(W_\lambda)$ of the V-bundle V_λ over Σ. This comes from the isomorphism of the two complexes:

$$\overline{\partial}_V : \Gamma(M,V) \cap \{iT = \lambda\} \to \Gamma(M, \Lambda^{0,1}H^* \otimes V) \cap \{iT = \lambda\}\,,$$

and

$$\overline{\partial}_{V_\lambda} : \Gamma(\Sigma, V_\lambda) \to \Omega^{0,1}V_\lambda = \Gamma(\Sigma, \Lambda^{0,1}H^* \otimes V_\lambda)\,.$$

Indeed the section spaces correspond, while using the connection (5.14) on $\mathbf{V}_\lambda$, one sees that

$$\overline{\partial}_V = \nabla^{0,1}_\rho = (\nabla^\lambda)^{0,1} = \overline{\partial}_{V_\lambda}\,.$$

As a conclusion, we can replace $\mathbf{V}_\lambda$ by V_λ, and $\mathbf{W}_\lambda$ by W_λ in the formulas (5.7)–(5.8) for the torsion and eta functions on M, and work with these V-bundles over Σ instead.

We can now apply Riemann–Roch–Kawasaki formula (see [20, 29, 7]) to the V-bundle W_λ. It states that

$$\begin{aligned}\chi_{\overline{\partial}}(W_\lambda) &= \dim(W_\lambda)(1-g) + \deg(W_\lambda) - \sum_{i,j}\left\{\frac{\beta_i(W_\lambda)}{\alpha_i(W_\lambda)}\right\} \\ &= \dim(V^x)\chi(\widetilde{\Sigma}) - \sum_{i,j}\left\{\frac{(n+k_{i,j})\beta_i}{\alpha_i}\right\} + \left\{\frac{-(n+k_{i,j})\beta_i}{\alpha_i}\right\}, \qquad (5.16)\end{aligned}$$

by (5.13). Here $\{a\} = a - [a] \in [0,1)$ is the fractional part of a and $\chi(\widetilde{\Sigma}) = 2-2g$ is the Euler characteristic of the smooth surface $\widetilde{\Sigma}$ associated to the orbifold Σ; see e.g. [29]. Observe that (5.16) does give integers since $\{a\}+\{-a\}$ is 0 when $a\in\mathbb{Z}$ and 1 otherwise. Recall also that, to ensure smoothness of the V-bundle $M = S(L)$, the numbers α_i and β_i are assumed relatively prime, thus giving a free action of $\mathbb{Z}/\alpha_i\mathbb{Z}$ at orbifold points. Hence the fractional part in (5.16) simplifies using

$$\delta(n,i,j) = \begin{cases} 1 & \text{if} \quad n+k_{i,j}\in\alpha_i\mathbb{Z} \\ 0 & \text{otherwise}. \end{cases} \qquad (5.17)$$

Then we have

$$\chi_{\overline{\partial}}(W_\lambda) = \dim(V^x)\chi(\Sigma^*) + \sum_{i,j}\delta(n,i,j)\,, \qquad (5.18)$$

where

$$\chi(\Sigma^*) = 2-2g-|I|$$

is the Euler characteristic of the punctured surface $\Sigma^* = \Sigma\setminus\cup_I\{x_i\}$ at the $|I|$ orbifold points.

For $a\in]0,1]$ let $\zeta(s,a) = \sum_{n\in\mathbb{N}}\frac{1}{(n+a)^s}$ be Hurwitz zeta function. We can now express κ as a combination of such functions. This is the first step towards the identification of the torsion function as a dynamical zeta function given in §5.4.

Theorem 5.4. — *Split V into irreducible V^x; then the torsion function spectrally decomposes as*

$$\kappa(s) = \sum_{V^x}\kappa_x(s)$$

such that :

• *On* V^x *with* $x \in]0,1[$, *i.e.* $\rho(f) = e^{2i\pi x} \neq \mathrm{Id}$, *we have*

$$\begin{aligned}\kappa_x(s) &= \dim(V^x)\chi(\Sigma^*)\big(\zeta(2s,x) + \zeta(2s,1-x)\big)\\ &\quad + \sum_{i,j} \frac{1}{\alpha_i^{2s}}\big(\zeta(2s,x_{i,j}) + \zeta(2s,1-x_{i,j})\big)\,.\end{aligned} \tag{5.19}$$

• *On* $V^0 = \ker(\mathrm{Id} - \rho(f))$ *let* $V^{0,i} = \ker(\mathrm{Id} - \rho(f_i))$*; then we have*

$$\begin{aligned}\kappa_0(s) &= \kappa(M,\rho)(2\zeta(2s)+1) + 2\zeta(2s)\sum_i \dim(V^{0,i})\big(\alpha_i^{-2s} - 1\big)\\ &\quad + \sum_{i,j \,|\, x_{i,j}\neq 0} \frac{1}{\alpha_i^{2s}}\big(\zeta(2s,x_{i,j}) + \zeta(2s,1-x_{i,j})\big)\,.\end{aligned} \tag{5.20}$$

This relates the torsion function to dynamical properties of the circle action here. Indeed apart from the cohomological term $\kappa(M,\rho)$, the expression is clearly built on the holonomy properties of ρ along the various closed primitive orbits of the flow: the generic orbit f of the action over Σ^*, and associated holonomy $\rho(f) = e^{2i\pi x}$ on V^x, and the exceptional orbits f_i of holonomy $\rho(f_i) = \{e^{2i\pi x_{i,j}}\}$.

Proof. —

• We compute first the contribution of V^x for $x \neq 0$, i.e. when $\rho(f) = e^{2i\pi x} \neq \mathrm{Id}$. Here $iT = \lambda = n - x \neq 0$ always, and by (5.7) and (5.18)

$$\begin{aligned}\kappa_x(s) &= \dim(V^x)\chi(\Sigma^*)\sum_{n\in\mathbb{Z}} \frac{1}{|n+x|^{2s}} + \sum_{i,j}\sum_{k\in\mathbb{Z}} \frac{1}{|k\alpha_i + k_{i,j} + x|^{2s}}\\ &= \dim(V^x)\chi(\Sigma^*)\Big(\sum_{n\geqslant 0} \frac{1}{|n+x|^{2s}} + \sum_{n>0} \frac{1}{|-n+x|^{2s}}\Big)\\ &\quad + \sum_{i,j} \frac{1}{\alpha_i^{2s}} \sum_{k\in\mathbb{Z}} \frac{1}{|k + x_{i,j}|^{2s}},\end{aligned}$$

by (5.12). This leads to (5.19).

• We compute now κ_0, including the cohomological term $\kappa(M,\rho)$ from (4.7) and (5.7), since harmonic forms only appear in $\ker(iT) \subset V^0$; see e.g. [37, Proposition 12]. We have

$$\kappa_0(s) = \dim(V^0)\chi(\Sigma^*)\sum_{n\in\mathbb{Z}^*} \frac{1}{|n|^{2s}} + \sum_{i,j}\sum_{n\in\mathbb{Z}^*} \frac{\delta(n,i,j)}{|n|^{2s}} + \kappa(M,\rho)\,. \tag{5.21}$$

We recall from (4.7) that

$$\kappa(M,\rho) = 2\dim H^0(M,\rho) - \dim H^1(M,\rho)$$

can be computed using contact-harmonic forms on M. By [37, Proposition 12] contact-harmonic forms are both holomorphic and T-invariant since the

Reeb flow preserves J here, i.e. Tanaka–Webster torsion vanishes. Then one gets

$$\begin{aligned}(5.22)\qquad \kappa(M,\rho) &= \chi_{\overline{\partial}}(W_0)\\ &= \dim(V^0)\chi(\Sigma^*) + \sum_{i,j}\delta(0,i,j) \quad \text{by (5.18)}\end{aligned}$$

$$(5.23)\qquad \kappa(M,\rho) = \dim(V^0)\chi(\Sigma^*) + \sum_{i}\dim(V^{0,i})\,,$$

since, by (5.12) and (5.17), $\delta(0,i,j)=1$ if and only if $x_{i,j}=0$. Then (5.21) reads

$$\kappa_0(s) = \kappa(M,\rho)(2\zeta(2s)+1) + \sum_{i,j}\sum_{n\in\mathbb{Z}^*}\frac{\delta(n,i,j)}{|n|^{2s}} - 2\zeta(2s)\sum_i \dim(V^{0,i})\,.$$

We observe now that if $x_{i,j}\in\,]0,1[$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}^*}\frac{\delta(n,i,j)}{|n|^{2s}} &= \sum_{k\in\mathbb{Z}}\frac{1}{|k\alpha_i + k_{i,j}|^{2s}}\\ &= \frac{1}{\alpha_i^{2s}}\big(\zeta(2s,x_{i,j}) + \zeta(2s,1-x_{i,j})\big) \quad \text{by (5.12).}\end{aligned}$$

On the other hand for $x_{i,j}=0$,

$$\sum_{n\in\mathbb{Z}^*}\frac{\delta(n,i,j)}{|n|^{2s}} = \sum_{k\in\mathbb{Z}^*}\frac{1}{|k\alpha_i|^{2s}} = \frac{2}{\alpha_i^{2s}}\zeta(2s)\,,$$

as needed in (5.20). □

The expression of κ_x given in Theorem 5.4 vanishes at $s=0$, as it ought to by Corollary 3.8. Here this follows from the classical result $\zeta(0,a) = 1/2 - a$; see [46, §13] for instance. The following observation will be useful in the sequel.

Corollary 5.5. — *The torsion function κ has a unique simple pole at $s=1/2$ with residue*

$$\mathrm{Res}_{1/2}(\kappa) = \chi(\Sigma)\dim V,$$

where $\chi(\Sigma) = 2-2g+\sum_i(\frac{1}{\alpha_i}-1)$ denotes the rational Euler class of the orbifold Σ.

Proof. — One knows (see [46, §13]) that the Hurwitz zeta function $\zeta(2s,a)$ has a unique simple pole at $s=1/2$ with residue $1/2$. Then (5.19) on V^x yields

$$\mathrm{Res}_{1/2}(\kappa_x) = \dim V^x\chi(\Sigma^*) + \dim V^x\sum_i\frac{1}{\alpha_i} = \dim V^x\chi(\Sigma)\,.$$

On the other hand by (5.20) in V^0,

$$\begin{aligned}\mathrm{Res}_{1/2}(\kappa_0) &= \kappa(M,\rho) + \sum_i \dim(V^{0,i})(\frac{1}{\alpha_i} - 1) + \sum_i \dim((V^{0,i})^\perp)\frac{1}{\alpha_i}\\ &= (\kappa(M,\rho) - \sum_i \dim V^{0,i}) + \dim V^0 \sum_i \frac{1}{\alpha_i}\\ &= \dim V^0\big(\chi(\Sigma^*) + \sum_i \frac{1}{\alpha_i}\big) = \dim V^0 \chi(\Sigma)\,,\end{aligned}$$

by (5.23). □

Remark 5.6. — We lastly observe that a similar treatment applies to handle the twisted eta function in Remark 5.3. Indeed by the Riemann–Roch–Kawasaki formula and (5.15) one has

$$\chi_{\overline{\partial}}(V_\lambda) - \chi_{\overline{\partial}}(\overline{V_\lambda}) = 2\dim(V^x)\lambda d(L) + \sum_{i,j}\left\{\frac{(n+k_{i,j})\beta_i}{\alpha_i}\right\} - \left\{\frac{-(n+k_{i,j})\beta_i}{\alpha_i}\right\},$$

and by (5.8) the contribution of V^x to eta is

$$\begin{aligned}\eta_x(P)(s) = 2\dim(V^x)d(L) &\sum_{\lambda\in \mathrm{spec}^*(iT)} \frac{1}{|\lambda|^{s-1}}\\ &+ \sum_{i,j}\sum_{\lambda\in\mathrm{spec}^*\, iT}\Big(2\left\{\frac{(n+k_{i,j})\beta_i}{\alpha_i}\right\} - 1 + \delta(n,i,j)\Big)\frac{\mathrm{sgn}(\lambda)}{|\lambda|^s}\,.\end{aligned}$$

The generic smooth contribution may be written as

$$2\dim(V^x)d(L) \times \begin{cases} 2\zeta(s-1) & \text{if} \quad x = 0\\ \zeta(s-1,x) + \zeta(s-1,1-x) & \text{if} \quad x \neq 0\,,\end{cases}$$

taking value

$$-\dim(V^x)d(L)\big(\frac{1}{6} + x(1-x)\big)$$

at $s = 0$; see [46, §13]. Following Nicolaescu's work [29], the remaining 'periodic' eta term can be handled using Dedekind–Rademacher sums and Hurwitz functions; see Proposition 1.10 and Lemma 1.11 in [29] for details.

5.3. A Lefschetz-type formula for the Ray–Singer metric

Using Theorem 4.2 we can now compute the Ray–Singer analytic torsion $T_{RS} = \exp(-\kappa'(0)/2) = (T_C)^{-1}$, which gives the associated Ray–Singer metric on $\det H^*(M,\rho)$,

$$\|\ \ \|_{RS} = (T_{RS})^{-1}\,|\ \ |_{L^2(\Omega^*M)}\,. \qquad (5.24)$$

In the acyclic case, i.e. $H^*(M,\rho)=0$, Fried [17] has shown that the Reidemeister–Franz torsion, and thus the analytic torsion by the works [15, 28] of Cheeger and Müller, may be nicely expressed 'à la Lefschetz' using determinants associated to the generic and exceptional holonomies along the primitive orbits of the circle action. For a general unitary representation $\rho : \pi_1(M) \to U(N)$ we obtain:

Theorem 5.7. — *Let $\rho(f)^\top$ and $\rho(f_i)^\top$ denote the restriction of these holonomies to respectively $(V^0)^\perp$ and $(V^{0,i})^\perp$ with*

$$V^0 = \ker(\mathrm{Id} - \rho(f)) \quad \text{and} \quad V^{0,i} = \ker(\mathrm{Id} - \rho(f_i))\,.$$

Then Ray–Singer analytic torsion $T_{RS}(M,\rho) = \exp(-\kappa'(0)/2)$ is given by

$$T_{RS}(M,\rho) = (2\pi)^{\kappa(M,\rho)} |\det(\mathrm{Id} - \rho(f)^\top)|^{\chi(\Sigma^*)} \prod_i \frac{|\det(\mathrm{Id} - \rho(f_i)^\top)|}{\alpha_i^{\dim(V^{0,i})}}\,. \tag{5.25}$$

Proof. — By Lerch's formula $\partial_s \zeta(s,x)_{s=0} = \ln\Gamma(x) - \frac{1}{2}\ln(2\pi)$, see [46, §13], we have

$$\begin{aligned}
\partial_s\zeta(0,x) + \partial_s\zeta(0,1-x) &= \ln\big(\Gamma(x)\Gamma(1-x)/2\pi\big) \\
&= -\ln(2\sin(\pi x)) \quad \text{by the Euler reflection formula,} \\
&= -\ln|1-e^{2i\pi x}|\,.
\end{aligned}$$

Hence by (5.19) on V^x,

$$-\kappa_x'(0)/2 = \dim(V^x)\chi(\Sigma^*)\ln|1-e^{2i\pi x}| + \sum_{i,j} \ln|1-e^{2i\pi x_{i,j}}|\,,$$

which gives the determinant contribution of V^x to (5.25).

By (5.20) on V^0 and Lerch's formula again, one finds

$$-\kappa_0'(0)/2 = -2\zeta'(0)\kappa(M,\rho) - \sum_i \dim(V^{0,i})\ln\alpha_i + \sum_{i,j\,|\,x_{i,j}\neq 0} \ln|1-e^{2i\pi x_{i,j}}|$$

similarly as above. This gives the needed contribution of V^0 to (5.25). □

As required, formula (5.25) coincides with that of Fried [17, p. 198] for acyclic representations. The only new factor in our case is the cohomological term $(2\pi)^{\kappa(M,\rho)}$. That the full expression for the torsion is 'quantized' here is due to the rigidity of volume in this CR Seifert case. Namely, the size of θ is fixed such that the circle action is generated by the Reeb field T in constant time 2π, hence the volume forms $\mathrm{dvol} = \theta\wedge d\theta$ on TM and $d\theta$ on H are also fixed. Thus, given θ, a variation of calibrated metrics (2.1) only comes from a variation of complex structure, and one sees easily that on horizontal forms

$$\alpha = *^{-1} *^{\bullet} = J^{-1} J^{\bullet}$$

with notations from Section 3.2. Now we recall that in vanishing torsion, J preserves harmonic forms in $\mathcal{H}^k(\mathcal{E}, d_H)$, see [37, II§3]. Then the L^2 metric induced on $\det \mathcal{H}^*(M, \rho)$ is constant in the CR Seifert case since by formula (3.13) one has

$$(\ln |\quad|_{L^2})^{\bullet} = \sum_{k=0}^{n} (-1)^k \operatorname{Tr}(\alpha \mathcal{P}_k) = 0\,,$$

because $\alpha J = -J\alpha$ and $\mathcal{P}_k J = J\mathcal{P}_k$.

5.4. The contact torsion function as a dynamical zeta function

In [16, 17], Fried proposed to express the torsion using the following basic dynamical objects.

For each free homotopical class C of periodic orbit of the Reeb field T, let $\ell(C)$ denote its length and $\operatorname{ind}(C)$ its Fuller index; see [19] or [16, §4] for an account of these notions.

Proposition 5.8 ([16, Lemma 5.3], [17]). — *The free homotopy classes of closed orbits of T are the following :*

(1) *f^n with $n \in \mathbb{N}^*$, of length $2\pi n$ and Fuller index $\chi(\Sigma)/n$, where*

$$\chi(\Sigma) = 2 - 2g - \sum_i (1 - 1/\alpha_i) = \chi(\Sigma^*) + \sum_i 1/\alpha_i$$

is the rational Euler class of the quotient orbifold $\Sigma = M/\langle T \rangle$;

(2) *the isolated f_i^n for $n \notin \alpha_i \mathbb{N}$, of length $2\pi n/\alpha_i$ and Fuller index $1/n$.*

Fried observed that for acyclic unitary representations one has

$$T_{RS}(M, \rho) = \left|\exp(Z_F(0))\right|\,, \tag{5.26}$$

where $Z_F(0)$ stands for the analytic continuation at $s = 0$ of the dynamical function

$$Z_F(s) = -\sum_C \operatorname{ind}(C) \operatorname{Tr}(\rho(C)) e^{-s\ell(C)}\,.$$

This can be checked directly from the calculation of torsion in Theorem 5.7 and Proposition 5.8, as in [17, §1]. Such a link between analytic torsion and flow dynamics is not coincidental; it has already been observed in many other geometric situations, see e.g. [16]. In particular it holds for the geodesic flow on hyperbolic manifolds, as proved by Fried in [18], or more generally on locally symmetric spaces of non-positive sectional curvature, as proved by Moscovici and Stanton in [27]. In such cases these results are rooted in Selberg's trace formula, expressing heat kernel traces as a sum of traces along closed geodesics.

We are not dealing with a geodesic flow here, but in view of Theorem 5.4, it is quite natural to try to express the contact torsion function $\kappa(s)$ itself using the same dynamical data as above. Indeed the *whole* spectral function κ may be nicely interpreted 'à la Selberg' as a purely dynamical zeta function of the Reeb flow.

THEOREM 5.9. — *Let*

$$f(s) = \Gamma(s)\cos(\frac{\pi s}{2})\,.$$

For C closed let ${}^R\mathrm{Tr}(\rho(C))$ denote the real part of the trace of $\rho(C)$ on V_ρ. Then

$$f(s)\big(\kappa(s/2) - \kappa(M,\rho)\big) = \sum_C \mathrm{ind}(C)\,{}^R\mathrm{Tr}(\rho(C))\,\ell(C)^s\,, \tag{5.27}$$

where the sum is taken over all free homotopical classes of closed orbits of the Reeb flow.

We first make some comments about this identity. First, to remove the real part in (5.27), one could also sum over all orbits C *and* C^{-1}, the latter corresponding to the opposite flow $-T$. Indeed the torsion function is not sensitive to a change of $\theta \mapsto -\theta$, together with $J \mapsto -J$, since κ is defined using real operators Δ_0 and Δ_1.

Also let

$$Z_\rho(s) = \sum_C \mathrm{ind}(C)\,{}^{\mathrm{R}}\mathrm{Tr}(\rho(C))\ell(C)^s \tag{5.28}$$

be the dynamical zeta side of (5.27). From Proposition 5.8 this converges for $\mathrm{Re}(s) < 0$. In contrast the spectral side

$$\kappa^*(s/2) = \kappa(s/2) - \kappa(M,\rho) = 2\,\mathrm{Tr}^*(\Delta_0^{-s/2}) - \mathrm{Tr}^*(\Delta_1^{-s/2})$$

are converging series for $\mathrm{Re}(s) > 2$. Hence, when seen as series, the spectral and dynamical sides of (5.27) don't converge for the same s, and the identity only holds through meromorphic continuation. When $s \to 0$ we get

$$\lim_{s\to 0}\Big(Z_\rho(s) + \frac{\kappa(M,\rho)}{s}\Big) = \kappa'(0)/2 = -\ln(T_{RS}(M,\rho))\,, \tag{5.29}$$

and (ln of) the analytic torsion may be seen as a topological regularisation of the *formal* dynamical series

$$-'' Z_\rho(0)'' = -\sum_C \operatorname{ind}(C)\, {}^{\mathrm{R}}\mathrm{Tr}(\rho(C))\,. \tag{5.30}$$

Comparing with (5.26) and Fried's dynamical function yields

$$'' Z_\rho(0)'' = Z_\rho(0) = -\operatorname{Re}(Z_F(0))\,,$$

in the acyclic case, and the dynamical functions Z_F and Z_ρ both provide analytic continuation of the same dynamical series in (5.30). This series has been interpreted in [16, 17] as being the total Fuller measure of periodic orbits, and has a formal invariance by deformation of the flow, as long as orbit periods stay bounded.

We note also that the trace formula (5.27) can be written in a more symmetric manner

$$\Gamma(s)\kappa^*(s) = \frac{2^{1-2s}}{\sqrt{\pi}}\Gamma(\frac{1}{2}-s)Z_\rho(2s)\,, \tag{5.31}$$

as follows from the classical identities (see [46])

$$\Gamma(s)\Gamma(s+\frac{1}{2}) = 2^{1-2s}\sqrt{\pi}\Gamma(2s) \quad \text{and} \quad \Gamma(s+\frac{1}{2})\Gamma(-s+\frac{1}{2}) = \frac{\pi}{\cos(\pi s)}\,.$$

This formulation will be useful in §5.5.

As a last comment, we observe that the trace formula (5.27) is homogeneous in the constant rescaling $\theta \mapsto K\theta$. Indeed, the metric here is $g = d\theta(\cdot, J\cdot) + \theta^2$, hence $\ell(C) = \int_C \theta$ changes to $K\ell(C)$, while the fourth-order contact Laplacians Δ_i are homogeneous and rescale to $K^{-2}\Delta_i$. Thus $\zeta^*(\Delta^{1/2})(s)$ rescales to $K^s\zeta^*(\Delta^{1/2})(s)$ as needed. Such a property does not hold for the Hodge–de Rham Laplacians.

Proof of Theorem 5.9. — We will start from the expression of $\kappa(s)$ by Hurwitz zeta functions as given in Theorem 5.4. First Hurwitz's formula (see [46, §13]) states that for $\operatorname{Re}(s) < 0$

$$\zeta(s,x) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\Big[\sin(\frac{\pi s}{2})\sum_{n=1}^{+\infty}\frac{\cos(2\pi x n)}{n^{1-s}} + \cos(\frac{\pi s}{2})\sum_{n=1}^{+\infty}\frac{\sin(2\pi x n)}{n^{1-s}}\Big]\,,$$

so that using $f(s)f(1-s) = \pi/2$ gives

$$f(s)(\zeta(s,x)+\zeta(s,1-x)) = \sum_{n=1}^{+\infty}\operatorname{Re}\big(e^{2i\pi x n}\big)\frac{(2\pi n)^s}{n}\,. \tag{5.32}$$

Note also the corresponding limit expression when $x \to 0^+$ with $\mathrm{Re}(s) < 0$:

$$2f(s)\zeta(s) = \sum_{n=1}^{+\infty} \frac{(2\pi n)^s}{n} \,. \tag{5.33}$$

• We study the contribution of κ_x on V^x with $x \in (0,1)$. By (5.32), formula (5.19) yields

$$f(s)\kappa_x(s/2) = \chi(\Sigma^*) \sum_{n\geqslant 1} \frac{{}^{\mathrm{R}}\mathrm{Tr}(\rho(f^n))}{n}(2\pi n)^s + \sum_i \sum_{n\geqslant 1} \frac{{}^{\mathrm{R}}\mathrm{Tr}(\rho(f_i^n))}{n}\Big(\frac{2\pi n}{\alpha_i}\Big)^s,$$

hence by Proposition 5.8,

$$\begin{aligned} f(s)\kappa_x(s/2) &= \sum_{n\geqslant 1}\Big(\mathrm{ind}(f^n) - \sum_i \frac{1}{n\alpha_i}\Big)\,{}^{\mathrm{R}}\mathrm{Tr}(\rho(f^n))\ell(f^n)^s \\ &\quad + \sum_i \sum_{n\notin\alpha_i\mathbb{N}} \mathrm{ind}(f_i^n)\,{}^{\mathrm{R}}\mathrm{Tr}(\rho(f_i^n))\ell(f_i^n)^s \\ &\quad + \sum_i \sum_{k\geqslant 1} \frac{{}^{\mathrm{R}}\mathrm{Tr}(\rho(f^k))}{k\alpha_i}\ell(f^k)^s \\ &= \sum_C \mathrm{ind}(C)\,{}^{\mathrm{R}}\mathrm{Tr}(\rho(C))\, l(C)^s\,, \end{aligned}$$

as needed in (5.27).

• We now study κ_0 on $V^0 = \ker(\mathrm{Id} - \rho(f))$. Recall that $\mathrm{spec}\,\rho(f_i) = \{e^{2i\pi x_{i,j}}\}$ and $V^{0,i} = \ker(\mathrm{Id} - \rho(f_i))$. By (5.32)–(5.33) formula (5.20) reads

$$\begin{aligned} f(s)\big(\kappa_0(s/2) - \kappa(M,\rho)\big) &= \kappa(M,\rho)\sum_{n\geqslant 1}\frac{(2\pi n)^s}{n} \\ &\quad + \sum_i \sum_{n\geqslant 1} \frac{{}^{\mathrm{R}}\mathrm{Tr}_{(V^{0,i})^\perp}(\rho(f_i^n))}{n}\Big(\frac{2\pi n}{\alpha_i}\Big)^s \\ &\quad + \sum_i \sum_{n\geqslant 1} \dim(V^{0,i})(\alpha_i^{-s} - 1)\frac{(2\pi n)^s}{n}\,. \end{aligned} \tag{5.34}$$

By (5.23), the first series reads

$$\begin{aligned} \kappa(M,\rho)\sum_{n\geqslant 1}\frac{(2\pi n)^s}{n} &= \Big[\dim(V^0)\big(\chi(\Sigma) - \sum_i \frac{1}{\alpha_i}\big) + \sum_i \dim(V_0^i)\Big]\sum_{n\geqslant 1}\frac{(2\pi n)^s}{n} \\ &= \sum_{n\geqslant 1}\mathrm{ind}(f^n)\,{}^{\mathrm{R}}\mathrm{Tr}(\rho(f^n))\ell(f^n)^s - \sum_i\sum_{n\geqslant 1}\dim(V^0)\frac{(2\pi n)^s}{n\alpha_i} \\ &\quad + \sum_i\sum_{n\geqslant 1}\dim(V^{0,i})\frac{(2\pi n)^s}{n}\,. \end{aligned}$$

Since $\rho(f_i) = \mathrm{Id}$ on $V^{0,i}$, the second series in (5.34) splits into

$$\begin{aligned}\sum_i \sum_{n\geqslant 1} \frac{{}^{\mathrm{R}}\mathrm{Tr}_{(V^{0,i})^\perp}(\rho(f_i^n))}{n}\Big(\frac{2\pi n}{\alpha_i}\Big)^s \\ &= \sum_i \sum_{n\geqslant 1} \frac{{}^{\mathrm{R}}\mathrm{Tr}(\rho(f_i^n))}{n}\Big(\frac{2\pi n}{\alpha_i}\Big)^s - \sum_i \sum_{n\geqslant 1} \frac{\dim V^{0,i}}{n}\Big(\frac{2\pi n}{\alpha_i}\Big)^s \\ &= \sum_i \sum_{n\notin \alpha_i\mathbb{N}} \mathrm{ind}(f_i^n)\, {}^{\mathrm{R}}\mathrm{Tr}(\rho(f_i^n))\, \ell(f_i^n)^s \\ &\quad + \sum_i \sum_{k\geqslant 1} \dim V^0 \frac{(2\pi k)^s}{k\alpha_i} - \sum_i \sum_{n\geqslant 1} \frac{\dim V^{0,i}}{n\alpha_i^s}(2\pi n)^s\end{aligned}$$

since $\rho(f_i^n) = \rho(f^k) = \mathrm{Id}$ on V^0 for $n = k\alpha_i$. Therefore after cancellations (5.34) yields

$$f(s)(\kappa_0(s/2) - \kappa(M,\rho)) = \sum_C \mathrm{ind}(C)\, {}^{\mathrm{R}}\mathrm{Tr}(\rho(C))\ell(C)^s\,,$$

as needed. □

5.5. The torsion heat trace as a dynamical theta function

In Theorem 5.9, spectral and dynamical aspects of analytic torsion are compared through zeta functions. One can also work at the level of heat kernels. Consider the heat operators of the fourth-order Laplacians Δ_0 and Δ_1 of the contact complex, and set

$$\begin{aligned}\mathrm{Tr}_\kappa(e^{-t\Delta}) &= 2\,\mathrm{Tr}(e^{-t\Delta_0}) - \mathrm{Tr}(e^{-t\Delta_1}) \\ &= \mathrm{Tr}^*_\kappa(e^{-t\Delta}) + \kappa(M,\rho)\,.\end{aligned} \tag{5.35}$$

Recall that for $\mathrm{Re}(s) > 1$,

$$\kappa^*(s) = 2\zeta^*(\Delta_0)(s) - \zeta^*(\Delta_1)(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\,\mathrm{Tr}^*_\kappa(e^{-t\Delta})dt\,. \tag{5.36}$$

Then the following trace formula holds in our CR Seifert setting.

Theorem 5.10. — *One has*

$$\mathrm{Tr}_\kappa(e^{-t\Delta}) = \dim V\frac{\sqrt{\pi}\chi(\Sigma)}{\sqrt{t}} + \frac{1}{\sqrt{\pi t}}\sum_C \ell(C)\,\mathrm{ind}(C)\,{}^{\mathrm{R}}\mathrm{Tr}(\rho(C))e^{-\ell(C)^2/4t}\,, \tag{5.37}$$

where C runs over free homotopical classes of closed orbits of the Reeb flow and $\chi(\Sigma)$ is the rational Euler class of the quotient orbifold.

Hence the torsion heat trace is closely related to the purely dynamical theta function

$$\vartheta(t) = \frac{1}{\sqrt{\pi t}} \sum_C \ell(C)\,\mathrm{ind}(C)\,{}^{\mathrm{R}}\mathrm{Tr}(\rho(C)) e^{-\ell(C)^2/4t}\,. \tag{5.38}$$

We will first need the following fact on the asymptotic heat development.

Proposition 5.11. — *As $t \searrow 0$, it holds that*

$$\mathrm{Tr}_\kappa(e^{-t\Delta}) = \dim V \frac{\sqrt{\pi}\chi(\Sigma)}{\sqrt{t}} + O(\sqrt{t})\,.$$

Proof. — By Corollary 5.5, the torsion function κ has a single simple pole at $s = 1/2$ with residue $\chi(\Sigma)\dim V$. On the other hand we know that, for a fourth-order hypoelliptic Laplacian in dimension 3, as $t \searrow 0$,

$$\mathrm{Tr}_\kappa(e^{-t\Delta}) = \frac{c_1}{t} + \frac{c_{1/2}}{\sqrt{t}} + c_0 + O(\sqrt{t})\,.$$

Mellin's transform (5.36) splits into $\int_0^1 + \int_1^{+\infty}$ providing

$$c_1 = \mathrm{Res}_{s=1}(\kappa(s)) = 0\,, \quad c_{1/2} = \Gamma(1/2)\mathrm{Res}_{1/2}(\kappa) = \sqrt{\pi}\chi(\Sigma)\dim V\,,$$

and

$$c_0 = \kappa(M,\rho) + \mathrm{Res}_0(\Gamma(s)\kappa^*(s)) = 0\,,$$

since $\kappa^*(0) = \kappa(0) - \kappa(M,\rho) = -\kappa(M,\rho)$. □

We can now prove Theorem 5.10.

Proof. — One takes Mellin transforms $\mathcal{M}$ of both sides in (5.37). For $\mathrm{Re}(s) < 0$ one finds

$$\mathcal{M}(\vartheta)(s) = \frac{2^{1-2s}}{\sqrt{\pi}}\Gamma(\frac{1}{2} - s) Z_\rho(2s)\,. \tag{5.39}$$

On the other hand (5.36) and Proposition 5.11 yield that, for $\mathrm{Re}(s) > 1/2$,

$$\mathcal{M}(\mathrm{Tr}^*_\kappa(e^{-t\Delta}))(s) = \Gamma(s)\kappa^*(s)\,.$$

In order to compare these identities we need first to extend them to a common domain. Indeed set

$$\mathrm{Tr}_0(e^{-t\Delta}) = \mathrm{Tr}^*_\kappa(e^{-t\Delta}) + \Big(\kappa(M,\rho) - \dim V \frac{\sqrt{\pi}\chi(\Sigma)}{\sqrt{t}}\Big)\chi_{]0,1]}(t);$$

by Proposition 5.11 this has a Mellin transform for $\mathrm{Re}(s) > -1/2$ and

$$\mathcal{M}(\mathrm{Tr}_0(e^{-t\Delta}))(s) = \Gamma(s)\kappa^*(s) + \frac{\kappa(M,\rho)}{s} + \dim V \frac{\sqrt{\pi}\chi(\Sigma)}{s - 1/2}\,. \tag{5.40}$$

Set also

$$\vartheta_0(t) = \vartheta(t) - \Big(\kappa(M,\rho) - \dim V \frac{\sqrt{\pi}\chi(\Sigma)}{\sqrt{t}}\Big)\chi_{[1,+\infty[}(t)\,,$$

so that (5.39) yields, for $\mathrm{Re}(s) < 0$,

$$\begin{aligned}\mathcal{M}(\vartheta_0)(s) &= \frac{2^{1-2s}}{\sqrt{\pi}}\Gamma(\frac{1}{2} - s)Z_\rho(2s) + \frac{\kappa(M,\rho)}{s} + \dim V\frac{\sqrt{\pi}\chi(\Sigma)}{s-1/2}\\ &= \mathcal{M}(\mathrm{Tr}_0(e^{-t\Delta}))(s)\,,\end{aligned}$$

for $-1/2 < \mathrm{Re}(s) < 0$, by (5.31) and (5.40). Hence by injectivity of the Mellin transform, coming from Fourier injectivity on integrable functions here, one concludes that $\vartheta_0(t) = \mathrm{Tr}_0(e^{-t\Delta})$, yielding the trace formula.

The authors thank Patrick Gérard for an enlightening discussion on this proof. □

Formula (5.37) is a typical Selberg-type trace formula, which holds in many other geometric situations, see [16, p. 57] for instance. Note that it holds here even in *variable* curvature. This may look unusual as Selberg's technique is algebraic and relies on group actions, hence makes sense on uniformised (locally symmetric) manifolds. However we know, by (5.7) or Theorem 5.7, that on CR Seifert manifolds the torsion function $\kappa(s)$ is 'topological', meaning independent of the complex structure J since θ is fixed by the circle action. Now except for some cases that fibre over the sphere S^2 with two singular points, all other CR Seifert manifolds can be uniformised, i.e. endowed with a constant curvature metric; see e.g. [5] or [20, Theorems 1.1, 1.2]. By Moser's lemma this can be done with a fixed volume. Thus, except for the special cases mentioned, one could have worked over a uniformised orbifold Σ, where Selberg's technique might also be applicable.

The trace formula (5.37) has a striking consequence for the small time behaviour of the torsion heat trace $\mathrm{Tr}_\kappa(e^{-t\Delta})$. Indeed, from its definition (5.38) and Proposition 5.8, the dynamical theta function ϑ clearly decays very fast as

$$\vartheta(t) = O(e^{-C/t}) \quad \text{when} \quad t \searrow 0\,,$$

so that instead of Proposition 5.11 we get actually the *full* heat development.

Corollary 5.12. — *On CR Seifert manifolds, as $t \searrow 0$ we have*

$$\mathrm{Tr}_\kappa(e^{-t\Delta}) = \dim V\frac{\sqrt{\pi}\chi(\Sigma)}{\sqrt{t}} + O(e^{-C/t})\,. \tag{5.41}$$

Thus on such manifolds the development of this torsion heat κ-trace does show a cancellation phenomenon, as encountered for heat supertraces in index theory. We recall that if P is an elliptic operator, then McKean–Singer formula states that for any $t > 0$

$$\operatorname{ind} P = \operatorname{Tr}(e^{-tP^*P}) - \operatorname{Tr}(e^{-tPP^*})\,,$$

by isospectrality of P^*P and PP^* except on kernels. The only remaining term here has to be a constant, while we have a $t^{-1/2}$ in (5.41). This explains as follows. By (4.5) and (5.5) it appears that Δ_1 is almost isospectral to two copies of Δ_0, except on remaining *infinite dimensional* spaces of (pluri)CR functions and forms. If working with heat instead of zeta functions, one finds using (5.5) and (5.22) that

$$\operatorname{Tr}_\kappa(e^{-t\Delta}) = \sum_{\lambda\in\operatorname{spec}(iT)} \chi_{\overline{\partial}}(W_\lambda) e^{-t\lambda^2} \tag{5.42}$$

in place of (5.7). Now by §4.1, the spectrum of iT splits into copies of $-x+\mathbb{Z}$. Hence from equation (5.18) for $\chi_{\overline{\partial}}(W_\lambda)$, the right side in (5.42) expresses using classical Jacobi theta functions, whose asymptotic behaviour near $t = 0$ is of type $t^{-1/2} + O(e^{-C/t})$ as in (5.41); see e.g. [46, §21]. From this viewpoint, the torsion heat trace appears as a theta regularized index of the infinite dimensional $\overline{\partial}$–cohomology of W.

Note that the only surviving term as $t \searrow 0$ in the development (5.41) is the integral of curvature data, as should be the case. Gauss–Bonnet reads here

$$\int_M R\,\theta\wedge d\theta = 2\pi \int_\Sigma R d\theta = 4\pi^2\chi(\Sigma)\,,$$

where R stands for the Tanaka–Webster scalar curvature, which coincides with the Riemannian scalar curvature of the base in this Seifert case; see e.g. [7]. Then, by universality of the coefficients of the heat development, they factorise in the following way on *any* 3-dimensional contact manifold.

Corollary 5.13. — *On any contact 3-manifold, the full development of* $\operatorname{Tr}_\kappa(e^{-t\Delta})$ *as* $t\searrow 0$ *is of type*

$$\operatorname{Tr}_\kappa(e^{-t\Delta}) \sim \frac{\dim V}{4\pi\sqrt{\pi t}} \int_M R\,\theta\wedge d\theta + \sum_{n\geqslant 0} t^{n/2} \int_M P_n(R,A)\,\theta\wedge d\theta\,, \tag{5.43}$$

where all invariant curvature polynomials $P_n(R,A)$ *involve at least one copy of Tanaka–Webster torsion* $A = \mathcal{L}_T J$.

In the opposite direction, when $t\to+\infty$, the trace formula (5.37) gives the asymptotic development of the dynamical theta function ϑ. Indeed, after removing the zero eigenspace, heat decays exponentially, yielding the following property.

Corollary 5.14. — *On CR Seifert manifolds, it holds as $t \to +\infty$ that*

$$\vartheta(t) = \kappa(M, \rho) - \dim V \frac{\sqrt{\pi}\chi(\Sigma)}{\sqrt{t}} + O(e^{-Ct}) . \tag{5.44}$$

This can also be seen from the explicit formula (5.38) and Proposition 5.8, which relate the dynamical theta function to the classical Jacobi theta function, whose decay at $+\infty$ is well known; see e.g. [46, §21.51].

BIBLIOGRAPHY

[1] C. Bär & S. Moroianu, "Heat kernel asymptotics for roots of generalized Laplacians", *Internat. J. Math.* **14** (2003), no. 4, p. 397-412.

[2] R. J. Baston & M. G. Eastwood, *The Penrose transform: Its interaction with representation theory*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1989.

[3] R. Beals & P. Greiner, *Calculus on Heisenberg manifolds*, Annals of Mathematics Studies, vol. 119, Princeton University Press, Princeton, NJ, 1988.

[4] R. Beals, P. C. Greiner & N. K. Stanton, "The heat equation and geometry of CR manifolds", *Bull. Amer. Math. Soc. (N.S.)* **10** (1984), no. 2, p. 275-276.

[5] F. A. Belgun, "Normal CR structures on S^3", *Math. Z.* **244** (2003), no. 1, p. 123-151.

[6] O. Biquard & M. Herzlich, "Burns-Epstein invariant for ACHE 4-manifolds", *Duke Math. J.* **126** (2005), no. 1, p. 53-100.

[7] O. Biquard, M. Herzlich & M. Rumin, "Diabatic limit, eta invariants and Cauchy-Riemann manifolds of dimension 3", *Ann. Sci. Ecole Norm. Sup. (4)* **40** (2007), no. 4, p. 589-631.

[8] J.-M. Bismut, "The hypoelliptic Laplacian on the cotangent bundle", *J. Amer. Math. Soc.* **18** (2005), no. 2, p. 379-476 (electronic).

[9] ———, "Loop spaces and the hypoelliptic Laplacian", *Comm. Pure Appl. Math.* **61** (2008), no. 4, p. 559-593.

[10] J.-M. Bismut, H. Gillet & S. C., "Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and analytic torsion", *Comm. Math. Phys.* **115** (1988), no. 1, p. 49-78.

[11] ———, "Analytic torsion and holomorphic determinant bundles. III. Quillen metrics on holomorphic determinants", *Comm. Math. Phys.* **115** (1988), no. 2, p. 301-351.

[12] J.-M. Bismut & G. Lebeau, *The hypoelliptic Laplacian and Ray-Singer metrics*, Annals of Mathematics Studies, vol. 167, Princeton University Press, Princeton, NJ, 2008.

[13] J.-M. Bismut & W. Zhang, *An extension of a theorem by Cheeger and Müller. With an appendix by François Laudenbach*, no. 205, Astérisque, 1992, 235 pages.

[14] T. Branson, "Q-curvature and spectral invariants", *Rend. Circ. Mat. Palermo (2) Suppl.* (2005), no. 75, p. 11-55.

[15] J. Cheeger, "Analytic torsion and the heat equation", *Ann. of Math. (2)* **109** (1979), no. 2, p. 259-322.

[16] D. Fried, "Lefschetz formulas for flows", in *The Lefschetz centennial conference, Part III* (Mexico City, 1984), vol. 58, Amer. Math. Soc., Providence, RI, 1987, p. 19-69.

[17] ———, “Counting circles”, in *Dynamical systems (College Park, MD, 1986–87)*, Lecture Notes in Math, vol. 1342, Springer, Berlin, 1988, p. 196-215.

[18] ———, “Torsion and closed geodesics on complex hyperbolic manifolds”, *Invent. Math.* **91** (1988), no. 1, p. 31-51.

[19] F. B. Fuller, “An index of fixed point type for periodic orbits”, *Amer. J. Math.* **89** (1967), p. 133-148.

[20] M. Furuta & B. Steer, “Seifert fibred homology 3-spheres and the Yang-Mills equations on Riemann surfaces with marked points”, *Adv. Math.* **96** (1992), no. 1, p. 38-102.

[21] E. Getzler, “An analogue of Demailly’s inequality for strictly pseudoconvex CR manifolds.”, *J. Differential Geom.* **29** (1989), no. 2, p. 231-244.

[22] P. B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, Mathematics Lecture Series, vol. 11, Publish or Perish Inc., Wilmington, DE, 1984.

[23] P. Julg & G. Kasparov, “Operator K-theory for the group $\mathrm{SU}(n,1)$”, *J. Reine Angew. Math.* **463** (1995), p. 99-152.

[24] C. Kassel, “Le résidu non commutatif (d’après M. Wodzicki)”, *Astérisque*, (177-178):Exp. No. 708, 199–229, 1989. Séminaire Bourbaki, Vol. 1988/89.

[25] F. F. Knudsen & D. Mumford, “The projectivity of the moduli space of stable curves. I. Preliminaries on “det” and “Div””, *Math. Scand.* **39** (1976), no. 1, p. 19-55.

[26] J. W. Milnor & J. D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, vol. 76, Princeton University Press, Princeton, NJ, 1974.

[27] H. Moscovici & R. J. Stanton, “R-torsion and zeta functions for locally symmetric manifolds”, *Invent. Math.* **105** (1991), no. 1, p. 185-216.

[28] W. Müller, “Analytic torsion and R-torsion of Riemannian manifolds”, *Adv. Math.* **28** (1978), no. 3, p. 233-305.

[29] L. I. Nicolaescu, “Finite energy Seiberg-Witten moduli spaces on 4-manifolds bounding Seifert fibrations”, *Comm. Anal. Geom.* **8** (2000), no. 5, p. 1027-1096.

[30] R. Ponge, “Noncommutative residue for Heisenberg manifolds. Applications in CR and contact geometry”, *J. Funct. Anal.* **252** (2007), p. 399-463.

[31] R. S. Ponge, *Heisenberg calculus and spectral theory of hypoelliptic operators on Heisenberg manifolds*, vol. 194, no 906, Mem. Amer. Math. Soc., 2008, viii + 134 pages.

[32] D. Quillen, “Determinants of Cauchy-Riemann operators on Riemann surfaces”, *Funct. Anal. Appl.* **214** (1985), p. 31-34.

[33] D. B. Ray & I. M. Singer, “R-torsion and the Laplacian on Riemannian manifolds”, *Adv. Math.* **7** (1971), p. 145-210.

[34] C. Rockland, “Hypoellipticity on the Heisenberg group–representation-theoretic criteria”, *Trans. Amer. Math. Soc.* **240** (1978), p. 1-52.

[35] S. Rosenberg, *The Laplacian on a Riemannian manifold. An introduction to analysis on manifolds*, London Mathematical Society Student Texts, vol. 31, Cambridge University Press, Cambridge, 1997.

[36] M. Rumin, “Un complexe de formes différentielles sur les variétés de contact”, *C. R. Acad. Sci. Paris Sér. I Math.* **310** (1990), no. 6, p. 101-404.

[37] ———, “Formes différentielles sur les variétés de contact”, *J. Differential Geom.* **39** (1994), no. 2, p. 281-330.

[38] ———, “Sub-Riemannian limit of the differential form spectrum of contact manifolds”, *Geom. Funct. Anal.* **10** (2000), no. 2, p. 407-452.

[39] P. Scott, “The geometries of 3-manifolds”, *Bull. London Math. Soc.* **15** (1983), no. 5, p. 401-487.

[40] N. Seshadri, "Approximately Einstein ACH metrics, volume renormalization, and an invariant for contact manifolds", *Bull. Soc. Math. France* **137** (2009), no. 1, p. 63-91.

[41] N. K. Stanton, "Spectral invariants of CR manifolds", *Michigan Math. J.* **36** (1989), no. 2, p. 267-288.

[42] N. Tanaka, *A differential geometric study on strongly pseudo-convex manifolds*, Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 9. Kinokuniya Book-Store Co. Ltd., Tokyo, 1975.

[43] S. Tanno, "Variational problems on contact Riemannian manifolds", *Trans. Amer. Math. Soc.* **314** (1989), no. 1, p. 349-379.

[44] M. E. Taylor, *Noncommutative microlocal analysis. I*, vol. 52, no 313, Mem. Amer. Math. Soc., 1984, iv + 182 pages.

[45] S. M. Webster, "Pseudo-Hermitian structures on a real hypersurface", *J. Differential Geom.* **13** (1978), no. 1, p. 25-41.

[46] E. T. Whittaker & G. N. Watson, *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions*, fourth ed., Reprinted. Cambridge University Press, New York, 1962.

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