



# ANNALES

DE

# L'INSTITUT FOURIER

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Tome 62, n° 1 (2012), p. 245-276.

[http://aif.cedram.org/item?id=AIF\\_2012\\_\\_62\\_1\\_245\\_0](http://aif.cedram.org/item?id=AIF_2012__62_1_245_0)

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# CONSTRUCTION OF COMPACT CONSTANT MEAN CURVATURE HYPERSURFACES WITH TOPOLOGY

by Mohamed JLELI (\*)

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ABSTRACT. — In this paper, we explain how the end-to-end construction together with the moduli space theory can be used to produce compact constant mean curvature hypersurfaces with nontrivial topology. For the sake of simplicity, the hypersurfaces we construct have a large group of symmetry but the method can certainly be used to provide many more examples with less symmetries.

RÉSUMÉ. — Dans cet article, nous expliquons comment la méthode de construction dite “recollement des surfaces bout-à-bout” avec des résultats sur l’ensemble des hypersurfaces complètes non compactes à courbure moyenne constante qui ont un nombre fini de bouts de type Delaunay peuvent être utilisées pour construire des nouvelles familles d’hypersurfaces compactes à courbure moyenne constante qui ont une topologie non triviale.

## 1. Introduction and statement of results

The only complete constant mean curvature surfaces known classically are the sphere, the cylinder, and the one-parameter family of rotationally invariant Delaunay surfaces [1]. The first modern breakthrough was Wente’s discovery, detailed in [20], of immersed constant mean curvature tori. Not long afterwards, Kapouleas used transcendental PDE methods to construct many new constant mean curvature surfaces, including compact ones with arbitrary genus [10], and noncompact ones with finitely many ends [9]. Große-Brauckman in [2] subsequently constructed certain of these noncompact surfaces with large discrete symmetry groups using more classical methods based on Schwarz reflection. The theory has developed in two

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*Keywords:* Mean curvature, Compact hypersurface.

*Math. classification:* 35J05, 53A07, 53C21.

(\*) The author extends their appreciation to the Deanship of Scientific Research at King Saud University for funding the work through the research group project No RGP-VPP-087.

fairly distinct directions. From the techniques used for Wente's construction ultimately has emerged the DPW (Dorfmeister-Pedit-Wu) method, which draws on the theory of integrable systems, and serves as a replacement for the Weierstraß representation; on the other hand, Kapouleas construction has engendered many new PDE approaches to the problem. These theories have distinct flavors, and in some senses illuminate quite different classes of surfaces. For example, the former method seems to be more successful in describing general immersed constant mean curvature surfaces, and is closely related to many of the computer experiments and simulations of constant mean curvature surfaces, see [12], while the latter has been particularly good in describing constant mean curvature surfaces which are nearly embedded. Then, using the last method, in the past years the theory of constant mean curvature surfaces in  $\mathbb{R}^3$  has been the object of intensive study. In the case of complete noncompact constant mean curvature surfaces, the moduli space of such surfaces is now fairly well understood (in the genus 0 case). Then, many examples of such surfaces are produced in [9] and [16] and a classification of embedded constant mean curvature surfaces with three ends is given in [3]. However, the set of compact constant mean curvature is not so well understood. After the examples of H. Wente and N. Kapouleas in [8] the authors give a new idea for the construction of a constant mean curvature compact surfaces of arbitrary genus ( $g \geq 3$ ). This construction is based on two important tools which has been developed for the understanding of complete noncompact constant mean curvature surfaces. The first is the moduli space theory which is developed in [15] and the second is the end-to-end gluing of constant mean curvature surfaces developed in [18].

In  $\mathbb{R}^{n+1}$ , for  $n \geq 3$ , there exists a one parameter family of hypersurfaces of revolution that will be denoted by  $\mathcal{D}_\tau$  for  $\tau \in (-\infty, 0) \cup (0, \tau_*]$  which are immersed or embedded and have constant mean curvature normalized to be equal to 1. These hypersurfaces, which were originally studied in [11], generalize the classical Delaunay surfaces in  $\mathbb{R}^3$ . The space of complete noncompact constant mean curvature 1 hypersurfaces in  $\mathbb{R}^{n+1}$  which have  $k$  ends, with  $k \in \mathbb{N}$ , asymptotic to the  $n$ -Delaunay  $\mathcal{D}_\tau$  will be denoted by  $\mathcal{M}_k$ . Let us define:

DEFINITION 1.1. — *Let  $\Sigma \in \mathcal{M}_k$  and  $\mathcal{L}_\Sigma$  its Jacobi operator. Then,  $\Sigma$  is said to be nondegenerate if*

$$\mathcal{L}_\Sigma : L^2(\Sigma) \longrightarrow L^2(\Sigma),$$

*is injective.*

Recently, in [5], we generalized the result of [15]. In order to find a description of the structure of the set  $\mathcal{M}_k$  near any nondegenerate element, we prove a maximum principle for  $\mathcal{L}_{\mathcal{D}_\tau}$ . However we need to impose a lower bound on the Delaunay parameter ( $\tau \in [\tau^*, 0) \cup (0, \tau_*]$ ), when the parameter  $\tau^*$  depends only in the dimension of the Euclidean space for the result to hold. In fact, for  $\tau$  tending to  $-\infty$  the number of negative eigenvalues of  $\mathcal{L}_{\mathcal{D}_\tau}$  changes which assures the existence of constant mean curvature hypersurfaces which are cylindrically bounded and which bifurcate from the family of constant mean curvature hypersurface of revolution which are immersed in  $\mathbb{R}^{n+1}$  (see [6]). More precisely, we prove in [5]:

**THEOREM 1.2 ([5]).** — *Assume that  $\Sigma$  is a nondegenerate element of  $\mathcal{M}_k$  and further assume that the Delaunay parameters  $\tau_\ell$  of the ends of  $\Sigma$  satisfy  $\tau_\ell \in [\tau^*, 0) \cup (0, \tau_*]$ . Then, there exists an open set  $\mathcal{U} \subset \mathcal{M}_k$  containing  $\Sigma$ , which is a smooth manifold of dimension  $k(n+1)$ .*

A generalization for the hypersurface of the result which is developed in [18] is given in [4]. In particular, starting with two nondegenerate hypersurfaces  $\Sigma_1 \in \mathcal{M}_{k_1}$  and  $\Sigma_2 \in \mathcal{M}_{k_2}$  and we assume that the ends  $E_1 \subset \Sigma_1$  and  $E_2 \subset \Sigma_2$  are asymptotic to the same  $n$ -Delaunay  $\mathcal{D}_\tau$  with parameter  $\tau \in [\tau^*, 0) \cup (0, \tau_*]$ . Then, we can align  $\Sigma_1$  and  $\Sigma_2$  such that the axis of  $E_1$  and of  $E_2$  are confused (with opposite directions). Finally, we can translate one of these hypersurface along this axis and we prove:

**THEOREM 1.3 ([4]).** — *Let  $\Sigma_1 \in \mathcal{M}_{k_1}$  and  $\Sigma_2 \in \mathcal{M}_{k_2}$  two nondegenerate constant mean curvature hypersurfaces described as above. There exists a family of hypersurfaces which is a connected sum of  $\Sigma_1$  and  $\Sigma_2$ . These hypersurfaces can be perturbed into a constant mean curvature hypersurface which is element of  $\mathcal{M}_{k_1+k_2-2}$ .*

The aim of this paper is the construction of compact constant mean curvature hypersurfaces with topology. More precisely, we explain how the end-to-end construction (Theorem 1.3) together with the moduli space theory (Theorem 1.2) can be used to produce compact constant mean curvature hypersurfaces with nontrivial topology. Starting with the study of the  $n$ -Delaunay hypersurface, we give in Section 2 a parameterization of such hypersurfaces and we study the asymptotic behavior of the physical period of this hypersurface as the parameter  $\tau$  tends to 0. Next, we define and study the Jacobi operator about an  $n$ -Delaunay hypersurface. In Section 3, we recall some well know results concerning the moduli space

theory of constant mean curvature hypersurface, the end-to-end construction of complete noncompact constant mean curvature hypersurfaces and the balancing formula.

Finally, in Section 4, we start with the description of two families of complete noncompact constant mean curvature hypersurfaces which will be used in the construction. The members of the first family are 3-ended hypersurfaces while the members of the second family are  $k$ -ended hypersurfaces. In particular, these hypersurfaces are constructed in [7] and obtained by gluing a finite number of half Delaunay hypersurfaces (with small Delaunay parameters) to the sphere or a Delaunay hypersurface. Then, these new hypersurfaces constructed in [7] are non degenerate if you add a half of Delaunay with parameters close to zero.

Next, we adopt the method given in [8], to construct many examples of compact constant mean curvature hypersurfaces by gluing together  $k$  elements of the first family and one element of the second family. These hypersurfaces we have obtained are immersed, not embedded and have the topology of a sphere with  $k$  handles attached.

## 2. Delaunay hypersurfaces

It will be more interesting to consider an isothermal type parametrization for which will be more convenient for analytical purposes. Hence, we parametrize hypersurfaces of revolution by

$$(2.1) \quad X(s, \theta) = (|\tau| e^{\sigma(s)} \theta, \kappa(s)),$$

for  $(s, \theta) \in \mathbb{R} \times S^{n-1}$ . The constant  $\tau$  being fixed, the functions  $\sigma$  and  $\kappa$  are determined by asking that the hypersurface parameterized by  $X$  has constant mean curvature equal to  $H$  and also by asking that the metric associated to the parametrization is conformal to the product metric on  $\mathbb{R} \times S^{n-1}$ , namely

$$(2.2) \quad (\partial_s \kappa)^2 = \tau^2 e^{2\sigma} (1 - (\partial_s \sigma)^2).$$

We choose the orientation of the hypersurface parameterized by  $X$  so that, the unit normal vector field is given by

$$(2.3) \quad N := \left( -\frac{\partial_s \kappa}{|\tau| e^\sigma} \theta, \partial_s \sigma \right).$$

This time, using (2.2) the first fundamental form  $g$  of the hypersurface parameterized by  $X$  is given by

$$g = \tau^2 e^{2\sigma} (ds \otimes ds + g_{S^{n-1}}),$$

where  $g_{S^{n-1}}$  denotes the first fundamental form of  $S^{n-1}$ . Then, the second fundamental form of the hypersurface parameterized by  $X$  is given by

$$b = (\partial_s^2 \kappa \partial_s \sigma - \partial_s \kappa (\partial_s^2 \sigma + (\partial_s \sigma)^2)) ds \otimes ds + \partial_s \kappa g_{S^{n-1}}.$$

Therefore, the mean curvature  $H$  of the hypersurface parameterized by  $X$  is given by

$$H = \frac{1}{n\tau^2 e^{2\sigma}} ((n-1) \partial_s \kappa - \partial_s \kappa (\partial_s^2 \sigma + (\partial_s \sigma)^2) + \partial_s^2 \kappa \partial_s \sigma).$$

This is a rather intricate second order ordinary differential equation in the functions  $\sigma$  and  $\tau$  which has to be complimented by the equation (2.2). In order to simplify our analysis, we use of (2.2) to get rid of the factor  $\tau^2 e^{2\sigma}$  in the above equation. This yields

$$\partial_s \sigma \partial_s^2 \kappa = \partial_s \kappa (1 - n + \partial_s^2 \sigma + (\partial_s \sigma)^2 + n H \partial_s \kappa (1 - (\partial_s \sigma)^2)^{-1}).$$

Now, we can differentiate (2.2) with respect to  $s$ , and we obtain

$$\partial_s \kappa \partial_s^2 \kappa = \tau^2 e^{2\sigma} \partial_s \sigma (1 - \partial_s^2 \sigma - (\partial_s \sigma)^2).$$

The difference between the last equation, multiplied by  $\partial_s \sigma$ , and the former equation, multiplied by  $\partial_s \kappa$ , yields

$$(2.4) \quad \partial_s^2 \sigma + (1 - n)(1 - (\partial_s \sigma)^2) + n H \partial_s \kappa = 0.$$

Hence, in order to find constant mean curvature hypersurfaces of revolution, we have to solve (2.2) together with (2.4).

Let us define  $\iota$  to be the sign of  $\tau$  and

$$\tau_* := \frac{1}{n} (n-1)^{\frac{n-1}{n}}.$$

For all  $\tau \in (-\infty, 0) \cup (0, \tau_*]$ , we define  $\sigma_\tau$  ( and we write  $\sigma$  for  $\sigma_\tau$  ) to be the unique smooth nonconstant solution of

$$(2.5) \quad (\partial_s \sigma)^2 + \tau^2 (e^\sigma + \iota e^{(1-n)\sigma})^2 = 1,$$

with initial condition  $\partial_s \sigma(0) = 0$  and  $\sigma(0) < 0$ . Next we define the function  $\kappa_\tau$  to be the unique solution of

$$(2.6) \quad \partial_s \kappa = \tau^2 (e^{2\sigma} + \iota e^{(2-n)\sigma}), \quad \text{with} \quad \kappa(0) = 0.$$

In particular, the hypersurface parameterized by

$$X_\tau(s, \theta) := (|\tau| e^{\sigma_\tau(s)} \theta, \kappa_\tau(s)),$$

for  $(s, \theta) \in \mathbb{R} \times S^{n-1}$ , is an embedded constant mean curvature hypersurface of revolution when  $\tau$  belongs  $(0, \tau_*]$ , this hypersurface will be referred to as the “ $n$ -unduloid” of parameter  $\tau$ . In the other case, if  $\tau < 0$ , this

hypersurfaces is only immersed and will be referred to as the “ $n$ -nodoid” of parameter  $\tau$ .

### 2.1. The physical period of a $n$ -Delaunay hypersurface

The Hamiltonian nature of the equation (2.5) implies that  $\sigma_\tau$  is periodic. We denote its period by a  $s_\tau$ . The physical period of this hypersurface is defined by

$$T_\tau = \kappa_\tau(s_\tau),$$

so that

$$X_\tau(s + s_\tau, \theta) = X_\tau(s, \theta) + T_\tau e_{n+1},$$

with  $e_{n+1} = (0_{\mathbb{R}^n}, 1)$ . We would like to study the behavior of  $T_\tau$  as  $\tau$  varies. When  $\tau \neq \tau_*$ , it follows from a simple analysis of the equation (2.5) that the function  $\sigma_\tau$  is strictly increasing for  $s \in (0, s_\tau/2)$ . Hence it can be used as a change of parameter and we can define

$$K := \kappa_\tau \circ \sigma_\tau^{-1}.$$

It follows from (2.5) and (2.6) that

$$\partial_\sigma K = \tau e^\sigma \frac{J(\sigma)}{\sqrt{1 - J^2(\sigma)}}$$

where

$$J(\sigma) := \tau \left( e^\sigma + \iota e^{(1-n)\sigma} \right).$$

Then, the expression of  $T_\tau$  takes the form

$$(2.7) \quad T_\tau = 2 \int_{\sigma_-}^{\sigma_+} \tau e^\sigma \frac{J(\sigma)}{\sqrt{1 - J^2(\sigma)}} d\sigma$$

where  $\sigma_- < 0 < \sigma_+$  are the roots of

$$J(\sigma)^2 = 1.$$

Now, we study the asymptotic behavior of  $T_\tau$  as the parameter  $\tau$  tends to 0.

PROPOSITION 2.1. — *As  $\tau$  tends to 0, we have*

$$T_\tau = 2 + \iota c_n |\tau|^{\frac{n}{n-1}} + \mathcal{O}(\tau^2),$$

with

$$c_n := \frac{2}{n-1} \int_0^1 \frac{x^{-\frac{1}{n-1}}}{\sqrt{1-x^2}} dx.$$

*Proof.* — We will distinguish two cases according to the sign of  $\tau$ .

**Case 1:** Assume that  $\tau < 0$  and let us define the function

$$h(y) := y - y^{1-n},$$

which is increasing on  $(0, \infty)$  and can be used as a change of variable. Writing  $h(e^\sigma) = \frac{x}{|\tau|}$  in (2.7), the physical period will be write as

$$(2.8) \quad T_\tau := 2 \int_{-1}^1 \frac{x}{\sqrt{1-x^2}} G\left(\frac{x}{|\tau|}\right) dx,$$

with

$$G(x) := \frac{1}{h' \circ h^{-1}(x)}.$$

Then, there exists  $c > 0$  such that for all  $x$  positive the function  $G$  satisfies the two estimates

$$|G(x) - 1| \leq c(1+x)^{-n}, \quad \text{and} \quad |xG'(x)| \leq c(1+x)^{-n}.$$

Indeed, for  $x > 0$  let  $y = h^{-1}(x) > 1$ , then we have

$$|G(x) - 1|(1+x)^n \leq (n-1)(y^{-1} + xy^{-1})^n = (n-1)(1+y^{-1} - y^{-n})^n \leq c.$$

Using these,

$$\mathcal{T}_1 := 2 \int_0^1 \frac{x}{\sqrt{1-x^2}} G\left(\frac{x}{|\tau|}\right) dx,$$

satisfies

$$|\mathcal{T}_1 - 2| \leq 2 \int_0^1 \frac{x}{\sqrt{1-x^2}} \left| G\left(\frac{x}{|\tau|}\right) - 1 \right| dx \leq c \int_0^1 \frac{x}{\sqrt{1-x^2}} \frac{dx}{\left(1 + \frac{x}{|\tau|}\right)^n}.$$

Witting  $x = \frac{2u}{1+u^2}$  and  $\alpha = \frac{1}{|\tau|}$ , we have

$$I_n = \int_0^1 \frac{x}{\sqrt{1-x^2}} \frac{dx}{\left(1 + \frac{x}{|\tau|}\right)^n} = 4 \int_0^{\frac{\sqrt{2}}{2}} \frac{u(1+u^2)^{n-2} du}{(u^2 + 2\alpha u + 1)^n}.$$

It is easy to see that for all  $n \geq 3$ ,  $I_n \leq I_3$ . Moreover,  $I_3$  can be written as

$$I_3 = \sum_{k=1}^3 \int_0^{\frac{\sqrt{2}}{2}} \frac{\alpha_k(\tau) du}{(u - a(\tau))^k} + \int_0^{\frac{\sqrt{2}}{2}} \frac{\beta_k(\tau) du}{(u - a^{-1}(\tau))^k} = \sum_{k=1}^3 J_k(\tau).$$

By an easy computation, we prove that

$$J_1 = \mathcal{O}(|\tau|^3 \log |\tau|) \quad \text{and} \quad J_2 + J_3 = \mathcal{O}(|\tau|^2).$$

Finally, we deduce that

$$|\mathcal{T}_1 - 2| \leq c\tau^2.$$

Using similar computations, we have also

$$|\partial_\tau \mathcal{T}_1| \leq c|\tau|.$$



Now, it is easy to verify that there exists a constant  $c > 0$  such that for all  $x < 0$ , we have

$$\left| G(x) - \frac{1}{n-1} |x|^{-\frac{n}{n-1}} \right| \leq c(1-x)^{-\frac{2n}{n-1}},$$

and

$$\left| x G'(x) \right| \leq c(1-x)^{-\frac{n}{n-1}}.$$

Hence

$$\mathcal{T}_2 := 2 \int_{-1}^0 \frac{x}{\sqrt{1-x^2}} G\left(\frac{x}{|\tau|}\right) dx,$$

satisfies

$$\left| \mathcal{T}_2 + \frac{2}{n-1} |\tau|^{\frac{n}{n-1}} \int_0^1 \frac{x^{-\frac{1}{n-1}}}{\sqrt{1-x^2}} dx \right| \leq c \tau^2 \int_0^{\frac{1}{|\tau|}} \frac{x^{-\frac{1}{n-1}}}{(1+x)^{\frac{n}{n-1}}} dx.$$

Finally, we see that the last integral converges. Which finishes the proof in this case.

**Case 2:** Assume that  $\tau \in (0, \tau_*)$  and let us define the function

$$f(y) := y + y^{1-n}.$$

Then, changing  $e^\sigma$  by  $y$ , (2.7) becomes

$$T_\tau := 2 \tau^2 \int_{y_-}^{y_+} \frac{f(y)}{\sqrt{1-\tau^2 f^2(y)}} dy,$$

with

$$f(y_-) = f(y_+) = \frac{1}{\tau}.$$

Now, it is clear that  $f_1$  the restriction of  $f$  in  $(y_-, (n-1)^{\frac{1}{n}})$  is strictly decreasing and  $f_2$  the other restriction of  $f$  in  $((n-1)^{\frac{1}{n}}, y_+)$  is strictly increasing. Hence, the physical period can be written as

$$T_\tau = 2 \left( \int_1^{\frac{\tau}{\tau_*}} \frac{x}{\sqrt{1-x^2}} G_1\left(\frac{x}{\tau}\right) dx + \int_{\frac{\tau}{\tau_*}}^1 \frac{x}{\sqrt{1-x^2}} G_2\left(\frac{x}{\tau}\right) dx \right),$$

where

$$G_1(x) := f'_1 \circ f_1^{-1}(x) \quad \text{and} \quad G_2(x) := f'_2 \circ f_2^{-1}(x).$$

As in the first case, we prove that

$$\int_1^{\frac{\tau}{\tau_*}} \frac{x}{\sqrt{1-x^2}} G_1\left(\frac{x}{\tau}\right) dx = \frac{\tau^{-\frac{n}{n-1}}}{n-1} \int_0^1 \frac{x^{-\frac{1}{n-1}}}{\sqrt{1-x^2}} dx + \mathcal{O}(\tau^2),$$

and

$$\int_{\frac{\tau}{\tau_*}}^1 \frac{x}{\sqrt{1-x^2}} G_2\left(\frac{x}{\tau}\right) dx = 1 + \mathcal{O}(\tau^2).$$

This completes the proof. Lebesgue’s dominated convergence Theorem implies that

$$\lim_{\tau \rightarrow 0} T_\tau = 2.$$

□

*Remark 2.2.* — In the case where  $\tau < 0$ ,  $T_\tau$  is monotone and  $\partial_\tau T_\tau > 0$ . Indeed, thanks to (2.8) we have

$$\partial_\tau T_\tau = 2 \int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}} \frac{n(n-1)z^{-n-1}}{(1+(n-1)z^{-n})^3} \frac{dx}{\tau^2} > 0, \text{ where } z := h^{-1}\left(\frac{x}{|\tau|}\right).$$

### 2.2. The Jacobi operator about a $n$ -Delaunay

It is well known [19] that the linearized mean curvature operator about  $\mathcal{D}_\tau$ , which is usually referred to as the Jacobi operator, is given by

$$\mathcal{L}_\tau := \Delta_\tau + |b_\tau|^2$$

where  $\Delta_\tau$  is the Laplace-Beltrami operator and  $|b_\tau|^2$  is the square of the norm of the second fundamental form on  $\mathcal{D}_\tau$ .

Let us define the function  $\varphi_\tau := |\tau|e^{\sigma_\tau}$ . We find the expression of the Jacobi operator in term of the function  $\varphi_\tau$

$$(2.9) \quad \mathcal{L}_\tau := \varphi_\tau^{-n} \partial_s(\varphi_\tau^{n-2} \partial_s) + \varphi_\tau^{-2} \Delta_{S^{n-1}} + n + n(n-1)\tau^{2n} \varphi_\tau^{-2n}.$$

It will be convenient to define the conjugate operator

$$(2.10) \quad L_\tau := \varphi_\tau^{\frac{n+2}{2}} \mathcal{L}_\tau \varphi_\tau^{\frac{2-n}{2}},$$

which is explicitly given in terms of the function  $\varphi_\tau$  by

$$(2.11) \quad L_\tau = \partial_s^2 + \Delta_{S^{n-1}} - \left(\frac{n-2}{2}\right)^2 + \frac{n(n+2)}{4} \varphi_\tau^2 + \frac{n(3n-2)}{4} \tau^{2n} \varphi_\tau^{2-2n}.$$

Since the operators  $\mathcal{L}_\tau$  and  $L_\tau$  are conjugate, the mapping properties of one of them will easily translate for the other one. With slight abuse of terminology, we shall refer to any of them as the Jacobi operator about  $\mathcal{D}_\tau$ .

Now, we give the following definition:

**DEFINITION 2.3.** — *Let us denote by  $\theta \mapsto e_j(\theta)$ , for  $j \in \mathbb{N}$ , the eigenfunctions of the Laplace-Beltrami operator on  $S^{n-1}$ , which will be normalized to have  $L^2$  norm equal to 1 and correspond to the eigenvalue  $\lambda_j$ . That is*

$$-\Delta_{S^{n-1}} e_j = \lambda_j e_j,$$

and

$$\lambda_0 = 0, \quad \lambda_1 = \dots = \lambda_n = n - 1, \quad \lambda_{n+1} = 2n, \dots \quad \text{and} \quad \lambda_j \leq \lambda_{j+1}.$$

We ended this section by giving only the expression of some Jacobi fields, i.e., solution of the homogeneous problem

$$\mathcal{L}_\tau \omega = 0,$$

since these Jacobi fields follow from a rigid motion or by changing the Delaunay parameter  $\tau$ . More details are given in [5].

- For  $\tau \in (-\infty, 0) \cup (0, \tau_*)$ , we define  $\Phi_\tau^{0,+}$  to be the Jacobi field corresponding to the translation of  $\mathcal{D}_\tau$  along the  $x_{n+1}$  axis

$$\Phi_\tau^{0,+} := \varphi^{\frac{n-4}{2}} \partial_s \varphi.$$

- Since we have  $n$  directions orthogonal to  $x_{n+1}$ , there are  $n$  linearly independent Jacobi fields which are obtained by translating  $\mathcal{D}_\tau$  in a direction orthogonal to its axis. We get for  $j = 1, \dots, n$

$$\Phi_\tau^{j,+} := \left( \varphi^{\frac{n}{2}} + |\tau|^n \varphi^{-\frac{n}{2}} \right) e_j.$$

- For  $j = 1, \dots, n$ , the Jacobi field corresponding to the rotation of the axis of  $\mathcal{D}_\tau$  takes the form

$$\Phi_\tau^{j,-}(s, \theta) := \varphi^{\frac{n-4}{2}} \left( \varphi \partial_s \varphi + \kappa \partial_s \kappa \right) e_j.$$

- Finally, The Jacobi field corresponding to a change of parameter  $\tau \in (-\infty, 0) \cup (0, \tau_*)$  is given by

$$\Phi_\tau^{0,-} := \varphi^{\frac{n-4}{2}} \left( \partial_\tau \kappa \partial_s \varphi - \partial_\tau \varphi \partial_s \kappa \right).$$

Because of the rotational invariance of the operator  $L_\tau$ , we can introduce the eigenfunction decomposition with respect to the cross-sectional Laplace-Beltrami operator  $\Delta_{S^{n-1}}$ . In this way, we obtain the sequence of operators

$$(2.12) \quad L_{\tau,j} = \partial_s^2 - \lambda_j - \left( \frac{n-2}{2} \right)^2 + \frac{n(n+2)}{4} \varphi^2 + \frac{n(3n-2)}{4} \tau^{2n} \varphi^{2-2n}$$

for  $j \in \mathbb{N}$ . By definition, for each  $\tau$  and  $j$ , the *indicial roots* of the operator  $L_{\tau,j}$  are the real numbers  $\pm \gamma_j(\tau)$  which characterize the exponential rate of growth (or rate of decay) of the solutions of the homogeneous equation

$$L_{\tau,j} \omega = 0$$

at infinity (see [15]). Observe that the explicit knowledge of some Jacobi fields yields some information about the indicial roots of the operator  $L_\tau$ . Indeed, since the Jacobi fields  $\Phi_\tau^{j,\pm}$  described below are at most linearly

growing (see [5] ) the associated indicial roots are all equal to 0. Hence we conclude that for all  $\tau \in (-\infty, 0) \cup (0, \tau_*]$

$$\gamma_j(\tau) = 0, \quad \text{for } j = 0, \dots, n.$$

The situation is completely different when  $j \geq n + 1$ . Indeed, the potential

$$\lambda_j + \left(\frac{n-2}{2}\right)^2 - \frac{n(n+2)}{4}\varphi^2 - \frac{n(3n-2)}{4}\tau^{2n}\varphi^{2-2n}$$

which appears in  $L_{\tau,j}$  can be seen to be bounded from below by a positive constant, when  $j \geq n + 1$  and  $\tau$  is negative close enough to 0 or positive (simply use the equation (2.5) to obtain an upper bound for  $\varphi^2$  and  $\tau^{2n}\varphi^{2-2n}$  ). Therefore, when  $j \geq n + 1$  and  $\tau$  is not too far away from 0, the maximum principle holds for  $L_{\tau,j}$  and the existence of solutions of the homogeneous problem  $L_{\tau,j} w = 0$  which either blows up exponentially or decays exponentially at  $\infty$  follows at once from the construction of barrier functions of the form  $s \mapsto e^s$  . Then, its proved in [5] that:

PROPOSITION 2.4. — *The exists  $\tau^* < 0$ , depending only on  $n$ , such that for all  $\tau \in [\tau^*, 0) \cup (0, \tau_*]$*

$$\gamma_j(\tau) > 0, \quad \text{for all } j \geq n + 1.$$

### 3. Well known results

Assume that we are given  $\Sigma$  a complete noncompact, constant mean curvature hypersurfaces in  $\mathbb{R}^{n+1}$ , with  $k$  ends which are modeled after  $n$ -Delaunay hypersurfaces. We denote by  $E_1, \dots, E_k$  the ends of the hypersurface  $\Sigma$ . We require that these ends are asymptotic to half  $n$ -unduloid or a half  $n$ -nodoid. More precisely, we require that, up to some rigid motion, each end can be parameterized as the normal graph of some exponentially decaying function over some Delaunay hypersurface, i.e., up to some rigid motion, the end  $E_\ell$  is parameterized by

$$(3.1) \quad Y_\ell := X_{\tau_\ell} + \omega_\ell N_{\tau_\ell},$$

where  $Y_\ell$  is defined in  $(0, +\infty) \times S^{n-1}$ ,  $\tau_\ell \in [\tau^*, 0) \cup (0, \tau_*]$  and where the function  $\omega_\ell$  is exponentially decaying as well as all its derivatives. More precisely, we assume that for all  $k \in \mathbb{N}$ , there exists  $c_k > 0$  such that

$$(3.2) \quad |\nabla^k \omega_\ell| \leq c_k e^{-\gamma_{n+1}(\tau_\ell) s}, \quad \text{on } (0, +\infty) \times S^{n-1}.$$

The Jacobi operator about the end  $E_\ell$  is close to the Jacobi operator about the  $n$ -Delaunay hypersurface  $\mathcal{D}_{\tau_\ell}$ . This follows at once from the fact that the coefficients of the first and second fundamental forms associated to the end  $E_\ell$  are equal to the coefficients of the first and second fundamental form of  $\mathcal{D}_{\tau_\ell}$  up to some functions which is exponentially decaying like  $e^{-\gamma_{n+1}(\tau_\ell)s}$ . The content of the following Lemma is to make this result quantitatively precise.

LEMMA 3.1. — *The Jacobi operator about  $\Sigma$ , restricted to the end  $E_\ell$  is given by*

$$\mathcal{L}_\Sigma := \Delta_\Sigma + |b_\Sigma|^2 = \mathcal{L}_{\tau_\ell} + L_\ell,$$

where  $\mathcal{L}_{\tau_\ell}$  is the operator given in (2.9) and  $L_\ell$  is a second order linear operator whose coefficients and their derivatives are bounded by a constant times  $e^{-\gamma_{n+1}(\tau_\ell)s}$ .

Now, we decompose  $\Sigma$  into slightly overlapping pieces which are a compact piece  $K$  and the ends  $E_\ell$ . Then, we define the following functional space

DEFINITION 3.2. — *For all  $r \in \mathbb{N}$ ,  $\delta \in \mathbb{R}$  and all  $\alpha \in (0, 1)$ , the function space  $\mathcal{C}_\delta^{r,\alpha}(\Sigma)$  is defined to be the space of functions  $w \in C_{\text{loc}}^{r,\alpha}(\Sigma)$  for which the following norm is finite*

$$\|w\|_{\mathcal{C}_\delta^{r,\alpha}(\Sigma)} := \sum_{\ell=1}^k \|w \circ Y_\ell\|_{\mathcal{E}_\delta^{r,\alpha}((0,+\infty) \times S^{n-1})} + \|w\|_{\mathcal{C}^{r,\alpha}(K)},$$

where the space

$$\mathcal{E}_\delta^{\ell,\alpha}([s_0, +\infty) \times S^{n-1})$$

to be the set of functions  $C_{\text{loc}}^{\ell,\alpha}$  which are defined on  $[s_0, +\infty) \times S^{n-1}$  and for which the following norm is finite :

$$\|\omega\|_{\mathcal{E}_\delta^{\ell,\alpha}(\mathbb{R} \times S^{n-1})} := \sup_{s \geq s_0} |e^{-\delta s} \omega|_{C^{\ell,\alpha}([s,s+1] \times S^{n-1})}.$$

Here,  $|\cdot|_{C^{\ell,\alpha}([s,s+1] \times S^{n-1})}$  denotes the usual Hölder norm in  $[s, s + 1] \times S^{n-1}$ .

Remark 3.3. — The nondegeneracy property introduced in definition 1.1 can be given in the last functional space by: the hypersurface  $\Sigma \in \mathcal{M}_k$  is said to be nondegenerate if

$$\mathcal{L}_\Sigma : \mathcal{C}_\delta^{2,\alpha}(\Sigma) \longrightarrow \mathcal{C}_\delta^{0,\alpha}(\Sigma),$$

is injective for all  $\delta < 0$ .

Let  $\chi_\ell$  be a cutoff function which is equal to 0 on  $E_\ell \cap K$  and equal to 1 on  $Y_\ell((c_\ell, +\infty) \times S^{n-1})$  for some  $c_\ell > 0$  chosen large enough. We define the deficiency space  $\mathcal{W}(\Sigma)$  by

$$\mathcal{W}(\Sigma) := \bigoplus_{\ell=1}^k \text{Span} \{ \chi_\ell \Phi_\ell^{j,\pm} : j = 0, \dots, n \}.$$

The analogue of the following result for  $n = 2$  is usually know as the “Linear Decomposition Lemma” (see [17] and [16]).

PROPOSITION 3.4 ([5]). — *We assume that  $\tau_\ell \in [\tau^*, 0) \cup (0, \tau_*]$ ,  $\delta \in (-\inf_\ell \gamma_{n+1}(\tau_\ell), 0)$ ,  $\alpha \in (0, 1)$  and  $\Sigma$  is nondegenerate. Let  $\mathcal{N}(\Sigma)$  the trace of the kernel of  $\mathcal{L}_\Sigma$  over the deficiency space  $\mathcal{W}_\Sigma$ , then  $\mathcal{N}(\Sigma)$  is a  $k(n + 1)$  dimensional subspace of  $\mathcal{W}(\Sigma)$  which satisfies*

$$\text{Ker}(\mathcal{L}_\Sigma) \subset \mathcal{C}_\delta^{2,\alpha}(\Sigma) \oplus \mathcal{N}(\Sigma).$$

If  $\mathcal{K}(\Sigma)$  is a  $k(n + 1)$  dimensional subspace of  $\mathcal{W}(\Sigma)$  such that

$$\mathcal{W}(\Sigma) = \mathcal{K}(\Sigma) \oplus \mathcal{N}(\Sigma),$$

we have

$$\mathcal{L}_\Sigma : \mathcal{C}_\delta^{2,\alpha}(\Sigma) \oplus \mathcal{K}(\Sigma) \longrightarrow \mathcal{C}_\delta^{0,\alpha}(\Sigma)$$

is an isomorphism.

### 3.1. The end-to-end gluing

We introduce the following

DEFINITION 3.5. — *Let  $\Sigma$  be a constant mean curvature 1 hypersurface with  $k$  ends of Delaunay type. We will say that the end  $E$  of  $\Sigma$ , which is asymptotic to some Delaunay hypersurface  $\mathcal{D}_\tau$ , is regular if there exists a Jacobi field  $\Psi$  which is globally defined on  $\Sigma$  and which is asymptotic to  $\Phi_\tau^{0,-}$  (the Jacobi field on  $\mathcal{D}_\tau$  which corresponds to the the change of the Delaunay parameter  $\tau$ ) on  $E$ .*

Remark 3.6. — The  $n$ -Delaunay hypersurfaces have regular ends and many other examples are constructed in [7].

It was observed by R. Kusner and proved by J. Ratzkin (see [18] ) that one can connect together two constant mean curvature surfaces having two ends with the same Delaunay parameter. This gluing procedure is known as a “end-to-end connected sum” and generalized for the hypersurfaces in [4]. To explain this let assume that we are given two nondegenerate, complete, noncompact, constant mean curvature hypersurfaces  $\Sigma_i \in \mathcal{M}_{k_i}$ ,

for  $i = 1, 2$ . We denote by  $E_{i,1}, \dots, E_{i,k_i}$  the ends of the hypersurface  $\Sigma_i$ , each of which are assumed to be asymptotic to a  $n$ - Delaunay hypersurfaces  $\mathcal{D}_{\tau_{i,j}}$ ,  $j = 1, \dots, k_i$ .

We further assume that  $\Sigma_1$  and  $\Sigma_2$  have one end with the same Delaunay parameter. For example, let us assume that the Delaunay parameter of  $E_{1,1}$  and  $E_{2,1}$  are the same, both equal to

$$\tau := \tau_{1,1} = \tau_{2,1}.$$

Further more we assume that  $\tau \neq \tau_*$  i.e.,  $E_{1,1}$  and  $E_{2,1}$  are not asymptotic to a cylinders.

Given  $m \in \mathbb{N}$ , we can use a rigid motion to ensure that the end  $E_{1,1}$  is parameterized by

$$Y_{1,1}(s, \theta) := X_\tau(s, \theta) + w_1(s + m s_\tau, \theta) N_\tau(s, \theta),$$

for all  $(s, \theta) \in (-m s_\tau, +\infty) \times S^{n-1}$  and that the end  $E_{2,1}$  is parameterized by

$$Y_{2,1}(s, \theta) := X_\tau(s, \theta) + w_2(s - m s_\tau, \theta) N_\tau(s, \theta),$$

for all  $(s, \theta) \in (-\infty, m s_\tau) \times S^{n-1}$ . Though this is not explicit in the notation,  $Y_{i,1}$  does depend on  $m$ . In addition, we know from (3.1) and (3.2) that the functions  $w_1$  and  $w_2$  are exponentially decaying as  $s$  tends to  $+\infty$  (resp.  $-\infty$ ). More precisely,

$$w_1 \in \mathcal{E}_{-\gamma_{n+1}(\tau)}^{2,\alpha}((0, +\infty) \times S^{n-1})$$

and also that

$$w_2 \in \mathcal{E}_{\gamma_{n+1}(\tau)}^{2,\alpha}((-\infty, 0) \times S^{n-1}),$$

where the function spaces  $\mathcal{E}_\delta^{2,\alpha}$  are introduced in Definition 3.2.

Given  $s > -m s_\tau$ , we define the truncated hypersurface

$$\Sigma_1(s) := \Sigma_1 - Y_{1,1}((s, +\infty) \times S^{n-1})$$

and given  $s < m s_\tau$  we define the truncated hypersurface

$$\Sigma_2(s) := \Sigma_2 - Y_{2,1}((-\infty, s) \times S^{n-1}).$$

Again,  $\Sigma_i(s)$  depends on  $m$ . Now let  $s \rightarrow \xi(s)$  be a cutoff function such that  $\xi \equiv 0$  for  $s \geq 1$  and  $\xi \equiv 1$  for  $s \leq -1$ . We define the hypersurface  $\tilde{\Sigma}_m$  to be

$$(3.3) \quad \tilde{\Sigma}_m := \Sigma_1(-1) \cup C(1) \cup \Sigma_2(1),$$

where, for all  $s \in (0, m s_\tau)$ , the cylindrical type hypersurface  $C(s)$  is the image of  $[-s, s] \times S^{n-1}$  by

$$(3.4) \quad (s, \theta) \rightarrow \xi(s) Y_{1,1}(s, \theta) + (1 - \xi(s)) Y_{2,1}(s, \theta).$$

By construction  $\tilde{\Sigma}_m$  is a smooth hypersurface whose mean curvature is identically equal to 1 except in the annulus  $C(1)$  where the mean curvature is close to 1 and tends to 1 as  $m$  tends to  $+\infty$ . Indeed, in  $C(m s_\tau)$ , the hypersurface  $\tilde{\Sigma}_m$  is a normal graph over the  $n$ -Delaunay hypersurface  $\mathcal{D}_\tau$  for some function  $w_3$ . But the estimates we have on  $w_1$  and  $w_2$  imply that

$$(3.5) \quad w_3 = \mathcal{O}_{\mathcal{C}^{2,\alpha}}(e^{-\gamma_{n+1}(\tau)(s+m s_\tau)}) + \mathcal{O}_{\mathcal{C}^{2,\alpha}}(e^{\gamma_{n+1}(\tau)(s-m s_\tau)})$$

on  $C(m s_\tau)$ . The mean curvature of  $\tilde{\Sigma}_m$  in  $C(m s_\tau)$  can then be computed using Taylor's expansion

$$H_{\tilde{\Sigma}_m} = H_{\mathcal{D}_\tau} + \mathcal{O}_{\mathcal{C}^{0,\alpha}}(w_3).$$

It then follows from (3.5) that the mean curvature of  $\tilde{\Sigma}_m$  can be estimated by

$$(3.6) \quad \|H_{\tilde{\Sigma}_m} - 1\|_{\mathcal{C}^{0,\alpha}(C(1))} \leq c e^{-m \gamma_{n+1}(\tau) s_\tau}.$$

Observe that, on a given end  $E_{i,j}$ , all Jacobi fields, except the Jacobi field  $\Phi_{E_{i,j}}^{0,-}$  which corresponds to a change of Delaunay parameter, come from the action of the group of rigid motions. Hence, these Jacobi fields are all globally defined on  $M_i$ . If we assume that  $E_{1,1}$  is a regular end then, the Jacobi field  $\Phi_{E_{1,1}}^{0,-}$  also corresponds to a globally defined Jacobi field on  $M_1$ . Using this and a fixed point argument we prove

**PROPOSITION 3.7** ([4]). — *Assume that  $E_{1,1}$  is a regular end. Then there exists  $m_0 \in \mathbb{N}$  such that, for all  $m \geq m_0$  the hypersurface  $\tilde{\Sigma}_m$  can be perturbed into a constant mean curvature 1 hypersurface element of  $\mathcal{M}_{k_1+k_2-2}$ .*

### 3.2. The balancing formula

In this section we recall the well known balancing formula for a constant mean curvature 1 hypersurface  $\Sigma \subset \mathbb{R}^{n+1}$ .

Assume that  $V$  is an open subset of  $\Sigma$  and that  $U \subset \mathbb{R}^{n+1}$  is an open set such that

$$\partial U \cap \Sigma = V.$$

We define

$$Q := \partial U - \Sigma.$$

With these notations, we have [13]

$$\int_{\partial V} \eta - \int_Q \nu = 0,$$



where  $\eta$  is the exterior conormal of  $\partial V$  relative to  $V$  and  $\nu$  be the exterior normal to  $\partial U$ .

In the case where the hypersurface  $\Sigma$  is a complete noncompact constant mean curvature hypersurface with finitely ends  $E_\ell$ , for  $\ell = 1, \dots, k$ , which are asymptotic to  $n$ -Delaunay hypersurfaces  $\mathcal{D}_{\tau_\ell}$ , its proved in [14] that the balancing formula reads :

$$(3.7) \quad \sum_{\ell=1}^k \iota_\ell |\tau_\ell|^n \vec{a}_\ell = 0$$

where  $\iota_\ell$  is the sign of  $\tau_\ell$  and  $\vec{a}_\ell$  is the direction of the axis of  $E_\ell$ , which is normalized by  $|\vec{a}_\ell| = 1$ .

#### 4. Construction of compact constant mean curvature hypersurfaces

We start with the description of two families of complete noncompact constant mean curvature hypersurfaces which will be used in the construction. The members of the first family are 3-ended hypersurfaces while the members of the second family are  $k$ -ended hypersurfaces.

##### Type-1 hypersurfaces

The members of the first family are denoted by  $\Sigma(\tau, \alpha, \ell)$ , where  $\tau$ ,  $\alpha$  and  $\ell$  are parameters. These hypersurfaces are assumed to enjoy the following properties :

- (1) Each  $\Sigma(\tau, \alpha, \ell)$  is a complete noncompact constant mean curvature hypersurface with 3 ends which are denoted by

$$E_{-1}(\tau, \alpha, \ell), E_0(\tau, \alpha, \ell), E_1(\tau, \alpha, \ell).$$

- (2) The hypersurface  $\Sigma(\tau, \alpha, \ell)$  is invariant under the action of the group

$$G := \{I, S\} \times O(n-1)$$

where  $S$  is the symmetry with respect to the hyperplane  $x_1 = 0$ .

- (3) Each  $\Sigma(\tau, \alpha, \ell)$  is nondegenerate and the parameters  $(\tau, \alpha, \ell)$  are local parameters on the moduli space of constant mean curvature hypersurfaces with 3 ends, which are invariant under the action of the group  $G$ .
- (4) The hypersurface  $\Sigma(\tau, \alpha, \ell)$  is obtained by translating  $\Sigma(\tau, \alpha, 0)$  by  $\ell \vec{e}_2$ .
- (5) Each end of  $\Sigma(\tau, \alpha, \ell)$  is regular.

- (6) Both  $E_{-1}(\tau, \alpha, \ell)$  and  $E_1(\tau, \alpha, \ell)$  are asymptotic to a  $n$ -Delaunay hypersurface of parameter  $\tau$ .
- (7) The end  $E_0(\tau, \alpha, \ell)$  is asymptotic to a  $n$ -Delaunay hypersurface whose parameter is denoted by  $\tilde{\tau}$ .
- (8) The parameter  $\alpha \in (0, \pi/2)$  and the axis of  $E_1(\tau, \alpha, 0)$  is the line of direction

$$\vec{a}_1 := \sin \alpha \vec{e}_1 - \cos \alpha \vec{e}_2$$

passing through the origin. The vector  $\vec{a}_1$  being directed toward the end of  $E_1(\tau, \alpha, 0)$ .

- (9) The axis of  $E_0(\tau, \alpha, 0)$  is the line of direction

$$\vec{a}_0 := -\vec{e}_2$$

passing through the origin. The vector  $\vec{a}_0$  being directed toward the end of  $E_0(\tau, \alpha, 0)$ .

Here  $(\vec{e}_1, \dots, \vec{e}_{n+1})$  is the canonical base of  $\mathbb{R}^{n+1}$ .

Observe that the image of  $E_1(\tau, \alpha, \ell)$  by  $S$  is  $E_{-1}(\tau, \alpha, \ell)$  and that  $E_0(\tau, \alpha, \ell)$  remains globally fixed under the action of  $S$ . The angle  $\alpha$  between the vectors  $\vec{a}_0$  and  $\vec{a}_1$  is given by

$$\cos \alpha = \vec{a}_0 \cdot \vec{a}_1$$

Applying the balancing formula (3.7), we conclude that the Delaunay parameter  $\tilde{\tau}$  is given in terms of  $\tau$  by the formula

$$\tilde{\tau} = -(2 \cos \alpha)^{1/n} \tau.$$

In particular the signs of  $\tau$  and  $\tilde{\tau}$  are different and this implies that the hypersurface  $\Sigma(\tau, \alpha, \ell)$  has always an end which is not embedded (a  $n$ -nodoid).

We now give two examples of such a family.

*Example 4.1.* — A first family can be obtained by gluing onto a sphere  $S^n \subset \mathbb{R}^{n+1}$ , three half  $n$ -Delaunay hypersurfaces of parameters  $\tau$ ,  $\tilde{\tau}$  and  $\tau$  (resp.) at the points of coordinates

$$(-\sin \alpha, -\cos \alpha, 0, \dots, 0) \quad (0, -1, 0, \dots, 0) \quad \text{and} \quad (\sin \alpha, -\cos \alpha, 0, \dots, 0).$$

This construction is explained in general in [7]. The same construction works equivariantly, namely if one imposes the hypersurfaces to be invariant under the action of the group

$$G := \{I, S\} \times O(n-1).$$

Given the symmetries of the hypersurfaces we want to construct there remains only three degrees of freedom which are the Delaunay parameter  $\tau$ ,

the angle  $\alpha$  between the ends and the translation parameter  $\ell$  along the  $x_2$  axis. The construction works for any  $\alpha \in (0, \pi/2)$  and any  $\tau \neq 0$  close enough to 0 and the hypersurface obtained is non degenerate for  $|\tau|$  small. The fact that the ends are regular follows at once from the construction itself since  $\tau$  can be used to parameterize this family of solutions and differentiation with respect to this parameter yields a Jacobi field with the right behavior on the ends.

*Example 4.2.* — A second family can be obtained by gluing on a  $n$ -Delaunay hypersurface of parameter  $\tau$  and axis  $x_1$ , a half  $n$ -Delaunay hypersurface of parameter  $\tilde{\tau}$  small enough and axis  $x_2$ . This time, the construction works for a small value of the parameter  $\tilde{\tau}$  and the point where the connected sum occurs can either be a point where  $\sigma_\tau$  is minimal or maximal. The point being that this Delaunay hypersurface has to be translated so that it is invariant under the action of the symmetry  $S$ .

### Type-2 hypersurfaces

Assume that  $k \geq 3$  is fixed. The members of the second family are denoted by  $\Sigma_k(\tau)$ , where  $\tau$  is a parameter. These hypersurfaces are assumed to enjoy the following properties :

- (1) Each  $\Sigma_k(\tau)$  is a complete noncompact constant mean curvature hypersurface with  $k$  ends which are denoted by  $E_1(\tau), \dots, E_k(\tau)$ .
- (2) The hypersurface is invariant under the action of the group

$$G_k := \{R_{2\pi/k}^j : j \in \mathbb{N}\} \times O(n-1),$$

where  $R_{2\pi/k}$  is the rotation of angle  $2\pi/k$  in the  $x_1, x_2$  plane.

- (3) Each  $\Sigma_k(\tau)$  is nondegenerate and the parameter  $\tau$  is local parameter on the moduli space of constant mean curvature hypersurfaces with  $k$  ends, which are invariant under the action of the group  $G_k$ .
- (4) Each end of  $\Sigma_k(\tau)$  is regular.
- (5) Each end  $\Sigma_k(\tau)$  is asymptotic to a  $n$ -Delaunay hypersurface of parameter  $\tau$ .
- (6) The axis of  $E_1(\tau)$  is the line of direction  $\vec{e}_2$  passing through the origin. The vector  $\vec{e}_2$  being directed toward the end of  $E_1(\tau)$ .

Observe that, for  $j = 1, \dots, k-1$ , the image of  $E_j(\tau)$  by  $R_{2\pi/k}$  is the end  $E_{j+1}(\tau)$  and the image of  $E_k(\tau)$  is the end  $E_1(\tau)$ . Hence the angle between two consecutive ends is given by  $2\pi/k$ .

Such a family can be obtained by gluing onto a sphere  $S^n \subset \mathbb{R}^{n+1}$ ,  $k$  copies of a half  $n$ -Delaunay hypersurface with small Delaunay parameter  $\tau \neq 0$  in such a way that the symmetries are preserved. Again this is the construction explained in [7]. Given the symmetries of the hypersurfaces

we construct, there remains only one degree of freedom which is the Delaunay parameter  $\tau$  of the ends. The construction works for any  $\tau \neq 0$  close enough to 0. The fact that the ends are regular follows at once from the construction itself since  $\tau$  can be used to parameterize this family of solutions and differentiation with respect to this parameter yields a Jacobi field which has the desired behavior at each end.

*Remark 4.3.* — Starting from a compact or a complete noncompact constant mean curvature hypersurface with  $k$  ends asymptotic to Delaunay hypersurfaces we construct in [7] a new constant mean curvature hypersurface with  $k+m$  ends which is obtained by performing a connected sum of  $m$  additional Delaunay ends of small Delaunay parameters at some  $m$  points  $p_1, \dots, p_m$  of the initial hypersurface. Then, hypersurfaces of type 1 and of type 2 are these new examples if the initial hypersurface is the sphere or the Delaunay hypersurface. In particular, if these  $m$  additional Delaunay ends used in the gluing are asymptotic to a Delaunay hypersurfaces with small parameters, then these new hypersurfaces are non degenerate and with regular ends.

We fix  $k \geq 3$  and define

$$\alpha_k := \frac{\pi}{2} - \frac{\pi}{k}.$$

We assume that, for  $\tau$  in some open interval  $I$ , we are given a family of hypersurfaces  $\Sigma(\alpha_k, \tau, \ell)$  of Type 1 and for  $\tilde{\tau}$  in some open interval  $\tilde{I}$ , we are given a family of hypersurfaces  $\Sigma_k(\tilde{\tau})$  of Type 2.

The parameter  $\tau$  being chosen in  $I$ , we define  $\tilde{\tau}$  by

$$(4.1) \quad \tilde{\tau} = -(2 \cos \alpha_k)^{1/n} \tau.$$

and we assume that  $\tilde{\tau} \in \tilde{I}$ .

It will be convenient to denote by  $X_{\tau}^{\vec{a}}$  the parameterization of the  $n$ -Delaunay hypersurface whose Delaunay parameter is  $\tau$ , whose axis is the line directed by  $\vec{a}$  passing through the origin, and having a neck passing through the hyperplane  $\vec{x} \cdot \vec{a} = 0$ .

Up to a translation along the  $x_2$  - axis, the end  $E_0(\tau, \alpha_k, 0)$  of  $\Sigma(\tau, \alpha_k, 0)$  can be parameterize as a normal graph over a  $n$ -Delaunay hypersurface  $\mathcal{D}_{\tilde{\tau}}$  whose axis is  $x_2$ . Using the fact that  $\mathcal{D}_{\tilde{\tau}}$  is periodic of period  $T_{\tilde{\tau}}$ , we conclude that there exists  $u_{\tilde{\tau}} \in (-T_{\tilde{\tau}}, T_{\tilde{\tau}})$  such that  $E_0(\tau, \alpha_k, 0)$  is a graph over the hypersurface parameterized by

$$(4.2) \quad (s, \theta) \longrightarrow X_{\tau, \alpha_k}^0(s, \theta) := X_{\tilde{\tau}}^{\vec{e}_2}(s, \theta) + u_{\tilde{\tau}} \vec{e}_2.$$

Reducing  $I$  if this is necessary, we can assume that  $\tilde{\tau} \longrightarrow u_{\tilde{\tau}}$  is smooth.

Similarly, there exists  $v_{\tilde{\tau}} \in (-T_{\tilde{\tau}}, T_{\tilde{\tau}})$  such that the end  $E_1(\tilde{\tau})$  of the hypersurface  $\Sigma(\tilde{\tau})$  is a graph over the  $n$ -Delaunay hypersurface  $\mathcal{D}_{\tilde{\tau}}$  whose axis is  $x_2$  parameterized by

$$(4.3) \quad (s, \theta) \longrightarrow X_{\tilde{\tau}}^1(s, \theta) := X_{\tilde{\tau}}^{\vec{e}_2}(s, \theta) + v_{\tilde{\tau}} \vec{e}_2.$$

Again, we can assume that  $\tilde{\tau} \longrightarrow v_{\tilde{\tau}}$  is smooth.

Finally, there exists  $w_{\tau} \in (-T_{\tau}, T_{\tau})$  such that the end  $E_1(\alpha, \tau, 0)$  of the hypersurface  $\Sigma(\tau, \alpha, 0)$  is a graph over the  $n$ -Delaunay hypersurface parameterized by

$$(4.4) \quad (s, \theta) \longrightarrow X_{\tau, \alpha_k}^1(s, \theta) := X_{\tau}^{\vec{a}_1}(s, \theta) + w_{\tau} \vec{a}_1$$

and we assume that  $\tau \longrightarrow w_{\tau}$  is smooth. In the same way, we can also denote by  $X_{\tau, \alpha_k}^{-1}$  the parametrization of the end  $E_{-1}(\alpha, \tau, 0)$ .

Given  $m \in \mathbb{N}$ , we set

$$\ell_m := u_{\tilde{\tau}} - v_{\tilde{\tau}} + m T_{\tilde{\tau}}$$

and we consider the hypersurface  $\Sigma(\tau, \alpha_k, \ell_m)$  which is truncated at its ends by cutting from this hypersurface the following three pieces

$$E_0^m(\tau, \alpha_k, \ell_m) := E_0(\tau, \alpha_k, \ell_m) \cap \left\{ x \in \mathbb{R}^{n+1} : x_2 < u_{\tilde{\tau}} + \frac{m}{2} T_{\tilde{\tau}} \right\},$$

$$E_1^m(\tau, \alpha_k, \ell_m) := E_1(\tau, \alpha_k, \ell_m) \cap \{ x \in \mathbb{R}^{n+1} : \sin(\pi/k) x_2 > \cos(\pi/k) x_1 \},$$

and

$$E_{-1}^m(\tau, \alpha_k, \ell_m) := E_{-1}(\tau, \alpha_k, \ell_m) \cap \{ x \in \mathbb{R}^{n+1} : \sin(\pi/k) x_2 < -\cos(\pi/k) x_1 \}.$$

The resulting piece of hypersurface will be extended by using the iterated action of  $R_{2\pi/k}$  and it can be considered as a hypersurface in  $\mathbb{R}^{n+1}/G_k$ .

Finally, we truncate the hypersurface  $\Sigma_k(\tilde{\tau})$  at its ends by considering the intersection of this hypersurface with

$$\left\{ x \in \mathbb{R}^{n+1} : \cos(2\pi j/k) x_2 + \sin(2\pi j/k) x_1 < u_{\tilde{\tau}} + \frac{m}{2} T_{\tilde{\tau}} \quad j = 0, \dots, k-1 \right\}.$$

The resulting piece of hypersurface is invariant under the action of  $R_{2\pi/k}$ .

We will apply the end-to-end construction to the union of these pieces of constant mean curvature hypersurfaces which can be connected together using suitable cutoff functions. Observe that the construction only depends on the continuous parameter  $\tau$  and the discrete parameter  $m$ . We denote by  $\Sigma(\tau, m)$  this hypersurface which is invariant under the action of the group  $G_k$ . Because of the symmetries we impose, it is only necessary to worry about the end-to-end gluing at two places. Namely where the end  $E_0(\tau, \alpha_k, \ell_m)$  meets the end  $E_1(\tilde{\tau})$  and where the end  $E_1(\tau, \alpha_k, \ell_m)$  meets the image of  $E_{-1}(\tau, \alpha_k, \ell_m)$  by  $R_{2\pi/k}$ .

The hypersurface  $\Sigma(\tau, m)$  has mean curvature close to 1. Then, this hypersurface will be used as an approximate solution in order to find a compact constant mean curvature hypersurface with nontrivial topology. Indeed, we consider hypersurfaces which can be written as a normal graph over  $\Sigma(\tau, m)$ , for some “small” function  $w$ . The equation guaranties that this new hypersurface has constant mean curvature equal to 1 takes the form:

$$\mathcal{L}_{\Sigma(\tau,m)} w + \mathcal{Q}_{\tau,m}(w) = H_{\Sigma(\tau,m)} - 1,$$

where  $\mathcal{L}_{\Sigma(\tau,m)}$  is the Jacobi operator about the hypersurface  $\Sigma(\tau, m)$ ,  $H_{\Sigma(\tau,m)}$  is the mean curvature of  $\Sigma(\tau, m)$  and the operator  $\mathcal{Q}_{\tau,m}$  collects all the nonlinear terms.

Technically, the proof of the gluing construction which is equivalent to solving the last equation is now identical to what we have done in the proof of the end-to-end construction we have performed in [4] and which is based on the use of the Schauder’s fixed point theorem. In the aim to use this we need to recover two important features :

- (1) The fact that the elements of the deficiency spaces, which are necessary to recover the surjectivity of the Jacobi operator about  $\Sigma(\tau, \alpha_k, \ell_m)$  and about  $\Sigma_k(\tilde{\tau})$ , can be extended to the hypersurface  $\Sigma(\tau, m)$ . Using this, we can prove the existence of an inverse of  $\mathcal{L}_{\Sigma(\tau,m)}$  denoted by  $G_{\tau,m}$  which is bounded and independent of  $\tau$  and  $m$ . Then our problem reduces to finding a fixed point for the mapping:

$$w \mapsto G_{\tau,m} \left( H_{\Sigma(\tau,m)} - 1 - \mathcal{Q}_{\tau,m}(w) \right).$$

- (2) The correct estimate of the mean curvature of the hypersurface  $\Sigma(\tau, m)$ . In particular, the quantity  $\|H_{\Sigma(\tau,m)} - 1\|_{C^{0,\alpha}(\Sigma(\tau,m))}$  represents the radius of the ball where the last map will be a contraction.

We now concentrate on the proof of these two properties.

### 4.1. The mean curvature of the approximate solution

By construction, the mean curvature of the hypersurface  $\Sigma(\tau, m)$  is equal to 1 except in two annular regions.

Since the end  $E_0(\tau, \alpha_k, \ell_m)$  and the end  $E_1(\tilde{\tau})$  are normal graphs over the same Delaunay hypersurface  $\mathcal{D}_{\tilde{\tau}}$ , using (4.2)-(4.3) we can connect the two pieces together by considering the parametrization

$$Y_{\tau,m}^0(s, \theta) := \varsigma(s) X_{\tilde{\tau}}^1(s + m s_{\tilde{\tau}}, \theta) + (1 - \varsigma(s)) (X_{\tau,\alpha_k}^0(m s_{\tilde{\tau}} - s, \theta) + \ell_m \vec{e}_2)$$

for  $(s, \theta) \in (-m s_{\tilde{\tau}}, s_{\tilde{\tau}}) \times S^{n-1}$ . Here  $s \mapsto \zeta(s)$  is a cutoff function equal to 1 for  $s \leq 1$ , equal to 0 for  $s \geq -1$  and satisfies

$$\zeta(-s) = 1 - \zeta(s).$$

We denote by  $A_1$  the image of  $(-1, 1) \times S^{n-1}$  by  $Y_{\tau, m}^0$ . In this region the end  $E_0(\tau, \alpha_k, \ell_m)$  meets the end  $E_1(\tilde{\tau})$ . Hence, the estimate (3.6) we have obtained in the end-to-end gluing still holds in  $A_1$ . Therefore, we already get

$$\|H_{\Sigma(\tau, m)} - 1\|_{C^{0, \alpha}(A_1)} \leq c e^{-m \gamma_{n+1}(\tilde{\tau}) s_{\tilde{\tau}}}.$$

It remains to estimate  $H_{\Sigma(\tau, m)}$  where the end  $E_1(\tau, \alpha_k, \ell_m)$  meets the image of  $E_{-1}(\tau, \alpha_k, \ell_m)$  by  $R_{2\pi/k}$ . This time, in order to get a similar estimate, we need to check that the  $n$ -Delaunay hypersurface  $\mathcal{D}_{\tau}^{\tilde{a}_1}$ , which has been translated by the vector

$$(u_{\tilde{\tau}} - v_{\tilde{\tau}} + m T_{\tilde{\tau}}) \vec{e}_2 + w_{\tau} \vec{a}_1$$

is invariant under the action of the symmetry by the hyperplane

$$\Pi_k := \{x \in \mathbb{R}^{n+1} : \sin(2\pi/k) x_2 = \cos(2\pi/k) x_1\}.$$

Hence the above  $n$ -Delaunay hypersurface should be symmetric with respect to the symmetry of the hyperplane  $\Pi_k$ . This in turn amounts to check that there exists an integer  $\tilde{m}$  such that

$$(4.5) \quad \sin(\pi/k) (v_{\tilde{\tau}} - u_{\tilde{\tau}} + m T_{\tilde{\tau}}) = w_{\tau} + \tilde{m} \frac{T_{\tau}}{2}.$$

We define

$$\mathfrak{G}(\tau) := \frac{2}{T_{\tau}} \left( \sin(\pi/k) (v_{\tilde{\tau}} - u_{\tilde{\tau}}) - w_{\tau} \right),$$

and

$$\mathfrak{F}(\tau) := 2 \sin(\pi/k) \frac{T_{\tilde{\tau}}}{T_{\tau}},$$

where we recall that  $\tilde{\tau}$  is defined by (4.1). We prove the :

PROPOSITION 4.4. — Assume that  $\tau \in I$  is fixed and that

$$(4.6) \quad \partial_{\tau} \left( \frac{T_{\tilde{\tau}}}{T_{\tau}} \right) \neq 0.$$

Then there exists  $m_0 \in \mathbb{N}$  and, for all  $m \geq m_0$  there exists  $\tau_m \in I$  such that

$$\mathfrak{G}(\tau_m) + m \mathfrak{F}(\tau_m) \in \mathbb{N}.$$

moreover,

$$|\tau - \tau_m| \leq c m^{-1}$$

for some constant  $c$  independent of  $m \geq m_0$ .

*Proof.* — Observe that, by construction the function  $\mathfrak{G}$  is bounded in a fixed neighborhood of  $\tau$ . Given  $\tau'$  close to  $\tau$  we evaluate

$$\mathfrak{h}(\tau') := (\mathfrak{G}(\tau') + m\mathfrak{F}(\tau')) - (\mathfrak{G}(\tau) + m\mathfrak{F}(\tau))$$

using Taylor's expansion. We obtain

$$\mathfrak{h}(\tau') = \left( m \partial_\tau \mathfrak{F}(\tau) + \partial_\tau \mathfrak{G}(\tau) \right) (\tau' - \tau) + m \mathcal{O}(|\tau - \tau'|^2).$$

Provided  $m$  is chosen large enough and

$$\begin{cases} c_0 \partial_\tau \mathfrak{F}(\tau) > 1 & \text{if } \partial_\tau \mathfrak{F}(\tau) > 0 \\ c_0 \partial_\tau \mathfrak{F}(\tau) < -1 & \text{if } \partial_\tau \mathfrak{F}(\tau) < 0 \end{cases}$$

we have  $\mathfrak{h}(\tau) = 0$  and  $|\mathfrak{h}(\tau + c_0/m)| > 1$ . Using the intermediate value theorem there would exist  $\tau_m \in (\tau, \tau + c_0 m^{-1})$  such that

$$\mathfrak{h}(\tau_m) = \begin{cases} 1 + E(\xi_\tau) - \xi_\tau & \text{if } \partial_\tau \mathfrak{F}(\tau) > 0 \\ E(\xi_\tau) - \xi_\tau & \text{if } \partial_\tau \mathfrak{F}(\tau) < 0 \end{cases}$$

with  $\xi_\tau = \mathfrak{G}(\tau) + m\mathfrak{F}(\tau)$  and  $E(\xi_\tau)$  is the entire part of  $\xi_\tau$ . This completes the proof. □

*Remark 4.5.* — A close inspection of the proof shows that, we even obtain many solutions of

$$\mathfrak{G}(\tau) + m\mathfrak{F}(\tau) \in \mathbb{N},$$

which are close to  $\tau$  since, given any integer  $j$ , we have  $|\mathfrak{h}(\tau + jc/m)| > j$  for  $m$  large enough.

Now, assume that (4.5) is fulfilled. Then, we can connect together the end  $E_1(\tau, \alpha_k, \ell_m)$  and the image of  $E_{-1}(\tau, \alpha_k, \ell_m)$  by  $R_{2\pi/k}$  using the following parametrization

$$Z_{\tau,m}^0(s, \theta) := \zeta(s) X_{\tau,\alpha_k}^1(s + \tilde{m}s_\tau, \theta) + (1 - \zeta(s)) R_{2\pi/k} \circ X_{\tau,\alpha_k}^{-1}(\tilde{m}s_\tau - s, \theta) + \ell_m \vec{e}_2$$

for  $(s, \theta) \in (-\tilde{m}s_\tau, \tilde{m}s_\tau) \times S^{n-1}$ . As above, we denote by  $A_2$  the image of  $(-1, 1) \times S^{n-1}$  by  $Z_{\tau,m}^0$ . Then, we also get the estimate

$$\|H_{\Sigma(\tau,m)} - 1\|_{C^{0,\alpha}(A_2)} \leq c e^{-\tilde{m} \gamma_{n+1}(\tau) s_\tau},$$

which is the desired estimate.



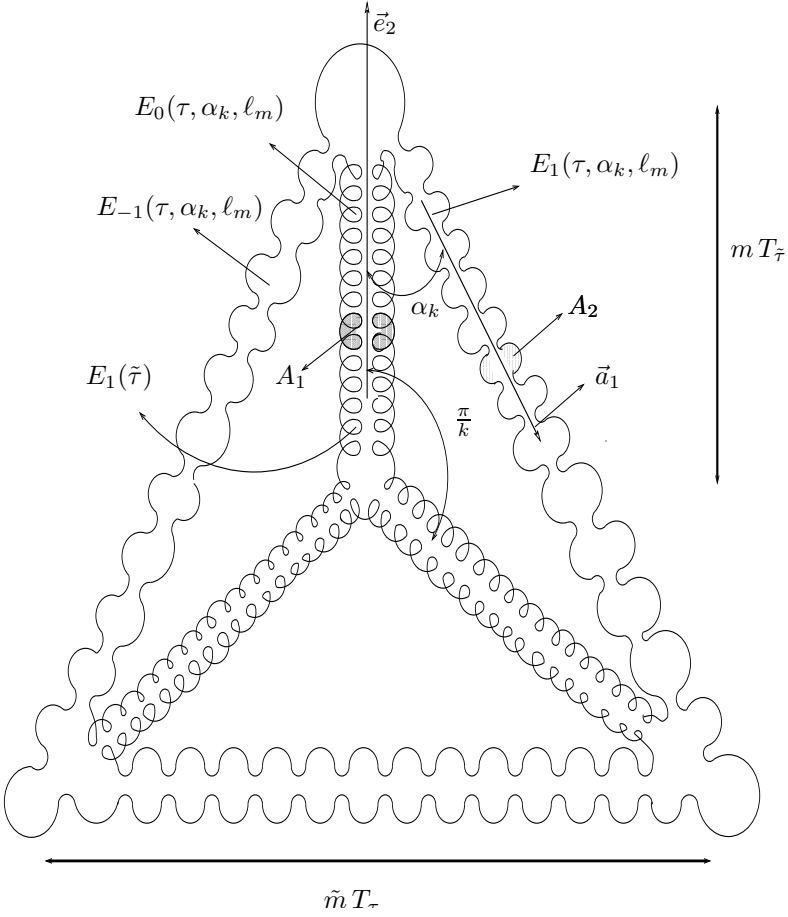


Figure 4.1. The hypersurface  $\Sigma(\tau, m)$  with  $k = 3$ .

**4.2. Extension of the elements of the deficiency spaces**

We give a precise description of the Jacobi fields on both  $\Sigma(\tau, \alpha, \ell)$  and on  $\Sigma_k(\tilde{\tau})$ . This description allows one to give a description of the elements of the deficiency subspaces associated to these hypersurfaces which are needed to ensure the surjectivity of the Jacobi operators about these hypersurfaces.

We keep the notations we have used in Section 2. Recall that on any end  $E$  asymptotic to a  $n$ -Delaunay hypersurface  $\mathcal{D}_\tau$ , there are  $2(n + 1)$  locally

defined Jacobi fields  $\Phi_E^{j,\pm}$  for  $j = 0, \dots, n$ , which are asymptotic to the corresponding Jacobi fields on  $\mathcal{D}_\tau$ .

By assumption,  $\Sigma(\tau, \alpha, \ell)$  is nondegenerate, therefore, the deficiency space  $\mathcal{W}_{\Sigma(\tau, \alpha, \ell)}$  is  $6(n + 1)$  dimensional. Now, recall that we are working in the space of hypersurfaces which are invariant under the action of the group  $G$  and this reduces the dimension of the corresponding moduli space to 3 and the deficiency space is now spanned by the 6 functions

$$\Psi_0^{0,\pm} := \chi_{E_0(\tau, \alpha, \ell)} \Phi_{E_0(\tau, \alpha, \ell)}^{0,\pm}$$

$$\Psi_1^{0,\pm} := \chi_{E_1(\tau, \alpha, \ell)} \Phi_{E_1(\tau, \alpha, \ell)}^{0,\pm} + (\chi_{E_1(\tau, \alpha, \ell)} \Phi_{E_1(\tau, \alpha, \ell)}^{0,\pm}) \circ S$$

and

$$\Psi_1^{1,\pm} := \chi_{E_1(\tau, \alpha, \ell)} \Phi_{E_1(\tau, \alpha, \ell)}^{1,\pm} + (\chi_{E_1(\tau, \alpha, \ell)} \Phi_{E_1(\tau, \alpha, \ell)}^{1,\pm}) \circ S$$

where  $\chi_E$  are cutoff functions which are equal to 1 on the end  $E$ , away from some compact set in  $\Sigma(\tau, \alpha, \ell)$  and  $\Phi_{E_1(\tau, \alpha, \ell)}^{1,\pm}$  are the two Jacobi field corresponding to the translation of the end  $E_1(\tau, \alpha, \ell)$  and the rotation of its axis in the direction of

$$\vec{a}_1^\perp = \cos \alpha_k \vec{e}_1 + \sin \alpha_k \vec{e}_2.$$

We now describe the Jacobi fields which are obtained by moving the three parameters  $\alpha, \tau$  and  $\ell$  and which span the nullspace  $\mathcal{N}(\Sigma(\tau, \alpha, \ell))$  defined in the beginning of Section 3.

- (1) Changing the  $\ell$  parameter (keeping  $\tau$  and  $\alpha$  fixed) yields a Jacobi field

$$\Phi^+ = \vec{N}_{\Sigma(\tau, \alpha, \ell)} \cdot \vec{e}_2$$

where,  $\vec{N}_{\Sigma(\tau, \alpha, \ell)}$  is the unit normal vector of  $\Sigma(\tau, \alpha, \ell)$ . This Jacobi fields is asymptotic to  $\Psi_0^{0,+}$  on  $E_0(\tau, \alpha, \ell)$  and is asymptotic to  $c_1 \Psi_1^{0,+} + c_2 \Psi_1^{1,+}$  on  $E_{-1}(\tau, \alpha, \ell) \cup E_1(\tau, \alpha, \ell)$ . In fact,  $c_1 = -\cos \alpha_k$  and  $c_2 = \sin \alpha_k$ .

- (2) Changing the  $\tau$  parameter (keeping  $\alpha$  and  $\ell$  fixed), yields a Jacobi field  $\Phi^-$  which is asymptotic to  $c_3 \Psi_1^{0,+} + c_4 \Psi_1^{0,-}$  on  $E_{-1}(\tau, \alpha, \ell) \cup E_1(\tau, \alpha, \ell)$  and which is asymptotic to  $\Psi_0^{0,+} + c_5 \Psi_0^{0,-}$  on  $E_0(\tau, \alpha, \ell)$ . In fact differentiation of (4.1) with respect to  $\tilde{\tau}$  we get

$$\partial_\tau \tilde{\tau} = -(2 \cos \alpha_k)^{1/n}$$

which implies that  $c_3 = -(2 \cos \alpha_k)^{1/n}$ .

- (3) Changing the  $\alpha$  parameter (keeping  $\tau$  and  $\ell$  fixed), yields a Jacobi field which is equal to  $\Psi_1^{1,-} + c_6 \Psi_1^{0,+} + c_7 \Psi_1^{0,-} + c_8 \Psi_1^{1,+}$  on  $E_{-1}(\tau, \alpha, \ell) \cup E_1(\tau, \alpha, \ell)$  and which is asymptotic to  $c_9 \Psi_0^{0,+}$  on  $E_0(\tau, \alpha, \ell)$ .

Recall that the space  $\mathcal{K}(\Sigma(\tau, \alpha, \ell))$  is a 3 dimensional subspace of the deficiency space  $\mathcal{W}_{\Sigma(\tau, \alpha, \ell)}$  chosen so that

$$\mathcal{W}_{\Sigma(\tau, \alpha, \ell)} = \mathcal{K}(\Sigma(\tau, \alpha, \ell)) \oplus \mathcal{N}(\Sigma(\tau, \alpha, \ell)).$$

It follows from the description of the elements of  $\mathcal{N}(\Sigma(\tau, \alpha, \ell))$  that we can choose

$$\mathcal{K}(\Sigma(\tau, \alpha, \ell)) = \text{Span}\{\Psi_0^{0,-}, \Psi_0^{0,+}, \Psi_1^{1,+}\}.$$

Similarly, since  $\Sigma_k(\tilde{\tau})$  is assumed to be nondegenerate, the deficiency space  $\mathcal{W}_{\Sigma_k(\tilde{\tau})}$  is  $2k(n+1)$  dimensional. However, recall that we are working in the space of hypersurfaces which are invariant under the action of the group  $G_k$  and this reduces the dimension of the corresponding moduli space to 1 and the deficiency space is now spanned by the 2 functions

$$\bar{\Psi}^{0,\pm} := \sum_{j=1}^k \chi_{E_j(\tilde{\tau})} \Phi_{E_j(\tilde{\tau})}^{0,\pm}$$

here  $\chi_E$  are also cutoff functions which are equal to 1 on the end  $E$ , away from some compact set in  $\Sigma_k(\tilde{\tau})$ . Since the end  $E_1(\tilde{\tau})$  is assumed to be regular there exists a globally Jacobi field  $\bar{\Phi}^-$  whose asymptotic on  $E_1(\tilde{\tau})$  has a nontrivial component on  $\Phi_{E_1(\tilde{\tau})}^{0,-}$ . This Jacobi field is obtained by changing the  $\tilde{\tau}$  parameter and is asymptotic to  $\bar{\Psi}^{0,-} + \bar{c}\bar{\Psi}^{0,+}$  on each  $E_j(\tilde{\tau})$ , were the constant  $\bar{c}$  depends on  $\tilde{\tau}$ . This implies that the space  $\mathcal{K}(\Sigma_k(\tilde{\tau}))$  can be chosen to be

$$\mathcal{K}(\Sigma_k(\tilde{\tau})) = \text{Span}\{\bar{\Psi}^{0,+}\}.$$

We apply the above result to the hypersurfaces  $\Sigma(\tau, \alpha_k, \ell_m)$  and  $\Sigma_k(\tilde{\tau})$ . We would like to extend the elements of  $\mathcal{K}(\Sigma(\tau, \alpha_k, \ell_m))$  and  $\mathcal{K}(\Sigma(\tilde{\tau}))$  to functions which are defined on  $\Sigma(\tau, m)$ .

- (i) First, the element  $\Psi_1^{1,+}$  of  $\mathcal{K}(\Sigma(\tau, \alpha_k, \ell_m))$  can be easily extended to  $\Sigma(\tau, m)$  using the fact that the end  $E_1(\tau, \alpha_k, \ell_m)$  and the image of  $E_{-1}(\tau, \alpha_k, \ell_m)$  by  $R_{2\pi/k}$  are symmetric with respect to  $\Pi_k$ , the function  $\Psi_1^{1,+} \circ Z_{\tau,m}^0$  is asymptotic to the function  $\Phi_1^{1,+} \circ Z_{\tau,m}^0$  which is even. We can then define a function  $\Psi^{1,+m}$  on the part of  $\Sigma(\tau, m)$  which is parameterized by  $Z_{\tau,m}^0$ , as follow:

$$\Psi^{1,+m} := \chi_m \Psi_1^{1,+} + (1 - \chi_m) \Psi_1^{1,+} \circ (R_{2\pi/k})^{-1}.$$

Here,  $\chi_m$  is a cutoff function such that in the part of  $\Sigma(\tau, m)$  parameterized by  $Y_{\tau,m}^0$  this function is equal to 1 for  $s \geq 1$  and equal to 0 for  $s \leq -1$  otherwise in the part of  $\Sigma(\tau, m)$  parameterized by  $Z_{\tau,m}^0$ ,  $\chi_m$  is equal to 0 for  $s \geq 1$  and equal to 1 for  $s \leq -1$ .

Then, we use the action of  $G_k$  to extend the function  $\Psi^{1,+m}$  to the

other components of  $\Sigma(\tau, m)$ .

- (ii) In the same way, the unique Jacobi field  $\bar{\Phi}^-$  defined on  $\Sigma(\tilde{\tau})$  can be extended along the end  $E_0(\tau, \alpha_k, \ell_m)$  ( which has been connected with the end  $E_1(\tilde{\tau})$  of  $\Sigma(\tilde{\tau})$  ) using a linear combination of the globally defined Jacobi fields  $\Phi^+$  and  $\Phi^-$ . Indeed, the function  $\bar{\Phi}^- \circ Y_{\tau,m}^0$  is asymptotic to

$$\Phi_{\tilde{\tau}}^{0,-}(\cdot + m s_{\tilde{\tau}}) + \bar{c} \Phi_{\tilde{\tau}}^{0,+}(\cdot + m s_{\tilde{\tau}}).$$

Similarly, on  $\Sigma(\tau, \alpha_k, \ell_m)$ , the globally defined Jacobi field  $\Phi^- \circ Y_{\tau,m}^0$  is asymptotic to the function

$$\Phi_{\tilde{\tau}}^{0,-}(\cdot - m s_{\tilde{\tau}}),$$

and the globally defined Jacobi field  $\Phi^+ \circ Y_{\tau,m}^0$  is asymptotic to the function

$$\Phi_{\tilde{\tau}}^{0,+}(\cdot - m s_{\tilde{\tau}}).$$

Now, we define  $\Psi^{0,-,m}$ , the extension of  $\bar{\Phi}^-$  on the connected sum of  $E_0^m(\tau, \alpha_k, \ell_m)$  and  $E_0(\tilde{\tau})$  by

$$\tilde{\Psi}^{0,-,m} := \bar{\chi}_m \bar{\Phi}^- + (1 - \bar{\chi}_m)(\Phi^- + (\bar{c} + m \partial_{\tilde{\tau}} T_{\tilde{\tau}}) \Phi^+).$$

Here,  $\bar{\chi}_m$  is a cutoff function defined on  $\Sigma(\tau, m)$  assumed to be invariant under the action of  $G_k$  and equal to

$$1 - \sum_{j=1}^k \chi_m \circ R_{2\pi j/k}.$$

As above, we use the action of  $G_k$  to extend the function  $\tilde{\Psi}^{0,-,m}$  to the other components of  $\Sigma(\tau, m)$ .

The key point is that we do not consider that the end  $E_1^m(\tau, \alpha_k, \ell_m)$  and the image of  $E_{-1}^m(\alpha_k, \tau, \ell_m)$  by  $R_{2\pi/k}$  are connected. In fact, on the part of  $\Sigma(\tau, m)$  parameterized by  $Z_{\tau,m}^0$  for  $s \in (-\tilde{m} s_{\tau}, 0)$  the function  $\Psi^{0,-,m}$  is equal to

$$\Phi^- + (\bar{c} + m \partial_{\tilde{\tau}} T_{\tilde{\tau}}) \Phi^+$$

and hence is asymptotic to

$$c_3 \Phi_{\tau}^{0,-}(\cdot + \tilde{m} s_{\tau}) + (c_4 + c_1 (\bar{c} + m \partial_{\tilde{\tau}} T_{\tilde{\tau}})) \Phi_{\tau}^{0,+} + c_2 (\bar{c} + m \partial_{\tilde{\tau}} T_{\tilde{\tau}}) \Phi_{\tau}^{1,+}$$

for  $s \in (-\tilde{m} s_{\tau}, \tilde{m} s_{\tau})$ . Since

$$\Phi_{\tau}^{0,-}(\cdot + \tilde{m} s_{\tau}) = \Phi_{\tau}^{0,-}(\cdot) + \frac{\tilde{m}}{2} \partial_{\tau} T_{\tau} \Phi_{\tau}^{0,+}(\cdot),$$

the function  $\tilde{\Psi}^{0,-,m}$  is also asymptotic to

$$c_3 \Phi_\tau^{0,-}(\cdot) + c_2 (\bar{c} + m \partial_{\bar{\tau}} T_{\bar{\tau}}) \Phi_\tau^{1,+} + \left( c_4 + c_1 (\bar{c} + m \partial_{\bar{\tau}} T_{\bar{\tau}}) + \frac{\tilde{m}}{2} \partial_\tau T_\tau \right) \Phi_\tau^{0,+}.$$

The part both  $\Phi_\tau^{0,-}$  and  $\Phi_\tau^{1,+}$  can be extended as we have done above because they are even functions of the variable  $s$ . However, observe that the coefficient in front of  $\Phi_\tau^{0,+}$  is certainly not zero when  $m$  and  $\tilde{m}$  are large enough and  $\Phi_\tau^{0,+}$  is not even. Hence, this function cannot be extended as we have already done.

To end this discussion, observe that the function  $\tilde{\Psi}^{0,-,m}$  is on the different summands a Jacobi field.

We set

$$\mu := c_4 + c_1 (\bar{c} + m \partial_{\bar{\tau}} T_{\bar{\tau}}) + \frac{\tilde{m}}{2} \partial_\tau T_\tau.$$

- (iii) The element  $\Psi_1^{0,+}$  of  $\mathcal{K}(\Sigma(\tau, \alpha_k, \ell_m))$  can be easily extended to  $\Sigma(\tau, m)$  after adding to it a suitable multiple of the function  $\Psi^{0,-,m}$  described above. Indeed, the function

$$\tilde{\Psi}_1^{0,+} := \Psi_1^{0,+} - \mu^{-1} \tilde{\Psi}^{0,-,m}$$

is now an even function of  $s$  on the image of  $Z_{\tau,m}^0$  and hence can be extended to a function  $\Psi^{0,+,m}$  on  $\Sigma(\tau, m)$  by defining, on the part of  $\Sigma(\tau, m)$  parameterized by  $Z_{\tau,m}^0$ ,

$$\Psi^{0,+,m} := \chi_m \tilde{\Psi}_1^{0,+} + (1 - \chi_m) \tilde{\Psi}_1^{0,+} \circ (R_{2\pi/k})^{-1}.$$

- (iv) The element  $\bar{\Psi}^{0,+}$  of  $\mathcal{K}(\Sigma_k(\tilde{\tau}))$  can be extended to  $\Sigma(\tau, m)$  using the unique Jacobi field  $\Phi^+$  defined on  $\Sigma(\tau, \alpha_k, \ell_m)$  which is asymptotic to  $\Phi_{E_0^m(\tau, \alpha, \ell)}^{0,+}$  on  $E_0^m(\tau, \alpha_k, \ell_m)$  to which a suitable multiple of a the function  $\Psi^{0,-,m}$  is added. We define a function  $\bar{\Psi}^{0,+}$  first by writing

$$\bar{\Psi}^{0,+} := \bar{\chi}_m \bar{\Psi}_1^{0,+} + (1 - \bar{\chi}_m) \Phi^+ - c_2 \mu^{-1} \tilde{\Psi}^{0,-,m},$$

on the part of  $\Sigma(\tau, m)$  which is parameterized by  $Y_{\tau,m}^0$ . Observe that  $\Phi^+$  is asymptotic to a linear combination of  $\Psi_1^{0,+}$  and  $\Psi_1^{1,+}$  on the other ends of  $\Sigma(\tau, \alpha_k, \ell_m)$  and we can use the type of extension described in (i) to extend the function to  $\Sigma(\tau, m)$ . For example,

$$\bar{\Psi}^{0,+} := \chi_m (\Phi^+ - c_2 \mu^{-1} \tilde{\Psi}^{0,-,m}) + (1 - \chi_m) (\Phi^+ - c_2 \mu^{-1} \tilde{\Psi}^{0,-,m}) \circ (R_{2\pi/k})^{-1},$$

on the part of  $\Sigma(\tau, m)$  which is parameterized by  $Z_{\tau,m}^0$ . Then, we use the action of  $G_k$  to extend the function  $\bar{\Psi}^{0,+}$  to the other components of  $\Sigma(\tau, m)$ .

(v) It remains to explain how to extend the element  $\Phi^-$  of  $\mathcal{K}(\Sigma_k(\tilde{\tau}))$  to  $\Sigma(\tau, m)$ . Observe that  $(\Phi^- + m\partial_{\tilde{\tau}}T_{\tilde{\tau}}\Phi^+) \circ Z_{\tau,m}^0$  is asymptotic to

$$\Phi_{\tilde{\tau}}^{0,-}(\cdot - m s_{\tilde{\tau}}) + m\partial_{\tilde{\tau}}T_{\tilde{\tau}}\Phi_{\tilde{\tau}}^{0,+}(\cdot - m s_{\tilde{\tau}})$$

and  $\Phi_{E_0(\tilde{\tau})}^{0,-} \circ Y_{\tau,m}^0$  is asymptotic to

$$\Phi_{\tilde{\tau}}^{0,-}(\cdot + m s_{\tilde{\tau}}).$$

Thanks to (1), we can add to these functions a suitable multiple of the function  $\tilde{\Psi}^{0,-,m}$  and connect these as we have already done above, to define the function  $\Psi^{0,-,m}$ . For example, we define

$$\Psi^{0,-,m} := \bar{\chi}_m(1 - \mu^{-1})\Phi^- + (1 - \bar{\chi}_m)\left(\Phi^+ + m\partial_{\tilde{\tau}}T_{\tilde{\tau}}\Phi^+ - \mu^{-1}\tilde{\Psi}^{0,-,m}\right)$$

on the part of  $\Sigma(\tau, m)$  which is parameterized by  $Y_{\tau,m}^0$ . And we define

$$\begin{aligned} \Psi^{0,-,m} &:= \bar{\chi}_m\left(\Phi^- + m\partial_{\tilde{\tau}}T_{\tilde{\tau}}\Phi^+ - \mu^{-1}\tilde{\Psi}^{0,-,m}\right) \\ &\quad + (1 - \bar{\chi}_m)\left(\Phi^- + m\partial_{\tilde{\tau}}T_{\tilde{\tau}}\Phi^+ - \mu^{-1}\tilde{\Psi}^{0,-,m}\right) \circ (R_{2\pi/k})^{-1} \end{aligned}$$

on the part of  $\Sigma(\tau, m)$  which is parameterized by  $Z_{\tau,m}^0$ . Then, we use the action of  $G_k$  to extend this function to the other components of  $\Sigma(\tau, m)$ .

As usual cutoff functions are used to interpolate smoothly between the different functions in the annulus  $A_1$  and  $A_2$ . We define

$$\tilde{\delta} = \inf\{\gamma_{n+1}(\tau), \gamma_{n+1}(\tilde{\tau})\} > 0.$$

As in [4], let  $\delta \in (-\tilde{\delta}, 0)$  and  $\tilde{m}$  is the integer defined by  $\tilde{m} = G(\tau) + mF(\tau)$ . We have

$$\mathcal{L}_{\Sigma(\tau,m)}\Psi = \mathcal{O}_{C^{0,\alpha}(A_1)}(e^{-\delta m s_{\tilde{\tau}}})$$

and

$$\mathcal{L}_{\Sigma(\tau,m)}\Psi = \mathcal{O}_{C^{0,\alpha}(A_1)}(e^{-\delta \tilde{m} s_{\tilde{\tau}}})$$

for  $\Psi = \tilde{\Psi}^{0,-,m}, \Psi^{0,+,m}, \bar{\Psi}^{0,+,m}, \Psi^{1,+,m}$ .

Then we define

$$\mathcal{K}(\Sigma(\tau, m)) = \text{Span}\{\tilde{\Psi}^{0,-,m}, \Psi^{0,+,m}, \bar{\Psi}^{0,+,m}, \Psi^{1,+,m}\}$$

and we define the weighted spaces  $\mathcal{D}_\delta^{k,\alpha}(\Sigma(\tau, m))$  such that the norm of a function in this space is the sum of the usual Hölder norm on compact parts of the pieces constituting  $\Sigma(\tau, m)$  and these on the annulus  $A_1$  and  $A_2$  times respectively  $e^{-\delta m s_{\tilde{\tau}}}$  and  $e^{-\delta \tilde{m} s_{\tilde{\tau}}}$ .

The analysis of the Jacobi operator about  $\Sigma(\tau, m)$  can now be performed following what we have done in [4]. We obtain the :

PROPOSITION 4.6. — Assume that  $\delta \in (-\tilde{\delta}, 0)$  is fixed. There exist  $m_0 > 0$  and  $c > 0$  and, for all  $m \geq m_0$ , one can find an operator

$$G_{\tau,m} : \mathcal{D}_\delta^{0,\alpha}(\Sigma(\tau, m)) \longrightarrow \mathcal{D}_\delta^{2,\alpha}(\Sigma(\tau, m)) \oplus \mathcal{K}(\Sigma(\tau, m)),$$

such that  $w := G_{\tau,m}(f)$  solves  $\mathcal{L}_{\Sigma(\tau,m)} w = f$  on  $\Sigma(\tau, m)$ . Furthermore,

$$\|w\|_{\mathcal{D}_\delta^{2,\alpha}(\Sigma(\tau,m)) \oplus \mathcal{K}(\Sigma(\tau,m))} \leq c \|f\|_{\mathcal{D}_\delta^{0,\alpha}(\Sigma(\tau,m))}.$$

### 4.3. Compact constant mean curvature hypersurfaces with topology

Using the end-to-end construction we obtain the :

THEOREM 4.7. — There exists  $m_0 > 0$  such that, for all  $m \geq m_0$  and  $\tau$  close enough to zero the hypersurface  $\Sigma(\tau, m)$  can be perturbed into a compact constant mean curvature 1 hypersurface.

*Proof.* — The proof of this result is identical to the proof of the corresponding result of Proposition 3.7. The only point being that we need to check that (4.6) is fulfilled. Observe that the function

$$P(\tau) := \frac{T_\tau}{T_{\tilde{\tau}}}$$

where  $\tau$  and  $\tilde{\tau}$  are related by (4.1), is real analytic. Hence either it is constant or its derivative has a isolated zeros. Using the result of Proposition 2.1, we see that  $P$  is not constant for  $\tau \neq 0$  close enough to 0. Indeed, we use the expansion of  $T_\tau$  and (4.1) we obtain

$$P(\tau) = 1 + \frac{c_n}{2} \left( 1 - (2^n \cos \alpha_k)^{\frac{1}{n}} \right) |\tau|^{\frac{n}{n-1}} + \mathcal{O}(|\tau|^{\frac{2n}{n-1}}).$$

Hence the function  $\partial_\tau P$  has isolated zeros and away from those zeros, the examples of hypersurfaces of type 1 and type 2 which are given in the beginning of Section 4 can be used for the construction.  $\square$

*Remark 4.8.* — These hypersurfaces are not embedded since the elements of Type 1 which have been used for their construction are never embedded.

*Acknowledgements.* It is pleasure to thank F. Pacard for insightful comments and assistance through this work. I would like to thank the referee for his careful reading of the paper, his valuable ideas and for fixing some details.

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Manuscrit reçu le 11 juillet 2008,  
révisé le 20 juillet 2010,  
accepté le 15 octobre 2010.

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