ANNALES

DE

L'INSTITUT FOURIER

Gloria MARÍ BEFFA

Moving frames, Geometric Poisson brackets and the KdV-Schwarzian evolution of pure spinors

MOVING FRAMES, GEOMETRIC POISSON BRACKETS AND THE KDV-SCHWARZIAN EVOLUTION OF PURE SPINORS

by Gloria MARÍ BEFFA

––––––––

ABSTRACT. — In this paper we describe a non-local moving frame along a curve of pure spinors in $\mathrm{O}(2m, 2m)/P$, and its associated basis of differential invariants. We show that the space of differential invariants of Schwarzian-type define a Poisson submanifold of the spinor Geometric Poisson brackets. The resulting restriction is given by a decoupled system of KdV Poisson structures. We define a generalization of the Schwarzian-KdV evolution for pure spinor curves and we prove that it induces a decoupled system of KdV equations on the invariants of projective type, when restricted to a certain level set. We also describe its associated Miura transformation and non-commutative modified KdV system.

RÉSUMÉ. — Nous décrivons un repère mobile non local pour les courbes de spineurs purs dans $\mathrm{O}(2m, 2m)/P$, et la base correspondante d'invariants différentiels. Nous montrons que l'espace des invariants différentiels de type Schwarzien définit une sous-variété de crochets de Poisson géométriques de spineurs purs. La restriction résultante est donnée par un systéme découplé de crochets de Poisson de KdV . Nous définissons une généralisation de l'évolution de Schwarz-KdV pour les courbes de spineurs purs et nous montrons que, en restriction à un niveau fixé, cela induit un système d'équations de KdV découplé pour les invariants de type projectif. Nous décrivons par ailleurs la transformation correspondante de Miura et le système non commutatif modifié de KdV.

–––––––––––

1. Introduction

Given a map $u : \mathbb{R} \to \mathbb{R}$, we define the *Schwarzian derivative* of u as

$$S(u) = \frac{u_{xxx}}{u_x} - \frac{3}{2}\left(\frac{u_{xx}}{u_x}\right)^2.$$

––––––––––––––

Locally, one can think of u as taking values in $\mathbb{RP}^1$. In that case, the Schwarzian derivative is the unique differential invariant for the projective action of $\mathrm{SL}(2)$ on $\mathbb{RP}^1$ (defined by fractional transformations). The evolution

$$u_t = u_x S(u) = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x}$$

is called the *Schwarzian-KdV evolution.* It is invariant under the projective action of $\mathrm{SL}(2)$ and, when written as an evolution on the invariant $S(u)$, it becomes the KdV equation, the well-known completely integrable equation

$$S(u)_t = S(u)_{xxx} + 3S(u)_x S(u).$$

The KdV equation is also bi-Hamiltonian, that is, Hamiltonian with respect to two different but *compatible* Poisson (or Hamiltonian) structures. These two Poisson structures can be generated using the background projective geometry of u, as in [30]. The author called these structures *geometric Poisson structures.* In particular, one of them is linked to evolutions of curves u, in the sense that any associated Hamiltonian evolution will come from an invariant evolution of curves $u(t,x)$ in identically the same way KdV is obtained from the Schwarzian-KdV equation. We say these evolution of curves are *projective geometric realizations* of the Hamiltonian systems. Thus, the Schwarzian-KdV equation is a projective realization of the KdV equation. For more information see [33].

   This relation between invariant curve flows and integrable systems occurs in a variety of different geometries for which both integrable systems and their Hamiltonian structures are generated by the background geometry of the flow. The best known example is the Vortex filament flow, an evolution of curves in $\mathbb{R}^3$, invariant under the Euclidean group. When written in terms of the *Euclidean natural curvatures*, the Vortex Filament flow becomes the Nonlinear Schrödinger equation, or NLS ([19], [24]). That is, the Vortex filament flow is an Euclidean realization of NLS. Other completely integrable systems also have geometric realizations, often in different geometries. For example, the Sawada-Koterra equation has both equi-centro affine and projective realizations and their Hamiltonian structures can be obtained using both equi-centro affine and projective invariants ([5]). The recent literature in this subject is extensive, with many authors finding many different geometric realizations and linking curve evolutions, Hamiltonian structures and background geometry to integrable systems, using many different points of view. See for example [2], [1], [8], [9], [16], [20], [21], [26], [25], [28], [39], [27], [37], [40], [41], a list that it is not meant to be, by any means, exhaustive.

The link between projective geometry and the periodic KdV follows from work of Kirillov [23], Segal [38] and others. The Schwarzian has the following cocycle property

$$S(f \circ g) = g_x^2 S(f) \circ g + S(g).$$

This allows us to use it in the definition of the action of the group $\text{diff}(S^1)$ on the space of projective connection, as defined by the kernel of Hill's operators $L = \frac{d}{dx^2} + p(x)$ (in fact, $p(x) = \frac{1}{2}S(\frac{\xi_1}{\xi_2})$ where ξ_i are independent solutions of $L\xi = 0$). The space of projective connections can be viewed in two different ways: as dual to the Virasoro algebra - itself a central extension of the algebra of $\text{diff}(S^1)$ by a cocycle related to $S(u)$; or as the space of projective invariants of curves in $\mathbb{RP}^1$. This invariant space can be obtained as in [30] choosing $G = \text{SL}(2)$ acting projectively on $\mathbb{RP}^1$. Two geometric Poisson brackets are defined on this space, namely the two Hamiltonian structures for KdV.

We will say that a differential invariant is of *projective or Schwarzian type* if

$$(1.1) \qquad \phi^* I = \left(\phi_x^2 I\right) \circ \phi^{-1} + S(\phi) \circ \phi^{-1}$$

where by ϕ^* we mean the pull-back, and where ϕ is a diffeomorphism of the line. That is, the invariant behaves like the Schwarzian under changes of variable.

In [31] the author described a conjecture by M. Eastwood about differential invariants of projective type. Eastwood conjectured that there exist invariants of projective type for curves in Hermitian symmetric manifolds (parabolic manifolds with |1|-gradation of the algebra) and hence a natural projective structure along curves given by the choice of a special type of parametrization. It was conjectured that the existence of these parametrizations indicates the existence of Hamiltonian structures of KdV type on these invariants. Indeed, in the conformal case $M = O(p + 1, q + 1)/P$, with P an appropriate parabolic subgroup, the two invariants of projective type are directly connected to invariant differential operators that appear in the work of Bailey and Eastwood (see [3] and [4]). There the authors used them to defined conformal circles and preferred parametrization that endow the conformal circles with a projective structure (theirs is an explicit proof of Cartan's observation that a curve in a conformal manifold inherits a natural projective structure, see [7]). Furthermore, the author of [15] stated that, if for a given curve all other non-projective invariants vanish, the curve is the orbit of the inversion subgroup of the conformal group. The author proved in [32] that, if we use a conformal *natural moving frame*, then two

compatible geometric conformal Poisson brackets can be restricted to the space of invariants of projective type as invariants of non-projective type vanish. The result is a complexly coupled system of KdV equations, while a different choice of orbit results in a decoupled system of KdV equations, as in [12]. That is, in the conformal case, the Poisson brackets linked to projective structures along the flow are restrictions of the geometric conformal Poisson brackets to an appropriate submanifold.

A similar situation seems to exist for the Lagrangian Grassmannian in $\mathbb{R}^{2n}$ where there are n invariants of projective type defining different projective structures on flows, much like the situation in the conformal case. In [33] the author defined these invariants as the eigenvalues of the *Schwarzian derivative of curves of Lagrangian planes*. The author proved that, if an appropriate moving frame is chosen along curves of Lagrangian planes, then as non-projective type invariants vanish, the Lagrangian Grassmannian geometric Poisson bracket restricts to the manifold of invariants of projective type to produce a bi-Hamiltonian structure for a decoupled system of n-KdV equations. She proved that the *Lagrangian Schwarzian-KdV* equation for curves of Lagrangian planes is a geometric realization for this decoupled KdV system whenever we restrict to curves for which invariants of non-projective type vanish. She also found the Lagrangian Grassmannian analog of a *Miura transformation* and an associated *non-commutative modified KdV system* of equations. The Schwarzian derivative for curves of Lagrangian planes was first defined by Ovsienko in [35] where the role of inversions in the generation of these invariants was also described. A definition can also be found in [36] where the interested reader can find a thorough description of projective geometry and many of its applications.

On the other hand, the case of curves of pure spinors, another Symmetric Hermitian case (case $M = O(n,n)/P$ for an appropriate *parabolic subgroup* P), remains elusive. Invariants of projective type for the even dimensional case were defined in [29], but the precise connection to KdV structures and projective structures on flows was not clear. In [29] the author proved that no local choice of moving frame would produce a result similar to that of the Lagrangian Grassmannian, in spite of their algebraic similarities (the Lagrangian moving frame in [29] was local). In the odd dimensional case the differential invariants are too involved to be effective for this type of study (this part of the study is not published).

In this paper we find a *non-local* moving frame along curves of even pure spinors and we prove that the situation in the Lagrangian Grassmannian manifold also takes place for pure even spinors. Recall that, although the

Lagrangian moving frame is local, natural moving frames for both Riemannian and conformal manifolds are non-local in nature. As this paper shows, the choice of an appropriate moving frame is fundamental to establishing the connection between the projectively-induced Poisson structures and the geometric ones. We use a group-based definition of moving frame introduced by Fels and Olver in [13, 14]. This definition is becoming well-known, but not so much as to make many in the area familiar with this new concept. Hence we have included a section on background information about group-based moving frames. This, information on geometric Poisson brackets and a brief description of the manifold of pure spinors is included in our second section.

In Section 3, Theorem 4.2, we describe an appropriate choice of non-local moving frame and their associated basis of differential invariants. This choice is produced using a non-local gauge of a moving frame appearing in [32], which is also described here. In Section 4, Theorem 4.3, we prove that the spinor geometric Poisson structure can be restricted to the manifold of invariants of projective type (the manifold where non-local invariants of non-projective type vanish) to produce a decoupled system of m KdV Poisson structures, where $n = 2m$. We also prove that a second compatible Poisson structure can be reduced to the submanifold of invariants of projective type to produce a system of decoupled and compatible Poisson structures for KdV. In Theorem 4.4 we describe a generalization of the Schwarzian-KdV evolution for spinors and we prove it is a spinor geometric realization for the decoupled system of KdV equations, as far as we restrict initial conditions to curves with vanishing non-projective invariants. Our last theorem (Theorem 4.5) describes the spinor Miura transformation and its associated non-commutative modified KdV for flows of skew-symmetric matrices. Notice that, although similar to Drinfeld and Sokolov's contruction, the reduction process here is not quite the same as the one presented in [10]. The idea of reducing the Poisson brackets to a certain quotient is the same, but they use a different gradation of the algebra (we use a |1|-gradation here, while they use the gradation induced by the natural gradation of $\mathfrak{gl}(n, \mathbb{R})$, that is, the finest possible one). It is still not clear if, when or how their construction produces a geometric realization in the group quotient.

Now, it seems natural to conjecture that what is true for conformal, Lagrangian and spinor cases, will also be true for general Symmetric Hermitian, and perhaps parabolic manifolds. A general proof is still not apparent: the methods used rely heavily on a proper choice of moving frame for which

the vanishing of non-projective invariants does not produce singularities in the geometric Hamiltonian vector field. It is not clear what such a choice must be in general, but one could predict that, once this point is cleared up, a general proof is within reach.

2. Definitions and background

2.1. Group-based moving frames and their associated differential invariants

The classical concept of moving frame was developed by Élie Cartan ([6]). A classical moving frame along a curve in a manifold M is a curve in the frame bundle of the manifold over the curve, invariant under the action of the transformation group under consideration. This method is a very powerful tool, but its explicit application relied on intuitive choices that were not clear on a general setting. Some ideas in Cartan's work and later work of Griffiths ([18]), Green ([17]) and others laid the foundation for the concept of a group-based moving frame, that is, an equivariant map between the jet space of curves in the manifold and the group of transformations. Recent work by Fels and Olver ([13, 14]) finally gave the precise definition of the group-based moving frame. In this section we will describe Fels and Olver's moving frame and its relation to the classical moving frame. We will also introduce some definitions that are useful to the study of Poisson brackets and biHamiltonian nonlinear PDEs. From now on we will assume $M = G/H$ with G acting on M via left multiplication on representatives of a class. We will also assume that curves in M are *parametrized* and, therefore, the group G does not act on the parameter.

DEFINITION 2.1. — *Let $J^k(\mathbb{R}, M)$ be the space of k-jets of curves, that is, the set of equivalence classes of curves in M up to k^{th} order of contact. If we denote by $u(x)$ a curve in M and by u_r the r derivative of u with respect to the parameter x, $u_r = \frac{d^r u}{dx^r}$, the jet space has local coordinates that can be represented by $u^{(k)} = (x, u, u_1, u_2, \ldots, u_k)$. The group G acts naturally on parametrized curves, therefore it acts naturally on the jet space via the formula*

$$g \cdot u^{(k)} = (x, g \cdot u, (g \cdot u)_1, (g \cdot u)_2, \ldots)$$

where by $(g \cdot u)_k$ we mean the formula obtained when one differentiates k times $g \cdot u$ and then writes the result in terms of g, u, u_1, etc. This is the natural way for G to act on jets and it is usually called the prolonged action of G on $J^k(\mathbb{R}, M)$.

DEFINITION 2.2. — *A function*

$$I : J^k(\mathbb{R}, M) \to \mathbb{R}$$

is called a kth order differential invariant if it is invariant with respect to the prolonged action of G.

DEFINITION 2.3. — *A map*

$$\rho : J^k(\mathbb{R}, M) \to G$$

is called a left (resp. right) moving frame if it is equivariant with respect to the prolonged action of G on $J^k(\mathbb{R}, M)$ and the left (resp. right) action of G on itself.

If a group acts (locally) *effectively on subsets*, then for k large enough a moving frame always exists on a neighborhood of a regular jet (for example, on a neighborhood of a generic curve, see [13, 14] for more details).

The group-based moving frame already appears in a familiar method for calculating the curvature of a curve $u(s)$ in the Euclidean plane. In this method one uses a translation to take $u(s)$ to the origin, and a rotation to make one of the axes tangent to the curve. The curvature can classically be found as the coefficient of the second order term in the Taylor expansion of the curve around $u(s)$. The crucial observation made by Fels and Olver is that *the element of the group* carrying out the translation and rotation depends on u and its derivatives and so it defines a map from the jet space to the group. *This map is a right moving frame*, and it carries all the geometric information of the curve. In fact, Fels and Olver developed a similar normalization process to find right moving frames (see [13, 14] and our next theorem).

THEOREM 2.4 ([13, 14]). — *Let $\cdot$ denote the prolonged action of the group on $u^{(k)}$ and assume we have normalization equations of the form*

$$g \cdot u^{(k)} = c_k$$

where c_k are constants (called normalization constants). Assume we have enough normalization equations to determine g as a function of $u, u_1, \ldots$. Then, the solution $g = \rho$ is a right moving frame.

The direct relation between classical moving frames and group-based moving frames is stated in the following theorem, whose proof can be found in [33].

THEOREM 2.5 ([33]). — *Let $\Phi_g : G/H \to G/H$ be defined by the action of $g \in G$. That is $\Phi_g([x]) = [gx]$. Let ρ be a group-based left moving frame*

with $\rho \cdot o = u$ where $o = [H] \in G/H$ is the base-point. Let e_i, $i = 1, \ldots, n$ be generators of the vector space $T_o G/H$. Then, $T_i = d\Phi_\rho(o)e_i$ form a *classical moving frame.*

This theorem illustrates how classical moving frames are described only by the action of the group-based moving frame on first order frames, while the action on higher order frames is left out. Accordingly, those invariants determined by the action on higher order frames will be not be found with the use of a classical moving frame.

We will next describe the equivalent to the classical Serret-Frenet equations. This concept if fundamental in our Poisson geometry study.

DEFINITION 2.6. — *Consider $K dx$ to be the horizontal component of the pullback of the left (resp. right) Maurer-Cartan form of the group G via a group-based left (resp. right) moving frame ρ. That is*

$$K = \rho^{-1}\rho_x \in \mathfrak{g}(resp.\ K = \rho_x\rho^{-1}).$$

We call K the Maurer-Cartan element of the algebra (or Maurer-Cartan matrix if $G \subset \mathrm{GL}(n, \mathbb{R})$), and $K = \rho^{-1}\rho_x$ the left (resp. right) Serret-Frenet equations for the moving frame ρ.

Notice that if ρ is a left moving frame, then ρ^{-1} is a right moving frame and their Serret-Frenet equations are the negative of each other. A complete set of generating differential invariants can always be found among the coefficients of group-based Serret-Frenet equations generated by normalization equations, a crucial difference with the classical picture. The following Theorem can be found in [22].

THEOREM 2.7. — *Let ρ be a (left or right) moving frame along a curve u, determined through a normalization process. Then, the coefficients of the (left or right) Serret-Frenet equations for ρ contain a basis for the space of differential invariants of the curve. That is, any other differential invariant for the curve is a function of the coordinates of K in some basis of the algebra (its entries if $G \subset \mathrm{GL}(n, \mathbb{R})$) and their derivatives with respect to x.*

There are formulas relating K directly to the invariantization of jet coordinates. They are called recurrence formulas in [13], and the theorem below is the adaptation of the results in [13] to our particular case.

THEOREM 2.8. — *Assume a right invariant moving frame is determined by the normalization equations*

$$\left(g \cdot u^{(r)}\right)^\alpha = c_r^\alpha$$

for some choices of α and r, where $c_r = (c_r^\alpha)$ (α indicates individual coordinates). Let $K = -\rho_x \rho^{-1}$ be the left invariant Serre-Frenet equations of ρ. Let $I_r^\alpha = \rho \cdot u_r^\alpha$ for any $r = 0, 1, 2, \ldots$ and any $\alpha = 1, \ldots, \dim M$. Then K satisfies the equations

(2.1) $$(K \cdot I_r)^\alpha = I_{r+1}^\alpha - (I_r^\alpha)_x,$$

where the dot in $K \cdot I_r$ denotes the prolonged infinitesimal action of the Lie algebra on $J^{(r)}(\mathbb{R}, \mathcal{M})$.

Notice that $I_r^\alpha = c_r^\alpha$ whenever r and α correspond to normalization equations, that is, I_r^α are either constant or differential invariants.

Finally, there is a formula that allow us to write any invariant evolution of curves in G/H in terms of a left group-based moving frame and the differential invariants they generate. When we say an invariant evolution, we mean invariant under the action of G (that is, G takes solutions to solutions). The theorem below is a simple consequence of Theorem 2.5 and results in [34] found in page 249 of that book.

THEOREM 2.9 ([30]). — Let $u(x, t)$ be a one parameter family of curves in G/H; let ρ be a left group-based moving frame and let $d\Phi_\rho(o)e_i = T_i$ be an associated classical moving frame. Then, any evolution of curves in G/H invariant under the action of G can be written as

(2.2) $$u_t = r_1 T_1 + \ldots r_n T_n = d\phi_\rho(o)\mathbf{r}$$

where r_i are differential invariants, that is, functions of the entries of K and their derivatives, and where $\mathbf{r} = (r_i)$.

2.2. Geometric Poisson structures

Assume $\mathfrak{g}$ is semisimple. One can define two natural Poisson brackets on $\mathcal{L}\mathfrak{g}^*$ (see [11] for more information): Let B be an invariant bilinear form that can be used to identify the algebra with its dual. If $\mathcal{H}, \mathcal{F} : \mathcal{L}\mathfrak{g}^* \to \mathbb{R}$ are two functionals defined on $\mathcal{L}\mathfrak{g}^*$ and if $L \in \mathcal{L}\mathfrak{g}^*$, we denote by $\frac{\delta\mathcal{H}}{\delta L}(L)$ and $\frac{\delta\mathcal{F}}{\delta L}(L)$ their variational derivatives at L identified, as usual, with an element of $\mathcal{L}\mathfrak{g}$. We can identify $\frac{\delta\mathcal{H}}{\delta L}(L) \in \mathcal{L}\mathfrak{g}$ with its dual counterpart using B and we can define

(2.3) $$\{\mathcal{H}, \mathcal{F}\}_1(L) = \int_{S^1} \left\langle B\left(\frac{\delta\mathcal{H}}{\delta L}(L)\right)_x + ad^*\left(\frac{\delta\mathcal{H}}{\delta L}(L)\right)(L), \frac{\delta\mathcal{F}}{\delta L}(L)\right\rangle dx$$

where $\langle, \rangle$ is the natural coupling between $\mathfrak{g}^*$ and $\mathfrak{g}$ (usually the trace of the product if we identify $\mathfrak{g}$ and $\mathfrak{g}^*$).

One also has a compatible family of second brackets, namely

$$(2.4) \qquad \{\mathcal{H}, \mathcal{F}\}_2(L) = \int_{S^1} \Big\langle \big(ad^* \big(\tfrac{\delta\mathcal{H}}{\delta L}(L)\big)\big)(L_0), \tfrac{\delta\mathcal{F}}{\delta L}(L)\Big\rangle dx$$

where $L_0 \in \mathfrak{g}^*$ is any constant element. Since $\mathfrak{g}$ is semisimple we can iden-tify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ and we will do so from now on. From now on we will also assume that our curves on homogeneous manifolds have a *group monodromy*, i.e., , there exists $m \in G$ such that

$$u(t+T) = m \cdot u(t)$$

where T is the period. Under these assumptions, the differential invariants will be periodic.

The following theorem is the foundation of the definition of Geometric Hamiltonian structures. It was proved in [33].

THEOREM 2.10. — *Let ρ be a left or right right moving frame along a curve u, determined by normalization equations. Let $\mathcal{K}$ be the manifold of Maurer-Cartan matrices K for nearby curves, generated using the same normalization equations. Then, $\mathcal{K} \cong U/\mathcal{L}H$ where $U \subset \mathcal{L}\mathfrak{g}^*$ is an open set, and where $\mathcal{L}H$ acts on U via a gauge transformation. Furthermore, the Poisson bracket defined on $\mathcal{L}\mathfrak{g}^*$ by (2.3) is reducible to the submanifold $\mathcal{K}$.*

We call this first reduced Poisson bracket a *Geometric Poisson bracket* on G/H.

The reduction of the Poisson bracket can be often found explicitly through algebraic manipulations. Indeed, if an extension $\mathcal{H}$ of h is con-stant on the gauge leaves of $\mathcal{L}H$, then its variational derivative will satisfy

$$(2.5) \qquad \Big(\tfrac{\delta\mathcal{H}}{\delta L}(K)\Big)_x + \Big[K, \tfrac{\delta\mathcal{H}}{\delta L}(K)\Big] \in \mathbf{h}^0$$

where $\mathbf{h}^0 \subset \mathfrak{g}^*$ is the annihilator of $\mathbf{h}$, and where K is any Maurer-Cartan element. This relation is often sufficient to determine $\tfrac{\delta\mathcal{H}}{\delta L}(K)$ completely and with it the reduced Poisson bracket. The reduced Poisson bracket will be defined through the application of (2.3) to two such extensions.

The Poisson bracket (2.4) does not reduce in general to this quotient. When it does, it indicates the existence of an associated completely in-tegrable system, as we will see. The geometric Poisson bracket above is directly related to invariant evolutions through our next theorem. Assume

$$(2.6) \qquad\qquad u_t = W(u, u_1, u_2, \dots)$$

is an evolution of curves invariant under the action of the group. Assume (2.6) induces an evolution of the form

$$(2.7) \qquad\qquad \mathbf{k}_t = Q(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \dots)$$

on a generating system of differential invariants of the flow $u(t, x)$. We say that *(2.6) is a G/H-geometric realization of the flow (2.7)*.

Assume that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and assume $\varsigma : G/H \to G$ to be a section that identifies T_oG/H with $\mathfrak{m}$. Our following theorem finds geometric realizations for any Geometric Hamiltonian flow, Hamiltonian with respect to the reduced Poisson bracket.

THEOREM 2.11 ([33]). — *Assume that $\mathcal{K}$ is described by an affine subspace of $\mathcal{L}\mathfrak{g}^*$. Let $h : \mathcal{K} \to \mathbb{R}$ be a Hamiltonian functional such that, if $\mathcal{H} : \mathcal{L}\mathfrak{g}^* \to \mathbb{R}$ is an extension of h, constant on the leaves of $\mathcal{L}H$ under the gauge action. Let $\frac{\delta \mathcal{H}}{\delta L}(\mathbf{k}) = \frac{\delta \mathcal{H}}{\delta L}(\mathbf{k})_{\mathfrak{m}} + \frac{\delta \mathcal{H}}{\delta L}(\mathbf{k})_{\mathfrak{h}}$ be the components defined by the splitting of the algebra. Then*

$$u_t = d\Phi_\rho(o) d\varsigma(o)^{-1} \frac{\delta \mathcal{H}}{\delta L}(\mathbf{k})_{\mathfrak{m}}$$

is a geometric realization of the reduced Hamiltonian system with Hamiltonian functional h. Notice that this evolution is of the form (2.2) with

$$(2.8) \qquad\qquad \frac{\delta \mathcal{H}}{\delta L}(\mathbf{k})_{\mathfrak{m}} = d\varsigma(o)\mathbf{r}.$$

Equation (2.8) is often referred to as the *compatibility condition*.

To finish this section we will give a brief description of the flat manifold of pure spinors.

2.3. The manifold of pure spinors

The manifold of pure spinors can be represented as the homogeneous space $O(n, n)/P = M$ where $O(n, n)$ is defined as the subgroup of $\mathrm{GL}(2n, \mathbb{R})$ preserving the matrix

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

In a neighborhood of the identity $O(n,n)$ can be described as matrices factoring as $g = g_1 g_0 g_{-1}$ with

(2.9)

$$g_1 = g_1(Z) = \begin{pmatrix} I+Z & -Z \\ Z & I-Z \end{pmatrix}, \quad g_{-1} = g_{-1}(Y) = \begin{pmatrix} I+Y & Y \\ -Y & I-Y \end{pmatrix}$$

$$g_0 = g_0(\Theta) = \tfrac{1}{2} \begin{pmatrix} \Theta + \Theta^{-T} & \Theta^{-T} - \Theta \\ \Theta^{-T} - \Theta & \Theta + \Theta^{-T} \end{pmatrix},$$

where Z and Y are skew symmetric matrices in $GL(n)$, and where $\Theta \in GL(n,\mathbb{R})$. This factorization follows the gradation of the algebra as in [32]. The parabolic subgroup P is given by $P = G_1 \cdot G_0$, where G_i is the subgroup of matrices of the form g_i. $i = 1, 0, -1$. Therefore, we can locally identify $O(n,n)/P$ with skew symmetric matrices (in our notation Y) in $GL(n,\mathbb{R})$, or with G_{-1}. This will be our section $\varsigma : M \to O(n,n)$ and under this section, a curve $u(x)$ in $O(n,n)/P$ can be identified with a curve in G_{-1} of the form

$$g_{-1}(u(x)) = \begin{pmatrix} I+u(x) & u(x) \\ -u(x) & I-u(x) \end{pmatrix}.$$

Since we are working on a homogeneous manifold, the action of $O(n,n)$ on M is determined by the relation $gg_{-1}(u) = g_{-1}(g \cdot u)h$, with $g \in O(n,n)$, $g_{-1}(u)$ and $g_{-1}(g \cdot u) \in G_{-1}$, and $h \in P$. With the factorization given above, this relation can be written as

$$\frac{1}{2} \begin{pmatrix} I+Z & -Z \\ Z & I-Z \end{pmatrix} \begin{pmatrix} \Theta+\Theta^{-T} & \Theta^{-T}-\Theta \\ \Theta^{-T}-\Theta & \Theta+\Theta^{-T} \end{pmatrix} \begin{pmatrix} I+Y & Y \\ -Y & I-Y \end{pmatrix} \begin{pmatrix} I+u & u \\ -u & I-u \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} I+g\cdot u & g\cdot u \\ -g\cdot u & I-g\cdot u \end{pmatrix} \begin{pmatrix} I+\widehat{Z} & -\widehat{Z} \\ \widehat{Z} & I-\widehat{Z} \end{pmatrix} \begin{pmatrix} \widehat{\Theta}+\widehat{\Theta}^{-T} & \widehat{\Theta}^{-T}-\widehat{\Theta} \\ \widehat{\Theta}^{-T}-\widehat{\Theta} & \widehat{\Theta}+\widehat{\Theta}^{-T} \end{pmatrix}$$

for some

$$h = \frac{1}{2} \begin{pmatrix} I+\widehat{Z} & -\widehat{Z} \\ \widehat{Z} & I-\widehat{Z} \end{pmatrix} \begin{pmatrix} \widehat{\Theta}+\widehat{\Theta}^{-T} & \widehat{\Theta}^{-T}-\widehat{\Theta} \\ \widehat{\Theta}^{-T}-\widehat{\Theta} & \widehat{\Theta}+\widehat{\Theta}^{-T} \end{pmatrix} \in P.$$

This relation uniquely determines the action. After some calculations we obtain the formula for the action to be

(2.10) $g \cdot u = \Theta(u+Y)\left(\Theta^{-T} + 4Z\Theta(u+Y)\right)^{-1}.$

The associated Lie algebra gradation is given by $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, with $\mathfrak{g}_i$ being the Lie algebra associated to G_i and $V_i \in \mathfrak{g}_i$, $i = 1, 0, -1$, given by

$$V_1 = V_1(z) = \begin{pmatrix} z & -z \\ z & -z \end{pmatrix}, \quad V_{-1} = V_{-1}(y) = \begin{pmatrix} y & y \\ -y & -y \end{pmatrix}$$

(2.11)

$$V_0 = V_0(C) = V_0(A - B) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

where z, y, A are skew symmetric matrices, and B is a symmetric matrix. Notice that $-B$ and A are the symmetric and skew-symmetric components, respectively, of $C = A - B$. The commutation relations of the algebra are given by

$$[V_{-1}(y), V_1(z)] = 4V_0(yz), \quad [V_1(z), V_0(C)] = V_1(zC + C^T z)$$

$$[V_0(C), V_{-1}(y)] = V_{-1}(Cy + yC^T).$$

3. Moving frame and differential invariants for even spinor curves

3.1. Spinor moving frame

Assume from now on that $n = 2m$. The study of differential invariants for even spinors was carried out in [32] through a process of normalization that can be summarized as follows. Assume $g = g_1(Z)g_0(\Theta)g_{-1}(Y)$.

Zeroth order normalization. The zero order normalization equation is simply

$$g \cdot u = c_0 = 0$$

which is readily solved choosing $Y = -u$.

First order normalization. The first order normalization equations are obtained differentiating the action $g \cdot u$ to obtain $g \cdot u_1$, and restricting to previous normalizations results. The equation is given by

$$g \cdot u^{(1)} = \Theta u_1 \left(\Theta^{-T} + 4Z\Theta(u + Y) \right)^{-1}$$

$$- \Theta(u+Y) \left(\Theta^{-T} + 4Z\Theta(u+Y) \right)^{-1} 4Z\Theta u_1 \left(\Theta^{-T} + 4Z\Theta(u + Y) \right)^{-1}$$

$$= \Theta u_1 \Theta^T = c_1 = J.$$

This equation determines Θ up to an element of the symplectic group $\text{Sp}(2m)$. We write Θ as $\Theta = \theta\mu$ for some $\theta \in \text{Sp}(2m)$ to be determined by later normalizations.

Second order normalizations. We will now skip some lengthy but straightforward calculations. If we differentiate once more the action and substitute the values we have obtained in previous normalizations we obtain the second normalization equation to be

$$(3.1) \qquad g \cdot u^{(2)} = \Theta u_2 \Theta^T - 8JZJ = c_2 = 0.$$

This equation solves for Z in terms of Θ, which we still have to determine completely. That is

$$(3.2) \qquad Z = \frac{1}{8} J \Theta u_2 \Theta^T J.$$

Third order normalizations. These equations are again obtained differentiating a third time and substituting previous values of the frame. It is given by

$$(3.3) \qquad g \cdot u^{(3)} = \theta \mu \left(u_3 - \frac{3}{2} u_2 u_1^{-1} u_2 \right) \mu^T \theta^T = c_3$$

We call

$$(3.4) \qquad S(u) = \mu \left(u_3 - \frac{3}{2} u_2 u_1^{-1} u_2 \right) \mu^T$$

the *skew-symmetric Schwarzian derivative of u*, unique up to the action of an element of the symplectic group. Notice that $\mu u_1 \mu^T = J$ and so μ can be viewed as a *skew-symmetric square root* of u_1. The expression $S(u)$ is the skew-symmetric version of the Lagrangian Schwarzian derivative, first introduced by V. Ovsienko in [35]. The normal form of a skew-symmetric matrix S under this action of $\mathrm{Sp}(2m)$ is

$$\theta S \theta^T = \mathcal{D} = \begin{pmatrix} 0 & \mathbf{D} \\ -\mathbf{D} & 0 \end{pmatrix}$$

where, if $\pm a_k i$ are the eigenvalues of S, then $\mathbf{D}$ is the diagonal matrix having a_k down the diagonal (see [32] for details). We choose $c_3 = \mathcal{D}$ for the choice $S = S(u)$.

The differential invariants of a generic curve in $O(2m, 2m)/H$ have all order three or higher and the entries of the matrix $\mathbf{D}$ generate all differential invariants of third order for u. We call the entries of $\mathbf{D}$ the *differential invariants of projective or Schwarzian Stype* for curves in $O(2m, 2m)/H$. They clearly satisfy property (1.1). For more details, please see [32].

Fourth order normalization equations. The isotropy subgroup of $\mathcal{D}$ is given by elements of $\mathrm{Sp}(2m)$ with a factorization of the form

$$d = \begin{pmatrix} I & \mathbf{D}_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} \mathbf{D}_2 & 0 \\ 0 & \mathbf{D}_2^{-T} \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathbf{D}_3 & I \end{pmatrix}$$

where the three matrices $\mathbf{D}_i$, $i = 1, 2, 3$ are diagonal. They are the part of the moving frame still to be determined. The fourth normalization equations, after simplifications, are given by

(3.5)
$$g \cdot u^{(4)} = d\theta \mu \left(u_4 - 2(u_3 u_1^{-1} u_2 + u_2 u_1^{-1} u_3) + 3 u_2 u_1^{-1} u_2 u_1^{-1} u_2 \right) \mu^T \theta^T d^T = c_4.$$

If $m > 3$, a total of $3m$ normalizations can be performed in (3.5). That means we will have $3m$ fifth order differential invariants appearing in the positions of $\rho \cdot u^{(5)}$ corresponding to the normalized positions chosen in $\rho \cdot u^{(4)}$. This implies the existence of m third order invariants, $2m(m-2)$ fourth order invariants and $3m$ fifth order invariants.

If $m \leqslant 3$ one needs to go higher to normalize entries in the fifth order normalization equations. In those cases we also obtain sixth order invariants corresponding to the normalized fifth order entries in $\rho \cdot u^{(5)}$ as located in $\rho \cdot u^{(6)}$. For $m = 1$ we are in the $\mathbb{RP}^1$ case. For $m = 2$ one can check that we have two differential invariants of projective type, two of fifth order and two of sixth order. For $m = 3$ one has three third order differential invariants, four fourth order ones, seven fifth order and one sixth order. For more details, please see [32].

The next section describes the Maurer-Cartan matrix associated to this moving frame. This matrix was also described in [32], but here we will use a different, more effective, method to find it.

3.2. Serret-Frenet equations
and a generating set of differential invariants

One could, in principle, attempt to find the Maurer-Cartan matrix directly; after all $\rho_x \rho^{-1} = K$ and we have determined ρ above. This is what we did in [32]. Instead we will relate the entries of K to the normalization equations and hence to the normalization constants. This way we obtain directly the matrix K without having to differentiate ρ, while we illustrate the use of the recursion relations (2.1) in the proof of the theorem. Different choices of normalization equations yield different shapes of the Maurer-Cartan matrix. If one wants to do any type of further study of invariants, having Serret-Frenet equations that are as simple as possible is essential (our choice of normalization constants was done with this in mind).

THEOREM 3.1. — *Let u be a generic curve in $O(2m, 2m)/H$. Let ρ be the moving frame determined above. Then, the (left-invariant) Serret-Frenet equations associated to ρ are given by $\rho(\rho^{-1})_x = -\rho_x \rho^{-1} = K$, with*

K equals

$$(3.6) \qquad K = V_{-1}(J) + \tfrac{1}{8}V_1(\mathcal{D}) + V_0(K_0),$$

and where K_0 is of the form

$$(3.7) \qquad K_0 = \begin{pmatrix} R_1 & R_2 \\ R_3 & -R_1^T \end{pmatrix} \in \mathrm{Sp}(2m)$$

with R_2 and R_3 symmetric. The matrix R contains in the entries off the diagonals of R_i, $i = 1, 2, 3$, a generating set of independent fourth order differential invariants and also $3m$ normalized (constant) entries. The diagonals of R_i, $i = 1, 2, 3$ contain a set of $3m$ independent and generating differential invariants of order 5 for $m > 3$ and of order 5 and higher if $m \leqslant 3$.

Proof. — Using Theorem 2.1 together with the normalizations used in the previous section we can completely determine K. Let $V = V_1(z) + V_0(C) + V_{-1}(y)$ be any element of the Lie algebra given as in (2.11). The action (2.10) and its prolongations induce an infinitesimal action of the Lie algebra on u given by

$$V \cdot u = Cu + uC^T - 4uzu + y$$
$$V \cdot u^{(1)} = Cu_1 + u_1 C^T - 4(u_1 zu + uzu_1).$$
$$V \cdot u^{(2)} = Cu_2 + u_2 C^T - 4 \cdot 2u_1 zu_1 + F_2$$
$$V \cdot u^{(3)} = Cu_3 + u_3 C^T - 4 \cdot 3(u_2 zu_1 + u_1 zu_2) + F_3,$$

where F_2 and F_3 are terms that vanish whenever $u = 0$. Assume $K = V_1(K_1) + V_0(K_0) + V_{-1}(K_{-1})$ and recall that $c_0 = 0$, $c_1 = J$, $c_2 = 0$, $c_3 = \mathcal{D}$ and c_4 has fourth order independent invariants off its diagonals, except for $3m$ normalized constant entries.

Then, according to (2.1) for $r = 0$, the matrix K must satisfy the equation

$$(3.8) \qquad K_0 c_0 + c_0 K_0^T - 4c_0 K_1 c_0 + K_{-1} = K_{-1} = c_1 = J.$$

For $r = 1$, K must satisfy

$$(3.9) \qquad K_0 c_1 + c_1 K_0^T = K_0 J + J K_0^T = c_2 = 0.$$

This requires K_0 to be symplectic.

For $r = 2$, K satifies

$$(3.10) \qquad K_0 c_2 + c_2 K_0^T - 8c_1 K_1 c_1 = -8J K_1 J = c_3 = \mathcal{D}.$$

From here we obtain $K_1 = \tfrac{1}{8}\mathcal{D}$.

If $r = 3$, we get

$$(3.11) \quad K_0 c_3 + c_3 K_0^T - 12(c_2 K_1 c_1 + c_1 K_1 c_2) = K_0 \mathcal{D} + \mathcal{D} K_0^T = c_4 - (c_3)_x.$$

If we denote

$$K_0 = \begin{pmatrix} R_1 & R_2 \\ R_3 & -R_1^T \end{pmatrix}$$

with R_2 and R_3 symmetric, then

$$K_0 \mathcal{D} + \mathcal{D} K_0^T = \begin{pmatrix} \mathbf{D} R_2 - R_2 \mathbf{D} & R_1 \mathbf{D} - \mathbf{D} R_1 \\ R_1^T \mathbf{D} - \mathbf{D} R_1^T & R_3 \mathbf{D} - \mathbf{D} R_3 \end{pmatrix} = c_4 - (c_3)_x.$$

This system allows us to solve for all entries in K_0, other than those in the diagonals of R_i. These determined entries are fourth order independent generators and $3m$ normalized entries. Notice that from this equation we can also conclude that the block diagonals of c_4 and $(c_3)_x$ are equal, since the LHS of the equation has vanishing diagonals. For simplicity we are using c_4 and I_4 interchangeably. Since not all entries in I_4 are normalized we should use I_4 instead of c_4. This should create no confusion, these entries are either constant or differential invariants.

We have now found all possible entries other than the diagonals of R_i, $i = 1, 2, 3$. But we know two important facts: there are $3m$ fifth (or higher) order (functionally) independent and generating differential invariants that have not been found yet, and the entries of K generate all possible differential invariants. Hence, we can conclude that the remaining entries in the block diagonals of K_0 are fifth (or higher) order independent generators.

The proof of the Theorem is now concluded. $\qquad\square$

4. De-coupled KdV Hamiltonian structures and the Schwarzian-KdV evolution of even pure spinors

The original Schwarzian-KdV evolution is an evolution for maps $u : \mathbb{R} \to \mathbb{R}$ described by the equation

$$(4.1) \qquad\qquad u_t = u_x S(u)$$

where $S(u) = \frac{u_{xxx}}{u_x} - \frac{3}{2}\left(\frac{u_{xx}}{u_x}\right)^2$ is the Schwarzian derivative of u. This evolution is invariant under the projective group $\mathrm{PSL}(2, \mathbb{R})$ and so it can be written as an evolution of $S(u)$, which is the generating projective differential invariant. If we call $k = S(u)$, then whenever u satisfies (4.1), k satisfies the KdV equation (hence the name)

$$k_t = k_{xxx} + 3kk_x.$$

This evolution was generalized to curves of Lagrangian subspaces in R^{2n} (under the action of the symplectic group) in [29]. In the Lagrangian case, Ovsienko ([35]) defined the Schwarzian derivative of a curve of Lagrangian planes. In [29] we proved that its eigenvalues generate all Lagrangian differential invariants of projective type. We defined the Schwarzian-KdV evolution for Lagrangian planes and we proved that, as non-projective type invariants vanished and if u evolved following the Schwarzian-KdV evolution, the projective-type invariants evolved following a decoupled system of KdV equations. Furthermore, for a particular choice of normalization constants (the equivalent of a particular choice of invariants), as non-projective differential invariants vanished, the Lagrangian geometric Poisson bracket restricts to a decoupled system of KdV Poisson structures. That is, Lagrangian projective differential invariants constitute a Poisson submanifold of $\mathcal{K}$.

In [32] we showed that no local choice of moving frame produces these integrable evolutions in the pure even spinor case (by local we mean depending on u and its derivatives, as are the results of a normalization process). Indeed, we showed that no matter what choice of normalization equations we have, the only constant values of non-projective differential invariants that could be preserved by the flow are zeroes. And as the non-projective invariants vanish, the evolution of the projective invariants blows up. In this section we will show that there exists a non-local choice of moving frame such that the Lagrangian situation can be replicated also in the manifold of pure spinors. This non-local moving frame will effectively remove the higher order invariants from the diagonals of the matrix K_0 and will place invariants in the current normalized entries. That way all entries of K_0, other than the constant diagonals, will be generators.

4.1. Non-local moving frame

In this section we will prove that there exists a non-local element of the group, we will call it g, such that when we gauge the matrix K by g, we obtain a Maurer-Cartan matrix K_n with vanishing diagonals in the R_i blocks. The entries of the Maurer-Cartan matrix will still form a basis for the differential invariants meaning that all other differential invariants are functions of the derivatives of the non-local ones. The fifth order invariants placed in those diagonals will be transformed and moved into the $3m$ normalized entries in R, while the other entries will also be modified. Furthermore, K_n will also form an affine subspace of $\mathfrak{g}$. We remark here that given a random

non-local gauge, there is no guarantee that the new Maurer-Cartan matrix will have generating entries (the result in [22] applies only to moving frames generated by local normalizing sections. Although the result could be also true for a general gauge, no such theorem has been proven yet). Therefore, once we gauge, generating properties need to be re-checked.

LEMMA 4.1. — *There exists an element g, not necessarily in $\mathcal{L}G$, but perhaps with a monodromy, such that*

$$g^{-1}g_x + g^{-1}Kg$$

has $\mathfrak{g}_0$-component with vanishing diagonals.

Proof. — Let Θ_d be a matrix of the form

$$(4.2) \qquad \Theta_d = \begin{pmatrix} I & 0 \\ d_3 & I \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_1^{-1} \end{pmatrix} \begin{pmatrix} I & d_2 \\ 0 & I \end{pmatrix}$$

where d_i are diagonal matrices to be determined. The element g will preserve both $V_{-1}(J)$ and $V_1(\mathcal{D})$, and a straightforward calculation shows that if K is given as in (3.6), and $g = g_0(\Theta_d)$ as in (2.9), then

$$g^{-1}g_x + g^{-1}Kg = V_{-1}(J) + \frac{1}{8}V_1(\mathcal{D}) + V_0(K_n)$$

where $K_n = \Theta_d^{-1}K_0\Theta_d + \Theta_d^{-1}(\Theta_d)_x$. The matrix K_n can be written explicitly as

$$K_n = \begin{pmatrix} Y & d_2Y + d_1^{-1}(d_1d_2)_x + d_1^{-1}\left(d_2(R_1d_1 + d_1d_3R_2) + d^{-1}R_2\right) \\ X & -Y \end{pmatrix}$$

where $X = -d_1^2\left((d_3)_x + R_3 + d_3(R_1 + R_1^T) + d_3^2 R_2\right)$ and $Y = -d_2X + d_1^{-1}(d_1)_x + R_1 + d_3R_2$.

If we want the diagonals of K_n to vanish, we need d_i to satisfy the differential equations

$$(d_1)_x + \left(R_1^d + R_2^d d_3\right) d_1 = 0$$
$$(d_2)_x + R_2^d d_1^{-2} = 0$$
$$(d_3)_x + R_3^d + 2R_1^d d_3 + R_2^d d_3^2 = 0$$

where R_i^d represents the diagonal component of the matrix R_i. This system is clearly (non-locally) solvable since the last equation is a Riccati equation, and once it is solved the other two can be trivially and explicitly integrated. This system can also we written as block-diagonals of the system $(\Theta_d)_x + K_0\Theta_d = 0$.   □

THEOREM 4.2. — *If d_i, $i = 1, 2, 3$, are chosen as solutions of the system above, the Maurer-Cartan matrix $K_n = g^{-1}g_x + g^{-1}Kg$ has vanishing diagonals and its entries generate all differential invariants of even spinor curves. Furthermore, K_n describes an affine subspace of $\mathfrak{o}(2m, 2m)$.*

Proof. — We chose the equations to ensure that the diagonals vanish, but we still need to show that the remaining matrix has generating entries and forms an affine subspace. Recall that

$$\rho^{-1} \cdot u_4 = c_4$$

and recall that c_4 has $3m$ normalizing constants in its entries. This normalizations were achieved through the equation (3.5)

$$dS_4 d^T = c_4$$

where S_4 depends uniquely on derivatives of u, and where d is of the same type as Θ_d (that is, block diagonal). The normalizations in c_4 were chosen to ensure that this equation has full rank on d, that is, to guarantee that d is uniquely determined by the normalizing equations. This also implies that if we use the normalization entries in $d^{-1}c_4 d^{-T} = Jd^T Jc_4 JdJ$, or in $d^T Jc_4 Jd$, one should be able to solve for all non-constant entries of d.

The recursion relation (3.11) can be re-written as

$$[K_0, J\mathcal{D}] = J(c_4 - \mathcal{D}_x).$$

If we conjugate by Θ_d as in (4.2), and using that $\Theta_d^{-1} = -J\Theta_d^T J$, after some short calculations we obtain

$$[\Theta_d^{-1}K_0\Theta_d, J\mathcal{D}]J = \mathcal{D}_x + J\Theta_d^T c_4\Theta_d J = \mathcal{D}_x + \Theta_d^{-1}Jc_4 J\Theta_d^{-T}.$$

Therefore, if the normalized entries in $\Theta_d^T Jc_4 J\Theta_d$ generate all entries of Θ_d, so do the ones located on the other side of the equality. Notice that the invariants in $\mathcal{D}$ are located in the diagonals of the Maurer-Cartan equation, while c_4 is normalized off-diagonals. They do not interfere in the functional generation of the entries of Θ_d. Notice also that we are choosing $d = \Theta_d^{-T}$ to make this argument. From here we conclude that using the normalized entries of $\Theta_d^{-1}K_0\Theta_d$ and the entries of $\mathcal{D}$, we can generate all the invariants that Θ_d generates.

Finally, we can readily check that using the non-normalized entries in the different blocks of $\Theta_d^{-1}K_0\Theta_d$, and using the already generated $\mathcal{D}$ and Θ_d, we can also functionally generate all other invariants. Indeed, if $\Theta_d^{-1}K_0\Theta_d$ and Θ_d are both known, we can certainly generate K_0 off the diagonals. Therefore, all off-diagonal entries will be generators, and since the are the same in number as the previous basis of non-projective invariants, they are

also functionally independent. Together with $\mathcal{D}$ they form a basis for the space of invariants and, therefore K_n generates an affine subspace. $\qquad\square$

Now that we have the proper Serret-Frenet equations, we can prove that the Geometric Poisson structure restricts to the manifold of projective-type differential invariants (defined by $\mathcal{D}$) as $K_0 \to 0$. Notice that we will understand this as a limit statement since $K_0 = 0$ is geometrically not well-defined. Additionally, once we solve for Θ_d, the resulting invariants are not periodic any longer, but they rather have a monodromy element associated to them. Therefore, the remaining results should be understood as algebraic formal statements. This same situation took place in the conformal case ([32]) when natural non-local frames were chosen and some of the invariants were made to vanish.

THEOREM 4.3. — Let $\{,\}_R$ be the Geometric Poisson bracket obtained when we reduce the bracket (2.3) to the affine subspace $\mathcal{K}$ defined by matrices K as in (3.6) with $R = R_n$ defined by the non-local gauge. Then, the bracket restricts to the $\mathcal{K}$-submanifold $K_0 = 0$ to produce a decoupled system of KdV-Hamiltonian structures. Furthermore, for the choice $L_0 = V_1(J)$, the bracket (2.4) reduces also to the $\mathcal{K}$-submanifold $K_0 = 0$ to produce a second compatible Hamiltonian structure for KdV.

Proof. — Since we do know that structure (2.3) reduces to $\mathcal{K}$, and any gauged transformation of it will do so, we simply need to show that the reduction can be restricted to $K_0 = 0$. For that we assume $f, h : \mathcal{K} \to \mathbb{R}$ to be two Hamiltonian functionals with f depending on the entries of $\mathcal{D}$ only and h being independent of $\mathcal{D}$. We will show that their bracket at $K_0 = 0$ vanishes. After this, we will calculate the bracket of two Hamiltonians that depend on the entries of $\mathcal{D}$ only. The result will be the restriction of the Geometric Poisson bracket to $\mathcal{D}$.

Using Theorem 2.10, we can describe how the geometric Poisson bracket is defined. As with any quotient reduction, since $\mathcal{K} \cong U/\mathcal{L}P$ the reduced Poisson bracket is explicitly obtained extending the Hamiltonian functionals f, h to operators on $U \subset \mathcal{L}\mathfrak{g}^*$ that are constant on the leaves of $\mathcal{L}P$. If $\mathcal{H}$ is such an extension of h, infinitesimally the extension property translates into

$$(4.3) \qquad \left(\frac{\delta\mathcal{H}}{\delta L}(L)\right)_x + [K, \frac{\delta\mathcal{H}}{\delta L}(L)] \in \mathfrak{p}^0$$

(similarly with f), where $\mathfrak{p}$ is the parabolic algebra (it corresponds to $y = 0$ in (2.11)) and $\mathfrak{p}^0$ is its annihilator. Here we will use the trace of the product as our invariant bilinear form, and hence we are identifying the dual of the

algebra with the algebra itself (the dual of the entry (i, j) will be the entry (j, i)). Therefore, $\mathfrak{p}^0$ corresponds to a vanishing -1 and 0 component; that is, $y = 0$, $C = 0$ in (2.11). If we split both K and $\frac{\delta \mathcal{H}}{\delta L}(L)$ according to the gradation, $\frac{\delta \mathcal{H}}{\delta L}(L) = V_{-1}(H_{-1}) + V_0(H_0) + V_1(H_1)$, then equation (4.3) becomes

$$(4.4) \qquad\qquad (H_{-1})_x - JH_0^T - H_0 J = 0$$

$$(4.5) \qquad\qquad (H_0)_x - \frac{1}{2}H_{-1}\mathcal{D} - 4JH_1 = 0$$

where we are already assuming $K_0 = 0$ already. Along the proof we will see that restricting to $K_0 = 0$ early on does not alter the result since the rank of our equations is maximal and remains so as K_0 vanishes.

Assume first that f depends only on $\mathcal{D}$. Let $\mathcal{F}$ is any extension constant on the leaves of $\mathcal{LP}$, and assume we decompose F_0 into symplectic and non-symplectic part ($F_0^s = \frac{1}{2}(F_0 + JF_0^T J)$ and $F_0^{ns} = \frac{1}{2}(F_0 - JF_0^T J)$). Then, since off-diagonal elements in K_0 are all of order higher than three invariants and f is independent from K_0, we can conclude that F_0^s is block-diagonal. Furthermore, we can also conclude that $F_{-1} = \tilde{\delta}f + f_{-1}$, where

$$\tilde{\delta}f = \begin{pmatrix} 0 & -\delta f \\ \delta f & 0 \end{pmatrix}$$

with $\delta f = \mathrm{diag}(\frac{\delta f}{\delta k_i})$, while f_{-1} has zero block-diagonals. Let us use (4.4) for F. The first equation can be written as

$$2F_0^{ns}J = (F_{-1})_x.$$

The second equation splits into symplectic and non symplectic components. They are given by

$$(F_0^{ns})_x + 4JF_1 - \frac{1}{4}(F_{-1}\mathcal{D} - J\mathcal{D}F_{-1}J) = 0$$

$$(F_0^s)_x - \frac{1}{4}(F_{-1}\mathcal{D} + J\mathcal{D}F_{-1}J) = 0.$$

Now, since F_0^s is block diagonal, the second equation implies $f_{-1} = 0$ (and as a consequence $F_0^{ns} = -\frac{1}{2}(\tilde{\delta}f)_x J$). The second equation also solves for the block-diagonal F_0^s, namely

$$F_0^s = \frac{1}{4}(\frac{d}{dx})^{-1}\left(\tilde{\delta}f\mathcal{D} - J\tilde{\delta}f\mathcal{D}J\right) = C$$

since all $\mathcal{D}$, $\tilde{\delta}f$ and J commute. The matrix C is constant, and block diagonal. Notice that $C^T\mathcal{D} - \mathcal{D}C = 0$. On the other hand, the first equation

solves for F_1, which is given by

$$F_1 = \frac{1}{8}(\tilde{\delta f})_{xx} - \frac{1}{8}J\tilde{\delta f}\mathcal{D}.$$

Notice that the $+1$ component of the equation (4.3) will now be given by

$$(F_1)_x + \frac{1}{8}(F_0\mathcal{D} + \mathcal{D}F_0^T) = -\frac{1}{8}\left((-\tilde{\delta f})_{xxx} + J(\tilde{\delta f}\mathcal{D})_x + J(\tilde{\delta f})_x\mathcal{D}\right).$$

This matrix is block-diagonal, with $(1,2)$ block given by $\frac{1}{8}\left((\delta f)_{xxx} + (\mathbf{D}\delta f)_x + (\delta f)_x\mathbf{D}\right)$.

Assume next that h does not depend on $\mathcal{D}$. This will imply that H_{-1} has zero diagonal in the $(1,2)$ and $(2,1)$ blocks. On the other hand, the reduced bracket is defined as

$$\{f,h\}_R(\mathbf{k}) = \int_{S^1} \mathrm{tr}\left(\left((F_1)_x + \frac{1}{8}(F_0\mathcal{D} + \mathcal{D}F_0^T)\right) H_{-1}\right) dx.$$

This bracket vanishes since H_{-1} has zero diagonals, while its companion is diagonal.

Finally, if both f and h depend on $\mathcal{D}$ only, then their bracket becomes

$$\{f,h\}_R(\mathbf{k}) = \int \mathrm{tr}\left(\left((F_1)_x + \frac{1}{16}(F_0\mathcal{D} + \mathcal{D}F_0^T)\right) H_{-1}\right) dx$$

$$= -\frac{1}{8}\int_{S^1}\left(-(\tilde{\delta f})_{xxx} + J\left(\tilde{\delta f}\mathcal{D}\right)_x + J(\tilde{\delta f})_x\mathcal{D}\right) \tilde{\delta h}\, dx.$$

whose block-diagonal defines the decoupled system of KdV structures $-\frac{1}{4}\left(\frac{d}{dx^3} + \mathbf{D}\frac{d}{dx} + \frac{d}{dx}\mathbf{D}\right)$ for the entries of $\mathbf{D}$.

The second bracket (2.4) can be seen to reduce directly. If we calculate the bracket of two functionals using the extensions as above, we obtain

$$\{f,h\}_0(\mathbf{k}) = \int \mathrm{tr}\left(\frac{\delta\mathcal{F}}{\delta L}(L)[V_1(J), \frac{\delta\mathcal{H}}{\delta L}(L)]\right) dx = 3\int \mathrm{tr}\left((\delta f)_x \delta h\right) dx.$$

This is a decoupled system of the KdV Poisson structures $\frac{d}{dx}$, companion to our previous Hamiltonian structure. $\qquad\square$

Finally, we will define the Schwarzian-KdV evolution for spinor curves and we will show that, as $K_0 \to 0$, the evolution induced on $\mathcal{D}$ by the curve evolution becomes indeed a decoupled system of KdV equations.

Consider the differential equation

(4.6) $$u_t = u_3 - \frac{3}{2}u_2 u_1^{-1} u_2.$$

We call this equation the *Spinor Schwarzian-KdV evolution.*

THEOREM 4.4. — *Let $u(t, x)$ be a flow solution of the Schwarzian-KdV evolution. Then, the level set $R_n = 0$ is preserved by this evolution. Furthermore, let $\mathbf{D}$ be the diagonal matrix representing the invariants of projective-type for the flow. If $R_n \to 0$, then $\mathbf{D}$ satisfies*

$$\mathbf{D}_t = -\frac{1}{2}\mathbf{D}_{xxx} + 3\mathbf{D}_x\mathbf{D},$$

that is, a decoupled system of KdV equations.

We can rephrase this theorem as stating that (4.6) is a spinor realization of a de-coupled system of KdV equations, as far as we restrict the initial conditions to the submanifold $K_0 = 0$.

Proof. — The first step of this proof is to check that the spinor Schwarzian-KdV equation can be written as

(4.7) $$\tilde{\rho}_{-1}^{-1}(\tilde{\rho}_{-1})_t = Ad(\tilde{\rho}_0)\mathbf{r}$$

where $\mathbf{r} = V_{-1}(\mathcal{D}) \in \mathfrak{g}_{-1}$, and where $\tilde{\rho} = \tilde{\rho}_{-1}\tilde{\rho}_0\tilde{\rho}_1$ is the splitting of the *left* moving frame according to the factorization inverse to (2.9). Indeed, a straightforward conjugation shows us that

$$\tilde{\rho}_{-1}^{-1}(\tilde{\rho}_{-1})_t = (V_{-1}(u))_t = Ad(\tilde{\rho}_0)\mathbf{r} = -V_{-1}(\Theta^{-1}\mathcal{D}\Theta^{-T})$$

where we have used that the left moving frame is the inverse to the right moving frame, and hence $\tilde{\rho}_0 = g_0(\Theta^{-1})$, Θ defining the right moving frame. Notice that Θ is found gauging the one we originally found using normalizations. We are abusing notation here when we denote both local and non-local factors with the same letter. We hope this will not be confusing.

Recall that Θ was chosen to satisfy the first and third order normalization equations

$$\Theta u_1 \Theta^T = J, \quad \Theta\left(u_3 - \frac{3}{2}u_2 u_1^{-1} u_2\right)\Theta^T = \mathcal{D}$$

And since the gauge preserved these components, the same condition is true for the non-local Θ. From here,

$$u_t = \Theta^{-1}\mathcal{D}\Theta^{-T} = u_3 - \frac{3}{2}u_2 u_1^{-1} u_2.$$

Finally, we need to use the compatibility conditions (2.8), proved in [30]. We can conclude that the evolution induced on the Maurer-Cartan matrix by (4.7) is the reduced Hamiltonian evolution with $V_{-1}(\tilde{\delta h}) = \mathbf{r} = V_{-1}(\mathcal{D})$, or $\delta h = \mathbf{D}$.

Finally, the level $K_0 = 0$ is preserved by the Schwarzian-KdV equation because it is preserved by the corresponding Hamiltonian evolution, as we

saw before. When $\mathbf{r} = V_{-1}(\mathcal{D})$ we obtain a decoupled system of KdV equations.

The process relating invariant evolutions to geometric Hamiltonians is general for any invariant evolution. But we can also calculate directly the evolution of the invariants. Although this is not necessary for the proof of this theorem, it will be convenient in the proof of our next theorem and it will clarify our previous comment.

Assume $N = \tilde{\rho}^{-1}\tilde{\rho}_t$, where $\tilde{\rho} = \rho^{-1}$ is the left moving frame (ρ is the right one as previously given), and where $\tilde{\rho}_t$ is induced on $\tilde{\rho}$ by (4.6). Since $K = \tilde{\rho}^{-1}\tilde{\rho}_x$, compatibility conditions (or the horizontal component of the pullback by $\tilde{\rho}$ of the structure equations) are given by

$$K_t = N_x + [K, N].$$

This equation splits according to gradation as

$$V_{-1}(J)_t = 0 = (V_{-1}(N_{-1}))_x + [V_{-1}(J), V_0(N_0)] + [V_0(K_0), V_{-1}(N_{-1})]$$
$$(V_0(K_0))_t = (V_0(N_0))_x + [V_0(K_0), V_0(N_0)] + [V_{-1}(J), V_1(N_1)]$$
$$+ [V_1(K_1), V_{-1}(N_{-1})]$$
$$(V_1(K_1))_t = (V_1(N_1))_x + [V_1(K_1), V_0(N_0)] + [V_0(K_0), V_1(N_1)].$$

As $K_0 \to 0$ (an assumption we make from now on), the first equation become

$$0 = (N_{-1})_x - (N_0 J + J N_0^T),$$

and if we split $N_0 = N_0^s + N_0^{ns}$ into symplectic and non-symplectic parts, then

(4.8) $$N_0^{ns} = -\frac{1}{2}(N_{-1})_x J.$$

The second equation can be written as

(4.9) $$(K_0)_t = (N_0)_x + [K_0, N_0] - \frac{1}{2}N_{-1}\mathcal{D} + 4JN_1.$$

Let us assume that N_{-1} is block diagonal (we will show shortly it indeed is). Using the non-symplectic part of (4.9), we get

$$0 = (N_0^{ns})_x - \frac{1}{2}N_{-1}\mathcal{D} + 4JN_1$$

which solves for N_1

(4.10) $$N_1 = -\frac{1}{8}\left(J(N_{-1})_{xx}J + JN_{-1}\mathcal{D}\right).$$

The third equation can be written as

(4.11) $$\frac{1}{8}\mathcal{D}_t = (N_1)_x + \frac{1}{8}(\mathcal{D}N_0 + N_0^T\mathcal{D}),$$

and from here we can conclude: 1) N_0^s is block diagonal; and 2) $\mathcal{D}$ evolves according to

$$\mathcal{D}_t = (N_{-1})_{xxx} - (N_{-1}\mathcal{D})_x - (N_{-1})_x\mathcal{D}.$$

This results on our decoupled system of KdV whenever $N_{-1} = \mathbf{r} = \mathcal{D}$. Indeed, the author proved in [30] that, if we use our formulation for the curve evolutions, then $N_{-1} = \mathbf{r}$ for any Hermitian symmetric case.

Using this data in the symplectic component of (4.9) we conclude that $(K_0)_t = 0$ and, therefore, $K_0 = 0$ is preserved by the evolution. We also conclude

$$(4.12) \qquad\qquad N_0^s = C$$

where C is constant and block diagonal. The value of C will depend on the initial conditions. $\qquad\square$

Notice that the proof of this theorem shows that, assuming initial conditions are restricted to vanishing non-projective invariants, for any choice of block-diagonal $\mathbf{r}$, the equation $u_t = \Theta^{-1}\mathbf{r}\Theta^{-T}$ is a spinor realization of the Hamiltonian system with Hamiltonian operator $\delta h = \mathbf{r}$. In this sense there is nothing special about the choice $\mathbf{r} = \mathcal{D}$. Notice also that, since both brackets preserve the level set $K_0 = 0$, the other brackets in the corresponding hierarchy will also preserve it. Indeed we obtained an entire hierarchy of Poisson brackets defined by the associated recursion operator, which is a decoupled system of recursion operators for KdV. Each evolution in the hierarchy is obtained from the first Poisson bracket using different Hamiltonians h_k, where these Hamiltonians are generated using the recursion operator. It suffices to choose $\mathbf{r}$ to be δh_k for those Hamiltonians to obtain the Hamiltonian evolution. It is clear that those evolutions all preserve $K_0 = 0$ because δh_k will all be block diagonal. Observe also that choosing $\mathbf{r} = \delta h_k$ means *we are defining spinor geometric realizations for the entire hierarchy of this decoupled system of KdV equations.*

THEOREM 4.5. — *Let Z be given by the right moving frame. Then, as $K_0 \to 0$*

$$(4.13) \qquad\qquad \mathcal{D} = 8(-Z_x + 4ZJZ),$$

and if u evolves according to (4.6), then Z satisfies the following system

$$(4.14) \qquad Z_t = Z_{xxx} - 4^2 \cdot 3(ZJZJZ_x + Z_xJZJZ) + [Z, C]$$

where C is given by (4.12).

If $[Z, C] = 0$ (for example, when $C = 0$), then Z evolves following a *noncommutative and skew-symmetric modified KdV system.* In that case,

we call the transformation (4.13) the *spinor Miura transformation*. Notice that the appearance of C cannot be avoided as $Z = \frac{1}{8} J \Theta u_2 \Theta^T J$ does not need to be block diagonal. C appears once non-local terms are introduced.

Proof. — We can use directly the fact that $K = -\rho_x \rho^{-1}$ and $N = -\rho_t \rho^{-1}$, and the definition of ρ to find the following relations

$$K_0 = -4JZ - \Theta_x \Theta^{-1}$$

$$K_1 = -Z_x - 4ZJZ - Z\Theta_x \Theta^{-1} - (\Theta_x \Theta^{-1})^T Z$$

$$N_{-1} = \mathbf{r}$$

$$N_0 = -4\mathbf{r}Z - \Theta_t \Theta^{-1}$$

$$N_1 = -Z_t - 4Z\mathbf{r}Z - Z\Theta_t \Theta^{-1} - (\Theta_t \Theta^{-1})^T Z.$$

As $K_0 \to 0$ we also have relations (4.8), (4.10) and (4.12). If $K_0 = 0$, we can use the first equation to obtain $\Theta_x \Theta^{-1} = -4JZ$ and from here get an expression for K_1, namely

$$K_1 = -Z_x + 4ZJZ.$$

Since $K_1 = \frac{1}{8} \mathcal{D}$, this gives the transformation (4.13). The evolution for Z can equally be found from the last equation using (4.10). After a short simplification one gets

$$Z_t = -4ZJ\mathbf{r}JZ - \frac{1}{2}(ZJ\mathbf{r}_x + \mathbf{r}_x JZ) - \frac{1}{8}(\mathbf{r}_{xx} - \mathbf{r}J\mathbf{r}) + [Z, C]$$

where we are assuming $\mathbf{r}$ is block diagonal so that it commutes with J. Now, one only needs to choose $\mathbf{r} = \mathcal{D} = 8K_1 = 8(-Z_x + 4ZJZ)$ and substitute it in this equation to obtain the final equation (4.14). □

Notice that our first geometric Poisson bracket, when written in terms of Z, is not equal to the expected bracket defined by the operator $\frac{d}{dx}$. In fact, if $\frac{d}{dx}$ was indeed the operator when written in terms of Z, the operator in terms of the invariants $\mathcal{D}$ would be given by

$$\frac{\delta \mathcal{D}}{\delta Z} \frac{d}{dx} \frac{\delta \mathcal{D}^*}{\delta Z}.$$

Using $\frac{\delta \mathcal{D}}{\delta Z} = 8(-\frac{d}{dx} + 4(ZJ + JZ))$ and after straightforward calculations, one can see that this operator becomes a multiple of

$$-\frac{d^3}{dx^3} + J\mathcal{D}\frac{d}{dx} + J\frac{d}{dx}\mathcal{D} + W[JZ - ZJ]$$

where the function $W[x]$ vanishes only at zero. That is, the Poisson bracket reduces to the expected Poisson bracket for Z only if Z commutes with J.

To understand how these brackets written in terms of invariants relate to the Lax representation of these integrable systems we refer the reader to

[5], where the relation between AKNS representations and moving frames is explained.

BIBLIOGRAPHY

[1] S. C. ANCO, "Bi-Hamiltonian operators, integrable flows of curves using moving frames and geometric map equations", *J. Phys. A* **39** (2006), no. 9, p. 2043-2072.

[2] ————, "Hamiltonian flows of curves in $G/\mathrm{SO}(N)$ and vector soliton equations of mKdV and sine-Gordon type", *SIGMA Symmetry Integrability Geom. Methods Appl.* **2** (2006), p. Paper 044, 17 pp. (electronic).

[3] T. N. BAILEY & M. G. EASTWOOD, "Conformal circles and parametrizations of curves in conformal manifolds", *Proc. Amer. Math. Soc.* **108** (1990), no. 1, p. 215-221.

[4] ————, "Complex paraconformal manifolds—their differential geometry and twistor theory", *Forum Math.* **3** (1991), no. 1, p. 61-103.

[5] A. CALINI, T. IVEY & G. MARÍ-BEFFA, "Remarks on KdV-type flows on star-shaped curves", *Phys. D* **238** (2009), no. 8, p. 788-797.

[6] É. CARTAN, "La Méthode du Repère Mobile, la Théorie des Groupes Continus et les Espaces Généralisés", in *Exposés de Géométrie*, vol. 5, Hermann, Paris, 1935.

[7] ————, "Les espaces à connexion conforme", in *Oeuvres Complètes*, vol. III.1, Gauthier-Villars, Paris, 1955, p. 747-797.

[8] K.-S. CHOU & C. QU, "Integrable equations arising from motions of plane curves", *Phys. D* **162** (2002), no. 1-2, p. 9-33.

[9] ————, "Integrable equations arising from motions of plane curves. II", *J. Nonlinear Sci.* **13** (2003), no. 5, p. 487-517.

[10] V. G. DRINFELD & V. V. SOKOLOV, "Lie algebras and equations of Korteweg-de Vries type", in *Current problems in mathematics, Vol. 24*, Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, p. 81-180.

[11] B. A. DUBROVIN & S. P. NOVIKOV, "Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory", *Uspekhi Mat. Nauk* **44** (1989), no. 6(270), p. 29-98, 203.

[12] M. EASTWOOD & G. MARÍ BEFFA, "Geometric Poisson brackets on Grassmannian and conformal spheres", submitted.

[13] M. FELS & P. J. OLVER, "Moving coframes. I. A practical algorithm", *Acta Appl. Math.* **51** (1998), no. 2, p. 161-213.

[14] ————, "Moving coframes. II. Regularization and theoretical foundations", *Acta Appl. Math.* **55** (1999), no. 2, p. 127-208.

[15] A. FIALKOV, "The Conformal Theory of Curves", *Transactions of the AMS* **51** (1942), p. 435-568.

[16] F. GAY-BALMAZ & T. S. RATIU, "Group actions on chains of Banach manifolds and applications to fluid dynamics", *Ann. Global Anal. Geom.* **31** (2007), no. 3, p. 287-328.

[17] M. L. GREEN, "The moving frame, differential invariants and rigidity theorems for curves in homogeneous spaces", *Duke Math. J.* **45** (1978), no. 4, p. 735-779.

[18] P. GRIFFITHS, "On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry", *Duke Math. J.* **41** (1974), p. 775-814.

[19] R. HASIMOTO, "A soliton on a vortex filament", *J. Fluid Mechanics* **51** (1972), p. 477-485.

[20] N. J. HITCHIN, G. B. SEGAL & R. S. WARD, *Integrable systems*, Oxford Graduate Texts in Mathematics, vol. 4, The Clarendon Press Oxford University Press, New York, 1999, Twistors, loop groups, and Riemann surfaces, Lectures from the Instructional Conference held at the University of Oxford, Oxford, September 1997, x+136 pages.

[21] R. HUANG & D. A. SINGER, "A new flow on starlike curves in $\mathbb{R}^3$", *Proc. Amer. Math. Soc.* **130** (2002), no. 9, p. 2725-2735 (electronic).

[22] E. HUBERT, "Generation properties of differential invariants in the moving frame methods", preprint http://hal.inria.fr/inria-00194528/en.

[23] A. A. KIRILLOV, "Infinite-dimensional groups, their representations, orbits, invariants", in *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)* (Helsinki), Acad. Sci. Fennica, 1980, p. 705-708.

[24] J. LANGER & R. PERLINE, "Poisson geometry of the filament equation", *J. Nonlinear Sci.* **1** (1991), no. 1, p. 71-93.

[25] ———, "Geometric realizations of Fordy-Kulish nonlinear Schrödinger systems", *Pacific J. Math.* **195** (2000), no. 1, p. 157-178.

[26] E. L. MANSFIELD & P. H. VAN DER KAMP, "Evolution of curvature invariants and lifting integrability", *J. Geom. Phys.* **56** (2006), no. 8, p. 1294-1325.

[27] G. MARÍ BEFFA & P. J. OLVER, "Poisson structures for geometric curve flows in semi-simple homogeneous spaces", *Regul. Chaotic Dyn.* **15** (2010), no. 4-5, p. 532-550.

[28] G. MARÍ BEFFA, J. A. SANDERS & J. P. WANG, "Integrable systems in three-dimensional Riemannian geometry", *J. Nonlinear Sci.* **12** (2002), no. 2, p. 143-167.

[29] G. MARÍ BEFFA, "On completely integrable geometric evolutions of curves of Lagrangian planes", *Proc. Roy. Soc. Edinburgh Sect. A* **137** (2007), no. 1, p. 111-131.

[30] ———, "Geometric Poisson brackets in flat semisimple homogenous spaces", *Asian Journal of Mathematics* **12** (2008), no. 1, p. 1-33.

[31] ———, "Geometric realizations of bi-Hamiltonian completely integrable systems", *SIGMA Symmetry Integrability Geom. Methods Appl.* **4** (2008), p. Paper 034, 23.

[32] ———, "Projective-type differential invariants and geometric curve evolutions of KdV-type in flat homogeneous manifolds", *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 4, p. 1295-1335.

[33] ———, "On bi-Hamiltonian flows and their realizations as curves in real homogeneous manifold", *Pacific Journal of Mathematics* **247** (2010), no. 1, p. 163-188.

[34] P. J. OLVER, *Equivalence, invariants, and symmetry*, Cambridge University Press, Cambridge, 1995, xvi+525 pages.

[35] V. OVSIENKO, "Lagrange Schwarzian derivative and symplectic Sturm theory", *Ann. Fac. Sci. Toulouse Math. (6)* **2** (1993), no. 1, p. 73-96.

[36] V. OVSIENKO & S. TABACHNIKOV, *Projective differential geometry old and new*, Cambridge Tracts in Mathematics, vol. 165, Cambridge University Press, Cambridge, 2005, From the Schwarzian derivative to the cohomology of diffeomorphism groups, xii+249 pages.

[37] J. A. SANDERS & J. P. WANG, "Integrable systems in n-dimensional Riemannian geometry", *Mosc. Math. J.* **3** (2003), no. 4, p. 1369-1393.

[38] G. SEGAL, "The geometry of the KdV equation", *Internat. J. Modern Phys. A* **6** (1991), no. 16, p. 2859-2869, Topological methods in quantum field theory (Trieste, 1990).

[39] S. A. SQUIRES & G. M. BEFFA, "Integrable systems associated to curves in flat Galilean and Minkowski spaces", *Appl. Anal.* **89** (2010), no. 4, p. 571-592.

[40] C.-L. TERNG & G. THORBERGSSON, "Completely integrable curve flows on adjoint orbits", *Results Math.* **40** (2001), no. 1-4, p. 286-309, Dedicated to Shiing-Shen Chern on his 90th birthday.

[41] C.-L. Terng & K. Uhlenbeck, "Schrödinger flows on Grassmannians", in *Integrable systems, geometry, and topology*, AMS/IP Stud. Adv. Math., vol. 36, Amer. Math. Soc., Providence, RI, 2006, p. 235-256.

Gloria MARÍ BEFFA
University of Wisconsin
Mathematics department
Madison, Wisconsin 53706 (USA)
maribeff@math.wisc.edu