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THE DIFFEOMORPHISM GROUP OF A LIE FOLIATION

by Gilbert HECTOR, Enrique MACÍAS-VIRGÓS & Antonio SOTELO-ARMESTO (*)

Abstract. — We describe explicitly the group of transverse diffeomorphisms of several types of minimal linear foliations on the torus T^n, $n \geqslant 2$. We show in particular that non-quadratic foliations are rigid, in the sense that their only transverse diffeomorphisms are $\pm$Id and translations. The description derives from a general formula valid for the group of transverse diffeomorphisms of any minimal Lie foliation on a compact manifold. Our results generalize those of P. Donato and P. Iglesias for T^2, P. Iglesias and G. Lachaud for codimension one foliations on T^n, $n \geqslant 2$, and B. Herrera for transcendent foliations. The theoretical setting of the paper is that of J. M. Souriau's diffeological spaces.

Résumé. — Nous décrivons explicitement le groupe des difféomorphismes transverses de plusieurs types de feuilletages linéaires minimaux sur le tore T^n, $n > 2$. En particulier, nous montrons que les feuilletages non quadratiques sont rigides, en ce sens que leurs seuls difféomorphismes sont $\pm$Id. La description découle d'une formule générale valable pour le groupe des difféomorphismes transverses de tout feuilletage de Lie minimal sur une variété compacte. Nos résultats généralisent ceux de P. Donato et P. Iglesias pour T^2, P. Iglesias et G. Lachaud pour les feuilletages de codimension un sur T^n, $n \geqslant 2$, et de B. Herrera pour les feuilletages transcendants. Le cadre théorique de l'article est celui des espaces difféologiques de J. M. Souriau.

1. Introduction

The category of Souriau's diffeological spaces [15] is a fruitful generalization of the category of manifolds, where there are natural constructions for subspaces, quotient spaces and functional spaces. In particular, the factor space G/Γ of the simply connected Lie group G by a totally disconnected

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subgroup Γ can be endowed with a diffeology, and its group of diffeomorphisms $\mathrm{Diff}(G/\Gamma)$ is a diffeological group (the notion corresponding to Lie group). When Γ is dense in G, we show that $\mathrm{Diff}(G/\Gamma)$ is isomorphic to $(\mathrm{Aut}_\Gamma(G) \ltimes G)/\Gamma$; this formula will be our main tool.

Let $\mathfrak{G}$ be the Lie algebra of G. For a $\mathfrak{G}$-Lie foliation [3, 4, 14] with global holonomy group Γ on the compact manifold M, the space of leaves $M/\mathcal{F}$ turns out to be diffeomorphic, as a diffeological space, to G/Γ hence its group of diffeomorphisms $\mathrm{Diff}(M/\mathcal{F})$ (*i.e.,* transverse diffeomorphisms of the foliation) can be described by means of the above formula.

A fundamental class is that of Lie foliations with abelian Lie algebra, in particular (minimal) linear foliations on tori. In this paper we describe explicitly the group $\mathrm{Diff}(M/\mathcal{F})$ for several types of such foliations on the torus T^n, $n \geqslant 2$, namely codimension one foliations, flows, and the so-called non-quadratic foliations. We show in particular that non-quadratic foliations are rigid, in the sense that they do not admit transverse diffeomorphisms other than $\pm\,\mathrm{Id}$ and translations. Our results generalize those of P. Donato and P. Iglesias [2] for T^2, P. Iglesias and G. Lachaud [11] for codimension one foliations on T^n, $n \geqslant 2$, and B. Herrera [8, 9] for transcendent foliations.

2. Diffeological spaces

We briefly review here some basic notations and constructions in the category of diffeological spaces (see also [6, 7, 12]).

Let M be a set. A map $\alpha\colon \mathbb{R}^n \supset U \to M$ defined on an open set U of some $\mathbb{R}^n$ will be called a *plot* of M.

Definition 2.1. — *A diffeology of class $\mathcal{C}^\infty$ on M is a collection $\mathcal{P}$ of plots $\alpha\colon \mathbb{R}^{n_\alpha} \supset U_\alpha \to M$ verifying the following axioms:*

1. *Any constant map $c\colon \mathbb{R}^n \to M$ belongs to $\mathcal{P}$;*
2. *For any plot $\alpha\colon \mathbb{R}^n \supset U \to M$ belonging to $\mathcal{P}$ and any $\mathcal{C}^\infty$ map $h\colon \mathbb{R}^m \supset V \to U$, the composition $\alpha \circ h$ belongs to $\mathcal{P}$;*
3. *Let $\alpha\colon \mathbb{R}^n \supset U \to M$ be a plot of M. If any $t \in U$ has a neighbourhood $U_t \subset U$ such that $\alpha_{|U_t}$ belongs to $\mathcal{P}$ then $\alpha \in \mathcal{P}$.*

Usually, a diffeology is defined by means of a *generating set* $\mathcal{G}$ of plots on M. We denote by $\langle\mathcal{G}\rangle$ the least diffeology on M containing $\mathcal{G}$ and all constant plots. Explicitly, $\langle\mathcal{G}\rangle$ is the collection of plots $\alpha\colon U \to M$ such that any point $t \in U$ has a neighbourhood U_t where α can be written as $\gamma \circ h$ for some $\mathcal{C}^\infty$ map h and some $\gamma \in \mathcal{G}$ [12].

Example 2.2. — An atlas of a finite dimensional manifold M generates the *manifold diffeology* of M.

2.1. Basic constructions

Let $F\colon M \to N$ be a map of sets.

a) For any diffeology $\mathcal{P}$ on M, we define the *final diffeology* $F_\star\mathcal{P}$ on N as the diffeology generated by the set of plots $F \circ \alpha$, $\alpha \in \mathcal{P}$. A particular case is the *quotient diffeology* associated to an equivalence relation on M.

b) If $(M, \mathcal{P})$ and $(N, \mathcal{Q})$ are two diffeological spaces, the *product diffeology* on the product $M \times N$ is the diffeology generated by the set of plots $\alpha \times \beta$, $\alpha \in \mathcal{P}$, $\beta \in \mathcal{Q}$.

c) A map $F\colon (M, \mathcal{P}) \to (N, \mathcal{Q})$ between diffeological spaces is *differentiable* if for any $\alpha \in \mathcal{P}$, the composition $F \circ \alpha \in \mathcal{Q}$. A *diffeomorphism* is a differentiable map with a differentiable inverse.

d) Finally, let $\mathcal{D}(M, N)$ be the space of differentiable maps between $(M, \mathcal{P})$ and $(N, \mathcal{Q})$. The *functional diffeology* on it is generated by the set of all plots $\alpha\colon U \to \mathcal{D}(M, N)$ such that the associated map $\widehat{\alpha}\colon U \times M \to N$ given by $\widehat{\alpha}(t, x) = \alpha(t)(x)$ is differentiable.

Definition 2.3. — *A diffeological group is a diffeological space $(G, \mathcal{P})$ endowed with a group structure such that the division map $\delta\colon G \times G \to G$, with $\delta(x, y) = xy^{-1}$, is differentiable.*

A typical example of diffeological group is the group of diffeomorphisms of a finite dimensional manifold M, endowed with the diffeology induced by $\mathcal{D}(M, M)$ [6]. It was proved in [7] that the group of diffeomorphisms of the leaf space of a "Lie foliation" is a diffeological group too. The aim of the present paper is to describe it precisely in several relevant cases.

3. Diffeological homogeneous spaces

Indeed we will focus on a very special class of diffeological spaces. Let $\Gamma \subset G$ be a totally disconnected subgroup (hence a discrete subgroup in diffeological sense) of a connected Lie group G. The quotient diffeology on G/Γ is the collection of plots $\alpha\colon U \to G/\Gamma$ which lift locally as a smooth map with values in G.

Definition 3.1. — *If Γ is dense in G, we call G/Γ a strongly homogeneous space (s.h.s. for short).*

Note that taking the universal cover $p\colon \widetilde{G} \to G$ of G and the pullback $\widetilde{\Gamma} = p^{-1}(\Gamma)$, it is easy to check that the induced map $\widetilde{G}/\widetilde{\Gamma} \to G/\Gamma$ is a diffeomorphism. Therefore, in what follows we will always assume that the Lie groups at hand are connected and simply connected.

3.1. Lifting maps and diffeomorphisms

Although it is possible to develop a general theory of fibre bundles and covering spaces in the diffeological category [10], we prove here directly, in seek of completeness, some lifting properties for the quotient map $\pi\colon G \to G/\Gamma$ which will be essential for our purposes. To do so, consider two s.h.s. G/Γ and G'/Γ' and the commutative diagram:

$$\begin{array}{ccc} G \supset U & \xrightarrow{\psi} & G' \\ {\scriptstyle\pi}\downarrow & \overset{\widetilde{\varphi}}{\searrow} & \downarrow{\scriptstyle\pi'} \\ G/\Gamma & \xrightarrow{\varphi} & G'/\Gamma' \end{array}$$

where φ is a differentiable map and $\widetilde{\varphi} = \varphi \circ \pi$.

The restriction of $\widetilde{\varphi}$ to a convenient neighborhood U of the identity $e \in G$ is a plot of G'/Γ' which lifts as a map $\psi\colon U \to G'$. Choose a smaller neighborhood W of e such that $W \cdot W \subset U$, then for any $\gamma \in \Gamma \cap W$ and any $g \in W$ we get

$$(\pi' \circ \psi)(g\gamma) = (\varphi \circ \pi)(g\gamma) = (\varphi \circ \pi)(g) = (\pi' \circ \psi)(g).$$

We call ψ a *(local) lift* of φ based at $\psi(e)$.

Given $h \in G'$ such that $\pi'(h) = (\varphi \circ \pi)(e)$, there exists a local lift ψ^h of φ based at h, namely $\psi^h = R_\gamma^{-1} \circ \psi$ for $\gamma = h^{-1} \circ \psi(e) \in \Gamma'$. Moreover two lifts based at the same point h have the same germ.

LEMMA 3.2. — *For any local lift ψ of φ, the map $\theta_\psi = L_{\psi(e)}^{-1} \circ \psi$ is a local group homomorphism.*

Proof. — The smooth map $\widetilde{\psi}_\gamma\colon W \to G'$ defined by $\widetilde{\psi}_\gamma(g) = \psi(g)^{-1}\psi(g\gamma)$ takes its values in Γ', hence it is constant and equal to $\widetilde{\psi}_\gamma(e) = \psi(e)^{-1}\psi(\gamma)$. For any $g \in W$ and $\gamma \in \Gamma \cap W$, we get

$$\psi(g\gamma) = \psi(g) \cdot \psi(e)^{-1} \cdot \psi(\gamma).$$

Thus, as Γ is dense in G, we conclude easily that θ_ψ is a local group homomorphism from G to G'. □

For two lifts ψ_1 and ψ_2, we get $\theta_{\psi_2} = i_\gamma \circ \theta_{\psi_1}$ where $\gamma = \psi_2(e)^{-1}\psi_1(e)$.

We globalize the previous results as follows. If G is simply connected, the local group homomorphism θ_ψ provided by Lemma 3.2 extends as a global homomorphism $\theta\colon G \to G'$, which, because Γ is generated by $\Gamma \cap W$, maps Γ into Γ'. Then a global lift of φ will be given by $\phi = L_{\psi(e)} \circ \theta$.

THEOREM 3.3. — *If G and G' are simply connected, any diffeomorphism $\varphi\colon G/\Gamma \to G'/\Gamma'$ has a global lift $\phi\colon G \to G'$ which is a diffeomorphism. Moreover $\theta_\phi = L^{-1}_{\phi(e)} \circ \phi$ is a Lie group isomorphism.*

As a consequence, we obtain the following simple characterization of diffeomorphic s.h.s. It will be made more precise for the abelian case in Theorem 4.4.

THEOREM 3.4. — *Two strongly homogeneous spaces G/Γ and G'/Γ' are diffeomorphic if and only if $G \cong G'$ and there exists an automorphism θ of G which conjugates Γ and Γ'.*

3.2. The group of diffeomorphisms

Here we apply the previous results to the description of $\mathrm{Diff}(G/\Gamma)$, the group of all diffeomorphisms of a strongly homogeneous space G/Γ. Theorem 3.6 was already announced in [7].

THEOREM 3.5 ([7]). — *For any s.h.s. G/Γ, the group $\mathrm{Diff}(G/\Gamma)$ is a diffeological group when endowed with the functional diffeology $\mathcal{D}(G/\Gamma, G/\Gamma)$.*

Now denote by $\mathrm{Aut}_\Gamma(G)$ the group of automorphisms of the simply connected group G which preserve Γ. We map Γ into the semidirect product $\mathrm{Aut}_\Gamma(G) \ltimes G$ by the map $\gamma \mapsto (i_\gamma, \gamma^{-1})$ where i_γ is the inner automorphism of G defined by γ. This map is injective and its image is an invariant subgroup isomorphic to Γ.

THEOREM 3.6. — *When G is simply connected, the map*

$$\Phi\colon \mathrm{Diff}(G/\Gamma) \to [\mathrm{Aut}_\Gamma(G) \ltimes G]/\Gamma$$

defined by $\Phi(\varphi) = [(\theta_\phi, \phi(e))]$, with ϕ any lift of φ, is a group isomorphism.

Proof. — The semidirect product is given by $(\theta_1, g_1)(\theta_2, g_2) = (\theta_1 \circ \theta_2, g_1 \cdot \theta_1(g_2))$. First note that Φ is well defined. Indeed, if ϕ, ψ are two different lifts of φ, we get

$$(\theta_\psi, \psi(e)) = (i_\gamma, \gamma^{-1})(\theta_\phi, \phi(e)), \quad \gamma = \psi(e)^{-1}\phi(e).$$

Next, $\Phi(L_g \circ \theta) = [(\theta, g)]$ for any $(\theta, g) \in \mathrm{Aut}_\Gamma(G) \times G$, which shows that Φ is onto. It is also injective because if $\Phi(\varphi)$ is the neutral element, then $\varphi = R_\gamma^{-1}$ for some $\gamma \in \Gamma$, and thus φ is the identity of G/Γ. It remains to show that Φ is a group morphism, which is immediate from the formula $\theta_{\phi\circ\psi} = \theta_\phi \circ \theta_\psi$. □

3.3. Lie foliations

Lie foliations, which play a central role in the study of Riemannian foliations [14], appear here as natural "desingularizations" of the spaces G/Γ.

Let $\mathfrak{G}$ be a finite dimensional Lie algebra. A $\mathfrak{G}$-*Lie foliation* on the compact manifold M is defined by a non-degenerate 1-form ω with values in $\mathfrak{G}$ verifying the Maurer-Cartan equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$. Once a basis of $\mathfrak{G}$ with structural constants c_{ij}^k has been fixed, this is equivalent to having n independent real 1-forms $\omega_1, \ldots, \omega_n$ on M such that $d\omega_k = \sum c_{ij}^k \omega_i \wedge \omega_j$.

Let G be the connected and simply connected Lie group corresponding to $\mathfrak{G}$. It is well known [3, 13] that there exist a group morphism $h\colon \pi_1(M) \to G$ defining a regular covering $p\colon \widetilde{M} \to M$ and a commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{D} & G \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\bar{D}} & G/\Gamma \end{array}$$

where the *developing map* D is h-equivariant and defines a locally trivial fibration which coincides with the lifted foliation $p^\star\mathcal{F}$.

The image Γ of h (the *holonomy group*) is a finitely generated group, which is dense in G if and only if $\mathcal{F}$ is minimal *i.e.,* all leaves are dense in M. All arrows in this square are differentiable (in the diffeological sense) and one deduces easily that the leaf space $M/\mathcal{F}$, endowed with the quotient diffeology, is diffeomorphic to the factor space G/Γ. In particular, $\mathrm{Diff}(M/\mathcal{F}) \cong \mathrm{Diff}(G/\Gamma)$. This group will be sometimes called the group of *transverse diffeomorphisms* of $\mathcal{F}$.

4. Abelian homogeneous spaces and Linear foliations

Here we specialize to the description of strongly homogeneous spaces G/Γ when G is an *abelian* simply connected Lie group $\mathbb{R}^n$.

A Lie foliation $\mathcal{F}$ of codimension n with abelian Lie algebra on a compact manifold M will be defined by a $\mathbb{R}^n$-valued closed 1-form $\omega = (\omega_1, \ldots, \omega_n)$. The holonomy group $\Gamma \subset \mathbb{R}^n$ is a free abelian group of rank k and the leaf space of $\mathcal{F}$ is diffeomorphic to the abelian group $\mathbb{R}^n/\Gamma$.

A particular class of such foliations is the family of *linear foliations* on tori. A linear foliation $\mathcal{F}$ of codimension $n \geqslant 1$ on $\mathbb{T}^{n+m}$ can also be determined by the natural action of some linear subgroup $V \subset \mathbb{R}^{n+m}$ of dimension m acting on the left on $\mathbb{T}^{n+m}$. The developing map $D\colon \widetilde{M} = \mathbb{R}^{n+m} \to G = \mathbb{R}^n$ is the natural projection onto the orthogonal subspace $V^\perp$ to V in $\mathbb{R}^{n+m}$ and the form ω is determined by the choice of some basis of $V^\perp$. We have $n+1 \leqslant k \leqslant n+m$.

4.1. Classifying foliations– Duality

Indeed we restrict to a special class of foliations.

DEFINITION 4.1. — *A linear foliation $\mathcal{F}$ on $\mathbb{T}^{n+m}$ will be called classifying if the group V acts freely on $\mathbb{T}^{n+m}$ or equivalently if all leaves of $\mathcal{F}$ are simply connected.*

Having contractible leaves, these foliations are the "classifying foliations" of abelian Lie foliations, in the sense of Haefliger's theory [5].

Now it is easy to see that $\mathcal{F}$ is classifying exactly when $k = n+m$ which is equivalent to $V \cap \mathbb{Z}^{n+m} = \{0\} = V^\perp \cap \mathbb{Z}^{n+m}$. These conditions mean that the two foliations $\mathcal{F}$ and its orthogonal $\mathcal{F}^\perp$, defined by the left action of $V^\perp$, are simultaneously classifying and minimal and that their holonomy groups Γ and $\Gamma^\perp$ have the same rank k. We will say that $(\mathcal{F}, \mathcal{F}^\perp)$ is a *dual pair* of classifying foliations on $\mathbb{T}^{n+m}$ and that the couple $(\mathbb{R}^n/\Gamma, \mathbb{R}^m/\Gamma^\perp)$ is a *dual pair* of abelian strongly homogeneous spaces.

PROPOSITION 4.2. — *Any abelian strongly homogeneous space $\mathbb{R}^n/\Gamma$ is the leaf space of a unique classifying foliation $\mathcal{F}$ on the torus $\mathbb{T}^k$, $k = \operatorname{rank}\Gamma$.*

Proof. — An isomorphism $D_0\colon \mathbb{Z}^k \to \Gamma$ extends linearly as a surjective group homomorphism $D\colon \mathbb{R}^k \to \mathbb{R}^n$ whose kernel V is a subgroup of dimension $m = k-n$. The action of V on the torus $\mathbb{T}^k$ will be free and defines the wanted classifying foliation $\mathcal{F}$. □

4.2. Classification

Let $\mathcal{F}$ be a linear foliation of codimension n on the torus $\mathbb{T}^k$, defined by the action of the linear group $V \subset \mathbb{R}^k$. Let $\mathrm{Gl}_V(k, \mathbb{Z})$ be the subgroup

of $\mathrm{Gl}(k,\mathbb{Z})$ which preserves the linear subgroup $V \subset \mathbb{R}^k$. An element A of $\mathrm{Gl}_V(k,\mathbb{Z})$ induces an automorphism of $\mathbb{R}^n$ which preserves Γ hence we get a morphism $\rho\colon \mathrm{Gl}_V(k,\mathbb{Z}) \to \mathrm{Aut}_\Gamma(\mathbb{R}^n)$.

PROPOSITION 4.3. — *ρ is an isomorphism*

$$\mathrm{Aut}_\Gamma(\mathbb{R}^n) \cong \mathrm{Gl}_V(k,\mathbb{Z}), \quad k = \mathrm{rank}\,\Gamma.$$

Proof. — Indeed ρ is onto because any automorphism in $\mathrm{Aut}_\Gamma(\mathbb{R}^n)$ restricts to an isomorphism of Γ which extends to an isomorphism of $\mathbb{R}^k = \Gamma\otimes\mathbb{R}$, the latter isomorphism belonging to $\mathrm{Gl}_V(k,\mathbb{Z})$. It is injective because the identity of $\mathbb{R}^n$ is the unique isomorphic lift of the identity of $\mathbb{R}^n/\Gamma$. □

THEOREM 4.4. — *Two abelian s.h.s. $\mathbb{R}^n/\Gamma$ and $\mathbb{R}^p/\Gamma'$, $\mathrm{rank}\,\Gamma = k$, $\mathrm{rank}\,\Gamma' = l$, are diffeomorphic if and only if the corresponding foliations $\mathcal{F}$ and $\mathcal{F}'$ are diffeomorphic or equivalently $(n,k) = (p,l)$ and there exists $A \in \mathrm{Gl}_V(k,\mathbb{Z})$ which conjugates Γ and Γ'.*

Proof. — According to Corollary 3.3, we can assume that a diffeomorphism $\varphi\colon \mathbb{R}^n/\Gamma \to \mathbb{R}^p/\Gamma'$ is a group isomorphism and lifts as a group isomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^p$ sending Γ onto Γ' and producing a commutative diagram of group isomorphisms:

$$\begin{array}{ccccccc}
V & \longrightarrow & \widehat{\Gamma} = \mathbb{R}^k & \xrightarrow{D} & \mathbb{R}^n & \xrightarrow{\pi} & \mathbb{R}^n/\Gamma \\
\downarrow & & \downarrow{\scriptstyle \widehat{\phi}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \varphi} \\
V' & \longrightarrow & \widehat{\Gamma}' = \mathbb{R}^l & \xrightarrow{D'} & \mathbb{R}^p & \xrightarrow{\pi'} & \mathbb{R}^p/\Gamma'
\end{array}$$

where $\widehat{\Gamma}$ is the linear group generated by $\Gamma \cong \mathbb{Z}^k$ and $\widehat{\phi}$ is the natural extension of the restriction of ϕ to Γ. It implies immediately that $(n,k) = (p,l)$ and the linear map $\widehat{\phi}\colon \mathbb{R}^k \to \mathbb{R}^l$ conjugating the kernels of D and D' conjugates the foliations as well as the corresponding linear subgroups. □

PROPOSITION 4.5. — *For any dual pair of abelian classifying s.h.s. $\mathbb{R}^n/\Gamma$ and $\mathbb{R}^m/\Gamma^\perp$, $k = n+m$, we have a sequence of isomorphisms:*

$$\mathrm{Aut}_\Gamma(\mathbb{R}^n) \cong \mathrm{Gl}_V(k,\mathbb{Z}) \overset{\tau}{\cong} \mathrm{Gl}_{V^\perp}(k,\mathbb{Z}) \cong \mathrm{Aut}_{\Gamma^\perp}(\mathbb{R}^m),$$

where τ is defined by transposition.

PROPOSITION 4.6. — *For an abelian s.h.s. $\mathbb{R}^n/\Gamma$, we get*

$$\mathrm{Diff}(\mathbb{R}^n/\Gamma) \cong \mathrm{Aut}_\Gamma(\mathbb{R}^n) \ltimes (\mathbb{R}^n/\Gamma).$$

5. Two fundamental particular cases

In the remainder of the paper we study some significant families of dual pairs. So let $(\mathcal{F}, \mathcal{F}^\perp)$ be a dual pair of linear classifying foliations on the torus $\mathbb{T}^k$, of complementary codimensions n and m, $k = n + m$.

5.1. Equations and forms

We fix a natural system of coordinates $(y^1, \dots, y^n, x^1, \dots, x^m)$ on $\mathbb{R}^{n+m}$. Consider $B = (\beta^i_j)$ a $n \times m$ matrix with real coefficients with n lines and m columns. If $X = (x^1, \dots, x^m)^T \in \mathbb{R}^m$, $Y = (y^1, \dots, y^n)^T \in \mathbb{R}^n$, then equation $Y + BX = 0$ defines an m-dimensional linear subspace V of $\mathbb{R}^{n+m}$ which generates a linear foliation $\mathcal{F}$ of the torus $\mathbb{T}^{n+m}$, defined by the linear $\mathbb{R}^n$-valued form $\omega = dY + BdX$. We call them the *reduced equations* of V and $\mathcal{F}$ respectively.

Conversely, any linear foliation can be described in this way.

Remark 5.1. — The orthogonal space $V^\perp$ and foliation $\mathcal{F}^\perp$ will be defined respectively by the reduced equation $X - B^T Y = 0$ and the $\mathbb{R}^m$-valued form $\Omega = dX - B^T dY$, where B^T is the transpose of B.

We get classification results as follows:

THEOREM 5.2. — *Two foliations $\mathcal{F}$ an $\mathcal{F}'$ of the same codimension n (and dimension m) corresponding to matrices B and B' respectively will be diffeomorphic if and only if there exists a matrix*

$$A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathrm{Gl}(n+m, \mathbb{Z}), \tag{5.1}$$

with $P \in \mathcal{M}_{n\times n}(\mathbb{Z})$ and $S \in \mathcal{M}_{m\times m}(\mathbb{Z})$, such that

1. *the $n \times n$-matrix $P + BR$ is invertible;*
2. *$(P + BR)B' = Q + BS$.*

The same conditions insure that the corresponding leaf spaces are diffeomorphic.

Proof. — We know that $\mathcal{F}$ and $\mathcal{F}'$ are diffeomorphic if and only if there exists a matrix $A \in \mathrm{Gl}(n + m, \mathbb{Z})$ which conjugates the equations of the corresponding subspaces V and V'.

Now let $Y + BX = 0$ be a reduced equation of V and consider the $n \times m$ matrix

$$\begin{pmatrix} I_n & B \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} P + BR & Q + BS \end{pmatrix}.$$

Then $(P+BR)Y+(Q+BS)X=0$ is an equation of V' equivalent to a reduced form $Y+B'X=0$ implying the two desired conditions. The converse is straightforward. □

As an immediate corollary, we obtain a corresponding result for diffeomorphisms of $\mathcal{F}$:

COROLLARY 5.3. — *A matrix $A\in \mathrm{Gl}(n+m,\mathbb{Z})$ as in (5.1) defines a diffeomorphism of the classifying foliation $\mathcal{F}$ of codimension n corresponding to the matrix B if and only if*

(1) *$P+BR$ is invertible;*
(2) *$(P+BR)B=Q+BS$.*

5.2. Linear flows and codimension one foliations on T^{n+1}

For a first concrete application, we consider a dual pair $(\mathcal{F},\mathcal{F}^{\perp})$ of linear classifying foliations on $\mathbb{T}^{n+1}$, where $\mathcal{F}$ is of dimension 1 and $\mathcal{F}^{\perp}$ of codimension 1. We make the following observations:

1) $\mathcal{F}$ is a flow defined by the $\mathbb{R}^n$-valued reduced closed form

$$\Omega=(dx^1-\beta_1 dy, dx^2-\beta_2 dy,\dots,dx^n-\beta_n dy);$$

2) $\mathcal{F}^{\perp}$ is a foliation by n-planes defined by a reduced closed 1-form

$$\omega=dy+\beta_1 dx^1+\cdots+\beta_n dx^n,$$

which is completely determined by Ω. It is a classifying foliation if the $(n+1)$-uple $(1,\beta_1,\dots,\beta_n)$ has rank $n+1$ over $\mathbb{Q}$.

The corresponding leaf spaces will be denoted by $\mathbb{R}/\Gamma_\omega$ and $\mathbb{R}^n/\Gamma_\Omega$ respectively. From the previous discussion in 5.1 we deduce:

PROPOSITION 5.4. — *Two such spaces $\mathbb{R}/\Gamma_\omega$ and $\mathbb{R}/\Gamma_{\omega'}$ are diffeomorphic if and only if there exists an integer matrix*

$$A=\begin{pmatrix} p & q_1 & \cdots & q_n \\ r^1 & s_1^1 & \cdots & s_n^1 \\ \vdots & \vdots & & \vdots \\ r^n & s_1^n & \cdots & s_n^n \end{pmatrix}\in GL(n+1,\mathbb{Z})$$

such that the associated reduced matrices B,B' verify the relations:

$$(p+\sum_{u=1}^{n}\beta_u r^u)\beta'_j=q_j+\sum_{u=1}^{n}\beta_u s_j^u,\quad j\in\{1,\dots,n\}.$$

Moreover in this case the two spaces $\mathbb{R}^n/\Gamma_\Omega$ and $\mathbb{R}^n/\Gamma_{\Omega'}$ are simultaneously diffeomorphic.

According to Proposition 4.5, the automorphism groups of a dual pair of s.h.s. are isomorphic. Thus using original work of P. Iglesias and G. Lachaud [11] about codimension 1 foliations, we get

THEOREM 5.5. — *For any dual pair $\mathbb{R}/\Gamma_\omega$ and $\mathbb{R}^n/\Gamma_\Omega$ there exists an integer $r \leqslant n$ such that*

$$\mathrm{Aut}_{\Gamma_\omega}(\mathbb{R}) \cong \mathrm{Aut}_{\Gamma_\Omega}(\mathbb{R}^n) \cong \mathbb{Z}_2 \times \mathbb{Z}^r.$$

Moreover

(1) $\mathrm{Diff}(\mathbb{R}/\Gamma_\omega) \cong \mathbb{Z}_2 \times \mathbb{Z}^r \ltimes (\mathbb{R}/\Gamma_\omega)$;
(2) $\mathrm{Diff}(\mathbb{R}^n/\Gamma_\Omega) \cong \mathbb{Z}_2 \times \mathbb{Z}^r \ltimes (\mathbb{R}^n/\Gamma_\Omega)$.

The integer r may take different values depending on the algebraic nature of the form ω, as described in [11]. For example, in case $n = 1$ we get $\omega = dy + \beta dx$ and, as P. Donato and P. Iglesias proved in [2], $\mathrm{Aut}_{\Gamma_\omega}(\mathbb{R})$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$ if β is a quadratic number and $\mathbb{Z}_2$ otherwise.

Let us recall that linear flows on tori have particular importance because of the following result of P. Caron and Y. Carrière.

THEOREM 5.6 ([1]). — *Let M be a compact manifold of dimension $n+1$, endowed with a dense G-Lie flow. Then $G = \mathbb{R}^n$, M is diffeomorphic to the torus T^{n+1}, and the given foliation is smoothly conjugate to a linear one.*

5.3. Non-quadratic and transcendent and foliations

Finally, we consider two special classes of classifying foliations and spaces.

DEFINITION 5.7. — *We will say that a matrix $B = (\beta_j^i)$ is non-quadratic [resp. transcendent] if the degree of any polynomial with rational coefficients and nm variables which annihilates the family of coefficients $\{\beta_j^i\}$ is strictly greater than 2 [resp. the coefficients $\{\beta_j^i\}$ are algebraically independent over $\mathbb{Q}$].*

According to Theorem 5.2, a linear foliation $\mathcal{F}$ will be called *non-quadratic* [resp. *transcendent*] if the corresponding matrix is non-quadratic [resp. transcendent].

Moreover a matrix B is non-quadratic [resp. transcendent] if and only if its transpose B^T is so. Thus, by Remark 5.1, dual foliations (or dual abelian s.h.s.) are simultaneously non-quadratic [resp. transcendent].

Example 5.8. — By definition, a transcendent matrix (or linear foliation) is also non-quadratic. The foliation by planes on $\mathbb{T}^3$ defined by the 1-form $\omega = dy + \sqrt[3]{2}dx_1 + \sqrt[3]{3}dx_2$ is non-quadratic and non-trascendent.

Transcendent foliations appeared first in B. Herrera's thesis [9]; our definition is easily seen to be equivalent to his. For non-quadratic foliations, we obtain the following rigidity result which generalizes [2] and recovers the main result of Herrera:

THEOREM 5.9. — *Let $(\mathcal{F}, \mathcal{F}^\perp)$ be a dual pair of non-quadratic classifying foliations on $\mathbb{T}^{n+m}$. Then*

$$\mathrm{Aut}_\Gamma(\mathbb{R}^n) \cong \mathbb{Z}_2 \cong \mathrm{Aut}_{\Gamma^\perp}(\mathbb{R}^m).$$

Furthermore, $\mathrm{Diff}(R^n/\Gamma) = \mathbb{Z}_2 \ltimes (\mathbb{R}^n/\Gamma)$ and $\mathrm{Diff}(R^m/\Gamma^\perp) = \mathbb{Z}_2 \ltimes (\mathbb{R}^m/\Gamma^\perp)$.

Proof. — According to Proposition 4.5, it will suffice to show that for a non-quadratic foliation $\mathcal{F}$ defined by the action of $V \subset \mathbb{R}^{n+m}$, the group $\mathrm{Gl}_V(n+m, \mathbb{Z})$ reduces to $\pm\mathrm{Id}$. Using Theorem 5.3, we see that a matrix $A \in \mathrm{Gl}(n+m, \mathbb{Z})$ as in (5.1) preserves the subgroup V if and only if

$$C = (P + BR)B - (Q + BS) = BRB + (PB - BS) - Q = 0. \tag{5.2}$$

The coefficients of C are polynomials with integer coefficients in the variables β^i_j, the coefficients of B. Their degree is less than or equal to 2 thus by definition of non-quadratic foliations all these polynomials have to be trivial. We make two observations:

a) the terms of degree 2 come exclusively from the matrix BRB and thus the vanishing condition implies that $R = 0$. Similarly the vanishing of the independent terms implies $Q = 0$. So our condition 5.2 reduces to $PB = BS$;

b) then by equaling the corresponding coefficients of PB and BS we obtain the following nm equations (with notations similar to those used in 5.4):

$$\sum_{u=1}^{n} p^j_u \beta^j_u = \sum_{v=1}^{m} \beta^v_j s^j_v,$$

where the only common variable of both sides is β^i_j. Thus for fixed (j, i) the vanishing condition implies $p^u_j = 0$ for $u \neq j$ and $s^i_v = 0$ for $v \neq i$, while $p^j_j = s^i_i$.

The last condition being valid for any (j, i), we conclude that both P and S and therefore also A are diagonal matrices. But having integer coefficients and being invertible, it follows that $A = \pm\mathrm{Id}$; the proof is complete. □

COROLLARY 5.10 ([9]). — *The only transverse automorphisms of a transcendent foliation are* $\pm \mathrm{Id}$.

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