ANNALES

DE

L'INSTITUT FOURIER

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Tome 59, nº 3 (2009), p. 1177-1229.

<http://aif.cedram.org/item?id=AIF_2009__59_3_1177_0>

Ann. Inst. Fourier, Grenoble
59, 3 (2009) 1177-1229

DECOMPOSITION NUMBERS FOR PERVERSE SHEAVES

by Daniel JUTEAU

Abstract. — The purpose of this article is to set foundations for decomposition numbers of perverse sheaves, to give some methods to calculate them in simple cases, and to compute them concretely in two situations: for a simple (Kleinian) surface singularity, and for the closure of the minimal non-trivial nilpotent orbit in a simple Lie algebra.

This work has applications to modular representation theory, for Weyl groups using the nilpotent cone of the corresponding semisimple Lie algebra, and for reductive algebraic group schemes using the affine Grassmannian of the Langlands dual group.

Résumé. — Le but de cet article est de poser les fondations pour les nombres de décomposition des faisceaux pervers, de donner quelques méthodes pour les calculer dans des cas simples et de les déterminer explicitement dans deux situations : pour une singularité simple (kleinienne) de surface et pour l'adhérence de l'orbite nilpotente non-triviale minimale dans une algèbre de Lie simple.

Ce travail a des applications dans la théorie des représentations modulaires, pour les groupes de Weyl en utilisant le cône nilpotent de l'algèbre de Lie semi-simple correspondante, et pour les schémas en groupes réductifs en utilisant la grassmannienne affine du dual de Langlands.

1. Introduction

The purpose of this article is to set foundations for decomposition numbers of perverse sheaves, to give some methods to calculate them in simple cases, and to compute them concretely for simple and minimal singularities.

We consider varieties over $\overline{\mathbb{F}}_p$, and perverse sheaves with coefficients in $\mathbb{E}$, where $\mathbb{E}$ is one of the rings in an ℓ-modular system $(\mathbb{K}, \mathbb{O}, \mathbb{F})$, where ℓ is a prime different from p. These notions are explained in Subsection 2.1.

Keywords: Perverse sheaves, intersection cohomology, integral cohomology, t-structures, torsion theories, decomposition matrices, simple singularities, minimal nilpotent orbits.
Math. classification: 55N33, 20C20.

Modular systems were introduced in modular representation theory of finite groups. The idea is that we use a ring of integers $\mathbb{O}$ to go from a field $\mathbb{K}$ of characteristic zero to a field $\mathbb{F}$ of characteristic ℓ. For a finite group W, we define the decomposition numbers d^W_{EF}, for $E \in \operatorname{Irr} \mathbb{K}W$ and $F \in \operatorname{Irr} \mathbb{F}W$, by $d^W_{EF} = [\mathbb{F} \otimes_{\mathbb{O}} E_{\mathbb{O}} : F]$, where $\mathbb{E}_{\mathbb{O}}$ is a W-stable $\mathbb{O}$-lattice in E (this multiplicity is well-defined). In many cases (for example, for the symmetric group), the ordinary irreducibles (over $\mathbb{K}$) are known, but the modular ones (over $\mathbb{F}$) are not. Then the problem of determining the modular characters is equivalent to the problem of determining the decomposition matrix $D^W = (d^W_{EF})$.

We can do the same for perverse sheaves on some variety X: we can define decomposition numbers $d^X_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}$, where $(\mathcal{O},\mathcal{L})$ and $(\mathcal{O}',\mathcal{L}')$ are pairs consisting of smooth irreducible locally closed subvariety and an irreducible $\mathbb{E}$-local system on it, for $\mathbb{E} = \mathbb{K}$ or $\mathbb{F}$ respectively. The simple perverse sheaves are indexed by such pairs (if we fix a stratification, we take strata for $\mathcal{O}$ and $\mathcal{O}'$). They are intersection cohomology complexes. As in modular representation theory, one can take an integral form and apply the functor of modular reduction $\mathbb{F} \otimes^L_{\mathbb{O}} -$.

In [12], it has been shown that the decomposition matrix of a Weyl group can be extracted from a decomposition matrix for equivariant perverse sheaves on the nilpotent cone. This required to define a modular Springer correspondence, using a Fourier-Deligne transform (I will explain this in a forthcoming article).

Thus it is very desirable to be able to calculate decomposition numbers for equivariant perverse sheaves on the nilpotent cone. The singularity of the nilpotent cone along the subregular orbit is a simple surface singularity [5, 20, 21]. At the other extreme, one can look at the singularity of the closure of a minimal non-trivial nilpotent orbit at the origin. These two cases are treated here.

On the other hand, by the results of [19], the decomposition numbers for a reductive algebraic group scheme can be interpreted as decomposition numbers for equivariant perverse sheaves on the affine Grassmannian of the Langlands dual group. Moreover, most of the minimal degenerations of this (infinite-dimensional) variety are simple or minimal singularities [18], so the calculations that we carry out in this article can be used to recover some decomposition numbers for reductive algebraic group schemes geometrically. This will be done in another article, where we will also explain that one can go in the other direction and prove geometric results using known decomposition numbers.

In the author's opinion, perverse sheaves over rings of integers and in positive characteristic, and their decomposition numbers, will prove to be useful in many ways. For simple and minimal singularities, we already have two different applications to modular representation theory. So it seemed desirable to show how to calculate these decomposition numbers independently of the framework of Springer correspondence.

Now let us give an outline of the article. Section 2 contains the technical preliminaries. First, we set the context and recall the definition of perverse sheaves over $\mathbb{K}$, $\mathbb{O}$, $\mathbb{F}$. The treatment of $\mathbb{O}$-coefficients in the standard reference [1] is done in two pages (§ 3.3). Over a field, the middle perversity p is self-dual, but here one has to consider two perversities, p and p_+, exchanged by the duality. The cause of the trouble is torsion. It seemed worthwhile to explain this construction in a more general context. Given an abelian category with a torsion theory, there is a known procedure to construct another abelian category lying inside the derived category [9]. Our point of view is slightly different: we start with a t-category, and we assume that its heart is endowed with a torsion theory. Then we can construct a new t-structure on the same triangulated category. After recalling the notion of t-structure, we study the interaction between torsion theories and t-structures. Then, we recall the notion of recollement and its properties (most can be found in [1]), and we see how it interacts with torsion theories. Then we see why the t-structure defining perverse sheaves is indeed a t-structure, thus justifying the definition we recalled before. In this context, we have functors of extension of scalars $\mathbb{K} \otimes_{\mathbb{O}} -$ and of modular reduction $\mathbb{F} \otimes^{L}_{\mathbb{O}} -$. One of the main technical points is that truncations do not commute with modular reduction. We study carefully the failure of commutativity of these functors, because this is precisely what will give rise to non-trivial decomposition numbers, in the setting of recollement. Then it is time to define these decomposition numbers for perverse sheaves, and finally we deal with equivariance.

Since we can translate some problems of modular representation theory in terms of decomposition numbers for perverse sheaves, it is very important to be able to compute them. In general, it should be very difficult. In Section 3, we give some techniques to compute them in certain cases. It is enough to determine the intersection cohomology stalks over $\mathbb{F}$ (in the applications, they are usually known over $\mathbb{K}$). In characteristic zero, a lot of information can be obtained from the study of semi-small and small proper separable morphisms. We explain what is still true in characteristic ℓ, but also why it is less useful, unless we have a small resolution of singularities.

Then we recall the notion of $\mathbb{E}$-smoothness, and we give some conditions which imply that some decomposition numbers are zero. This is the simplest case, where the intersection cohomology complex is just the constant sheaf (suitably shifted). In general, we do not have many tools at our disposal, so Deligne's construction, which works in any case, is very important to do calculations in the modular setting. When we have an isolated cone singularity, or more generally an isolated singularity on an affine variety endowed with a $\mathbb{G}_m$-action contracting to the origin, it is much more likely to be handled. Finally, we recall the notion of smooth equivalence of singularities. We can use the results about a singularity to study a smoothly equivalent one. When we deal only with constant local systems, this even gives all the information. In general, one has to get extra information to determine all the decomposition numbers.

In Section 4, we determine the decomposition numbers for simple (or Kleinian) surface singularities. Their geometry has been studied a lot. It is a nice illustration of the theory and techniques described earlier to do this calculation, using geometrical results in the literature. By a famous theorem of Brieskorn and Slodowy [5, 20, 21], the singularity of the nilpotent cone of a simple Lie algebra along the subregular orbit is a simple singularity. This is an instance where we can determine all the decomposition numbers, even for non constant local systems, using a smooth equivalence of singularities, thanks to Slodowy's study of the symmetries of the minimal resolutions of simple singularities (thus giving a meaning to simple singularities of non-homogeneous type).

Finally, in Section 5, we determine the decomposition numbers for closures of minimal non-trivial nilpotent orbits in simple Lie algebras. Again, this is a nice illustration of the previous parts (it is an isolated cone singularity). This result uses the determination of the integral cohomology of the minimal orbit, which we obtained in a previous article [13].

2. Perverse sheaves over $\mathbb{K}$, $\mathbb{O}$ and $\mathbb{F}$

2.1. Context

In all this article, we fix on the one hand a prime number p and an algebraic closure $\overline{\mathbb{F}}_p$ of the prime field with p elements, and for each power q of p, we denote by $\mathbb{F}_q$ the unique subfield of $\overline{\mathbb{F}}_p$ with q elements. On the other hand, we fix a prime number ℓ distinct from p, and a finite extension $\mathbb{K}$ of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers, whose valuation ring we denote by $\mathbb{O}$.

Let $\mathfrak{m} = (\varpi)$ be the maximal ideal of $\mathbb{O}$, and let $\mathbb{F} = \mathbb{O}/\mathfrak{m}$ be its residue field (which is finite of characteristic ℓ). In modular representation theory, a triple such as $(\mathbb{K}, \mathbb{O}, \mathbb{F})$ is called an ℓ-modular system. The letter $\mathbb{E}$ will often be used to denote either of these three rings.

Let k denote $\mathbb{F}_q$ or $\overline{\mathbb{F}}_p$ (we could have taken the field $\mathbb{C}$ of complex numbers instead, and then we could have used arbitrary coefficients; however, for future applications, we will need to treat the positive characteristic case, with the étale topology). We will consider only separated k-schemes of finite type, and morphisms of k-schemes. Such schemes will be called varieties. If X is a variety, we will say "$\mathbb{E}$-sheaves on X" for "constructible $\mathbb{E}$-sheaves on X". We will denote by $\mathrm{Sh}(X, \mathbb{E})$ the Noetherian abelian category of $\mathbb{E}$-sheaves on X, and by $\mathrm{Loc}(X, \mathbb{E})$ the full subcategory of $\mathbb{E}$-local systems on X. If X is connected, these correspond to the continuous representations of the étale fundamental group of X at any base point.

Let $D^b_c(X, \mathbb{E})$ be the bounded derived category of $\mathbb{E}$-sheaves as defined by Deligne. The category $D^b_c(X, \mathbb{E})$ is triangulated, and endowed with a t-structure whose heart is equivalent to the abelian category of $\mathbb{E}$-sheaves, because the following condition is satisfied [1, 6].

(2.1) For each finite extension k' of k contained in $\overline{\mathbb{F}}_p$, the groups $H^i(\mathrm{Gal}(\overline{\mathbb{F}}_p/k'), \mathbb{Z}/\ell)$, $i \in \mathbb{N}$, are finite.

We call this t-structure the *natural* t-structure on $D^b_c(X, \mathbb{E})$. The notion of t-structure will be recalled in the next section. For triangulated categories and derived categories, we refer to [14, 24].

We have internal operations $\otimes^{\mathbb{L}}_{\mathbb{E}}$ and $\mathrm{R}\underline{\mathrm{Hom}}$ on $D^b_c(X, \mathbb{E})$, and, if Y is another scheme, for $f : X \to Y$ a morphism we have triangulated functors

$$f_!,\ f_* : D^b_c(X, \mathbb{E}) \to D^b_c(Y, \mathbb{E})$$
$$f^*,\ f^! : D^b_c(Y, \mathbb{E}) \to D^b_c(X, \mathbb{E})$$

We omit the letter R which is normally used (*e.g.* Rf_*, $Rf_!$) meaning that we consider derived functors. For the functors between categories of sheaves, we will use a 0 superscript, as in ${}^0 f_*$ and ${}^0 f_!$, following [1].

We will denote by

$$\mathcal{D}_{X,\mathbb{E}} : D^b_c(X, \mathbb{E})^{\mathrm{op}} \to D^b_c(X, \mathbb{E})$$

the dualizing functor $\mathcal{D}_{X,\mathbb{E}}(-) = \mathrm{R}\underline{\mathrm{Hom}}(-, a^!\mathbb{E})$, where $a : X \to \operatorname{Spec} k$ is the structural morphism.

We have a modular reduction functor $\mathbb{F} \otimes^{\mathbb{L}}_{\mathbb{O}} (-) : D^b_c(X, \mathbb{O}) \to D^b_c(X, \mathbb{F})$, which we will simply denote by $\mathbb{F}(-)$. It is triangulated, and it commutes with the functors $f_!$, f_*, f^*, $f^!$ and the duality. Moreover, it maps a torsion-free sheaf to a sheaf, and a torsion sheaf to a complex concentrated in degrees -1 and 0.

By definition, we have $D^b_c(X,\mathbb{K}) = \mathbb{K}\otimes_{\mathbb{O}} D^b_c(X,\mathbb{O})$, and $\mathrm{Sh}(X,\mathbb{K}) = \mathbb{K}\otimes_{\mathbb{O}} \mathrm{Sh}(X,\mathbb{O})$. The functor $\mathbb{K}\otimes_{\mathbb{O}}(-) : D^b_c(X,\mathbb{O}) \to D^b_c(X,\mathbb{K})$ is exact.

In this section, we are going to recall the construction of the perverse t-structure on $D^b_c(X,\mathbb{E})$ for the middle perversity p (with two versions over $\mathbb{O}$, where we have two perversities p and p_+ exchanged by the duality). We will recall the main points of the treatment of t-structures and recollement of [1], to which we refer for the details. However, in this work we emphasize the aspects concerning $\mathbb{O}$-sheaves, and we give some complements.

Before going through all these general constructions, let us already see what these perverse sheaves are. They form an abelian full subcategory ${}^p\mathcal{M}(X,\mathbb{E})$ of $D^b_c(X,\mathbb{E})$. For $j : V \to X$ the inclusion of a smooth irreducible subvariety, and $\mathcal{L}$ an irreducible locally constant constructible $\mathbb{E}$-sheaf on V, we have a perverse sheaf that we will denote by $\mathcal{IC}(\overline{V},\mathcal{L})$, which is the intersection cohomology complex orginally defined (topologically) by Goresky and McPerson [7, 8], shifted by the dimension of V (so that it is concentrated in degrees $\leqslant 0$), and extended by zero outside $\overline{V}$. Deligne gave an algebraic construction of this complex, using a functor of intermediate extension ${}^pj_{!*}$, which satisfies ${}^pj_{!*}(\mathcal{L}[\dim V)]) = \mathcal{IC}(\overline{V},\mathcal{L})$.

If $\mathbb{E}$ is $\mathbb{K}$ or $\mathbb{F}$, then this abelian category is Artinian and Noetherian, and all its simple objects are of this form. If $\mathbb{E} = \mathbb{O}$, this abelian category is only Noetherian. In any case, ${}^p\mathcal{M}(X,\mathbb{E})$ is the intersection of the full subcategories ${}^pD^{\leqslant 0}(X,\mathbb{E})$ and ${}^pD^{\geqslant 0}(X,\mathbb{E})$ of $D^b_c(X,\mathbb{E})$, where, if A is a complex in $D^b_c(X,\mathbb{E})$, we have

$$A \in {}^pD^{\leqslant 0}(X,\mathbb{E}) \iff \text{for all points } x \text{ in } X, \quad \mathcal{H}^n i_x^* A = 0 \text{ for all } n > -\dim(x) \tag{2.2}$$

$$A \in {}^pD^{\geqslant 0}(X,\mathbb{E}) \iff \text{for all points } x \text{ in } X, \quad \mathcal{H}^n i_x^! A = 0 \text{ for all } n < -\dim(x). \tag{2.3}$$

Here the points are not necessarily closed, i_x is the inclusion of x into X, and $\dim(x) = \dim\overline{\{x\}} = \deg\operatorname{tr}(k(x)/k)$.

The pair $({}^pD^{\leqslant 0}, {}^pD^{\geqslant 0})$ is a t-structure on $D^b_c(X,\mathbb{E})$, and ${}^p\mathcal{M}(X,\mathbb{E})$ is its *heart*.

When $\mathbb{E}$ is a field (*i.e.*, $\mathbb{E} = \mathbb{K}$ or $\mathbb{F}$), the duality functor $\mathcal{D}_{X,\mathbb{E}}$ exchanges ${}^pD^{\leqslant 0}(X,\mathbb{E})$ and ${}^pD^{\geqslant 0}(X,\mathbb{E})$, so it induces a self-duality on ${}^p\mathcal{M}(X,\mathbb{E})$.

However, when $\mathbb{E} = \mathbb{O}$, this is no longer true. The perversity p is not self-dual in this case. The duality exchanges the t-structure defined by the middle perversity p with the t-structure $({}^{p_+}D^{\leqslant 0}(X, \mathbb{O}), {}^{p_+}D^{\geqslant 0}(X, \mathbb{O}))$ defined by

$$(2.4) \quad A \in {}^{p_+}D^{\leqslant 0}(X, \mathbb{O}) \Longleftrightarrow \text{for all points } x \text{ in } X,$$
$$\begin{cases} \mathcal{H}^n i_x^* A = 0 \text{ for all } n > -\dim(x) + 1 \\ \mathcal{H}^{-\dim(x)+1} i_x^* A \text{ is torsion} \end{cases}$$

$$(2.5) \quad A \in {}^{p_+}D^{\geqslant 0}(X, \mathbb{O}) \Longleftrightarrow \text{for all points } x \text{ in } X,$$
$$\begin{cases} \mathcal{H}^n i_x^! A = 0 \text{ for all } n < -\dim(x) \\ \mathcal{H}^{-\dim(x)} i_x^! A \text{ is torsion-free.} \end{cases}$$

The definition of torsion (resp. torsion-free) objects is given in Definition 2.10. We say that this t-structure is defined by the perversity p_+, and that the duality exchanges p and p_+. We denote by ${}^{p_+}\mathcal{M}(X, \mathbb{O}) = {}^{p_+}D^{\leqslant 0}(X, \mathbb{O}) \cap {}^{p_+}D^{\geqslant 0}(X, \mathbb{O})$ the heart of the t-structure defined by p_+, and we call its objects p_+-perverse sheaves, or dual perverse sheaves. This abelian category is only Artinian. The t-structures defined by p and p_+ determine each other (see [1, §3.3]). We have:

$$(2.6) \quad A \in {}^{p_+}D^{\leqslant 0}(X, \mathbb{O}) \Longleftrightarrow A \in {}^{p}D^{\leqslant 1}(X, \mathbb{O}) \text{ and } {}^{p}H^1 A \text{ is torsion}$$

$$(2.7) \quad A \in {}^{p_+}D^{\geqslant 0}(X, \mathbb{O}) \Longleftrightarrow A \in {}^{p}D^{\geqslant 0}(X, \mathbb{O}) \text{ and } {}^{p}H^0 A \text{ is torsion-free}$$

$$(2.8) \quad A \in {}^{p}D^{\leqslant 0}(X, \mathbb{O}) \Longleftrightarrow A \in {}^{p_+}D^{\leqslant 0}(X, \mathbb{O}) \text{ and } {}^{p_+}H^0 A \text{ is divisible}$$

$$(2.9) \quad A \in {}^{p}D^{\geqslant 0}(X, \mathbb{O}) \Longleftrightarrow A \in {}^{p_+}D^{\geqslant -1}(X, \mathbb{O}) \text{ and } {}^{p_+}H^{-1} A \text{ is torsion.}$$

If A is p-perverse, then it is also p_+-perverse if and only if A is torsion-free in ${}^{p}\mathcal{M}(X, \mathbb{O})$. If A is p_+-perverse, then A is also p-perverse if and only if A is divisible in ${}^{p_+}\mathcal{M}(X, \mathbb{O})$. Thus, if A is both p- and p_+-perverse, then A is torsion-free in ${}^{p}\mathcal{M}(X, \mathbb{O})$ and divisible in ${}^{p_+}\mathcal{M}(X, \mathbb{O})$. The modular reduction of a p-perverse sheaf A over $\mathbb{O}$ will be a perverse over $\mathbb{F}$ if and only if A is also p_+-perverse, and *vice versa*.

In the following, we will recall why $({}^{p}D^{\leqslant 0}, {}^{p}D^{\geqslant 0})$ (resp. the two versions with p and p_+ if $\mathbb{E} = \mathbb{O}$) is indeed a t-structure on $D^b_c(X, \mathbb{E})$. We refer to [1] for more details, however their treatment of the case $\mathbb{E} = \mathbb{O}$ is quite brief, so we give some complements. The rest of the section is organized as follows.

First, we recall the definition of t-categories and their main properties. Then we see how they can be combined with torsion theories. Afterwards, we recall the notion of recollement of t-categories, stressing on some important properties, such as the construction of the perverse extensions ${}^p j_!$, ${}^p j_{!*}$ and ${}^p j_*$ with functors of truncation on the closed part. We also study the tops and socles of the extensions ${}^p j_!$, ${}^p j_{!*}$ and ${}^p j_*$, and show that the intermediate extension preserves multiplicities. Then again, we study the connection with torsion theories. Already at this point, we have six possible extensions (the three just mentioned, in the two versions p and p_+).

Then we leave the general context of t-structures and recollement and we focus on perverse sheaves over $\mathbb{E} = \mathbb{K}$, $\mathbb{O}$, $\mathbb{F}$. First, we see how the preceding general constructions show that the definitions of perverse t-structures given above actually give t-structures on the triangulated categories $D^b_c(X, \mathbb{E})$, first fixing a stratification, and then taking the limit. Now we have functors $\mathbb{K} \otimes^{\mathbb{L}}_{\mathbb{O}} (-)$ and $\mathbb{F} \otimes^{\mathbb{L}}_{\mathbb{O}} (-)$ (it would be nice to treat this situation in an axiomatic framework, maybe including duality). We study the connection between modular reduction and truncation. If we take a complex A over $\mathbb{O}$, for each degree we have three places where we can truncate its reduction modulo ϖ, because $\mathcal{H}^i(\mathbb{F}A)$ has pieces coming from $\mathcal{H}^i_{\mathrm{tors}}(A)$, $\mathcal{H}^i_{\mathrm{free}}(A)$ and $\mathcal{H}^{i+1}_{\mathrm{tors}}(A)$. So, in a recollement situation, we have nine natural ways to truncate $\mathbb{F}A$.

Finally, we introduce decomposition numbers for perverse sheaves, and particularly in the G-equivariant setting. We have in mind, for example, G-equivariant perverse sheaves on the nilpotent cone.

The relation between modular reduction and truncation is really one of the main technical points. For example, the fact that the modular reduction does not commute with the intermediate extension means that the reduction of a simple perverse sheaf will not necessarily be simple, that is, that we have can have non-trivial decomposition numbers.

2.2. t-categories

Let us begin by recalling the notion of t-structure on a triangulated category, introduced in [1].

Definition 2.1. — *A t-category is a triangulated category $\mathcal{D}$, endowed with two strictly full subcategories $\mathcal{D}^{\leqslant 0}$ and $\mathcal{D}^{\geqslant 0}$, such that, if we let $\mathcal{D}^{\leqslant n} = \mathcal{D}^{\leqslant 0}[-n]$ and $\mathcal{D}^{\geqslant n} = \mathcal{D}^{\geqslant 0}[-n]$, we have*

(i) *For X in $\mathcal{D}^{\leqslant 0}$ and Y in $\mathcal{D}^{\geqslant 1}$, we have* $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$.

(ii) $\mathcal{D}^{\leqslant 0} \subset \mathcal{D}^{\leqslant 1}$ *and* $\mathcal{D}^{\geqslant 0} \supset \mathcal{D}^{\geqslant 1}$.

(iii) *For each X in $\mathcal{D}$, there is a distinguished triangle (A, X, B) in $\mathcal{D}$ with A in $\mathcal{D}^{\leqslant 0}$ and B in $\mathcal{D}^{\geqslant 1}$.*

We also say that $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ *is a t-structure on* $\mathcal{D}$. *Its* heart *is the full subcategory* $\mathcal{C} := \mathcal{D}^{\leqslant 0} \cap \mathcal{D}^{\geqslant 0}$.

PROPOSITION 2.2. — *Let* $\mathcal{D}$ *be a t-category.*

(i) *The inclusion of* $\mathcal{D}^{\leqslant n}$ *(resp.* $\mathcal{D}^{\geqslant n}$*) in* $\mathcal{D}$ *has a right adjoint* $\tau_{\leqslant n}$ *(resp. a left adjoint* $\tau_{\geqslant n}$*).*

(ii) *For all* X *in* $\mathcal{D}$*, there is a unique* $d \in \operatorname{Hom}(\tau_{\geqslant 1}X, \tau_{\leqslant 0}X[1])$ *such that the triangle*

$$\tau_{\leqslant 0}X \longrightarrow X \longrightarrow \tau_{\geqslant 1}X \xrightarrow{d}$$

is distinguished. Up to unique isomorphism, this is the unique triangle (A, X, B) *with* A *in* $\mathcal{D}^{\leqslant 0}$ *and* B *in* $\mathcal{D}^{\geqslant 1}$.

(iii) *Let* $a \leqslant b$. *Then, for any* X *in* $\mathcal{D}$*, there is a unique morphism* $\tau_{\geqslant a}\tau_{\leqslant b}X \to \tau_{\leqslant b}\tau_{\geqslant a}X$ *such that the following diagram is commutative.*

$$\begin{tikzcd}
\tau_{\leqslant b}X \arrow[r] \arrow[d] & X \arrow[r] & \tau_{\geqslant a}X \\
\tau_{\geqslant a}\tau_{\leqslant b}X \arrow[rr, "\sim"] & & \tau_{\leqslant b}\tau_{\geqslant a}X \arrow[u]
\end{tikzcd}$$

It is an isomorphism.

For example, if $\mathcal{A}$ is an abelian category and $\mathcal{D}$ is its derived category, the natural t-structure on $\mathcal{D}$ is the one for which $\mathcal{D}^{\leqslant n}$ (resp. $\mathcal{D}^{\geqslant n}$) is the full subcategory of the complexes K such that $H^iK = 0$ for $i > n$ (resp. $i < n$). For $K = (K^i, d^i : K^i \to K^{i+1})$ in $\mathcal{D}$, the truncated complex $\tau_{\leqslant n}K$ is the subcomplex $(\cdots \to K^{n-1} \to \operatorname{Ker} d^n \to 0 \to \cdots)$ of K. The heart is equivalent to the abelian category $\mathcal{A}$ we started with. Note that, in this case, the cone of a morphism $f : A \to B$ between two objects of $\mathcal{A}$ is a complex concentrated in degrees -1 and 0. More precisely, we have $H^{-1}(\operatorname{Cone} f) \simeq \operatorname{Ker} f$ and $H^0(\operatorname{Cone} f) \simeq \operatorname{Coker} f$. In particular, we have a triangle $(\operatorname{Ker} f[1],\ \operatorname{Cone} f,\ \operatorname{Coker} f)$.

If we abstract the relations between $\mathcal{A}$ and $\mathcal{D}(\mathcal{A})$, we get the notion of admissible abelian subcategory of a triangulated category $\mathcal{D}$, and a t-structure on $\mathcal{D}$ precisely provides an admissible abelian subcategory by taking the heart.

More precisely, let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ a full subcategory of $\mathcal{D}$ such that $\operatorname{Hom}^i(A, B) := \operatorname{Hom}(A, B[i])$ is zero for $i < 0$ and A, B in $\mathcal{C}$. We have the following proposition, which results from the octahedron axiom.

Proposition 2.3. — *Let $f : X \to Y$ in $\mathcal{C}$. We can complete f into a distinguished triangle (X, Y, S). Suppose S is in a distinguished triangle $(N[1], S, C)$ with N and C in $\mathcal{C}$. Then the morphisms $N \to S[-1] \to X$ and $Y \to S \to C$, obtained by composition from the morphisms in the two triangles above, are respectively a kernel and a cokernel for the morphism f in $\mathcal{C}$.*

Such a morphism will be called *$\mathcal{C}$-admissible*. In a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{d}$ on objects in $\mathcal{C}$, the morphisms f and g are admissible, f is a kernel of g, g is a cokernel of f, and d is uniquely determined by f and g. A short exact sequence in $\mathcal{C}$ will be called *admissible* if it can be obtained from a distinguished triangle in $\mathcal{D}$ by suppressing the degree one morphism.

Proposition 2.4. — *Suppose $\mathcal{C}$ is stable by finite direct sums. Then the following conditions are equivalent.*

(i) *$\mathcal{C}$ is abelian, and its short exact sequences are admissible.*
(ii) *Every morphism of $\mathcal{C}$ is $\mathcal{C}$-admissible.*

Now we can state the theorem that says that t-structures provide admissible abelian categories.

Theorem 2.5. — *The heart $\mathcal{C}$ of a t-category $\mathcal{D}$ is an admissible abelian subcategory of $\mathcal{D}$, stable by extensions. The functor $H^0 := \tau_{\geqslant 0}\tau_{\leqslant 0} \simeq \tau_{\leqslant 0}\tau_{\geqslant 0} : \mathcal{D} \to \mathcal{C}$ is a cohomological functor.*

We have a chain of morphisms

$$\cdots \longrightarrow \tau_{\leqslant i-2} \longrightarrow \tau_{\leqslant i-1} \longrightarrow \tau_{\leqslant i} \longrightarrow \tau_{\leqslant i+1} \longrightarrow \cdots$$

which can be seen as a “filtration” of the identity functor, with “successive quotients” the $H^i[-i]$. Thus we have distinguished triangles:

$$\tau_{\leqslant i-1} \longrightarrow \tau_{\leqslant i} \longrightarrow H^i[-i] \rightsquigarrow .$$

An object A in $\mathcal{D}$ can be seen as “made of” its cohomology objects H^iA (by successive extensions). We depict this by the following diagram:

$\cdots$	H^{i-1}	H^i	H^{i+1}	$\cdots$
$\cdots$	$\tau_{\leqslant i-1}$	$\tau_{\leqslant i}$	$\tau_{\leqslant i+1}$	$\cdots$

In the next sections, when we study the interplay between t-structures and other structures (torsion theories, modular reduction...), we will see refinements of this “filtration”, and there will be more complicated pictures.

Now let $\mathcal{D}_i$ $(i = 1, 2)$ be two t-categories, and let $\varepsilon_i : \mathcal{C}_i \to \mathcal{D}_i$ denote the inclusion functors of their hearts. Let $T : \mathcal{D}_1 \to \mathcal{D}_2$ be a triangulated functor. Then we say that T is right t-exact if $T(\mathcal{D}_1^{\leqslant 0}) \subset \mathcal{D}_2^{\leqslant 0}$, left t-exact if $T(\mathcal{D}_1^{\geqslant 0}) \subset \mathcal{D}_2^{\geqslant 0}$, and t-exact if it is both left and right t-exact.

PROPOSITION 2.6 (Exactness and adjunction properties of the ${}^pT$). —

(1) *If T is left (resp. right) t-exact, then the additive functor ${}^pT := H^0 \circ T \circ \varepsilon_1$ is left (resp. right) exact.*

(2) *Let (T^*, T_*) be a pair of adjoint triangulated functors, with $T^* : \mathcal{D}_2 \to \mathcal{D}_1$ and $T_* : \mathcal{D}_1 \to \mathcal{D}_2$. Then T^* is right t-exact if and only if T_* is left t-exact, and in that case $({}^pT^*, {}^pT_*)$ is a pair of adjoint functors between $\mathcal{C}_1$ and $\mathcal{C}_2$.*

2.3. Torsion theories and t-structures

We will give some variations of known results [9].

DEFINITION 2.7. — *Let $\mathcal{A}$ be an abelian category. A torsion theory on $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories such that*

(i) *for all objects T in $\mathcal{T}$ and F in $\mathcal{F}$, we have*

$$\mathrm{Hom}_{\mathcal{A}}(T, F) = 0,$$

(ii) *for any object A in $\mathcal{A}$, there are objects T in $\mathcal{T}$ and F in $\mathcal{F}$ such that there is a short exact sequence*

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0.$$

Let us first give some elementary properties of torsion theories.

PROPOSITION 2.8. — *Let $\mathcal{A}$ be an abelian category endowed with a torsion theory $(\mathcal{T}, \mathcal{F})$. Then the following hold:*

(i) *The inclusion of $\mathcal{T}$ (resp. $\mathcal{F}$) in $\mathcal{A}$ has a right adjoint $(-)_{\mathrm{tors}} : \mathcal{A} \to \mathcal{T}$ (resp. a left adjoint $(-)_{\mathrm{free}} : \mathcal{A} \to \mathcal{F}$).*

(ii) *We have*

$$\mathcal{F} = \mathcal{T}^{\perp} = \{F \in \mathcal{C} \mid \forall T \in \mathcal{T},\ \mathrm{Hom}_{\mathcal{C}}(T, F) = 0\}$$
$$\mathcal{T} = {}^{\perp}\mathcal{F} = \{T \in \mathcal{C} \mid \forall F \in \mathcal{F},\ \mathrm{Hom}_{\mathcal{C}}(T, F) = 0\}.$$

(iii) *The torsion class $\mathcal{T}$ (resp. the torsion-free class $\mathcal{F}$) is closed under quotients and extensions (resp. under subobjects and extensions).*

DEFINITION 2.9. — *A torsion theory $(\mathcal{T}, \mathcal{F})$ on an abelian category $\mathcal{A}$ is said to be* hereditary *(resp.* cohereditary*) if the torsion class $\mathcal{T}$ (resp. the torsion-free class $\mathcal{F}$) is closed under subobjects (resp. under quotients).*

Examples of torsion theories arise with $\mathbb{O}$-linear abelian categories.

Definition 2.10. — *Let $\mathcal{A}$ be an $\mathbb{O}$-linear abelian category. An object A in $\mathcal{A}$ is* torsion *if $\varpi^N 1_A$ is zero for some $N \in \mathbb{N}$, and it is* torsion-free *(resp.* divisible*) if $\varpi.1_A$ is a monomorphism (resp. an epimorphism).*

Proposition 2.11. — *Let $\mathcal{A}$ be an $\mathbb{O}$-linear abelian category.*

(i) *If $T \in \mathcal{A}$ is torsion and $F \in \mathcal{A}$ is torsion-free, then we have*

$$\mathrm{Hom}_{\mathcal{A}}(T, F) = 0.$$

(ii) *If $Q \in \mathcal{A}$ is divisible and $T \in \mathcal{A}$ is torsion, then we have*

$$\mathrm{Hom}_{\mathcal{A}}(Q, T) = 0.$$

Proof.

(i) Let $f \in \mathrm{Hom}_{\mathcal{A}}(T, F)$. Let $N \in \mathbb{N}$ such that $\varpi^N 1_T = 0$. Then we have $(\varpi^N 1_F) \circ f = f \circ (\varpi^N.1_T) = 0$, and consequently $f = 0$, since $\varpi^N 1_F$ is a monomorphism.

(ii) Let $g \in \mathrm{Hom}_{\mathcal{A}}(Q, T)$. Let $N \in \mathbb{N}$ such that $\varpi^N 1_T = 0$. Then we have $g \circ (\varpi^N 1_Q) = (\varpi^N 1_T) \circ g = 0$, and consequently $g = 0$, since $\varpi^N 1_Q$ is an epimorphism. □

Proposition 2.12. — *Let $\mathcal{A}$ be an $\mathbb{O}$-linear abelian category. Then subobjects and quotients of torsion objects are torsion objects.*

Proof. — Let T be a torsion object in $\mathcal{A}$. We can choose an integer N such that $\varpi^N 1_T = 0$.

If $i : S \hookrightarrow T$ is a subobject, then we have $i \circ (\varpi^N 1_S) = (\varpi^N 1_T) \circ i = 0$, hence $\varpi^N 1_S = 0$ since i is a monomorphism. Thus S is torsion.

If $q : T \twoheadrightarrow U$ is a quotient, then we have $(\varpi^N 1_U) \circ q = q \circ (\varpi^N 1_T) = 0$, hence $\varpi^N 1_U = 0$ since q is an epimorphism. Thus U is torsion. □

Proposition 2.13. — *Let A be an object in an $\mathbb{O}$-linear abelian category $\mathcal{A}$.*

(i) *If A is Noetherian, then A has a greatest torsion subobject A_{tors}, the quotient A/A_{tors} is torsion-free and $\mathbb{K}A \simeq \mathbb{K}A/A_{\mathrm{tors}}$.*

(ii) *If A is Artinian, then A has a greatest divisible subobject A_{div}, the quotient A/A_{div} is a torsion object and we have $\mathbb{K}A \simeq \mathbb{K}A_{\mathrm{div}}$.*

Proof. — In the first case, the increasing sequence $\mathrm{Ker}\,\varpi^n 1_A$ of subobjects of A must stabilize, so there is an integer N such that $\mathrm{Ker}\,\varpi^n 1_A = \mathrm{Ker}\,\varpi^N 1_A$ for all $n \geqslant N$. We set $A_{\mathrm{tors}} := \mathrm{Ker}\,\varpi^N 1_A$. This is clearly a torsion object, since it is killed by ϖ^N. Now let T be a torsion subobject of A. It is killed by some ϖ^k, and we can assume $k \geqslant N$. Thus

$T \subset \operatorname{Ker} \varpi^k 1_A = \operatorname{Ker} \varpi^N 1_A = A_{\text{tors}}$. This shows that A_{tors} is the greatest torsion subobject of A. We have

$$\operatorname{Ker} \varpi 1_{A/A_{\text{tors}}} = \operatorname{Ker} \varpi^{N+1} 1_A / \operatorname{Ker} \varpi^N 1_A = 0$$

which shows that A/A_{tors} is torsion-free. Applying the exact functor $\mathbb{K} \otimes_{\mathbb{O}} -$ to the short exact sequence $0 \to A_{\text{tors}} \to A \to A/A_{\text{tors}} \to 0$, we get $\mathbb{K}A \simeq \mathbb{K}A/A_{\text{tors}}$.

In the second case, the decreasing sequence $\operatorname{Im} \varpi^n 1_A$ of subobjects of A must stabilize, so there is an integer N such that $\operatorname{Im} \varpi^n 1_A = \operatorname{Im} \varpi^N 1_A$ for all $n \geqslant N$. We set $A_{\text{div}} := \operatorname{Im} \varpi^N 1_A$. We have $\operatorname{Im} \varpi 1_{A_{\text{div}}} = \operatorname{Im} \varpi^{N+1} 1_A = \operatorname{Im} \varpi^N 1_A = A_{\text{div}}$, thus A_{div} is divisible. We have

$$\operatorname{Im} \varpi^n 1_{A/A_{\text{div}}} = \operatorname{Im} \varpi^n 1_A / \operatorname{Im} \varpi^N 1_A = 0$$

for $n \geqslant N$. Hence A/A_{div} is a torsion object. Applying the exact functor $\mathbb{K} \otimes_{\mathbb{O}} -$ to the short exact sequence $0 \to A_{\text{div}} \to A \to A/A_{\text{div}} \to 0$, we get $\mathbb{K}A_{\text{div}} \simeq \mathbb{K}A$. ☐

PROPOSITION 2.14. — *Let $\mathcal{A}$ be an $\mathbb{O}$-linear abelian category. We denote by $\mathcal{T}$ (resp. $\mathcal{F}$, $\mathcal{Q}$) the full subcategory of torsion (resp. torsion-free, divisible) objects in $\mathcal{A}$. If $\mathcal{A}$ is Noetherian (resp. Artinian), then $(\mathcal{T}, \mathcal{F})$ (resp. $(\mathcal{Q}, \mathcal{T})$) is an hereditary (resp. cohereditary) torsion theory on $\mathcal{A}$.*

Proof. — This follows from Propositions 2.11, 2.12 and 2.13. ☐

We want to discuss the combination of t-structures with torsion theories.

PROPOSITION 2.15. — *Let $\mathcal{D}$ be a triangulated category endowed with a t-structure $({}^p\mathcal{D}^{\leqslant 0}, {}^p\mathcal{D}^{\geqslant 0})$. Let us denote its heart by $\mathcal{C}$, the truncation functors by ${}^p\tau_{\leqslant i}$ and ${}^p\tau_{\geqslant i}$, and the cohomology functors by ${}^pH^i : \mathcal{D} \to \mathcal{C}$. Suppose that $\mathcal{C}$ is endowed with a torsion theory $(\mathcal{T}, \mathcal{F})$. Then we can define a new t-structure $({}^{p+}\mathcal{D}^{\leqslant 0}, {}^{p+}\mathcal{D}^{\geqslant 0})$ on $\mathcal{D}$ by*

$${}^{p+}\mathcal{D}^{\leqslant 0} = \{A \in {}^p\mathcal{D}^{\leqslant 1} \mid {}^pH^1(A) \in \mathcal{T}\}$$

$${}^{p+}\mathcal{D}^{\geqslant 0} = \{A \in {}^p\mathcal{D}^{\geqslant 0} \mid {}^pH^0(A) \in \mathcal{F}\}.$$

Proof. — Let us check the three axioms for t-structures given in Definition 2.1.

(i) Let $A \in {}^{p+}\mathcal{D}^{\leqslant 0}$ and $B \in {}^{p+}\mathcal{D}^{\geqslant 1}$. Then we have

$$\operatorname{Hom}_{\mathcal{D}}(A, B) = \operatorname{Hom}_{\mathcal{D}}({}^p\tau_{\geqslant 1} A, {}^p\tau_{\leqslant 1} B) = \operatorname{Hom}_{\mathcal{C}}({}^pH^1 A, {}^pH^1 B) = 0.$$

The first equality follows from the adjunctions of Proposition 2.2 (i), since we have $A \in {}^{p+}\mathcal{D}^{\leqslant 0} \subset {}^p\mathcal{D}^{\leqslant 1}$ and $B \in {}^{p+}\mathcal{D}^{\geqslant 1} \subset {}^p\mathcal{D}^{\geqslant 1}$. The second equality follows since ${}^p\tau_{\geqslant 1}A \simeq {}^pH^1A[-1]$ and ${}^p\tau_{\leqslant 1}B \simeq {}^pH^1B[-1]$. The last equality follows from the first axiom in the definition of torsion theories, since ${}^pH^1A \in \mathcal{T}$ and ${}^pH^1B \in \mathcal{F}$ (see Definition 2.7 (i)).

(ii) We have ${}^{p+}\mathcal{D}^{\leqslant 0} \subset {}^p\mathcal{D}^{\leqslant 1} \subset {}^{p+}\mathcal{D}^{\leqslant 1}$ and ${}^{p+}\mathcal{D}^{\geqslant 0} \supset {}^p\mathcal{D}^{\geqslant 1} \supset {}^{p+}\mathcal{D}^{\geqslant 1}$.

(iii) Let $A \in \mathcal{D}$. By Definition 2.7 (ii), there are objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$ such that we have a short exact sequence

$$0 \longrightarrow T \longrightarrow {}^pH^1A \longrightarrow F \longrightarrow 0.$$

By [1, Proposition 1.3.15] there is a distinguished triangle

$$A' \xrightarrow{a} A \xrightarrow{b} A'' \xrightarrow{d} A'[1]$$

such that $A' \in {}^p\mathcal{D}^{\leqslant 1}$ and $A'' \in {}^p\mathcal{D}^{\geqslant 1}$, ${}^pH^1A' \simeq T$ and ${}^pH^1A'' \simeq F$, and thus $A' \in {}^{p+}\mathcal{D}^{\leqslant 0}$ and $A'' \in {}^{p+}\mathcal{D}^{\geqslant 1}$. □

We denote by $\mathcal{C}^+$ the heart of this new t-structure, by ${}^{p+}H^n : \mathcal{D} \to \mathcal{C}^+$ the new cohomology functors, and by ${}^{p+}\tau_{\leqslant n}$, ${}^{p+}\tau_{\geqslant n}$ the new truncation functors.

We may also use the following notation. For the notions attached to the initial t-structure, we may drop all the p, and for the new t-structure one may write n_+ instead of n, as follows: $(\mathcal{D}^{\leqslant n_+}, \mathcal{D}^{\geqslant n_+})$, H^{n_+}, $\tau_{\leqslant n_+}$, $\tau_{\geqslant n_+}$.

Note that $\mathcal{C}^+$ is endowed with a torsion theory, namely $(\mathcal{F}, \mathcal{T}[-1])$. We can do the same construction, and we find that $\mathcal{C}^{++} = \mathcal{C}[-1]$. We recover the usual shift of t-structures.

We have the following chain of morphisms:

$$\cdots \longrightarrow \tau_{\leqslant (n-2)_+} \longrightarrow \tau_{\leqslant n-1} \longrightarrow \tau_{\leqslant (n-1)_+} \longrightarrow \tau_{\leqslant n} \longrightarrow \tau_{\leqslant n_+} \longrightarrow \tau_{\leqslant n+1} \longrightarrow \cdots$$

and the following distinguished triangles:

$$\tau_{\leqslant n} \longrightarrow \tau_{\leqslant n_+} \longrightarrow H^{n+1}_{\mathrm{tors}}(-)[-n-1] \rightsquigarrow \tag{2.10}$$

$$\tau_{\leqslant n_+} \longrightarrow \tau_{\leqslant n+1} \longrightarrow H^{n+1}_{\mathrm{free}}(-)[-n-1] \rightsquigarrow . \tag{2.11}$$

This follows from [1, Prop. 1.3.15], which is proved using the octahedron axiom. These triangles can be read off the following diagram:

<table>
<tr><td colspan="2">H^{n-1}</td><td colspan="2">H^n</td><td colspan="2">H^{n+1}</td><td>$\cdots$</td></tr>
<tr><td>H^{n-1}_{tors}</td><td>H^{n-1}_{free}</td><td>H^{n}_{tors}</td><td>H^{n}_{free}</td><td>H^{n+1}_{tors}</td><td>H^{n+1}_{free}</td><td>H^{n+2}_{tors}</td></tr>
<tr><td>$\cdots$</td><td colspan="2">$H^{(n-1)_+}$</td><td colspan="2">H^{n_+}</td><td colspan="2">$H^{(n+1)_+}$</td></tr>
<tr><td>$\cdots$</td><td>$\tau_{\leqslant n-1}$</td><td>$\tau_{\leqslant (n-1)_+}$</td><td>$\tau_{\leqslant n}$</td><td>$\tau_{\leqslant n_+}$</td><td>$\tau_{\leqslant n+1}$</td><td>$\cdots$</td></tr>
</table>

If $\mathcal{D}$ is an $\mathbb{O}$-linear t-category, then its heart $\mathcal{C}$ is also $\mathbb{O}$-linear. If $\mathcal{C}$ is Noetherian (resp. Artinian), then it is naturally endowed with a torsion theory by Proposition 2.14, and the preceding considerations apply.

Assume, for example, that $\mathcal{C}$ is Noetherian, endowed with the torsion theory $(\mathcal{T}, \mathcal{F})$, where $\mathcal{T}$ (resp. $\mathcal{F}$) is the full subcategory of torsion (resp. torsion-free) objects in $\mathcal{C}$. For L in $\mathcal{F}$, $\varpi 1_L$ is a monomorphism in $\mathcal{C}$, and we have a short exact sequence in $\mathcal{C}$

$$0 \longrightarrow L \xrightarrow{\varpi 1_L} L \longrightarrow \operatorname{Coker}_{\mathcal{C}} \varpi 1_L \longrightarrow 0.$$

Since $\mathcal{C}$ is an admissible abelian subcategory of $\mathcal{D}$, this short exact sequence comes from a distinguished triangle in $\mathcal{D}$

$$L \xrightarrow{\varpi 1_L} L \longrightarrow \operatorname{Coker}_{\mathcal{C}} \varpi 1_L \rightsquigarrow .$$

Rotating it (by the TR 2 axiom), we get a distinguished triangle

$$\operatorname{Coker}_{\mathcal{C}} \varpi 1_L[-1] \longrightarrow L \xrightarrow{\varpi 1_L} L \rightsquigarrow$$

all of whose objects are in $\mathcal{C}^+$. Since this abelian subcategory is also admissible, we have the following short exact sequence in $\mathcal{C}^+$

$$0 \longrightarrow \operatorname{Coker}_{\mathcal{C}} \varpi 1_L[-1] \longrightarrow L \xrightarrow{\varpi 1_L} L \longrightarrow 0$$

showing that $\varpi 1_L$ is an epimorphism in $\mathcal{C}^+$ (that is, L is divisible in $\mathcal{C}^+$), and that $\operatorname{Ker}_{\mathcal{C}^+} \varpi 1_L = \operatorname{Coker}_{\mathcal{C}} \varpi 1_L[-1]$.

Example 2.16. — Let us consider $\mathcal{D} = D^b_c(\mathbb{O})$, the full subcategory of the bounded derived category of $\mathbb{O}$-modules, whose objects are the complexes all of whose cohomology groups are finitely generated over $\mathbb{O}$. We can take the natural t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$. The heart $\mathcal{C}$ is then the abelian category of finitely generated $\mathbb{O}$-modules (we identify such a module with the corresponding complex concentrated in degree zero). The category $\mathcal{C}$ is Noetherian but not Artinian: the object $\mathbb{O}$ has an infinite decreasing sequence of subobjects $(\mathfrak{m}^n)$. In $\mathcal{C}$, it is a torsion-free object: $\varpi^n 1_{\mathbb{O}}$ is a monomorphism in $\mathcal{C}$, with cokernel $\mathbb{O}/\mathfrak{m}^n$.

Now, we can look at $\mathbb{O}$ as an object of the abelian category $\mathcal{C}^+$ obtained as above. Then $\mathbb{O}$ is a divisible object in $\mathcal{C}^+$: $\varpi^n 1_{\mathbb{O}}$ is an epimorphism, with kernel $\mathbb{O}/\mathfrak{m}^n[-1]$. This provides an infinite increasing sequence of subobjects of $\mathbb{O}$ in $\mathcal{C}^+$, showing that $\mathcal{C}^+$ is not Noetherian.

Remark 2.17. — The preceding example is just about perverse sheaves on a point, for the perversities p and p_+.

2.4. Recollement

The recollement (gluing) construction consists roughly in a way to construct a t-structure on some derived category of sheaves on a topological space (or a ringed topos) X, given t-structures on derived categories of sheaves on U and on F, where $j : U \to X$ is an open subset of X, and $i : F \to X$ its closed complement. This can be done in a very general axiomatic framework [1, §1.4], which can be applied to both the complex topology and the étale topology. The axioms can even be applied to non-topological situations, for example for representations of algebras. Let us recall the definitions and main properties of the recollement procedure.

So let $\mathcal{D}$, $\mathcal{D}_U$ and $\mathcal{D}_F$ be three triangulated categories, and let $i_* : \mathcal{D}_F \to \mathcal{D}$ and $j^* : \mathcal{D} \to \mathcal{D}_U$ be triangulated functors. It is convenient to set $i_! = i_*$ and $j^! = j^*$. We assume that the following conditions are satisfied.

Assumption 2.18 (Recollement situation). —

(i) *i_* has triangulated left and right adjoints, denoted by i^* and $i^!$ respectively.*
(ii) *j^* has triangulated left and right adjoints, denoted by $j_!$ and j_* respectively.*
(iii) *We have $j^* i_* = 0$. By adjunction, we also have $i^* j_! = 0$ and $i^! j_* = 0$. Moreover, for A in $\mathcal{D}_F$ and B in $\mathcal{D}_U$, we have*

$$\operatorname{Hom}(j_! B, i_* A) = 0 \text{ and } \operatorname{Hom}(i_* A, j_* B) = 0.$$

(iv) *For all K in $\mathcal{D}$, there exists $d : i_* i^* K \to j_! j^* K[1]$ (resp. $d : j_* j^* K \to i_* i^! K[1]$), necessarily unique, such that the triangle $j_! j^* K \to K \to i_* i^* K \xrightarrow{d}$ (resp. $i_* i^! K \to K \to j_* j^* K \xrightarrow{d}$) is distinguished.*
(v) *The functors i_*, $j_!$ and j_* are fully faithful: the adjunction morphisms $i^* i_* \to \mathrm{Id} \to i^! i_*$ and $j^* j_* \to \mathrm{Id} \to j^* j_!$ are isomorphisms.*

Whenever we have a diagram

$$\begin{tikzcd} \mathcal{D}_F \arrow[r, "i_*" description] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*" description] & \mathcal{D}_U \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{2.12}$$

such that the preceding conditions are satisfied, we say that we are in a situation of recollement.

Note that for each recollement situation, there is a dual recollement situation on the opposite triangulated categories. Recall that the opposite category of a triangulated category $\mathcal{T}$ is also triangulated, with translation functor $[-1]$, and distinguished triangles the triangles (Z, Y, X), where (X, Y, Z) is a distinguished triangle in $\mathcal{T}$. One can check that the conditions in Assumption 2.18 are satisfied for the following diagram, where the roles of i^* and $i^!$ (resp. $j_!$ and j_*) have been exchanged.

$$(2.13)\qquad \mathcal{D}_F^{\mathrm{op}} \underset{\underset{i^*}{\longleftarrow}}{\overset{\overset{i^!}{\longleftarrow}}{\xrightarrow{i_*}}} \mathcal{D}^{\mathrm{op}} \underset{\underset{j_!}{\longleftarrow}}{\overset{\overset{j_*}{\longleftarrow}}{\xrightarrow{j^*}}} \mathcal{D}_U^{\mathrm{op}}$$

We can say that there is a "formal duality" in the axioms of a recollement situation, exchanging the symbols ! and $*$. Note that, in the case of $D^b_c(X,\mathbb{E})$, the duality $\mathcal{D}_{X,\mathbb{E}}$ really exchanges these functors.

If $\mathcal{U} \xrightarrow{u} \mathcal{T} \xrightarrow{q} \mathcal{V}$ is a sequence of triangulated functors between triangulated categories such that u identifies $\mathcal{U}$ with a thick subcategory of $\mathcal{T}$, and q identifies $\mathcal{V}$ with the quotient of $\mathcal{T}$ by the thick subcategory $u(\mathcal{U})$, then we say that the sequence $0 \to \mathcal{U} \xrightarrow{u} \mathcal{T} \xrightarrow{q} \mathcal{V} \to 0$ is exact.

Proposition 2.19. — *The sequences*

$$0 \longleftarrow \mathcal{D}_{\mathcal{F}} \xleftarrow{i^*} \mathcal{D} \xleftarrow{j_!} \mathcal{D}_U \longleftarrow 0$$

$$0 \longrightarrow \mathcal{D}_{\mathcal{F}} \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{D}_U \longrightarrow 0$$

$$0 \longleftarrow \mathcal{D}_{\mathcal{F}} \xleftarrow{i^!} \mathcal{D} \xleftarrow{j_*} \mathcal{D}_U \longleftarrow 0$$

are exact.

Suppose we are given a t-structure $(\mathcal{D}_U^{\leqslant 0}, \mathcal{D}_U^{\geqslant 0})$ on $\mathcal{D}_U$, and a t-structure $(\mathcal{D}_F^{\leqslant 0}, \mathcal{D}_F^{\geqslant 0})$ on $\mathcal{D}_F$. Let us define

$$(2.14)\qquad \mathcal{D}^{\leqslant 0} := \{K \in \mathcal{D} \mid j^*K \in \mathcal{D}_U^{\leqslant 0} \text{ and } i^*K \in \mathcal{D}_F^{\leqslant 0}\}$$

$$(2.15)\qquad \mathcal{D}^{\geqslant 0} := \{K \in \mathcal{D} \mid j^*K \in \mathcal{D}_U^{\geqslant 0} \text{ and } i^!K \in \mathcal{D}_F^{\geqslant 0}\}.$$

Theorem 2.20. — *With the preceding notations, $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is a t-structure on $\mathcal{D}$.*

We say that it is obtained from those on $\mathcal{D}_U$ and $\mathcal{D}_F$ by *recollement* (gluing).

Now suppose we are just given a t-structure on $\mathcal{D}_F$. Then we can apply the recollement procedure to the degenerate t-structure $(\mathcal{D}_U, 0)$ on $\mathcal{D}_U$ and to the given t-structure on $\mathcal{D}_F$. The functors $\tau_{\leqslant n}$ $(n \in \mathbb{Z})$ relative to the t-structure obtained on $\mathcal{D}$ will be denoted $\tau^F_{\leqslant n}$. The functor $\tau^F_{\leqslant n}$ is right adjoint to the inclusion of the full subcategory of $\mathcal{D}$ whose objects are the X such that i^*X is in $\mathcal{D}_F^{\leqslant n}$. We have a distinguished triangle $(\tau^F_{\leqslant n} X, X, i_*\tau_{>n} i^* X)$. The H^n cohomology functors for this t-structure are the $i_* H^n i^*$. Thus we have a chain of morphisms:

$$(2.16)\qquad \cdots \longrightarrow \tau^F_{\leqslant n-1} \longrightarrow \tau^F_{\leqslant n} \longrightarrow \tau^F_{\leqslant n+1} \longrightarrow \cdots$$

and distinguished triangles:

$$\tau^F_{\leqslant n} \longrightarrow \tau^F_{\leqslant n+1} \longrightarrow i_* H^{n+1} i^*[-n-1] \rightsquigarrow \ . \tag{2.17}$$

We summarize this by the following diagram:

$\cdots$	$i_* H^{n-1} i^*$	$i_* H^n i^*$	$i_* H^{n+1} i^*$	$\cdots$
$\cdots$	$\tau^F_{\leqslant n-1}$	$\tau^F_{\leqslant n}$	$\tau^F_{\leqslant n+1}$	$\cdots$

One has to keep in mind, though, that this t-structure is degenerate, so an object should not be thought as "made of" its "successive quotients" $i_* H^n i^*$ (an object in $j_! \mathcal{D}_U$ will be in $\mathcal{D}^{\leqslant n}$ for all n).

Dually, one can define the functor $\tau^F_{\geqslant n}$ using the degenerate t-structure $(0, \mathcal{D}_U)$ on $\mathcal{D}_U$. It is left adjoint to the inclusion of $\{X \in \mathcal{D} \mid i^! X \in \mathcal{D}_F^{\geqslant n}\}$ in $\mathcal{D}$, we have distinguished triangles $(i_* \tau_{<n} i^! X, X, \tau^F_{\geqslant n} X)$, and the H^n are the $i_* H^n i^!$.

Similarly, if we are just given a t-structure on $\mathcal{D}_U$, and if we endow $\mathcal{D}_F$ with the degenerate t-structure $(\mathcal{D}_F, 0)$ (resp. $(0, \mathcal{D}_F)$), we can define a t-structure on $\mathcal{D}$ for which the functors $\tau_{\leqslant n}$ (resp. $\tau_{\geqslant n}$), denoted by $\tau^U_{\leqslant n}$ (resp. $\tau^U_{\geqslant n}$), yield distinguished triangles $(\tau^U_{\leqslant n}, X, j_* \tau_{>n} j^* X)$ (resp. $(j_! \tau_{<n} j^* X, X, \tau^U_{\geqslant n} X)$), and for which the H^n functors are the $j_* H^n j^*$ (resp. $j_! H^n j^*$).

Moreover, we have

$$\tau_{\leqslant n} = \tau^F_{\leqslant n} \tau^U_{\leqslant n} \quad \text{and} \quad \tau_{\geqslant n} = \tau^F_{\geqslant n} \tau^U_{\geqslant n}. \tag{2.18}$$

An *extension* of an object Y of $\mathcal{D}_U$ is an object X of $\mathcal{D}$ endowed with an isomorphism $j^* X \xrightarrow{\sim} Y$. Such an isomorphism induces morphisms $j_! Y \to X \to j_* Y$ by adjunction. If an extension X of Y is isomorphic, as an extension, to $\tau^F_{\geqslant n} j_! Y$ (resp. $\tau^F_{\leqslant n} j_* Y$), then the isomorphism is unique, and we just write $X = \tau^F_{\geqslant n} j_! Y$ (resp. $\tau^F_{\leqslant n} j_* Y$).

Proposition 2.21. — *Let Y in $\mathcal{D}_U$ and n an integer. There is, up to unique isomorphism, a unique extension X of Y such that $i^* X$ is in $\mathcal{D}_F^{\leqslant n-1}$ and $i^! X$ is in $\mathcal{D}_F^{\geqslant n+1}$. It is $\tau^F_{\leqslant n-1} j_* Y$, and this extension of Y is canonically isomorphic to $\tau^F_{\geqslant n+1} j_! Y$.*

Let $\mathcal{D}_m$ be the full subcategory of $\mathcal{D}$ consisting of the objects X such that $i^* X \in \mathcal{D}_F^{\leqslant n-1}$ and $i^! X \in \mathcal{D}_F^{\leqslant n+1}$. The functor j^* induces an equivalence $\mathcal{D}_m \to \mathcal{D}_U$, with quasi-inverse $\tau^F_{\leqslant n-1} j_* = \tau^F_{\geqslant n+1} j_!$, which will be denoted $j_{!*}$.

Let $\mathcal{C}$, $\mathcal{C}_U$ and $\mathcal{C}_F$ denote the hearts of the t-categories $\mathcal{D}$, $\mathcal{D}_U$ and $\mathcal{D}_F$. We will use the notation ${}^p T$ of Proposition 2.6, where T is one of the functors of the recollement diagram (2.12). By definition of the t-structure of $\mathcal{D}$, j^*

is t-exact, i^* is right t-exact, and $i^!$ is left t-exact. Applying Proposition 2.6, we get the first two points of the following proposition.

PROPOSITION 2.22. — *The functors ${}^p j_!$, ${}^p j^*$, ${}^p j_*$, ${}^p i^*$, ${}^p i_*$, ${}^p i^!$ have the following properties:*

(i) *The functor ${}^p i_*$ has left and right adjoints ${}^p i^*$ and ${}^p i^!$. Hence ${}^p i_*$ is exact, ${}^p i^*$ is right exact and ${}^p i^!$ is left exact.*

(ii) *The functor ${}^p j^*$ has left and right adjoints ${}^p j_!$ and ${}^p j_*$. Hence ${}^p j^*$ is exact, ${}^p j_!$ is right exact and ${}^p j_*$ is left exact.*

(iii) *The compositions ${}^p j^* {}^p i_*$, ${}^p i^* {}^p j_!$ and ${}^p i^! {}^p j_*$ are zero. For A in $\mathcal{C}_F$ and B in $\mathcal{C}_U$, we have*

$$\operatorname{Hom}({}^p j_! B, {}^p i_* A) = 0 \text{ and } \operatorname{Hom}({}^p i_* A, {}^p j_* B) = 0.$$

(iv) *For any object A in $\mathcal{C}$, we have exact sequences*

$$0 \longrightarrow {}^p i_* H^{-1} i^* A \longrightarrow {}^p j_! {}^p j^* A \longrightarrow A \longrightarrow {}^p i_* {}^p i^* A \longrightarrow 0 \tag{2.19}$$

$$0 \longrightarrow {}^p i_* {}^p i^! A \longrightarrow A \longrightarrow {}^p j_* {}^p j^* A \longrightarrow {}^p i_* H^1 i^! A \longrightarrow 0. \tag{2.20}$$

(v) *The functors ${}^p i_*$, ${}^p j_!$ and ${}^p j_*$ are fully faithful: the adjunction morphisms ${}^p i^* {}^p i_* \to \mathrm{Id} \to {}^p i^! {}^p i_*$ and ${}^p j^* {}^p j_* \to \mathrm{Id} \to {}^p j^* {}^p j_!$ are isomorphisms.*

(vi) *The essential image of the fully faithful functor ${}^p i_*$ is a thick subcategory of $\mathcal{C}$. For any object A in $\mathcal{C}$, ${}^p i_* {}^p i^* A$ is the largest quotient of A in ${}^p i_* \mathcal{C}_F$, and ${}^p i_* {}^p i^! A$ is the largest subobject of A in ${}^p i_* \mathcal{C}_F$.*

(vii) *The functor ${}^p j^*$ identifies $\mathcal{C}_U$ with the quotient of $\mathcal{C}$ by the thick subcategory ${}^p i_* \mathcal{C}_F$.*

Since j^* is a quotient functor of triangulated categories, the composition of the adjunction morphisms $j_! j^* \to \mathrm{Id} \to j_* j^*$ comes from a unique morphism of functors $j_! \to j_*$. Applying j^*, we get the identity automorphism of the identity functor.

Similarly, since the functor ${}^p j^*$ is a quotient functor of abelian categories, the composition of the adjunction morphisms ${}^p j_! {}^p j^* \to \mathrm{Id} \to {}^p j_* {}^p j^*$ comes from a unique morphism of functors ${}^p j_! \to {}^p j_*$. Applying ${}^p j^*$, we get the identity automorphism of the identity functor.

Let ${}^p j_{!*}$ be the image of ${}^p j_!$ in ${}^p j_*$. We have a factorization

$$j_! \longrightarrow {}^p j_! \longrightarrow {}^p j_{!*} \longrightarrow {}^p j_* \longrightarrow j_* \,. \tag{2.21}$$

The following characterization of the functors ${}^p j_!$, ${}^p j_{!*}$ and ${}^p j_*$ will be very useful.

PROPOSITION 2.23. — *We have*

$$ {}^p j_! = \tau^F_{\geqslant 0}\, j_! = \tau^F_{\leqslant -2}\, j_* \tag{2.22} $$

$$ {}^p j_{!*} = \tau^F_{\geqslant 1}\, j_! = \tau^F_{\leqslant -1}\, j_* \tag{2.23} $$

$$ {}^p j_* = \tau^F_{\geqslant 2}\, j_! = \tau^F_{\leqslant 0}\, j_*\,. \tag{2.24} $$

So (2.16) and (2.17) now read: we have a chain of morphisms:

$$ {}^p j_! \longrightarrow {}^p j_{!*} \longrightarrow {}^p j_* $$

and distinguished triangles:

$$ {}^p j_! \longrightarrow {}^p j_{!*} \longrightarrow i_* H^{-1} i^* j_*[1] \rightsquigarrow \tag{2.25} $$

$$ {}^p j_{!*} \longrightarrow {}^p j_* \longrightarrow i_* H^0 i^* j_* \rightsquigarrow\ . \tag{2.26} $$

In other words, for A in $\mathcal{C}$, the kernel and cokernel of ${}^p j_! A \to {}^p j_* A$ are in ${}^p i_* \mathcal{C}_F$, and we have the following Yoneda splice of two short exact sequences:

$$
\begin{tikzcd}[column sep=small]
0 \arrow[r] & i_* H^{-1} i^* j_* A \arrow[r] & {}^p j_! A \arrow[rr] \arrow[dr] & & {}^p j_* A \arrow[r] & i_* H^0 i^* j_* A \arrow[r] & 0 \\
& & & {}^p j_{!*} A \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0\,. & &
\end{tikzcd}
$$

COROLLARY 2.24. — *For A in $\mathcal{C}_U$, ${}^p j_{!*} A$ is the unique extension X of A in $\mathcal{D}$ such that $i^* X$ is in $\mathcal{D}_F^{\leqslant -1}$ and $i^! X$ is in $\mathcal{D}_F^{\geqslant 1}$. Thus it is the unique extension of A in $\mathcal{C}$ with no non-trivial subobject or quotient in ${}^p i_* \mathcal{C}_F$.*

Similarly, ${}^p j_! A$ (resp. ${}^p j_* A$) is the unique extension X of A in $\mathcal{D}$ such that $i^* X$ is in $\mathcal{D}_F^{\leqslant -2}$ (resp. $\mathcal{D}_F^{\leqslant 0}$) and $i^! X$ is in $\mathcal{D}_F^{\geqslant 0}$ (resp. $\mathcal{D}_F^{\geqslant 2}$). In particular, ${}^p j_! A$ (resp. ${}^p j_* A$) has no non-trivial quotient (resp. subobject) in ${}^p i_* \mathcal{C}_F$.

Building on the preceding results, it is now easy to get the following description of the simple objects in $\mathcal{C}$.

PROPOSITION 2.25. — *The simple objects in $\mathcal{C}$ are the ${}^p i_* S$, with S simple in $\mathcal{C}_F$, and the ${}^p j_{!*} S$, for S simple in $\mathcal{C}_U$.*

Let $\mathcal{S}$ (resp. $\mathcal{S}_U$, $\mathcal{S}_F$) denote the set of (isomorphisms classes of) simple objects in $\mathcal{C}$ (resp. $\mathcal{C}_U$, $\mathcal{C}_F$). So we have $\mathcal{S} = {}^p j_{!*} \mathcal{S}_U \cup {}^p i_* \mathcal{S}_F$. Let us assume that $\mathcal{C}$, $\mathcal{C}_U$ and $\mathcal{C}_F$ are Noetherian and Artinian, so that the multiplicities of the simple objects and the notion of composition length are well-defined. Thus, if B is an object in $\mathcal{C}$, then we have the following relation in the Grothendieck group $K_0(\mathcal{C})$:

$$ [B] = \sum_{T \in \mathcal{S}} [B : T] \cdot [T]. \tag{2.27} $$

We will now show that ${}^p j_{!*}$ preserves multiplicities.

PROPOSITION 2.26. — *If B is an object in $\mathcal{C}$, then we have*

$$[B : {}^p j_{!*} S] = [j^* B : S] \tag{2.28}$$

for all simple objects S in $\mathcal{C}_U$. In particular, if A is an object in $\mathcal{C}_U$, then we have

$$[{}^p j_! A : {}^p j_{!*} S] = [{}^p j_{!*} A : {}^p j_{!*} S] = [{}^p j_* A : {}^p j_{!*} S] = [A : S]. \tag{2.29}$$

Proof. — The functor j^* is exact, and sends a simple object T on a simple a simple object if $T \in {}^p j_{!*} \mathcal{S}_U$, or on zero if $T \in {}^p i_* \mathcal{S}_F$. Moreover, it sends non-isomorphic simple objects in ${}^p j_{!*} \mathcal{S}_U$ on non-isomorphic simple objects in $\mathcal{S}_U$. Thus, applying j^* to the relation (2.27), we get

$$[j^* B] = \sum_{S \in \mathcal{S}_U} [j^* B : S] \cdot [S] = \sum_{S \in \mathcal{S}_U} [B : {}^p j_{!*} S] \cdot [S]$$

hence (2.28), and (2.29) follows. □

PROPOSITION 2.27. — *The functor ${}^p j_{!*}$ preserves monomorphisms and epimorphisms.*

Proof. — Let $u : A \to B$ be a monomorphism in $\mathcal{C}_U$. Let K be the kernel of the morphism ${}^p j_{!*} u : {}^p j_{!*} A \to {}^p j_{!*} B$ in $\mathcal{C}$. Since this morphism becomes a monomorphism after applying ${}^p j^*$ (restriction to U), K is in ${}^p i_* \mathcal{C}_F$. But K is a subobject of ${}^p j_{!*} A$, which has no non-trivial subobject in ${}^p i_* \mathcal{C}_F$. Hence $K = 0$ and ${}^p j_{!*} u$ is a monomorphism.

Dually, let $v : A \to B$ be an epimorphism in $\mathcal{C}_U$. Let C be the cokernel of the morphism ${}^p j_{!*} v : {}^p j_{!*} A \to {}^p j_{!*} B$ in $\mathcal{C}$. Since this morphism becomes an epimorphism after applying ${}^p j^*$ (restriction to U), C is in ${}^p i_* \mathcal{C}_F$. But C is a quotient of ${}^p j_{!*} B$, which has no non-trivial quotient in ${}^p i_* \mathcal{C}_F$. Hence $C = 0$ and ${}^p j_{!*} v$ is an epimorphism. □

PROPOSITION 2.28. — *Let A be an object of $\mathcal{C}_U$. Then we have*

$$\operatorname{Soc} {}^p j_{!*} A \simeq \operatorname{Soc} {}^p j_* A \simeq {}^p j_{!*} \operatorname{Soc} A$$
$$\operatorname{Top} {}^p j_! A \simeq \operatorname{Top} {}^p j_{!*} A \simeq {}^p j_{!*} \operatorname{Top} A.$$

Proof. — By definition, ${}^p j_{!*} A$ is a subobject of ${}^p j_* A$. Taking socles, we get $\operatorname{Soc} {}^p j_{!*} A \subset \operatorname{Soc} {}^p j_* A$ as subobjects of ${}^p j_* A$.

By applying the exact functor ${}^p j^*$ to the monomorphism $\operatorname{Soc} {}^p j_* A \subset {}^p j_* A$, we get a monomorphism ${}^p j^* \operatorname{Soc} {}^p j_* A \subset A$. But ${}^p j^* \operatorname{Soc} {}^p j_* A$ is semisimple, so we get ${}^p j^* \operatorname{Soc} {}^p j_* A \subset \operatorname{Soc} A$ as subobjects of A. Thus, by Proposition 2.27, we have ${}^p j_{!*} {}^p j^* \operatorname{Soc} {}^p j_* A \subset {}^p j_{!*} \operatorname{Soc} A$ as subobjects of ${}^p j_* A$. Now, we have ${}^p j_{!*} {}^p j^* \operatorname{Soc} {}^p j_* A = \operatorname{Soc} {}^p j_* A$ because $\operatorname{Soc} {}^p j_* A$ is a semisimple object with no simple constituent in ${}^p i_* \mathcal{C}_F$. Hence $\operatorname{Soc} {}^p j_* A \subset {}^p j_{!*} \operatorname{Soc} A$ as subobjects of ${}^p j_* A$.

By Proposition 2.27, if we apply the functor ${}^p j_{!*}$ to the monomorphism $\operatorname{Soc} A \subset A$, we get a monomorphism ${}^p j_{!*} \operatorname{Soc} A \subset {}^p j_{!*} A$. But ${}^p j_{!*} \operatorname{Soc} A$ is semisimple, so we get ${}^p j_{!*} \operatorname{Soc} A \subset \operatorname{Soc} {}^p j_{!*} A$ as subobjects of ${}^p j_* A$.

This proves the first relation, and the second one is dual. □

Proposition 2.29. — *The functor ${}^p j_{!*}$ is fully faithful.*

Proof. — Let A and B be two objects in $\mathcal{C}_U$. Applying the left exact functor $\operatorname{Hom}_{\mathcal{C}}(-, {}^p j_{!*} B)$ to the short exact sequence

$$0 \longrightarrow {}^p i_* {}^p i^! A \longrightarrow {}^p j_! A \longrightarrow {}^p j_{!*} A \longrightarrow 0$$

we get an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{C}}({}^p j_{!*} A, {}^p j_{!*} B) \longrightarrow \operatorname{Hom}_{\mathcal{C}}({}^p j_! A, {}^p j_{!*} B) \longrightarrow \operatorname{Hom}_{\mathcal{C}}({}^p i_* {}^p i^! A, {}^p j_{!*} B).$$

Since ${}^p j_{!*} B$ has no non-trivial subobject in ${}^p i_* \mathcal{C}_F$, we deduce that

$$\operatorname{Hom}_{\mathcal{C}}({}^p i_* {}^p i^! A, {}^p j_{!*} B) = 0$$

and thus we have

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}({}^p j_{!*} A, {}^p j_{!*} B) &\simeq \operatorname{Hom}_{\mathcal{C}}({}^p j_! A, {}^p j_{!*} B) \\ &\simeq \operatorname{Hom}_{\mathcal{C}_U}(A, {}^p j^* {}^p j_{!*} B) \simeq \operatorname{Hom}_{\mathcal{C}_U}(A, B) \end{aligned}$$

using Proposition 2.22 (ii) and the fact that ${}^p j_{!*} B$ is an extension of B. Thus the functor ${}^p j_{!*}$ is fully faithful. □

2.5. Torsion theories and recollement

We will see now how to glue torsion theories in the recollement procedure.

Proposition 2.30. — *Suppose we are in a recollement situation as in Subsection 2.4, and that we are given torsion theories $(\mathcal{T}_F, \mathcal{F}_F)$ and $(\mathcal{T}_U, \mathcal{F}_U)$ of $\mathcal{C}_F$ and $\mathcal{C}_U$. Then we can define a torsion theory $(\mathcal{T}, \mathcal{F})$ on $\mathcal{C}$ by*

$$\mathcal{T} = \{T \in \mathcal{C} \mid {}^p i^* T \in \mathcal{T}_F \text{ and } {}^p j^* T \in \mathcal{T}_U\} \tag{2.30}$$

$$\mathcal{F} = \{L \in \mathcal{C} \mid {}^p i^! L \in \mathcal{F}_F \text{ and } {}^p j^* L \in \mathcal{F}_U\}. \tag{2.31}$$

Let us begin by some lemmas.

Lemma 2.31. — *The subcategory $\mathcal{T}$ (resp. $\mathcal{F}$) of $\mathcal{C}$ is closed under quotients and extensions (resp. under subobjects and extensions).*

Proof. — Let us consider a short exact sequence in $\mathcal{C}$

$$0 \longrightarrow S \longrightarrow A \longrightarrow Q \longrightarrow 0.$$

Applying the functors ${}^p i^*$, ${}^p j^*$ and ${}^p i^!$, we get three exact sequences:

$$ {}^p i^* S \longrightarrow {}^p i^* A \longrightarrow {}^p i^* Q \longrightarrow 0 \tag{2.32}$$

$$0 \longrightarrow {}^p j^* S \longrightarrow {}^p j^* A \longrightarrow {}^p j^* Q \longrightarrow 0 \tag{2.33}$$

$$0 \longrightarrow {}^p i^! S \longrightarrow {}^p i^! A \longrightarrow {}^p i^! Q. \tag{2.34}$$

Let us first assume that A is in $\mathcal{T}$, and let us show that Q is also in $\mathcal{T}$. We have to show that ${}^p i^* Q$ is in $\mathcal{T}_F$ and that ${}^p j^* Q$ is in $\mathcal{T}_U$. This follows from Proposition 2.8, since ${}^p i^* Q$ is a quotient of ${}^p i^* A$ and ${}^p j^* Q$ is quotient of ${}^p j^* A$.

Secondly, suppose that S and Q are in $\mathcal{T}$, and let us show that A is also in $\mathcal{T}$. We have to show that ${}^p i^* A$ is in $\mathcal{T}_F$ and that ${}^p j^* A$ is in $\mathcal{T}_U$. This follows also from Proposition 2.8, since ${}^p i^* A$ is an extension of ${}^p i^* Q$ by a quotient of ${}^p i^* S$, and ${}^p j^* A$ is an extension of ${}^p j^* Q$ by ${}^p j^* S$.

The proofs for the statements about $\mathcal{F}$ are dual. □

Lemma 2.32. — *We have*

$$\begin{array}{ccc} {}^p i_*(\mathcal{T}_F) \subset \mathcal{T} & {}^p j_!(\mathcal{T}_U) \subset \mathcal{T} & {}^p j_{!*}(\mathcal{T}_U) \subset \mathcal{T} \\ {}^p i_*(\mathcal{F}_F) \subset \mathcal{F} & {}^p j_*(\mathcal{F}_U) \subset \mathcal{F} & {}^p j_{!*}(\mathcal{F}_U) \subset \mathcal{F}. \end{array}$$

Proof. — This follows from Proposition 2.22 (iii) and (v), the definition of $(\mathcal{T}, \mathcal{F})$, the definition of ${}^p j_{!*}$, and Lemma 2.31. □

Lemma 2.33. — *If $T \in \mathcal{T}$ and $L \in \mathcal{F}$, then we have* $\mathrm{Hom}_{\mathcal{C}}(T, L) = 0$.

Proof. — By Proposition 2.22 (iv), we have an exact sequence (2.19)

$${}^p j_! {}^p j^* T \longrightarrow T \longrightarrow {}^p i_* {}^p i^* T \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_{\mathcal{C}}(-, L)$, which is left exact, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}({}^p i_* {}^p i^* T, L) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(T, L) \longrightarrow \mathrm{Hom}_{\mathcal{C}}({}^p j_! {}^p j^* T, L).$$

By the adjunctions of Proposition 2.22 (i) and (ii), this becomes

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}_F}({}^p i^* T, {}^p i^! L) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(T, L) \longrightarrow \mathrm{Hom}_{\mathcal{C}_U}({}^p j^* T, {}^p j^* L).$$

Now, we have $\mathrm{Hom}_{\mathcal{C}_F}({}^p i^* T, {}^p i^! L) = 0$ because ${}^p i^* T \in \mathcal{T}_F$ and ${}^p i^! L \in \mathcal{F}_F$, and similarly $\mathrm{Hom}_{\mathcal{C}_U}({}^p j^* T, {}^p j^* L) = 0$ because ${}^p j^* T \in \mathcal{T}_U$ and ${}^p j^* L \in \mathcal{F}_U$. Thus $\mathrm{Hom}_{\mathcal{C}}(T, L) = 0$. □

We are now ready to prove the Proposition. To check the second axiom for torsion theories, the idea is the following: given an object A, of $\mathcal{C}$, we construct a filtration $0 \subset S \subset B \subset A$ where S is in $\mathcal{T}$, A/B is in $\mathcal{F}$, and $M := B/S$ is in ${}^p i_* \mathcal{C}_F$. Then we use the torsion theory on $\mathcal{C}_F$ to cut M into a torsion part and a torsion-free part. Taking the inverse image in B of the torsion part of M, we get the torsion subobject of A. Now let us give the details.

Proof of Proposition 2.30. — The first axiom for torsion theories has been checked in Lemma 2.33.

Secondly, given A in $\mathcal{C}$, we have to find T in $\mathcal{T}$ and L in $\mathcal{F}$ such that we have a short exact sequence $0 \to T \to A \to L \to 0$.

Since $(\mathcal{T}_U, \mathcal{F}_U)$ is a torsion theory on $\mathcal{C}_U$, we have a short exact sequence

$$0 \longrightarrow ({}^p j^* A)_{\mathrm{tors}} \longrightarrow {}^p j^* A \longrightarrow ({}^p j^* A)_{\mathrm{free}} \longrightarrow 0. \tag{2.35}$$

By adjunction, we have morphisms

$${}^p j_! ({}^p j^* A)_{\mathrm{tors}} \xrightarrow{f} A \xrightarrow{g} {}^p j_* ({}^p j^* A)_{\mathrm{free}}$$

and the morphisms of (2.35) are ${}^p j^* f$ and ${}^p j^* g$. Let S and Q denote the images of f and g. We have canonical factorizations

$$\begin{tikzcd} {}^p j_! ({}^p j^* A)_{\mathrm{tors}} \arrow[rr, "f"] \arrow[dr, "q_S"'] & & A \arrow[rr, "g"] \arrow[dr, "q_Q"'] & & {}^p j_* ({}^p j^* A)_{\mathrm{free}} \\ & S \arrow[ur, hook, "i_S"'] & & Q \arrow[ur, hook, "i_Q"'] & \end{tikzcd}$$

By Lemma 2.32, since $({}^p j^* A)_{\mathrm{tors}}$ is in $\mathcal{T}_U$, the object ${}^p j_! ({}^p j^* A)_{\mathrm{tors}}$ is in $\mathcal{T}$, so by Lemma 2.31, its quotient S is also in $\mathcal{T}$. Similarly, ${}^p j_* ({}^p j^* A)_{\mathrm{free}}$ is in $\mathcal{F}$ so its subobject Q is also in $\mathcal{F}$. By Lemma 2.33, it follows that $\mathrm{Hom}_{\mathcal{C}}(S, Q) = 0$. Thus $q_Q i_S = 0$, and i_S factors through the kernel $b : B \hookrightarrow A$ of $q_Q : A \twoheadrightarrow Q$ as $i_S = b\iota$, for some monomorphism $\iota : S \hookrightarrow B$, and we can identify S with a subobject of B. Now let $M = B/S$, and let $\pi : B \twoheadrightarrow M$ be the canonical quotient morphism.

The morphism ${}^p j^* f$ is a monomorphism, hence ${}^p j^* q_S$, which is an epimorphism since ${}^p j^*$ is exact, is actually an isomorphism. Similarly, ${}^p j^* i_Q$ is an isomorphism. Thus ${}^p j^* b$ is the kernel of ${}^p j^* g$, and ${}^p j^* \iota$ is an isomorphism as well. Applying ${}^p j^*$ to the short exact sequence

$$0 \longrightarrow S \longrightarrow B \longrightarrow M \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow {}^p j^* S \longrightarrow {}^p j^* B \longrightarrow {}^p j^* M \longrightarrow 0$$

where the first morphism is an isomorphism, hence ${}^p j^* M = 0$, and M is in ${}^p i_* \mathcal{C}_F$. We have a short exact sequence

$$0 \longrightarrow ({}^p i^* M)_{\mathrm{tors}} \longrightarrow {}^p i^* M \longrightarrow ({}^p i^* M)_{\mathrm{free}} \longrightarrow 0.$$

Applying the exact functor ${}^p i_*$, we get a short exact sequence

$$0 \longrightarrow {}^p i_* ({}^p i^* M)_{\mathrm{tors}} \longrightarrow M \longrightarrow {}^p i_* ({}^p i^* M)_{\mathrm{free}} \longrightarrow 0$$

and, by Lemma 2.32, ${}^p i_* ({}^p i^* M)_{\mathrm{tors}}$ is in $\mathcal{T}$ and ${}^p i_* ({}^p i^* M)_{\mathrm{free}}$ is in $\mathcal{F}$.

Let T denote the inverse image $\pi^{-1}({}^p i_* ({}^p i^* M)_{\mathrm{tors}})$ in B (recall that $\pi : B \to M = B/S$ is the quotient morphism), and let $L = A/T$.

We have a filtration $0 \subset S \subset T \subset B \subset A$ of A, and the following short exact sequences:

$$0 \longrightarrow S \longrightarrow T \longrightarrow {}^p i_* ({}^p i^* M)_{\mathrm{tors}} \longrightarrow 0$$

which shows that T is in $\mathcal{T}$ by Lemma 2.31, and

$$0 \longrightarrow {}^p i_* ({}^p i^* M)_{\mathrm{free}} \longrightarrow L \longrightarrow Q \longrightarrow 0$$

which shows that L is in $\mathcal{F}$ (by the same lemma), and

$$0 \longrightarrow T \longrightarrow A \longrightarrow L \longrightarrow 0$$

which completes the proof. □

Using these torsion theories on $\mathcal{C}$, $\mathcal{C}_F$ and $\mathcal{C}_U$, one can define, as in Subsection 2.3, new t-structures on $\mathcal{D}$, $\mathcal{D}_F$ and $\mathcal{D}_U$, denoted with the superscript p_+. Then the t-structure for p_+ on $\mathcal{D}$ is obtained by recollement from the t-structures for p_+ on $\mathcal{D}_F$ and $\mathcal{D}_U$.

Moreover, we have six interesting functors from $\mathcal{C}_U \cap \mathcal{C}_U^+$ to $\mathcal{D}$:

$$\begin{aligned}
&(2.36) & {}^p j_! &= {}^p\tau^F_{\leqslant -2}\, j_* = {}^p\tau^F_{\geqslant 0}\, j_! \\
&(2.37) & {}^{p_+} j_! &= {}^p\tau^F_{\leqslant -2_+}\, j_* = {}^p\tau^F_{\geqslant 0_+}\, j_! \\
&(2.38) & {}^p j_{!*} &= {}^p\tau^F_{\leqslant -1}\, j_* = {}^p\tau^F_{\geqslant 1}\, j_! \\
&(2.39) & {}^{p_+} j_{!*} &= {}^p\tau^F_{\leqslant -1_+}\, j_* = {}^p\tau^F_{\geqslant 1_+}\, j_! \\
&(2.40) & {}^p j_* &= {}^p\tau^F_{\leqslant 0}\, j_* = {}^p\tau^F_{\geqslant 2}\, j_! \\
&(2.41) & {}^{p_+} j_* &= {}^p\tau^F_{\leqslant 0_+}\, j_* = {}^p\tau^F_{\geqslant 2_+}\, j_!\, .
\end{aligned}$$

The first of these functors has image in $\mathcal{C}$, the last one in $\mathcal{C}^+$, and the other four in $\mathcal{C} \cap \mathcal{C}^+$.

We have a chain of morphisms

$${}^p j_! \longrightarrow {}^{p_+} j_! \longrightarrow {}^p j_{!*} \longrightarrow {}^{p_+} j_{!*} \longrightarrow {}^p j_* \longrightarrow {}^{p_+} j_*$$

and distinguished triangles:

$$(2.42)\qquad {}^p j_! \longrightarrow {}^{p_+} j_! \longrightarrow {}^p i_* {}^p H^{-1}_{\mathrm{tors}} i^* j_* [1] \rightsquigarrow$$

$$(2.43)\qquad {}^{p_+} j_! \longrightarrow {}^p j_{!*} \longrightarrow {}^p i_* {}^p H^{-1}_{\mathrm{free}} i^* j_* [1] \rightsquigarrow$$

$$(2.44)\qquad {}^p j_{!*} \longrightarrow {}^{p_+} j_{!*} \longrightarrow {}^p i_* {}^p H^{0}_{\mathrm{tors}} i^* j_* \rightsquigarrow$$

$$(2.45)\qquad {}^{p_+} j_{!*} \longrightarrow {}^p j_* \longrightarrow {}^p i_* {}^p H^{0}_{\mathrm{free}} i^* j_* \rightsquigarrow$$

$$(2.46)\qquad {}^p j_* \longrightarrow {}^{p_+} j_* \longrightarrow {}^p i_* {}^p H^{1}_{\mathrm{tors}} i^* j_* [-1] \rightsquigarrow$$

summarized by:

$\cdots$	${}^p i_* {}^p H^{-1}_{\mathrm{tors}} i^* j_*$	${}^p i_* {}^p H^{-1}_{\mathrm{free}} i^* j_*$	${}^p i_* {}^p H^{0}_{\mathrm{tors}} i^* j_*$	${}^p i_* {}^p H^{0}_{\mathrm{free}} i^* j_*$	${}^p i_* {}^p H^{1}_{\mathrm{tors}} i^* j_*$
${}^p j_!$	${}^{p_+} j_!$	${}^p j_{!*}$	${}^{p_+} j_{!*}$	${}^p j_*$	${}^{p_+} j_*$

2.6. Perverse t-structures

Let us go back to the setting of 2.1. We want to define the t-structure defining the $\mathbb{E}$-perverse sheaves on X for the middle perversity p (and, in case $\mathbb{E} = \mathbb{O}$, also for the perversity p_+), following [1]. Let us start with the case $\mathbb{E} = \mathbb{F}$. We will consider pairs $(\mathfrak{X}, \mathfrak{L})$ satisfying the following conditions:

Assumption 2.34. —

(i) *$\mathfrak{X}$ is a partition of X into finitely many locally closed smooth pieces, called strata, and the closure of a stratum is a union of strata.*

(ii) *$\mathfrak{L}$ consists in the following data: for each stratum S in $\mathfrak{X}$, a finite set $\mathfrak{L}(S)$ of isomorphism classes of irreducible locally constant sheaves of $\mathbb{F}$-modules over S.*

(iii) *For each S in $\mathfrak{X}$ and for each $\mathcal{F}$ in $\mathcal{L}(S)$, if j denotes the inclusion of S into X, then the $R^n j_* \mathcal{F}$ are $(\mathfrak{X}, \mathfrak{L})$-constructible, with the definition below.*

A sheaf of $\mathbb{F}$-modules is $(\mathfrak{X}, \mathfrak{L})$-constructible if and only if its restriction to each stratum S in $\mathfrak{X}$ is locally constant and a finite iterated extension of irreducible locally constant sheaves whose isomorphism class is in $\mathfrak{L}(S)$. We denote by $D^b_{\mathfrak{X},\mathfrak{L}}(X, \mathbb{F})$ the full subcategory of $D^b(X, \mathbb{F})$ consisting of the $(\mathfrak{X}, \mathfrak{L})$-constructible complexes, that is, whose cohomology sheaves are $(\mathfrak{X}, \mathfrak{L})$-constructible.

We say that $(\mathfrak{X}', \mathfrak{L}')$ refines $(\mathfrak{X}, \mathfrak{L})$ if each stratum S in $\mathfrak{X}$ is a union of strata in $\mathfrak{X}'$, and all the $\mathcal{F}$ in $\mathfrak{L}(S)$ are $(\mathfrak{X}', \mathfrak{L}')$-constructible, that is, $(\mathfrak{X}'_{|S}, \mathfrak{L}_{|\mathfrak{X}'_{|S}})$-constructible.

The condition (iii) ensures that for $U \overset{j}{\hookrightarrow} V \subset X$ locally closed and unions of strata, the functors j_*, $j_!$ (resp. j^*, $j^!$) send $D^b_{\mathfrak{X},\mathfrak{L}}(U,\mathbb{F})$ into $D^b_{\mathfrak{X},\mathfrak{L}}(V,\mathbb{F})$ (resp. $D^b_{\mathfrak{X},\mathfrak{L}}(V,\mathbb{F})$ into $D^b_{\mathfrak{X},\mathfrak{L}}(U,\mathbb{F})$). It follows from the constructibility theorem for j_* (SGA $4\frac{1}{2}$) that any pair $(\mathfrak{X}',\mathfrak{L}')$ satisfying (i) and (ii) can be refined into a pair $(\mathfrak{X},\mathfrak{L})$ satisfying (i), (ii) and (iii) (see [1, §2.2.10]).

So let us fix a pair $(\mathfrak{X},\mathfrak{L})$ as above. Then we define the full subcategories ${}^pD^{\leqslant 0}_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{F})$ and ${}^pD^{\geqslant 0}_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{F})$ of $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{F})$ by

$$A \in {}^pD^{\leqslant 0}_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{F}) \Longleftrightarrow \forall S \in \mathfrak{X},\ \forall n > -\dim S,\ \mathcal{H}^n i_S^* A = 0$$
$$A \in {}^pD^{\geqslant 0}_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{F}) \Longleftrightarrow \forall S \in \mathfrak{X},\ \forall n < -\dim S,\ \mathcal{H}^n i_S^! A = 0$$

for any A in $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{F})$, where i_S is the inclusion of the stratum S.

One can show by induction on the number of strata that this defines a t-structure on $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{F})$, by repeated applications of Theorem 2.20. On a stratum, we consider the natural t-structure shifted by $\dim S$, and we glue these t-structures successively.

The t-structure on $D^b_{\mathfrak{X}',\mathfrak{L}'}(X,\mathbb{F})$ for a finer pair $(\mathfrak{X},\mathfrak{L})$ induces the same t-structure on $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{F})$, so passing to the limit we obtain a t-structure on $D^b_c(X,\mathbb{F})$, which is described by the conditions (2.2) and (2.3) of Subsection 2.1.

Over $\mathbb{O}/\varpi^n$, we proceed similarly. An object K of $D^b_c(X,\mathbb{O}/\varpi^n)$ is $(\mathfrak{X},\mathfrak{L})$-constructible if all the $\varpi^i\mathcal{H}^j K/\varpi^{i+1}\mathcal{H}^j K$ are $(\mathfrak{X},\mathfrak{L})$-constructible as $\mathbb{F}$-sheaves.

Over $\mathbb{O}$, since our field k is finite or algebraically closed, we can use Deligne's definition of $D^b_c(X,\mathbb{O})$ as the projective 2-limit of the triangulated categories $D^b_c(X,\mathbb{O}/\varpi^n)$. The assumption insures that it is triangulated. We have triangulated functors $\mathbb{O}/\varpi^n \otimes^{\mathbb{L}}_{\mathbb{O}} (-) : D^b_c(X,\mathbb{O}) \to D^b_c(X,\mathbb{O}/\varpi^n)$, and in particular $\mathbb{F} \otimes^{\mathbb{L}}_{\mathbb{O}} (-)$. We will often omit from the notation $\otimes^{\mathbb{L}}_{\mathbb{O}}$ and simply write $\mathbb{F}(-)$. The functor $\mathcal{H}^i : D^b_c(X,\mathbb{O}) \to \mathrm{Sh}(X,\mathbb{O})$ is defined by sending an object K to the projective system of the $\mathcal{H}^i(\mathbb{O}/\varpi^n \otimes^{\mathbb{L}}_{\mathbb{O}} K)$. We have exact sequences:

$$0 \longrightarrow \mathbb{O}/\varpi^n \otimes_{\mathbb{O}} \mathcal{H}^i(K) \longrightarrow \mathcal{H}^i(\mathbb{O}/\varpi^n \otimes^{\mathbb{L}}_{\mathbb{O}} K) \longrightarrow \mathrm{Tor}_1^{\mathbb{O}}(\mathbb{O}/\varpi^n, \mathcal{H}^{i+1}(K)) \longrightarrow 0.$$

Let $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{O})$ be the full subcategory of $D^b_c(X,\mathbb{O})$ consisting of the objects K such that for some (or any) n, $\mathbb{O}/\varpi^n \otimes^{\mathbb{L}}_{\mathbb{O}} K$ is in $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{O}/\varpi^n)$, or equivalently, such that the $\mathbb{F} \otimes_{\mathbb{O}} \mathcal{H}^i K$ are $(\mathfrak{X},\mathfrak{L})$-constructible. We define the t-structure for the perversity p on $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{O})$ as above. Its heart is the abelian category ${}^p\mathcal{M}_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{O})$. Since it is $\mathbb{O}$-linear, it is endowed with a natural torsion theory, and we can define another t-structure as in Subsection 2.3, and we will say that it is associated to the perversity p_+.

By Subsection 2.5, it can also be obtained by recollement. Passing to the limit, we get two t-structures on $D^b_c(X,\mathbb{O})$, for the perversities p and p_+, which can be characterized by the conditions (2.2), (2.3), (2.4) and (2.5) of Subsection 2.1.

An object A of $D^b_c(X,\mathbb{O})$ is in ${}^pD^{\leqslant 0}(X,\mathbb{O})$ (resp. ${}^{p_+}D^{\geqslant 0}(X,\mathbb{O})$) if and only if $\mathbb{F}A$ is in ${}^pD^{\leqslant 0}(X,\mathbb{F})$ (resp. ${}^pD^{\geqslant 0}(X,\mathbb{F})$).

If A is an object in ${}^p\mathcal{M}(X,\mathbb{O})$, then $\mathbb{F}A$ is in ${}^p\mathcal{M}(X,\mathbb{F})$ if and only if A is torsion-free (that is, if and only if A is also p_+-perverse). Then we have $\mathbb{F}A = \operatorname{Coker}\varpi.1_A$ (the cokernel being taken in ${}^p\mathcal{M}(X,\mathbb{O})$).

Similarly, if A is an object in ${}^{p_+}\mathcal{M}(X,\mathbb{O})$, then $\mathbb{F}A$ is in ${}^p\mathcal{M}(X,\mathbb{F})$ if and only if A is divisible (that is, if and only if A is also p-perverse). Then we have $\mathbb{F}A = \operatorname{Ker}\varpi.1_A[1]$ (the kernel being taken in ${}^{p_+}\mathcal{M}(X,\mathbb{O})$).

To pass from $\mathbb{O}$ to $\mathbb{K}$, we simply apply $\mathbb{K}\otimes_{\mathbb{O}}(-)$. Thus $D^b_c(X,\mathbb{K})$ is the category with the same objects as $D^b_c(X,\mathbb{O})$, and morphisms

$$\operatorname{Hom}_{D^b_c(X,\mathbb{K})}(A,B) = \mathbb{K}\otimes_{\mathbb{O}} \operatorname{Hom}_{D^b_c(X,\mathbb{O})}(A,B).$$

We write $D^b_c(X,\mathbb{K}) = \mathbb{K}\otimes_{\mathbb{O}} D^b_c(X,\mathbb{O})$. We also have $\operatorname{Sh}(X,\mathbb{K}) = \mathbb{K}\otimes_{\mathbb{O}} \operatorname{Sh}(X,\mathbb{O})$. Then we define the full subcategory $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{K})$ of $D^b_c(X,\mathbb{K})$ as the image of $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{O})$. The t-structures p and p_+ on $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{O})$ give rise to a single t-structure p on $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{K})$, because torsion objects are killed by $\mathbb{K}\otimes_{\mathbb{O}}(-)$. This perverse t-structure can be defined by recollement. Passing to the limit, we get the perverse t-structure on $D^b_c(X,\mathbb{K})$ defined by (2.2) and (2.3). We have ${}^p\mathcal{M}(X,\mathbb{K}) = \mathbb{K}\otimes_{\mathbb{O}} {}^p\mathcal{M}(X,\mathbb{O})$.

2.7. Modular reduction and truncation functors

Modular reduction does not commute with truncation functors. We will now study the failure of commutativity between these functors. Recall that, to simplify the notation, we write $\mathbb{F}(-)$ for $\mathbb{F}\otimes^{\mathbb{L}}_{\mathbb{O}}(-)$.

Proposition 2.35. — *For $A\in D^b_c(X,\mathbb{O})$ and $n\in\mathbb{Z}$, we have distinguished triangles:*

$$\mathbb{F}\,\tau_{\leqslant n}A \longrightarrow \tau_{\leqslant n}\mathbb{F}A \longrightarrow \mathcal{H}^{-1}(\mathbb{F}\,\mathcal{H}^{n+1}_{\mathrm{tors}}A)[-n] \rightsquigarrow \tag{2.47}$$

$$\tau_{\leqslant n}\mathbb{F}A \longrightarrow \mathbb{F}\,\tau_{\leqslant n_+}A \longrightarrow \mathcal{H}^{0}(\mathbb{F}\,\mathcal{H}^{n+1}_{\mathrm{tors}}A)[-n-1] \rightsquigarrow \tag{2.48}$$

$$\mathbb{F}\,\tau_{\leqslant n_+}A \longrightarrow \mathbb{F}\,\tau_{\leqslant n+1}A \longrightarrow \mathbb{F}\,\mathcal{H}^{n+1}_{\mathrm{free}}A[-n-1] \rightsquigarrow . \tag{2.49}$$

In particular,

$$\mathcal{H}^{n+1}_{\mathrm{tors}}A = 0 \implies \mathbb{F}\,\tau_{\leqslant n}A \xrightarrow{\sim} \tau_{\leqslant n}\mathbb{F}A \xrightarrow{\sim} \mathbb{F}\,\tau_{\leqslant n_+}A \tag{2.50}$$

$$\mathcal{H}^{n+1}_{\mathrm{free}}A = 0 \implies \mathbb{F}\,\tau_{\leqslant n_+}A \xrightarrow{\sim} \mathbb{F}\,\tau_{\leqslant n+1}A. \tag{2.51}$$

Proof. — We have a distinguished triangle (2.11)

$$\tau_{\leqslant n_+} A \to \tau_{\leqslant n+1} A \to \mathcal{H}^{n+1}_{\mathrm{free}} A[-n-1] \rightsquigarrow$$

in $D^b_c(X,\mathbb{O})$. Applying $\mathbb{F}(-)$, we get the triangle (2.49). If $\mathcal{H}^{n+1}_{\mathrm{free}} A = 0$, this reduces to the isomorphism (2.51).

We also have a distinguished triangle (2.10)

$$\tau_{\leqslant n} A \to \tau_{\leqslant n_+} A \to \mathcal{H}^{n+1}_{\mathrm{tors}} A[-n-1] \rightsquigarrow$$

in $D^b_c(X,\mathbb{O})$. Applying $\mathbb{F}(-)$, we get a distinguished triangle in $D^b_c(X,\mathbb{F})$

$$\mathbb{F}\,\tau_{\leqslant n} A \to \mathbb{F}\,\tau_{\leqslant n_+} A \to \mathbb{F}\,\mathcal{H}^{n+1}_{\mathrm{tors}} A[-n-1] \rightsquigarrow . \tag{2.52}$$

On the other hand, we have a distinguished triangle

$$\mathrm{Tor}_1^{\mathbb{O}}(\mathbb{F},\mathcal{H}^{n+1}_{\mathrm{tors}} A)[-n] \to \mathbb{F}\,\mathcal{H}^{n+1}_{\mathrm{tors}} A[-n-1] \to \mathbb{F}\otimes_{\mathbb{O}} \mathcal{H}^{n+1}_{\mathrm{tors}} A[-n-1]. \tag{2.53}$$

By the dual octahedron axiom of triangulated categories (the TR 4' axiom, see [1]), we have an octahedron diagram

(Ω)

$$\begin{tikzcd}[column sep=small]
 & & & {} & & \\
 & & \mathrm{Tor}_1^{\mathbb{O}}(\mathbb{F},\mathcal{H}^{n+1}_{\mathrm{tors}} A)[-n] \arrow[ur] \arrow[dr] & & {} & \\
 & B \arrow[ur] \arrow[dr] & & \mathbb{F}\,\mathcal{H}^{n+1}_{\mathrm{tors}} A[-n-1] \arrow[ur] \arrow[ddr] & & \\
 & & \mathbb{F}\,\tau_{\leqslant n_+} A \arrow[ur] \arrow[drr] & & & \\
 \mathbb{F}\,\tau_{\leqslant n} A \arrow[uur] \arrow[urr] & & & & \mathbb{F}\otimes_{\mathbb{O}} \mathcal{H}^{n+1}_{\mathrm{tors}} A[-n-1] \arrow[dr] \arrow[ddr] & \\
 & & & & & {} \\
 & & & & & {}
\end{tikzcd}$$

for some B in $D^b_c(X,\mathbb{O})$.

The triangle $(\mathbb{F}\,\tau_{\leqslant n} A,\ B,\ \mathrm{Tor}_1^{\mathbb{O}}(\mathbb{F},\mathcal{H}^{n+1}_{\mathrm{tors}} A)[-n])$ shows that B lies in $D^{\leqslant n}_c(X,\mathbb{F})$, and then the triangle $(B,\ \mathbb{F}\,\tau_{\leqslant n_+} A,\ \mathbb{F}\otimes_{\mathbb{O}} \mathcal{H}^{n+1}_{\mathrm{tors}} A[-n-1])$ shows that B is (uniquely) isomorphic to $\tau_{\leqslant n}\,\mathbb{F}\,\tau_{\leqslant n_+} A$. Let us now show that $\tau_{\leqslant n}\,\mathbb{F}\,\tau_{\leqslant n_+} A \simeq \tau_{\leqslant n}\,\mathbb{F}\, A$.

By the TR 4 axiom [1], we have an octahedron diagram

$$\begin{tikzcd}
 & & & {} \\
 & & \tau_{\geqslant n+1}\, \mathbb{F}\, \tau_{\leqslant n_+} A \arrow[ur] \arrow[dr] & & {} \\
 & \mathbb{F}\, \tau_{\leqslant n_+} A \arrow[ur] \arrow[dr] & & C \arrow[ur] \arrow[ddr] \\
 & & \mathbb{F}\, A \arrow[ur] \arrow[drr] \\
\tau_{\leqslant n}\, \mathbb{F}\, \tau_{\leqslant n_+} A \arrow[uur] \arrow[urr] & & & & \mathbb{F}\, \tau_{\geqslant (n+1)_+} A \arrow[r] \arrow[dr] & {} \\
 & & & & & {}
\end{tikzcd}$$

for some C in $D^b_c(X,\mathbb{O})$.

First, the triangle $(\tau_{\geqslant n+1}\, \mathbb{F}\, \tau_{\leqslant n_+} A,\ C,\ \mathbb{F}\, \tau_{\geqslant (n+1)_+} A)$ shows that C lies in $D^{\geqslant n+1}_c(X,\mathbb{F})$. Secondly, the triangle $(\tau_{\leqslant n}\, \mathbb{F}\, \tau_{\leqslant n_+} A,\ \mathbb{F}\, A,\ C)$ shows that $B \simeq \tau_{\leqslant n}\, \mathbb{F}\, \tau_{\leqslant n_+} A \simeq \tau_{\leqslant n}\, \mathbb{F}\, A$ and $C \simeq \tau_{\geqslant n+1}\, \mathbb{F}\, A$.

Hence the octahedron diagram (Ω) contains the triangles (2.47) and (2.48). If $\mathcal{H}^{n+1}_{\mathrm{tors}} A = 0$, the diagram reduces to the isomorphisms (2.50). $\square$

We can summarize the Proposition by the following diagram:

$\mathcal{H}^n \mathbb{F}$			$\mathcal{H}^{n+1} \mathbb{F}$		
$\mathcal{H}^0 \mathbb{F} \mathcal{H}^n_{\mathrm{tors}}$	$\mathbb{F} \mathcal{H}^n_{\mathrm{free}}$	$\mathcal{H}^{-1} \mathbb{F} \mathcal{H}^{n+1}_{\mathrm{tors}}$	$\mathcal{H}^0 \mathbb{F} \mathcal{H}^{n+1}_{\mathrm{tors}}$	$\mathbb{F} \mathcal{H}^{n+1}_{\mathrm{free}}$	$\mathcal{H}^{-1} \mathbb{F} \mathcal{H}^{n+2}_{\mathrm{tors}}$
$\cdots$	$\mathbb{F}\tau_{\leqslant n}$	$\tau_{\leqslant n}\mathbb{F}$	$\mathbb{F}\tau_{\leqslant n_+}$	$\mathbb{F}\tau_{\leqslant n+1}$	$\cdots$

We have the same result if we replace $\tau_{\leqslant n}$ by ${}^p\tau_{\leqslant n}$, and $\mathcal{H}^n$ by ${}^p\mathcal{H}^n$.

2.8. Modular reduction and recollement

Let us fix an open subvariety $j : U \to X$, with closed complement $i : F \to X$. We want to see how the modular reduction behaves with respect to this recollement situation.

For A in ${}^p\mathcal{M}(U,\mathbb{O}) \cap {}^{p_+}\mathcal{M}(U,\mathbb{O})$, we have nine interesting extensions of $\mathbb{F}A$, out of which seven are automatically perverse. These correspond to nine ways to truncate $\mathbb{F}j_* A = j_*\mathbb{F}A$, three for each degree between -2 and 0. Indeed, each degree is "made of" three parts: the ${}^p\mathcal{H}^0\, \mathbb{F}(-)$ of the torsion part of the cohomology of A of the same degree, the reduction of the torsion-free part of the cohomology of A of the same degree, and the ${}^p\mathcal{H}^{-1}\, \mathbb{F}(-)$ of the torsion part of the next degree (like a Tor_1).

There is a variant of Proposition 2.35 for the functors $\tau^F_{\leqslant i}$ instead of $\tau_{\leqslant i}$. We get the following diagram:

$i_* \mathcal{H}^n i^* \mathbb{F}$			$i_* \mathcal{H}^{n+1} i^* \mathbb{F}$		
$\mathcal{H}^0 \mathbb{F} i_* \mathcal{H}^n_{\text{tors}} i^*$	$\mathbb{F} i_* \mathcal{H}^n_{\text{free}} i^*$	$\mathcal{H}^{-1} \mathbb{F} i_* \mathcal{H}^{n+1}_{\text{tors}} i^*$	$\mathcal{H}^0 \mathbb{F} i_* \mathcal{H}^{n+1}_{\text{tors}} i^*$	$\mathbb{F} i_* \mathcal{H}^{n+1}_{\text{free}} i^*$	$\mathcal{H}^{-1} \mathbb{F} i_* \mathcal{H}^{n+2}_{\text{tors}} i^*$
$\cdots$	$\mathbb{F}\tau^F_{\leqslant i}$	$\tau^F_{\leqslant i}\mathbb{F}$	$\mathbb{F}\tau^F_{\leqslant i_+}$	$\mathbb{F}\tau^F_{\leqslant i+1}$	$\cdots$

The same remark applies if we use ${}^p\tau^F_{\leqslant n}$ instead of $\tau^F_{\leqslant n}$. Using Proposition 2.23, we obtain a chain of morphisms:

$$\mathbb{F}\,{}^p j_! \to {}^p j_!\,\mathbb{F} \to \mathbb{F}\,{}^{p+} j_! \to \mathbb{F}\,{}^p j_{!*} \to {}^p j_{!*}\,\mathbb{F} \to \mathbb{F}\,{}^{p+} j_{!*} \to \mathbb{F}\,{}^p j_* \to {}^p j_*\,\mathbb{F} \to \mathbb{F}\,{}^{p+} j_*$$

and distinguished triangles:

$$\mathbb{F}\,{}^p j_! \longrightarrow {}^p j_!\,\mathbb{F} \longrightarrow {}^p\mathcal{H}^{-1}\,\mathbb{F}\,{}^p i_* {}^p\mathcal{H}^{-1}_{\text{tors}}\, i^* j_*[2] \rightsquigarrow \tag{2.54}$$

$${}^p j_!\,\mathbb{F} \longrightarrow \mathbb{F}\,{}^{p+} j_! \longrightarrow {}^p\mathcal{H}^{0}\,\mathbb{F}\,{}^p i_* {}^p\mathcal{H}^{-1}_{\text{tors}}\, i^* j_*[1] \rightsquigarrow \tag{2.55}$$

$$\mathbb{F}\,{}^{p+} j_! \longrightarrow \mathbb{F}\,{}^p j_{!*} \longrightarrow \mathbb{F}\,{}^p i_* {}^p\mathcal{H}^{-1}_{\text{free}}\, i^* j_*[1] \rightsquigarrow \tag{2.56}$$

$$\mathbb{F}\,{}^p j_{!*} \longrightarrow {}^p j_{!*}\,\mathbb{F} \longrightarrow {}^p\mathcal{H}^{-1}\,\mathbb{F}\,{}^p i_* {}^p\mathcal{H}^{0}_{\text{tors}}\, i^* j_*[1] \rightsquigarrow \tag{2.57}$$

$${}^p j_{!*}\,\mathbb{F} \longrightarrow \mathbb{F}\,{}^{p+} j_{!*} \longrightarrow {}^p\mathcal{H}^{0}\,\mathbb{F}\,{}^p i_* {}^p\mathcal{H}^{0}_{\text{tors}}\, i^* j_* \rightsquigarrow \tag{2.58}$$

$$\mathbb{F}\,{}^{p+} j_{!*} \longrightarrow \mathbb{F}\,{}^p j_* \longrightarrow \mathbb{F}\,{}^p i_* {}^p\mathcal{H}^{0}_{\text{free}}\, i^* j_* \rightsquigarrow \tag{2.59}$$

$$\mathbb{F}\,{}^p j_* \longrightarrow {}^p j_*\,\mathbb{F} \longrightarrow {}^p\mathcal{H}^{-1}\,\mathbb{F}\,{}^p i_* {}^p\mathcal{H}^{1}_{\text{tors}}\, i^* j_* \rightsquigarrow \tag{2.60}$$

$${}^p j_*\,\mathbb{F} \longrightarrow \mathbb{F}\,{}^{p+} j_* \longrightarrow {}^p\mathcal{H}^{0}\,\mathbb{F}\,{}^p i_* {}^p\mathcal{H}^{1}_{\text{tors}}\, i^* j_*[-1] \rightsquigarrow\ . \tag{2.61}$$

In particular, for A in ${}^p\mathcal{M}(U,\mathbb{O}) \cap {}^{p+}\mathcal{M}(U,\mathbb{O})$, we have:

$${}^p\mathcal{H}^{-1}_{\text{tors}}\, i^* j_* A = 0 \implies \mathbb{F}\,{}^p j_!\, A \xrightarrow{\sim} {}^p j_!\,\mathbb{F}\, A \xrightarrow{\sim} \mathbb{F}\,{}^{p+} j_!\, A \tag{2.62}$$

$${}^p\mathcal{H}^{-1}_{\text{free}}\, i^* j_* A = 0 \implies \mathbb{F}\,{}^{p+} j_!\, A \xrightarrow{\sim} \mathbb{F}\,{}^p j_{!*}\, A \tag{2.63}$$

$${}^p\mathcal{H}^{0}_{\text{tors}}\, i^* j_* A = 0 \implies \mathbb{F}\,{}^p j_{!*}\, A \xrightarrow{\sim} {}^p j_{!*}\,\mathbb{F}\, A \xrightarrow{\sim} \mathbb{F}\,{}^{p+} j_{!*}\, A \tag{2.64}$$

$${}^p\mathcal{H}^{0}_{\text{free}}\, i^* j_* A = 0 \implies \mathbb{F}\,{}^{p+} j_{!*}\, A \xrightarrow{\sim} \mathbb{F}\,{}^p j_*\, A \tag{2.65}$$

$${}^p\mathcal{H}^{1}_{\text{tors}}\, i^* j_* A = 0 \implies \mathbb{F}\,{}^p j_*\, A \xrightarrow{\sim} {}^p j_*\,\mathbb{F}\, A \xrightarrow{\sim} \mathbb{F}\,{}^{p+} j_*\, A. \tag{2.66}$$

2.9. Decomposition numbers

Let X be endowed with a pair $(\mathfrak{X}, \mathfrak{L})$ satisfying the conditions (i), (ii) and (iii) of Section 2.6. Let $\mathfrak{P}$ be the set of pairs $(\mathcal{O}, \mathcal{L})$ where $\mathcal{O} \in \mathfrak{X}$ and $\mathcal{L} \in \mathfrak{L}(\mathcal{O})$. Let $K_0^{\mathfrak{X},\mathfrak{L}}(X, \mathbb{F})$ be the Grothendieck group of the triangulated category $D^b_{\mathfrak{X},\mathfrak{L}}(X, \mathbb{F})$.

For $\mathcal{O} \in \mathfrak{X}$, let $j_{\mathcal{O}} : \mathcal{O} \to X$ denote the inclusion. For $(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}$, let us denote by

$$ {}^0\mathcal{J}_!(\mathcal{O}, \mathcal{L}) = {}^0 j_{\mathcal{O}!}\ (\mathcal{L}[\dim \mathcal{O}]) \tag{2.67} $$

the extension by zero of the local system $\mathcal{L}$, shifted by $\dim \mathcal{O}$. We also introduce the following notation for the three perverse extensions.

$$ {}^p\mathcal{J}_!(\mathcal{O}, \mathcal{L}) = {}^p j_{\mathcal{O}!}\ (\mathcal{L}[\dim \mathcal{O}]) \tag{2.68} $$

$$ {}^p\mathcal{J}_{!*}(\mathcal{O}, \mathcal{L}) = {}^p j_{\mathcal{O}!*}(\mathcal{L}[\dim \mathcal{O}]) \tag{2.69} $$

$$ {}^p\mathcal{J}_*(\mathcal{O}, \mathcal{L}) = {}^p j_{\mathcal{O}*}\ (\mathcal{L}[\dim \mathcal{O}]). \tag{2.70} $$

We have

$$ K_0^{\mathfrak{X},\mathfrak{L}}(X, \mathbb{F}) \simeq K_0(\mathrm{Sh}_{\mathfrak{X},\mathfrak{L}}(X, \mathbb{F})) \simeq K_0({}^p\mathcal{M}_{\mathfrak{X},\mathfrak{L}}(X, \mathbb{F})). \tag{2.71} $$

If $K \in D^b_{\mathfrak{X},\mathfrak{L}}(X, \mathbb{F})$, then we have

$$ [K] = \sum_{i \in \mathbb{Z}} (-1)^i [\mathcal{H}^i(K)] = \sum_{j \in \mathbb{Z}} (-1)^j [{}^p\mathcal{H}^j(K)] $$

in $K_0^{\mathfrak{X},\mathfrak{L}}(X, \mathbb{F})$.

This Grothendieck group is free over $\mathbb{Z}$, and admits the following bases

$$ \begin{aligned} \mathcal{B}_0 &= ({}^0\mathcal{J}_!(\mathcal{O}, \mathcal{L}))_{(\mathcal{O},\mathcal{L}) \in \mathfrak{P}} \\ \mathcal{B}_! &= ({}^p\mathcal{J}_!(\mathcal{O}, \mathcal{L}))_{(\mathcal{O},\mathcal{L}) \in \mathfrak{P}} \\ \mathcal{B}_{!*} &= ({}^p\mathcal{J}_{!*}(\mathcal{O}, \mathcal{L}))_{(\mathcal{O},\mathcal{L}) \in \mathfrak{P}} \\ \mathcal{B}_* &= ({}^p\mathcal{J}_*(\mathcal{O}, \mathcal{L}))_{(\mathcal{O},\mathcal{L}) \in \mathfrak{P}}. \end{aligned} $$

For $C \in K_0^{\mathfrak{X},\mathfrak{L}}(X, \mathbb{F})$, let us define the integers $\chi_{(\mathcal{O},\mathcal{L})}(C)$, for $(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}$, by the relations

$$ C = \sum_{(\mathcal{O},\mathcal{L}) \in \mathfrak{P}} \chi_{(\mathcal{O},\mathcal{L})}(C)\ [{}^0\mathcal{J}_!(\mathcal{O}, \mathcal{L})]. $$

For $? \in \{!, !*, *\}$, the complex ${}^p\mathcal{J}_?(\mathcal{O}, \mathcal{L})$ extends the shifted local system $\mathcal{L}[\dim \mathcal{O}]$, and is supported on $\overline{\mathcal{O}}$. This implies

$$ \chi_{(\mathcal{O}',\mathcal{L}')}({}^p\mathcal{J}_?(\mathcal{O}, \mathcal{L})) = 0 \text{ unless } \overline{\mathcal{O}}' \subsetneq \overline{\mathcal{O}} \text{ or } (\mathcal{O}', \mathcal{L}') = (\mathcal{O}, \mathcal{L}) \tag{2.72} $$

and

$$ \chi_{(\mathcal{O},\mathcal{L})}({}^p\mathcal{J}_?(\mathcal{O}, \mathcal{L})) = 1. \tag{2.73} $$

In other words, the three bases $\mathcal{B}_!$, $\mathcal{B}_{!*}$ and $\mathcal{B}_*$ are unitriangular with respect to the basis $\mathcal{B}_0$. This implies that they are also unitriangular with respect to each other. In fact, we already knew it by Proposition 2.28, since ${}^p\mathcal{J}_!(\mathcal{O}, \mathcal{L})$ (resp. ${}^p\mathcal{J}_*(\mathcal{O}, \mathcal{L})$) has a top (resp. socle) isomorphic to ${}^p\mathcal{J}_{!*}(\mathcal{O}, \mathcal{L})$, and the radical (resp. the quotient by the socle) is supported on $\overline{\mathcal{O}} \setminus \mathcal{O}$. In particular, for $? \in \{!, *\}$, we have

$$[{}^p\mathcal{J}_?(\mathcal{O},\mathcal{L}) : {}^p\mathcal{J}_{!*}(\mathcal{O}',\mathcal{L}')] = 0 \text{ unless } \overline{\mathcal{O}}' \subsetneq \overline{\mathcal{O}} \text{ or } (\mathcal{O}',\mathcal{L}') = (\mathcal{O},\mathcal{L}) \tag{2.74}$$

and

$$[{}^p\mathcal{J}_?(\mathcal{O},\mathcal{L}) : {}^p\mathcal{J}_{!*}(\mathcal{O},\mathcal{L})] = 1. \tag{2.75}$$

Let $K_0^{\mathfrak{X},\mathfrak{L}}(X,\mathbb{K})$ be the Grothendieck group of the triangulated category $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{K})$. As for the case $\mathbb{E} = \mathbb{F}$, it can be identified with the Grothendieck groups of $\mathrm{Sh}_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{K})$ and ${}^p\mathcal{M}_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{K})$.

Now, let K be an object of $D^b_{\mathfrak{X},\mathcal{L}}(X,\mathbb{K})$. If $K_{\mathbb{O}}$ is an object of $D^b_{\mathfrak{X},\mathcal{L}}(X,\mathbb{O})$ such that $\mathbb{K} \otimes_{\mathbb{O}} K_{\mathbb{O}} \simeq K$, we can consider $[\mathbb{F}K_{\mathbb{O}}]$ in $K_0^{\mathfrak{X},\mathcal{L}}(X,\mathbb{F})$. This class does not depend on the choice of $K_{\mathbb{O}}$ (note that the modular reduction of a torsion object has a zero class in the Grothendieck group: if we assume, for simplicity, that we have only finite monodromy, then by dévissage we can reduce to the analogue result for finite groups). In fact, it depends only on the class $[K]$ of K in $K_0^{\mathfrak{X},\mathcal{L}}(X,\mathbb{K})$. So we have a well-defined morphism

$$d : K_0^{\mathfrak{X},\mathcal{L}}(X,\mathbb{K}) \longrightarrow K_0^{\mathfrak{X},\mathcal{L}}(X,\mathbb{F}). \tag{2.76}$$

For $(\mathcal{O},\mathcal{L}) \in \mathfrak{P}$, we can consider the decomposition number $[\mathbb{F}K_{\mathbb{O}} : {}^p\mathcal{J}_{!*}(\mathcal{O},\mathcal{L})]$, where $K_{\mathbb{O}}$ is any object of $D^b_{\mathfrak{X},\mathcal{L}}(X,\mathbb{O})$ such that $\mathbb{K}K_{\mathbb{O}} \simeq K$.

2.10. Equivariance

We now introduce G-equivariant perverse sheaves in the sense of [17, §0], [16, §4.2].

Let G be a *connected* algebraic group acting on a variety X. Let $\rho : G \times X \to X$ be the morphism defining the action, and let $p : G \times X \to X$ be the second projection. A sheaf F on X is *G-equivariant* if there is an isomorphism $\alpha : p^*F \xrightarrow{\sim} \rho^*F$. In that case, we can choose α in a unique way such that the induced isomorphism $i^*(\alpha) : F \to F$ is the identity, where $i : X \to G \times X$ is defined by $i(x) = (1_G, x)$.

If $f : X \to Y$ is a G-equivariant morphism, the functors ${}^0f^*$, ${}^0f_*$ and ${}^0f_!$ take G-equivariant sheaves to G-equivariant sheaves.

Let $\mathrm{Sh}_G(X,\mathbb{E})$ be the category whose objects are the G-equivariant $\mathbb{E}$-sheaves on X, and such that the morphisms between two objects F_1 and F_2

are the morphisms ϕ in $\mathrm{Sh}(X,\mathbb{E})$ such that the following diagram commutes

$$\begin{tikzcd} p^*F_1 \arrow[r, "p^*\phi"] \arrow[d, "\alpha_1"'] & p^*F_2 \arrow[d, "\alpha_2"] \\ \rho^*F_1 \arrow[r, "\rho^*\phi"'] & \rho^*F_2 \end{tikzcd}$$

where α_j is the unique isomorphism such that $i^*(\alpha_j)$ is the identity for $j=1,2$. Then it turns out that $\mathrm{Sh}_G(X,\mathbb{E})$ is actually a full subcategory of $\mathrm{Sh}(X,\mathbb{E})$.

For a general complex in $D^b_c(X,\mathbb{E})$, the notion of G-equivariance is more delicate. However, for a perverse sheaf we can take the same definition as above, and again the isomorphism α can be normalized with the same condition. If f is a G-equivariant morphism, then the functors ${}^p\mathcal{H}^j f^*$, ${}^p\mathcal{H}^j f^!$, ${}^p\mathcal{H}^j f_*$ and ${}^p\mathcal{H}^j f_!$ take G-equivariant perverse sheaves to G-equivariant perverse sheaves.

We define in the same way the category ${}^p\mathcal{M}_G(X,\mathbb{E})$ of G-equivariant perverse $\mathbb{E}$-sheaves, and again it is a full subcategory of ${}^p\mathcal{M}(X,\mathbb{E})$. Moreover, it is stable by subquotients. The simple objects in ${}^p\mathcal{M}_G(X,\mathbb{E})$ are the intermediate extensions of irreducible G-equivariant $\mathbb{E}$-local systems on G-stable locally closed smooth irreducible subvarieties of X.

Suppose $\mathbb{E}$ is a field. If $\mathcal{O}$ is a homogeneous space for G, let x be a point in $\mathcal{O}$, and let $A_G(x)=C_G(x)/C^0_G(x)$. Then the set of isomorphism classes of irreducible G-equivariant $\mathbb{E}$-local systems on G is in bijection with the set $\mathrm{Irr}\,\mathbb{E}A_G(x)$ of isomorphism classes of irreducible representations of the group algebra $\mathbb{E}A_G(x)$.

Suppose X is a G-variety with finitely many orbits. Then we can take the stratification $\mathfrak{X}$ of X by its G-orbits. The orbits are indeed locally closed by [22, Lemma 2.3.1], and they are smooth. For each G-orbit $\mathcal{O}$ in X, let $x_\mathcal{O}$ be a closed point in $\mathcal{O}$. For $\mathfrak{L}(\mathcal{O})$ we take all the irreducible G-equivariant $\mathbb{F}$-local systems, so that we can identify $\mathfrak{L}(\mathcal{O})$ with $\mathrm{Irr}\,\mathbb{F}A_G(x_\mathcal{O})$.

Suppose $\mathbb{E}$ is a field. Let $K^G_0(X,\mathbb{E})$ be the Grothendieck group of the triangulated category $D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{E})$. Then we have

(2.77)
$$K^G_0(X,\mathbb{E}) = K_0({}^p\mathcal{M}_G(X,\mathbb{E})) = K_0(\mathrm{Sh}_G(X,\mathbb{E})) \simeq \bigoplus_{\mathcal{O}} K_0(\mathrm{Irr}\,\mathbb{E}A_G(x_\mathcal{O})).$$

If $K\in D^b_{\mathfrak{X},\mathfrak{L}}(X,\mathbb{E})$, then we have

$$[K] = \sum_{i\in\mathbb{Z}}(-1)^i[\mathcal{H}^i(K)] = \sum_{j\in\mathbb{Z}}(-1)^j[{}^p\mathcal{H}^j(K)]$$

in $K_0^G(X,\mathbb{E})$.

Let $\mathfrak{P}_\mathbb{E}$ be the set of pairs $(\mathcal{O},\mathcal{L})$ with $\mathcal{O}\in\mathfrak{X}$ and $\mathcal{L}$ an irreducible G-equivariant $\mathbb{E}$-local system on $\mathcal{O}$ (corresponding to an irreducible representation L of $\mathbb{E}A_G(x_\mathcal{O})$). We will sometimes identify $\mathfrak{P}_\mathbb{E}$ with the set of pairs (x,ρ), where $x\in X$ and $\rho\in\operatorname{Irr}\mathbb{E}A_G(x)$, up to G-conjugacy.

Then we have bases $\mathcal{B}_0^\mathbb{E}=({}^0\mathcal{J}_!(\mathcal{O},\mathcal{L}))_{(\mathcal{O},\mathcal{L})\in\mathfrak{P}_\mathbb{E}}$, $\mathcal{B}_!^\mathbb{E}=({}^p\mathcal{J}_!(\mathcal{O},\mathcal{L}))_{(\mathcal{O},\mathcal{L})\in\mathfrak{P}_\mathbb{E}}$, $\mathcal{B}_{!*}^\mathbb{E}=({}^p\mathcal{J}_{!*}(\mathcal{O},\mathcal{L}))_{(\mathcal{O},\mathcal{L})\in\mathfrak{P}_\mathbb{E}}$, $\mathcal{B}_*^\mathbb{E}=({}^p\mathcal{J}_*(\mathcal{O},\mathcal{L}))_{(\mathcal{O},\mathcal{L})\in\mathfrak{P}_\mathbb{E}}$. Note that, if ℓ does not divide the $|A_G(x_\mathcal{O})|$, then we can identify $\mathfrak{P}_\mathbb{K}$ with $\mathfrak{P}_\mathbb{F}$.

The transition matrices from $\mathcal{B}_0^\mathbb{E}$ to $\mathcal{B}_?^\mathbb{E}$ (for $?\in\{!,!*,*\}$) are unitriangular, and also the transition matrices from $\mathcal{B}_{!*}^\mathbb{E}$ to $\mathcal{B}_?^\mathbb{E}$ (for $?\in\{!,*\}$).

As in the last section, we have a morphism

$$d: K_0^G(X,\mathbb{K})\longrightarrow K_0^G(X,\mathbb{F}).$$

The matrix of d with respect to the bases $\mathcal{B}_0^\mathbb{E}$ is just a product of blocks indexed by the orbits $\mathcal{O}$, the block corresponding to $\mathcal{O}$ being the decomposition matrix of the finite group $A_G(x_\mathcal{O})$. If ℓ does not divide the $|A_G(x_\mathcal{O})|$, this is just the identity matrix.

We are interested in the matrix of d in the bases $\mathcal{B}_{!*}^\mathbb{E}$. That is, we want to study the decomposition numbers

$$d^X_{(\mathcal{O},\mathcal{L}),(\mathcal{O}',\mathcal{L}')}=[\mathbb{F}\mathcal{J}_{!*}(\mathcal{O},\mathcal{L}_\mathbb{O}):\mathcal{J}_{!*}(\mathcal{O}',\mathcal{L}')]$$

for $(\mathcal{O},\mathcal{L})\in\mathfrak{P}_\mathbb{K}$ and $(\mathcal{O}',\mathcal{L}')\in\mathfrak{P}_\mathbb{F}$, where $\mathcal{L}_\mathbb{O}$ is an integral form for $\mathcal{L}$. Recall that, if ℓ does not divide the $|A_G(x)|$, then we can identify $\mathfrak{P}_\mathbb{K}$ with $\mathfrak{P}_\mathbb{F}$.

3. Some techniques

By the results in Subsection 2.10, to compute decomposition numbers in a G-equivariant setting, it is enough to compute the stalks of the intersection cohomology complexes over $\mathbb{K}$ and $\mathbb{F}$, with the actions of the groups $A_G(x)$ (then we just have to solve a triangular linear system). In the applications, these are usually known over $\mathbb{K}$ but not over $\mathbb{F}$. It is harder to compute over $\mathbb{F}$: for example, one cannot use arguments involving counting points, or the Decomposition Theorem. We are going to see some methods that can be used in the modular case. Some of them will be illustrated in the next sections. The results about $\mathbb{E}$-smoothness will be illustrated in [11] (see [12]), in relation with the special pieces of the nilpotent cone.

3.1. Semi-small morphisms

The classical results about semi-small and small projective morphisms still apply in the modular case. Nevertheless, unless we have a small resolution, they are less useful to determine the stalks of the intersection cohomology complexes, because the Decomposition Theorem [1] does not hold in this case.

Definition 3.1. — *A morphism $\pi : \tilde{X} \to X$ is* semi-small *if there is a stratification $\mathfrak{X}$ of X such that the for all strata S in $\mathfrak{X}$, and for all closed points s in S, we have $\dim \pi^{-1}(s) \leqslant \frac{1}{2} \operatorname{codim}_X(S)$. If moreover these inequalities are strict for all strata of positive codimension, we say that π is* small.

Recall that $\operatorname{Loc}(S, \mathbb{E})$ is the full subcategory of $\operatorname{Sh}(X, \mathbb{E})$ consisting of the $\mathbb{E}$-local systems. It is the heart of the t-category $D^b_{\operatorname{Loc}}(S, \mathbb{E})$ which is the full subcategory of $D^b_c(S, \mathbb{E})$ of objects A such that all the $\mathcal{H}^i A$ are local systems, with the t-structure induced by the natural t-structure on $D^b_c(S, \mathbb{E})$. For $\mathbb{E} = \mathbb{O}$, according to the definition given after Proposition 2.15, we have an abelian category $\operatorname{Loc}^+(S, \mathbb{O})$, which is the full subcategory of $D^b_c(S, \mathbb{O})$ consisting of the objects A such that $\mathcal{H}^0 A$ is a torsion-free $\mathbb{O}$-local system, and $\mathcal{H}^1 A$ is a torsion $\mathbb{O}$-local system.

Proposition 3.2. — *Let $\pi : \tilde{X} \to X$ be a surjective, proper and separable morphism, with $\tilde{X}$ smooth irreducible of dimension d. Let $\mathcal{L}$ be in $\operatorname{Loc}(\tilde{X}, \mathbb{E})$. Let us consider the complex $K = \pi_! \mathcal{L}[d]$.*

(i) *If π is semi-small, then $\dim X = d$ and K is p-perverse.*
(ii) *If π is small, then $K = {}^p j_{!*} {}^p j^* K$ for any inclusion $j : U \to X$ of a smooth open dense subvariety over which π is étale.*

In the case $\mathbb{E} = \mathbb{O}$, we can take $\mathcal{L}$ in $\operatorname{Loc}^+(X, \mathbb{O})$ and replace p by p_+.

In the case $\mathbb{E} = \mathbb{K}$, the Decomposition Theorem [1] says that K is the direct sum of its shifted perverse cohomology sheaves and that each ${}^p H^i K$ is a semi-simple perverse sheaf. If π is semi-small, then only ${}^p H^0 K$ can be non-zero. So, in the characteristic zero case, if π is semi-small, the intersection cohomology complex will be a direct summand of the direct image of the constant perverse sheaf; if moreover π is birational, then the other summands will have strictly smaller support. These simple summands correspond to the relevant pairs [2, 3]. If π is small, then the only relevant stratum is the open stratum.

In the favorable case where we have a small resolution, to compute the intersection cohomology stalks over any $\mathbb{E}$, we are reduced to compute the stalks of the direct image of the constant sheaf, that is, the cohomology with $\mathbb{E}$ coefficients of the fibers.

However, in the case of a semi-small resolution, the situation is less favorable in characteristic ℓ than in characteristic zero. We can only say that the intersection cohomology complex of X is a subquotient of K. For example, it can have non-zero stalks in odd degree, even if K has non-zero stalks only in even degree.

Now let us say what can happen when π is a semi-small morphism which is not a resolution. Since it is assumed to be separable, there is a smooth open dense subvariety $j : X_0 \hookrightarrow X$ over which the pullback $\pi_0 : \tilde{X}_0 \to X_0$ is finite étale. We can find a Galois finite étale covering Y of X_0, with Galois group G, such that $\pi_0 : \tilde{X}_0 \to X$ is the subcovering corresponding to a subgroup H of G. Then the direct image under π_0 of the constant perverse sheaf on $\tilde{X}_0$, which is just ${}^p j^* K$, is the local system corresponding to the permutation representation $\mathbb{E}[G/H]$ of $\mathbb{E}G$. If ℓ does not divide the index $|G : H|$, then the trivial module $\mathbb{E}$ is a direct summand of $\mathbb{E}[G/H]$, and ${}^p j_{!*}\underline{\mathbb{E}}$ is a direct summand of ${}^p j_{!*} {}^p j^* K$. Otherwise, $\mathbb{E}$ is both a submodule and a quotient of $\mathbb{E}[G/H]$, so ${}^p j_{!*} {}^p j^* K$ will have ${}^p j_{!*}\underline{\mathbb{E}}$ both as a subobject and as a quotient, but, besides the other composition factors coming from X_0, there can be new composition factors coming from the closed complement F (thus illustrating the non-exactness of ${}^p j_{!*}$). If π is small, then we have $K = {}^p j_{!*} {}^p j^* K$, but otherwise K can have composition factors coming from F as subobjects and as quotients, and ${}^p j_{!*} {}^p j^* K$ is just a subquotient of K.

3.2. $\mathbb{E}$-smoothness

Suppose X is an irreducible variety. Recall that, if $j : V \to X$ the inclusion of a smooth open dense subvariety and $\mathcal{L}$ is a local system on V, then we denote by $\mathcal{IC}(X, \mathcal{L})$ the intermediate extension ${}^p j_{!*}(\mathcal{L}[\dim X])$, see Subsection 2.1. We say that X is $\mathbb{E}$-smooth if $\mathcal{IC}(X, \mathbb{E})$ is reduced to $\underline{\mathbb{E}}_X[\dim X]$. When $\mathbb{E} = \mathbb{O}$, we require this condition for both perversities, p and p_+. This property ensures that X satisfies Poincaré duality with $\mathbb{E}$ coefficients. The notion of rational smoothness was introduced by Deligne in [6].

A smooth variety is $\mathbb{E}$-smooth in all cases. If X is not $\mathbb{K}$-smooth, then it is not $\mathbb{F}$-smooth. Moreover, X is $\mathbb{O}$-smooth if and only if it is $\mathbb{F}$-smooth.

For a moment, let us consider the case where X is a complex algebraic variety, so that we can use an arbitrary commutative ring as coefficients. Then we have the following implications: if X is rationally smooth, then it is $\mathbb{F}_\ell$-smooth for all but finitely many ℓ. If X is not rationally smooth, then it is $\mathbb{F}_\ell$-smooth for no ℓ. On the other hand, X is $\mathbb{Z}$-smooth if and only if it is $\mathbb{F}_\ell$-smooth for all ℓ.

The next proposition provides examples of varieties that are $\mathbb{F}_\ell$-smooth for some but not necessarily all primes ℓ. For example, all the simple singularities are rationally smooth. However, they are not smooth, as there is a double point. We will see that, in all types but E_8, there is always some prime number ℓ for which it is not $\mathbb{F}_\ell$-smooth. So, considering intersection cohomology complexes with $\mathbb{F}_\ell$ coefficients, for all primes ℓ (different from p), is a finer invariant that just the case of rational coefficients. It detects more cases of non-smoothness.

Proposition 3.3. — *Let H be a finite group of order prime to ℓ. If X is an $\mathbb{F}_\ell$-smooth H-variety, then X/H is also $\mathbb{F}_\ell$-smooth.*

Looking at the stalks, we can deduce the following information about decomposition numbers: if a locally closed irreducible union of strata is $\mathbb{F}_\ell$-smooth, then the decomposition numbers involving the intermediate extension of the constant perverse sheaf on the open stratum and a simple perverse sheaf associated to any irreducible modular local system on a smaller stratum in this union are all zero.

3.3. Deligne's construction

Initially, intersection homology was defined topologically, using chains satisfying certain conditions with respect to the stratification. This construction was sheafified: intersection cohomology can be computed as the hypercohomology of a complex, the intersection cohomology complex. Deligne found a purely algebraic construction of the intersection cohomology complex, making sense also when the base field has field positive characteristic p (in the étale topology). Then this was included in the theory of perverse sheaves [1]. The abstract setting is that of a recollement situation. The intersection cohomology complexes of irreducible closed subvarieties Y, with coefficients in any irreducible local system on a smooth open dense subvariety of Y, are the simple perverse sheaves (if the stratification is fixed, one takes for Y the closure of a stratum). These intersection cohomology complexes coincide with the intermediate extensions of the (shifted) local systems. This works both with $\mathbb{K}$-sheaves and $\mathbb{F}$-sheaves.

In the examples we will compute over $\mathbb{F}$, Deligne's construction will be the main tool, because most other approaches fail (we do not have weights nor the Decomposition Theorem). So let us recall the procedure to calculate these intermediate extensions.

Assume we have a pair $(\mathfrak{X}, \mathfrak{L})$ as in Assumption 2.34. Let U_k be the union of the strata of dimension at least $-k$ (it is an open subvariety of X). Let $j_k : U_{k-1} \hookrightarrow U_k$ denote the open inclusion. We have

$$U_{-d} \subset \cdots \subset U_{-1} \subset U_0 = X.$$

Proposition 3.4. — *Let A be a p-perverse $\mathbb{E}$-sheaf on U_k. Let j denote the inclusion of U_k into X. Then we have*

$${}^p j_{!*} A = \tau_{\leqslant -1} j_{0*} \cdots \tau_{\leqslant k} j_{k+1*} A.$$

If $\mathbb{E} = \mathbb{O}$, we also have a similar formula with p replaced by p_+ and $\tau_{\leqslant i}$ replaced by $\tau_{\leqslant i_+}$.

The proof uses the transitivity of ${}^p j_{!*}$, (2.18) and Proposition 2.23. See [1], Proposition 2.1.11, Proposition 2.2.4 and 3.3.4.

Actually, in the examples we will compute, there will be only one step (to go from one stratum to the union of two strata), so what we will really use here is Proposition 2.23.

3.4. Cones

Let $Y \subset \mathbb{P}^{N-1}$ be a smooth projective variety of dimension $d-1$. We denote by $\pi : \mathbb{A}^N \setminus \{0\} \to \mathbb{P}^{N-1}$ the canonical projection. Let $U = \pi^{-1}(Y) \subset \mathbb{A}^N \setminus \{0\}$ and $X = \overline{U} = U \cup \{0\} \subset \mathbb{A}^N$. They have dimension d.

We have a smooth open immersion $j : U \hookrightarrow X$ and a closed immersion $i : \{0\} \hookrightarrow X$. If $d > 1$, then j is not affine.

Proposition 3.5. — *With the preceding notations, we have*

$$i^* j_* \mathbb{E} \simeq R\Gamma(U, \mathbb{E}).$$

Truncating appropriately, one deduces the fiber at 0 of the complexes ${}^p j_? \, \mathbb{E}[d]$, where $? \in \{!, !*, *\}$, and similarly for p_+ if $\mathbb{E} = \mathbb{O}$.

More generally, we have the following result, which is contained in [15, Lemma 4.5 (a)]. As indicated there, in the complex case, this follows easily from topological considerations.

Proposition 3.6. — *Let X be an irreducible closed subvariety of $\mathbb{A}^N$ stable under the $\mathbb{G}_m$-action defined by $\lambda(z_1, \ldots, z_N) = (\lambda^{a_1} z_1, \ldots, \lambda^{a_N} z_N)$, where $a_1 > 0, \ldots, a_N > 0$. Let $j : U = X \setminus \{0\} \to X$ be the open immersion, and $i : \{0\} \to X$ the closed immersion. Then we have*

$$i^* j_* \mathbb{E} \simeq R\Gamma(U, \mathbb{E}).$$

So, if U is smooth, the calculation of the intersection cohomology complex stalks for X is reduced to the calculation of the cohomology of U.

3.5. Equivalent singularities

Definition 3.7. — *Given X and Y two varieties, and two points $x \in X$ and $y \in Y$, we say that the singularity of X at x and the singularity of Y at y are smoothly equivalent, and we write* $\mathrm{Sing}(X, x) = \mathrm{Sing}(Y, y)$, *if there exist a variety Z, a point $z \in Z$, and two maps $\varphi : Z \to X$ and $\psi : Z \to Y$, smooth at z, with $\varphi(z) = x$ and $\psi(z) = y$.*

If an algebraic group G acts on X, then $\mathrm{Sing}(X, x)$ *depends only on the orbit $\mathcal{O}$ of x. In that case, we write* $\mathrm{Sing}(X, \mathcal{O}) := \mathrm{Sing}(X, x)$.

In fact, there is an open subset U of Z containing z where φ and ψ are smooth, so after replacing Z by U, we can assume that φ and ψ are smooth on Z.

We have the following result (it follows from the remarks after Lemma 4.2.6.1. in [1]).

Proposition 3.8. — *Suppose that* $\mathrm{Sing}(X, x) = \mathrm{Sing}(Y, y)$. *Then the complexes of $\mathbb{E}$-modules $\mathcal{IC}(X, \mathbb{E})_x[-\dim X]$ and $\mathcal{IC}(Y, \mathbb{E})_y[-\dim Y]$ are isomorphic.*

We recall that our convention is that $\mathcal{IC}(X, \mathbb{E})$ is concentrated in degrees between $-\dim X$ and 0, so that $\mathcal{IC}(X, \mathbb{E})_x[-\dim X]$ is concentrated in degrees between 0 and $\dim X$.

Remark 3.9. — Suppose we have a stratification $\mathfrak{X}$ of X adapted to $\mathcal{IC}(X, \mathbb{E})$ and a stratification $\mathfrak{Y}$ of Y adapted to $\mathcal{IC}(Y, \mathbb{E})$, and let $\mathcal{O}(x)$ and $\mathcal{O}(y)$ denote the respective strata of x and y. Suppose that we know $\mathcal{H}^{i-\dim X}\mathcal{IC}(X, \mathbb{E})_x$ as an $\mathbb{E}$-module with continuous action of $\pi_1(\mathcal{O}(x), x)$. The proposition then gives us $\mathcal{H}^{i-\dim Y}\mathcal{IC}(Y, \mathbb{E})_y$ as an $\mathbb{E}$-module, but it does not give the action of $\pi_1(\mathcal{O}(y), y)$. To determine the latter structure, one needs more information.

4. Simple singularities

In this section, we will calculate the intersection cohomology complexes over $\mathbb{K}$, $\mathbb{O}$ and $\mathbb{F}$ for rational double points, and the corresponding decomposition numbers. We will also consider the case of simple singularities of inhomogeneous type, that is, rational double points with an associated group of symmetries. It is necessary to keep track of the action of this finite group for the final application, which is the calculation of the decomposition numbers for equivariant perverse sheaves on the nilpotent cone of a simple Lie algebra, involving the regular orbit and the subregular orbit.

For the convenience of the reader, we will recall the main points in the theory of simple singularities, following [21], to which we refer for more details. The application to the nilpotent cone uses the result of Brieskorn and Slodowy [5, 20, 21], showing that the singularity of the nilpotent cone along the subregular class is a simple singularity of the corresponding type.

4.1. Rational double points

We assume that k is algebraically closed. Let (X, x) be the spectrum of a two-dimensional normal local k-algebra, where x denotes the closed point of X. Then (X, x) is *rational* if there is a resolution $\pi : \widetilde{X} \to X$ of the singularities of X such that the higher direct images of the structural sheaf of $\widetilde{X}$ vanish, that is, $R^q\pi_*(\mathcal{O}_{\widetilde{X}}) = 0$ for $q > 0$. In fact, this property is independent of the choice of a resolution. The rationality property is stronger than the Cohen-Macaulay property.

If $\pi : \widetilde{X} \to X$ is a resolution, then the reduced exceptional divisor $E = \pi^{-1}(x)_{\mathrm{red}}$ is a finite union of irreducible curves (in particular, π is semi-small). Since X is a surface, there is a minimal resolution, unique up to isomorphism, through which all other resolutions must factor. For the minimal resolution of a simple singularity, these curves will have a very special configuration.

Let Γ be an irreducible homogeneous Dynkin diagram, with set of vertices Δ. We recall that a Dynkin diagram is *homogeneous*, or *simply-laced*, when the corresponding root system Φ has only roots of the same length. Thus Γ is of type A_n $(n \geqslant 1)$, D_n $(n \geqslant 4)$, E_6, E_7 or E_8. The Cartan matrix $C = (n_{\alpha,\beta})_{\alpha,\beta\in\Delta}$ of Γ satisfies $n_{\alpha,\alpha} = 2$ for all α in Δ, and $n_{\alpha,\beta} \in \{0, -1\}$ for all $\alpha \neq \beta$ in Δ.

A resolution $\pi : \widetilde{X} \to X$ of the surface X, as above, has an *exceptional configuration of type* Γ if all the irreducible components of the exceptional divisor E are projective lines, and if there is a bijection $\alpha \mapsto E_\alpha$ from Δ to the set $\mathrm{Irr}(E)$ of these components such that the intersection numbers $E_\alpha \cdot E_\beta$ are given by the opposite of the Cartan matrix C, that is, $E_\alpha \cdot E_\beta = -n_{\alpha,\beta}$ for α and β in Δ. Thus we have a union of projective lines whose normal bundles in $\widetilde{X}$ are isomorphic to the cotangent bundle $T^*\mathbb{P}^1$, and two of them intersect transversely in at most one point.

The minimal resolution is characterized by the fact that it has no exceptional curves with self-intersection -1. Therefore, if the resolution π of the surface X has an exceptional configuration of type Γ, then it is minimal.

THEOREM 4.1. — *The following properties of a normal surface* (X, x) *are equivalent.*

(i) (X, x) *is rational of embedding dimension* 3 *at* x.
(ii) (X, x) *is rational of multiplicity* 2 *at* x.
(iii) (X, x) *is of multiplicity* 2 *at* x *and it can be resolved by successive blowing up of points.*
(iv) *The minimal resolution of* (X, x) *has the exceptional configuration of an irreducible homogeneous Dynkin diagram.*

DEFINITION 4.2. — *If any (hence all) of the properties of the preceding theorem is satisfied, then* (X, x) *is called a* rational double point *or a* simple singularity.

THEOREM 4.3. — *Let the characteristic of* k *be good for the irreducible homogeneous Dynkin diagram* Γ. *Then there is exactly one rational double point of type* Γ *up to isomorphism of Henselizations. Representatives of the individual classes are given by the local varieties at* $0 \in \mathbb{A}^3$ *defined by the equations in the table below.*

In each case, this equation is the unique relation (syzygy) between three suitably chosen generators X, Y, Z *of the algebra* $k[\mathbb{A}^2]^H$ *of the invariant polynomials of* $\mathbb{A}^2$ *under the action of a finite subgroup* H *of* SL_2, *given in the same table.*

H		$\lvert H \rvert$	*equation of* $\mathbb{A}^2/H \subset \mathbb{A}^3$	Γ
$\mathfrak{C}_{n+1}$	*cyclic*	$n+1$	$X^{n+1} + YZ = 0$	A_n
$\mathfrak{D}_{4(n-2)}$	*dihedral*	$4(n-2)$	$X^{n-1} + XY^2 + Z^2 = 0$	D_n
$\mathfrak{T}$	*binary tetrahedral*	24	$X^4 + Y^3 + Z^2 = 0$	E_6
$\mathfrak{O}$	*binary octahedral*	48	$X^3Y + Y^3 + Z^2 = 0$	E_7
$\mathfrak{I}$	*binary icosahedral*	120	$X^5 + Y^3 + Z^2 = 0$	E_8

Moreover, if k *is of characteristic* 0, *these groups are, up to conjugation, the only finite subgroups of* SL_2.

Thus, in good characteristic, every rational double point is, after Henselization at the singular point, isomorphic to the corresponding quotient $\mathbb{A}^2/H$. When p divides $n+1$ (resp. $4(n-2)$), the group $\mathfrak{C}_{n+1}$ (resp. $\mathfrak{D}_{4(n-2)}$)

is not reduced. We have the following exact sequences

$$1 \longrightarrow \mathfrak{D}_8 \longrightarrow \mathfrak{T} \longrightarrow \mathfrak{C}_3 \longrightarrow 1 \tag{4.1}$$

$$1 \longrightarrow \mathfrak{T} \longrightarrow \mathfrak{O} \longrightarrow \mathfrak{C}_2 \longrightarrow 1 \tag{4.2}$$

$$1 \longrightarrow \mathfrak{D}_8 \longrightarrow \mathfrak{O} \longrightarrow \mathfrak{S}_3 \longrightarrow 1 \tag{4.3}$$

when the characteristic of k is good for the Dynkin diagram attached to each of the groups involved.

4.2. Symmetries on rational double points

To each inhomogeneous irreducible Dynkin diagram Γ we associate a homogeneous diagram $\widehat{\Gamma}$ and a group $A(\Gamma)$ of automorphisms of $\widehat{\Gamma}$, as follows.

Γ	B_n	C_n	F_4	G_2
$\widehat{\Gamma}$	A_{2n-1}	D_{n+1}	E_6	D_4
$A(\Gamma)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathfrak{S}_3$

In general, there is a unique (in case $\Gamma = C_3$ or G_2 : up to conjugation by $\mathrm{Aut}(\widehat{\Gamma}) = \mathfrak{S}_3$) faithful action of $A(\Gamma)$ on $\widehat{\Gamma}$. One can see Γ as the quotient of $\widehat{\Gamma}$ by $A(\Gamma)$.

In all cases but $\Gamma = C_3$, the group $A(\Gamma)$ is the full group of automorphisms of $\widehat{\Gamma}$. Note that D_4 is associated to C_3 and G_2. For a homogeneous diagram, it will be convenient to set $\widehat{\Gamma} = \Gamma$ and $A(\Gamma) = 1$.

A rational double point may be represented as the quotient $\mathbb{A}^2/H$ of $\mathbb{A}^2$ by a finite subgroup H of SL_2 provided the characteristic of k is good for the corresponding Dynkin diagram. If $\widehat{H}$ is another finite subgroup of SL_2 containing H as a normal subgroup, then the quotient $\widehat{H}/H$ acts naturally on $\mathbb{A}^2/H$.

Definition 4.4. — *Let Γ be an inhomogeneous irreducible Dynkin diagram and let the characteristic of k be good for Γ. A couple (X, A) consisting of a normal surface singularity X and a group A of automorphisms of X is called a* simple singularity of type Γ *if it is isomorphic (after Henselization) to a couple $(\mathbb{A}^2/H, \widehat{H}/H)$ according to the following table.*

Γ	B_n	C_n	F_4	G_2
H	$\mathfrak{C}_{2n}$	$\mathfrak{D}_{4(n-1)}$	$\mathfrak{T}$	$\mathfrak{D}_8$
$\widehat{H}$	$\mathfrak{D}_{4n}$	$\mathfrak{D}_{8(n-1)}$	$\mathfrak{O}$	$\mathfrak{O}$

Then X is a rational double point of type $\widehat{\Gamma}$ and A is isomorphic to $A(\Gamma)$. The action of A on X lifts in a unique way to an action of A on the resolution $\widetilde{X}$ of X. As A fixes the singular point of X, the exceptional divisor in $\widetilde{X}$ will be stable under A. In this way, we recover the action of A on $\widehat{\Gamma}$. The simple singularities of inhomogeneous type can be characterized in the following way.

PROPOSITION 4.5. — *Let Γ be a Dynkin diagram of type B_n, C_n, F_4 or G_2, and let the characteristic of k be good for Γ. Let X be a rational double point of type $\widehat{\Gamma}$ endowed with an action of $A(\Gamma)$, free on the complement of the singular point, and such that the induced action on the dual diagram of the minimal resolution of X coincides with the associated action of $A(\Gamma)$ on $\widehat{\Gamma}$. Then (X, A) is a simple singularity of type Γ.*

4.3. Perverse extensions and decomposition numbers

Let Γ be any irreducible Dynkin diagram, and suppose the characteristic of k is good for Γ. Let $\widehat{\Gamma}$ be the associated homogeneous Dynkin diagram, $A(\Gamma)$ the associated symmetry group, and $H \subset \widehat{H}$ the corresponding finite subgroups of SL_2. We recall that, if Γ is already homogeneous, then we take $\widehat{\Gamma} = \Gamma$, $A(\Gamma) = 1$ and $\widehat{H} = H$. We stratify the simple singularity $X = \mathbb{A}^2/H$ into two strata: the origin $\{0\}$ (the singular point), and its complement U, which is smooth since H acts freely on $\mathbb{A}^2 \setminus \{0\}$. We want to determine the stalks of the three perverse extensions of the (shifted) constant sheaf $\mathbb{E}$ on U, for $\mathbb{E}$ in $(\mathbb{K}, \mathbb{O}, \mathbb{F})$, and for the two perversities p and p_+ in the case $\mathbb{E} = \mathbb{O}$. By the results of Section 2, this will allow us to determine a decomposition number.

By the quasi-homogeneous structure of the equation defining X in $\mathbb{A}^3$, we have a $\mathbb{G}_m$-action on X contracting X to the origin. We are in the situation of Proposition 3.6. Thus it is enough to calculate the cohomology of U with $\mathbb{O}$ coefficients. The cases $\mathbb{E} = \mathbb{K}$ or $\mathbb{F}$ will follow.

Let $\widehat{\Phi}$ be the root system corresponding to $\widehat{\Gamma}$, in a real vector space $\widehat{V}$ of dimension equal to the rank n of $\widehat{\Gamma}$. We identify the set $\widehat{\Delta}$ of vertices of $\widehat{\Gamma}$ with a basis of $\widehat{\Phi}$. We denote by $P(\widehat{\Phi})$ and $Q(\widehat{\Phi})$ the weight lattice and the root lattice of $\widehat{V}$. The finite abelian group $P(\widehat{\Phi})/Q(\widehat{\Phi})$ is the fundamental group of the corresponding adjoint group, and also the center of the corresponding simply-connected group. Its order is called the connection index of $\widehat{\Phi}$. The coweight lattice $P^\vee(\widehat{\Phi})$ (the weight lattice of the dual root system $\widehat{\Phi}^\vee$ in $\widehat{V}^*$) is in duality with $Q(\widehat{\Phi})$, and the coroot lattice $Q^\vee(\widehat{\Phi})$ is in duality with $P(\widehat{\Phi})$. Thus the finite abelian group $P^\vee(\widehat{\Phi})/Q^\vee(\widehat{\Phi})$ is dual to $P(\widehat{\Phi})/Q(\widehat{\Phi})$.

Let $\pi : \widetilde{X} \to X$ be the minimal resolution of X. The exceptional divisor E is the union of projective lines E_α, $\alpha \in \widehat{\Delta}$. Then we have an isomorphism $H^2(\widetilde{X}, \mathbb{O}) \xrightarrow{\sim} \mathbb{O} \otimes_{\mathbb{Z}} P(\widehat{\Phi})$ such that, for each α in $\widehat{\Delta}$, the cohomology class of the subvariety E_α is identified with $1 \otimes \alpha$, and such that the intersection pairing is the opposite of the pullback of the W-invariant pairing on $P(\widehat{\Phi})$ normalized by the condition $(\alpha, \alpha) = 2$ for α in $\widehat{\Delta}$ [10]. Thus the natural map $H^2_c(\widetilde{X}, \mathbb{O}) \to H^2_c(E, \mathbb{O})$ is identified with the opposite of the map $\mathbb{O} \otimes_{\mathbb{Z}} Q^\vee(\widehat{\Phi}) \to \mathbb{O} \otimes_{\mathbb{Z}} P^\vee(\widehat{\Phi})$ induced by the inclusion.

By Poincaré duality (U is smooth), it is enough to compute the cohomology with proper support of U, and to do this we will use the long exact sequence in cohomology with proper support for the open subvariety U with closed complement E in $\widetilde{X}$. The following table gives the $H^i_c(-, \mathbb{O})$ of the three varieties (the first column is deduced from the other two).

i	U	$\widetilde{X}$	E
0	0	0	$\mathbb{O}$
1	$\mathbb{O}$	0	0
2	0	$\mathbb{O} \otimes_{\mathbb{Z}} Q^\vee(\widehat{\Phi})$	$\mathbb{O} \otimes_{\mathbb{Z}} P^\vee(\widehat{\Phi})$
3	$\mathbb{O} \otimes_{\mathbb{Z}} P^\vee(\widehat{\Phi})/Q^\vee(\widehat{\Phi})$	0	0
4	$\mathbb{O}$	$\mathbb{O}$	0

By (derived) Poincaré duality, we obtain the cohomology of U.

Proposition 4.6. — *The cohomology of U is given by*

$$R\Gamma(U, \mathbb{O}) \simeq \mathbb{O} \oplus \mathbb{O} \otimes_{\mathbb{Z}} P(\widehat{\Phi})/Q(\widehat{\Phi})[-2] \oplus \mathbb{O}[-3]. \tag{4.4}$$

The closed stratum is a point, and for complexes on the point the perverse t-structures for p and p_+ are the usual ones (there is no shift since the point is 0-dimensional). With the notations of Subsection 3.4, we have

$$H^{-1} i^* j_*(\mathbb{O}[2]) \simeq H^1(U, \mathbb{O}) = 0 \tag{4.5}$$

$$H^0 i^* j_*(\mathbb{O}[2]) \simeq H^2(U, \mathbb{O}) \simeq \mathbb{O} \otimes_{\mathbb{Z}} P(\widehat{\Phi})/Q(\widehat{\Phi}) \tag{4.6}$$

$$H^1 i^* j_*(\mathbb{O}[2]) \simeq H^3(U, \mathbb{O}) \simeq \mathbb{O}. \tag{4.7}$$

By our analysis in Subsections 2.5 and 2.8, we obtain the following results.

Proposition 4.7. — *We keep the preceding notation. In particular, X is a simple singularity of type Γ.*

Over $\mathbb{K}$, we have canonical isomorphisms

$$ {}^p j_!(\mathbb{K}[2]) \simeq {}^p j_{!*}(\mathbb{K}[2]) \simeq {}^p j_*(\mathbb{K}[2]) \simeq \mathbb{K}_X[2]. \tag{4.8}$$

In particular, X is $\mathbb{K}$-smooth.

Over $\mathbb{O}$, we have canonical isomorphisms

$$ {}^p j_!(\mathbb{O}[2]) \simeq {}^{p+}j_!(\mathbb{O}[2]) \simeq {}^p j_{!*}(\mathbb{O}[2]) \simeq \mathbb{O}_X[2] \tag{4.9}$$

$$ {}^{p+}j_{!*}(\mathbb{O}[2]) \simeq {}^p j_*(\mathbb{O}[2]) \simeq {}^{p+}j_*(\mathbb{O}[2]) \tag{4.10}$$

and a short exact sequence in ${}^p\mathcal{M}(X,\mathbb{O})$

$$ 0 \longrightarrow {}^p j_{!*}(\mathbb{O}[2]) \longrightarrow {}^{p+}j_{!*}(\mathbb{O}[2]) \longrightarrow i_*\mathbb{O} \otimes_{\mathbb{Z}} (P(\widehat{\Phi})/Q(\widehat{\Phi})) \longrightarrow 0. \tag{4.11}$$

Over $\mathbb{F}$, we have canonical isomorphisms

$$ \mathbb{F}\,{}^p j_!\ (\mathbb{O}[2]) \xrightarrow{\sim} {}^p j_!\ (\mathbb{F}[2]) \xrightarrow{\sim} \mathbb{F}\,{}^{p+}j_!\ (\mathbb{O}[2]) \xrightarrow{\sim} \mathbb{F}\,{}^p j_{!*}\ (\mathbb{O}[2]) \xrightarrow{\sim} \mathbb{F}_X[2] \tag{4.12}$$

$$ \mathbb{F}\,{}^{p+}j_{!*}\ (\mathbb{O}[2]) \xrightarrow{\sim} \mathbb{F}\,{}^p j_*\ (\mathbb{O}[2]) \xrightarrow{\sim} {}^p j_*\ (\mathbb{F}[2]) \xrightarrow{\sim} \mathbb{F}\,{}^{p+}j_*\ (\mathbb{O}[2]) \tag{4.13}$$

and short exact sequences

$$ 0 \longrightarrow i_*\mathbb{F} \otimes_{\mathbb{Z}} \Big(P(\widehat{\Phi})/Q(\widehat{\Phi})\Big) \longrightarrow \mathbb{F}\,{}^p j_{!*}(\mathbb{O}[2]) \longrightarrow {}^p j_{!*}(\mathbb{F}[2]) \longrightarrow 0 \tag{4.14}$$

$$ 0 \longrightarrow {}^p j_{!*}(\mathbb{F}[2]) \longrightarrow \mathbb{F}\,{}^{p+}j_{!*}(\mathbb{O}[2]) \longrightarrow i_*\mathbb{F} \otimes_{\mathbb{Z}} \Big(P(\widehat{\Phi})/Q(\widehat{\Phi})\Big) \longrightarrow 0. \tag{4.15}$$

We have

$$ [\mathbb{F}\,{}^p j_{!*}\ (\mathbb{O}[2]) : i_*\mathbb{F}] = [\mathbb{F}\,{}^{p+}j_{!*}\ (\mathbb{O}[2]) : i_*\mathbb{F}] = \dim_{\mathbb{F}} \mathbb{F} \otimes_{\mathbb{Z}} \Big(P(\widehat{\Phi})/Q(\widehat{\Phi})\Big)$$

In particular, $\mathbb{F}\,{}^p j_{!*}\ (\mathbb{O}[2])$ is simple (and equal to $\mathbb{F}\,{}^{p+}j_{!*}\ (\mathbb{O}[2])$) if and only if ℓ does not divide the connection index $|P(\widehat{\Phi})/Q(\widehat{\Phi})|$ of $\widehat{\Phi}$. The variety X is $\mathbb{F}$-smooth under the same condition.

Proof. — Taking into account (4.5), (4.6) and (4.7), the statements over $\mathbb{K}$ follow from the triangles (2.25) and (2.26), the statements over $\mathbb{O}$ follow from the triangles (2.42) to (2.46), and the statements over $\mathbb{F}$ follow from the triangles (2.54) to (2.61). The determination of the decomposition number follows. □

Let us give this decomposition number in each type:

$\widehat{\Gamma}$	$P(\widehat{\Phi})/Q(\widehat{\Phi})$	$[\mathbb{F}\,{}^p j_{!*}\ (\mathbb{O}[2]) : i_*\ \mathbb{F}]$
A_n	$\mathbb{Z}/(n+1)$	1 if $\ell \mid n+1,\ 0$ otherwise
D_n (n even)	$(\mathbb{Z}/2)^2$	2 if $\ell = 2,\ 0$ otherwise
D_n (n odd)	$\mathbb{Z}/4$	1 if $\ell = 2,\ 0$ otherwise
E_6	$\mathbb{Z}/3$	1 if $\ell = 3,\ 0$ otherwise
E_7	$\mathbb{Z}/2$	1 if $\ell = 2,\ 0$ otherwise
E_8	0	0

Let us note that for $\Gamma = E_8$, the variety X is $\mathbb{F}$-smooth for any ℓ. However, it is not smooth, since it has a double point.

In the preceding calculations, the closed stratum was just a point, and local systems on a point can be considered as $\mathbb{E}$-modules. However, for the next application (to the subregular orbit), non-trivial local systems may occur. For that reason, we have to keep track of the action of $A(\Gamma)$.

Let us first recall some facts from [4]. Let $\mathrm{Aut}(\widehat{\Phi})$ denote the group of automorphisms of $\widehat{V}$ stabilizing $\widehat{\Phi}$. The subgroup of $\mathrm{Aut}(\widehat{\Phi})$ of the elements stabilizing $\widehat{\Delta}$ is identified with $\mathrm{Aut}(\widehat{\Gamma})$. The Weyl group $W(\widehat{\Phi})$ is a normal subgroup of $\mathrm{Aut}(\widehat{\Phi})$, and $\mathrm{Aut}(\widehat{\Phi})$ is the semi-direct product of $\mathrm{Aut}(\widehat{\Gamma})$ and $W(\widehat{\Phi})$ [4, Chap. VI, §1.5, Prop. 16].

The group $\mathrm{Aut}(\widehat{\Phi})$ stabilizes $P(\widehat{\Phi})$ and $Q(\widehat{\Phi})$, thus it acts on the quotient $P(\widehat{\Phi})/Q(\widehat{\Phi})$. By [4, Chap. VI, §1.10, Prop. 27], the group $W(\widehat{\Phi})$ acts trivially on $P(\widehat{\Phi})/Q(\widehat{\Phi})$. Thus, the quotient group $\mathrm{Aut}(\widehat{\Phi})/W(\widehat{\Phi}) \simeq \mathrm{Aut}(\widehat{\Gamma})$ acts canonically on $P(\widehat{\Phi})/Q(\widehat{\Phi})$.

Now $A(\Gamma)$ acts on X, $\widetilde{X}$, E and U, and hence on their cohomology (with or without supports). Moreover, the action of $A(\Gamma)$ on $H^2_c(E, \mathbb{O}) \simeq \mathbb{O} \otimes_{\mathbb{Z}} P^\vee(\widehat{\Phi})$ is the one induced by the inclusions $A(\Gamma) \subset \mathrm{Aut}(\widetilde{\Gamma}) \subset \mathrm{Aut}(\widehat{\Phi})$. The inclusions of E and U in $\widetilde{X}$ are $A(\Gamma)$-equivariant, hence the maps in the long exact sequence in cohomology with compact support that we considered earlier (to calculate $H^3_c(U, \mathbb{O})$) are $A(\Gamma)$-equivariant. Thus the action of $A(\Gamma)$ on $H^3_c(U, \mathbb{O}) \simeq \mathbb{O} \otimes_{\mathbb{Z}} P^\vee(\widehat{\Phi})/Q^\vee(\widehat{\Phi})$ is induced by the inclusion $A(\Gamma) \subset \mathrm{Aut}(\Gamma) \simeq \mathrm{Aut}(\widehat{\Phi})/W(\widehat{\Phi})$ from the canonical action. It follows that the action of $A(\Gamma)$ on $H^2(U, \mathbb{O}) \simeq \mathbb{O} \otimes_{\mathbb{Z}} P(\widehat{\Phi})/Q(\widehat{\Phi})$ also comes from the canonical action of $\mathrm{Aut}(\widehat{\Phi})/W(\widehat{\Phi})$.

4.4. Subregular class

Let G be a simple and adjoint algebraic group over k of type Γ. We will recall some facts about the geometry of the subregular orbit from [21]. We assume that the characteristic of k is 0 or greater than $4h - 2$ (where h is the Coxeter number). This is a serious restriction on p, but it does not matter so much for our purposes. Note that, on the other hand, we make no assumption on ℓ (the only restriction is $\ell \neq p$).

Let $\mathcal{N}$ denote the nilpotent cone in the Lie algebra $\mathfrak{g}$ of G. Let $\mathcal{O}_{\mathrm{reg}}$ (resp. $\mathcal{O}_{\mathrm{subreg}}$) be the regular (resp. subregular) orbit in $\mathcal{N}$. The orbit $\mathcal{O}_{\mathrm{subreg}}$ is the unique open dense orbit in $\mathcal{N} \setminus \mathcal{O}_{\mathrm{reg}}$ (we assume that $\mathfrak{g}$ is simple). It is of codimension 2 in $\mathcal{N}$. Let $x_{\mathrm{reg}} \in \mathcal{O}_{\mathrm{reg}}$ and $x_{\mathrm{subreg}} \in \mathcal{O}_{\mathrm{subreg}}$.

The centralizer of x_{reg} in G is a connected unipotent subgroup, hence $A_G(x_{\mathrm{reg}}) = 1$. The unipotent radical of the centralizer in G of x_{subreg} has a reductive complement C given by the following table.

Γ	A_n $(n > 1)$	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
C	$\mathbb{G}_m$	$\mathbb{G}_m \rtimes \mathbb{Z}/2$	$\mathbb{Z}/2$	1	1	1	1	$\mathbb{Z}/2$	$\mathfrak{S}_3$

In type A_1, the subregular class is just the trivial class, so in this case the centralizer is $G = PSL_2$ itself, which is reductive.

We have $A_G(x_{\mathrm{subreg}}) \simeq C/C^0$. This group is isomorphic to the associated symmetry group $A(\Gamma)$ introduced in Subsection 4.2.

Let X be the intersection $X = S \cap \mathcal{N}$ of a transverse slice S to the orbit $\mathcal{O}_{\mathrm{subreg}}$ of x_{subreg} with the nilpotent variety $\mathcal{N}$. The group C acts on X. We can find a section A of $C/C^0 \simeq A_G \simeq A(\Gamma)$ in C. In homogeneous types, A is trivial. If $\Gamma = C_n$, F_4 or G_2, then $A = C$. If $\Gamma = B_n$, take $\{1, s\}$ where s is a nontrivial involution (in this case, A is well-defined up to conjugation by $C^0 = \mathbb{G}_m$).

Theorem 4.8. — [5, 20, 21] *We keep the preceding notation. The surface X has a rational double point of type $\widehat{\Gamma}$ at x_{subreg}. Thus*

$$\mathrm{Sing}(\overline{\mathcal{O}}_{\mathrm{reg}}, \mathcal{O}_{\mathrm{subreg}}) = \widehat{\Gamma}.$$

Moreover the couple (X, A) is a simple singularity of type Γ.

In fact, the first part of the theorem is already true when the characteristic of k is very good for G. This part is enough to calculate the decomposition numbers $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},1)}$ for homogeneous types (then $A = 1$), and even some more decomposition numbers $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\rho)}$ for the other types. Here, we identify $\mathbb{F}$-local systems on $\mathcal{O}_{\mathrm{subreg}}$ with modular characters of $A \simeq A_G(x_{\mathrm{subreg}})$. Actually, what can be deduced in all types is the following relation:

$$\sum_{\rho \in \mathrm{Irr}\, \mathbb{F}A} \rho(1) \cdot d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\rho)} = \dim_{\mathbb{F}} \mathbb{F} \otimes_{\mathbb{Z}} P(\widehat{\Phi})/Q(\widehat{\Phi}).$$

This is enough, for example, to determine for which ℓ we have

$$\forall \rho \in \mathrm{Irr}\, \mathbb{F}A,\ d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\rho)} = 0$$

(those ℓ are the ones which do not divide the connection index of $\widehat{\Phi}$).

Anyway, the second part of the theorem will allow us to deal with the local systems involved on $\mathcal{O}_{\mathrm{subreg}}$.

Let $j_{\mathrm{reg}} : \mathcal{O}_{\mathrm{reg}} \hookrightarrow \mathcal{O}_{\mathrm{reg}} \cup \mathcal{O}_{\mathrm{subreg}}$ be the open immersion, and $i_{\mathrm{subreg}} : \mathcal{O}_{\mathrm{subreg}} \hookrightarrow \mathcal{O}_{\mathrm{reg}} \cup \mathcal{O}_{\mathrm{subreg}}$ the closed complement. Finally, let j be the open inclusion of $\mathcal{O}_{\mathrm{subreg}} \cup \mathcal{O}_{\mathrm{reg}}$ into $\mathcal{N}$. Applying the functor j^*, we see that

$$\begin{aligned} d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\rho)} &:= [\mathbb{F}{}^p\mathcal{J}_{!*}(\mathcal{O}_{\mathrm{reg}}, \mathbb{O}) : {}^p\mathcal{J}_{!*}(\mathcal{O}_{\mathrm{subreg}}, \rho)] \\ &= [\mathbb{F} j_{\mathrm{reg}\,!*}(\mathbb{O}[2\nu]) : i_{\mathrm{subreg}\,*}(\rho[2\nu - 2])]. \end{aligned}$$

where ν is the number of positive roots in Φ.

By Slodowy's theorem and the analysis of Subsection 4.3, we obtain the following result:

Theorem 4.9. — *We have*

$$d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\rho)} = [\mathbb{F}\otimes_{\mathbb{Z}} P(\widehat{\Phi})/Q(\widehat{\Phi}) : \rho]$$

for all ρ *in* $\operatorname{Irr}\mathbb{F}A$.

For homogeneous types, we recover the decomposition numbers described in Subsection 4.3. Let us describe in detail all the other possibilities. The action of $\mathrm{Aut}(\widehat{\Phi})/W(\widehat{\Phi})$ on $P(\widehat{\Phi})/Q(\widehat{\Phi})$ is described in all types in [4, Chap. VI, §4].

In the types B_n, C_n and F_4, we have $A \simeq \mathbb{Z}/2$. When $\ell = 2$, we have $\operatorname{Irr}\mathbb{F}A = \{1\}$. In this case, we would not even need to know the actual action, since for our purposes we only need the class in the Grothendieck group $K_0(\mathbb{F}A) \simeq \mathbb{Z}$, that is, the dimension. When ℓ is not 2, we have $\operatorname{Irr}\mathbb{F}A = \{1,\varepsilon\}$, where ε is the unique non-trivial character of $\mathbb{Z}/2$.

4.4.1. Case $\Gamma = B_n$

We have $\widehat{\Gamma} = A_{2n-1}$ and $P(\widehat{\Phi})/Q(\widehat{\Phi}) \simeq \mathbb{Z}/2n$. The non-trivial element of $A \simeq \mathbb{Z}/2$ acts by -1. Thus we have

If $\ell = 2$, then $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},1)} = 1$.
If $2 \neq \ell \mid n$, then $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},1)} = 0$ and $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\varepsilon)} = 1$.
If $2 \neq \ell \nmid n$, then $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\rho)} = 0$ for $\rho = 1,\ \varepsilon$.

4.4.2. Case $\Gamma = C_n$

We have $\widehat{\Gamma} = D_{n+1}$.

If n is even, then we have $P(\widehat{\Phi})/Q(\widehat{\Phi}) \simeq \mathbb{Z}/4$, and the nontrivial element of $A \simeq \mathbb{Z}/2$ acts by -1.

If n is odd, then we have $P(\widehat{\Phi})/Q(\widehat{\Phi}) \simeq (\mathbb{Z}/2)^2$, and the nontrivial element of $A \simeq \mathbb{Z}/2$ acts by exchanging two nonzero elements.

Thus we have

If $\ell = 2$ and n is even, then $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},1)} = 1$.
If $\ell = 2$ and n is odd, then $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},1)} = 2$.
If $\ell \neq 2$, then $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\rho)} = 0$ for $\rho = 1,\ \varepsilon$.

4.4.3. Case $\Gamma = F_4$

We have $\widehat{\Gamma} = E_6$ and $P(\widehat{\Phi})/Q(\widehat{\Phi}) \simeq \mathbb{Z}/3$. The nontrivial element of $A \simeq \mathbb{Z}/2$ acts by -1. Thus we have:

If $\ell = 2$, then $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},1)} = 0$.
If $\ell = 3$, then $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},1)} = 0$ and $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\varepsilon)} = 1$.
If $\ell > 3$, then $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\rho)} = 0$ for $\rho = 1,\ \varepsilon$.

4.4.4. Case $\Gamma = G_2$

We have $\widehat{\Gamma} = D_4$ and $P(\widehat{\Phi})/Q(\widehat{\Phi}) \simeq (\mathbb{Z}/2)^2$. The group $A \simeq \mathfrak{S}_3$ acts by permuting the three non-zero elements. Let us denote the sign character by ε (it is nontrivial when $\ell \neq 2$), and the degree two character by ψ (it remains irreducible for $\ell = 2$, but for $\ell = 3$ it decomposes as $1 + \varepsilon$). We have:

If $\ell = 2$, then $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},1)} = 0$ and $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\psi)} = 1$.
If $\ell = 3$, then $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\rho)} = 0$ for $\rho = 1,\ \varepsilon$.
If $\ell > 3$, then $d^{\mathcal{N}}_{(\mathcal{O}_{\mathrm{reg}},1),(\mathcal{O}_{\mathrm{subreg}},\rho)} = 0$ for $\rho = 1,\ \varepsilon,\ \psi$.

5. Minimal singularities

Let G be as in the last section. We assume that p is good. We consider the unique (non-trivial) minimal nilpotent orbit $\mathcal{O}_{\mathrm{min}}$ in $\mathfrak{g}$ (it is the orbit of a highest weight vector for the adjoint representation). It is of dimension $d = 2h^\vee - 2$, where $h^\vee$ is the dual Coxeter number [23].

Its closure $\overline{\mathcal{O}}_{\mathrm{min}} = \mathcal{O}_{\mathrm{min}} \cup \{0\}$ is a cone with origin 0. Let $j_{\mathrm{min}} : \mathcal{O}_{\mathrm{min}} \to \overline{\mathcal{O}}_{\mathrm{min}}$ be the open immersion, and $i_0 : \{0\} \to \overline{\mathcal{O}}_{\mathrm{min}}$ the closed complement. By Proposition 3.5, we have

$$i_0{}^* j_{\mathrm{min}*}(\mathbb{O}[d]) \simeq \bigoplus_i H^{i+d}(\mathcal{O}_{\mathrm{min}}, \mathbb{O})[-i].$$

Let Φ denote the root system of $\mathfrak{g}$ and let us choose some basis of Φ. Let Φ' be the root subsystem of Φ generated by the long simple roots. In [13], we computed the cohomology of $\mathcal{O}_{\mathrm{min}}$ over $\mathbb{O}$. In particular, we obtained the following results:

$$H^{-1} i_0{}^* j_{\mathrm{min}*}(\mathbb{O}[d]) = H^{d-1}(\mathcal{O}_{\mathrm{min}}, \mathbb{O}) = 0 \tag{5.1}$$

$$H^{0} i_0{}^* j_{\mathrm{min}*}(\mathbb{O}[d]) = H^{d}(\mathcal{O}_{\mathrm{min}}, \mathbb{O}) = \mathbb{O} \otimes_{\mathbb{Z}} \big(P^\vee(\Phi')/Q^\vee(\Phi')\big) \tag{5.2}$$

$$H^{1} i_0{}^* j_{\mathrm{min}*}(\mathbb{O}[d]) = H^{d+1}(\mathcal{O}_{\mathrm{min}}, \mathbb{O}) \text{ is torsion-free.} \tag{5.3}$$

By the distinguished triangles in Subsections 2.5 and 2.8, we obtain the following:

Theorem 5.1. — *Over $\mathbb{O}$, we have canonical isomorphisms*

$$ {}^p j_{\min !}(\mathbb{O}[d]) \simeq {}^{p+} j_{\min !}(\mathbb{O}[d]) \simeq {}^p j_{\min !*}(\mathbb{O}[d]) $$
$$ {}^{p+} j_{\min !*}(\mathbb{O}[d]) \simeq {}^p j_{\min *}(\mathbb{O}[d]) \simeq {}^{p+} j_{\min *}(\mathbb{O}[d]) $$

and a short exact sequence

$$ 0 \longrightarrow {}^p j_{\min !*}(\mathbb{O}[d]) \longrightarrow {}^{p+} j_{\min !*}(\mathbb{O}[d]) \longrightarrow i_{0*}\mathbb{O} \otimes_{\mathbb{Z}} \left(P^\vee(\Phi')/Q^\vee(\Phi')\right) \longrightarrow 0. $$

Over $\mathbb{F}$, we have canonical isomorphisms

$$ \mathbb{F}\, {}^p j_{\min !}(\mathbb{O}[d]) \xrightarrow{\sim} {}^p j_{\min !}(\mathbb{F}[d]) \xrightarrow{\sim} \mathbb{F}\, {}^{p+} j_{\min !}(\mathbb{O}[d]) \xrightarrow{\sim} \mathbb{F}\, {}^p j_{\min !*}(\mathbb{O}[d]) $$
$$ \mathbb{F}\, {}^{p+} j_{\min !*}(\mathbb{O}[d]) \xrightarrow{\sim} \mathbb{F}\, {}^p j_{\min *}(\mathbb{O}[d]) \xrightarrow{\sim} {}^p j_{\min *}(\mathbb{F}[d]) \xrightarrow{\sim} \mathbb{F}\, {}^{p+} j_{\min *}(\mathbb{O}[d]) $$

and short exact sequences

$$ 0 \longrightarrow i_{0*}\mathbb{F} \otimes_{\mathbb{Z}} \left(P^\vee(\Phi')/Q^\vee(\Phi')\right) \longrightarrow \mathbb{F}\, {}^p j_{\min !*}(\mathbb{O}[d]) \longrightarrow {}^p j_{\min !*}(\mathbb{F}[d]) \longrightarrow 0 $$
$$ 0 \longrightarrow {}^p j_{\min !*}(\mathbb{F}[d]) \longrightarrow \mathbb{F}\, {}^{p+} j_{\min !*}(\mathbb{O}[d]) \longrightarrow i_{0*}\mathbb{F} \otimes_{\mathbb{Z}} \left(P^\vee(\Phi')/Q^\vee(\Phi')\right) \longrightarrow 0. $$

We have

$$ [\mathbb{F}\, {}^p j_{\min !*}(\mathbb{O}[d]) : i_{0*}\mathbb{F}] = [\mathbb{F}\, {}^{p+} j_{\min !*}(\mathbb{O}[d]) : i_{0*}\mathbb{F}] = \dim_{\mathbb{F}} \mathbb{F} \otimes_{\mathbb{Z}} \left(P^\vee(\Phi')/Q^\vee(\Phi')\right). $$

In particular, $\mathbb{F}\, {}^p j_{\min !*}(\mathbb{O}[d])$ is simple (and equal to $\mathbb{F}\, {}^{p+} j_{\min !*}(\mathbb{O}[d])$) if and only if ℓ does not divide the connection index of Φ'.

Let us give this decomposition number in each type. We denote the singularity of $\overline{\mathcal{O}}_{\min}$ at the origin by the lower case letter corresponding to the type Γ of $\mathfrak{g}$. We denote by Γ' the subdiagram of Γ consisting of the long simple roots.

Singularity	Γ'	$P^\vee(\Phi')/Q^\vee(\Phi')$	$d^{\mathcal{N}}_{(\mathcal{O}_{\min},1),(0,1)}$
a_n	A_n	$\mathbb{Z}/(n+1)$	1 if $\ell \mid n+1,\ 0$ otherwise
b_n	A_{n-1}	$\mathbb{Z}/n$	1 if $\ell \mid n,\ 0$ otherwise
c_n	A_1	$\mathbb{Z}/2$	1 if $\ell = 2,\ 0$ otherwise
d_n (n even)	D_n	$(\mathbb{Z}/2)^2$	2 if $\ell = 2,\ 0$ otherwise
d_n (n odd)	D_n	$\mathbb{Z}/4$	1 if $\ell = 2,\ 0$ otherwise
e_6	E_6	$\mathbb{Z}/3$	1 if $\ell = 3,\ 0$ otherwise
e_7	E_7	$\mathbb{Z}/2$	1 if $\ell = 2,\ 0$ otherwise
e_8	E_8	0	0
f_4	A_2	$\mathbb{Z}/3$	1 if $\ell = 3,\ 0$ otherwise
g_2	A_1	$\mathbb{Z}/2$	1 if $\ell = 2,\ 0$ otherwise

Here we only used the $H^i(\mathcal{O}_{\min}, \mathbb{O})$ for $i = d-1$, d, $d+1$, but in [13], we computed all of the cohomology of $\mathcal{O}_{\min}$, so if the reader is interested, one can deduce from that all the stalks of the perverse extensions. In particular, there is torsion in the stalks of ${}^p j_{\min !}(\mathbb{O}[d])$ only if ℓ is bad for G. Note that the singularities c_n (for $n \geqslant 1$, including $c_1 = a_1 = A_1$ and $c_2 = b_2$) and g_2 are $\mathbb{K}$-smooth but not $\mathbb{F}_2$-smooth (actually the latter is not $\mathbb{F}_3$-smooth either).

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Manuscrit reçu le 13 mars 2008,
accepté le 17 juillet 2008.

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