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ON THE DISTRIBUTION OF THE FREE PATH LENGTH OF THE LINEAR FLOW IN A HONEYCOMB

by Florin P. BOCA & Radu N. GOLOGAN

Abstract. — Consider the region obtained by removing from $\mathbb{R}^2$ the discs of radius ε, centered at the points of integer coordinates (a,b) with $b \not\equiv a \pmod{\ell}$. We are interested in the distribution of the free path length (exit time) $\tau_{\ell,\varepsilon}(\omega)$ of a point particle, moving from $(0,0)$ along a linear trajectory of direction ω, as $\varepsilon \to 0^+$. For every integer number $\ell \geqslant 2$, we prove the weak convergence of the probability measures associated with the random variables $\varepsilon\tau_{\ell,\varepsilon}$, explicitly computing the limiting distribution. For $\ell = 3$, respectively $\ell = 2$, this result leads to asymptotic formulas for the exit time of a billiard with pockets of radius $\varepsilon \to 0^+$ centered at the corners and trajectory starting at the center in a regular hexagon, respectively in a square.

Résumé. — Nous considérons la région obtenue en enlevant de $\mathbb{R}^2$ les disques de rayon ε, centrés aux points de coordonnées entières (a,b) avec $b \not\equiv a \pmod{\ell}$. Nous étudions la répartition de la longueur du libre parcours (temps de sortie) $\tau_{\ell,\varepsilon}(\omega)$ d'une particule ponctuelle, partant de $(0,0)$ sur une trajectoire rectiligne de direction ω quand $\varepsilon \to 0^+$. Pour tout nombre entier $\ell \geqslant 2$, on montre la convergence faible des mesures de probabilité attachées aux variables aléatoires $\varepsilon\tau_{\ell,\varepsilon}$, en calculant la distribution limite d'une manière explicite. Pour $\ell = 3$, respectivement $\ell = 2$, ce résultat mène à des formules asymptotiques pour le temps de sortie d'un billard avec des poches de rayon $\varepsilon \to 0^+$ centrés aux coins dans un hexagone régulier, respectivement dans un carré.

1. Introduction

Recent progress led to a better understanding of the statistics of the free path length of the periodic Lorentz gas in the small scatterer limit [2, 3, 5, 6, 7, 8, 9, 10]. The aim of this paper is to study situations where periodicity conditions are altered by imposing certain congruence conditions on the integer lattice points where scatterers are placed. The case of

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the honeycomb lattice arises as a particular example in this context. In this paper we only consider the situation where motion originates at the origin, extending the results from [2] and [3]. The case where the initial position is randomly chosen is more intricate and will not be treated here.

Let $\ell \geqslant 2$ be an integer. Consider the set $\mathbb{Z}^2_{(\ell)}$ of pairs (a,b) of integers with $a \not\equiv b \pmod{\ell}$. For every $\varepsilon > 0$ consider the "fat lattice points" (*scatterers*) given by small discs of radius ε centered at all points of $\mathbb{Z}^2_{(\ell)}$ and the region

$$Z_{\ell,\varepsilon} = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \mathbb{Z}^2_{(\ell)}) \geqslant \varepsilon\}$$

obtained by removing all scatterers. In $\mathbb{R}^2$ consider a point-like particle moving at constant unit speed along a linear trajectory originating at $(0,0)$. The *free path length* (*exit time*) is defined as

$$\tau_{\ell,\varepsilon}(\omega) = \inf\{\tau > 0 : \tau\vec{\omega} \in \partial Z_{\ell,\varepsilon}\},$$

the distance traveled to reach the first scatterer along the direction $\vec{\omega} = e^{i\omega} \in \mathbb{T}$, $\omega \in [0,2\pi)$, and as $+\infty$ when the particle escapes to infinity without reaching any scatterer. The Lebesgue measure of a measurable set $A \subseteq \mathbb{R}$ is denoted by $|A|$. This paper is concerned with the study, in the small scatterer limit ($\varepsilon \to 0^+$), of the asymptotic behavior of the repartition function of $\varepsilon\tau_{\ell,\varepsilon}$ defined by

$$\mathbb{P}_{\ell,\varepsilon}(\lambda) = \frac{1}{2\pi}\left|\left\{\omega \in [0,2\pi) : \tau_{\ell,\varepsilon}(\omega) > \frac{\lambda}{\varepsilon}\right\}\right|.$$

To accomplish this we first consider the situation where scatterers are obtained by translating the vertical segment $V_\varepsilon = \{0\} \times [-\varepsilon,\varepsilon]$ by $(a,b) \in \mathbb{Z}^2_{(\ell)}$ and estimate, for any interval $I \subseteq [0,1]$ of length $|I| \asymp \varepsilon^c$ with fixed $c \in (0,1)$, the repartition

$$\mathbb{G}_{\ell,I,\varepsilon}(\lambda) = \left|\left\{\omega \in \arctan I : q_{\ell,\varepsilon}(\omega) > \frac{\lambda}{\varepsilon}\right\}\right|, \qquad \varepsilon \to 0^+,$$

of the *horizontal free path length*

$$q_{\ell,\varepsilon}(\omega) = \inf\left\{q : (q, q\tan\omega) \in \mathbb{Z}^2_{(\ell)} + V_\varepsilon\right\}, \qquad \omega \in \left[0, \frac{\pi}{4}\right].$$

In this paper φ denotes Euler's totient function. The dilogarithm is defined by

$$\operatorname{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\ln(1-t)}{t}\,dt, \qquad x \in [0,1].$$

Clearly $\operatorname{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$.

The main result of this paper shows that the limit of $\mathbb{P}_{\ell,\varepsilon}$ exists as $\varepsilon \to 0^+$ and this limit is explicitly computed.

THEOREM 1.1. — (i) *For every* $0 < c_1 < 1$ *and* $\delta > 0$, *as* $\varepsilon \to 0^+$,

$$\mathbb{G}_{\ell,I,\varepsilon}(\lambda) = c_I G_\ell(\lambda) + O_{\delta,\lambda,\ell}\left(\varepsilon^{-\delta+\theta(c,c_1)}\right), \qquad \lambda > 0, \tag{1.1}$$

where

$$c_I = \int_I \frac{du}{1+u^2}, \qquad \theta(c,c_1) = \min\left\{c + c_1, \frac{1}{2} - 2c_1\right\},$$

and the limiting repartition function G_ℓ *is given by*

$$G_\ell(\lambda) = \begin{cases} 1 - \left(\frac{1}{\zeta(2)} + A(\ell)\right)\lambda & \text{if } \lambda \in \left(0, \frac{1}{2}\right], \\ 1 - \frac{\lambda}{\zeta(2)} + A(\ell) H_2(\lambda) & \text{if } \lambda \in \left[\frac{1}{2}, 1\right], \\ \frac{2C(\ell)}{\ell} H_3(\lambda) & \text{if } \lambda \in [1, \infty), \end{cases}$$

with

$$C(\ell) = \frac{\varphi(\ell)}{\zeta(2)\ell} \prod_{\substack{p \mid \ell \\ p\,\text{prime}}} \left(1 - \frac{1}{p^2}\right)^{-1} = \frac{\varphi(\ell)}{\ell} \prod_{\substack{p \nmid \ell \\ p\,\text{prime}}} \left(1 - \frac{1}{p^2}\right),$$

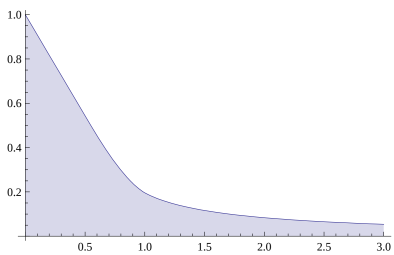
$$A(\ell) = \frac{1}{\zeta(2)} - \frac{2C(\ell)}{\ell}, \tag{1.2}$$

$$H_2(\lambda) = 3\lambda - 2 + \zeta(2) - (\ln\lambda)^2 + 2(1-\lambda)\ln\left(\frac{1}{\lambda} - 1\right) - 2\,\mathrm{Li}_2(\lambda),$$

$$H_3(\lambda) = \mathrm{Li}_2\left(\frac{1}{\lambda}\right) - (\lambda - 1)\ln\left(1 - \frac{1}{\lambda}\right) - 1.$$

(ii) *For every* $\delta > 0$, *as* $\varepsilon \to 0^+$,

$$\mathbb{P}_{\ell,\varepsilon}(\lambda) = G_\ell(\lambda) + O_{\delta,\lambda,\ell}\left(\varepsilon^{\frac{1}{8}+\delta}\right), \qquad \lambda > 0.$$



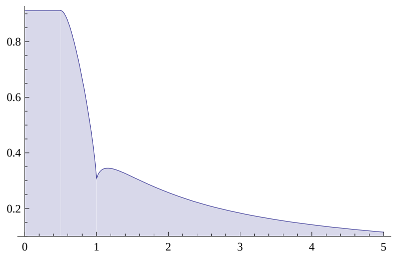


Figure 1.1. The repartition function G_3 *and the density function* g_3

The continuity of G_ℓ at $\lambda=\frac{1}{2}$ is equivalent with the well known dilogarithm identity

$$\sum_{n=1}^{\infty}\frac{1}{2^n n^2}=\mathrm{Li}_2\left(\frac{1}{2}\right)=\int_0^{1/2}\frac{1}{u}\ \ln\frac{1}{1-u}\ du=\frac{\zeta(2)-(\ln 2)^2}{2}.$$

Since $\frac{C(\ell)}{\ell}\to 0$ and $A(\ell)\to\frac{1}{\zeta(2)}$ as $\ell\to\infty$, the compactly supported limiting repartition $H(\lambda)$ from [2, Theorem 1.1] is being recovered as $\lim_{\ell\to\infty} G_\ell(\lambda)$.

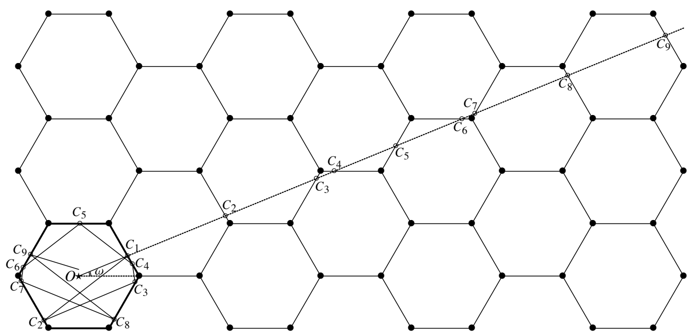


Figure 1.2. The free path in a hexagonal billiard and in a honeycomb

Our original motivation for considering this problem comes from the study of the exit time of the linear motion with specular cushion collisions on a hexagonal (open) billiard table with (small) circular open pockets of radius ε removed from its corners (see Figure 1.2). The starting remark here is that, after unfolding the hexagon to a honeycomb in $\mathbb{R}^2$, one can deform the later to $\mathbb{Z}^2_{(3)}$. This process converts the problem on the hexagonal billiard with pockets into one concerning the free path length of a Lorentz gas in $\mathbb{R}^2$ with small identical ellipses centered at the points from $\mathbb{Z}^2_{(3)}$ as scatterers (see Figure 5.1).

Let $\tau_\varepsilon^{\mathrm{hex}}(\omega)$ denote the exit time in the hexagonal billiard with discs of radius ε removed from the corners and motion starting at the center, and let

$$\mathbb{P}_\varepsilon^{\mathrm{hex}}(\lambda)=\frac{1}{2\pi}\left|\left\{\omega\in[0,2\pi]:\tau_\varepsilon^{\mathrm{hex}}(\omega)>\frac{\lambda}{\varepsilon}\right\}\right|$$

denote the repartition function of $\varepsilon\tau_\varepsilon^{\mathrm{hex}}$. We prove

THEOREM 1.2. — *For every $\delta > 0$, as $\varepsilon \to 0^+$,*

$$\mathbb{P}_\varepsilon^{\rm hex}(\lambda) = G_3\left(\frac{2\lambda}{\sqrt{3}}\right) + O_\delta\left(\varepsilon^{\frac{1}{8}-\delta}\right), \qquad \lambda > 0.$$

Scaling ε to $\varepsilon\sqrt{2}$ one can apply Theorem 1.1 (ii) with $\ell = 2$ to estimate the repartition function $\mathbb{P}_\varepsilon^\square(\lambda) = \frac{1}{2\pi}|\{\omega \in [0, 2\pi] : \tau_\varepsilon^\square(\omega) > \frac{\lambda}{\varepsilon}\}|$ of the exit time $\tau_\varepsilon^\square(\omega)$ of a billiard in the unit square with pockets of radius ε at the corners and trajectory starting at the center (see Figure 1.3), getting

THEOREM 1.3. — *For every $\delta > 0$, as $\varepsilon \to 0^+$,*

$$\mathbb{P}_\varepsilon^\square(\lambda) = G_2\left(\frac{\lambda}{\sqrt{2}}\right) + O_\delta\left(\varepsilon^{\frac{1}{8}-\delta}\right), \qquad \lambda > 0.$$

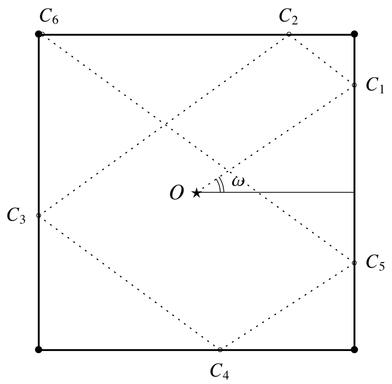


Figure 1.3. The free path in a square billiard with trajectory starting at the center

Theorem 1.3 should be compared with the situation where the initial position is at one of the four vertices, and where the limiting distribution has compact support [2, 3]. It is not clear whether these methods would directly extend to other concrete initial positions, such as $(\frac{1}{n}, 0)$, $n \in \mathbb{N}$. However, it looks likely that further refinements could lead to "space-phase average" results similar to those from [5]. The main difficulty seems to arise from the increasing complexity and number of cases that need to be analyzed in detail, leading to integrals in the main terms of the asymptotic formula which are manifestly more intricate than in the case of the square.

2. The contribution of consecutive Farey fractions $\gamma < \gamma'$ with $\gamma, \gamma' \in \mathcal{F}_Q \setminus \mathcal{F}^{(\ell)}$

Let $\varepsilon > 0$ and $Q = \left[\frac{1}{\varepsilon}\right]$, the integer part of $\frac{1}{\varepsilon}$. Denote by $\mathcal{F}_Q$ the set of Farey fractions $\gamma = \frac{a}{q}$ in lowest terms with $0 < a \leqslant q \leqslant Q$. For any interval $I \subseteq [0,1]$, set $\mathcal{F}_{I,Q} = I \cap \mathcal{F}_Q$. Denote by $\mathcal{F}_Q^{(\ell)}$ the set of Farey fractions $\gamma = \frac{a}{q} \in \mathcal{F}_Q$ with $\ell \mid (q-a)$, and set $\mathcal{F}_{I,Q}^{(\ell)} = I \cap \mathcal{F}_Q^{(\ell)}$. Set also $\mathcal{F}^{(\ell)} = \cup_{Q=1}^{\infty} \mathcal{F}_Q^{(\ell)}$. It is well known that $\gamma = \frac{a}{q} < \gamma' = \frac{a'}{q'}$ are consecutive elements in $\mathcal{F}_Q$ if and only if

$$a'q - aq' = 1 \quad \text{and} \quad q + q' > Q \geqslant \max\{q, q'\}.$$

In particular we have

$$\varepsilon(q+q') \geqslant \varepsilon(Q+1) > 1 \geqslant \max\{\varepsilon q, \varepsilon q'\}. \tag{2.1}$$

For convenience consider

$$\tilde{\mathbb{G}}_{\ell,I,\varepsilon}(\lambda) := \left|\{\omega \in \arctan I : q_{\ell,\varepsilon}(\omega) > \lambda Q\}\right| = \mathbb{G}_{\ell,I,\varepsilon}\left(\lambda\varepsilon\left[\frac{1}{\varepsilon}\right]\right), \quad \lambda > 0.$$

For any interval $I \subseteq [0,1]$ with $|I| \asymp Q^{-c}$, $0 < c < 1$, we will prove a formula of type

$$\tilde{\mathbb{G}}_{\ell,I,\varepsilon}(\lambda) = c_I G_\ell(\lambda) + O_\delta\left(\varepsilon^{-\delta+\theta(c,c_1)}\right),$$

which will immediately imply (1.1), because $\varepsilon\left[\frac{1}{\varepsilon}\right] = 1 + O(\varepsilon)$ and for any compact set $K \subseteq \mathbb{R}_+$ there is $C_K > 0$ such that

$$|G_\ell(x') - G_\ell(x'')| \leqslant C_K |x' - x''|, \qquad x', x'' \in K. \tag{2.2}$$

Given $\gamma < \gamma'$ consecutive elements in $\mathcal{F}_Q$, denote $t_0 = \gamma' - \frac{\varepsilon}{q'} = \frac{a'-\varepsilon}{q'}$, $u_0 = \gamma + \frac{\varepsilon}{q} = \frac{a+\varepsilon}{q}$. Employing (2.1) and

$$t_0 - \gamma = \frac{1-\varepsilon q}{qq'}, \qquad u_0 - t_0 = \frac{\varepsilon(q+q')-1}{qq'}, \qquad \gamma' - u_0 = \frac{1-\varepsilon q'}{qq'}, \tag{2.3}$$

we find $\gamma \leqslant t_0 < u_0 \leqslant \gamma'$. In particular if $\gamma < \gamma' < \gamma''$ are consecutive elements in $\mathcal{F}_Q$, then $\gamma + \frac{\varepsilon}{q} \leqslant \gamma' < \gamma'' - \frac{\varepsilon}{q''}$. Therefore the intervals $[\gamma - \frac{\varepsilon}{q}, \gamma + \frac{\varepsilon}{q}]$, $\gamma = \frac{a}{q} \in \mathcal{F}_Q$, cover the interval $[0,1]$ in such a way that every element in $[0,1]$ belongs to at most two of these intervals. As a result any trajectory with slope $\tan\omega \in (\gamma, \gamma')$ will intersect, as in the case of the square lattice [2, 3], one of the scatterers $(q,a) + V_\varepsilon$ or $(q',a') + V_\varepsilon$.

Since $a'q - aq' = 1$, only two situations can occur here: $\gamma \notin \mathcal{F}^{(\ell)}$ and $\gamma' \notin \mathcal{F}^{(\ell)}$, respectively $\gamma \in \mathcal{F}^{(\ell)}$ or $\gamma' \in \mathcal{F}^{(\ell)}$. The later will be discussed in Section 3. In the remainder of this section we assume that $\gamma \notin \mathcal{F}^{(\ell)}$ and

$\gamma' \notin \mathcal{F}^{(\ell)}$, situation where the horizontal free path is given (see Figure 2.1) by

$$q_{\ell,\varepsilon}(\omega) = \begin{cases} q & \text{if } \gamma < \tan\omega < t_0, \\ \min\{q,q'\} & \text{if } t_0 < \tan\omega < u_0, \\ q' & \text{if } u_0 < \tan\omega < \gamma'. \end{cases}$$

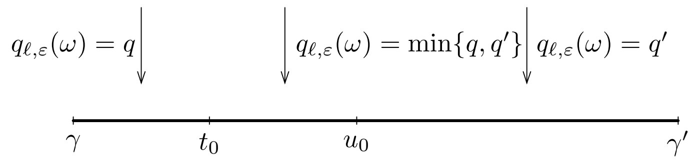


Figure 2.1. The horizontal free path when $\gamma, \gamma' \in \mathcal{F}_{I,Q} \setminus \mathcal{F}^{(\ell)}$

In this way the contribution of the interval $[\gamma,\gamma']$ to $\tilde{\mathbb{G}}_{\ell,I,\varepsilon}(\lambda)$ is given by

$$\begin{cases} 0 & \text{if } \max\{q,q'\} \leqslant \lambda Q, \\ \arctan\gamma' - \arctan\gamma & \text{if } \min\{q,q'\} > \lambda Q, \\ \arctan\gamma' - \arctan u_0 & \text{if } q \leqslant \lambda Q < q', \\ \arctan t_0 - \arctan\gamma & \text{if } q' \leqslant \lambda Q < q. \end{cases}$$

This contribution is zero whenever $\lambda \geqslant 1$, so we next assume $0 < \lambda < 1$.

Using (2.3), the estimates

$$\begin{aligned} \arctan(x+h) - \arctan x &= \frac{h}{1+x^2} + O(h^2), \\ \frac{1}{1+\gamma^2} - \frac{1}{1+\gamma'^2} &\leqslant 2(\gamma'-\gamma) = \frac{2}{qq'}, \end{aligned} \tag{2.4}$$

and the inequality

$$\sum_{\gamma\in\mathcal{F}_Q} \frac{1}{q^2q'^2} \leqslant \sum_{\gamma\in\mathcal{F}_Q} \frac{1}{Qqq'} = \frac{1}{Q},$$

we infer that the contribution to $\tilde{\mathbb{G}}_{\ell,I,\varepsilon}(\lambda)$ of all intervals $[\gamma,\gamma'] \subseteq I$ with $\gamma < \gamma'$ consecutive elements in $\mathcal{F}_Q$ and $\gamma,\gamma' \notin \mathcal{F}^{(\ell)}$ is given by

$$\mathbb{G}^{(1)}_{\ell,I,\varepsilon}(\lambda) = A_{I,Q}(\lambda) + B_{I,Q}(\lambda) + C_{I,Q}(\lambda) + O(\varepsilon),$$

with

$$A_{I,Q}(\lambda) = \sum_{\substack{\gamma,\gamma' \in \mathcal{F}_{I,Q} \setminus \mathcal{F}^{(\ell)} \\ \min\{q,q'\} > \lambda Q}} \frac{1}{qq'} \cdot \frac{1}{1+\gamma^2}, \tag{2.5}$$

$$\begin{aligned} B_{I,Q}(\lambda) &= \sum_{\substack{\gamma,\gamma' \in \mathcal{F}_{I,Q} \setminus \mathcal{F}^{(\ell)} \\ q \leqslant \lambda Q < q'}} \frac{1-\varepsilon q'}{qq'} \cdot \frac{1}{1+\gamma'^2}, \\ C_{I,Q}(\lambda) &= \sum_{\substack{\gamma,\gamma' \in \mathcal{F}_{I,Q} \setminus \mathcal{F}^{(\ell)} \\ q' \leqslant \lambda Q < q}} \frac{1-\varepsilon q}{qq'} \cdot \frac{1}{1+\gamma^2}. \end{aligned} \tag{2.6}$$

Since there are no consecutive elements in $\mathcal{F}_Q$ which belong both to $\mathcal{F}^{(\ell)}$ we have

$$A_{I,Q}(\lambda) = A^{+}_{I,Q}(\lambda) - A^{-,1}_{I,Q}(\lambda) - A^{-,2}_{I,Q}(\lambda), \tag{2.7}$$

with

$$\begin{aligned} A^{+}_{I,Q}(\lambda) &= \sum_{\substack{\gamma,\gamma' \in \mathcal{F}_{I,Q} \\ \min\{q,q'\} > \lambda Q}} \frac{1}{qq'} \cdot \frac{1}{1+\gamma^2}, \\ A^{-,1}_{I,Q}(\lambda) &= \sum_{\substack{\gamma \in \mathcal{F}^{(\ell)}_{I,Q},\, \gamma' \in \mathcal{F}_{I,Q} \\ \min\{q,q'\} > \lambda Q}} \frac{1}{qq'} \cdot \frac{1}{1+\gamma^2}, \\ A^{-,2}_{I,Q}(\lambda) &= \sum_{\substack{\gamma \in \mathcal{F}_{I,Q},\, \gamma' \in \mathcal{F}^{(\ell)}_{I,Q} \\ \min\{q,q'\} > \lambda Q}} \frac{1}{qq'} \cdot \frac{1}{1+\gamma^2}, \end{aligned}$$

and respectively

$$\begin{aligned} B_{I,Q}(\lambda) &= B^{+}_{I,Q}(\lambda) - B^{-,1}_{I,Q}(\lambda) - B^{-,2}_{I,Q}(\lambda), \\ C_{I,Q}(\lambda) &= C^{+}_{I,Q}(\lambda) - C^{-,1}_{I,Q}(\lambda) - C^{-,2}_{I,Q}(\lambda), \end{aligned} \tag{2.8}$$

with

$$B^{+}_{I,Q}(\lambda) = \sum_{\substack{\gamma,\gamma' \in \mathcal{F}_{I,Q} \\ q \leqslant \lambda Q < q'}} \frac{1-\varepsilon q'}{qq'} \cdot \frac{1}{1+\gamma'^2},$$

$$C^{+}_{I,Q}(\lambda) = \sum_{\substack{\gamma,\gamma' \in \mathcal{F}_{I,Q} \\ q' \leqslant \lambda Q < q}} \frac{1-\varepsilon q}{qq'} \cdot \frac{1}{1+\gamma^2},$$

$$B^{-,1}_{I,Q}(\lambda) = \sum_{\substack{\gamma\in\mathcal{F}^{(\ell)}_{I,Q},\,\gamma'\in\mathcal{F}_{I,Q}\\ q\leqslant\lambda Q<q'}} \frac{1-\varepsilon q'}{qq'}\cdot\frac{1}{1+\gamma'^2},$$

$$C^{-,1}_{I,Q}(\lambda) = \sum_{\substack{\gamma\in\mathcal{F}^{(\ell)}_{I,Q},\,\gamma'\in\mathcal{F}_{I,Q}\\ q'\leqslant\lambda Q<q}} \frac{1-\varepsilon q}{qq'}\cdot\frac{1}{1+\gamma^2},$$

$$B^{-,2}_{I,Q}(\lambda) = \sum_{\substack{\gamma\in\mathcal{F}_{I,Q},\,\gamma'\in\mathcal{F}^{(\ell)}_{I,Q}\\ q\leqslant\lambda Q<q'}} \frac{1-\varepsilon q'}{qq'}\cdot\frac{1}{1+\gamma'^2},$$

$$C^{-,2}_{I,Q}(\lambda) = \sum_{\substack{\gamma,\gamma'\in\mathcal{F}_{I,Q}\\ q'\leqslant\lambda Q<q}} \frac{1-\varepsilon q}{qq'}\cdot\frac{1}{1+\gamma^2}.$$

Lemma 2.1. — *For any function $V\in C^1[0,N]$ with total variation $T_0^N V$ and $C(\ell)$ as in* (1.2)*,*

$$\sum_{\substack{1\leqslant k\leqslant N\\ \gcd(\ell,k)=1}} \frac{\varphi(k)}{k}\,V(k) = C(\ell)\int_0^N V + O_\ell\Big((\|V\|_\infty + T_0^N V)\ln N\Big).$$

Proof. — The left-hand side $L_{\ell,N}$ can be expressed as

$$\sum_{\substack{1\leqslant k\leqslant N\\ \gcd(\ell,k)=1}} \sum_{d\mid k} \frac{\mu(d)}{d}\,V(k).$$

Writing $k=dk'$ with $1\leqslant k'\leqslant[\frac{N}{d}]$ and $\gcd(\ell,d)=\gcd(\ell,k')=1$, denoting $V_n(x)=V(nx)$, $\sigma_0(\ell):=\#\{d\geqslant 1 : d\mid\ell\}$, and using Möbius summation (as in [1, Lemma 2.2]) we have

$$
\begin{aligned}
&L_{\ell,N}\\
&= \sum_{\substack{1\leqslant d\leqslant N\\ \gcd(\ell,d)=1}} \frac{\mu(d)}{d} \sum_{\substack{1\leqslant k'\leqslant [N/d]\\ \gcd(\ell,k')=1}} V_d(k')\\
&= \sum_{\substack{1\leqslant d\leqslant N\\ \gcd(\ell,d)=1}} \frac{\mu(d)}{d} \left(\frac{\varphi(\ell)}{\ell} \int_0^{[N/d]} V_d + O\Big(\big(\|V_d\|_\infty + T_0^{[N/d]} V_d \big) \sigma_0(\ell) \Big) \right)\\
&= \frac{\varphi(\ell)}{\ell} \sum_{\substack{1\leqslant d\leqslant N\\ \gcd(\ell,d)=1}} \frac{\mu(d)}{d} \cdot \frac{1}{d} \left(\int_0^N V + O\big(d \|V\|_\infty \big) \right)\\
&\qquad + O_\ell \left(\big(\|V\|_\infty + T_0^N V \big) \sum_{d=1}^N \frac{1}{d} \right)\\
&= \frac{\varphi(\ell)}{\ell} \sum_{\substack{1\leqslant d\leqslant N\\ \gcd(\ell,d)=1}} \frac{\mu(d)}{d^2} \int_0^N V + O\big(\|V\|_\infty \ln N \big) + O_\ell \Big(\big(\|V\|_\infty + T_0^N V \big) \ln N \Big)\\
&= \frac{\varphi(\ell)}{\ell} \left(\sum_{\substack{d\geqslant 1\\ \gcd(\ell,d)=1}} \frac{\mu(d)}{d^2} + O\Big(\sum_{d>N} \frac{1}{d^2} \Big) \right) \int_0^N V + O_\ell \Big(\big(\|V\|_\infty + T_0^N V \big) \ln N \Big)\\
&= C(\ell) \int_0^N V + O_\ell \Big((\|V\|_\infty + T_0^N V) \ln N \Big),
\end{aligned}
$$

with

$$
\begin{aligned}
C(\ell) = \frac{\varphi(\ell)}{\ell} \sum_{\substack{d\geqslant 1\\ \gcd(\ell,d)=1}} \frac{\mu(d)}{d^2} &= \frac{\varphi(\ell)}{\ell} \prod_{\substack{p\nmid \ell\\ p\,\text{prime}}} \left(1 - \frac{1}{p^2} \right)\\
&= \frac{\varphi(\ell)}{\zeta(2)\ell} \prod_{\substack{p\mid \ell\\ p\,\text{prime}}} \left(1 - \frac{1}{p^2} \right)^{-1},
\end{aligned}
$$

which gives the desired estimate. □

LEMMA 2.2. — *For any function $V \in C^1[0,N]$ and any $\delta > 0$,*

$$
\sum_{n=1}^N \frac{\varphi(\ell n)}{n} V(n) = \ell C(\ell) \int_0^N V + O_{\ell,\delta} \left((\|V\|_\infty + T_0^N V) N^\delta \right).
$$

Proof. — Let $\ell = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_1, \dots, p_r$ distinct primes and $\alpha_1, \dots, \alpha_r \in \mathbb{N}$ (so $r = \omega(\ell)$, the number of prime divisors of ℓ).

Writing $n = p_1^{k_1} \cdots p_r^{k_r} m$ with $k_i \geqslant 0$ and $\gcd(\ell, m) = 1$, we obtain

$$(2.9)\quad \sum_{1\leqslant n\leqslant N} \frac{\varphi(\ell n)}{n}\, V(n) = \sum_{\substack{k_1,\dots,k_r\geqslant 0\\ p_1^{k_1}\cdots p_r^{k_r}\leqslant N}} \prod_{1\leqslant i\leqslant r} \frac{\varphi(p_i^{k_i+\alpha_i})}{p_i^{k_i}} \sum_{\substack{1\leqslant m\leqslant N/(p_1^{k_1}\cdots p_r^{k_r})\\ \gcd(\ell,m)=1}} \frac{\varphi(m)}{m}\, V_{p_1^{k_1}\cdots p_r^{k_r}}(m).$$

According to Lemma 2.1 the inner sum above can be expressed as

$$C(\ell)\int_0^{N/(p_1^{k_1}\cdots p_r^{k_r})} V(p_1^{k_1}\cdots p_r^{k_r} x)\, dx + O_\ell\big((\|V\|_\infty + T_0^N V)\ln N\big)$$
$$= \frac{C(\ell)}{p_1^{k_1}\cdots p_r^{k_r}} \int_0^N V + O_\ell\big((\|V\|_\infty + T_0^N V)\ln N\big).$$

Inserting this into (2.9) and using the fact that the number of terms in the first sum in (2.9) is $\ll (\ln N)^{\omega(\ell)}$ and $\varphi(p_i^{k_i+\alpha_i}) = p_i^{k_i+\alpha_i}(1-p_i^{-1})$ we infer that the expression in (2.9) is given by

$$(2.10)\quad \ell C(\ell) \prod_{\substack{p\mid\ell\\ p\,\text{prime}}} \left(1-\frac{1}{p}\right) \sum_{\substack{k_1,\dots,k_r\geqslant 0\\ p_1^{k_1}\cdots p_r^{k_r}\leqslant N}} \frac{1}{p_1^{k_1}\cdots p_r^{k_r}} \int_0^N V + O_\ell\Big((\|V\|_\infty + T_0^N V)(\ln N)^{1+\omega(\ell)}\Big).$$

The statement now follows from (2.10), using

$$\sum_{k_1,\dots,k_r\geqslant 0} \frac{1}{p_1^{k_1}\cdots p_r^{k_r}} = \prod_{\substack{p\mid\ell\\ p\,\text{prime}}} \left(1-\frac{1}{p}\right)^{-1}$$

and the bound

$$\sum_{\substack{k_1,\dots,k_r\geqslant 0\\ p_1^{k_1}\cdots p_r^{k_r}> N}} \frac{1}{p_1^{k_1}\cdots p_r^{k_r}} \ll_{p_1,\dots,p_r} \frac{(\ln N)^{r-1}}{N}.$$

□

The following estimate [4, Proposition A4] will be employed several times.

Lemma 2.3. — *Assume that $q \geqslant 1$ and h are two given integers, $\mathcal{I}$ and $\mathcal{J}$ are intervals of length less than q, and $f : \mathcal{I}\times\mathcal{J} \to \mathbb{R}$ is a C^1 function.*

Then for any integer $T > 1$ and any $\delta > 0$

$$\sum_{\substack{a\in\mathcal{I},b\in\mathcal{J}\\ ab=h \pmod q\\ \gcd(b,q)=1}} f(a,b) = \frac{\varphi(q)}{q^2}\iint_{\mathcal{I}\times\mathcal{J}} f(x,y)\,dxdy + \mathcal{E},$$

with

$$\mathcal{E} \ll_\delta T^2\|f\|_\infty q^{\frac{1}{2}+\delta}\gcd(h,q)^{\frac{1}{2}} + T\|\nabla f\|_\infty q^{\frac{3}{2}+\delta}\gcd(h,q)^{\frac{1}{2}} + \frac{\|\nabla f\|_\infty |\mathcal{I}||\mathcal{J}|}{T},$$

where $\|f\|_\infty$, respectively $\|\nabla f\|_\infty$, denotes the sup-norm of f, respectively of $\left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial y}\right|$, on $\mathcal{I}\times\mathcal{J}$.

Lemma 2.3 will be typically applied to the following situations: Let I be a subinterval of $[0,1]$. For every $q\in[1,Q]$ consider the intervals $\mathcal{I} = qI$ and $\mathcal{J} = J_{\lambda,q} = (\max\{\lambda Q, Q-q\}, Q]$, and the functions f_q and g_q defined on $qI\times J_{\lambda,q}$ by

$$f_q(u,v) := \frac{1}{qv}\cdot\frac{1}{1+\left(\frac{u}{q}\right)^2}, \qquad g_q(u,v) = \frac{1-\varepsilon v}{qv}\cdot\frac{1}{1+\left(\frac{u}{q}\right)^2}.$$

We clearly have

(2.11)
$$\max\{\|f_q\|_\infty, \|g_q\|_\infty\} \ll_\lambda \frac{1}{Qq}, \qquad \max\{\|\nabla f_q\|_\infty, \|\nabla g_q\|_\infty\} \ll_\lambda \frac{1}{Qq^2}.$$

Proposition 2.4. — *For every $\lambda\in(0,1]$ and $c_1\in(0,1)$,*

$$A_{I,Q}(\lambda) = c_I A(\ell) I_1(\lambda) + O_{\delta,\lambda,\ell}(Q^{\delta+\theta_1(c_1)}),$$

where

$$I_1(\lambda) = \int_\lambda^1 \frac{1}{x}\ln\frac{1}{\max\{\lambda,1-x\}}\,dx = \begin{cases} \ln(1-\lambda)\ln\lambda + \int_\lambda^{1-\lambda}\frac{1}{x}\ln\frac{1}{1-x}\,dx & \text{if } \lambda\in\left(0,\frac{1}{2}\right],\\ (\ln\lambda)^2 & \text{if } \lambda\in\left[\frac{1}{2},1\right],\end{cases}$$

and $\theta_1(c_1) = \max\left\{2c_1 - \frac{1}{2}, -c_1, -1\right\}$.

Proof. — There is at most one $\gamma\in\mathcal{F}_{I,Q}$ with $\gamma'\notin I$. Since $\frac{1}{|I|}\ll Q^c$ and $\frac{1}{qq'}\leqslant\frac{1}{Q}$, the total contribution to the final asymptotic results from Theorem 1.1 resulting from replacing the two conditions $\gamma,\gamma'\in\mathcal{F}_{I,Q}$ by $a\in qI$ or by $a'\in q'I$ will be $\ll Q^{-1}$ and respectively $\ll Q^{\frac{1}{8}-1}$, thus negligible. As a result we shall tacitly do this in formulas (2.12), (2.15), (2.18), (2.22), (2.24), and (2.27).

Furthermore, the summation constraints in $\gamma = \frac{a}{q} \in \mathcal{F}_{I,Q}$ and $\gamma' = \frac{a'}{q'} \in \mathcal{F}_Q$ will translate, using standard properties of Farey fractions, into the following constraints on the triplet (q,a,q'):

$$\begin{cases} q \in (\lambda Q, Q], \\ a \in qI, \quad q' \in J_{\lambda,q}, \quad \gcd(q',q)=1, \quad aq' = -1 \pmod{q}. \end{cases}$$

Summing first over q and then after integer pairs $(a,q') \in qI \times J_{\lambda,q}$ as above (note that the number of such pairs is $\leqslant q|I|$) we infer

$$A^{+}_{I,q}(\lambda) = \sum_{q\in(\lambda Q,Q]} \sum_{\substack{(a,q')\in qI\times J_{\lambda,q} \\ \gcd(q',q)=1 \\ aq'\equiv -1 \pmod{q}}} f_q(a,q') + O\left(\frac{1}{Q}\right). \tag{2.12}$$

Applying Lemma 2.3 with $T = [Q^{c_1}]$ and (2.11), the inner sum in (2.12) can be expressed as

$$\begin{aligned} &\frac{\varphi(q)}{q^2}\int_{J_{\lambda,q}} \frac{dv}{qv}\int_{qI} \frac{du}{1+\left(\frac{u}{q}\right)^2} \\ &\qquad + O_{\delta,\lambda}\left(Q^{2c_1} q^{\frac{1}{2}+\delta} Q^{-2} + Q^{c_1} q^{\frac{3}{2}+\delta} Q^{-3} + q^2 Q^{-3} Q^{-c_1}\right) \\ &\quad = c_I \frac{\varphi(q)}{q^2} \ln \frac{1}{\max\left\{\lambda, 1-\frac{q}{Q}\right\}} + O_{\delta,\lambda}\left(Q^{2c_1-\frac{3}{2}+\delta} + Q^{-1-c_1}\right). \end{aligned} \tag{2.13}$$

Summing over $q \in (\lambda Q, Q]$ in (2.13) and employing (2.12) and [1, Lemma 2.3] we find

$$A^{+}_{I,Q}(\lambda) = \frac{c_I}{\zeta(2)} I_1(\lambda) + O_{\delta,\lambda}\left(Q^{\delta+\theta_1(c_1)}\right). \tag{2.14}$$

To estimate $A^{-,1}_{I,Q}(\lambda)$, note first that $\gcd(\ell,q)=1$ because $\ell \mid q-a$ and $\gcd(q,a)=1$. As a result, putting $w = q-a = \ell u$, $v = q'$ and using $a'q - aq' = 1$ and $q' > Q-q$, we find (with $\bar{\ell}$ denoting the multiplicative inverse of $\ell \pmod{q}$)

$$\begin{aligned} A^{-,1}_{I,Q}(\lambda) &= \sum_{\substack{\lambda Q<q\leqslant Q \\ \gcd(\ell,q)=1}} \sum_{\substack{(w,v)\in q(1-I)\times J_{\lambda,q} \\ \ell|w,\, \gcd(v,q)=1 \\ wv\equiv 1 \pmod{q}}} \frac{1}{qv}\cdot\frac{1}{1+\left(\frac{q-w}{q}\right)^2} + O\left(\frac{1}{Q}\right) \\ &= \sum_{\substack{\lambda Q<q\leqslant Q \\ \gcd(\ell,q)=1}} \sum_{\substack{(u,v)\in(q/\ell)(1-I)\times\in J_{\lambda,q} \\ \gcd(v,q)=1 \\ uv=\bar{\ell} \pmod{q}}} f_q(q-\ell u, v) + O\left(\frac{1}{Q}\right). \end{aligned} \tag{2.15}$$

By Lemma 2.3 (with $T=[Q^{c_1}]$) and (2.11) the inner sum in (2.15) can be expressed as

(2.16)
$$\begin{aligned}\frac{\varphi(q)}{q^2}\int_{J_{\lambda,q}}\frac{dv}{qv}\int_{\frac{q}{\ell}(1-I)}\frac{du}{1+\left(\frac{q-\ell u}{q}\right)^2} \;+\; O_{\delta,\lambda,\ell}\left(Q^{2c_1-\frac{3}{2}+\delta}+Q^{-1-c_1}\right)\\ = c_I\frac{\varphi(q)}{\ell q^2}\,\ln\frac{1}{\max\left\{\lambda,1-\frac{q}{Q}\right\}}+O_{\delta,\lambda,\ell}\left(Q^{2c_1-\frac{3}{2}+\delta}+Q^{-1-c_1}\right).\end{aligned}$$

Summing over q in (2.16) and employing (2.15) and Lemma 2.1 we find

$$A^{-,1}_{I,Q}(\lambda)\;=\frac{c_IC(\ell)}{\ell}\,I_1(\lambda)+O_{\delta,\lambda,\ell}\left(Q^{\delta+\theta_1(c_1)}\right). \tag{2.17}$$

Using the second inequality in (2.4) and $\sum_{\gamma\in\mathcal{F}_{I,Q}}\frac{1}{qq'}\leqslant 1$ we infer

$$A^{-,2}_{I,Q}=\sum_{\substack{\gamma\in\mathcal{F}_{I,Q},\,\gamma'\in\mathcal{F}^{(\ell)}_{I,Q}\\ \min\{q,q'\}>\lambda Q}}\frac{1}{qq'}\cdot\frac{1}{1+\gamma'^2}\;+\;O\left(\frac{1}{Q}\right). \tag{2.18}$$

We now proceed as for $A^{-,1}_{I,Q}$, noting that $\gcd(\ell,q')=1$ because $\ell\mid q'-a'$ and $\gcd(q',a')=1$. As above, with $w=q'-a'=\ell u$, $v=q$, we infer

(2.19)
$$\begin{aligned}A^{-,2}_{I,Q}(\lambda)&=\sum_{\substack{q'\in(\lambda Q,Q]\\ \gcd(\ell,q')=1}}\;\sum_{\substack{(w,v)\in q'(1-I)\times J_{\lambda,q'}\\ \ell\mid w,\,\gcd(v,q')=1\\ wv=-1\ (\mathrm{mod}\ q')}}\frac{1}{q'v}\cdot\frac{1}{1+\left(\frac{q'-w}{q'}\right)^2}+O\left(\frac{1}{Q}\right)\\ &=\sum_{\substack{q'\in(\lambda Q,Q]\\ \gcd(\ell,q')=1}}\;\sum_{\substack{(u,v)\in(q'/\ell)(1-I)\times J_{\lambda,q'}\\ \gcd(v,q')=1\\ uv=-\bar{\ell}\ (\mathrm{mod}\ q')}}f_{q'}(q'-\ell u,v)+O\left(\frac{1}{Q}\right)\\ &=\sum_{\substack{q'\in(\lambda Q,Q]\\ \gcd(\ell,q')=1}}\frac{\varphi(q')}{q'^2}\int_{J_{\lambda,q'}}\frac{dv}{q'v}\int_{\frac{q'}{\ell}(1-I)}\frac{du}{1+\left(\frac{q'-\ell u}{q'}\right)^2}\\ &\qquad+O_{\delta,\lambda,\ell}\left(Q(Q^{2c_1-\frac{3}{2}+\delta}+Q^{-1-c_1})\right)\\ &=\frac{c_I}{\ell}\sum_{\substack{q'\in(\lambda Q,Q]\\ \gcd(\ell,q')=1}}\frac{\varphi(q')}{q'^2}\,\ln\frac{1}{\max\left\{\lambda,1-\frac{q'}{Q}\right\}}+O_{\delta,\lambda,\ell}\left(Q^{\delta+\theta_1(c_1)}\right)\\ &=\frac{c_IC(\ell)}{\ell}\,I_1(\lambda)+O_{\delta,\lambda,\ell}\left(Q^{\delta+\theta_1(c_1)}\right).\end{aligned}$$

The estimate for $A_{I,Q}(\lambda)$ follows from (2.7), (2.14), (2.17) and (2.19). □

PROPOSITION 2.5. — *For every* $\lambda \in (0,1]$ *and* $c_1 \in (0,1)$,

$$B_{I,Q}(\lambda) = c_I A(\ell) I_2(\lambda) + O_{\delta,\lambda,\ell}\left(Q^{\delta+\theta_1(c_1)}\right) = C_{I,Q}(\lambda),$$

where

$$I_2(\lambda) = \int_{\max\{\lambda,1-\lambda\}}^1 \frac{1-x}{x} \ln \frac{\lambda}{1-x}\, dx.$$

Proof. — Using the argument leading to (2.18) we see that

$$B^+_{I,Q}(\lambda) = \tilde{B}^+_{I,Q}(\lambda) + O\left(\frac{1}{Q}\right) = \sum_{\substack{\gamma,\gamma' \in \mathcal{F}_{I,Q} \\ q \leqslant \lambda Q < q'}} \frac{1-\varepsilon q'}{qq'} \cdot \frac{1}{1+\gamma^2} + O\left(\frac{1}{Q}\right).$$

Using customary properties of Farey fractions we infer (setting $u = q-a$, $v = q'$)

$$\tilde{B}^+_{I,Q}(\lambda) = \sum_{q \leqslant \lambda Q} \sum_{\substack{(u,v) \in q(1-I) \times J_{\lambda,q} \\ uv \equiv 1 \pmod q}} g_q(q-u, v) + O\left(\frac{1}{Q}\right).$$

Applying Lemma 2.3 to g_q and $T = [Q^{c_1}]$ to the inner sum above and using (2.11) we find

$$\begin{aligned} B^+_{I,Q}(\lambda) = \sum_{q \leqslant \lambda Q} \Bigg(\frac{\varphi(q)}{q^2} \int_{q(1-I)} \frac{du}{1+\frac{(q-u)^2}{q^2}} \int_{J_{\lambda,q}} \frac{1-\varepsilon v}{qv}\, dv \\ + O_\delta\left(\frac{Q^{2c_1} q^{\frac12+\delta}}{Qq} + \frac{Q^{c_1} q^{\frac32+\delta}}{Qq^2} + \frac{q^2 \frac{1}{Qq^2}}{Q^{c_1}} \right)\Bigg) \\ = c_I \sum_{q \leqslant \lambda Q} \frac{\varphi(q)}{q} V(q) + O_\delta\left(Q^{\max\{2c_1-\frac12+\delta,-c_1\}}\right), \end{aligned}$$

where

$$V(n) = \frac{1}{n} \int_{\max\{\lambda,1-\frac{n}{Q}\}}^1 \frac{1-y}{y}\, dy, \qquad n \in [1, \lambda Q].$$

Using

$$\|V\|_\infty \leqslant \frac{1}{\lambda Q} \qquad \text{and} \qquad T_0^{\lambda Q} V \ll_\lambda \frac{1}{Q}, \tag{2.20}$$

and applying Möbius summation to V (e.g. [1, Lemma 2.3]) and Tonelli's theorem we find

$$\begin{aligned} B^{+}_{I,Q}(\lambda) &= \frac{c_I}{\zeta(2)} \int_0^{\lambda Q} \frac{du}{u} \int_{\max\{\lambda, 1-\frac{u}{Q}\}}^1 \frac{1-y}{y}\, dy + O_\delta\left(Q^{\delta+\theta_1(c_1)}\right) \\ &= \frac{c_I}{\zeta(2)} \int_0^{\lambda} \frac{dx}{x} \int_{\max\{\lambda, 1-x\}}^1 \frac{1-y}{y}\, dy + O_\delta\left(Q^{\delta+\theta_1(c_1)}\right) \\ &= \frac{c_I}{\zeta(2)}\, I_2(\lambda) + O_\delta\left(Q^{\delta+\theta_1(c_1)}\right). \end{aligned} \tag{2.21}$$

The condition $\ell \mid q-a$ gives $\gcd(\ell,q)=1$. Taking $q-a=\ell u$ and $v=q'$ and proceeding as in the case of $A^{-,1}_{I,Q}$ from the proof of Proposition 2.4 we have

(2.22)
$$\begin{aligned} B^{-,1}_{I,Q}(\lambda) &= \sum_{\substack{\gamma\in\mathcal{F}^{(\ell)}_{I,Q},\, \gamma'\in\mathcal{F}_{I,Q} \\ q\leqslant \lambda Q<q'}} \frac{1-\varepsilon q'}{qq'} \cdot \frac{1}{1+\gamma^2} \\ &= \sum_{\substack{q\leqslant\lambda Q \\ \gcd(\ell,q)=1}} \sum_{\substack{(u,v)\in(q/\ell)(1-I)\times J_{\lambda,q} \\ \gcd(v,q)=1 \\ uv\equiv\bar{\ell} \pmod q}} \frac{1-\varepsilon v}{qv} \cdot \frac{1}{1+\left(\frac{q-\ell u}{q}\right)^2} + O\left(\frac{1}{Q}\right) \\ &= \frac{c_I}{\ell} \sum_{\substack{q\leqslant\lambda Q \\ \gcd(\ell,q)=1}} \frac{\varphi(q)}{q}\, V(q) + O_{\delta,\ell}\left(Q^{\delta+\theta_1(c_1)}\right). \end{aligned}$$

Applying Lemma 2.1 to the last sum in (2.22) we find

$$\begin{aligned} B^{-,1}_{I,Q} &= \frac{c_I C(\ell)}{\ell} \int_0^{\lambda Q} \frac{du}{u} \int_{\max\{\lambda,1-\frac{u}{Q}\}}^1 \frac{1-y}{y}\, dy + O_{\delta,\ell}\left(Q^{\delta+\theta_1(c_1)}\right) \\ &= \frac{c_I C(\ell)}{\ell}\, I_2(\lambda) + O_{\delta,\ell}\left(Q^{\delta+\theta_1(c_1)}\right). \end{aligned} \tag{2.23}$$

To estimate $B^{-,2}_{I,Q}$ we first fix $q\in(\lambda Q, Q]$, then set $u=q-a$, $v=q'$. Since $\ell \mid q'-a'$ and $\gcd(q',a')=1$ we have $\gcd(\ell,q')=1$. Moreover,

$q'-a'=\frac{q'u-1}{q}$ is divisible by ℓ, so $uv\equiv 1 \pmod{(\ell q)}$ and we have

$$\begin{aligned}B^{-,2}_{I,Q}(\lambda) &= \sum_{\substack{\gamma\in\mathcal{F}_{I,Q},\,\gamma'\in\mathcal{F}^{(\ell)}_{I,Q}\\ q\leqslant\lambda Q<q'}} \frac{1-\varepsilon q'}{qq'}\cdot\frac{1}{1+\left(\frac{a}{q}\right)^2}\\ &= \sum_{q\leqslant\lambda Q}\ \sum_{\substack{(u,v)\in q(1-I)\times J_{\lambda,q}\\ \gcd(\ell q,v)=1\\ uv\equiv 1 \pmod{(\ell q)}}} g_q(q-u,v)+O\left(\frac{1}{Q}\right).\end{aligned} \tag{2.24}$$

By (2.24), Lemma 2.3 and (2.11) it follows that $B^{-,2}_{I,Q}(\lambda)$ can be expressed as

$$\begin{aligned}&\sum_{q\leqslant\lambda Q}\left(\frac{\varphi(\ell q)}{\ell^2 q^2}\ qc_I\,\frac{1}{q}\int_{J_{\lambda,q}}\frac{1-\frac{v}{Q}}{v}\ dv+O_\delta\left(Q^{2c_1-1}q^{-\frac12+\delta}+Q^{-c_1-1}\right)\right)\\ &\qquad=\frac{c_I}{\ell^2}\sum_{q\leqslant\lambda Q}\frac{\varphi(\ell q)}{q}\ V(q)+O_\delta\left(Q^{\max\{2c_1-\frac12+\delta,-c_1\}}\right).\end{aligned} \tag{2.25}$$

From (2.25), Lemma 2.2 and (2.20) we infer

$$\begin{aligned}B^{-,2}_{I,Q}(\lambda) &= \frac{c_I C(\ell)}{\ell}\int_0^{\lambda Q} V+O_{\delta,\ell}\left(Q^{\delta+\theta_1(c_1)}\right)\\ &= \frac{c_I C(\ell)}{\ell}\ I_2(\lambda)+O_{\delta,\ell}\left(Q^{\delta+\theta_1(c_1)}\right).\end{aligned} \tag{2.26}$$

The desired estimate on $B_{I,Q}(\lambda)$ follows from (2.8), (2.21), (2.23) and (2.26).

In similar fashion one gets

(2.27)

$$\begin{aligned}
C^{+}_{I,Q}(\lambda) &= \sum_{\substack{\gamma,\gamma'\in\mathcal{F}_{I,Q}\\ q'\leqslant\lambda Q<q}} \frac{1-\varepsilon q}{qq'}\cdot\frac{1}{1+\gamma'^{2}} \\
&= \sum_{q'\leqslant\lambda Q} \sum_{\substack{(u,v)=(q'-a',q)\in q'(1-I)\times J_{\lambda,q'}\\ \gcd(v,q')=1,\, uv\equiv -1 \pmod{q'}}} g_{q'}(q'-u,v) + O\left(\frac{1}{Q}\right) \\
&= \frac{c_I}{\zeta(2)}\ I_2(\lambda) + O_\delta\left(Q^{\delta+\theta_1(c_1)}\right), \\
C^{-,1}_{I,Q}(\lambda) &= \sum_{\substack{\gamma\in\mathcal{F}^{(\ell)}_{I,Q},\,\gamma'\in\mathcal{F}_{I,Q}\\ q'\leqslant\lambda Q<q}} \frac{1-\varepsilon q}{qq'}\cdot\frac{1}{1+\gamma'^{2}} \\
&= \sum_{q'\leqslant\lambda Q} \sum_{\substack{(u,v)=(q'-a',q)\in q'(1-I)\times J_{\lambda,q'}\\ \gcd(v,q')=1\; uv\equiv -1 \pmod{(\ell q')}}} g_{q'}(q'-u,v) + O\left(\frac{1}{Q}\right) \\
&= \sum_{q'\leqslant\lambda Q} \left(\frac{\varphi(\ell q')}{\ell^2 q'^2}\ q' c_I V(q') + O_\delta\left(Q^{\max\{2c_1-\frac{3}{2}+\delta,-c_1-1\}}\right)\right) \\
&= \frac{c_I C(\ell)}{\ell}\ I_2(\lambda) + O_{\delta,\lambda,\ell}\left(Q^{\delta+\theta_1(c_1)}\right), \\
C^{-,2}_{I,Q}(\lambda) &= \sum_{\substack{\gamma\in\mathcal{F}_{I,Q},\,\gamma'\in\mathcal{F}^{(\ell)}_{I,Q}\\ q'\leqslant\lambda Q<q}} \frac{1-\varepsilon q}{qq'}\cdot\frac{1}{1+\gamma'^{2}} \\
&= \sum_{\substack{q'\leqslant\lambda Q\\ \gcd(\ell,q')=1}} \sum_{\substack{(u,v)=\left(\frac{q'-a'}{\ell},q\right)\in\frac{q'}{\ell}(1-I)\times J_{\lambda,q'}\\ \gcd(v,q')=1,\, uv\equiv -\bar{\ell} \pmod{q'}}} g_{q'}(q'-\ell u,v) + O\left(\frac{1}{Q}\right) \\
&= \sum_{\substack{q'\leqslant\lambda Q\\ \gcd(\ell,q')=1}} \frac{\varphi(q')}{q'^2}\cdot\frac{q'c_I}{\ell}\int_{J_{\lambda,q'}} \frac{1-\varepsilon v}{q'v}\ dv + O_{\delta,\lambda,\ell}\left(Q^{\delta+\theta_1(c_1)}\right) \\
&= \frac{c_I C(\ell)}{\ell}\ I_2(\lambda) + O_{\delta,\lambda,\ell}\left(Q^{\delta+\theta_1(c_1)}\right).
\end{aligned}$$

The desired estimate on $C_{I,Q}(\lambda)$ follows from (2.8) and (2.27). □

Corollary 2.6. — *For every $\lambda > 0$ and $\delta > 0$,*

$$\mathbb{G}^{(1)}_{\ell,I,\varepsilon}(\lambda) = c_I A(\ell) G^{(1)}(\lambda) + O_{\delta,\ell}\left(\varepsilon^{-\delta+\theta(c,c_1)}\right),$$

with $G^{(1)}$ *given by*

$$\begin{cases} \ln(1-\lambda)\ln\lambda + \int_\lambda^{1-\lambda} \frac{1}{u}\ln\frac{1}{1-u}\,du + 2\int_{1-\lambda}^1 \frac{1-u}{u}\ln\frac{\lambda}{1-u}\,du & \textit{if } \lambda \in \left(0,\frac{1}{2}\right], \\ (\ln\lambda)^2 + 2\int_\lambda^1 \frac{1-u}{u}\ln\frac{\lambda}{1-u}\,du & \textit{if } \lambda \in \left[\frac{1}{2},1\right], \\ 0 & \textit{if } \lambda \in [1,\infty). \end{cases}$$

3. The contribution of consecutive Farey fractions $\gamma < \gamma'$ with $\gamma \in \mathcal{F}_Q^{(\ell)}$ or $\gamma' \in \mathcal{F}_Q^{(\ell)}$

Suppose first that $\gamma < \gamma'$ are consecutive in $\mathcal{F}_{I,Q}$ and $\gamma \in \mathcal{F}_{I,Q}^{(\ell)}$, where again $Q = \left[\frac{1}{\varepsilon}\right]$. Consider $t_{-1} = \gamma'$ and for $k \geqslant 0$ set

$$a_k = ka + a', \qquad q_k = kq + q', \qquad \gamma_k = \frac{a_k}{q_k}, \qquad t_k = \gamma_k - \frac{\varepsilon}{q_k} = \frac{a_k - \varepsilon}{q_k}.$$

Inequalities (2.1) show that

(3.1)
$$\gamma \xleftarrow{k} \gamma_{k+1} < \gamma_k < \cdots < \gamma_1 < \gamma_0 = \gamma' \quad \text{and} \quad t_{k+1} < t_k < \gamma_{k+1} < \gamma_k.$$

Since $q \equiv a \pmod{\ell}$ and $a'q - aq' = 1$, we cannot have $q' \equiv a' \pmod{\ell}$, and so $\gamma' \in \mathcal{F}_{I,Q} \backslash \mathcal{F}^{(\ell)}$. Moreover, since $\ell \mid (q-a)$ we must have $\gcd(\ell, q) = 1$ and $q_k \not\equiv a_k \pmod{\ell}$ for all $k \geqslant 0$.

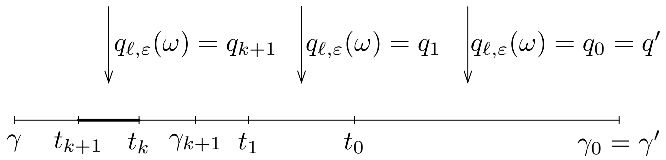


Figure 3.1. The horizontal free path when $\gamma \in \mathcal{F}_{I,Q}^{(\ell)}$ *and* $\gamma' \in \mathcal{F}_{I,Q} \setminus \mathcal{F}^{(\ell)}$

The number $K = \left[\frac{\lambda Q - q'}{q}\right]$ is the unique integer $K \geqslant 1$ for which $q_K = Kq + q' \leqslant \lambda Q < q_{K+1} = (K+1)q + q'$. The presence of the sink at γ, (3.1) and $\gamma_k \notin \mathcal{F}^{(\ell)}$ show that (see also Figure 3.1)

$$q_{\ell,\varepsilon}(\omega) = q_{k+1} \quad \text{if } t_{k+1} < \tan\omega < t_k, \;\; k \geqslant -1,$$

and the contribution to $\mathbb{G}_{I,Q}(\lambda)$ of the interval $[\gamma, \gamma']$ is given by

$$\text{(3.2)} \quad \arctan t_K - \arctan\gamma = \frac{1 - \varepsilon q}{q(Kq+q')} \cdot \frac{1}{1+\gamma^2} + O\left(\frac{1}{q^2(Kq+q')^2}\right).$$

Since $qq' \geqslant Q$ and $\sum_{\gamma\in\mathcal{F}_{I,Q}} \frac{1}{qq'} \leqslant 1$, it follows from (3.2) that the total contribution to $\tilde{\mathbb{G}}_{\ell,I,\varepsilon}(\lambda)$ of intervals $[\gamma,\gamma'] \subseteq I$ with $\gamma < \gamma'$ consecutive in $\mathcal{F}_Q$ and $\gamma \in \mathcal{F}^{(\ell)}$ is given by

$$\mathbb{G}^{(2)}_{\ell,I,\varepsilon}(\lambda) = \sum_{k=0}^{\infty} \sum_{\substack{\gamma\in\mathcal{F}^{(\ell)}_{I,Q} \\ q_k\leqslant\lambda Q<q_{k+1}}} \frac{1-\varepsilon q}{q(kq+q')} \cdot \frac{1}{1+\gamma^2} + O\left(\frac{1}{Q}\right) = S_{I,Q}(\lambda) + O\left(\frac{1}{Q}\right).$$

When $\gamma' \in \mathcal{F}^{(\ell)}_{I,Q}$ similar bookkeeping with $q'_k = kq' + q$, $a'_k = ka' + a$, $u_k = \frac{a'_k+\varepsilon}{q'_k}$, provides the contribution

$$\begin{aligned}\mathbb{G}^{(3)}_{\ell,I,\varepsilon}(\lambda) &= \sum_{k=0}^{\infty} \sum_{\substack{\gamma'\in\mathcal{F}^{(\ell)}_{I,Q} \\ q'_k\leqslant\lambda Q<q'_{k+1}}} (\arctan\gamma' - \arctan u_k) \\ &= \sum_{k=0}^{\infty} \sum_{\substack{\gamma'\in\mathcal{F}^{(\ell)}_{I,Q} \\ q'_k\leqslant\lambda Q<q'_{k+1}}} \frac{1-\varepsilon q'}{q'(kq'+q)} \cdot \frac{1}{1+\gamma'^2} + O\left(\frac{1}{Q}\right).\end{aligned}$$

The situation is analogous to the one encountered in [5, Section 5]. Consider the "Farey triangle" $\mathcal{T} = \{(x,y)\in(0,1]^2 : x+y>1\}$, the sets

$$\Omega_k = \left\{(x,y)\in\mathbb{R}^2 : \left[\frac{\lambda-y}{x}\right] = k\right\}, \quad k\geqslant 0,$$
$$I_k = \left[\frac{\lambda-1}{k}, \frac{\lambda-1}{k-1}\right) \cap [0,1), \quad k\geqslant 1,$$

and for $q\in QI_k$, $k\geqslant 1$, the intervals

$$\begin{aligned}J^{(0)}_{k,q} &= (\lambda - k\varepsilon q, 1] = \{\varepsilon q' : (\varepsilon q, \varepsilon q') \in \Omega_{k-1}\cap\mathcal{T}\} \subseteq (1-\varepsilon q, 1], \\ J^{(1)}_{k,q} &= (1-\varepsilon q, \lambda - k\varepsilon q] = \{\varepsilon q' : (\varepsilon q, \varepsilon q') \in \Omega_k\cap\mathcal{T}\} \subseteq (1-\varepsilon q, 1].\end{aligned}$$

Denote

$$f_{k,q}(q',a) = \frac{1-\varepsilon q}{q(kq+q')} \cdot \frac{1}{1+\left(\frac{a}{q}\right)^2}, \quad g_{k,q}(x,y) := f_{k,q}(x, q-y), \quad k\geqslant 0.$$

When $\lambda\geqslant 2$, we have

$$\min\{k : \Omega_k\cap\mathcal{T}\neq\emptyset\} = [\lambda]-1\geqslant 1, \min\{k : I_k\neq\emptyset\} = [\lambda]\geqslant 2,$$

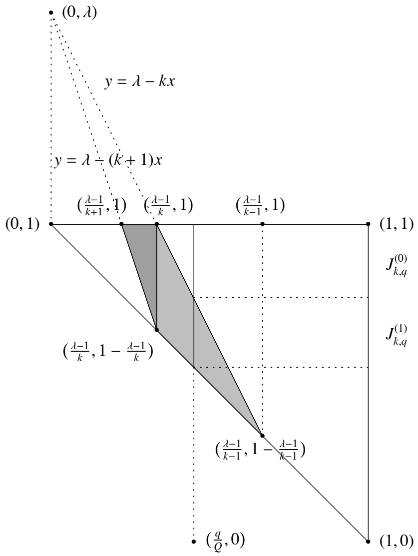


Figure 3.2. The set $\Omega_k \cap \mathcal{T}$

and

$$S_{I,Q}(\lambda) = \sum_{k=1}^{\infty} \sum_{\substack{\gamma \in \mathcal{F}_{I,Q}^{(\ell)} \\ (q,q') \in Q(\Omega_k \cap \mathcal{T})}} f_{k,q}(q',a) = \sum_{k=2}^{\infty} \sum_{\substack{q \in QI_k \\ \gcd(\ell,q)=1}} \Big(S_{I,Q,k}(\lambda,q) + T_{I,Q,k}(\lambda,q) \Big),$$

with

$$S_{I,Q,k}(\lambda,q) = \sum_{\substack{(a,q') \in qI \times QJ_{k,q}^{(1)} \\ -aq' \equiv 1 \pmod{q},\ \ell | (q-a)}} f_{k,q}(q',a),$$

$$T_{I,Q,k}(\lambda,q) = \sum_{\substack{(a,q') \in qI \times QJ_{k,q}^{(0)} \\ -aq' \equiv 1 \pmod{q},\, \ell | (q-a)}} f_{k-1,q}(q',a).$$

Taking $x = q'$, $y = q - a = \ell z \in q(1-I)$, and $\bar{\ell}$ the multiplicative inverse of $\ell \,(\mathrm{mod}\ q)$, we can write

$$S_{I,Q,k}(\lambda, q) = \sum_{\substack{(x,y)\in QJ^{(1)}_{k,q}\times q(1-I),\\ xy\equiv 1 \pmod q,\, \ell | y}} f_{k,q}(x, q-y) = \sum_{\substack{(x,z)\in QJ^{(1)}_{k,q}\times (q/\ell)(1-I),\\ \gcd(x,q)=1,\, xz\equiv \bar{\ell} \pmod q}} g_{k,q}(x, \ell z),$$

$$T_{I,Q,k}(\lambda, q) = \sum_{\substack{(x,z)\in QJ^{(0)}_{k,q}\times (q/\ell)(1-I),\\ \gcd(x,q)=1,\, xz\equiv \bar{\ell} \pmod q}} g_{k-1,q}(x, \ell z).$$

Since $(q,x) \in Q\Omega_k$, we have $Q-(k+1)q < x \leqslant \lambda - kq$, so $(\lambda-1)Q \leqslant \lambda Q - q < kq + x \leqslant t$ and one finds

$$\|g_{k,q}\|_\infty \leqslant \frac{1}{qQ}, \qquad \|\nabla g_{k,q}\|_\infty \leqslant \frac{3}{q^2 Q}, \qquad \forall k \geqslant 1.$$

Since $\ell \mid (q-a)$ and $\gcd(q, q-a) = 1$ we have $\gcd(\bar{\ell}, q) = 1$. The length of each of the intervals $QJ^{(0)}_{k,q}$ and $QJ^{(1)}_{k,q}$ is less than q, so we can apply Lemma 2.3 with $T = [Q^{c_1}]$ to find

$$(3.3)\quad \begin{aligned} S_{I,Q,k}(\lambda, q) &= \frac{\varphi(q)}{q^2} \iint_{QJ^{(1)}_{k,q}\times \frac{q}{\ell}(1-I)} g_{k,q}(x, \ell z)\, dx\, dz + \mathcal{E}^{(1)}_{k,q,\ell} \\ &= \frac{c_I}{\ell} \cdot \frac{\varphi(q)}{q^2} \cdot \frac{Q-q}{Q} \int_{Q-q}^{\lambda Q - kq} \frac{dx}{x+kq} + \mathcal{E}^{(1)}_{k,q,\ell}, \end{aligned}$$

with

$$(3.4)\quad \mathcal{E}^{(1)}_{k,q,\ell} \ll_{\delta,\ell} T^2 \frac{1}{qQ}\, q^{\frac{1}{2}+\delta} + T\, \frac{1}{q^2 Q}\, q^{\frac{3}{2}+\delta} + \frac{q\, \frac{Q}{Q^c} \cdot \frac{1}{q^2 Q}}{T} \ll Q^{2c_1 - 1} q^{-\frac{1}{2}+\delta} + \frac{Q^{-c-c_1}}{q}.$$

A similar argument shows that $T_{I,Q,k}(\lambda, q)$ can be expressed as

$$(3.5)\quad \frac{c_I}{\ell} \cdot \frac{\varphi(q)}{q^2} \cdot \frac{Q-q}{Q} \int_{\lambda Q - kq}^{Q} \frac{dx}{x+(k-1)q} + O_{\delta,\ell}\left(Q^{2c_1-1} q^{-\frac{1}{2}+\delta} + \frac{Q^{-c-c_1}}{q} \right).$$

But

$$\int_{Q-q}^{\lambda Q-kq} \frac{dx}{x+kq} + \int_{\lambda Q-kq}^{Q} \frac{dx}{x+(k-1)q} = \int_{Q+(k-1)q}^{\lambda Q} \frac{du}{u} + \int_{\lambda Q-q}^{Q+(k-1)q} \frac{du}{u} = \ln \frac{\lambda Q}{\lambda Q - q},$$

hence (3.3)–(3.5) show that $S_{I,Q,k}(\lambda, q) + T_{I,Q,k}(\lambda, q)$ can be expressed as

$$\frac{c_I}{\ell} \cdot \frac{\varphi(q)}{q^2} \cdot \frac{Q-q}{Q} \ln \frac{\lambda Q}{\lambda Q - q} + O_{\delta,\ell}\left(Q^{2c_1-1} q^{-\frac{1}{2}+\delta} + \frac{Q^{-c-c_1}}{q} \right). \tag{3.6}$$

The intervals I_k are disjoint, so when summing over k and $q \in QI_k$ (or in a smaller range) we are actually summing over $q \in [1, Q]$. This way in (3.6) the error will sum up to

$$O_{\delta,\ell}\left(Q^{2c_1-1} \sum_{q \leqslant Q} q^{-\frac{1}{2}+\delta} + Q^{-c-c_1} \sum_{q \leqslant Q} \frac{1}{q} \right) = O_{\ell,\delta}\Big(Q^{\delta - \theta(c,c_1)} \Big),$$

while the main term will sum up to

$$M_{\ell,I}(Q) = \frac{c_I}{\ell} \sum_{\substack{q \leqslant Q \\ \gcd(\ell,q)=1}} \frac{\varphi(q)}{q} W(q),$$

where

$$W(q) = \frac{Q-q}{qQ} \ln \frac{\lambda Q}{\lambda Q - q}, \qquad q \in [1, Q],$$

with $\|W\|_\infty \ll \frac{\lambda}{Q}$ and $T_0^Q W \ll \frac{\lambda}{Q}$. Lemma 2.1 now provides

$$M_{\ell,I}(Q) = \frac{c_I C(\ell)}{\ell} \int_0^Q W(q)\, dq + O\left(\frac{\lambda \ln Q}{Q} \right),$$

and we proved

Proposition 3.1. — *For any $\delta > 0$, uniformly in λ on compacts of $[2, \infty)$,*

$$\mathbb{G}^{(2)}_{\ell,I,\varepsilon}(\lambda) = \frac{c_I C(\ell)}{\ell} \int_0^1 \frac{1-u}{u} \ln \frac{\lambda}{\lambda - u}\, du + O_{\delta,\ell}\Big(\varepsilon^{-\delta + \theta(c,c_1)} \Big). \tag{3.7}$$

An identical formula holds for $\mathbb{G}^{(3)}_{\ell,I,\varepsilon}(\lambda)$.

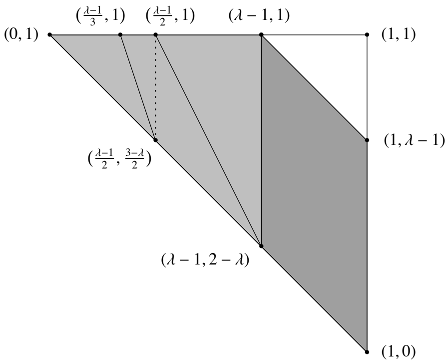


Figure 3.3. The set $\cup_{k=1}^{\infty}\Omega_k \cap \mathcal{T}$ *when* $1 < \lambda < 2$

When $1 < \lambda < 2$ the contribution to $S_{I,Q}(\lambda)$ of γ with $q + q' \leqslant \lambda Q$ is

$$\begin{aligned} A_{I,Q}(\lambda) &= \sum_{k=1}^{\infty} \sum_{\substack{\gamma\in\mathcal{F}_{I,Q}^{(\ell)} \\ (q,q')\in Q(\Omega_k\cap\mathcal{T})}} f_{k,q}(q',a) \\ &= \sum_{k=2}^{\infty} \sum_{\substack{q\in QI_k \\ \gcd(\ell,q)=1}} \sum_{\substack{(a,q')\in qI\times QJ_{k,q}^{(1)} \\ aq'\equiv -1 \pmod{q},\, \ell|(q-a)}} f_{k,q}(q',a) \\ &\quad + \sum_{k=2}^{\infty} \sum_{\substack{q\in QI_k \\ \gcd(\ell,q)=1}} \sum_{\substack{(a,q')\in qI\times QJ_{k,q}^{(0)} \\ aq'\equiv -1 \pmod{q},\, \ell|(q-a)}} f_{k-1,q}(q',a) \\ &\quad + \sum_{\substack{(\lambda-1)Q\leqslant q\leqslant Q \\ \gcd(\ell,q)=1}} \sum_{\substack{(a,q')\in qI\times QJ_{1,q}^{(1)} \\ aq'\equiv -1 \pmod{q},\, \ell|(q-a)}} f_{1,q}(q',a), \end{aligned}$$

and the contribution of γ with $q + q' > \lambda Q$ is

$$B_{I,Q}(\lambda) = \sum_{\substack{(\lambda-1)Q<q\leqslant Q \\ \gcd(\ell,q)=1}} \sum_{\substack{(a,q')\in qI\times(\lambda Q-q,Q] \\ aq'\equiv -1 \pmod{q},\, \ell|(q-a)}} f_{0,q}(q',a).$$

Using $\|f_{k,q}\|_\infty \leqslant \frac{1}{qQ}$ and $\|\nabla f_{k,q}\|_\infty \leqslant \frac{3}{q^2Q}$ on $(QJ_{k,q}^{(0)} \cup QJ_{k+1,q}^{(1)}) \times qI$ if $k \geqslant 1$, $\|f_{0,q}\|_\infty \leqslant \frac{1}{(\lambda-1)qQ}$ and $\|\nabla f_{0,q}\|_\infty \leqslant \frac{1}{(\lambda-1)q^2Q}$ on $QJ_{1,q}^{(0)} \times qI$, and summing as in the case $\lambda > 2$ we also obtain

PROPOSITION 3.2. — *For every $\delta > 0$, uniformly in λ on compacts of $(1,2]$,*

$$\mathbb{G}^{(2)}_{\ell,I,\varepsilon}(\lambda) = \frac{c_I C(\ell)}{\ell} \int_0^1 \frac{1-u}{u} \ \ln \frac{\lambda}{\lambda - u}\ du + O_{\delta,\ell}\Big(\varepsilon^{-\delta+\theta(c,c_1)}\Big). \tag{3.8}$$

An identical asymptotic formula holds for $\mathbb{G}^{(3)}_{\ell,I,\varepsilon}(\lambda)$.

As a result formula (3.7) will also hold when $1 < \lambda \leqslant 2$.

Consider finally the case $0 < \lambda \leqslant 1$. When $q' > \lambda Q$ we have $q_k > \lambda Q$ for all $k \geqslant 0$. So the contribution of each $\gamma \in \mathcal{F}^{(\ell)}_{I,Q}$ with $q' > \lambda Q$ is in this case

$$\arctan \gamma' - \arctan \gamma = \frac{1}{qq'} \cdot \frac{1}{1+\gamma^2} + O\left(\frac{1}{q^2 q'^2}\right),$$

summing up to

$$C_{I,Q}(\lambda) = \sum_{\substack{1\leqslant q\leqslant Q \\ \gcd(\ell,q)=1}} \ \sum_{\substack{(a,q')\in qI\times J_{\lambda,q} \\ \gcd(q',q)=1,\, \ell|(q-a) \\ aq'\equiv -1 \pmod q}} \frac{1}{qq'} \cdot \frac{1}{1+\left(\frac{a}{q}\right)^2} + O\left(\frac{1}{Q}\right). \tag{3.9}$$

Using the same estimates as in case $\lambda > 2$, (3.9) leads to

(3.10)

$$\begin{aligned} C_{I,Q}(\lambda) &= \frac{c_I C(\ell)}{\ell} \int_0^1 \frac{1}{u} \ln \frac{1}{\max\{1-u,\lambda\}}\, du + O_{\delta,\ell}\Big(Q^{\delta-\theta(c,c_1)}\Big) \\ &= \frac{c_I C(\ell)}{\ell} \left(\int_0^{1-\lambda} \frac{1}{u} \ \ln \frac{1}{1-u}\ du + \ln(1-\lambda)\ln\lambda \right) + O_{\delta,\ell}\Big(Q^{\delta-\theta(c,c_1)}\Big). \end{aligned}$$

Finally the contribution of each $\gamma \in \mathcal{F}^{(\ell)}_{I,Q}$ with $q' \leqslant \lambda Q$ is

$$\arctan t_0 - \arctan \gamma = \frac{1-\varepsilon q}{qq'} \cdot \frac{1}{1+\gamma^2} + O\left(\frac{1}{q^2 q'^2}\right),$$

summing up to

(3.11)

$$\begin{aligned} D_{I,Q}(\lambda) &= \sum_{\substack{(1-\lambda)Q<q\leqslant Q \\ \gcd(\ell,q)=1}} \ \sum_{\substack{(a,q')\in qI\times(Q-q,\lambda Q] \\ \gcd(q',q)=1,\, \ell|(q-a) \\ aq'\equiv -1 \pmod q}} \frac{1-\varepsilon q}{qq'} \cdot \frac{1}{1+\left(\frac{a}{q}\right)^2} + O\left(\frac{1}{Q}\right) \\ &= \frac{c_I C(\ell)}{\ell} \int_{1-\lambda}^1 \frac{1-u}{u} \ \ln \frac{\lambda}{1-u}\, du + O_{\delta,\ell}\left(\lambda Q^{-1+\delta} + Q^{\delta+\max\{2c_1-\frac{1}{2},-c-c_1\}}\right). \end{aligned}$$

From (3.10) and (3.11) we infer, uniformly in λ on compacts of $(0,1]$,

$$\begin{aligned}\mathbb{G}^{(2)}_{\ell,I,\varepsilon}(\lambda) &= C_{I,Q}(\lambda) + D_{I,Q}(\lambda) \\ &= \frac{c_I C(\ell)}{\ell}\left(\int_0^1 \frac{1}{u}\ \ln\frac{1}{1-u}\,du - \lambda\right) \\ &\qquad + O_{\delta,\ell}\Big(\lambda Q^{-1+\delta} + Q^{\delta+\max\{2c_1-\frac{1}{2},-c-c_1\}}\Big),\end{aligned} \tag{3.12}$$

whence

Proposition 3.3. — *For every $\delta > 0$, uniformly in λ on compacts of $(0,1]$,*

$$\mathbb{G}^{(2)}_{\ell,I,\varepsilon}(\lambda) = \frac{c_I C(\ell)(\zeta(2)-\lambda)}{\ell} + O_{\delta,\ell}\Big(\varepsilon^{-\delta+\theta(c,c_1)}\Big).$$

An identical formula holds for $\mathbb{G}^{(3)}_{\ell,I,\varepsilon}(\lambda)$.

4. End of the proof of Theorem 1.1

From Corollary 2.6 and Propositions 3.1, 3.2, 3.3 we gather

$$\tilde{\mathbb{G}}_{\ell,I,\varepsilon}(\lambda) = c_I G_\ell(\lambda) + O_{\delta,\ell}\Big(\varepsilon^{-\delta+\theta(c,c_1)}\Big),$$

with repartition function G_ℓ given by

$$G_\ell(\lambda) = \begin{cases} 1 - \frac{\lambda}{\zeta(2)} + A(\ell)H_1(\lambda) & \text{if } \ \lambda \in \big(0,\frac{1}{2}\big], \\ 1 - \frac{\lambda}{\zeta(2)} + A(\ell)H_2(\lambda) & \text{if } \ \lambda \in \big[\frac{1}{2},1\big], \\ \frac{2C(\ell)}{\ell}\, H_3(\lambda) & \text{if } \ \lambda \in [1,\infty), \end{cases}$$

where

$$\begin{aligned}
H_1(\lambda) &= \lambda - \zeta(2) + (\ln\lambda)\ln(1-\lambda) + 2\int_{\lambda}^{1-\lambda} \frac{1}{u}\ \ln\frac{1}{1-u}\ du \\
&\qquad + 2\int_{1-\lambda}^{1} \frac{1-u}{u}\ \ln\frac{\lambda}{1-u}\ du, \\
&= -\lambda - (\ln\lambda)\ln(1-\lambda) + \int_{1-\lambda}^{1} \frac{1}{u}\ \ln\frac{1}{1-u}\ du - \int_{0}^{\lambda} \frac{1}{u}\ \ln\frac{1}{1-u}\ du \\
&= -\lambda, \\
H_2(\lambda) &= \lambda - \zeta(2) + (\ln\lambda)^2 + 2\int_{\lambda}^{1} \frac{1-u}{u}\ \ln\frac{\lambda}{1-u}\ du \\
&= 3\lambda - 2 - \zeta(2) - (\ln\lambda)^2 \\
&\qquad + 2(1-\lambda)\ln\left(\frac{1}{\lambda}-1\right) + 2\int_{\lambda}^{1} \frac{1}{u}\ \ln\frac{1}{1-u}\ du, \\
H_3(\lambda) &= \int_0^1 \frac{1-u}{u}\ \ln\frac{\lambda}{\lambda-u}\ du.
\end{aligned}$$

This establishes part (i) in Theorem 1.1. Using also

$$\begin{aligned}
H_3'(\lambda) &= \int_0^1 \frac{1-u}{u}\left(\frac{1}{\lambda}-\frac{1}{\lambda-u}\right)du = -\frac{1}{\lambda}\int_0^1 \frac{1-u}{\lambda-u}\,du \\
&= -\frac{1}{\lambda} + \left(1-\frac{1}{\lambda}\right)\ln\left(1-\frac{1}{\lambda}\right), \quad \lambda > 1,
\end{aligned}$$

it follows that the density of the repartition function G_ℓ is

$$\begin{aligned}
g_\ell(\lambda) &= -G_\ell'(\lambda) \\
&= \begin{cases}
\frac{1}{\zeta(2)} + A(\ell) & \text{if } \lambda \in \big(0,\frac{1}{2}\big], \\
\frac{1}{\zeta(2)} + A(\ell)\Big(-3+\frac{2}{\lambda} - 2\big(\frac{1}{\lambda}-1\big)\ln\big(\frac{1}{\lambda}-1\big)\Big) & \text{if } \lambda \in \big[\frac{1}{2},1\big), \\
\frac{2C(\ell)}{\ell}\Big(\frac{1}{\lambda} - \big(1-\frac{1}{\lambda}\big)\ln\big(1-\frac{1}{\lambda}\big)\Big) & \text{if } \lambda \in (1,\infty).
\end{cases}
\end{aligned}$$

Part (ii) of Theorem 1.1 can now be deduced from part (i) by a standard approximation argument [1, 2, 3, 6] based on $\tau_{\ell,\varepsilon}(\omega) \approx \frac{q_{\ell,\varepsilon}(\omega)}{\cos\omega}$ for $\tan\omega \in I$ with $I \subseteq [0,1]$ interval of length $|I| \asymp \varepsilon^c$. We skip the detailed proof, which can be easily reconstructed from some of the arguments in the next section.

5. The conversion from a honeycomb to a square lattice with congruence constraints

In the situation of the honeycomb it suffices to consider $\omega \in [0, \frac{\pi}{6}]$. The linear transformation

$$\mathbb{R}^2 \ni (x, y) \quad \xrightarrow{T} \quad (x', y') = \left(x - \frac{y}{\sqrt{3}}, \frac{2y}{\sqrt{3}} \right) \in \mathbb{R}^2$$

maps the first sextant $\Gamma_+ := \{(q + \frac{a}{2}, \frac{a\sqrt{3}}{2}) : q, a \in \mathbb{Z}, 0 \leqslant a \leqslant q\}$ of the grid of equilateral triangles of side 1 onto the first quadrant of the square lattice $\mathbb{Z}^2 = \{(q, a) : a, q \in \mathbb{Z}\}$. Elements of the subset $\Gamma_+^{\text{hex}} \subseteq \Gamma_+$ of vertices from the honeycomb grid map to integer lattice points (q, a) with $q \not\equiv a \,(\text{mod } 3)$ (see Figure 5.1).

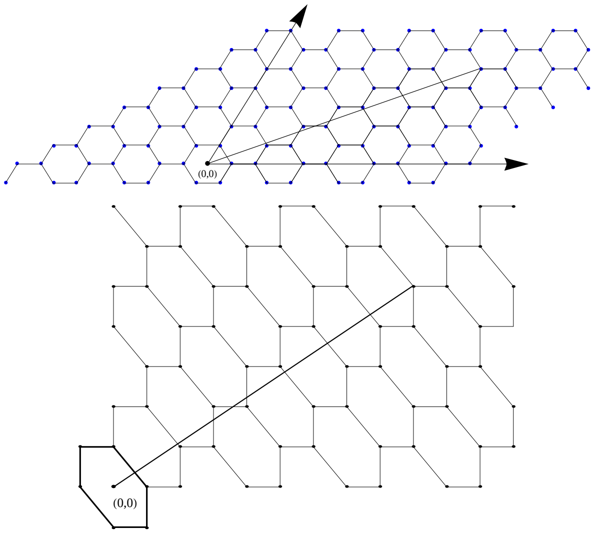


Figure 5.1. The free path length in the honeycomb and in the deformed honeycomb

T also maps circular scatterers $(q + \frac{a}{2}, \frac{a\sqrt{3}}{2}) + \varepsilon(\cos\theta, \sin\theta)$ centered at $(x_0, y_0) = (q + \frac{a}{2}, \frac{a\sqrt{3}}{2})$ to ellipsoidal scatterers $(q, a) + \varepsilon(\cos\theta - \frac{\sin\theta}{\sqrt{3}}, \frac{2\sin\theta}{\sqrt{3}})$ centered at $(x_0', y_0') = (q, a)$. Denote $\omega_0 = \arctan \frac{y_0}{x_0}$ and $\omega_0' = \arctan \frac{y_0'}{x_0'}$. Denote also $S_\varepsilon = \{\delta(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}) : |\delta| \leqslant \varepsilon\}$.

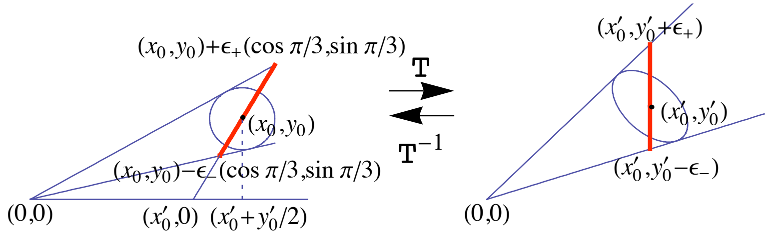


Figure 5.2. Change of scatterers under the linear transformation T

In the honeycomb with scatterers $\{(x_0,y_0)+S_\varepsilon : (x_0,y_0)\in\Gamma_+^{\mathrm{hex}}\}$ of same length, the "local" repartition function

$$\tilde{\mathbb{G}}^{\mathrm{hex}}_{I,\varepsilon}(\lambda)=\left|\left\{\omega\in\arctan I:\tilde{q}^{\mathrm{hex}}_\varepsilon(\omega)>\frac{\lambda}{\varepsilon}\right\}\right|,\qquad I\subseteq\left[0,\frac{\pi}{6}\right]\ \text{interval},$$

of the "horizontal" free path length $\tilde{q}^{\mathrm{hex}}_\varepsilon(\omega)$ defined as the infimum of the set

$$\left\{q\in\mathbb{N}:\exists a\in\mathbb{N},\exists(x_0,y_0)\in\Gamma_+^{\mathrm{hex}},\left(\frac{q+a/2}{\cos\omega},\frac{a\sqrt{3}/2}{\sin\omega}\right)\in(x_0,y_0)+S_\varepsilon\right\},$$

turns out to be closely related with $\mathbb{G}_{3,I,\varepsilon}(\lambda)$, being estimated through the same approximation procedure as for the later when $I\subseteq[0,1]$ is a short interval of length $|I|\asymp\varepsilon^c$, $0<\varepsilon<1$. Indeed, the equality

$$T^{-1}(x_0',y_0'+\delta)=\left(x_0'+\frac{y_0'+\delta}{2},\frac{(y_0'+\delta)\sqrt{3}}{2}\right),$$

shows that T maps the oblique scatterer $(x_0,y_0)+S_\varepsilon$ from the honeycomb onto the vertical scatterer $(x_0',y_0')+V_\varepsilon$ from the square lattice with $x_0'\equiv y_0'$ (mod 3), and the line through the origin with slope $\frac{y_0'+\delta}{x_0'}$ onto the line through the origin with slope $\Phi\big(\frac{y_0'+\delta}{x_0'}\big)$, where Φ is the bijection

$$\Phi:[0,1]\to\left[0,\frac{\sqrt{3}}{3}\right],\qquad\Phi(\mu)=\frac{\mu\sqrt{3}}{2+\mu},\qquad\Phi^{-1}(x)=\frac{2x}{\sqrt{3}-x}.$$

In the process the condition $\tilde{q}^{\mathrm{hex}}_\varepsilon(\omega)>\frac{\lambda}{\varepsilon}$ above is being replaced by $q_{3,\varepsilon}(\omega')>\frac{\lambda}{\varepsilon}$ in the square lattice. The only difference arises from replacing expressions (with $x_0'=q$, $y_0'=a$ and $0\leqslant\delta\leqslant\frac{1}{q}$)

$$\arctan\frac{y_0'+\delta}{x_0'}-\arctan\frac{y_0'}{x_0'}=\frac{\delta}{x_0'}\cdot\frac{1}{1+\frac{y_0'^2}{x_0'^2}}+O\left(\frac{\delta^2}{x_0'^2}\right)=\frac{\delta q}{q^2+a^2}+O\left(\frac{\delta^2}{q^2}\right)$$

that collect the contribution of angles ω for $\mathbb{G}_{3,I,\varepsilon}(\lambda)$, by

$$\begin{aligned}
\arctan\Phi\left(\frac{y_0'+\delta}{x_0'}\right) &- \arctan\Phi\left(\frac{y_0'}{x_0'}\right) \\
&= \arctan\frac{(y_0'+\delta)\sqrt{3}}{2x_0'+y_0'+\delta} - \arctan\frac{y_0'\sqrt{3}}{2x_0'+y_0'} \\
&= \left(\frac{(y_0'+\delta)\sqrt{3}}{2x_0'+y_0'+\delta} - \frac{y_0'\sqrt{3}}{2x_0'+y_0'}\right)\frac{1}{1+\left(\frac{y_0'\sqrt{3}}{2x_0'+y_0'}\right)^2} \\
&= \frac{2x_0'\delta\sqrt{3}}{(2x_0'+y_0')^2}\left(1+O\left(\frac{\delta}{2x_0'+y_0'}\right)\right)\frac{1}{1+\left(\frac{y_0'\sqrt{3}}{2x_0'+y_0'}\right)^2} \\
&= \frac{\delta q\sqrt{3}}{2(q^2+aq+a^2)} + O\left(\frac{\delta^2}{q^2}\right).
\end{aligned}$$

The effect will only be on the main term, where $c_J = \int_J \frac{dx}{1+x^2}$, $J \subseteq [0,1]$, will be replaced by $c_I^{\mathrm{hex}} := \int_{\Phi^{-1}(I)} \frac{\sqrt{3}}{2(1+x+x^2)}\,dx$, $I \subseteq \left[0, \frac{1}{\sqrt{3}}\right]$. In this way we obtain, uniformly in λ on compacts in $\mathbb{R}_+ \setminus \{1\}$,

$$\tilde{\mathbb{G}}_{I,\varepsilon}^{\mathrm{hex}}(\lambda) = c_I^{\mathrm{hex}} G_3(\lambda) + O_\delta\left(\varepsilon^{-\delta+\theta(c,c_1)}\right). \tag{5.1}$$

The change of variable $x = \Phi(\mu)$ gives

$$c^{\Delta}_{[\tan\omega_0,\tan\omega_1]} = \frac{\sqrt{3}}{2}\int_{\Phi^{-1}(\tan\omega_0)}^{\Phi^{-1}(\tan\omega_1)} \frac{dx}{x^2+x+1} = \int_{\tan\omega_0}^{\tan\omega_1} \frac{d\mu}{\mu^2+1} = \omega_1 - \omega_0.$$

In this case $\tilde{q}_\varepsilon^{\mathrm{hex}}(\omega)$ and the free path length $\tilde{\tau}_\varepsilon^{\mathrm{hex}}(\omega)$ are related (by the rule of Sines) by

$$\tilde{\tau}_\varepsilon^{\mathrm{hex}}(\omega) = \frac{\sin\frac{2\pi}{3}}{\sin\left(\frac{\pi}{3}-\omega\right)}\,\tilde{q}_\varepsilon^{\mathrm{hex}}(\omega) = \frac{\sqrt{3}}{2}\cdot\frac{\tilde{q}_\varepsilon^{\mathrm{hex}}(\omega)}{\cos\left(\frac{\pi}{6}+\omega\right)},$$

so that

$$\begin{aligned}
\tilde{\mathbb{P}}_{I,\varepsilon}^{\mathrm{hex}}(\lambda) :&= \left|\left\{\omega \in \arctan I : \tilde{\tau}_\varepsilon^{\mathrm{hex}}(\omega) > \frac{\lambda}{\varepsilon}\right\}\right| \\
&= \left|\left\{\omega \in \arctan I : \tilde{q}_\varepsilon^{\mathrm{hex}}(\omega) > \frac{2\lambda\cos\left(\frac{\pi}{6}+\omega\right)}{\varepsilon\sqrt{3}}\right\}\right|.
\end{aligned} \tag{5.2}$$

Fix $\omega_I \in \arctan I$. Using $\left|\cos\left(\frac{\pi}{6}+\omega\right) - \cos\left(\frac{\pi}{6}+\omega_I\right)\right| \leqslant |\omega-\omega_I| \ll \varepsilon^c$, (2.2), and $c_I^{\mathrm{hex}} \asymp \varepsilon^c$, equalities (5.1) and (5.2) yield

$$\tilde{\mathbb{P}}_{I,\varepsilon}^{\mathrm{hex}}(\lambda) = c_I^{\mathrm{hex}} G_3\left(\frac{2\lambda\cos\left(\frac{\pi}{6}+\omega_I\right)}{\sqrt{3}}\right) + O_\delta\left(\varepsilon^{2c} + \varepsilon^{-\delta+\theta(c,c_1)}\right). \tag{5.3}$$

Let $\varepsilon_\pm = \varepsilon_\pm(\omega_0,\varepsilon)$ as in Figure 5.2. Consider the case of scatterers $(x_0,y_0)+S_{\omega_0,\varepsilon}$, $(x_0,y_0)\in\Gamma_+^{\rm hex}$, where $S_{\omega_0,\varepsilon}$ denotes the segment

$$\{\delta(\cos\frac{\pi}{3},\sin\frac{\pi}{3}) : -\varepsilon_- \leqslant \delta \leqslant \varepsilon_+\}.$$

Let $\tilde{\tilde{\tau}}_\varepsilon^{\rm hex}(\omega)$ denote the free path length and $\tilde{\tilde{\mathbb{P}}}_{I,\varepsilon}^{\rm hex}(\lambda) = |\{\omega\in\arctan I : \tilde{\tilde{\tau}}_\varepsilon^{\rm hex}(\omega) > \frac{\lambda}{\varepsilon}\}|$. Since $|\cos\omega_\pm - \cos\omega_I| \ll \varepsilon^c$ and $\cos(\frac{\pi}{6}+\omega_I) > \frac12$ there exists $C_1>0$ such that

$$\begin{aligned}(5.4)\quad \varepsilon' := \varepsilon\left(\frac{1}{\cos(\frac{\pi}{6}+\omega_I)} - C_1\varepsilon^c\right) \leqslant \varepsilon_- \leqslant \varepsilon_+ \leqslant \varepsilon'' \\ := \varepsilon\left(\frac{1}{\cos(\frac{\pi}{6}+\omega_I)} + C_1\varepsilon^c\right).\end{aligned}$$

The inequalities $\tilde{\tau}_{\varepsilon''}^\Delta(\omega) \leqslant \tilde{\tilde{\tau}}_\varepsilon^\Delta(\omega) \leqslant \tilde{\tau}_{\varepsilon'}^\Delta(\omega)$ and (5.4) combined with formula (5.3) and (2.2) lead to

$$\begin{aligned}\tilde{\tilde{\mathbb{P}}}_{I,\varepsilon}^{\rm hex}(\lambda) &\leqslant \left|\left\{\omega\in\arctan I : \tilde{\tau}_{\varepsilon'}^{\rm hex}(\omega) > \frac{\lambda}{\varepsilon}\right\}\right| \\ &= \left|\left\{\omega\in\arctan I : \tilde{\tau}_{\varepsilon'}^{\rm hex}(\omega) > \frac{\lambda}{\varepsilon'\big(\cos(\frac{\pi}{6}+\omega_I) - C_1\varepsilon^c\big)}\right\}\right| \\ &= \tilde{\mathbb{P}}_\varepsilon^{\rm hex}\left(\frac{\lambda}{\cos(\frac{\pi}{6}+\omega_I) - C_1\varepsilon^c}\right) \\ &= c_I^{\rm hex} G_3\left(\frac{2\lambda}{\sqrt3}\cdot\frac{\cos(\frac{\pi}{6}+\omega_I)}{\cos(\frac{\pi}{6}+\omega_I) - C_1\varepsilon^c}\right) + O_\delta\Big(\varepsilon^{2c} + \varepsilon^{-\delta+\theta(c,c_1)}\Big) \\ &= c_I^{\rm hex} G_3\left(\frac{2\lambda}{\sqrt3}\right) + O_\delta\Big(\varepsilon^{2c} + \varepsilon^{-\delta+\theta(c,c_1)}\Big),\end{aligned}$$

and to a similar lower bound for $\tilde{\tilde{\mathbb{P}}}_{I,\varepsilon}^{\rm hex}(\lambda)$, and so we get

$$(5.5)\qquad \tilde{\tilde{\mathbb{P}}}_{I,\varepsilon}^{\rm hex}(\lambda) = c_I^{\rm hex} G_3\left(\frac{2\lambda}{\sqrt3}\right) + O_\delta\Big(\varepsilon^{2c} + \varepsilon^{-\delta+\theta(c,c_1)}\Big).$$

The trivial inequality $|\tau_\varepsilon^\Delta(\omega) - \tilde{\tilde{\tau}}_\varepsilon^\Delta(\omega)| \leqslant 2\varepsilon$ and (5.5) now provide the formula

$$(5.6)\qquad \mathbb{P}_{I,\varepsilon}^{\rm hex}(\lambda) = c_I^{\rm hex} G_3\left(\frac{2\lambda}{\sqrt3}\right) + O_\delta\Big(\varepsilon^{2c} + \varepsilon^{-\delta+\theta(c,c_1)}\Big)$$

for the repartition function $\mathbb{P}_{I,\varepsilon}^{\rm hex}(\lambda) = |\{\omega\in\arctan I : \tau_\varepsilon^{\rm hex}(\omega) > \frac{\lambda}{\varepsilon}\}|$ of $\varepsilon\tau_\varepsilon^{\rm hex}$.

Finally we choose a partition $\big[0,\frac{1}{\sqrt3}\big] = \cup_{j=1}^N I_j$ with intervals I_j of equal size $\frac1N \asymp \varepsilon^c$. Applying (5.6) to each individual interval I_j with $c=c_1=\frac18$

and summing over j we find

$$\text{(5.7)}\quad \mathbb{P}^{\text{hex}}_{[0,1/\sqrt{3}],\varepsilon}(\lambda) = \sum_{j=1}^{N} c^{\text{hex}}_{I_j} G_3\left(\frac{2\lambda}{\sqrt{3}}\right) + O_\delta\left(\varepsilon^{\frac{1}{8}-\delta}\right) = \frac{\pi}{6} G_3\left(\frac{2\lambda}{\sqrt{3}}\right) + O_\delta\left(\varepsilon^{\frac{1}{8}-\delta}\right).$$

Theorem 1.2 now follows immediately from (5.7) and obvious symmetry properties of the honeycomb.

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