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CODIMENSION 3 ARITHMETICALLY GORENSTEIN SUBSCHEMES OF PROJECTIVE N-SPACE

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ABSTRACT. — We study the lowest dimensional open case of the question whether every arithmetically Cohen–Macaulay subscheme of $\mathbb{P}^N$ is glicci, that is, whether every zero-scheme in $\mathbb{P}^3$ is glicci. We show that a general set of $n \geqslant 56$ points in $\mathbb{P}^3$ admits no strictly descending Gorenstein liaison or biliaison. In order to prove this theorem, we establish a number of important results about arithmetically Gorenstein zero-schemes in $\mathbb{P}^3$.

RÉSUMÉ. — Nous étudions le problème de savoir si tous les sous-schémas arithmétiquement de Cohen-Macaulay de $\mathbb{P}^N$ sont "glicci" dans le cas de plus petite dimension, c'est-à-dire le cas de sous-schémas de dimension zéro de $\mathbb{P}^3$. Nous prouvons qu'il n'y a pas de liaisons ni de biliaisons de Gorenstein descendantes d'un ensemble d'au moins 56 points généraux de $\mathbb{P}^3$. Pour démontrer ce théorème, nous établissons plusieurs résultats concernant les sous-schémas arithmétiquement de Gorenstein de $\mathbb{P}^3$.

1. Introduction

There has been considerable interest recently in the notion of Gorenstein liaison for subschemes of $\mathbb{P}^N$. In particular, a question that has attracted a lot of attention, by analogy with the case of complete intersection liaison in codimension 2, is whether every ACM subscheme of $\mathbb{P}^N$ is in the Gorenstein liaison class of a complete intersection (glicci for short). Many special classes of ACM subschemes have been found that are glicci, but the question in general remains open [4],[5],[17],[23],[28],[29].

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Our motivation for this research was to consider the lowest dimensional case of this question, namely zero-dimensional subschemes of $\mathbb{P}^3$. In this case, since every zero-dimensional scheme is ACM, the question becomes simply: is every zero-scheme in $\mathbb{P}^3$ glicci? One of us has shown in an earlier paper [17] that a general set of $n \leqslant 19$ points is glicci (we give a new proof of this in Proposition 7.1), while for $n \geqslant 20$ it is unknown. In this paper we show

THEOREM 1.1 (Theorems 7.2 and 7.4). — *A general set of $n \geqslant 56$ points in $\mathbb{P}^3$ admits no strictly descending Gorenstein liaison or biliaison.*

The theorem does not imply that a general set of $n \geqslant 56$ points is not glicci, but it shows that, if such a set is glicci, it must first be linked upwards to a larger set before eventually linking down to a point.

In order to prove the theorem, since Gorenstein liaisons are performed with arithmetically Gorenstein (AG) subschemes, we need to establish a number of results about these.

It is known that the family $\mathrm{PGor}(h)$ of codimension three AG subschemes of $\mathbb{P}^N$ with a fixed Hilbert function, encoded in the h-vector h (see section 2), is irreducible [9]. The dimension of $\mathrm{PGor}(h)$ is computed in [21] and [24], but not explicitly as a function of the h-vector. In the case of zero-dimensional AG subschemes of $\mathbb{P}^3$, a formula for $\dim \mathrm{PGor}(h)$ in terms of the h-vector is given in [6, §5]; following a different approach, we derive in Corollary 5.3 a formula which allows to compute $\dim \mathrm{PGor}(h)$ inductively.

In all cases we are aware of where a class of ACM subschemes of $\mathbb{P}^N$ has been proved to be glicci, the proof was actually accomplished using strict Gorenstein liaisons, *i.e.,* using only those AG schemes of the form $mH_X - K_X$ on some ACM scheme X, where H_X and K_X denote respectively the hyperplane and the canonical divisor class. So we ask whether all AG subschemes can be obtained in this way, or only some. This analysis was performed in an earlier paper [18] for curves in $\mathbb{P}^4$, and we extend the results to any $\mathbb{P}^N$, $N \geqslant 3$. One may consult [22] for a deformation theoretic approach to this problem, and Boij in [1, Theorem 3.4] proves a result related to ours in an algebraic context. To give a precise statement, we use two numerical invariants $b(h)$ and $\overline{b}(h)$ of the Hilbert function, which satisfy $b(h) \geqslant 2\overline{b}(h)$. Our result is

THEOREM 1.2 (Theorem 3.4). — *Given an h-vector arising as the h-vector of a codimension 3 AG subscheme of $\mathbb{P}^N$, there is always in* $\mathrm{PGor}(h)$ *a scheme of the form $mH_X - K_X$ on some codimension* 2 ACM *subscheme $X \subset \mathbb{P}^N$. Furthermore, if $b(h) \geqslant 2\overline{b}(h) + 2$, the general element of* $\mathrm{PGor}(h)$ *has this form.*

It looks as if the most interesting h-vectors will be those with $b(h)$ equal to $2\overline{b}(h)$ or $2\overline{b}(h)+1$, in which case it is possible the most general element of $\mathrm{PGor}(h)$ is not of the form $mH_X - K_X$ on any ACM X. For points in $\mathbb{P}^3$, we give strong evidence in Proposition 6.8 that this happens in degree 30, by giving an example of a family of AG sets of 30 points whose general element is not of the form $mH_C - K_C$ on any integral ACM curve C. This phenomenon may occur earlier in degrees 20, 28 or 29.

Watanabe [35] showed that codimension 3 AG subschemes are licci, that is, in the complete intersection liaison class of a complete intersection. In section 5 we sharpen this result in the case of general AG sets of points in $\mathbb{P}^3$ by showing that

THEOREM 1.3 (Theorem 5.2). — *For a given h-vector, a general AG zero dimensional subscheme of $\mathbb{P}^3$ with that h-vector can be obtained by ascending complete intersection biliaisons from a point.*

This was proved for curves in $\mathbb{P}^4$ in [18]. To establish the result, we need a version of the well known Cayley–Bacharach property for a zero-scheme in $\mathbb{P}^N$ to be AG, of which we give a new proof in section 4.

In section 6, we study the Hilbert schemes $\mathrm{PGor}(h)$ of *AG* zero-schemes Z in $\mathbb{P}^3$ keeping in mind the two crucial questions

(1) When does the general element of $\mathrm{PGor}(h)$ have the form $mH - K$ on some ACM curve C ?
(2) As the AG scheme Z varies in the Hilbert scheme $\mathrm{PGor}(h)$, how many general points can we assign to a general Z ? This is important in order to understand the possible Gorenstein liaisons one can perform on a set of general points.

We don't have a general answer to these questions, but we have made computations up to degree 30, and summarized our results in Table 8.1. The notations and methods used for these computations are explained in section 6.

We hope that this ground work will be the foundation for an eventual solution of our motivating problems discussed above.

We would like to thank the referee for his careful reading of the paper and his several helpful comments.

2. The h-vector of ACM subschemes of $\mathbb{P}^N$

NOTATION. — *$R = \mathrm{H}^0_*(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n))$ denotes the homogeneous coordinate ring of $\mathbb{P}^N$; $R_X = R/I_X$ the coordinate ring of a closed subscheme X, where $I_X = \mathrm{H}^0_*(\mathbb{P}^N, \mathcal{I}_{X,\mathbb{P}^N})$.*

Throughout the paper, X denotes an arithmetically Cohen-Macaulay (ACM for short) subscheme of $\mathbb{P}^N$: recall X is called ACM if the coordinate ring R_X is Cohen-Macaulay (of dimension $t+1$ where $t = \dim X$). We denote by Ω_X the graded canonical module of R_X. When $\dim X > 0$, we have

$$\Omega_X \cong \mathrm{H}^0_*(\mathbb{P}^N, \omega_X) = \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^0(\mathbb{P}^N, \omega_X(n))$$

where ω_X is the Grothendieck dualizing sheaf of X.

A subscheme $Z \subseteq \mathbb{P}^N$ is *arithmetically Gorenstein* (AG for short) if its homogeneous coordinate ring is a Gorenstein ring. This is equivalent to saying Z is ACM and the canonical module Ω_Z is isomorphic to $R_Z(m)$ for some $m \in \mathbb{Z}$.

Denote by $H(n) = \dim_k(R_X)_n$ the Hilbert function of R_X. The difference function $h_X(n) = \partial^{t+1}H(n)$ is called the *h-vector* of X [28, §1.4]: it is nonnegative and with finite support. We let $b = b(X)$ denote the largest integer n such that $h_X(n) > 0$. One can show $h_X(n) > 0$ for $0 \leqslant n \leqslant b$. It is convenient to represent the h-vector in the form

$$h_X = \{1 = h_X(0), h_X(1), \ldots, h_X(b)\}. \tag{2.1}$$

We now recall how other numerical invariants can be computed in terms of the h-vector. First of all, the degree of X is given by the formula $d = \sum_{n=0}^{b} h_X(n)$. If X is nondegenerate, $h_X(1) = \mathrm{codim}(X)$.

For any closed subscheme $X \subset \mathbb{P}^N$, we denote by $s(X)$ the least degree of a hypersurface containing X. If X is ACM of codimension $\mathrm{codim}(X)$, the number $s(X)$ is the least positive integer n such that $h_X(n) < \binom{n+\mathrm{codim}(X)-1}{\mathrm{codim}(X)-1}$.

The integer $b(X)$ is related to the regularity and to the index of speciality of X. A sheaf $\mathcal{F}$ on $\mathbb{P}^N$ is n-regular in the sense of Castelnuovo-Mumford if

$$H^i(\mathbb{P}^N, \mathcal{F}(n-i)) = 0 \quad \text{for } i > 0.$$

If $\mathcal{F}$ is n-regular, then it is also $n+1$-regular. Thus one defines the regularity of a sheaf $\mathcal{F}$ as the least integer r such that $\mathcal{F}$ is r-regular. The regularity of a subscheme $X \subseteq \mathbb{P}^N$ is the regularity of its ideal sheaf.

We let $m(X)$ denote the largest integer n such that $(\Omega_X)_{-n} \neq 0$ (when $\dim X > 0$, this is the index of speciality of X).

Proposition 2.1. — *Let X be an ACM subscheme of $\mathbb{P}^N$ of dimension $t < N$. Then*

(1) *the regularity of X is $b(X)+1$;*
(2) *we have $m(X) = b(X) - t - 1$;*
(3) *we have $m(X) = \max\{n \ : \ h^{t+1}(\mathbb{P}^N, \mathcal{I}_X(n)) \neq 0\}$.*

Proof. — The first statement [28, p.8 and p.30] follows computing both the h-vector and the regularity of X out of the minimal free resolution of I_X over R. The second statement follows from the isomorphism

$$\Omega_X \cong \mathrm{Hom}_{R_L}(R_X, \Omega_L)$$

where $L \cong \mathbb{P}^t$, the map $R_L \to R_X$ is induced by projection from a general linear subspace of codimension $t+1$, so that R_X is a free finitely generated graded R_L module with $h_X(n)$ minimal generators in degree n.

The third statement is immediate when X is a hypersurface. If $0 \leqslant t \leqslant N-2$ it is a consequence of the fact that the canonical module Ω_X is the graded k-dual of $\mathrm{H}^{t+1}_*(\mathbb{P}^N, \mathcal{I}_X)$. In fact, by the local duality theorem for graded modules (see *e.g.*, [2, Theorem 3.6.19]), the canonical module Ω_X is the graded k-dual of $\mathrm{H}^{t+1}_{\mathfrak{m}}(R_X)$, where $\mathrm{H}^i_{\mathfrak{m}}$ denotes the i-th local cohomology group with respect to the irrelevant maximal ideal $\mathfrak{m}$ of R, and moreover $\mathrm{H}^{t+1}_{\mathfrak{m}}(R_X) \cong \mathrm{H}^{t+1}_*(\mathbb{P}^N, \mathcal{I}_X)$ (see [14, p.137]). □

For AG subschemes we have more ways to compute the invariant m:

PROPOSITION 2.2. — *Let Z be an AG subscheme of $\mathbb{P}^N$. Then $m(Z)$ is the integer m such that $\Omega_Z \cong R_Z(m)$. Furthermore, if the minimal free resolution of R_Z over R has the form*

$$0 \to R(-c) \to \cdots \to R \to R_Z \to 0,$$

then $m(Z) = c - N - 1$ and $b(Z) = c - \mathrm{codim}(Z)$.

Proof. — It is clear that $\Omega_Z \cong R_Z(m)$ implies $m = m(Z)$. To relate the integer c appearing at the last step of the resolution with m, we compute $\Omega_Z \cong \mathrm{Ext}^{\mathrm{codim}(Z)}(R_Z, \Omega_{\mathbb{P}^N})$ using the minimal free resolution of R_Z, and we find

$$\Omega_Z \cong R_Z(c - N - 1).$$

Hence $m = c - N - 1$. □

Remark 2.3. — For an AG subscheme Z, the integer $b(Z)$ is called the *socle degree* of R_Z in Migliore's book [28] and is denoted by r; the integer $m(Z)$ is sometimes referred to as the a-invariant of the Gorenstein graded algebra R_Z.

COROLLARY 2.4. — *Let X be a codimension 2 ACM subscheme of $\mathbb{P}^N$. Then in the minimal free resolution of I_X:*

$$0 \longrightarrow \oplus R(-b_j) \longrightarrow \oplus R(-a_i) \longrightarrow I_X \longrightarrow 0$$

we have $\max\{b_j\} = b(X) + 2$.

Proof. — The regularity of $\mathcal{I}_X$ equals $\max\{b_j\} - 1$ by [28, p. 8] and $b(X) + 1$ by Proposition 2.1. □

Macaulay has characterized the possible h-vectors for ACM subschemes of $\mathbb{P}^N$ [26]. We will only need this result for subschemes of codimension 2 (Proposition 2.6) and its analogue for AG subschemes of codimension 3 (Proposition 2.8), which is due to Stanley.

Definition 2.5. — *An h-vector is said to be $C2$-admissible if there exists an $s \geqslant 1$ such that*

$$\begin{cases} h(n) = n+1 & \text{if } 0 \leqslant n \leqslant s-1, \\ h(n) \geqslant h(n+1) & \text{if } n \geqslant s-1. \end{cases}$$

A $C2$-admissible h-vector is said to be of decreasing type if $h(a) > h(a+1)$ implies that for each $n \geqslant a$ either $h(n) > h(n+1)$ or $h(n) = 0$.

Proposition 2.6.

(1) *A finitely supported numerical function $h : \mathbb{N} \to \mathbb{N}$ is the h-vector of a codimension 2 ACM subscheme X of $\mathbb{P}^N$ ($N \geqslant 2$) if and only if h is $C2$-admissible. Furthermore, X can be taken reduced, and a locally complete intersection in codimension $\leqslant 2$.*

(2) *If X is an integral codimension 2 ACM subscheme of $\mathbb{P}^N$, then h_X is of decreasing type. Conversely, if h is of decreasing type and $N \geqslant 3$, there exists an integral codimension 2 ACM subscheme $X \subset \mathbb{P}^N$ with $h_X = h$ which is smooth in codimension $\leqslant 2$ (thus smooth if $N = 3, 4$).*

Proof. — The first statement is a very special case of a theorem by Macaulay (see [26],[33]), and is equivalent, in case $N = 3$, to Theorem V.1.3 of [27], and, in general, to Proposition 1.3 of [30]. The fact that X can be taken reduced, and a locally complete intersection in codimension $\leqslant 2$ is proven in [30, 3.2]. The second statement is proven in [30, 3.3]; it was first proven over the complex numbers and for $N = 3$ in [13, 2.2 and 2.5]. □

We have also the result of Ellingsrud [12] about the Hilbert scheme:

Theorem 2.7 (Ellingsrud). — *The set of all ACM codimension 2 subschemes of $\mathbb{P}^N$ ($N \geqslant 3$) with a given h-vector is a smooth, open, irreducible subset of the Hilbert scheme of all closed subschemes of $\mathbb{P}^N$. (There is also an explicit formula for its dimension).*

Stanley [33], drawing on Macaulay's Theorem, and applying the structure theorem of Buchsbaum and Eisenbud [3], characterized the possible

h-vectors of AG codimension 3 subschemes. Before we state his result, we define the "first half" k_h of an h-vector $\{1, h(1), \ldots, h(b)\}$ setting:

$$k_h(n) = \begin{cases} \partial h(n) & \text{for } 0 \leqslant n \leqslant [b/2], \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

where $[b/2]$ denotes the integral part of $b/2$. We have (see [33, 4.2])

PROPOSITION 2.8. — *A finitely supported numerical function $h : \mathbb{N} \to \mathbb{N}$ is the h-vector of an AG codimension 3 subscheme of $\mathbb{P}^N$ ($N \geqslant 3$) if and only if*

(1) *h is symmetric, meaning that $h(n) = h(b-n)$ for all $0 \leqslant n \leqslant b$;*
(2) *the first half of h is $C2$-admissible.*

We say such an h-vector is $G3$-admissible.

Remark 2.9. — If Z is a codimension 3 AG-subscheme of $\mathbb{P}^N$, we let k_Z denote the first half of h_Z and call it the k-vector of Z. Note that the degree of k_Z is precisely $h_Z([b(Z)/2])$. Furthermore, because of its symmetry, the h-vector of Z is determined by its first half k_Z and by $b(Z)$.

Example 2.10. — The h-vectors

$$\{1, 3, 3, \ldots, 3, 3, 1\}$$

all are $G3$-admissible with first half $\{1, 2\}$ - no matter how many 3's appear in the string.

Remark 2.11. — There is an analogue of Ellingsrud's Theorem in the case of codimension 3 AG-subschemes. If we fix the h-vector, the set $\mathrm{PGor}(h)$ of codimension 3 AG-subschemes of $\mathbb{P}^N$ with the given h-vector carries a natural scheme structure, which makes it into a smooth irreducible subscheme of the Hilbert scheme; furthermore, the dimension of $\mathrm{PGor}(h)$ can be computed, see [9],[21],[24]. Corollary 5.3 below allows one to compute $\dim \mathrm{PGor}(h)$ inductively as a function of the h-vector.

We will need formulas for the variation of the h-vector under liaison and biliaison. Recall the definition of Gorenstein liaison [28, 5.1.2] and biliaison [19]:

DEFINITION 2.12. — *Let V_1, V_2, X be equidimensional subschemes of $\mathbb{P}^N$, all of the same dimension and without embedded components, with V_1 and V_2 contained in X. We say that V_2 is G-*linked *to V_1 by X if X is AG and $\mathcal{I}_{V_2,X} \cong \mathrm{Hom}(\mathcal{O}_{V_1}, \mathcal{O}_X)$; in this case, it is also true that V_1 is G-linked to V_2 by X.*

DEFINITION 2.13. — *Let V_1 and V_2 be equidimensional closed subschemes of dimension t of $\mathbb{P}^N$. We say that V_2 is obtained by an* elementary biliaison *of height h from V_1 if there exists an ACM scheme X in $\mathbb{P}^N$, of dimension $t+1$, containing V_1 and V_2 so that $\mathcal{I}_{V_2,X} \cong \mathcal{I}_{V_1,X}(-h)$. (In the language of generalized divisors on X, this says that $V_2 \sim V_1 + hH$, where H denotes the hyperplane class).*

If $h \geqslant 0$ (respectively $h \leqslant 0$), we will speak of an ascending (respectively descending) biliaison. If we restrict the scheme X in the definition to be a complete intersection scheme, we will speak of *CI-biliaison*.

The following proposition is well known and shows how the h-vector of an ACM scheme changes under G-liaison and elementary biliaison.

PROPOSITION 2.14.

(1) *Suppose V_1 and V_2 are two ACM subschemes of $\mathbb{P}^N$ linked by the AG scheme X. Let $b = b(X)$. Then*

$$h_{V_2}(n) = h_X(n) - h_{V_1}(b-n) \quad \text{for all} \quad n \in \mathbb{Z}.$$

(2) *Suppose V_1 and V_2 are ACM subschemes of $\mathbb{P}^N$ such that V_2 is obtained by an elementary biliaison of height 1 from V_1 on the ACM scheme X. Then*

$$h_{V_2}(n) = h_X(n) + h_{V_1}(n-1) \quad \text{for all} \quad n \in \mathbb{Z}.$$

Proof. — The first statement is proven in [28, 5.2.19]; the second follows from the isomorphism $\mathcal{I}_{V_2,X} \cong \mathcal{I}_{V_1,X}(-1)$. □

Example 2.15. — We will later need the h-vector of a zero dimensional subscheme $W \subset \mathbb{P}^3$ consisting of d general points. Since points in W impose independent conditions on surfaces of degree n, the least degree $s = s(W)$ of a surface containing W is the unique positive integer such that $\binom{s+2}{3} \leqslant d < \binom{s+3}{3}$, and we have

$$h_W(n) = \begin{cases} \binom{n+2}{2} & \text{if } 0 \leqslant n < s \\ d - \binom{s+2}{3} & \text{if } n = s \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

3. AG divisors on codimension 2 ACM subschemes

If Z is an AG subscheme of codimension 3 in $\mathbb{P}^N$, we know that its "first half" k_Z is the h-vector of a codimension 2 ACM subscheme $X \subset \mathbb{P}^N$. Thus

given a pair of h-vectors h and k, with k equal to the first half of h, it is natural to ask whether we can find a pair of subschemes Z and X of $\mathbb{P}^N$ with $h_Z = h$ and $h_X = k$ satisfying further $Z \subset X$. Theorem 3.4 gives a positive answer.

Consider an ACM scheme $X \subset \mathbb{P}^N$ which is generically Gorenstein, so that in the language of generalized divisors [19] one can speak of the anticanonical divisor $-K$ on X. Then it is well known that any divisor Z on X linearly equivalent to $-K + mH$, that is, such that $\mathcal{I}_{Z,X} \cong \omega_X(-m)$, is AG ([23, 5.2], [19, 3.4]). In the following proposition we recall this fact, and describe the relation between the h-vectors of X and Z.

PROPOSITION 3.1. — *Let X be a generically Gorenstein ACM subscheme $\mathbb{P}^N$ of dimension $t \geqslant 1$, and let H be a hyperplane section of X. Suppose $Z \subset X$ is divisor linearly equivalent to $-K + mH$, for some integer m. Then*

(a) *Z is an AG subscheme of $\mathbb{P}^N$ with $\Omega_Z \cong R_Z(m)$; in particular, $m = m(Z)$.*

(b) *The h-vector of Z is determined by the integer m and the h-vector of X by the formula*

$$\partial h_Z(n) = h_X(n) - h_X(m + t + 1 - n).$$

In particular,

$$b(Z) \geqslant b(X)$$

and, if k_Z denotes the first half of h_Z, we have $h_X(n) \geqslant k_Z(n)$ for every n.

(c) *The equality $h_X = k_Z$ holds if and only if $b(Z) \geqslant 2b(X)$ (or equivalently $m \geqslant 2b(X) - t$), so that in this case the h-vector of X is determined by that of Z.*

Remark 3.2. — Since $b(Z) \geqslant b(X)$, we see that if Z is fixed, there are at most finitely many h-vectors $\tilde{h}$ with the property that there exists an ACM X with $h_X = \tilde{h}$ on which Z is a divisor linearly equivalent to $-K_X + mH_X$.

Proof of 3.1. — >From the exact sequence

$$0 \to \Omega_X(-m) \to R_X \to R_Z \to 0 \tag{3.1}$$

it follows

$$\operatorname{depth} R_Z \geqslant \operatorname{Min}(\operatorname{depth} \Omega_X - 1, \operatorname{depth} R_X) = \dim R_X - 1.$$

Since Z is a divisor on X, $\dim Z = \dim X - 1$, so we must have equality, and thus Z is ACM.

To see Z is in fact AG, we apply the functor $\mathrm{Hom}_{R_X}(-, \Omega_X)$ to the exact sequence (3.1) to get

$$0 = \mathrm{Hom}_{R_X}(R_Z, \Omega_X) \to \Omega_X \cong I_{Z,X}(m) \to R_X(m) \to \Omega_Z \to 0.$$

This implies $\Omega_Z \cong R_Z(m)$ because Ω_Z is a faithful R_Z-module.

To prove (b), we use the exact sequence (3.1) and Serre duality to obtain

$$\partial h_Z(n) = h_X(n) - \partial^{t+1}(h^t(\mathcal{O}_X(m-n)). \tag{3.2}$$

To see $\partial^{t+1}(h^t(\mathcal{O}_X(m-n)) = h_X(m+t+1-n)$, we use the fact that $\partial^{t+1}h^t\mathcal{O}_X(n) = (-1)^{t+1}\partial^{t+1}h^0\mathcal{O}_X(n)$ because $\chi\mathcal{O}_X(n) = h^0\mathcal{O}_X(n)+(-1)^t h^t\mathcal{O}_X(n)$ is a polynomial of degree t in n, together with the fact that for any numerical function $f(n)$, if $g(n) = f(m-n)$, then $\partial^{t+1}g(n) = (-1)^{t+1}\partial^{t+1}f(m+t+1-n)$.

Setting $n = 0$ in the formula, we see that $h_X(m+t+1) = 0$, hence $b(X) < m+t+1$. But $m+t = b(Z)$ because $m = m(Z)$ and $\dim Z = t-1$. Thus $b(X) \leqslant b(Z)$.

As for part (c), it is clear that $h_X = k_Z$ implies $b(Z) \geqslant 2b(X)$. Conversely, suppose $b(Z) \geqslant 2b(X)$. Then $[b(Z)/2] \geqslant b(X)$, which implies $h_X(n) = 0 = k_Z(n)$ for $n > [b(Z)/2]$. On the other hand, for $n \leqslant [b(Z)/2]$, we have $m+t+1-n = b(Z)+1-n > b(X)$, hence by part (b)

$$k_Z(n) = \partial h_Z(n) = h_X(n).$$

□

Proposition 3.3. — *Let X be a codimension 2 ACM subscheme of $\mathbb{P}^N$, $N \geqslant 3$. Assume X is a locally complete intersection in codimension $\leqslant 2$. Then*

(a) *if $m \geqslant 2b(X)-N+1$, there is an effective generalized divisor $Z \subset X$ linearly equivalent to $mH - K$;*
(b) *if $m \geqslant 2b(X) - N + 2$ and furthermore X is a locally complete intersection everywhere, the invertible sheaf $\mathcal{L}(mH-K) = \omega_X^\vee(m)$ is very ample.*

Proof. — We will follow the proof of Theorem 3.5 in [18]. First of all note that the assumption X is a locally complete intersection in codimension $\leqslant 2$ assures that X satisfies the conditions "$G_1 + S_2$" under which the theory of generalized divisors is developed in [16]; also K_X is an almost Cartier divisor on X, and $\mathcal{L}(mH-K) = \omega_X^\vee(m)$.

Let us begin by proving part (a): since $\mathcal{I}_X(b+1)$ is generated by global sections, so are $(\mathcal{I}/\mathcal{I}^2)(b+1)$ and $\bigwedge^2(\mathcal{I}/\mathcal{I}^2)(2b+2)$. By assumption, off a closed subscheme Y of X of codimension at least 3, X is a locally complete

intersection, so $\bigwedge^2(\mathcal{I}/\mathcal{I}^2) \cong \omega_X^\vee(-N-1)$ on $X \setminus Y$, and $\omega_X^\vee(2b-N+1)$ is generated by its global sections on $X \setminus Y$. In particular, for $m \geqslant 2b-N+1$ we can find a section of $\omega_X^\vee(m)$ over $X \setminus Y$ which generates $\omega_X^\vee(m)$ at the generic points of X. Since codim $Y > 1$, we can extend this section to all of X [16, 1.11] . We conclude that for $m \geqslant 2b-N+1$ there is an effective divisor $Z \subset X$ linearly equivalent to $mH-K$ [16, 2.9].

If X is a locally complete intersection everywhere, then the above argument shows that for $m \geqslant 2b-N+1$ the invertible sheaf $\omega_X^\vee(m)$ is globally generated. Statement (b) now follows from this and the fact that the tensor product of a very ample invertible sheaf and a globally generated invertible sheaf is very ample. □

THEOREM 3.4. — *Let h be a $G3$-admissible h-vector, and let k be its first half as defined in section 2 formula (2.2). Denote by b and $\overline{b}$ the largest integers such that $h(n) \neq 0$ and $k(n) \neq 0$ respectively, and let $m = b-N+2$, $N \geqslant 3$. Then:*

(a) *there is an AG subscheme Z of $\mathbb{P}^N$ of codimension 3 with h-vector $h_Z = h$ lying on an ACM scheme X of codimension 2 with h-vector $h_X = k$;*

(b) *if the first half k of h is of decreasing type, we may take X to be integral, and in case $N = 3$ or 4, both X and Z to be smooth;*

(c) *if $b \geqslant 2\overline{b}+2$ (that is, $m \geqslant 2\overline{b}-N+4$), then there is an open subset V of the Hilbert scheme $\mathrm{PGor}(h)$ such that every $Z \in V$ is of the form $Z \sim mH-K$ on a codimension 2 ACM scheme X with h-vector k.*

Remark 3.5. — Note that in the statement the integers b and $\overline{b}$ depend only on h, and by definition of the "first half" of h we always have $b \geqslant 2\overline{b}$. Thus the hypothesis in (c) holds unless $b = 2\overline{b}$ or $b = 2\overline{b}+1$.

Remark 3.6. — In [8] it is shown that an integral AG subscheme of codimension 3 of $\mathbb{P}^N$ exists with given h-vector if and only if the first half of that h-vector is of decreasing type. A result related to statement (c) above is Theorem 3.4 of [1]; see also Remark 5.3 of [23]. For a deformation theoretic approach to the question of determining those AG schemes Z of the form $Z \sim mH_X - K_X$ see [22].

Proof. — (a) Suppose given a $G3$-admissible h-vector h. Then its first half k is $C2$-admissible, hence there exists [30, 3.2] an ACM subscheme $X \subset \mathbb{P}^N$ of codimension 2 with h-vector $h_X = k$ which is reduced and a locally complete intersection in codimension $\leqslant 2$. Then $b(X) = \overline{b}$, and by definition of the "first half" function $b \geqslant 2\overline{b}$, that is, $m \geqslant 2\overline{b}-N+2$. By

Proposition 3.3, there is a divisor $Z \subset X$ linearly equivalent to $mH - K$, and, by Proposition 3.1, Z is AG with $m(Z) = m$, hence $b(Z) = b$ and the first half of the h-vector h_Z equals h_X. Thus h_Z and h both have first half k and are last nonzero at b, hence by symmetry $h_Z = h$.

(b) If k is of decreasing type, then X can be taken to be integral [30, 3.3]. If $N = 3$ or 4, then X can be taken even smooth [30, 3.3], and, since $mH - K$ is very ample by Proposition 3.3, we can take Z to be smooth by the usual Bertini theorem.

(c) It is known that $\mathrm{PGor}(h)$ is smooth, and the tangent space to $\mathrm{PGor}(h)$ at the point corresponding to the subscheme Z is isomorphic to the degree zero piece ${}_0\operatorname{Hom}_R(I_Z, R_Z)$ of the graded R-module $\operatorname{Hom}_R(I_Z, R_Z)$: see for example [24], and [21] for the zero-dimensional case. We remark that, in case $\dim Z > 0$, the tangent space ${}_0\operatorname{Hom}_R(I_Z, R_Z)$ is isomorphic to the space of global sections of the normal sheaf $\mathcal{N}_Z$ of Z in $\mathbb{P}^N$.

It is therefore sufficient to show that the dimension of the family in statement (c) is greater or equal than the dimension of ${}_0\operatorname{Hom}_R(I_Z, R_Z)$.

We will restrict X to vary in the open subset of locally complete intersections in codimension $\leqslant 2$. Therefore we can use the theory of generalized divisors on X. Since X is an ACM scheme and since the linear system $|Z| = |mH - K|$ is effective, the dimension of the linear system $|Z|$ on X is equal to $\dim {}_0\operatorname{Hom}_{R_X}(I_{Z,X}, R_X) - 1$ [16, 2.9].

Since $I_{Z,X} \cong \Omega_X(-m)$, we have $\operatorname{Hom}_{R_X}(I_{Z,X}, I_{Z,X}) \cong R_X$ and $\operatorname{Ext}^1_{R_X}(I_{Z,X}, I_{Z,X}) = 0$. Thus applying the functor $\operatorname{Hom}_{R_X}(I_{Z,X}, -)$ to the sequence

$$0 \longrightarrow I_{Z,X} \longrightarrow R_X \longrightarrow R_Z \longrightarrow 0 \tag{3.3}$$

we obtain an exact sequence

$$0 \longrightarrow R_X \longrightarrow \operatorname{Hom}_{R_X}(I_{Z,X}, R_X) \longrightarrow \operatorname{Hom}_{R_X}(I_{Z,X}, R_Z) \longrightarrow 0.$$

Hence

$$\dim {}_0\operatorname{Hom}_{R_X}(I_{Z,X}, R_Z) = \dim {}_0\operatorname{Hom}_{R_X}(I_{Z,X}, R_X) - 1 \tag{3.4}$$

is the dimension of the linear system $|Z|$ on X.

The family of ACM schemes in which X varies has dimension

$$\dim {}_0\operatorname{Hom}_R(I_X, R_X) \tag{3.5}$$

because codimension 2 ACM subschemes of positive dimension are unobstructed [12].

Next we remark that if Z is as in (c) then by Proposition 3.1 the h-vector h_X equals k_Z. The ACM scheme X is unique. Indeed, the equality of the Hilbert functions of X and Z for $n \leqslant \bar{b}$ guarantees that the map

$H^0(\mathcal{I}_X(n)) \to H^0(\mathcal{I}_Z(n))$ is an isomorphism for $n \leqslant \overline{b}$. Since $\mathcal{I}_X$ is generated in degrees less or equal than $\overline{b}+1$, the homogeneous ideal of X is uniquely determined by the ideal of Z.

Therefore the dimension of the family in statement (c) is the sum of the dimension of the linear system $|Z|$ on X and of dimension of the Hilbert scheme in which X varies. By (3.4) and (3.5) this dimension is equal to

$$\dim\ {}_0\operatorname{Hom}_R(I_X, R_X) + \dim\ {}_0\operatorname{Hom}_{R_X}(I_{Z,X}, R_Z).$$

Thus to complete the proof it suffices to show

(3.6)
$$\dim\ {}_0\operatorname{Hom}_R(I_Z, R_Z) \leqslant \dim\ {}_0\operatorname{Hom}_R(I_X, R_X) + \dim\ {}_0\operatorname{Hom}_{R_X}(I_{Z,X}, R_Z).$$

To this end, we apply $\operatorname{Hom}_R(-, R_Z)$ to the exact sequence

$$0 \longrightarrow I_X \longrightarrow I_Z \longrightarrow I_{Z,X} \longrightarrow 0 \tag{3.7}$$

and obtain a new exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(I_{Z,X}, R_Z) \longrightarrow \operatorname{Hom}_R(I_Z, R_Z) \longrightarrow \operatorname{Hom}_R(I_X, R_Z),$$

from which we deduce:

(3.8)
$$\dim\ {}_0\operatorname{Hom}_R(I_Z, R_Z) \leqslant \dim\ {}_0\operatorname{Hom}_R(I_X, R_Z) + \dim\ {}_0\operatorname{Hom}_R(I_{Z,X}, R_Z).$$

Comparing (3.6) and (3.8), we see we will be done if we can show $\operatorname{Hom}_R(I_X, R_X) \cong \operatorname{Hom}_R(I_X, R_Z)$. For this, we apply the $\operatorname{Hom}_R(I_X, -)$ to the exact sequence (3.3) and obtain

$$0 \longrightarrow \operatorname{Hom}_R(I_X, I_{Z,X}) \longrightarrow \operatorname{Hom}_R(I_X, R_X) \longrightarrow \operatorname{Hom}_R(I_X, R_Z) \longrightarrow \operatorname{Ext}^1_R(I_X, I_{Z,X}).$$

Since $I_{Z,X} \cong \Omega_X(-m))$, we need to show that $\operatorname{Hom}_R(I_X, \Omega_X(-m)) = \operatorname{Ext}^1_R(I_X, \Omega_X(-m)) = 0$.

To prove these vanishings, we consider the minimal presentation of I_X over R:

$$\bigoplus R(-b_j) \longrightarrow \bigoplus R(-a_i) \longrightarrow I_X \longrightarrow 0. \tag{3.9}$$

The module $\operatorname{Hom}_R(I_X, \Omega_X(-m))$ is a submodule of $\bigoplus \Omega_X(a_i - m)$. By Proposition 2.4 we have $\overline{b}+2 = \operatorname{Max}\{b_j\} \geqslant a_i$ for every i. Since by assumption $m \geqslant 2\overline{b} - N + 4$, we have

$$m - a_i \geqslant \overline{b} - N + 2 = \overline{b} + \dim X.$$

By Proposition 2.1 it follows $\Omega_X(a_i - m) = 0$ for every i, hence $\operatorname{Hom}_R(I_X, \Omega_X(-m)) = 0$.

To deal with $\mathrm{Ext}^1_R(I_X, \Omega_X(-m))$, let M denote the kernel of the surjection $\bigoplus R(-a_i) \to I_X$. Then $\mathrm{Ext}^1_R(I_X, \Omega_X(-m))$ is a quotient of $\mathrm{Hom}_R(M, \Omega_X(-m))$, which in turn is a submodule of $\bigoplus \Omega_X(b_j - m))$. Since $m \geqslant 2\overline{b} - N + 4$, the same argument as above shows $\Omega_X(b_j - m) = 0$ for every j. Hence $\mathrm{Hom}_R(M, \Omega_X(-m)) = 0$, and, a fortiori, $\mathrm{Ext}^1_R(I_X, \Omega_X(-m)) = 0$, which is what was left to prove. □

4. Arithmetically Gorenstein zero-dimensional subschemes

A natural problem is to look for geometric conditions which allow one to decide whether or not a subscheme of $\mathbb{P}^N$ is AG. In the case of zero-dimensional subschemes there is a characterization in terms of the so called Cayley–Bacharach property: it is the implication $(a) \Longleftrightarrow (b)$ in the following proposition, and it was originally proven in [7] in the reduced case and then generalized to the non reduced case in [25]. Below we give an alternative proof.

Recall a zero dimensional subscheme $Z \subset \mathbb{P}^N$ is ACM, and by Proposition 2.1 its m-invariant is

$$m(Z) = \max\{n \mid (\Omega_Z)_{-n} \neq 0\} = \max\{n : h^1(\mathbb{P}^N, \mathcal{I}_Z(n)) \neq 0\},$$

and the regularity of Z is $m(Z) + 2$.

Proposition 4.1. — *Let Z be a zero-dimensional scheme in $\mathbb{P}^N$, let $m = m(Z)$ and suppose the h–vector of Z is symmetric, i.e.,*

$$h(n) = h(m + 1 - n) \quad \textit{for every } n \in \mathbb{Z}.$$

Then the following are equivalent:

(a) *Z is AG;*
(b) *Z satisfies the Cayley–Bacharach property: for every subscheme $W \subset Z$ with $\deg(W) = \deg(Z) - 1$, we have*
$$h^0(\mathbb{P}^N, \mathcal{I}_W(m)) = h^0(\mathbb{P}^N, \mathcal{I}_Z(m));$$
(c) *for any proper subscheme W_0 of Z, $m(W_0) < m$;*
(d) *for any proper subscheme W_0 of Z, $h^1(\mathbb{P}^N, \mathcal{I}_{W_0}(m)) = 0$.*

Proof.

(c)$\Longleftrightarrow$ (d) Since $\bigoplus_{n\in\mathbb{Z}} \mathrm{H}^1(\mathbb{P}^N, \mathcal{I}_{W_0}(n))$ is the graded k-dual of Ω_{W_0}, the vanishing

$$h^1(\mathbb{P}^N, \mathcal{I}_{W_0}(m)) = 0$$

implies $h^1(\mathbb{P}^N, \mathcal{I}_{W_0}(n)) = 0$ for every $n \geqslant m$, that is, $m(W_0) < m$.

(b)$\Longleftrightarrow$ (d) Since $m(W_0) \leqslant m(W)$ whenever $W_0 \subseteq W \subseteq Z$, we may assume in (d) that $W_0 = W$ has length $\deg Z - 1$.

Looking at Hilbert polynomials we see

$$h^0(\mathcal{I}_W(m)) - h^1(\mathcal{I}_W(m)) = h^0(\mathcal{I}_Z(m)) - h^1(\mathcal{I}_Z(m)) + 1.$$

The assumption on the h-vector implies that $h^1(\mathbb{P}^N, \mathcal{I}_Z(m)) = h_Z(m+1) = 1$. Thus $h^0(\mathcal{I}_W(m)) = h^0(\mathcal{I}_Z(m))$ if and only if $h^1(\mathcal{I}_W(m)) = 0$.

(c)$\Longrightarrow$ (a) The symmetry of the h-vector implies that R_Z and $\Omega_Z(-m)$ have the same Hilbert function.

We can define a function $\varphi : R_Z \to \Omega_Z(-m)$ sending 1 to α where α is a non zero element of degree zero in $\Omega_Z(-m)$. If φ had a nontrivial kernel, then it would define a proper subscheme W of Z together with a non zero map $R_W \to \Omega_Z(-m)$ and so $m(W) = m(Z)$, contradicting (c). Thus φ has trivial kernel, hence it is an isomorphism because R_Z and $\Omega_Z(-m)$ have the same Hilbert function. Therefore Z is AG.

(a)$\Longrightarrow$ (c) Suppose $m(W) = m$. Then we have a nontrivial map

$$R_W \to \Omega_W(-m) \to \Omega_Z(-m) \cong R_Z$$

lifting the natural map $R_Z \to R_W$, hence $W = Z$. □

For later use we need to rephrase this criterion in the case Z is a subscheme of an AG curve. Thus suppose we are given an AG curve C and a zero dimensional subscheme $Z \subset C$. Then Z can be thought of as an effective generalized divisor on C [16], and it corresponds to a nondegenerate section of the reflexive sheaf of $\mathcal{O}_C$ modules

$$\mathcal{O}_C(Z) = \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{I}_{Z,C}, \mathcal{O}_C).$$

Analogously we define

$$R_C(Z) = \mathrm{Hom}_{R_C}(I_{Z,C}, R_C).$$

Note that, since C is ACM, we have $I_{Z,C} \cong \mathrm{H}^0_*(\mathcal{I}_{Z,C})$ and $R_C \cong \mathrm{H}^0_*(\mathcal{O}_C)$, hence the n-th graded piece $R_C(Z)_n$ of $R_C(Z)$ is isomorphic to $\mathrm{H}^0(\mathcal{O}_C(Z + nH))$, which at the level of graded modules we may rewrite as

$$R_C(Z) \cong \mathrm{H}^0_*(\mathcal{O}_C(Z)).$$

Applying the functor $\mathrm{Hom}_{R_C}(-, R_C)$ to the exact sequence

$$0 \to I_{Z,C} \to R_C \to R_Z \to 0$$

we obtain another exact sequence, analogous to [16, 2.10]

$$0 \to R_C \to R_C(Z) \to \mathrm{Ext} \to 0^1_{R_C}(R_Z, R_C) \cong \Omega_Z \otimes \Omega_C^\vee \tag{4.1}$$

where the last isomorphism can be derived as follows:

$$\mathrm{Ext}^1_{R_C}(R_Z, R_C) \cong \mathrm{Ext}^1_{R_C}(R_Z, \Omega_C) \otimes \Omega_C^\vee \cong \Omega_Z \otimes \Omega_C^\vee.$$

COROLLARY 4.2. — *Suppose Z is an effective divisor on the AG curve C. Let $m = m(Z)$ and $e = m(C)$. Suppose the h–vector of Z is symmetric. Then Z is AG if and only if*

$$\dim(R_C(W))_{e-m} < \dim(R_C(Z))_{e-m}$$

for every subscheme $W \subset Z$ with $\deg(W) = \deg(Z) - 1$.

Proof. — Pick a subscheme $W \subset Z$ with $\deg(W) = \deg(Z) - 1$. Since $\Omega_C \cong R_C(e)$, applying (4.1) to W and then to Z, we see that $m(W) < m$ is equivalent to

$$\dim(R_C(W))_{e-m} < \dim(R_C(Z))_{e-m}.$$

□

5. Complete Intersection Biliaison

In this section, we will show that a general AG set of points in $\mathbb{P}^3$ is obtained by ascending complete intersection biliaisons from a point (see Definition 2.13). We follow closely section 4 of [18].

LEMMA 5.1. — *Let Z be a codimension 3 AG subscheme of $\mathbb{P}^N$ with h-vector h. Let s denote the minimum degree of a hypersurface containing Z, $b = b(Z) = \max\{n \mid h(n) > 0\}$, and let m be the integer for which $\Omega_Z \cong R_Z(m)$. Then*

a) $s = \min\{n > 0 \mid h(n) < \binom{n+2}{2}\}$.
b) $m = b - \dim(Z) - 1$.
c) $\mathcal{I}_Z(b + 2 - s)$ *is generated by global sections.*

Proof. — Part a) follows from the definition of the h-vector. Part b) we recall for memory from Proposition 2.1. For part c) we use the theorem of Buchsbaum–Eisenbud [3] in the notation of [20, §5, pp. 62-63]. Let R be the homogeneous coordinate ring of $\mathbb{P}^N$. Then the homogeneous ideal I_Z of Z has a resolution of the form

$$0 \to R(-c) \to \oplus R(-b_i) \to \oplus R(-a_i) \to I_Z \to 0$$

with $i = 1, 2, \ldots, 2r+1$ for some positive integer r. Moreover, this resolution is symmetric in the sense that if we order $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{2r+1}$ and

$b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{2r+1}$, then $b_i = c - a_i$ for each i. Furthermore, if we let $u_{ij} = b_i - a_j$ be the associated degree matrix, then $u_{ij} > 0$ for $i+j = 2r+3$.

To relate this to the invariants s and b of the h-vector, first note that the a_i are the degrees of a minimum set of generators of I_Z. Hence $a_1 = s$, which is the least degree of a generator. By symmetry, $b_1 = c - s$. On the other hand, $b = c-3$ by Proposition 2.2. From the inequality $u_{2,2r+1} > 0$ we find $b_2 > a_{2r+1} = \max\{a_i\}$. But $b_1 = c - s \geqslant b_2$, so we find $\max\{a_i\} < c - s$. Hence $\max\{a_i\} \leqslant b + 2 - s$, and $\mathcal{I}_Z(b + 2 - s)$ is generated by global sections. □

THEOREM 5.2. — *For any h-vector h corresponding to an AG zero dimensional subscheme of $\mathbb{P}^3$ (as in (2.8)), there is a nonempty open subset V_h of the Hilbert scheme* $\mathrm{PGor}(h)$ *such that any $Z \in V_h$ can be obtained by strictly ascending CI-biliaisons from a point in $\mathbb{P}^3$.*

Proof. — We will prove, by induction on the degree, the following slightly more precise statement. For each h, let $s = s(h)$ and $t = t(h)$ denote the value of the invariants $s(Z)$ and $m(Z) + 3 - s(Z)$ for $Z \in \mathrm{PGor}(h)$. We claim there is an open set $V_h \subseteq \mathrm{PGor}(h)$ such that for any $Z \in V_h$

(i) There is a reduced complete intersection curve $C = F_s \cap F_t$ such that Z is contained in the smooth locus C^{sm} of C and intersects every irreducible component of C^{sm}.
(ii) There is an AG zero-scheme $Z' \sim Z - H$ on C, with h-vector h', such that $Z' \in V_{h'}$.

To begin with, by Lemma 5.1 a zero dimensional AG subscheme $Z \subseteq \mathbb{P}^3$ is contained in the complete intersections C of two surfaces of degree $s = s(Z)$ and $t = m(Z)+3-s(Z)$ respectively. Thus property (i) is an open condition on $\mathrm{PGor}(h)$.

We start the induction with AG subschemes Z having $s = 1$. These are contained in a $\mathbb{P}^2$, so they are complete intersections, and for these the theorem is well known.

So now we assume $s \geqslant 2$. Suppose for a moment that $Z \subseteq C$ satisfies condition (i). We will show that the linear system $|Z - H|$ is nonempty and contains an AG subscheme Z'. We use the exact sequence (4.1) twisted by $-H$:

$$0 \to R_C(-H) \to R_C(Z - H) \to \Omega_Z \otimes \Omega_C^{\vee}(-H) \to 0. \tag{5.1}$$

Now $\Omega_C \cong R_C(s + t - 4) = R_C(m - 1)$, and $\Omega_Z \cong R_Z(m)$, so the sheaf on the right is just R_Z. Therefore

$$h^0(\mathcal{O}_C(Z - H)) = \dim(R_C(Z - H))_0 = \dim(R_Z)_0 = 1 \tag{5.2}$$

so the sheaf $\mathcal{O}_C(Z-H)$ has a unique section σ whose restriction to Z is 1. From the condition that Z meets every irreducible component of C^{sm}, and C being reduced, we conclude that σ is nondegenerate, and defines an effective divisor $Z' \sim Z - H$ [16, 2.9]. Furthermore, since σ restricted to Z is 1, we find that $\mathrm{Supp}(Z) \cap \mathrm{Supp}(Z') = \emptyset$.

We claim that Z' is AG. First of all from the equality

$$h_{Z'}(n) = h_Z(n+1) - h_C(n+1) \tag{5.3}$$

and the fact that both h_Z and h_C are h-vectors satisfying $h(n) = h(m+1-n)$ for all $n \in \mathbb{Z}$, we see that $h_{Z'}$ is symmetric and $m(Z') = m-2$. Therefore to show Z' is AG it is enough by Corollary 4.2 to show that for every $D \subset Z'$ with $\deg(D) = \deg(Z') - 1$ we have

$$\dim(R_C(D))_1 < \dim(R_C(Z'))_1.$$

Since $Z' \sim Z - H$ is a Cartier divisor on C, the divisor $Z' - D$ is effective and has degree 1, therefore it is a point Q in the support of Z'. Now $D + H \sim Z - Q$ and $Z' + H \sim Z$, so what we have to prove is that

$$h^0(\mathcal{O}_C(Z-Q)) = \dim(R_C(Z-Q))_0 < \dim(R_C(Z))_0 = h^0(\mathcal{O}_C(Z)).$$

This follows from the fact that $\mathrm{Supp}(Z) \cap \mathrm{Supp}(Z') = \emptyset$, hence the section of $\mathcal{O}_C(Z)$ corresponding to Z is not the image of a section of $\mathcal{O}_C(Z-Q)$.

Now we explain the induction step of the proof. Given an admissible h-vector h with $s \geqslant 2$, define h' as in (5.3) above. By the induction hypothesis there exists an open set $V_{h'} \subseteq \mathrm{PGor}(h')$ of AG schemes satisfying (i) and (ii). Let Z' be such a scheme, and let $Z' \subseteq C' = F_{s'} \cap F_{t'}$ satisfy (i). Note that we either have $s' = s-1$ and $t' = t-1$ or $s' = s$ and $t' = t-2$. So define a curve $C = (F_{s'} + H_1) \cap (F_{t'} + H_2)$ or $C = F_{s'} \cap (F_{t'} + H_1 + H_2)$, where H_1, H_2 are planes in general position. Then C is a reduced complete intersection curve of two surfaces of degree s and t respectively.

On this curve C, we will show, by an argument analogous to the one above, that a general divisor Z in the linear system $Z' + H$ on C is AG.

We first prove that an effective divisor $Z \in |Z' + H|$ is AG if and only if $\mathrm{Supp}(Z) \cap \mathrm{Supp}(Z') = \emptyset$. By construction the h-vector of Z is symmetric. Therefore by Corollary 4.2 the divisor Z is AG if and only if for every $W \subset Z$ of degree $\deg(Z) - 1$ we have

$$\dim(R_C(W))_{-1} < \dim(R_C(Z))_{-1}.$$

Since Z is Cartier on C, we may write $W = Z - Q$ where Q is a point in the support in Z, and then the inequality above is equivalent to

$$h^0(\mathcal{O}_C(Z'-Q)) < h^0(\mathcal{O}_C(Z')).$$

Now from the exact sequence

$$0 \to R_C \to R_C(Z') \to \Omega_{Z'} \otimes \Omega_C^\vee \cong R_{Z'}(-1) \to 0 \tag{5.4}$$

we deduce $h^0\mathcal{O}_C(Z') = 1$. Thus the inequality above is satisfied if and only if for every $Q \in \mathrm{Supp}(Z)$ we have $h^0\mathcal{O}_C(Z' - Q) = 0$, that is, Q is not in $\mathrm{Supp}(Z')$.

This completes the proof of the claim that Z is AG if and only if $\mathrm{Supp}(Z) \cap \mathrm{Supp}(Z') = \emptyset$. Now we show that a general Z in $|Z' + H|$ satisfies this property. For this, note that $I_{Z,C} \cong I_{Z',C}(-1)$, hence twisting by one the exact sequence (5.4) we obtain

$$0 \to R_C(1) \to R_C(Z) \to R_{Z'} \to 0. \tag{5.5}$$

The degree zero piece is

$$0 \to \mathrm{H}^0\mathcal{O}_C(H) \to \mathrm{H}^0\mathcal{O}_C(Z) \to (R_{Z'})_0 \to 0$$

which shows that a general section of $\mathrm{H}^0\mathcal{O}_C(Z)$ maps to a unit in $(R_{Z'})_0$, hence does not vanish at any point P of Z'.

Furthermore, since the trivial biliaison $Z' + H$ satisfies (i), and this is an open condition, we can choose Z in $|Z' + H|$ so that it is AG and satisfies (i).

Thus there exists an open subset of AG subschemes $Z \in \mathrm{PGor}(h)$ satisfying (i). Since the procedures of constructing Z' from Z and Z from Z' are reversible, we can find an open subset $V_h \subseteq \mathrm{PGor}(h)$ of AG schemes Z satisfying (i) with the associated scheme Z' lying in $V_{h'}$.

This completes the inductive proof of (i) and (ii). To prove the theorem, we take a $Z \in V_h$, and by (ii) find a $Z' \in V_{h'}$ with smaller degree. We continue this process until either the degree is 1 or $s = 1$, which we have discussed above. □

As a corollary we now derive a formula that allows one to compute the dimension of $\mathrm{PGor}(h)$ inductively (cf. [6] for a different approach). Fix a *G3*-admissible h-vector h, and let $s = s(h) = \min\{n > 0 : h(n) < \binom{n+2}{2}\}$ and $t = m(h) + 3 - s(h)$. Denote by h' the h-vector defined as in (5.3) by the formula $h'(n) = h(n+1) - h_{s,t}(n+1)$, where $h_{s,t}$ denotes the h-vector of a a complete intersection of two surfaces of degree s and t.

Corollary 5.3. — *With the above notation, if $s \geqslant 2$, then*

$$\dim \mathrm{PGor}(h) = \dim \mathrm{PGor}(h') - h'(s) - h'(t) + st + 3s + 3 - \varepsilon,$$

where $\varepsilon = 0$ if $t > s$ and $\varepsilon = 1$ if $t = s$.

Proof. — As in the proof of Theorem 5.2, given a general $Z \in \mathrm{PGor}(h)$, there are $Z' \in \mathrm{PGor}(h')$ and a curve C, complete intersection of surfaces of degree s and t, such that Z is linearly equivalent to $Z' + H$ on C. The

h-vector $h_C = h_{s,t}$ of C is given by $h_C(n) = n+1$ for $0 \leqslant n \leqslant s-1$, $h_C(n) = s$ for $s-1 \leqslant n \leqslant t-1$ and $h_C(n) = h_C(m+1-n)$ for $n \geqslant t$. Let us denote by $\mathcal{F}$ the family of complete intersection curves $C = F_s \cap F_t$ containing the AG zero-scheme Z as a divisor $Z \sim Z' + H$ and by $\mathcal{F}'$ the family of complete intersection curves $C = F_s \cap F_t$ containing the AG zero-scheme Z'. We have

$$\dim \mathrm{PGor}(h) = \dim \mathrm{PGor}(h') + \dim \mathcal{F}' + \dim_C |Z' + H| - \dim \mathcal{F}, \tag{5.6}$$

because $\dim_C |Z'| = 0$ by formula (5.2). Let us begin by supposing $t > s$ and let us compute $\dim \mathcal{F}$ and $\dim \mathcal{F}'$. We have

$$\begin{aligned}\dim \mathcal{F} &= (h^0\mathcal{I}_Z(s) - 1) + (h^0\mathcal{I}_Z(t) - h^0\mathcal{O}_{\mathbb{P}^3}(t-s) - 1)\\ &= \left(\binom{s+3}{3} - \sum_{n=0}^{s} h(n) - 1\right) + \left(\binom{t+3}{3} - \sum_{n=0}^{t} h(n) - \binom{t-s+3}{3} - 1\right).\end{aligned}$$

To determine $\dim \mathcal{F}'$ we need the Hilbert function of Z' which equals $\sum_{r=1}^{n+1}(h(r) - h_C(r))$.

$$\begin{aligned}\dim \mathcal{F}' = &\left(\binom{s+3}{3} - \sum_{n=1}^{s+1} h(n) + \sum_{n=1}^{s+1} h_C(n) - 1\right)\\ &\left(\binom{t+3}{3} - \sum_{n=1}^{t+1} h(n) + \sum_{n=1}^{t+1} h_C(n) - \binom{t-s+3}{3} - 1\right).\end{aligned}$$

The exact sequence (5.1) shows that $\dim_C |Z' + H| = 4$. By substituting the various pieces in (5.6) we obtain:

$$\dim \mathrm{PGor}(h) = \dim \mathrm{PGor}(h') - h(s+1) - h(t+1) + \sum_{n=0}^{s+1} h_C(n) + \sum_{n=0}^{t+1} h_C(n) + 4.$$

Using the properties of the h-vectors h_C, h' we obtain

$$\begin{aligned}\dim \mathrm{PGor}(h) &= \dim \mathrm{PGor}(h') - h(s+1) - h(t+1) + h_C(s) + h_C(s+1) + st + 3s + 1\\ &= \dim \mathrm{PGor}(h') - h'(s) - h'(t) + h_C(s) - h_C(t+1) + st + 3s + 1.\end{aligned}$$

Moreover $h_C(s) = s-1$ if $t = s$, $h_C(s) = s$ if $t \geqslant s+1$, so $h_C(s) - h_C(t+1) = 1$ if $t = s$, $h_C(s) - h_C(t+1) = 2$ if $t \geqslant s+1$, from which the statement follows.

When $t = s$, the calculation is similar, the only difference being that in this case we have $\dim \mathcal{F}' = \dim \mathrm{Grass}(2, H^0(\mathcal{I}_{Z'}(s)))$ and analogously for $\dim \mathcal{F}$. □

Remark 5.4. — Using Corollary 5.3 we can compute the dimension of $\mathrm{PGor}(h)$ by induction on s, once we know the dimensions for all h with $s = 1$. Now a zero dimensional subscheme Z with $s = 1$ is the complete intersection of two plane curves of degree p and q, $q \geqslant p$, and thus the dimension count is immediate:

$$\dim \mathrm{PGor}(h) = \begin{cases} 3, & \text{if } p = q = 1 \\ q + 4, & \text{if } p = 1, q > 1 \\ p^2 + 3p + 1, & \text{if } q = p \geqslant 2 \\ pq + 3p + 2, & \text{if } q > p \geqslant 2. \end{cases}$$

Given a $C2$-admissible h-vector h (see Definition 2.5), an argument similar to the one in Corollary 5.3 can be used to determine the dimension of the family $\mathrm{ACM}(h)$ of ACM curves in $\mathbb{P}^3$ with a fixed h-vector h. This dimension was first computed by Ellingsrud [12]. Here we give a formula which allows to compute $\dim \mathrm{ACM}(h)$ inductively (compare [27], Proposition 6.8 p. 176). Let $s = s(h)$ denote the least degree of a surface containing a curve C in $\mathrm{ACM}(h)$. Let h' be the h-vector defined by

$$h'(n) = \begin{cases} h(n) & \text{for } n \leqslant s - 2 \\ h(n+1) & \text{for } n \geqslant s - 1. \end{cases} \tag{5.7}$$

Note that h' is the h-vector of an ACM curve C' obtained from $C \in \mathrm{ACM}(h)$ by performing an elementary descending biliaison of height one on a surface of degree s. Then:

PROPOSITION 5.5. — *Let h, h' as above. Then*

$$\dim \mathrm{ACM}(h) = \dim \mathrm{ACM}(h') + 4s + \sum_{n \geqslant s+1} h'(n). \tag{5.8}$$

COROLLARY 5.6. — *If $h(n) = 0$ for $n \geqslant s + 2$, then $\dim \mathrm{ACM}(h) = 4d$ where $d = \sum_{n \geqslant 0} h(n)$ is the degree of a curve in $\mathrm{ACM}(h)$.*

Proof. — By induction on s, beginning with $s = 1$, in which case we have a plane curve of degree 1, 2, or 3 and the result is known. Alternatively, one can observe that the hypothesis on the h-vector implies that any curve $C \in \mathrm{ACM}(h)$ has index of speciality $m(C) \leqslant s - 1$; hence the normal bundle N_C satisfies $h^1(N_C) = 0$ by [34, Lemma 4.2], and this implies the statement by deformation theory. □

6. AG zero dimensional subschemes of $\mathbb{P}^3$

In this section we study AG zero dimensional subschemes of $\mathbb{P}^3$ of low degree and their Hilbert schemes $\mathrm{PGor}(h)$. The results are summarized in Table 8.1.

We begin by addressing the question of how many general points one can impose on a general $Z \in \mathrm{PGor}(h)$. We denote by $\mu = \mu(h)$ this number.

More generally, when $\mathcal{F}$ is an irreducible flat family of subschemes of $\mathbb{P}^N$, we define $\mu(\mathcal{F})$ (or $\mu(Y)$) to be the maximum number of general points one can impose on a general $Y \in \mathcal{F}$. Since d general points in $\mathbb{P}^N$ depend on dN parameters, one has $\mu(\mathcal{F}) \cdot \mathrm{codim}(Y, \mathbb{P}^N) \leqslant \dim(\mathcal{F})$. For complete intersections one has

PROPOSITION 6.1. — *Let $\mathcal{F}$ be the family of complete intersections of hypersurfaces Y of degrees $d_1, \ldots, d_r$ in $\mathbb{P}^N$. Let $s = \min\{d_i\}$. Then*

$$\mu(\mathcal{F}) = \binom{s+N}{N} - \#\{i \mid d_i = s\}.$$

In particular, $\mu(\mathcal{F}) \cdot \mathrm{codim}(Y, \mathbb{P}^N) = \dim(\mathcal{F})$ if and only if $d_i = s$ for every i.

We let $\mu(h)$ denote $\mu(\mathrm{PGor}(h))$. There are two obvious upper bounds for $\mu(h)$:

PROPOSITION 6.2. — *Let μ denote the maximum number of general points one can impose on a general $Z \in \mathrm{PGor}(h)$. Then*

(a)
$$\mu \leqslant \frac{1}{3} \dim \mathrm{PGor}(h).$$

(b) *if $s = s(h)$, then*
$$\mu \leqslant \binom{s+3}{3} - h^0(\mathcal{I}_Z(s)) = h(s) + \binom{s+2}{3}.$$

Proof. — For part (b), note that whenever $W \subset Z$, we have $h_Z(n) \geqslant h_W(n)$. Hence, if W is a set of d general points contained in Z, we must have

$$h_Z(n) \geqslant h_W(n) = \begin{cases} \binom{n+2}{2} & \text{if } n < s(W) \\ d - \binom{s(W)+2}{3} & \text{if } n = s(W). \end{cases}$$

Thus either $s(W) < s$, which means W is contained in a surface of degree $s-1$ and hence $d < \binom{s+2}{3}$, or $s(W) = s$, in which case $h(s) \geqslant d - \binom{s+2}{3}$. Thus in any case $d \leqslant h(s) + \binom{s+2}{3}$ as claimed. □

Lower bounds for $\mu(h)$ are provided by the numbers $\tilde{\mu}$ which we now introduce. Given a pair $(\tilde{h}, m)$ consisting of a $C2$-admissible h-vector $\tilde{h}$ and an integer m, we let $h = h(\tilde{h}, m)$ denote the h-vector determined by the formula

$$\partial h(n) = \tilde{h}(n) - \tilde{h}(m + 2 - n). \tag{6.1}$$

By Proposition 3.1, if Z is a divisor linearly equivalent to $mH_C - K_C$ on some ACM curve C with h-vector $\tilde{h}$, then $h = h_Z$. Assuming this linear system is nonempty for a general curve C in $\mathrm{ACM}(\tilde{h})$, we denote by U the open dense subset of $\mathrm{ACM}(\tilde{h})$ consisting of reduced and locally complete intersection curves, smooth if $\tilde{h}$ is of decreasing type, on which the linear system $mH_C - K_C$ has the smallest dimension. We can also require curves in U to have any other general property we may need – for example the normal bundle being stable – as long this does not make U empty. Let $\mathcal{B} = \mathcal{B}(\tilde{h}, m)$ be the subscheme of $\mathrm{PGor}(h)$ defined as

$$\mathcal{B} = \{Z \in \mathrm{PGor}(h) | \, \exists\, C \in U \text{ such that } Z \sim mH_C - K_C \text{ on } C\}.$$

By construction $\mathcal{B}$ is irreducible of dimension

$$\dim \mathcal{B} = D(\tilde{h}, m) + \dim \mathrm{ACM}(\tilde{h}) - \delta \tag{6.2}$$

where $D(\tilde{h}, m)$ is the dimension of the linear system $|mH_C - K_C|$ for $C \in U$, and δ is the dimension of the family of curves $C \in U$ containing a fixed general Z in $\mathcal{B}$.

We let $\tilde{\mu} = \tilde{\mu}(\tilde{h}, m) = \mu(\mathcal{B})$ denote the maximum number of general points one can impose on a general $Z \in \mathcal{B}$. By definition $\mu(h) \geqslant \tilde{\mu}(\tilde{h}, m)$. We will later see one can often compute $\tilde{\mu}$.

It is interesting to know when $\mathcal{B}(\tilde{h}, m)$ contains a open set of $\mathrm{PGor}(h)$, in which case $\mu = \tilde{\mu}$. By Theorem 3.4 this is the case when $\tilde{h}$ is the first half of h and $m \geqslant 2b(\tilde{h}) + 1$. It is also the case when $\mathcal{B}(\tilde{h}, m)$ has the same dimension of $\mathrm{PGor}(h)$. The latter is known (see Remark 5.4), while to compute the former we can use formula (6.2). Note that $\dim \mathrm{ACM}(\tilde{h})$ is known (see Proposition 5.5), and, when the linear system $|mH_C - K_C|$ is nonspecial for a general C in $\mathrm{ACM}(\tilde{h})$, we have $D(\tilde{h}, m) = \deg(h) - g(C)$. Then to determine $\dim \mathcal{B}$ we still need to know the number δ, although this seems more difficult to compute in general. In several cases one can show δ is zero, and of course this will be the case if the degree of Z is large enough. We give a precise statement in Corollary 6.4 below.

Following Ellia [11], we denote by $G_{CM}(d, s)$ the maximum genus of an ACM irreducible curve of degree d not lying on a surface of degree $s - 1$. When $d > s(s - 1)$, dividing d by s we write $d = st - r$ with $0 \leqslant r < s$.

Then by [13]

$$G_{CM}(d,s) = 1 + \frac{d}{2}\left(s + \frac{d}{s} - 4\right) - \frac{r(s-r)(s-1)}{2s}, \tag{6.3}$$

which is the genus of a curve linked by two surfaces of degrees s and t to a plane curve of degree r.

PROPOSITION 6.3. — *Assume the base field has characteristic zero. Fix a $C2$-admissible h-vector $\tilde{h}$ of decreasing type. Let C_1 and C_2 be two distinct irreducible curves in* $\mathrm{ACM}(\tilde{h})$. *Let $d = \deg(C_i)$ and $s = s(C_i)$. If $d \geqslant 3$, then*

$$g(C_1 \cup C_2) \leqslant G_{CM}(2d, s).$$

Furthermore, if equality holds, then $C_1 \cup C_2$ is an ACM curve linked to plane curve of degree r by two surfaces of degree s and t respectively, where r and t are defined by the relation $2d = st - r$ with $0 \leqslant r < s$.

Proof. — We follow closely the arguments of [11] which require characteristic zero. Note that the numerical character $\chi(X)$ used by Ellia is an invariant equivalent to the h-vector h_X when X is ACM. Let $C = C_1 \cup C_2$. Since C_1 and C_2 are the irreducible components of C and have the same numerical character, the proof of [11, Theorem 10] shows the character of C is connected.

Furthermore, since C_1 is an ACM curve with $s(C_1) = s$, we have $h_{C_1}(n) = n + 1$ for $0 \leqslant n \leqslant s - 1$, thus

$$d = \sum h_{C_1}(n) \geqslant \frac{1}{2}s(s+1).$$

As in the proof of [11, Theorem 13], if we had $g(C) > G_{CM}(2d, s)$, we would have $\sigma \leqslant s - 1$, where σ is the length of the character $\chi(C)$. Then $2d \geqslant s(s+1) > \sigma^2 + 1$. Since $d \geqslant 3$, C contains no curve of degree 2, hence by [11, Lemma 12] C is contained in a surface of degree σ, which is absurd since $\sigma \leqslant s - 1$.

We conclude $g(C) \leqslant G_{CM}(2d, s)$. Suppose equality holds and write $2d = st - r$ with $0 \leqslant r < s$. Then the argument in [11, Theorem 10] together with [13, Theorem 2.7] shows that C is ACM with the same Hilbert function as a curve linked by two surfaces of degrees s and t to a plane curve of degree r. In particular, C is contained in a surface S of degree $s = s(C_1)$. Therefore S is of minimal degree among surfaces containing C_1, and, since C_1 is irreducible, S must also be irreducible. But then looking at the Hilbert function of C we see C is contained in a surface T of degree t that cuts S properly, and the curve linked to C by the complete intersection $S \cap T$ is a plane curve of degree r. □

COROLLARY 6.4. — *Assume the base field has characteristic zero. Fix an h-vector of decreasing type $\tilde{h}$ and let U_0 be the dense open subset of* $\mathrm{ACM}(\tilde{h})$ *consisting of smooth irreducible curves. Let Z be a divisor in the linear system $|mH_C - K_C|$ on some curve $C \in U_0$, and let $d = \deg(C)$ and $s = s(C)$.*

If $d \geqslant 3$ and $md \geqslant G_{CM}(2d, s)$, then C is the unique curve in U_0 containing Z. In particular, δ is zero in this case.

Proof. — If there was another $C' \in U_0$ containing Z, then we would have $\deg(Z) \leqslant \deg(C \cap C') = g(C \cup C') - 2g(C) + 1 \leqslant G_{CM}(2d, s) - 2g(C) + 1$. Since $\deg(Z) = md - 2g(C) + 2$, this would imply

$$m \deg(C) \leqslant G_{CM}(2d, s) - 1$$

contradicting the assumptions. □

We now give an example in which δ is zero, but there is more than one curve in $\mathrm{ACM}(\tilde{h})$ containing Z. Let C be a curve of type $(a, a-1)$ on a smooth quadric surface Q. One knows C is ACM with h-vector $\tilde{h} = \{1, 2, \ldots, 2\}$ ending in degree $b(C) = a - 1$

LEMMA 6.5. — *Suppose Z is a zero dimensional scheme on a smooth quadric surface that is the intersection of two curves C_1 and C_2 of type $(a, a-1)$ and $(a-1, a)$ respectively. Then Z is AG, $\deg(Z) = 2a^2 - 2a + 1$ and $m(Z) = 2a - 3$. If $a \geqslant 3$ and the curves C_1 and C_2 are smooth, then these are the only irreducible curves in* $\mathrm{ACM}(\tilde{h})$ *that contain Z. In particular, the dimension δ of the family of irreducible curves $C \in \mathrm{ACM}(\tilde{h})$ containing Z is zero.*

Proof. — The fact Z is AG is well known and can be seen as follows: since $Z = C_1 \cap C_2$, we have $\mathcal{O}_{C_1}(Z) = \mathcal{O}_Q(a-1, a) \otimes \mathcal{O}_{C_1}$, and by the adjunction formula $Z \sim (2a-3)H - K$ on C_1. Therefore Z is AG with $m = 2a - 3$.

Suppose now C is an irreducible curve in $ACM(\tilde{h})$ which contains Z. Then Z is contained in $C \cap C_1$. By Proposition 6.3 and Corollary 6.4, we must have $Z = C \cap C_1$, and $C \cup C_1$ must be the complete intersection of a quadric surface Q' and a surface of degree $2a - 1$. Since $a \geqslant 3$, we must have $Q' = Q$, hence C is a curve type $(a-1, a)$ on Q. To finish, observe that there is a one to one correspondence between curves C of type $(a-1, a)$ on Q and effective divisors linearly equivalent to Z on C_1, because $h^0\mathcal{O}_Q(-1, 1) = h^1\mathcal{O}_Q(-1, 1) = 0$. □

The following results deal with some particular questions which arise in degree 14, 21 and 30.

PROPOSITION 6.6. — *Assume the base field has characteristic zero. The general zero-dimensional arithmetically Gorenstein scheme of degree 14 with h-vector $h = \{1, 3, 6, 3, 1\}$ is a divisor $Z \sim 3H_C - K_C$ on some smooth ACM curve C of degree 6 and genus 3.*

Proof. — The h-vector of an ACM curve C of degree 6 and genus 3 is $\tilde{h} = \{1, 2, 3\}$, the first half of h, and $h = h(\tilde{h}, 3)$, hence a divisor in the linear system $3H_C - K_C$ has h-vector $\{1, 3, 6, 3, 1\}$. Thus it is enough to show $\dim \mathcal{B}(\tilde{h}, 3) = \dim \mathrm{PGor}(h)$, which is 35. By formula (6.2) we have $\dim \mathcal{B} = \dim_C |3H_C - K_C| + \dim \mathrm{ACM}(\tilde{h}) - \delta = 11 + 24 - \delta$, so it suffices to show $\delta = 0$.

Assume by way of contradiction that $\delta > 0$: this means that, having fixed a general Z in $\mathcal{B}$, one can find a positive dimensional family $\mathcal{C}$ of curves C in U containing Z. By the deformation theory of the pair (Z, C) in $\mathbb{P}^3$, the infinitesimal deformations of C that leave Z fixed are sections of the normal bundle N_C that vanish on Z. Thus a tangent vector to the family $\mathcal{C}$ gives a nonzero section of $N_C(-Z)$. Therefore we obtain a contradiction, and the proposition will be proven, if we can show $\mathrm{H}^0(C, N_C(-Z))$ is zero.

For this, we claim that, if C is a general ACM curve of degree 6 and genus 3 and Z is any divisor of degree 14 on C, then $\mathrm{H}^0(C, N_C(-Z)) = 0$. Since there are curves of type $(6, 3)$ whose normal bundle N_C is stable by [10], and stability is an open property, then for a general such C the normal bundle is stable. The rank two bundle N_C has degree 28 while $\deg Z = 14$, so we have $\deg N(-Z) = 0$ and, by stability, $\mathrm{H}^0(C, N_C(-Z)) = 0$. □

PROPOSITION 6.7. — *The general zero-dimensional arithmetically Gorenstein scheme of degree 21 with h-vector $h = \{1, 3, 4, 5, 4, 3, 1\}$ is a divisor $Z \sim 5H_C - K_C$ on an ACM curve C of degree 5 and genus 3.*

Proof. — An ACM curve of degree 5 and genus 3 has h-vector $\tilde{h} = \{1, 2, 1, 1\}$. Since $\dim \mathrm{ACM}(\tilde{h}) = 20$ by Corollary 5.6, and $\dim \mathrm{PGor}(h) = 37$, the statement follows if $\dim \mathrm{PGor}(h)$ equals $\dim \mathcal{B}(\tilde{h}, 5) = 38 - \delta$, *i.e.*, if $\delta = 1$. The general curve of degree 5 and genus 3 is the union of a plane curve D of degree 4 and a line L intersecting D in one point P. We claim that a general divisor $Z \sim 5H_C - K_C$ consists of 15 points on the plane quartic and 6 points on the line. In fact, from the exact sequences (see [32, 11-10])

$$0 \to \omega_D \to \omega_C \to \omega_L(P) \to 0 \tag{6.4}$$

and

$$0 \to \omega_L \to \omega_C \to \omega_D(P) \to 0 \tag{6.5}$$

we see $\mathcal{O}_L(5H_C - K_C) = \mathcal{O}_L(6)$ and $\mathcal{O}_D(5H_C - K_C) = \mathcal{O}_D(4H_D - P)$. Let W denote the 15 points of Z lying on the quartic D. Then W is the divisor $4H_D - P$ on D, so $h^0(\mathcal{I}_{W,D}(4)) = 1$ and $h^0(\mathcal{I}_{W,\mathbb{P}^3}(4)) = 2$. Therefore $\delta = 1$. □

PROPOSITION 6.8. — *The general zero-dimensional arithmetically Gorenstein scheme Z of degree 30 with h-vector $h = \{1, 3, 6, 10, 6, 3, 1\}$ is not of the form $mH_C - K_C$ on any integral ACM curve.*

Proof. — >From the h-vector we see $m(Z) = 5$. We claim that, if $Z \sim 5H_C - K_C$ on an integral ACM curve C, then C has h vector equal to the first half $k_Z = \{1, 2, 3, 4\}$ of h_Z. In fact, by Proposition 3.1 the h-vector of C must be of the form $h_C = \{1, 2, 3, 4, h_C(4), ..., h_C(b)\}$. The curve C then has degree $d_C = 10 + \sum_{n=4}^{b} h_C(n) = 10 + \tilde{d}$ and genus $g_C = 11 + \sum_{n=4}^{b}(n-1)h_C(n) \geqslant 11 + 3\tilde{d}$. Thus

$$30 = \deg Z = 5d_C - 2g_C + 2 \leqslant 50 + 5\tilde{d} - 22 - 6\tilde{d} + 2 = 30 - \tilde{d},$$

hence $\tilde{d} = 0$ and C has h-vector $\{1, 2, 3, 4\}$. Now the family of such curves C has dimension 40, while for fixed C, we have $\dim_C |Z| = h^0\mathcal{O}_C(5H_C - K_C) - 1 = 19$ because $\deg Z > 2g_C - 2$ (here we use the hypothesis C integral). Thus the family of schemes Z of the form $5H_C - K_C$ on some integral ACM curve C has dimension at most 59. However, if we compute $\dim \mathrm{PGor}(h_Z)$, as explained in Remark 5.4 we find $\dim \mathrm{PGor}(h_Z) = 63$. Thus a general Z cannot be of the form $mH_C - K_C$ on any integral ACM curve C. □

Remark 6.9. — It seems unlikely that Z could be of the form $-K + mH$ on any ACM curve (possibly reducible), but we do not have a complete proof.

In Table 8.1 we list, for every degree $d \leqslant 30$, all possible h-vectors of zero-schemes of degree d in $\mathbb{P}^3$ not contained in a plane. The list is constructed using Proposition 2.8. For every h vector in the table we record

- the dimension A of $\mathrm{PGor}(h)$, which can be computed as explained in Remark 5.4, or applying the formula of [21];
- the invariant m of Propositions 2.1 and 2.2;
- at least one h-vector $\tilde{h}$ such that $h = h(\tilde{h}, m)$ as in formula (6.1);
- the degree and genus of an ACM curve with h-vector $\tilde{h}$;
- the dimension B of $\mathcal{B}(\tilde{h}, m)$; we note that:
  (a) when h is the h-vector of a complete intersection of type (a, b, c) and $\tilde{h}$ is the h-vector of a complete intersection of type (a, b), then $B = A$.

(b) when Theorem 3.4(c) holds, that is, $m \geqslant 2b(\tilde{h}) + 1$, one has $B = A$.
(c) in all other cases, since in the table we always have $\deg(h) \geqslant 2g(\tilde{h})-1$, formula (6.2) gives $B = \deg(h)-g(\tilde{h})+\dim \mathrm{ACM}(\tilde{h})-\delta$;

- the maximum number $\tilde{\nu}$ of general points on a general curve $C \in \mathrm{ACM}(\tilde{h})$; by results of Perrin [31] and Ellia [11], it happens that in all cases in our table one has

$$\tilde{\nu} = \mathrm{Min}\left(\frac{1}{2}\dim \mathrm{ACM}(\tilde{h}),\ \alpha\right)$$

where α is the dimension of the family of surfaces of degree $s(C)$ that contain curves in $\mathrm{ACM}(\tilde{h})$. When $s(C) \leqslant 3$, we have $\alpha = \binom{s(C)+3}{3} - 1$. Note that the problem of determining $\tilde{\nu}$ for ACM curves of higher degree remains open.
- the maximum number $\tilde{\mu}$ of general points on a general $Z \in \mathcal{B}(\tilde{h}, m)$; for C a general curve in $\mathrm{ACM}(\tilde{h})$, we have $\tilde{\mu} = \tilde{\nu}$ if $\dim |mH_C - K_C| \geqslant \tilde{\nu}$, while, if $\dim |mH_C - K_C| < \tilde{\nu}$, we can only say

$$\dim |mH_C - K_C| \leqslant \tilde{\mu} \leqslant \mu;$$

- the maximum number μ of general points on a general $Z \in \mathrm{PGor}(h)$; note that $\mu = \tilde{\mu}$ when $A = B$.

We indicate with a check "✓" whether a general AG zero-scheme is in the linear system $|mH - K|$ on some ACM curve C with h-vector $\tilde{h}$.

7. General sets of points in $\mathbb{P}^3$

PROPOSITION 7.1. — *A set of $n \leqslant 19$ general points in $\mathbb{P}^3$ can be obtained by a sequence of ascending (i.e., degree increasing) Gorenstein liaisons from a point. In particular, it is glicci.*

Proof. — If we have a set W of n general points and we can find a family of AG schemes Z of degree d containing μ general points, and if $\frac{1}{2}d < n \leqslant \mu$, then we can perform a descending Gorenstein liaison from W using Z to get a new set W' of $d - n < \frac{1}{2}d < n$ points. Since the process is reversible, the set W' also consists of general points, and we can continue the process. For one or two points the result is trivial. For $n \geqslant 3$ general points, using

Table 8.1, we choose Z of degree d and h-vector h as follows

n	d	h	μ
$3,4$	5	$\{1,3,1\}$	5
$5,6,7$	8	$\{1,3,3,1\}$	7
$8,9,10,11$	14	$\{1,3,6,3,1\}$	11
$12,13,14$	20	$\{1,3,6,6,3,1\}$	14
$15,16,17$	27	$\{1,3,6,7,6,3,1\}$	17
$18,19$	30	$\{1,3,6,10,6,3,1\}$	$\geqslant 19$.

□

At present, this is as far as we can go, because for $n = 20$ we do not know if an AG zero scheme of degree 30 and $h = \{1,3,6,10,6,3,1\}$ has $\mu \geqslant 20$.

THEOREM 7.2. — *A set of $n \geqslant 56$ general points in $\mathbb{P}^3$ admits no strictly descending Gorenstein liaison.*

Proof. — Let W be a set of n general points, with $s(W) = s$, so that its h-vector is

$$h_W = \left\{1, 3, 6, \ldots, \binom{s+1}{2}, a\right\}$$

with $0 \leqslant a < \binom{s+2}{2}$ and $n = \binom{s+2}{3} + a$. If W is contained in an AG scheme Z then $s(Z) \geqslant s$. On the other hand, if the residual scheme W' has degree less than n, then looking at the h-vectors and using Proposition 2.14, we see that there are only three possibilities for h_Z:

Type 1 $h_Z = \{1,3,6,\ldots,\binom{s+1}{2},\ldots,6,3,1\}$.
Type 2 $h_Z = \{1,3,6,\ldots,\binom{s+1}{2},\binom{s+1}{2},\ldots,6,3,1\}$ and $a > 0$.
Type 3 $h_Z = \{1,3,6,\ldots,\binom{s+1}{2},b,\binom{s+1}{2},\ldots,6,3,1\}$, with $\binom{s+1}{2} \leqslant b \leqslant \binom{s+2}{2}$ and $a > \dfrac{1}{2}b$.

A necessary condition for Z to contain n general points is that $\dim \mathrm{PGor}(h) \geqslant 3n$. We compute $\dim \mathrm{PGor}(h)$ for each of the types above, using Corollary 5.3 and induction on s. Setting $b = \binom{s+1}{2} + c$, so that $0 \leqslant c \leqslant s+1$, we find

Type 1 $\dim \mathrm{PGor}(h) = 4s^2 - 1$.
Type 2 $\dim \mathrm{PGor}(h) = 4s^2 + 3s - 1$.
Type 3 $\dim \mathrm{PGor}(h) = 4s^2 + 4s + 4c - 1$.

Now writing the inequality $\dim \mathrm{PGor}(h) \geqslant 3n$, we find for Type 1

$$4s^2 - 1 \geqslant 3\binom{s+2}{3} + 3a,$$

and using $a \geqslant 0$, this implies $s < 5$.

For Type 2, we find

$$4s^2 + 3s - 1 \geqslant 3\binom{s+2}{3} + 3a.$$

Again using $a > 0$, this implies $s < 6$.
For Type 3, we have

$$4s^2 + 4s + 4c - 1 \geqslant 3\binom{s+2}{3} + 3a.$$

Using $a > \frac{1}{2}b = \frac{1}{2}\binom{s+1}{2} + \frac{1}{2}c$ this gives

$$4s^2 + 4s + \frac{5}{2}c - 1 \geqslant 3\binom{s+2}{3} + \frac{3}{2}\binom{s+1}{2}.$$

Now using $c \leqslant s + 1$ we get

$$4s^2 + 4s + \frac{5}{2}(s+1) - 1 \geqslant 3\binom{s+2}{3} + \frac{3}{2}\binom{s+1}{2}$$

which implies $s < 6$.

Thus for $s \geqslant 6$, and hence for $n \geqslant 56$, a set of n general points has no descending Gorenstein liaison. □

Remark 7.3. — Checking possible values of a and c for $s = 5$, the same method applies for all $n \geqslant 35$, except possibly 36, 37, 38, 45, 46, 47.

One of us [17, Proposition 2.7] has shown that a set of $n < 19$, $n \neq 17$, general points in $\mathbb{P}^3$ can be obtained by a sequence of ascending biliaisons from a point. On the other hand, we can prove:

THEOREM 7.4. — *A set of $n \geqslant 56$ general points in $\mathbb{P}^3$ admits no strictly descending elementary biliaison.*

Proof. — Suppose a set Z of $n \geqslant 56$ general points admits a descending biliaison on an ACM curve C, *i.e.*, , $Z \sim W + H$ on C. Then by Proposition 2.14 the h-vectors satisfy

$$h_Z(l) = h_C(l) + h_W(l+1)$$

for all l. Let $s = s(Z)$, so that $h_Z(s-1) = \frac{1}{2}s(s+1)$. It follows that $h_C(s-1)$ and $h_W(s-2)$ must each achieve their maximum values, namely s and $\frac{1}{2}s(s-1)$ respectively. It follows also that $h_C(l) = 0$ for $l \geqslant s+1$, since this is true also for h_Z by formula (2.3). Thus C satisfies the hypothesis of Corollary 5.6 and the dimension of the family ACM(h_C) is $4d$, where d is the degree of C.

Now in order for C to contain n general points, we must have $\dim \mathrm{ACM}\,(h_C) \geqslant 2n$. Let $h_C(s) = a$. Then $d = \deg\, C = \frac{1}{2}s(s+1)+a$. Let $h_Z(s) = b$. Then $n = \deg\, Z = \binom{s+2}{3} + b$. Furthermore, $a \leqslant s+1$ and $a \leqslant b$.

>From $4d \geqslant 2n$, we thus obtain $2d \geqslant n$, or

$$s(s+1) + 2a \geqslant \binom{s+2}{3} + b.$$

Writing $2a \leqslant s+1+b$, we get

$$(s+1)^2 \geqslant \binom{s+2}{3}$$

which implies $s \leqslant 2 + \sqrt{10} < 6$. So for $s \geqslant 6$, and hence for any $n \geqslant 56$, a set Z of n general points admits no descending biliaison. □

Remark 7.5. — The same argument, taking into account the exact values of a and b, applies to all $n \geqslant 31$, except for $n = 40, 41, 42$. Using a slightly more sophisticated argument, we can treat smaller values of n such as the following case of $n = 20$.

Example 7.6. — A set Z of 20 general points admits no descending elementary biliaison. Indeed, by the analysis in the proof above, the only possibility would be on an ACM curve C with h-vector $h_C = \{1, 2, 3, 4\}$. This is a curve of degree 10 and genus 11. Since the family of all Z's has dimension 60, and the family of pairs (C, Z) with $Z \subset C$ has dimension 60 also, we conclude that a general Z must lie on a general C, and that the points in Z are also general on C. Thus for a general Z, the divisor $W = Z - H$ on C is a general divisor of degree 10. But since the genus of C is 11, this general W cannot be effective. Thus the general Z has no descending biliaison.

On the other hand, we do not know if Z admits a strictly descending Gorenstein liaison, because we cannot answer the question whether there are AG zero-schemes of degree 30 containing 20 general points. If so, we could link 20 general points to 10 general points and thus 20 general points would be glicci. In our notation, the question is whether $\mu \geqslant 20$. We have only been able to show $19 \leqslant \mu \leqslant 21$. There are AG schemes of degree 30 in the linear system $5H - K$ on an ACM curve with $(d, g) = (10, 11)$, but these can contain at most 19 general points. The more general AG schemes of degree 30 are not of the form $mH - K$ on any integral ACM curve by Proposition 6.8, so some new technique will be necessary to answer this question.

Example 7.7. — We can show by an analogous but more complicated argument that a set Z of 31 general points on a nonsingular cubic surface in $\mathbb{P}^3$ does not admit any descending Gorenstein biliaison in $\mathbb{P}^3$. Since Z is in the strict Gorenstein liaison equivalence class of a point [17, 2.4], it is glicci. Furthermore, since in codimension 3, even strict Gorenstein liaison gives the same equivalence relation as Gorenstein biliaison [19, 5.1], Z is even in the Gorenstein biliaison equivalence class of a point. This is the first example we know of of a scheme Z that is glicci but cannot be obtained by a sequence of ascending Gorenstein biliaisons from a linear space.

8. Conclusion

We have established a number of fundamental results about arithmetically Gorenstein zero-dimensional schemes in $\mathbb{P}^3$. In particular, we have investigated those that occur in the form $mH - K$ on an ACM curve and we studied the number of general points that one can impose on an AG scheme with given h-vector, in order to understand the possible Gorenstein liaisons that one can perform on a set of general points. In all cases we are aware of where a class of zero-dimensional subschemes of $\mathbb{P}^3$ has been proved to be glicci, the proof was actually accomplished using strict Gorenstein liaisons, *i.e.,* using only those AG schemes of the form $mH - K$ on some ACM curve (see [17, § 1], for the terminology of strict G-liaison). Remembering that a Gorenstein biliaison is a composition of two strict G-liaisons, this remark applies to the determinantal schemes of [23, 3.6], to any zero-scheme on a non singular quadric surface or a quadric cone [5, 5.1 and 6.1] and to n general points on a non singular cubic surface in $\mathbb{P}^3$ [17, 2.4]. For us, this underlines the importance of studying those more general AG zero-dimensional schemes not of the form $mH - K$ on any ACM curve, and by making use of them either to prove or disprove the assertion that "Every zero-scheme in $\mathbb{P}^3$ is glicci".

In the course of this work we have been led to reconsider some old problems whose solution would be helpful in furthering the work of this paper. One is the problem of Perrin's thesis [31] to find how many general points one can impose to a curve of given degree and genus in $\mathbb{P}^3$. For us it is the ACM curves that are relevant, so we ask: is it true that for a general smooth ACM curve with h-vector $\tilde{h}$, the number $\tilde{\nu}$ of general points one can impose on the curve is given by the formula mentioned in section 6:

$$\tilde{\nu} = \min\left(\frac{1}{2}\dim \mathrm{ACM}(\tilde{h}), \alpha\right)?$$

The other old question, which appeared in a special case in the proof of Proposition 6.6, concerns the stability of the normal bundle of a space curve. The problem was stated in [15], and has been more recently studied by Ellia [11]: if C is a general smooth ACM curve of degree d, genus g, $s = s(C)$ and

$$g < d(s-2) + 1 \qquad (\text{resp. } \leqslant)$$

then is the normal bundle of C stable (resp. semistable)?

Added in Proof. Recently J. Migliore and U. Nagel (math.arXiv:0804.0590) have expanded and generalized our Theorem 5.2 by showing that every codimension three AG subscheme of $\mathbb{P}^N$ admits descending CI-biliaisons to a complete intersection.

Table 8.1: Nondegenerate AG zero-dimensional subschemes in $\mathbb{P}^3$ of degree $d \leqslant 30$ (char. $k = 0$)

d	h	A	m	$\tilde{h}$	$(\tilde{d}, \tilde{g})$	B		$\tilde{\nu}$	$\tilde{\mu}$	μ
5	$\{1,3,1\}$	15	1	$\{1,2\}$	(3,0)	15	✓	6	5	5^a
8	$\{1,3,3,1\}$	21	2	$\{1,2\}$	(3,0)	20	no	6	6	7^e
				$\{1,2,1\}$	(4,1)	21^e	✓	8	7	
11	$\{1,3,3,3,1\}$	23	3	$\{1,2\}$	(3,0)	23^d	✓	6	6	6^d
12	$\{1,3,4,3,1\}$	27	3	$\{1,2,1\}$	(4,1)	27^e	✓	8	8	8^e
13	$\{1,3,5,3,1\}$	31	3	$\{1,2,2\}$	(5,2)	31	✓	9	9	9^b
14	$\{1,3,3,3,3,1\}$	26	4	$\{1,2\}$	(3,0)	26^d	✓	6	6	6^d
14	$\{1,3,6,3,1\}$	35	3	$\{1,2,3\}$	(6,3)	35	✓	12	11	11^a
16	$\{1,3,4,4,3,1\}$	31	4	$\{1,2,1\}$	(4,1)	31^e	✓	8	8	8^e
17	$\{1,3,3,3,3,3,1\}$	29	5	$\{1,2\}$	(3,0)	29^d	✓	6	6	6^d
18	$\{1,3,5,5,3,1\}$	37	4	$\{1,2,2\}$	(5,2)	36	no	9	9	9^e
				$\{1,2,2,1\}$	(6,4)	37^e	✓	9	9	
20	$\{1,3,3,3,3,3,3,1\}$	32	6	$\{1,2\}$	(3,0)	32^d	✓	6	6	6^d
20	$\{1,3,4,4,4,3,1\}$	35	5	$\{1,2,1\}$	(4,1)	35^e	✓	8	8	8^e
20	$\{1,3,6,6,3,1\}$	44	4	$\{1,2,3\}$	(6,3)	41	no	12	12	14^a
				$\{1,2,3,1\}$	(7,5)	43	no	14	14	
				$\{1,2,3,2\}$	(8,7)	$\leqslant 44$	?	16	?	
21	$\{1,3,4,5,4,3,1\}$	37	5	$\{1,2,1,1\}$	(5,3)	37	✓	4	4	4
23	$\{1,3,3,3,3,3,3,3,1\}$	35	7	$\{1,2\}$	(3,0)	35^d	✓	6	6	6^d
23	$\{1,3,5,5,5,3,1\}$	41	5	$\{1,2,2\}$	(5,2)	41^d	✓	9	9	9^d
24	$\{1,3,4,4,4,4,3,1\}$	39	6	$\{1,2,1\}$	(4,1)	39^d	✓	8	8	8^d
24	$\{1,3,5,6,5,3,1\}$	44	5	$\{1,2,2,1\}$	(6,4)	44^e	✓	9	9	9^e

d	h	A	m	$\tilde{h}$	$(\tilde{d},\tilde{g})$	B		$\tilde{\nu}$	$\tilde{\mu}$	μ
25	$\{1,3,5,7,5,3,1\}$	47	5	$\{1,2,2,2\}$	(7,6)	47	✓	9	9	$9^{a,b}$
26	$\{1,3,3,3,3,3,3,3,3,1\}$	38	8	$\{1,2\}$	(3,0)	38^{d}	✓	6	6	6^{d}
26	$\{1,3,4,5,5,4,3,1\}$	43	6	$\{1,2,1,1\}$	(5,3)	43	✓	4	4	4
26	$\{1,3,6,6,6,3,1\}$	47	5	$\{1,2,3\}$	(6,3)	47^{d}	✓	12	12	12^{d}
27	$\{1,3,6,7,6,3,1\}$	51	5	$\{1,2,3,1\}$	(7,5)	50	no	14	14	17^{e}
				$\{1,2,3,2,1\}$	(9,10)	51^{e}	✓	18	17	
28	$\{1,3,4,4,4,4,4,3,1\}$	43	7	$\{1,2,1\}$	(4,1)	43^{d}	✓	8	8	8^{d}
28	$\{1,3,5,5,5,5,3,1\}$	46	6	$\{1,2,2\}$	(5,2)	46^{d}	✓	9	9	9^{d}
28	$\{1,3,6,8,6,3,1\}$	55	5	$\{1,2,3,2\}$	(8,7)	53	no	16	16	$16\leqslant\mu\leqslant18$
				$\{1,2,3,3,1\}$	(10,12)	$\leqslant 55$	?	19	16	
29	$\{1,3,3,3,3,3,3,3,3,3,1\}$	41	9	$\{1,2\}$	(3,0)	41^{d}	✓	6	6	6^{d}
29	$\{1,3,6,9,6,3,1\}$	59	5	$\{1,2,3,3\}$	(9,9)	$\leqslant 56$	no	18	18	$18\leqslant\mu\leqslant19$
				$\{1,2,3,4,1\}$	(11,14)	$\leqslant 59$	?	22	?	
30	$\{1,3,5,6,6,5,3,1\}$	50	6	$\{1,2,2,1\}$	(6,4)	50^{e}	✓	9	9	9^{e}
30	$\{1,3,6,10,6,3,1\}$	63	5	$\{1,2,3,4\}$	(10,11)	$\leqslant 59^{f}$	?	20	19	$19\leqslant\mu\leqslant21$

a: $\mu=\tilde{\mu}$ because $A=B$ or $\tilde{\mu}=[A/3]$
b: the upper bound on μ is given by Proposition 6.2
c: $\delta=0$ by 6.4, 6.5 or 6.6
d: Theorem 3.4(c) applies, and $\tilde{\mu}=\tilde{\nu}$ because $\dim|mH_C-K_C|\geqslant\tilde{\nu}$
e: complete intersection case
f: see Proposition 6.8

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