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A BOCHNER TYPE THEOREM FOR INDUCTIVE LIMITS OF GELFAND PAIRS

by Marouane RABAOUI

Abstract. — In this article, we prove a generalisation of Bochner-Godement theorem. Our result deals with Olshanski spherical pairs (G,K) defined as inductive limits of increasing sequences of Gelfand pairs $(G(n),K(n))_{n\geqslant 1}$. By using the integral representation theory of G. Choquet on convex cones, we establish a Bochner type representation of any element φ of the set $\mathcal{P}^{\natural}(G)$ of K-biinvariant continuous functions of positive type on G.

Résumé. — Dans cet article, on démontre une généralisation du théorème de Bochner-Godement. Ce résultat concerne les paires sphériques d'Olshanski qui sont définies comme des limites inductives de suites croissantes de paires de Guelfand $\big(G(n),K(n)\big)_{n\geqslant 1}$. En utilisant la théorie de la représentation intégrale de G. Choquet sur les cônes convexes, on établit une représentation intégrale de type Bochner pour tout élément φ de l'ensemble $\mathcal{P}^{\natural}(G)$ des fonctions continues sur G, de type positif et biinvariantes par K.

1. Introduction

One of the main problems in harmonic analysis is to decompose a unitary representation by means of irreducible ones. The classical Bochner theorem provides an answer for this problem by giving a decomposition of a continuous function of positive type on $\mathbb{R}$ as an integral of indecomposable ones.

In harmonic analysis on groups of the type $G=\bigcup_{n=1}^{\infty} G(n)$, where $G(n)$ is a sequence of classical groups, with a subgroup K of the same type, i.e. $K=\bigcup_{n=1}^{\infty} K(n),\ K(n)\subset G(n)$, several extensions of the Bochner theorem had been proved. For example, E. Thoma in 1964 and S. Kerov, G.

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Olshanski and A. Vershik in 2004 studied the case of the infinite symmetric group $\mathfrak{S}_\infty = \bigcup_{n=1}^\infty \mathfrak{S}_n$, with $G = \mathfrak{S}_\infty \times \mathfrak{S}_\infty$ and $K = \mathrm{diag}(\mathfrak{S}_\infty \times \mathfrak{S}_\infty)$ (cf. [19], [13]). D. Voiculescu in 1976 and G. Olshanski in 2003 treated the pair $G = U(\infty) \times U(\infty)$, $K = \mathrm{diag}(U(\infty) \times U(\infty)) \simeq U(\infty)$, where $U(\infty) = \bigcup_{n=1}^\infty U(n)$ is the infinite dimensional unitary group (cf. [15], [21]).

G. Olshanski proved that the inductive limit of an increasing sequence of Gelfand pairs is a spherical pair. Hence, the cited examples and many others are part of G. Olshanski's theory for spherical pairs which was elaborated in 1990 (cf. [14]). However, a Bochner type decomposition in this setting has not been established yet. In this paper, by using Choquet's theorem, we prove such generalisation, answering a question asked by J. Faraut in*Infinite Dimensional Harmonic Analysis and Probability* (cf. [8]).

This paper consists of 4 sections devoted to the following topics : in section 2 we begin by recalling some definitions and results concerning continuous functions of positive type, then we prove that, for a classical Gelfand pair (H, M), the commutant $\pi^\varphi(H)^{'}$ is commutative and use this to give a direct proof of the fact that the set $\mathcal{P}^\natural(H)$ of M-biinvariant continuous functions of positive type on H is a lattice. In section 3, we move to the general setting of an increasing sequence of Gelfand pairs $(G(n), K(n))_{n\geqslant 1}$. Our main tool for establishing the generalised Bochner type decomposition is Choquet's theorem. In order to prove the existence of the decomposition, we embed $\mathcal{P}^\natural(G)$, for $G = \bigcup_{n=1}^\infty G(n)$, and $K = \bigcup_{n=1}^\infty K(n)$, into a bigger set $\mathcal{Q}$. For the uniqueness, we prove that the commutant $\pi^\varphi(G)^{'}$ remains commutative, and that $\mathcal{P}^\natural(G)$ is a lattice too. At the end of this paper, we present some remarks and open questions.

We have tried to keep notations and proofs to a minimum in order to make the presentation as clear as possible, we refer to [1], [9], [10] and [11] for more details on functions of positive type and Bochner theorem. The method we follow in our proof is a generalisation of E. Thoma's method in the case of a countable discrete group (cf. [20]), with some modifications inspired from Olshanski's work on the space of infinite dimensional hermitian matrices (cf. [16]).

2. Definitions and results for continuous functions of positive type

We first recall some definitions and results about functions of positive type. Let G be a Hausdorff topological group having e as unit, and K a closed subgroup of G.

Definition 2.1. — A function $\varphi : G \longrightarrow \mathbb{C}$ is said to be *of positive type* if the kernel defined on $G \times G$ by $(g_1, g_2) \longmapsto \varphi(g_2^{-1}g_1)$ is of positive type, i.e. for all $g_1, g_2, \ldots, g_n \in G$ and all $c_1, c_2, \ldots, c_n \in \mathbb{C}$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j}\varphi(g_j^{-1}g_i) \geqslant 0.$$

Proposition 2.2. — *Every function φ of positive type on G is hermitian, i.e. for all $g \in G$, $\overline{\varphi(g)} = \varphi(g^{-1})$. In addition, φ is bounded : $|\varphi(g)| \leqslant \varphi(e)$.*

A function φ defined on G is said to be *K-biinvariant* if it verifies $\varphi(k_1gk_2) = \varphi(g)$, for all k_1, $k_2 \in K$ and all $g \in G$. For a unitary representation $(\pi, \mathcal{H})$, we denote by $\mathcal{H}_K$ the subspace of K-invariant vectors in $\mathcal{H}$.

Proposition 2.3. — *Let $(\pi, \mathcal{H})$ be a unitary representation of G and ξ a vector in $\mathcal{H}_K$. Then, the function $\varphi : G \longrightarrow \mathbb{C}$, $g \longmapsto \langle\pi(g)\xi, \xi\rangle_{\mathcal{H}}$ is K-biinvariant of positive type.*

Using the G.N.S. (Gelfand-Naimark-Segal) construction, we can prove that every K-biinvariant function of positive type on G can be represented by a unitary representation on G.

Proposition 2.4 (**G.N.S. construction**). — *Let φ be a K-biinvariant continuous function of positive type on G. Then, there exists a triplet $(\pi^\varphi, \mathcal{H}^\varphi, \xi^\varphi)$ consisting of a unitary representation π^φ on a Hilbert space $(\mathcal{H}^\varphi, \langle ., . \rangle_\varphi)$, and a cyclic vector $\xi^\varphi \in \mathcal{H}_K^\varphi$ such that, for all $g \in G$,*

$$\varphi(g) = \langle\pi^\varphi(g)\xi^\varphi, \xi^\varphi\rangle_\varphi.$$

Moreover, this triplet is unique in the following sense : if $(\pi, \mathcal{H}, \xi)$ is another triplet, then there exists an interwining isomorphism $T : \mathcal{H}^\varphi \to \mathcal{H}$ between π^φ and π such that $T\xi^\varphi = \xi$.

Let $\mathcal{P}(G)$ be the set of continuous functions of positive type on G. $\mathcal{P}(G)$ is a convex cone which is invariant under product and complex conjugation.

For a convex set E, we denote by $\mathrm{Ext}(E)$ its subset of extremal points. We also denote by $\mathcal{P}_{\leqslant 1}(G)$ (respectively $\mathcal{P}_1(G)$) the set of elements φ of $\mathcal{P}(G)$ verifying $\varphi(e) \leqslant 1$ (respectively $\varphi(e) = 1$).

Lemma 2.5. — $\mathrm{Ext}\,(\mathcal{P}_{\leqslant 1}(G)) = \mathrm{Ext}\,(\mathcal{P}_1(G)) \cup \{0\}$.

Next, we will prove some algebraic characterizations which will be used to establish the uniqueness of the decomposition given by the generalized Bochner theorem.

Let Γ be a convex cone in a topological vector space E. This cone is equipped with its proper order : $\gamma_1 \ll \gamma_2$ if $\gamma_2 - \gamma_1 \in \Gamma$. The cone Γ is said to be a *lattice* if each couple of elements γ_1, γ_2 in Γ have (for the order defined by the cone) *a least upper bound* in Γ, denoted by $\gamma_1 \vee \gamma_2$, and *a greatest lower bound* in Γ, denoted by $\gamma_1 \wedge \gamma_2$.
For $\gamma_0 \in \Gamma$, we denote by Γ^{γ_0} *the face of* Γ defined as:

$$\Gamma^{\gamma_0} = \{\gamma \in \Gamma \mid \exists\ \lambda \geqslant 0\ ;\ \gamma \ll \lambda\gamma_0\}.$$

The order of Γ^{γ_0} coincides with the one induced by Γ. The cone Γ is a lattice if and only if, for every γ_0, the face Γ^{γ_0} is a lattice.

Let now $\Gamma = \mathcal{P}^{\natural}(G)$ be the subcone of $\mathcal{P}(G)$ which consists of K-biinvariant elements. On this convex cone, and similarly on $\mathcal{P}^{\natural}_{\leqslant 1}(G)$, the proper order $\ll$ is given by:

$$\varphi \ll \psi \quad \text{if and only if} \quad \psi - \varphi \in \mathcal{P}^{\natural}(G) \quad (\varphi, \psi \in \mathcal{P}^{\natural}(G)).$$

Recall that every function $\varphi \in \mathcal{P}^{\natural}(G)$ is associated to a triplet $(\pi^{\varphi}, \mathcal{H}^{\varphi}, \xi^{\varphi})$. Let $\mathcal{A} = \pi^{\varphi}(G)'$ be *the commutant* of $\pi^{\varphi}(G)$. It is a selfadjoint subalgebra of $\mathcal{L}(\mathcal{H}^{\varphi})$. We will prove that each face Γ^{φ} of $\mathcal{P}^{\natural}(G)$ is lineary isomorphic to the cone $\mathcal{A}^{+} = \{T \in \mathcal{A} \mid \forall\ v \in \mathcal{H}^{\varphi}, \langle Tv, v\rangle_{\varphi} \geqslant 0\}$ of positive operators of $\mathcal{A}$ on which we define an order, denoted $\prec$:

$$P \prec Q \quad \text{if and only if} \quad \langle Pv, v\rangle_{\varphi} \ \leqslant\ \langle Qv, v\rangle_{\varphi} \quad (v \in \mathcal{H}^{\varphi},\ P, Q \in \mathcal{A}^{+}).$$

Theorem 2.6. — *Let K be a closed subgroup of a Hausdorff topological group G. For all $\varphi \in \mathcal{P}^{\natural}(G)$ the face Γ^{φ} is lineary isomorphic to the cone $\mathcal{A}^{+}$ of positive operator of the algebra $\mathcal{A} = \pi^{\varphi}(G)'$. This bijective correspondence identifies an element $\psi \in \Gamma^{\varphi}$ with an element $T \in \mathcal{A}^{+}$ such that*

$$\psi(g) = \langle T\pi^{\varphi}(g)\xi^{\varphi}, \xi^{\varphi}\rangle_{\varphi},\ g \in G. \tag{2.1}$$

Proof. — Let $T \in \mathcal{A}^{+}$. The operator $T^{\frac{1}{2}}$ exists and belongs to $\mathcal{A}^{+}$ ([5], page 430, 11.17). So, for all $g \in G$,

$$\begin{aligned} \psi(g) = \langle T\pi^{\varphi}(g)\xi^{\varphi}, \xi^{\varphi}\rangle_{\varphi} &= \langle T^{\frac{1}{2}}\pi^{\varphi}(g)\xi^{\varphi}, (T^{\frac{1}{2}})^{*}\xi^{\varphi}\rangle_{\varphi} \\ &= \langle \pi^{\varphi}(g)T^{\frac{1}{2}}\xi^{\varphi}, T^{\frac{1}{2}}\xi^{\varphi}\rangle_{\varphi}. \end{aligned}$$

The function ψ is of positive type (Proposition 2). It is also continuous since the map $\xi \longmapsto \pi^{\varphi}(g)\xi$ is continuous for every $g \in G$. It is also K-biinvariant. Hence, $\psi \in \mathcal{P}^{\natural}(G)$.

If we put $\lambda_0 = ||T||$, where $||.||$ is the usual operator norm defined on $\mathcal{L}(\mathcal{H}^\varphi)$, then $\lambda_0\varphi - \psi \in \mathcal{P}^\natural(G)$. In fact

$$\begin{aligned}(\lambda_0\varphi - \psi)(g) &= ||T||\langle \pi^\varphi(g)\xi^\varphi, \xi^\varphi\rangle_\varphi - \langle \pi^\varphi(g)T\xi^\varphi, \xi^\varphi\rangle_\varphi \\ &= \langle \pi^\varphi(g)C\xi^\varphi, \xi^\varphi\rangle_\varphi,\end{aligned}$$

where $C = ||T||I - T$. As, for all $v \in \mathcal{H}^\varphi$, $0 \leqslant \langle Tv, v\rangle_\varphi \leqslant ||T||\langle v, v\rangle_\varphi$, the operator $C \in \mathcal{A}^+$. Hence $C = D^2$ with $D \in \mathcal{A}^+$, and so

$$(\lambda_0\varphi - \psi)(g) = \langle \pi^\varphi(g)D^2\xi^\varphi, \xi^\varphi\rangle_\varphi = \langle \pi^\varphi(g)D\xi^\varphi, D\xi^\varphi\rangle_\varphi.$$

This proves, by Proposition 2, that $\lambda_0\varphi - \psi$ is of positive type. It is also continuous and K-biinvariant. Hence, $\lambda_0\varphi - \psi \in \mathcal{P}^\natural(G)$.

One can also remark that ψ uniquely determine T. In fact, for every $g, h \in G$,

$$\psi(h^{-1}g) = \langle \pi^\varphi(h^{-1}g)T\xi^\varphi, \xi^\varphi\rangle_\varphi = \langle T\pi^\varphi(g)\xi^\varphi, \pi^\varphi(h)\xi^\varphi\rangle_\varphi.$$

If $\widetilde{T}$ is another operator in $\mathcal{A}^+$ verifying (2.1), then for every $g, h \in G$,

$$\langle \pi^\varphi(g)(T - \widetilde{T})\xi^\varphi, \pi^\varphi(h)\xi^\varphi\rangle_\varphi = 0.$$

Since $V_\varphi = Vect\{\pi^\varphi(g)\xi^\varphi\ ,\ g \in G\}$ is dense in $\mathcal{H}^\varphi$,

$$T = \widetilde{T}.$$

It remains to prove that, for every $\psi \in \Gamma^\varphi$, there exists $T \in \mathcal{A}^+$ verifying (2.1). Let us denote by

$$\mathfrak{M}^\circ(G) := \{\mu = \sum_{i=1}^m a_i\delta_{x_i} \mid (a_i)_i \subset \mathbb{C}\ ,\ (x_i)_i \subset G\},$$

the space of measures with finite support. For a function of positive type φ and $\mu, \nu \in \mathfrak{M}^\circ(G)$, put

$$(\varphi, \nu^* * \mu) = \sum_{i=1}^m \sum_{j=1}^n \overline{b_j} a_i \varphi(x_j^{-1}x_i) \geqslant\ 0.$$

We can also define the function

$$\mu * \varphi(x) = \int_G \varphi(y^{-1}x)d\mu(y) = \sum_{i=1}^m a_i\varphi(x_i^{-1}x),$$

it is continuous and right K-invariant. With the previous notation and definitions, the vector space V_φ can also be given by :

$$V_\varphi := \{\varphi^\mu = \mu * \check{\varphi} = \sum_{i=1}^m a_i\pi^\varphi(g_i)\xi^\varphi,\ \mu \in \mathfrak{M}^\circ(G)\},$$

where $\check{\varphi}(g) = \varphi(g^{-1})$, for all $g \in G$. For $\varphi^\mu, \varphi^\nu \in V_\varphi$, put

$$\langle \varphi^\mu, \varphi^\nu \rangle_\varphi = (\varphi, \nu^* * \mu).$$

The map $(\varphi^\mu, \varphi^\nu) \longmapsto \langle \varphi^\mu, \varphi^\nu \rangle_\varphi$ is a hermitian positive form on V_φ, which is in addition definite as it verifies, for all $g \in G$,

$$|\varphi^\mu(g)|^2 = |\mu * \varphi(g)|^2 \leqslant \varphi(e) \langle \varphi^\mu, \varphi^\mu \rangle_\varphi.$$

Now, let $\psi \in \Gamma^\varphi$, there exists $\lambda_0 \geqslant 0$ such that

$$\lambda_0 \varphi - \psi \in \mathcal{P}^\natural(G).$$

So, for all $\mu \in \mathfrak{M}^\mathrm{o}(G)$,

$$(\lambda_0 \varphi - \psi, \mu^* * \mu) \geqslant 0 \quad \text{or equivalently} \quad (\psi, \mu^* * \mu) \leqslant (\varphi, \mu^* * \mu).$$

Hence

$$\langle \psi^\mu, \psi^\mu \rangle_\psi \ \leqslant \lambda_0 \langle \varphi^\mu, \varphi^\mu \rangle_\varphi.$$

Consequently, we can define on $V_\varphi \times V_\varphi$ a hermitian form ω given, for every $\mu, \nu \in \mathfrak{M}^\mathrm{o}(G)$, by

$$\omega(\varphi^\mu, \varphi^\nu) = (\psi, \nu^* * \mu) = \langle \psi^\mu, \psi^\nu \rangle_\psi.$$

In fact

$$|\omega(\varphi^\mu, \varphi^\nu)|^2 = |\langle \psi^\mu, \psi^\nu \rangle_\psi|^2 \leqslant \lambda_0^2 \langle \varphi^\mu, \varphi^\mu \rangle_\varphi \langle \varphi^\nu, \varphi^\nu \rangle_\varphi.$$

In addition

$$\omega(\varphi^\mu, \varphi^\nu) = (\psi, \nu^* * \mu) = \overline{(\psi, \mu^* * \nu)} = \overline{\omega(\varphi^\nu, \varphi^\mu)}.$$

So, ω is a well-defined hermitian form which is continuous on $V_\varphi \times V_\varphi$. It is also positive as, for all $\mu \in \mathfrak{M}^\mathrm{o}(G)$,

$$\omega(\varphi^\mu, \varphi^\mu) = (\psi, \mu^* * \mu) \geqslant 0.$$

As V_φ is dense in $\mathcal{H}^\varphi$, ω may be extended to a positive hermitian continuous form on $\mathcal{H}^\varphi \times \mathcal{H}^\varphi$. So, by Riesz's theorem, there exists an unique positive hermitian operator T in $\mathcal{L}(\mathcal{H}^\varphi)$ such that, for every $v_1, v_2 \in \mathcal{H}^\varphi$,

$$\langle T v_1, v_2 \rangle_\varphi = \omega(v_1, v_2).$$

In particular, for $\varphi^\mu, \varphi^\nu \in V_\varphi$,

$$\langle T \varphi^\mu, \varphi^\nu \rangle_\varphi = \omega(\varphi^\mu, \varphi^\nu) = (\psi, \nu^* * \mu).$$

Consequently, for $\mu_0 = \delta_g$, $g \in G$ and $\nu_0 = \delta_e$,

$$\langle T \varphi^{\mu_0}, \varphi^{\nu_0} \rangle_\varphi = \langle T \varphi^{\delta_g}, \varphi^{\delta_e} \rangle_\varphi = (\psi, \delta_e^* * \delta_g) = \psi(g).$$

But, $\varphi^{\delta_g} = \pi^\varphi(g)\xi^\varphi$ and $\varphi^{\delta_e} = \xi^\varphi$. Hence $\psi(g) = \langle T\pi^\varphi(g)\xi^\varphi, \xi^\varphi\rangle_\varphi$. The operator T is also selfadjoint and positive. In fact, as ψ is of positive type, for every $g, h \in G$, $\psi(g^{-1}h) = \overline{\psi(h^{-1}g)}$. Hence

$$\langle T\pi^\varphi(h)\xi^\varphi, \pi^\varphi(g)\xi^\varphi\rangle_\varphi = \overline{\langle T\pi^\varphi(g), \pi^\varphi(h)\xi^\varphi\rangle_\varphi},$$

and so

$$\langle \pi^\varphi(h)\xi^\varphi, T^*\pi^\varphi(g)\xi^\varphi\rangle_\varphi = \langle \pi^\varphi(h)\xi^\varphi, T\pi^\varphi(g)\rangle_\varphi.$$

Since V_φ is dense in $\mathcal{H}^\varphi$,

$$T = T^*.$$

The positivity of T follows from ω's one. The operator T also commutes with $\pi^\varphi(g)$, for all $g \in G$. □

Next, we give a necessary and sufficient condition for the cone $\mathcal{P}^\natural(G)$ to be a lattice.

Lemma 2.7. — *The cone $\mathcal{A}^+$ is a lattice if and only if the algebra $\mathcal{A}$ is commutative.*

Proof. — The proof is similar to the one given in ([7], Theorem III.2.4, page 129). □

By Theorem 2.6 and this last lemma, we prove the following theorem,

Theorem 2.8. — *Let K be a closed subgroup of a Hausdorff topological group G. The cone $\mathcal{P}^\natural(G)$ is a lattice if and only if, for every function φ of this cone, the algebra $\mathcal{A} = \pi^\varphi(G)'$ is commutative.*

Proof. — From Theorem 2.6, we deduce that, for every function $\varphi \in \mathcal{P}^\natural(G)$, the face Γ^φ is lineary isomorphic to the cone $\mathcal{A}^+$, which is a lattice if and only if $\mathcal{A}$ is commutative. So, for every function $\varphi \in \mathcal{P}^\natural(G)$, Γ^φ is a lattice if and only if $\mathcal{A}$ is commutative. □

Definition 2.9. — A pair (G, K), where G is a locally compact group and K a compact subgroup of G, is said to be a *Gelfand pair* if the convolution algebra of K-biinvariant integrable functions is commutative.

We will prove by using some elements of von Neumann algebra theory that, in the case of a Gelfand pair (G, K), the algebra $\pi^\varphi(G)'$ is commutative, for all $\varphi \in \mathcal{P}^\natural(G)$.

Proposition 2.10. — *Let (G, K) be a Gelfand pair and P the orthogonal projection onto $\mathcal{H}_K^\varphi$ defined by*

$$P = \int_K \pi^\varphi(k)\ \alpha(dk),$$

where α is the normalized Haar measure of the subgroup K. Then P is an element of $\pi^{\varphi}(G)''$, and the algebra $P\pi^{\varphi}(G)''P$ is commutative.

Proof. — Let us prove that $P \in \pi^{\varphi}(G)''$. In fact, for every $T \in \pi^{\varphi}(G)'$ and every $v, w \in \mathcal{H}^{\varphi}$,

$$\langle PTv, w\rangle = \langle \pi^{\varphi}(\alpha)Tv, w\rangle = \langle \pi^{\varphi}(\alpha)v, T^* w\rangle = \langle TPv, w\rangle.$$

So, for every v in $\mathcal{H}^{\varphi}$, $PTv = TPv$. Hence $P \in \pi^{\varphi}(G)''$. As (G, K) is a Gelfand pair, for every $\mu,\ \nu \in \mathfrak{M}^{\circ}(G)$, the K-biinvariant measures $\alpha * \mu * \alpha$ and $\alpha * \nu * \alpha$ commute. Thus, for every $\mu,\ \nu \in \mathfrak{M}^{\circ}(G)$,

$$P\pi^{\varphi}(\mu)P\pi^{\varphi}(\nu)P = P\pi^{\varphi}(\nu)P\pi^{\varphi}(\mu)P.$$

As $\pi^{\varphi}(\mathfrak{M}^{\circ}(G))$ is a selfadjoint subalgebra containing the identity of $\mathcal{L}(\mathcal{H}^{\varphi})$, it is dense in $\pi^{\varphi}(G)''$ in the strong topology of operators ([3], Theorem 2 and Corollary 1, page 45). Hence, for every $A,\ B \in \pi^{\varphi}(G)''$,

$$PAPBP = PBPAP.$$

Put $S = PAP$ and $T = PBP$. The operators S and T are two arbitrary elements of the algebra $P\pi^{\varphi}(G)''P$ and they verify

$$ST = PAPPBP = PAPBP = PBPAP = TS.$$

It follows that the algebra $P\pi^{\varphi}(G)''P$ is commutative. $\square$

For an operator A of the von Neumann algebra $\pi^{\varphi}(G)'$, let us denote by A_P the restriction of the operator PA to $\mathcal{H}^{\varphi}_K$. Put

$$[\pi^{\varphi}(G)']_P = \{A_P,\ A \in \pi^{\varphi}(G)'\}.$$

By ([3], Proposition 1, page 18), the algebras $[\pi^{\varphi}(G)']_P$ and $[\pi^{\varphi}(G)'']_P$ are von Neumann algebras and they verify

$$([\pi^{\varphi}(G)'']_P)' = [\pi^{\varphi}(G)']_P.$$

Since ξ^{φ} is a cyclic vector for the algebra $\pi^{\varphi}(\mathfrak{M}^{\circ}(G))$, by ([4], Appendice A, A14), it is a separating vector for the von Neumann algebra $\pi^{\varphi}(\mathfrak{M}^{\circ}(G))' = \pi^{\varphi}(G)'$. Thus it is also separating for the von Neumann algebra $[\pi^{\varphi}(G)']_P$. Hence it is cyclic for the von Neumann algebra $[\pi^{\varphi}(G)'']_P$.

By using the fact that every von Neumann algebra $\mathcal{M}$ which is commutative and possesses a cyclic vector verifies $\mathcal{M}' = \mathcal{M}$ ([3], Corollaire 2, page 89), and by noticing that the algebra $[\pi^{\varphi}(G)'']_P$ is nothing but $P\pi^{\varphi}(G)''P$, we obtain $([\pi^{\varphi}(G)'']_P)' = [\pi^{\varphi}(G)'']_P$. Hence

$$[\pi^{\varphi}(G)']_P = [\pi^{\varphi}(G)'']_P.$$

Now, to get the commutativity of $\pi^{\varphi}(G)'$, it is sufficient to prove the following proposition,

Proposition 2.11. — *Let (G,K) be a Gelfand pair. The commutant $\pi^\varphi(G)^{'}$, seen as a von Neumann algebra, is isomorphic to the algebra $[\pi^\varphi(G)^{'}]_P$.*

Proof. — Let $\Psi : \pi^\varphi(G)^{'} \to [\pi^\varphi(G)^{'}]_P$, $A \longmapsto A_P$. Ψ is well-defined, it is also a homomorphism of algebras, since for every $S,\ T \in \pi^\varphi(G)^{'}$,

$$\Psi(ST) = [ST]_P = PSTP = PSPPTP = S_P T_P = \Psi(S)\Psi(T),$$

$$\Psi(T^*) = PT^*P = P^*T^*P^* = (PTP)^* = (T_P)^* = \Psi(T)^*.$$

It is evident that Ψ is onto by construction. Let us prove that it is one to one. Let $S \in \pi^\varphi(G)^{'}$ such that $\Psi(S) = 0$. Then,

$$\Psi(S) = 0 \Rightarrow PS\xi^\varphi = 0 \Rightarrow SP\xi^\varphi = 0 \Rightarrow S\xi^\varphi = 0.$$

Hence, for every $g \in G$, $S\pi^\varphi(g)\xi^\varphi = \pi^\varphi(g)S\xi^\varphi = 0$. And since ξ^φ is cyclic, we get immediately $S = 0$. Therefore, Ψ is one to one. □

Theorem 2.12. — *Let (G,K) be a Gelfand pair and φ a K-biinvariant continuous function of positive type on G. Then, the algebra $\pi^\varphi(G)^{'}$ is commutative.*

Proof. — By the previous proposition, $\pi^\varphi(G)^{'}$ is isomporphic to $[\pi^\varphi(G)^{'}]_P$. Also we know that $[\pi^\varphi(G)^{'}]_P = [\pi^\varphi(G)^{''}]_P = P\pi^\varphi(G)^{''}P$. The result follows since the algebra $P\pi^\varphi(G)^{''}P$ is commutative. □

Corollary 2.13. — *Let (G,K) be a Gelfand pair. Then, the cone $\mathcal{P}^\natural(G)$ is a lattice.*

Proof. — By Theorem 2.8, $\mathcal{P}^\natural(G)$ is a lattice if and only if, for every element φ in this cone, the algebra $\pi^\varphi(G)^{'}$ is commutative, which is satisfied in this case as shown by the previous theorem. Hence $\mathcal{P}^\natural(G)$ is a lattice. □

We know that every function of positive type is bounded. Since G is a locally compact topological group, $\mathcal{P}(G)$ can be seen as a subset of $L^\infty(G)$ for a left invariant Haar measure on G. We add, from now on, the condition that G is separable and we consider on $\mathcal{P}(G)$ the topology induced by the weak-$*$ topology $\sigma(L^\infty(G), L^1(G))$, denoted by $\tau^*(L^\infty(G))$. By the Banach-Alaoglu theorem (cf. [18]), the unit ball of $L^\infty(G)$ is compact in this topology. In addition, $\mathcal{P}^\natural_{\leqslant 1}(G)$ considered as a subset of $L^\infty(G)$, is closed in this topology(cf. [18], [6]). Therefore, $\mathcal{P}^\natural_{\leqslant 1}(G)$ is compact. Furthermore, the unit ball of $L^\infty(G)$, for G separable, is metrisable in the weak-$*$ topology $\tau^*(L^\infty(G))$ (cf. [4], [18]). Hence $\mathcal{P}^\natural_{\leqslant 1}(G)$ is metrisable. Thus $\mathcal{P}^\natural_{\leqslant 1}(G)$ is convex, compact and metrisable in the topological space $L^\infty(G)$ which is locally convex in the weak-$*$ topology $\tau^*(L^\infty(G))$. Furthermore, by Corollary 1, the cone generated by $\mathcal{P}^\natural_{\leqslant 1}(G)$, namely $\mathcal{P}^\natural(G)$, is a lattice. Therefore, we get by applying Choquet's theorem that every element $\varphi \in \mathcal{P}^\natural(G)$ has an integral representation :

$$\varphi(g) = \int_{\mathrm{Ext}(\mathcal{P}^\natural_1(\mathrm{G}))} \omega(g)\mu(d\omega).$$

This last formula constitutes Bochner-Godement's theorem. It is evident now that Choquet's theorem is fundamental for the proof. Because of its importance, we finish this section by giving its statement.

Theorem 2.14 (**Choquet's theorem**, see [17] sections 3 and 10). — *Let $\mathcal{U}$ be a convex subset of a locally convex topological vector space E. If $\mathcal{U}$ is compact and metrisable, then*

(i) $\mathrm{Ext}(\mathcal{U})$ *is a Borel subset of $\mathcal{U}$.*

(ii) *For every $a \in \mathcal{U}$, there exists a probability measure μ on* $\mathrm{Ext}(\mathcal{U})$, *such that for all continuous linear form L on E,*

$$L(a) = \int_{b \in \mathrm{Ext}(\mathcal{U})} L(b)\mu(db).$$

(iii) *μ is unique if and only if the cone generated by $\mathcal{U}$ is a lattice.*

3. A Bochner type theorem for Olshanski spherical pairs

Definition 3.1. — *Let H be a Hausdorff topological group and M a closed subgroup of H. The pair (H, M) is said to be* spherical *if, for every irreducible unitary representation π of H on a Hilbert space $\mathcal{H}$,*

$$\dim \mathcal{H}_M \leqslant 1.$$

If H is locally compact, and M compact, then the pair (H, M) is spherical if and only if it is a Gelfand pair.

Let $\big(G(n), K(n)\big)_{n\geqslant 1}$ be a sequence of Gelfand pairs such that $G(n)$ is a locally compact topological group which is in addition a closed subgroup of $G(n+1)$. Also $K(n)$ is a closed subgroup of $K(n+1)$ and $K(n) = K(n+1) \cap G(n)$. The family of Gelfand pairs $\big(G(n), K(n)\big)_{n\geqslant 1}$, equiped with the system of canonical continuous embeddings from $G(i)$ to $G(j)$ with $i \leqslant j$, constitute an inductive countable system of topological groups (cf. [2]). Hence we may define the following inductive limit groups : $G = \bigcup_{n=1}^\infty G(n)$ and $K = \bigcup_{n=1}^\infty K(n)$. The topology defined on G is the inductive limit topology. It is the finest topology such that all the canonical embeddings from $G(n)$ into G are continuous. Olshanski proved that (G,K) is a spherical pair (cf. [8], [14]). Hence we can introduce the following definition:

Definition 3.2. — *Let $\big(G(n),K(n)\big)_{n\geqslant 1}$ be an increasing sequence of Gelfand pairs as above. The inductive limit pair (G,K) is called an Olshanski spherical pair.*

The group G equipped with the inductive limit topology is Hausdorff. But, such topology does not make G locally compact. Therefore we can not directly apply Choquet's theorem to $\mathcal{P}^\natural(G)$ as in the classical case. In order to solve this problem, we embed $\mathcal{P}^\natural(G)$ in the cone of subprojective systems :

$$\mathcal{Q} := \left\{ \varphi = \{\varphi^{(i)}\}_i \in \prod_{i=1}^{\infty} \mathcal{P}^\natural(G(i)) \mid \boldsymbol{Res}_i^{i+1}\big(\varphi^{(i+1)}\big) \ll \varphi^{(i)} \ i=1,2,... \right\}.$$

$\boldsymbol{Res}_n^{n+1}$ is the restriction to $G(n)$ of a function defined on $G(n+1)$. Choquet's theory of integral representation applied to $\mathcal{Q}$ will give us a Bochner type theorem for the spherical pairs of Olshanski. Let $\boldsymbol{Res}_n$ be the restriction to $G(n)$ of a function defined on G, and put $\mathcal{P}_m^n = \prod_{k=m}^n \mathcal{P}^\natural(G(k))$, where $1 \leqslant m \leqslant n \leqslant \infty$.

Remark 3.3. — If $G_1 \subset G_2$ are two locally compact groups the set of pairs $\{(\varphi,\psi) \in \mathcal{P}(G_1)\times\mathcal{P}(G_2) \mid \boldsymbol{Res}(\psi)=\varphi\}$, where $\boldsymbol{Res}$ is the restriction to G_1 of a function on G_2, is not closed in general, and in some cases it can be shown that it is dense in $\{(\varphi,\psi) \in \mathcal{P}(G_1)\times\mathcal{P}(G_2) \mid \boldsymbol{Res}(\psi)\ll\varphi\}$.

Next we will prove that $\mathcal{Q}$ is closed in $\mathcal{P}_1^\infty$ in the product topology $\tau^* = \prod_{n=1}^\infty \tau^*(L^\infty(G(n)))$. To establish this, it is sufficient to prove that the set

$$\mathcal{R}_k = \left\{ (\varphi^{(k)},\varphi^{(k+1)}) \in \mathcal{P}_k^{k+1} \mid \boldsymbol{Res}_k^{k+1}(\varphi^{(k+1)}) \ll \varphi^{(k)} \right\}$$

is closed in the topology $\tau^*(L^\infty(G(k)))\times\tau^*(L^\infty(G(k+1)))$.

Let H be a locally compact group, α its left invariant Haar measure, and M a compact subgroup of H such that (H,M) is a Gelfand pair.

Lemma 3.4. — *For every function $\varphi \in \mathcal{P}^\natural(H)$ and $f \in L^1(H)^\natural$ such that $||f||_1 \leqslant 1$, one has*

$$f^* * \varphi * f \ll \varphi.$$

Proof. — Let $(\pi^\varphi,\mathcal{H}^\varphi)$ be the unitary representation associated to φ :

$$\varphi(h) = \langle \pi^\varphi(h)\xi^\varphi,\xi^\varphi\rangle_\varphi \quad (h\in H).$$

Since (H,M) is a Gelfand pair, the operator $\pi^\varphi(f)$ commutes, for every $h\in H$, with $\pi^\varphi(h)$, and

$$f^* * \varphi * f(h) = \langle \pi^\varphi(h)\pi^\varphi(f)\xi^\varphi, \pi^\varphi(f)\xi^\varphi\rangle_\varphi.$$

Therefore

$$\begin{aligned}\sum_{i,j=1}^N f^* * \varphi * f(h_j^{-1}h_i)c_i\overline{c_j} &= \Big|\Big|\sum_{i=1}^N c_i\pi^\varphi(h_i)\pi^\varphi(f)\xi^\varphi\Big|\Big|_\varphi^2 \\ &= \Big|\Big|\pi^\varphi(f)\sum_{i=1}^N c_i\pi^\varphi(h_i)\xi^\varphi\Big|\Big|_\varphi^2 \\ &\leqslant ||\pi^\varphi(f)||^2\Big|\Big|\sum_{i=1}^N c_i\pi^\varphi(h_i)\xi^\varphi\Big|\Big|_\varphi^2 \\ &\leqslant \Big|\Big|\sum_{i=1}^N c_i\varphi(h_i)\xi^\varphi\Big|\Big|_\varphi^2 \\ &= \sum_{i,j=1}^N \varphi(h_j^{-1}h_i)c_i\overline{c_j}.\end{aligned}$$

□

Under the same assumptions as Lemma 3.4, we prove the following lemma,

Lemma 3.5. — *The linear form L defined, for every bounded measure μ on H, by*

$$L(\varphi) = \int_{H\times H}\varphi(y^{-1}x)\mu(dx)\overline{\mu(dy)}$$

is lower-semicontinuous on $\mathcal{P}^\natural(H)$ in the weak-$*$ topology $\tau^*(L^\infty(H))$.

Proof. — Firstly, let us remark that L is positive on $\mathcal{P}^\natural(H)$ and that if $\mu=\delta$, then $L(\varphi)=\varphi(e)$. We will prove that, for every constant $C\geqslant 0$, the set

$$\{\varphi\in\mathcal{P}^\natural(H)\mid L(\varphi)\leqslant C\}$$

is closed. Let (φ_n) be a sequence of $\mathcal{P}^\natural(H)$ that converges to φ, i.e. for every $f\in L^1(H)$,

$$\lim_{n\to\infty}\int_H \varphi_n(h)f(h)\alpha(dh) = \int_H\varphi(h)f(h)\alpha(dh).$$

Suppose that, for every n, $L(\varphi_n)\leqslant C$. We know that, for every bounded measure μ and $f\in L^1(H)^\natural$, $f*\mu\in L^1(H)$. Suppose $||f||_1\leqslant 1$. By hypothesis, for every n,

$$\mu^* * \varphi_n * \mu(e)\leqslant C.$$

Therefore, by Lemma 3.4,

$$\mu^* * f^* * \varphi_n * f * \mu(e) \leqslant C,$$

and since

$$\lim_{n\to\infty} \mu^* * f^* * \varphi_n * f * \mu(e) = \mu^* * f^* * \varphi * f * \mu(e),$$

it follows that

$$\mu^* * f^* * \varphi * f * \mu(e) \leqslant C.$$

By considering an approximation of the identity $(f_k) : f_k \in L^1(H)^\natural$, $f_k \geqslant 0$,

$$\int_H f_k(h)\alpha(dh) = 1,$$

and observing that for every continuous bounded function ψ :

$$\lim_{k\to\infty} \int_H \psi(h) f_k(h)\alpha(dh) = \psi(e),$$

we deduce that

$$\mu^* * \varphi * \mu(e) \leqslant C.$$

□

Proposition 3.6. — *Let U be a closed unimodular subgroup of H, α_U its left invariant Haar measure and* ***Res*** *the application that for a function on H associates its restriction to U. The set*

$$\{(\phi,\psi) \in \mathcal{P}^\natural(H) \times \mathcal{P}^\natural(U) \mid \boldsymbol{Res}(\phi) \ll \psi\}$$

is closed.

Proof. — Let (ϕ_n, ψ_n) be a sequence in $\mathcal{P}^\natural(H) \times \mathcal{P}^\natural(U)$ that converges to (ϕ,ψ), and suppose that, for every n and every function $f \in L^1(U)$,

$$\int_{U\times U} \phi_n(y^{-1}x) f(x)\overline{f(y)}\alpha_U(dx)\alpha_U(dy) \leqslant \int_{U\times U} \psi_n(y^{-1}x) f(x)\overline{f(y)}\alpha_U(dx)\alpha_U(dy).$$

Let

$$C > \int_{U\times U} \psi(y^{-1}x) f(x)\overline{f(y)}\alpha_U(dx)\alpha_U(dy).$$

There exists n_0 such that, if $n \geqslant n_0$

$$\int_{U\times U} \psi_n(y^{-1}x) f(x)\overline{f(y)}\alpha_U(dx)\alpha_U(dy) \leqslant C,$$

and thus

$$\int_{U\times U} \phi_n(y^{-1}x) f(x)\overline{f(y)}\alpha_U(dx)\alpha_U(dy) \leqslant C.$$

Lemma 3.5 applied to the measure $\mu(dx) = f(x)\alpha_U(dx)$ gives

$$\int_{U\times U} \phi(y^{-1}x)f(x)\overline{f(y)}\alpha_U(dx)\alpha_U(dy) \leqslant C.$$

This being true for every constant C verifying

$$C > \int_{U\times U} \psi(y^{-1}x)f(x)\overline{f(y)}\alpha_U(dx)\alpha_U(dy),$$

we can deduce that

$$\int_{U\times U} \phi(y^{-1}x)f(x)\overline{f(y)}\alpha_U(dx)\alpha_U(dy) \leqslant \int_{U\times U} \psi(y^{-1}x)f(x)\overline{f(y)}\alpha_U(dx)\alpha_U(dy).$$

Therefore $\boldsymbol{Res}(\phi) \ll \psi$. It follows that the set

$$\{(\phi,\psi) \in \mathcal{P}^\natural(H) \times \mathcal{P}^\natural(U) \mid \boldsymbol{Res}(\phi) \ll \psi\}$$

is closed. □

Since, for all n, the pair $(G(n), K(n))$ is supposed to be a Gelfand pair, the groups $G(n)$ are all unimodular (see [6], Proposition I.1). Hence we can apply the previous proposition in the case where $H = G(k+1)$ and $U = G(k)$. Then, one gets that $\mathcal{R}_k$ is closed, for every k, and hence $\mathcal{Q}$ is closed in $\mathcal{P}_1^\infty$. As a consequence, the set

$$\mathcal{Q}_{\leqslant 1} := \left\{ \varphi = \{\varphi^{(i)}\}_i \in \prod_{i=1}^{\infty} \mathcal{P}^\natural_{\leqslant 1}(G(i)) | \boldsymbol{Res}_i^{i+1}\left(\varphi^{(i+1)}\right) \ll \varphi^{(i)} i = 1, 2, \ldots \right\},$$

is compact. In order to get the metrisability of $\mathcal{Q}_{\leqslant 1}$, it is sufficient to suppose that all the $G(n)$ are separable.

It remains to prove that the cone $\mathcal{Q}$ is a lattice in order to apply Choquet's theorem.

Let $(\pi^\varphi, \mathcal{H}^\varphi, \xi^\varphi)$ be the triplet associated to a function $\varphi \in \mathcal{P}^\natural(G)$. We are going to prove that the algebra $\pi^\varphi(G)'$ is commutative. Since $G(n)$ is a subgroup of G, the representation π^φ of G remains a continuous unitary representation of $G(n)$ on $\mathcal{H}^\varphi$. Put $\mathcal{H}_n^\varphi = \overline{Vect\{\pi^\varphi(g)\xi^\varphi\ ,\ g \in G(n)\}}$. It is a $G(n)$-invariant closed subspace of $\mathcal{H}^\varphi$. Hence we may restrict, for every $g \in G(n)$, the operator $\pi^\varphi(g)$ to $\mathcal{H}_n^\varphi$. We obtain a continuous unitary representation of $G(n)$ on $\mathcal{H}_n^\varphi$ that will be denoted by π_n^φ.

Let P_n be the orthogonal projection onto $\mathcal{H}_n^\varphi$, LEMMA 3.7. —

(i) $\bigcup_{n=1}^{\infty} \mathcal{H}_n^\varphi$ *is dense in* $\mathcal{H}^\varphi$.

(ii) P_n *converges strongly to the identity* I *of* $\mathcal{H}^\varphi$.

PROPOSITION 3.8. — *Let* (G, K) *be an Olshanski spherical pair. For every* $\varphi \in \mathcal{P}^\natural(G)$, *the commutant* $\mathcal{A} = \pi^\varphi(G)'$ *of the representation* π^φ *which is associated to* φ *by the G.N.S. construction, is a commutative algebra.*

Proof. — Let B be an arbitrary operator of $\mathcal{A}$. Then, for every g in G, B commutes with $\pi^\varphi(g)$. This is also true on $G(n)$, for every $n \in \mathbb{N}^*$. On the other hand, for every $n \in \mathbb{N}^*$, $P_n B P_n$ which is an operator of $\mathcal{L}(\mathcal{H}_n^\varphi)$ commutes with the representation π_n^φ of $G(n)$ on $\mathcal{H}_n^\varphi$.

Since $\mathcal{H}_n^\varphi$ is $G(n)$-invariant, for every $g \in G(n)$, P_n commutes with $\pi^\varphi(g)$. Therefore, for every $g \in G(n)$,

$$P_n B P_n \pi_n^\varphi(g) = P_n B \pi_n^\varphi(g) P_n = P_n \pi_n^\varphi(g) B P_n = \pi_n^\varphi(g) P_n B P_n.$$

By Theorem 2.12, the algebra $\pi_n^\varphi(G(n))'$ is commutative. So, for two operators B_1 and B_2 of $\pi^\varphi(G)'$, and for every $n \in \mathbb{N}^*$,

$$P_n B_1 P_n P_n B_2 P_n = P_n B_2 P_n P_n B_1 P_n,$$

$$P_n B_1 P_n B_2 P_n = P_n B_2 P_n B_1 P_n.$$

Since $K_n \subset K_{n+1}$, then $\mathcal{H}_{K_{n+1}} \subset \mathcal{H}_{K_n}$, and therefore

$$P_{n+1} = P_n P_{n+1} = P_{n+1} P_n.$$

Also, for every $n, m \geqslant 1$,

$$P_{n+m} = P_{n+m} P_n = P_n P_{n+m}.$$

Hence, for every $m, m', n \geqslant 1$,

$$P_{n+m} B_1 P_n B_2 P_{n+m'} = P_{n+m} B_2 P_n B_1 P_{n+m'}.$$

By using the fact that P_n converges strongly to I and by pushing m, m' to ∞, one obtains

$$B_1 P_n B_2 = B_2 P_n B_1.$$

Finally, by pushing n to ∞, one gets

$$B_1 B_2 = B_2 B_1.$$

□

THEOREM 3.9. — *For an Olshanski spherical pair* (G, K), *the cone* $\mathcal{P}^\natural(G)$ *is a lattice.*

Proof. — By the previous proposition, the algebra $\mathcal{A} = \pi^\varphi(G)'$ is commutative. Hence, by Theorem 2.8, the cone $\mathcal{P}^\natural(G)$ is a lattice. $\square$

Let us prove that $\mathcal{Q}$ is a lattice. We start by giving a decomposition of the elements of $\mathcal{Q}$.

LEMMA 3.10. — *Let H be a locally compact topological group having e as unit, L a closed subgroup of H and $(u_n)_n$ a sequence of L-biinvariant continuous functions of positive type on H.*

(a) *If*

$$\sum_{n=1}^{\infty} u_n(e) < \infty,$$

then the series $\sum_{n=1}^{\infty} u_n$ converges uniformly on H and its sum is a L-biinvariant continuous function of positive type.

(b) *Furthermore if, for $n \geqslant 1$,*

$$\sum_{k=1}^{n} u_k \ll \varphi,$$

where φ is a L-biinvariant continuous function of positive type, then

$$\sum_{n=1}^{\infty} u_n \ll \varphi.$$

(c) *If v_n is another sequence such that $v_n \ll u_n$, then*

$$\sum_{n=1}^{\infty} v_n \ll \sum_{n=1}^{\infty} u_n.$$

PROPOSITION 3.11. — *For every subprojective system $\varphi = \{\varphi^{(k)}\}_k$ in $\mathcal{Q}$, there exists a projective system $\Phi = \{\Phi^{(k)}\}_k$ and functions $\psi^{(k)}$ in $\mathcal{P}^\natural(G(k))$ such that, for every k,*

$$\varphi^{(k)} = \Phi^{(k)} + \sum_{j=0}^{\infty} \boldsymbol{Res}_k^{k+j}(\psi^{(k+j)}). \tag{3.1}$$

The functions $\Phi^{(k)}$ and $\psi^{(k)}$ are unique.

Proof. — Let $\varphi \in \mathcal{Q}$. Put, for every $k \geqslant 1$,

$$\psi^{(k)} = \varphi^{(k)} - \boldsymbol{Res}_k^{k+1}(\varphi^{(k+1)}). \tag{3.2}$$

By the definition of $\mathcal{Q}$, for every $k \geqslant 1$, $\psi^{(k)}$ is a function of positive type on $G(k)$. By iteration, equality (3.2) gives, for every $k \geqslant 1$,

$$\varphi^{(k)} = \psi^{(k)} + \boldsymbol{Res}_k^{k+1}(\psi^{(k+1)}) + \dots \\ + \boldsymbol{Res}_k^{k+n-1}(\psi^{(k+n-1)}) + \boldsymbol{Res}_k^{k+n}(\varphi^{(k+n)}).$$

Put $\Psi^{(k,n)} = \sum_{j=0}^{n-1} \boldsymbol{Res}_k^{k+j}(\psi^{(k+j)})$, then for every $k \geqslant 1$,

$$\varphi^{(k)} = \Psi^{(k,n)} + \boldsymbol{Res}_k^{k+n}(\varphi^{(k+n)}).$$

It follows that, for every $n \geqslant 1$, $\Psi^{(k,n)} \ll \varphi^{(k)}$. This implies, by (b) of Lemma 3.10, that the sequence $\{\Psi^{(k,n)}\}_n$ converges uniformly on $G(k)$ to $\Psi^{(k)} \in \mathcal{P}^\natural(G(k))$, where $\Psi^{(k)} = \sum_{j=0}^{\infty} \boldsymbol{Res}_k^{k+j}(\psi^{(k+j)})$. Hence the sequence $\boldsymbol{Res}_k^{k+n}(\varphi^{(k+n)})$ converges uniformly on $G(k)$. Let us denote by $\Phi^{(k)}$ its limit. Since $\boldsymbol{Res}_k^{k+1}$ is continuous in the topology of uniform convergence on $G(k)$,

$$\begin{aligned}\Phi^{(k)} = \lim_{n\to+\infty} \boldsymbol{Res}_k^{k+n}(\varphi^{(k+n)}) &= \lim_{n\to+\infty} \boldsymbol{Res}_k^{k+1+n}(\varphi^{(k+1+n)}) \\ &= \lim_{n\to+\infty} (\boldsymbol{Res}_k^{k+1} \circ \boldsymbol{Res}_{k+1}^{k+1+n})(\varphi^{(k+1+n)}) \\ &= \boldsymbol{Res}_k^{k+1}\big(\lim_{n\to+\infty} \boldsymbol{Res}_{k+1}^{k+1+n}(\varphi^{(k+1+n)})\big) \\ &= \boldsymbol{Res}_k^{k+1}(\Phi^{(k+1)}).\end{aligned}$$

Then $\{\Phi^{(k)}\}_{k\geqslant 1}$ is a projective system. In order to prove the uniqueness, let us suppose that, for every $k \geqslant 1$, $\varphi^{(k)}$ is given by another decomposition

$$\varphi^{(k)} = \Phi_1^{(k)} + \sum_{j=0}^{\infty} \boldsymbol{Res}_k^{k+j}(\psi_1^{(k+j)}),$$

then

$$\begin{aligned}\psi^{(k)} &= \varphi^{(k)} - \boldsymbol{Res}_k^{k+1}(\varphi^{(k+1)}) \\ &= \Phi_1^{(k)} + \sum_{j=0}^{\infty} \boldsymbol{Res}_k^{k+j}(\psi_1^{(k+j)}) \\ &\quad - \boldsymbol{Res}_k^{k+1}\left(\Phi_1^{(k+1)} + \sum_{j=0}^{\infty} \boldsymbol{Res}_{k+1}^{k+1+j}(\psi_1^{(k+1+j)})\right) \\ &= \sum_{j=0}^{\infty} \boldsymbol{Res}_k^{k+j}(\psi_1^{(k+j)}) - \sum_{j=1}^{\infty} \boldsymbol{Res}_k^{k+j}(\psi_1^{(k+j)}) = \psi_1^{(k)}.\end{aligned}$$

□

Corollary 3.12. — *Let $\varphi_1 = \{\varphi_1^{(n)}\}_n$ and $\varphi_2 = \{\varphi_2^{(n)}\}_n$ be two sub-projective systems of $\mathcal{Q}$ such that $\varphi_1 \lll \varphi_2$, in the sense that, for every n, $\varphi_1^{(n)} \ll \varphi_2^{(n)}$. Then, for every n, $\Phi_1^{(n)} \ll \Phi_2^{(n)}$ and $\psi_1^{(n)} \ll \psi_2^{(n)}$.*

Proof. — We may write

$$\varphi_2 = \varphi_1 + \varphi_0, \text{ with } \varphi_0 \in \mathcal{Q}.$$

By the uniqueness of the decomposition given by formula (3.1),

$$\Phi_2 = \Phi_1 + \Phi_0,$$

and for every n,

$$\psi_2^{(n)} = \psi_1^{(n)} + \psi_0^{(n)}.$$

Since $\Phi_0^{(n)}$ and $\psi_0^{(n)}$ are in $\mathcal{P}^\natural(G(n))$, we can deduce that, for every n, $\Phi_1^{(n)} \ll \Phi_2^{(n)}$ and $\psi_1^{(n)} \ll \psi_2^{(n)}$. □

By Corollary 2.13, for every $n \geqslant 1$, $\mathcal{P}^\natural(G(n))$ is a lattice. Moreover, by Theorem 3.9, $\mathcal{P}^\natural(G)$ is a lattice. Using the previous decomposition, we prove the following proposition,

Proposition 3.13. — *The cone $\mathcal{Q}$ is a lattice.*

Proof. — Let $\varphi_1 = \{\varphi_1^{(n)}\}_n$, $\varphi_2 = \{\varphi_2^{(n)}\}_n$ be two subprojective systems of $\mathcal{Q}$. By Proposition 3.11,

$$\varphi_1^{(n)} = \Phi_1^{(n)} + \sum_{j=0}^{\infty} \boldsymbol{Res}_n^{n+j}(\psi_1^{(n+j)}),$$

$$\varphi_2^{(n)} = \Phi_2^{(n)} + \sum_{j=0}^{\infty} \boldsymbol{Res}_n^{n+j}(\psi_2^{(n+j)}).$$

Put $\Phi_{Min}^{(n)} = \Phi_1^{(n)} \wedge \Phi_2^{(n)}$ and $\psi_{Min}^{(n)} = \psi_1^{(n)} \wedge \psi_2^{(n)}$. Let $\varphi = \{\varphi^{(n)}\}_n \in \mathcal{Q}$. If $\varphi \lll \varphi_1$ and $\varphi \lll \varphi_2$, then by Corollary 3.12, $\Phi^{(n)} \ll \Phi_1^{(n)}$, $\Phi^{(n)} \ll \Phi_2^{(n)}$, and thus $\Phi^{(n)} \ll \Phi_{Min}^{(n)}$. Also $\psi^{(n)} \ll \psi_1^{(n)}$, $\psi^{(n)} \ll \psi_2^{(n)}$, which implies that $\psi^{(n)} \ll \psi_{Min}^{(n)}$. Since, for every n, $\psi_{Min}^{(n)} \ll \psi_1^{(n)}$, then by (c) of Lemma 3.10, $\sum_{j=0}^{\infty} \boldsymbol{Res}_n^{n+j}(\psi_{Min}^{(n+j)})$ converges in $\mathcal{P}^\natural(G(n))$ uniformly on $G(n)$. We put then, for every n,

$$\varphi_{Min}^{(n)} = \Phi_{Min}^{(n)} + \sum_{j=0}^{\infty} \boldsymbol{Res}_n^{n+j}(\psi_{Min}^{(n+j)}).$$

We get, for every n, $\varphi^{(n)} \ll \varphi_{Min}^{(n)}$, and so (φ_1, φ_2) has a greatest lower bound $\varphi_{Min} = \{\varphi_{Min}^{(n)}\}_n$. Now, put for every n, $\Phi_{Max}^{(n)} = \Phi_1^{(n)} \vee \Phi_2^{(n)}$, and $\psi_{Max}^{(n)} = \psi_1^{(n)} \vee \psi_2^{(n)}$. Since, for every n, $\psi_{Max}^{(n)} \ll \psi_1^{(n)} + \psi_2^{(n)}$, then by (c) of Lemma 3.10, we can put, for every n,

$$\varphi_{Max}^{(n)} = \Phi_{Max}^{(n)} + \sum_{j=0}^{\infty} \boldsymbol{Res}_n^{n+j}(\psi_{Max}^{(n+j)}).$$

Thus, (φ_1, φ_2) has a least upper bound $\varphi_{Max} = \{\varphi_{Max}^{(n)}\}_n$. As a consequence, $\mathcal{Q}$ is a lattice. □

Next, we will determine the set of extremal points of $\mathcal{Q}_{\leqslant 1}$. We need to define, for $n \geqslant 1$, the following subset :

$$\mathcal{P}^n = \{\varphi \in \prod_{i=1}^{\infty} \mathcal{P}^{\natural}_{\leqslant 1}(G(i)) \mid \varphi^{(i)} = \boldsymbol{Res}^n_i\big(\varphi^{(n)}\big),\ \text{for } 1 \leqslant i \leqslant n$$

$$\text{and } \varphi^{(i)} = 0,\ \text{for } i \geqslant n+1\},$$

where, for every $i = 1, \ldots, n-1$,

$$\boldsymbol{Res}^n_i = \boldsymbol{Res}^{i+1}_i \circ \boldsymbol{Res}^{i+2}_{i+1} \circ \cdots \circ \boldsymbol{Res}^n_{n-1}.$$

The set $\mathcal{P}^n$, with finite n, consists of projective systems of finite order n obtained via the following linear isomorphism :

$$\iota : \mathcal{P}^{\natural}_{\leqslant 1}(G(n)) \to \mathcal{P}^n$$

$$\varphi^{(n)} \longmapsto (\boldsymbol{Res}^n_1(\varphi^{(n)}), \boldsymbol{Res}^n_2(\varphi^{(n)}), \ldots, \boldsymbol{Res}^n_{n-1}(\varphi^{(n)}), \varphi^{(n)}, 0, \ldots).$$

Since $\boldsymbol{Res}^{n+1}_n(\mathcal{P}^{\natural}_{\leqslant 1}(G(n+1))) \subset \mathcal{P}^{\natural}_{\leqslant 1}(G(n))$, the set $\mathcal{P}^{\natural}_{\leqslant 1}(G)$ can be identified with the projective limit of $\{\mathcal{P}^{\natural}_{\leqslant 1}(G(n))\}_{n \geqslant 1}$ and an element φ in $\mathcal{P}^{\natural}_{\leqslant 1}(G)$ determines a projective system $\{\varphi^{(n)}\}$ with $\varphi^{(n)} = \boldsymbol{Res}_n(\varphi)$. The same holds for an element ω of the set $\mathcal{E}_\infty$ of non zero extremal points of $\mathcal{P}^{\natural}_{\leqslant 1}(G)$, i.e. $\mathcal{E}_\infty = \mathrm{Ext}(\mathcal{P}^{\natural}_1(G))$.

Let $\mathcal{E}_n$ denote the set of non zero extremal points of $\mathcal{P}^n$. An element φ in $\mathcal{E}_n$ is the image by the isomorphism ι of an element $\varphi^{(n)} \in \mathrm{Ext}(\mathcal{P}^{\natural}_1(G(n))$.

Theorem 3.14. — *The set of extremal points of $\mathcal{Q}_{\leqslant 1}$ consists of two types of elements :*

$$\textit{type } \infty\ :\ \mathcal{E}_\infty,\ \textit{and type } n\ :\ \mathcal{E}_n,$$

and we have

$$\mathrm{Ext}(\mathcal{Q}_{\leqslant 1}) = \{0\} \cup \mathcal{E}_\infty \cup \Big(\bigcup_{n=1}^{\infty} \mathcal{E}_n\Big). \tag{3.3}$$

The sets $\mathcal{E}_\infty$, $\mathcal{E}_n$ $(n \geqslant 1)$ are disjoint.

Proof. — (a) Let us prove that every φ in $\mathcal{E}_n$ is extremal. Suppose that $\varphi = \varphi_1 + \varphi_2$, $\varphi_1, \varphi_2 \in \mathcal{Q}_{\leqslant 1}$. Then, for every n,

$$\varphi^{(n)} = \varphi_1^{(n)} + \varphi_2^{(n)}.$$

So, $\varphi_1^{(n)} = \lambda_1 \varphi^{(n)}, \varphi_2^{(n)} = \lambda_2 \varphi^{(n)}$. On the other hand,

$$\begin{aligned}\varphi^{(n-1)} = \boldsymbol{Res}^n_{n-1}\varphi^{(n)} &= \varphi_1^{(n-1)} + \varphi_2^{(n-1)} \\ &\gg \lambda_1 \boldsymbol{Res}^n_{n-1}\varphi^{(n)} + \lambda_2 \boldsymbol{Res}^n_{n-1}\varphi^{(n)} = \boldsymbol{Res}^n_{n-1}\varphi^{(n)}.\end{aligned}$$

Therefore

$$\varphi_1^{(n-1)} = \lambda_1 \boldsymbol{Res}_{n-1}^n \varphi^{(n)},\ \varphi_2^{(n-1)} = \lambda_2 \boldsymbol{Res}_{n-1}^n \varphi^{(n)},$$

and hence

$$\varphi_1 = \lambda_1 \varphi,\ \varphi_2 = \lambda_2 \varphi.$$

(b) Let us prove that $\varphi \in \mathcal{E}_\infty$ is extremal. Suppose that $\varphi = \varphi_1 + \varphi_2$, $\varphi_1, \varphi_2 \in \mathcal{Q}_{\leqslant 1}$. Since φ is a projective system, for every n, $\psi^{(n)} = 0$. Thus, $\psi_1^{(n)} = 0$, $\psi_2^{(n)} = 0$, and hence $\varphi_1, \varphi_2 \in \mathcal{P}_1^\natural(G)$. Therefore

$$\varphi_1 = \lambda_1 \varphi,\ \varphi_2 = \lambda_2 \varphi.$$

(c) Let φ be a non zero extremal point of $\mathcal{Q}_{\leqslant 1}$, we can write

$$\varphi^{(n)} = \Phi^{(n)} + \sum_{j=0}^{\infty} \boldsymbol{Res}_n^{n+j}(\psi^{(n+j)}),$$

it's a decomposition into two elements of $\mathcal{Q}_{\leqslant 1}$:

First case : $\psi^{(n)} = 0$, for every n, and so $\varphi \in \mathcal{E}_\infty$.

Second case : $\Phi^{(n)} = 0$, for every n, and hence

$$\varphi = \Phi + \Psi_1 + \Psi_2 + \dots,$$

where

$$\begin{aligned} \Psi_n^{(j)} &= \boldsymbol{Res}_j^n(\psi^{(n)}) && \text{if } j \leqslant n, \\ &= 0 && \text{if } j > n. \end{aligned}$$

As a result, there exists n_0 such that $\varphi = \Psi_{n_0}$, with $\psi^{(n_0)} \in \mathrm{Ext}(\mathcal{P}_1^\natural(G(n_0)))$. We can then conclude that $\varphi \in \mathcal{E}_{n_0}$. □

Assuming all $G(n)$ separable, we can now state a Bochner type theorem for the corresponding Olshanski spherical pairs.

Theorem 3.15. — *Let (G, K) be an Olshanski spherical pair defined as inductive limit of an increasing sequence of Gelfand pairs $(G(n), K(n))_n$, with the assumption that all $G(n)$ are separable. Then, for every function $\varphi \in \mathcal{P}^\natural(G)$, there exists, on the Borel set $\Omega = \mathrm{Ext}(\mathcal{P}_1^\natural(G))$, a unique bounded and positive measure μ such that*

$$\varphi(g) = \int_\Omega \omega(g) \mu(d\omega).$$

Proof. — The set $\mathcal{Q}_{\leqslant 1}$ being convex, compact and metrisable in $\mathcal{Q}$, it satisfies the hypothesis of Choquet's theorem. Hence $\mathrm{Ext}(\mathcal{Q}_{\leqslant 1})$ is a Borel set and every element of $\mathcal{Q}_{\leqslant 1}$ can be represented via a probability measure ν on $\mathrm{Ext}(\mathcal{Q}_{\leqslant 1})$ such that, for every continuous linear form L on $\mathcal{Q}$,

$$L(q) = \int_{\mathrm{Ext}(\mathcal{Q}_{\leqslant 1})} L(p)\nu(dp). \tag{3.4}$$

Moreover, as $\mathcal{Q}$ is a lattice (Proposition 3.13), by (iii) of Choquet's theorem, the measure ν is unique. Furthermore, we can deduce from formula (3.3) that

$$\Omega = \mathrm{Ext}(\mathcal{Q}_{\leqslant 1}) \setminus \big(\bigcup_{\mathrm{n}=1}^{\infty} \mathcal{E}_{\mathrm{n}} \cup \{0\} \big).$$

Hence Ω is a Borel set.

Let φ be an element of $\mathcal{P}^{\natural}_{\leqslant 1}(G)$. We know that φ determines a sequence $\{\varphi^{(n)}\}_{n\geqslant 1}$ where $\varphi^{(n)} = \boldsymbol{Res}_n(\varphi)$. Let us take, for L in (3.4), the linear form

$$\varphi^{(n)} \mapsto (\varphi^{(n)}, f) = \int_{G(n)} \varphi^{(n)}(h) f(h) \alpha_n(dh),$$

where $f \in L^1(G(n))$ and α_n is the left invariant Haar measure of $G(n)$. By considering, for every n, the approximation $(f_k) : f_k \in L^1(G(n))$, $f_k \geqslant 0$,

$$\int_{G(n)} f_k(h)\alpha_n(dh) = 1,$$

and for every continuous bounded function ψ :

$$\lim_{k\to\infty} \int_{G(n)} \psi(h) f_k(h)\alpha_n(dh) = \psi(g),$$

we get that, for every $n \geqslant 1$,

$$\varphi^{(n)}(g) = \int_{\Omega} \omega(g)\ \nu^{(\infty)}(d\omega) + \sum_{k=n}^{\infty} \int_{\mathcal{E}_n} \omega(g)\ \nu^{(k)}(d\omega),$$

where $\nu^{(\infty)}$ (respectively $\{\nu^{(k)}\}_{k\geqslant n}$), are the restrictions of ν to Ω (respectively $\{\mathcal{E}_k\}_{k\geqslant n}$). Therefore we obtain, for $g \in G(n)$,

$$\varphi^{(n)}(g) - \varphi^{(n+1)}(g) = \int_{\mathcal{E}_n} \omega(g)\ \nu^{(n)}(d\omega).$$

Since $\{\varphi^{(n)}\}_{n\geqslant 1}$ is a projective system, for every $g \in G(n)$ and every $n \geqslant 1$,

$$\int_{\mathcal{E}_n} \omega(g)\ \nu^{(n)}(d\omega) = 0.$$

As $\omega(e) = 1$ we get, for every $n \geqslant 1$,

$$\nu^{(n)}(\mathcal{E}_n) = 0.$$

Hence ν is concentrated on $\mathcal{E}_\infty = \Omega$. It follows that every element φ in $\mathcal{P}^\natural_{\leqslant 1}(G)$ has the following integral representation :

$$\varphi(g) = \int_\Omega \omega(g)\nu^{(\infty)}(d\omega), \ (g \in G).$$

Finally, every element φ in $\mathcal{P}^\natural(G)$ can be uniquely written as $\varphi(g) = \lambda\varphi_0(g)$ with φ_0 in $\mathcal{P}^\natural_{\leqslant 1}(G)$ and $\lambda = \varphi(e) \geqslant 0$. Hence φ is represented via a measure μ equal to $\lambda\nu_0^{(\infty)}$, where $\nu_0^{(\infty)}$ verifies

$$\varphi_0(g) = \int_\Omega \omega(g)\nu_0^{(\infty)}(d\omega).$$

□

4. Remarks and open questions

(1) We do not know a topology making $\mathcal{P}^\natural_{\leqslant 1}(G)$ compact and enabling in consequence a direct application of Choquet's theorem without using $\mathcal{Q}$. T. Hirai and E. Hirai had studied this problem in [12].

(2) Given a generalized Gelfand pair, i.e. an Olshanski spherical pair, one problem is to find the set of extremal points Ω. This is known in several cases. Another problem is, given $\varphi \in \mathcal{P}^\natural(G)$, to find the representing measure μ.

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BIBLIOGRAPHY

[1] S. Bochner, *Lectures on Fourier integrals.*, Princeton University Press, Princeton, N.J., 1959.

[2] N. Bourbaki, *Éléments de mathématique. Fasc. XX. Livre I: Théorie des ensembles. Chapitre 3: Ensembles ordonnés cardinaux, nombres entiers*, Hermann, Paris, 1963.

[3] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann)*, Gauthier-Villars, Paris, 1957.

[4] ———, *Les C^*-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964.

[5] R. S. Doran & J. M. G. Fell, *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles. Vol. 1*, Academic Press Inc., Boston, MA, 1988, Basic representation theory of groups and algebras.

[6] J. Faraut, "Analyse harmonique sur les paires de Guelfand et les espaces hyperboliques", in *Analyse harmonique. Les cours du C.I.M.P.A.*, P. Eymard, Nice, 1982, p. 315-446.

[7] ———, "Espaces hilbertiens invariants de fonctions holomorphes", in *Analyse sur les groupes de Lie et théorie des représentations (Kénitra, 1999)*, Sémin. Congr., vol. 7, Soc. Math. France, Paris, 2003, p. 101-167.

[8] ———, "Infinite dimensional harmonic analysis and probability", in *Probability measures on groups: recent directions and trends*, Tata Inst. Fund. Res., Mumbai, 2006, p. 179-254.

[9] R. Godement, "Sur les fonctions de type positif", *C. R. Acad. Sci. Paris* **221** (1945), p. 69-71.

[10] ———, "Sur quelques propriétés des fonctions de type positif définies sur un groupe quelconque", *C. R. Acad. Sci. Paris* **222** (1946), p. 529-531.

[11] ———, "Les fonctions de type positif et la théorie des groupes", *Trans. Amer. Math. Soc.* **63** (1948), p. 1-84.

[12] T. Hirai & E. Hirai, "Positive definite class functions on a topological group and characters of factor representations", *J. Math. Kyoto Univ.* **45** (2005), no. 2, p. 355-379.

[13] S. Kerov, G. Olshanski & A. Vershik, "Harmonic analysis on the infinite symmetric group", *Invent. Math.* **158** (2004), no. 3, p. 551-642.

[14] G. Olshanski, "Unitary representations of infinite-dimensional pairs (G, K) and the formalism of R. Howe", in *Representation of Lie groups and related topics*, Adv. Stud. Contemp. Math., vol. 7, Gordon and Breach, New York, 1990, p. 269-463.

[15] ———, "The problem of harmonic analysis on the infinite-dimensional unitary group", *J. Funct. Anal.* **205** (2003), no. 2, p. 464-524.

[16] G. Olshanski & A. Vershik, "Ergodic unitarily invariant measures on the space of infinite Hermitian matrices", in *Contemporary mathematical physics*, Amer. Math. Soc. Transl. Ser. 2, vol. 175, Amer. Math. Soc., Providence, RI, 1996, p. 137-175.

[17] R. Phelps, *Lectures on Choquet's theorem*, second ed., Lecture Notes in Mathematics, vol. 1757, Springer-Verlag, Berlin, 2001.

[18] W. Rudin, *Functional analysis*, McGraw-Hill Book Co., New York, 1973, McGraw-Hill Series in Higher Mathematics.

[19] E. Thoma, "Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe", *Math. Z.* **85** (1964), p. 40-61.

[20] ———, "Über positiv-definite Klassenfunctionen abzählbarer Gruppen", *Math. Z.* **84** (1964), p. 389-402.

[21] D. Voiculescu, "Représentations factorielles de type II1 de $U(\infty)$", *J. Math. Pures Appl. (9)* **55** (1976), no. 1, p. 1-20.

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