# ANNALES

## DE

# L'INSTITUT FOURIER

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**Semi-classical formula beyond the Ehrenfest time in quantum chaos. (I) Trace formula**

# SEMI-CLASSICAL FORMULA BEYOND THE EHRENFEST TIME IN QUANTUM CHAOS. (I) TRACE FORMULA

### by Frédéric FAURE

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ABSTRACT. — We consider a nonlinear area preserving Anosov map $M$ on the torus phase space, which is the simplest example of a fully chaotic dynamics. We are interested in the quantum dynamics for long time, generated by the unitary quantum propagator $\hat{M}$. The usual semi-classical Trace formula expresses $\mathrm{Tr}\left(\hat{M}^t\right)$ for finite time $t$, in the limit $\hbar \to 0$, in terms of periodic orbits of $M$ of period $t$. Recent work reach time $t \ll t_E/6$ where $t_E = \log(1/\hbar)/\lambda$ is the Ehrenfest time, and $\lambda$ is the Lyapounov coefficient. Using a semi-classical normal form description of the dynamics uniformly over phase space, we show how to extend the trace formula for longer time of the form $t = C.t_E$ where $C$ is any constant, with an arbitrary small error.

RÉSUMÉ. — On considère une application $M$, Anosov non linéaire qui conserve l'aire sur le tore $T^2$. C'est un des exemples les plus simples d'une dynamique chaotique. On s'intéresse à la dynamique quantique pour les temps longs, générée par un opérateur unitaire $\hat{M}$. La formule des traces semi-classique habituelle exprime $\mathrm{Tr}\left(\hat{M}^t\right)$ pour $t$ fini, dans la limite $\hbar \to 0$, en termes d'orbites périodiques de $M$ de période $t$. Des travaux récents atteignent des temps $t \ll t_E/6$ où $t_E = \log(1/\hbar)/\lambda$ est le temps d'Ehrenfest, et $\lambda$ est le coefficient de Lyapounov. En utilisant une description uniforme de la dynamique au moyen d'une forme normale semi-classique, nous montrons comment étendre la formule des traces pour des temps plus longs, de la forme $t = C.t_E$, où $C$ est une constante arbitraire, et avec une erreur arbitrairement petite.

## 1. Introduction

Semi-classical analysis is a fruitful approach to understand wave equations in the regime where the wave length is small in comparison with the size of the domain, or with the size of the typical variation of the potential,

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where the wave evolves. In that regime, one shows that the evolution of a wave can be described in terms of Hamiltonian classical dynamics in the same domain (or with the same potential). For example, the Van-Vleck formula (1928) expresses the evolved wave as a sum of the initial wave transported along several classical trajectories. Because the wave formalism enters in many area of physics (acoustic waves, seismic waves, electromagnetic waves, quantum waves ...), semi-classical analysis is an important mathematical tool to understand physical phenomena. Wave equations, and more generally Partial Differential Equations is an important domain of mathematics, so semi-classical analysis has also been extremely studied and developed in mathematics.

## 1.1. Problematics of quantum chaos

A common way to express the semi-classical limit or short wave-length limit, is to introduce a dimensionless parameter $\hbar$ in the wave equation, called the "Planck constant", which corresponds to $\hbar \simeq l/L$ where $l$ is the wave length and $L$ the typical size of the domain. The semi-classical limit is then $\hbar \to 0$.

One possible way to understand semi-classical correspondences is through wave packets (or coherent states), which are waves localized as much as possible in phase space, so that to mimic the motion of a classical particle. Because of the uncertainty principle $\Delta q \Delta p \geqslant \hbar$, in the phase space of position $q$ and momentum $p$, a wave packet can not be localized better than a small domain of surface $\hbar$, called the Planck cell. This dispersion of the wave packet in phase space is responsible for its spreading during time evolution (in a similar way as the spreading of a classical disk of surface $\hbar$). After a finite time (compared to $\hbar \to 0$), the spreading is finite, the wave is still localized. For that reason, one can derive the Van-Vleck formula giving the matrix elements of the propagator or the semi-classical trace formula [16, 15][34], and obtain the Weyl formula giving the averaged density of states.

If the classical dynamics takes place in a compact energy surface, and has exponential instabilities (e.g. hyperbolic dynamical systems), the evolution of a wave packet is much harder to understand for longer times. This is one of the goals of quantum chaos studies. If $\lambda$ denotes the classical Lyapounov exponent of instabilities, an heuristic approach suggests that the wave packet which has initial length $\Delta q, \Delta p \geqslant \hbar$ should spreads into a long and thin distribution of length $L_t \geqslant \hbar e^{\lambda t}$, which rolls up in the energy

surface, as soon as $L_t \gg constant \Leftrightarrow t \gg \frac{1}{\lambda} \log (1/\hbar)$. This introduces the Ehrenfest time $t_E = \frac{1}{\lambda} \log (1/\hbar)$ which is an important characteristic time scale in quantum chaos. For longer times, $t \gg t_E$, the distribution may intersect all the other elementary cells of surface $\hbar$ (Planck cells) in many distinct components, whose number grows exponentially fast with $t$. Each component is identified with a classical trajectory. As this description suggests (and as been suggested by many works in the physical literature [56]), the matrix elements of the propagator operator $\hat{M}_t$ between localized wave packets, for $t \gg t_E$, could be expressed as a (exponentially large) sum over these distinct trajectories. In particular the trace of the propagator could be expressed as a sum over $\mathcal{N}_t \propto e^{\lambda t}$ periodic orbits. This is the content of the Gutzwiller trace formula [33][34]. Although there are numerical calculations that suggest its validity, such a semi-classical formula has not yet been proved to be valid for time greater than $\frac{1}{6} t_E$. The difficulty relies in the control of the errors of individual terms which add together.

A major goal of quantum chaos would be to describe the long time dynamics of wave packets up to time $t_H \simeq 1/\hbar$ (or more), called the Heisenberg time[1], which would allow semi-classical formula to resolve individual energy levels. But at time $t \simeq t_H$ the number of classical trajectories entering in a semi-classical formula is of order $\mathcal{N}_{t_H} \simeq \exp (\lambda/\hbar)$, extremely large (!), compared to the dimension of the effective Hilbert space which is $1/\hbar$. One famous observation and conjecture in quantum chaos is that correlations between closed energy levels behave as eigenvalues of random matrix ensembles [6][5]. Many works in physical literature assume that semi-classical expressions such as the Gutzwiller formula are valid up to time of order $t_H \simeq 1/\hbar$, and deduce some heuristic explanations for the random matrix theory, or other important quantum chaos phenomena [22][35]. For recent reviews on the mathematical aspects of quantum chaos, see [18],[17], [58][61].

In this paper we consider a simple model of quantum chaotic dynamics, namely a non linear uniform hyperbolic map on the torus, and we show that semi-classical formula extends to time scales $t = C.t_E$, where $C > 0$ is *any* constant. We do not reach the Heisenberg time. The model is particular, but we believe that the methods presented here could be extended to more general uniform hyperbolic dynamical systems. In this paper, the objective is to discuss the Gutzwiller trace formula, and in a second paper we

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[1] For a time independent dynamics with $d$ degrees of freedom, $t_H \simeq 1/\hbar^d$. For the model developed in this paper, $t_H \simeq 1/\hbar$.

will discuss a second application, the "Van-Vleck formula" which expresses matrix elements of the propagator, also for times $t = Ct_E$.

## 1.2. Characteristic time in quantum evolution of wave packets discussed with a numerical example

In this Section, in order to motivate the importance of the Ehrenfest time, as a qualitative frontier in semi-classical evolution problems in quantum chaos, we present and discuss some recent results concerning the evolution of quantum states in a hyperbolic flow. One of the main challenges in quantum chaos is to deal with both the long time limit $t \to \infty$, and the semi-classical limit $\hbar \to 0$. Usual semi-classical results, such as the Ehrenfest or Egorov theorem, concerns $\hbar \to 0$ first, and $t \to \infty$ after. The challenge is to try to reverse this order (in order to get information on individual eigenfunctions and eigenvalues), or more modestly, make $t$ depending on $\hbar$.

The present discussion is made around the observation of the evolution of a wave packet with a numerical example. We refer to Section 2 for a complete definition of the model. Let $M$ denotes a non linear hyperbolic map on the torus phase space $\mathbb{T}^2$, and $\hat{M}$ the corresponding quantized map . A wave packet (a coherent state $\psi_{x_0}$) is launched at time $t = 0$, at a generic position $x_0 = (q_0, p_0) \in \mathbb{T}^2$. Figure 1.1 shows the Husimi distribution (i.e. phase space representation) of the evolved state $\psi(t) = \hat{M}^t \psi_{x_0}$ at different time $t \in \mathbb{N}$. We remind that the Husimi distribution of the initial state $\psi_{x_0}$ has typical width $\Delta_0 \simeq \sqrt{\hbar}$, (due to the uncertainty principle $\Delta q \Delta p \simeq \hbar$, and the specific choice $\Delta q = \Delta p = \Delta_0 \simeq \sqrt{\hbar}$). During time evolution, the wave packet center is moving, and its distribution spreads due to instabilities of the trajectories. Figure 1.2 summarizes the main effects we discuss below.

**Finite time regime with "no dispersion":** We first consider a fixed value of $t = Cste$, and $\hbar \to 0$ (of course $t$ can be arbitrary large in principle). The evolved state $\psi(t)$ is localized at the classical position $x(t) = M^t x_0$. In more precise terms, the semi-classical measure of $\psi(t)$ is a Dirac measure at $x(t) = M^t x_0$. The evolved state $\psi(t)$ spreads but its width is $\Delta_t \simeq e^{\lambda t} \Delta_0 \simeq \sqrt{\hbar}$, still of order $\hbar^{1/2}$ [39][38][48]. Because $t$ can be chosen arbitrary large a priori, the ergodic nature of the dynamics may have importance if $x(t)$ follows a dense trajectory for example. Some well known semi-classical results such as the semi-classical Egorov theorem [23], or the Schnirelman quantum ergodicity theorem [11][57] use these finite range of time.
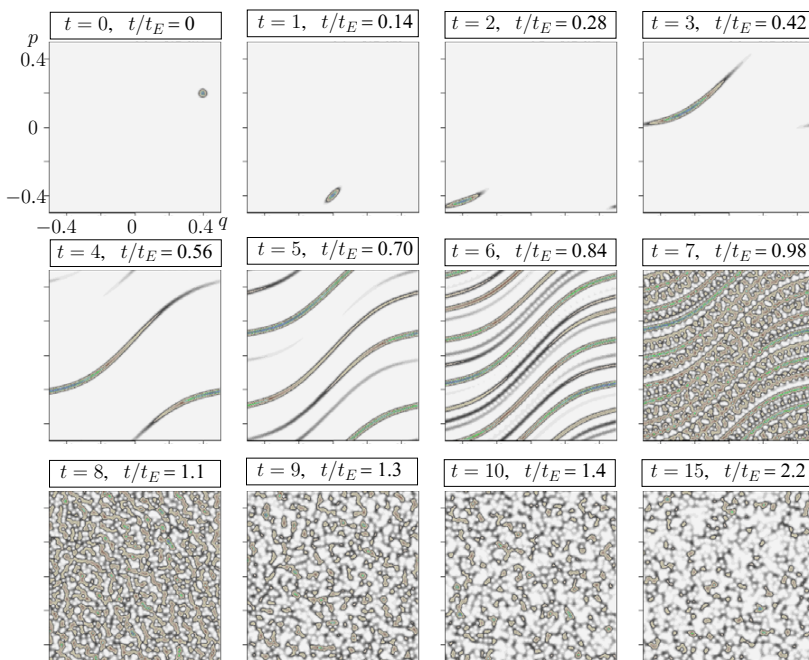
Figure 1.1. Phase space (Husimi) distribution of the evolution of a (generic) coherent state at initial position $x_0 = (0.4, 0.2)$, for different times $t = 0, 1, 2, \ldots$ and $2\pi\hbar = 10^{-3}$.
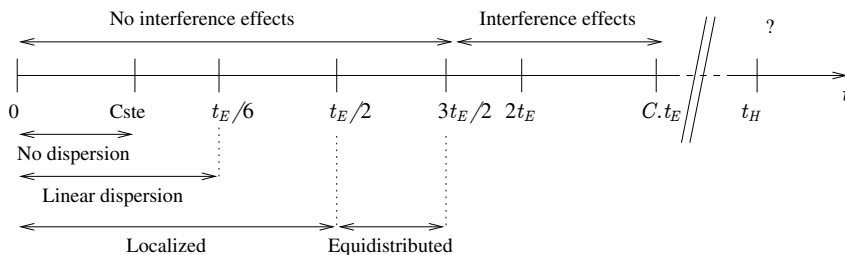


Figure 1.2. Characteristic times which appear in the semi-classical limit $\hbar \to 0$, for the evolution of an initial coherent state. $t_E = \frac{1}{\lambda} \log (1/\hbar)$ is the Ehrenfest time, (very small "in practice") compared to the Heisenberg time $t_H = 1/\hbar$.

**Linear dispersion regime:** Some recent and very general results [16][4][36][9] describe the evolved quantum state $\psi(t)$, in the linear dispersion regime, which means that non linear effects on the dispersion of the coherent state are supposed to be negligible with respect to the linear effects. Because the first non linear effects correspond to cubic terms in the Hamiltonian, this imposes that $\Delta_t^3 \ll \hbar$, equivalently $e^{\lambda t}\hbar^{1/2} \ll \hbar^{1/3}$, or $t \ll \frac{1}{6}t_E$. In our numerical example $\frac{1}{6}t_E = 1.2$.

**Localized regime:** After that time, the coherent state spreads more and more. But its width is still of microscopic size if $\Delta_t \ll 1$, i.e. $t \ll \frac{1}{2}t_E$. In more precise terms, the semi-classical measure of $\psi(t)$ is still a Dirac measure at $x(t)$ in that range of time. In our example $\frac{1}{2}t_E = 3.6$. At a time around $t \simeq \frac{1}{2}t_E$, the quantum state has size of order 1, and can be described as a "Lagrangian W.K.B state" [50].

**Equidistribution regime:** For time $t$ larger than $\frac{1}{2}t_E$, the wave packet spreads and wraps around the torus phase space, along unstable manifolds, like a classical probability measure. Thanks to classical mixing, a smooth classical probability distribution is known to converge towards the uniform Liouville measure for large time. The Husimi distribution is expected to behave like a classical measure, and equidistributes, if the different branches do not "interfere" with each other on phase space. After the time $\frac{1}{2}t_E$, we evaluate that the distance between consecutive branches get smaller and smaller like $d \sim e^{-\lambda(t-t_E/2)}$ until the critical value $\hbar$ is obtained at time $t = \frac{1}{2}t_E + t_E = \frac{3}{2}t_E$ (This is indeed the ultimate value, because if $d \gg \hbar$, one can still insert a (squeezed) localized wave packet between two consecutive branches, which means that the branches do not yet interfere). Correspondingly to this description, in [7], the authors show that for the *linear* map, the semi-classical measure $\psi(t)$ converges towards the Liouville measure, in the range of time $\frac{1}{2}t_E \ll t \ll \frac{3}{2}t_E$. J.M. Bouclet and S. De Bièvre in [8] obtain a similar result for a non linear hyperbolic map, but for $t \ll \frac{2}{3}t_E$. S. Nonnenmacher in [50] reaches the time $t \ll \frac{3}{2}t_E$. In [53], R. Schubert has described evolution of an initial Lagrangian state under a hyperbolic flow. He has obtained similar results, namely equidistribution up to time $t \ll t_E$. This is indeed similar, because an initial coherent state becomes a Lagrangian state at time $\frac{1}{2}t_E$. This range of time is also considered and controlled in [1, 2].

**Longer time and interference effects:** For longer time very little is known. Some arguments and numerical observations in [56] suggest that

semi-classical formula applies for longer time. This is the subject of the present work and the second paper to come [24]. We show that in the range of time $t \in [0, C.t_E]$, where $C$ is any constant, the evolved state $\psi(t)$ described by its Husimi distribution $Hus(x) = |\langle x|\psi(t)\rangle|^2$, or by its Bargmann distribution $\langle x|\psi(t)\rangle$, can be expressed in general as a (finite) sum over different classical trajectories starting from the vicinity of the initial state $x_0$, and ending at time $t$ in the vicinity of the point $x$ (similarly to the semi-classical Van-Vleck formula). These trajectories give unavoidable interferences effects for time $t \geqslant \frac{3}{2}t_E$, because one can estimate that the sum involves more than one trajectory. Similarly the trace formula expresses $\mathrm{Tr}\left(\hat{M}^t\right)$ as a sum over periodic trajectories of period $t$. It is known that in specific cases, revival may occur[2] at time $t \simeq 2t_E$ [27], however it is expected that at least generically, these different contributions are somehow uncorrelated, and as a result, the state $\psi(t)$ is "generically" equidistributed over phase space as can be observed on figure 1.1.

An important characteristic time which is not considered here, because far much larger than the actual semi-classical approach could reach, is the **Heisenberg time** $t_H = 1/h$ (= 1000 in our example). This time is related with the mean separation between eigenvalues of $\hat{M}$. Some important effect of quantum chaos are numerically observed at this range of time, and explained by a **Random Matrix Theory** approach [5]. It allows to describe statistical properties of individual eigenfunctions and eigenvalues. Note that contrary to the mathematical works which are "stopped" by the Ehrenfest time, the Heisenberg time is extremely discussed in the physical literature, essentially with the random matrix theory.

## 1.3. Results and organisation of the paper

The main result of this paper is Theorem 6.1 page 2559, which shows that Gutzwiller semi-classical trace formula is valid for long time $t \simeq C \log(1/\hbar)$, with any $C > 0$. This formula expresses $\mathrm{Tr}\left(\hat{M}^t\right)$ in terms of semi-classical invariants associated to periodic orbits of period $t$, up to an arbitrary small

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[2] For a linear map, these interferences effects are responsible for exact revival of quantum states at time $t = 2t_E$, and existence of non ergodic invariant semi-classical measure (strong scars), as shown in [27, 26]. Reaching the time $2t_E$ for the semi-classical description of a *non linear* hyperbolic map, was the main motivation of this work, although we have not yet any precise idea if strong scarring effect may exist for general non linear hyperbolic systems.

error: for any $K$, there exists $D_{C,k} > 0$, such that

$$\left| \mathrm{Tr} \left( \hat{M}^t \right) - \mathrm{T}_{semi,t,J} \right| \leqslant D_{C,K} \hbar^K$$

with

$$\mathrm{T}_{semi,t,J} = \sum_{x=M^t x} \left| \mathrm{Det} \left( D_x M^t - 1 \right) \right|^{-1/2}$$
$$\exp \left( i \mathcal{A}_{x,t} / \hbar \right) \left( 1 + \hbar E_{1,x,t} + \hbar^2 E_{2,x,t} + \ldots \hbar^J E_{J,x,t} \right)$$

with $J > 2 \left( K + C \right)$, and where $\mathcal{A}_{x,t} \equiv \oint p \, dq - H \, dt$ is the action of the periodic orbit, and $E_{j,x,t}$ depends on semi-classical Normal forms coefficients of the periodic orbit.

This formula has already been derived in a similar form (i.e. with Normal form invariants) many time [32][60][43][55], but for finite time $t$ (with respect to $\hbar$) giving:

$$\left| \mathrm{Tr} \left( \hat{M}^t \right) - \mathrm{T}_{semi,t,K} \right| \leqslant C_{t,K} \hbar^K.$$

The main difficulty to extend this last formula for larger time was twice: how to control that neither the error term $C_{t,K}$, nor the normal form coefficients $E_{j,x,t}$ do diverge, when $t \simeq C \log \left( 1/\hbar \right)$ is so large. Such divergences could a priori be expected, due to exponential increase of the complexity of the dynamics with $t$. For a linear hyperbolic map, there are no semiclassical corrections, the trace formula is exact, and has been derived by J.P. Keating [47].

In Section 2, we define the dynamics and recall important properties of uniform hyperbolicity. In Section 3, we describe the periodic points and prepare the Trace Formula. In Section 4, we show how to control the semiclassical dynamics over large time, which will give us a control of the above error term $C_{t,K}$. In Section 5, we give a global semi-classical Normal form description of the hyperbolic dynamics, derived from a work of David De-Latte [19] in the classical case, and as a result, this gives us a control of the above coefficients $E_{j,x,t}$ uniformly over $x$ and $t$. Considering the results together, we deduce the semi-classical Trace Formula in Section 6. The appendices, that can be skipped in a first reading, give proofs of the Propositions used in this paper. The results are illustrated with numerical examples in Section 6.2. It is observed there that our bounds on the errors are not sharp at all, and that the validity of semi-classical Gutzwiller formula seems to be much better, as already suggested in [56]. We discuss again this fact in the conclusion, where we suggest an alternative approach with prequantum dynamics, and some perspectives.

## 2. Quantum non linear Anosov map on the torus

In this section we describe the dynamical system considered in this paper, a non linear area preserving map $M$ on the torus $\mathbb{T}^2$. This map is constructed as a perturbation of a linear hyperbolic map $M_0$, and is supposed to be uniformly hyperbolic (which is true for small enough perturbations because of structural stability theorem). In order to quantize the map $M$ in a natural way, we construct it from a Hamiltonian flow. This section recalls some well known results about this construction.

### 2.1. Classical Dynamics

#### 2.1.1. The linear map $M_0$

Consider a quadratic Hamiltonian on phase space $x = (q, p) \in \mathbb{R}^2$ with symplectic two form $\omega = dq \wedge dp$:

$$(2.1) \qquad H_0(q, p) = \frac{1}{2}\alpha q^2 + \frac{1}{2}\beta p^2 + \gamma qp,$$

with coefficients $\alpha, \beta, \gamma \in \mathbb{R}$. The Hamilton equation of motion for the trajectory $x(t)$ are $dq(t)/dt = \partial H_0/\partial p = \gamma q + \beta p$ and $dp(t)/dt = -\partial H_0/\partial q = -\alpha q - \gamma p$. We deduce that the flow after time 1 is $x(1) = M_0 x(0)$ with the matrix

$$(2.2) \qquad M_0 \overset{\text{def}}{=} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \exp\begin{pmatrix} \gamma & \beta \\ -\alpha & -\gamma \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),$$

*i.e.* $\det(M_0) = AD - BC = 1$. We assume that $\gamma^2 > \alpha\beta$ (equivalently $\mathrm{Tr}(M_0) = A + D > 2$), so that $M_0$ is a **hyperbolic map** with two real eigenvalues $e^{\pm\lambda_0}$ where $\lambda_0 = \sqrt{\gamma^2 - \alpha\beta} > 0$ is called the **Lyapounov exponent**. The two associated real eigenvectors denoted by $u_0, s_0 \in \mathbb{R}^2$, correspond to an unstable and a stable direction for the dynamics. Suppose moreover that $A, B, C, D \in \mathbb{Z}$, *i.e.* $M_0 \in SL(2, \mathbb{Z})$. Then for any $x \in \mathbb{R}^2$, $n \in \mathbb{Z}^2$,

$$M_0(x + n) = M_0(x) + M_0(n) \equiv M_0(x) \bmod 1$$

so $M_0$ induces a map on the torus phase space $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, see figure 2.1. This map is Anosov (uniformly hyperbolic), with strong chaotic properties, such as ergodicity and mixing, see [45] p. 154.
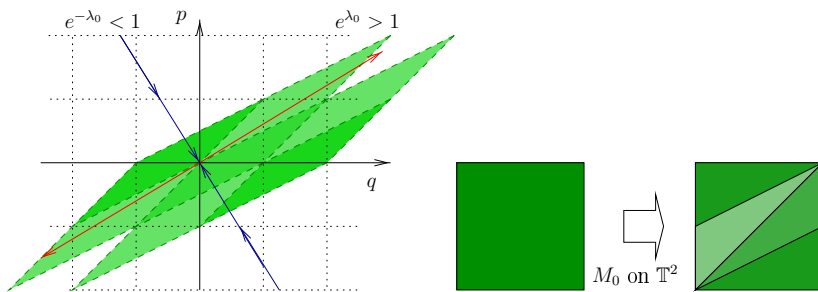


Figure 2.1. Dynamics of $M_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL\,(2,\mathbb{Z})$ on $\mathbb{R}^2$ and on $\mathbb{T}^2$.

### 2.1.2. Hamiltonian perturbation

We introduce a non linear perturbation of the previous map. Consider a $C^\infty$ function on the torus $H_1 : \mathbb{T}^2 \to \mathbb{R}$ (i.e. $H_1$ is a periodic function on $\mathbb{R}^2$). Let $M_1$ be the map on $\mathbb{R}^2$ given by the Hamiltonian flow generated from $H_1$ after time 1. We compose the two maps and define

$$(2.3) \qquad \boxed{M \overset{\text{def}}{=} M_1.M_0}$$

Which induces a map on $\mathbb{T}^2$ also denoted by $M$.

*Remarks.* —

    (1) The final dynamics $M^t : \mathbb{R}^2 \to \mathbb{R}^2$, $t \in \mathbb{Z}$, might be seen as generated by a time dependant Hamiltonian $H\,(x,t)$ (periodic in time $t \in \mathbb{R}$ but discontinuous).

    (2) There is a useful relation:

$$(2.4) \qquad M\,(x+n) = M\,(x) + M_0\,(n), \qquad \forall x \in \mathbb{R}^2, n \in \mathbb{Z}^2.$$

*Proof.* — From $M_0\,(x+n) = M_0\,(x) + M_0\,(n)$, and $M_1\,(x+n) = M_1\,(x) + n$, one deduces $M\,(x+n) = M_1 M_0\,(x+n) = M_1\big(M_0\,(x) + M_0\,(n)\big) = M\,(x) + M_0\,(n)$, because $M_0\,(n) \in \mathbb{Z}^2$. $\qquad\qquad\square$

### 2.1.3. Hyperbolic dynamics and structural stability

**Structural stability theorem:** The structural stability theorem states that if the perturbation $H_1$ is small enough with respect to the $C^2$ norm, then $M$ is an hyperbolic map on $\mathbb{T}^2$ conjugated to $M_0$ by a Hölder continuous map $\mathbf{H}$ ([3],[45] p. 89):

THEOREM 2.1. — *There exists $\varepsilon > 0$, such that if $\|H_1\|_{C^2} < \varepsilon$, then*

$$(2.5) \qquad M = \mathbf{H}.M_0.\mathbf{H}^{-1}$$

*with $\mathbf{H} : \mathbb{T}^2 \to \mathbb{T}^2$ Hölder continuous, and $M$ is uniformly hyperbolic in the following sense.*

*Let $DM_x : T_x\mathbb{T}^2 \to T_{M(x)}\mathbb{T}^2$ be the tangential map at $x \in \mathbb{T}^2$. For any $x \in \mathbb{T}^2$, there exists a frame of tangent vectors $(u_x, s_x) \in \big(T_x\mathbb{T}^2\big)$, called* **respectively unstable** *and* **stable directions**, *(chosen such that*

$$(2.6) \qquad u_x \wedge s_x = 1$$

*i.e. they form a symplectic basis. We can also impose that $\|u_x\| = \|s_x\|$, in order to fix the choice) and an* **expansion rate** *$\lambda_x$ such that:*

$$(2.7) \qquad DM_x\left(u_x\right) = e^{\lambda_x} u_{M(x)}, \quad DM_x\left(s_x\right) = e^{-\lambda_x} s_{M(x)},$$

*and more important: $u_x, s_x, \lambda_x$ are Hölder continuous functions of $x \in \mathbb{T}^2$. For any $x \in \mathbb{T}^2$,*

$$(2.8) \quad 0 < \lambda_{min} \leqslant \lambda_x \leqslant \lambda_{max}, \quad \text{with } \lambda_{min} \stackrel{\text{def}}{=} \min_{x \in \mathbb{T}^2} \lambda_x, \quad \lambda_{max} \stackrel{\text{def}}{=} \max_{x \in \mathbb{T}^2} \lambda_x.$$

In all the paper, we will suppose the perturbation to be weak enough so that the previous theorem holds. See Figure 2.2.

*Remarks.* —

- Of course, for a null perturbation $H_1 = 0$, $M = M_0$, then $\lambda_x = \lambda_0$ for every $x$.
- The conjugation Eq.(2.5) between $M$ and $M_0$ is very useful for topological purpose. For example, it implies that the periodic trajectories of $M$ (and more generally all the symbolic dynamics) are in one-to-one correspondence with those of $M_0$. As a consequence, $M$ has also strong chaotic properties such as topological mixing. However some more refined characteristic quantities of the dynamical systems $M_0$ and $M$ are different. For example the Lyapounov coefficients of corresponding trajectories of $M$ and $M_0$ are different, and this is related to the fact that the transformation map $\mathbf{H}$ is not $C^1$, and does not conserve area. To understand the quantum

(or semi-classical) dynamics of $M$, the equivalence Eq.(2.5) is not strong enough, because in semiclassical analysis, we need symplectic conjugations (here, $\mathbf{H}$ does not conserve area and can not be quantized).

• In [40], it is shown that $u_x, s_x, \lambda_x$ are $C^{2-\delta}$ functions of $x$, but we will not use it.

• For practical purpose, we introduce $Q_x \in SL(2, \mathbb{R})$, $x \in \mathbb{R}^2$, the symplectic matrix which transforms the canonical basis $(e_q, e_p)$ of $\mathbb{R}^2$ to the basis $(u_x, s_x)$:

$$(2.9) \qquad Q_x = \begin{pmatrix} (u_x)_q & (s_x)_q \\ (u_x)_p & (s_x)_p \end{pmatrix}.$$

$Q_x$ depends continuously on $x \in \mathbb{T}^2$. We can write:

$$(2.10) \qquad DM_x = Q_{Mx} \begin{pmatrix} \exp(\lambda_x) & 0 \\ 0 & \exp(-\lambda_x) \end{pmatrix} Q_x^{-1}$$

• **Explicit expression of $u_x$ and $s_x$:** Let $u_0 \in \mathbb{R}^2$ be any given vector, and denote $[u_0] \in P(\mathbb{R}^2)$ its direction in the projective space and $[DM_x]$ the action of $DM_x$ in the projective space. Then the direction of $u_x$ is given by

$$[u_x] = \lim_{t \to -\infty} \left[ DM^t_{M^{-t}(x)} \right] [u_0]$$

which converges for almost every $u_0$. Similarly the stable direction is given by

$$[s_x] = \lim_{t \to +\infty} \left[ DM^{-t}_{M^t(x)} \right] [s_0]$$

where $s_0$ is a given vector.

### 2.1.4. Example for numerical illustrations

In this paper we will illustrate the results by numerical calculations in a specific example. The linear non perturbed map is:

$$(2.11) \qquad M_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

with Lyapounov coefficient $\lambda_0$, given by $e^{\lambda_0} = \frac{3+\sqrt{5}}{2} = 2,62\ldots$. The perturbation $M_1$ is generated by the Hamiltonian

$$H_1(q, p) = a \cos(2\pi q), \qquad a = 0.01$$

giving a hyperbolic map on $\mathbb{T}^2$

(2.12) $$M \overset{\text{def}}{=} M_1 M_0$$

Figure 2.2 shows the torus phase space foliated by stable and unstable manifolds.
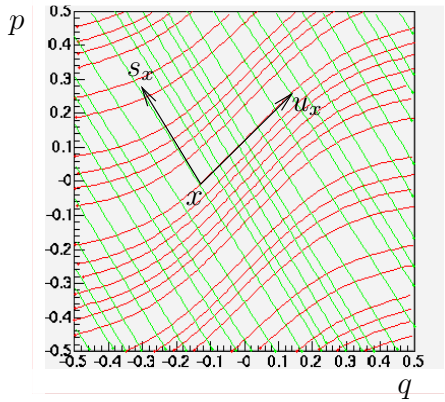


Figure 2.2. *Part of Stable and unstable manifolds of the Anosov map M, Eq.(2.12), passing through the origin* $(0,0)$. *At every point* $x \in \mathbb{T}^2$, $u_x, s_x$ *are vectors tangent to the foliation.*

## 2.2. Quantum mechanics

### 2.2.1. Quantum mechanics on the plane

The Hilbert space associated to the plane phase space $\mathbb{R}^2$ is $\mathcal{H}_{plane} \overset{\text{def}}{=} L^2(\mathbb{R})$. The Hamiltonian operator $\hat{H}_0$ is obtained by Weyl quantization of $H_0$, Eq.(2.1):

$$\hat{H}_0 = Op_{Weyl}(H_0) = \frac{\alpha}{2}\hat{q}^2 + \frac{\beta}{2}\hat{p}^2 + \frac{\gamma}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}),$$

Where the operators are defined by $(\hat{q}\varphi)(q) \overset{\text{def}}{=} q\varphi(q)$ and $(\hat{p}\varphi)(q) \overset{\text{def}}{=} -i\hbar\left(\frac{d\varphi}{dq}\right)(q)$, with $\varphi \in L^2(\mathbb{R})$ in suitable domains, and where the "Planck constant" $\hbar > 0$ has been introduced. The Schrödinger equation defines the evolution of $\varphi(t) \in \mathcal{H}_{plane} = L^2(\mathbb{R})$:

$$\frac{d\varphi(t)}{dt} = -\frac{i}{\hbar}\hat{H}_0\varphi(t),$$

and generates a unitary evolution operator between time $t = 0 \to 1$, written $\hat{M}_0$:

$$(2.13) \qquad \varphi(1) = \hat{M}_0 \varphi(0), \quad \hat{M}_0 = \exp\left(-\frac{i}{\hbar}\hat{H}_0\right).$$

Similarly, we define $\hat{H}_1 = Op_{Weyl}(H_1)$ to be the Weyl quantization[3] of $H_1$, and $\hat{M}_1 = \exp\left(-\frac{i}{\hbar}\hat{H}_1\right)$ the evolution operator after time 1. Finally the quantum map is

$$(2.15) \qquad \hat{M} = \hat{M}_1.\hat{M}_0 : \qquad \mathcal{H}_{plane} \to \mathcal{H}_{plane}$$

also written $\hat{M}_{plane}$ in the following, and not to be confused with the quantum map $\hat{M}_{torus}$ for the torus phase space introduced below.

**Unitary translation operators:** For $v = (v_1, v_2) \in \mathbb{R}^2$, let

$$(2.16) \qquad T_v : \begin{cases} \mathbb{R}^2 & \to \mathbb{R}^2 \\ x & \mapsto T_v(x) = x + v \end{cases}$$

be the translation on classical phase space. This translation can be expressed as the flow of the Hamiltonian function $f(q, p) = (v_1 p - v_2 q)$ after time 1. The corresponding quantum translation operator is defined by:

$$(2.17) \qquad \hat{T}_v \overset{\text{def}}{=} \exp\left(-\frac{i}{\hbar}(v_1 \hat{p} - v_2 \hat{q})\right).$$

These quantum translations satisfy the algebraic relation of the Weyl-Heisenberg group[28][51]:

$$(2.18) \qquad \hat{T}_v \hat{T}_{v'} = e^{iS/\hbar} \hat{T}_{v+v'},$$

with $S = \frac{1}{2}(v_2 v_1' - v_1 v_2') = -\frac{1}{2} v \wedge v'$. For any matrix $M_0 \in \mathrm{SL}(2, \mathbb{R})$, one trivially has $M_0(x + v) = M_0(x) + M_0(v)$ which rewrites $M_0 T_v = T_{M_0 v} M_0$. This intertwining persists at the quantum level:

$$(2.19) \qquad \hat{M}_0 \hat{T}_v = \hat{T}_{M_0 v} \hat{M}_0.$$

---

[3] Explicitly, if $H_1$ is expanded in Fourier series:

$$H_1(q, p) = \sum_{n=(n_1, n_2) \in \mathbb{Z}^2} c_n e^{i2\pi(n_1 q + n_2 p)}$$

(with $c_{-n} = \overline{c_n}$, so that $H_1$ is real valued), then

$$(2.14) \qquad \hat{H}_1 = Op_{Weyl}(H_1) = \sum_{n=(n_1, n_2) \in \mathbb{Z}^2} c_n e^{i2\pi(n_1 \hat{q} + n_2 \hat{p})}$$

We deduce a relation for the non linear quantum map $\hat{M}$ analogous to the classical relation Eq.(2.4) :

$$(2.20) \qquad \hat{M}\hat{T}_n = \hat{T}_{M_0(n)}\hat{M}, \qquad \forall n \in \mathbb{Z}^2$$

*Proof.* — From Eq.(2.14) and Eq.(2.16), we can write

$$\hat{H}_1 = \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} c_n \hat{T}_{-2\pi\hbar n}.$$

From Eq.(2.18), one has $\left[\hat{T}_{-2\pi\hbar n}, \hat{T}_{n'}\right] = 0$. We deduce that $\left[\hat{T}_{n'}, \hat{H}_1\right] = 0$, $\left[\hat{T}_{n'}, \hat{M}_1\right] = 0$, for any $n' \in \mathbb{Z}^2$, which reflects the periodicity of the function $H_1$ at the quantum level. Then using Eq.(2.19), $\hat{M}\hat{T}_n = \hat{M}_1\hat{M}_0\hat{T}_n = \hat{M}_1\hat{T}_{M_0(n)}\hat{M}_0 = \hat{T}_{M_0(n)}\hat{M}$. $\qquad\square$

## 2.2.2. Quantum mechanics on the torus

At the classical level, the torus phase space was obtained by introducing periodicity on $\mathbb{R}^2$ with respect to the $\mathbb{Z}^2$ lattice, generated by translations $T_{(1,0)}$ and $T_{(0,1)}$. The same construction can be done in quantum mechanics. The difference is that we have to check that the quantum translation operators $\hat{T}_{(1,0)}$ and $\hat{T}_{(0,1)}$ commute before we consider their common eigenspaces. From Eq.(2.18), we have $\hat{T}_{(1,0)}\hat{T}_{(0,1)} = e^{-i/\hbar}\hat{T}_{(0,1)}\hat{T}_{(1,0)}$, so $\left[\hat{T}_{(1,0)}, \hat{T}_{(0,1)}\right] = 0$ if and only if $\hbar$ is such that

$$(2.21) \qquad \boxed{N \stackrel{\text{def}}{=} \frac{1}{2\pi\hbar} \in \mathbb{N}^*}$$

**We will suppose condition Eq.(2.21) from now on**. We define

$$\boxed{\mathcal{H}_{torus} \stackrel{\text{def}}{=} \left\{\varphi \in \mathcal{S}'(\mathbb{R}) \,/\, \hat{T}_{(1,0)}\varphi = \varphi, \quad \hat{T}_{(0,1)}\varphi = \varphi\right\}}$$

In order to have a concrete expression of a wave function $\varphi \in \mathcal{H}_{torus}{}^{(4)}$, remark that using $\hbar$-Fourier-Transform defined by

$$\tilde{\varphi}(p) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi\hbar}} \int dq\, \varphi(q)\, e^{-ipq/\hbar},$$

then $\varphi \in \mathcal{H}_{torus}$ is characterized by the periodicity conditions $\tilde{\varphi}(p+1) = \tilde{\varphi}(p)$ and $\varphi(q+1) = \varphi(q)$. The periodicity of $\tilde{\varphi}$ implies that $\varphi(q) = \sum_{n\in\mathbb{Z}} a_n \delta\left(q - \frac{n}{N}\right)$, with $a_n \in \mathbb{C}$. The periodicity of $\varphi(q)$ implies that

---

[4] $\mathcal{H}_{torus}$ is the well known space for Finite Fourier Transform.

$a_{n+N} = a_n$ for any $n$. So $\varphi \in \mathcal{H}_{torus}$ is specified by $(a_n)_{n=1 \to N} \in \mathbb{C}^N$. We deduce that:

$$\boxed{\dim\mathcal{H}_{torus} = N = \frac{1}{2\pi\hbar}.}$$

For simplicity we also assume, that $N$ **is even**, so that, using Eq.(2.18), $\hat{T}_{n=(n_1,n_2)} = \hat{T}_{(1,0)}^{n_1}\hat{T}_{(0,1)}^{n_2}$ for any $n \in \mathbb{Z}^2$. We define a onto projector $\hat{\mathcal{P}}$ : $\mathcal{H}_{plane} \to \mathcal{H}_{torus}$ (with a dense domain which contains $\mathcal{S}(\mathbb{R})$), which makes a quantum state periodic with respect to the lattice $\mathbb{Z}^2$ in phase space:

$$(2.22) \qquad \hat{\mathcal{P}} \overset{\text{def}}{=} \sum_{(n_1,n_2)\in\mathbb{Z}^2} \hat{T}_{(1,0)}^{n_1}\,\hat{T}_{(0,1)}^{n_2} = \sum_{n\in\mathbb{Z}^2} \hat{T}_n.$$

From Eq.(2.19), we deduce:

$$(2.23) \qquad\qquad \hat{M}\hat{\mathcal{P}} = \hat{\mathcal{P}}\hat{M}$$

In other words we have a commutative diagram:

$$\begin{array}{ccc}
\mathcal{H}_{plane} & \xrightarrow{\hat{M}} & \mathcal{H}_{plane} \\
\downarrow \hat{\mathcal{P}} & & \downarrow \hat{\mathcal{P}} \\
\mathcal{H}_{torus} & \xrightarrow{\hat{M}} & \mathcal{H}_{torus}
\end{array}$$

Which means that $\hat{M}$ induces an endomorphism:

$$(2.24) \qquad \boxed{\hat{M}_{torus} = \hat{M}_1.\hat{M}_0 : \ \mathcal{H}_{torus} \to \mathcal{H}_{torus}}$$

When no confusing is possible, we will write $\hat{M}$ for this operator $\hat{M}_{torus}$.

### 2.2.3. Standard coherent states

We will use in many instances some particular quantum states, the standard coherent states [51][62], which are semi-classically localized on phase space. For $x = (q,p) \in \mathbb{R}^2$, the coherent state $\varphi_x \in \mathcal{H}_{plane}$ is $\varphi_x(q') \overset{\text{def}}{=} \frac{1}{(\pi\hbar)^{1/4}} \exp\left(-i\frac{pq}{2\hbar}\right) \exp\left(i\frac{pq'}{\hbar}\right) \exp\left(-\frac{(q'-q)^2}{2(\sqrt{\hbar})^2}\right)$. Semi-classical localization of $\varphi_x$ at $x \in \mathbb{R}^2$ in phase space comes from the fact that its modulus is a Gaussian "localized around" $q$ with width $\Delta q \simeq \sqrt{\hbar}$ (which vanishes for $\hbar \to 0$). Its $\hbar$-Fourier Transform is

$$\tilde{\varphi}_x(p') = \frac{1}{(\pi\hbar)^{1/4}} \exp\left(-i\frac{pq}{2\hbar}\right) \exp\left(-i\frac{qp'}{\hbar}\right) \exp\left(-\frac{(p'-p)^2}{2\left(\sqrt{\hbar}\right)^2}\right),$$

similarly localized around $p$. More algebraically, $\varphi_0 \in \mathcal{H}_{plane}$ (with $x = 0$), is the ground state of the Harmonic Oscillator and is defined by $a\varphi_0 =$

0, with $a = (\hat{q} + i\hat{p}) / \sqrt{2\hbar}$. The coherent state $\varphi_x$ is then obtained by translation

$$(2.25) \qquad \varphi_x \overset{\text{def}}{=} \hat{T}_x \varphi_0, \quad x = (q, p) \in \mathbb{R}^2.$$

For short, we will also write $|x\rangle \overset{\text{def}}{=} \varphi_x$ for a coherent state. The **Husimi distribution** of a quantum state $\psi \in \mathcal{H}_{plane}$ is the positive measure on phase space:

$$\text{Hus}_\psi (x) \overset{\text{def}}{=} \frac{1}{2\pi\hbar} |\langle x | \psi \rangle|^2$$

The closure relation is ([51] p. 15)

$$(2.26) \qquad \hat{\text{Id}}_{/\mathcal{H}_{plane}} = \int_{\mathbb{R}^2} \frac{dx}{2\pi\hbar} |x\rangle\langle x|$$

A **coherent state on the torus** is defined by periodicity, using Eq. (2.22):

$$|x\rangle_{torus} \overset{\text{def}}{=} \hat{\mathcal{P}}|x\rangle \in \mathcal{H}_{torus}$$

They provide the closure relation:

$$(2.27) \qquad \hat{\text{Id}}_{/\mathcal{H}_{torus}} = \int_{[0,1]^2} \frac{dx}{2\pi\hbar} |x\rangle_{torus}\langle x|$$

## 3. The map as a sum over periodic points

In the first part of this section we characterize and count the number of periodic points of the map $M$ with a given period $t \in \mathbb{Z}$. These points are shown to play an important role in the second part, where we will express $\hat{M}_{torus}^t$, defined in Eq.(2.24), and its trace in terms of them. Some parts of this section can be found in [46] or [52].

### 3.1. Periodic points of $M$

Consider a discrete time $t \in \mathbb{Z}\backslash\{0\}$. A point $x \in \mathbb{R}^2$ gives a periodic point $[x] \in \mathbb{T}^2$ of $M$ with period $t$ if

$$
\begin{aligned}
M^t [x] &= [x] \\
&\Leftrightarrow &\exists n \in \mathbb{Z}^2 \, / \, M^t x = x + n \\
&\Leftrightarrow &\exists n \in \mathbb{Z}^2 \, / \, \mathcal{T}_t (x) = n
\end{aligned}
$$

with the map

$$
\mathcal{T}_t \overset{\text{def}}{=} \left\{
\begin{array}{ccc}
\mathbb{R}^2 & \to & \mathbb{R}^2 \\
x & \to & \mathcal{T}_t (x) = M^t (x) - x
\end{array}
\right.
$$

**Periodicity of $\mathcal{T}_t$:** From Eq.(2.4), we get

$$\mathcal{T}_t\left(x+m\right) = \mathcal{T}_t\left(x\right) + \mathcal{T}_{0,(t)}\left(m\right), \qquad \forall x \in \mathbb{R}^2, \ \forall m \in \mathbb{Z}^2$$

Where $\mathcal{T}_{0,(t)}\left(x\right) \stackrel{\text{def}}{=} M_0^t\left(x\right) - x$. In particular $\mathcal{T}_{0,(t)}\left(m\right) \in \mathbb{Z}^2$ for any $m \in \mathbb{Z}^2$. We introduce the sub-lattice of $\mathbb{Z}^2$ :

(3.1) $$\Lambda_t \stackrel{\text{def}}{=} \mathcal{T}_{0,(t)}\left(\mathbb{Z}^2\right) \subset \mathbb{Z}^2$$

generated by $e_1 = \mathcal{T}_{0,(t)}\left(1,0\right) \in \mathbb{Z}^2$ and $e_2 = \mathcal{T}_{0,(t)}\left(0,1\right) \in \mathbb{Z}^2$. We denote by $\mathcal{C}_t \subset \mathbb{Z}^2$ the elements which belong to the origin cell of $\Lambda_t$ :

$$\mathcal{C}_t \stackrel{\text{def}}{=} \left\{ n \in \mathbb{Z}^2 \ \ / \ \ \mathcal{T}_{0,(t)}^{-1}\left(n\right) \in [0,1]^2 \right\}$$

Of course $\mathcal{C}_t \equiv \mathbb{Z}^2/\Lambda_t$.

PROPOSITION 3.1. — *For a small enough perturbation, $\mathcal{T}_t$ is a diffeomorphism, for every $t$. Periodic points with period $t$ can be labeled by $n \in \mathbb{Z}^2$:*

(3.2) $$x_{n,t} \stackrel{\text{def}}{=} \mathcal{T}_t^{-1}\left(n\right), \qquad n \in \mathbb{Z}^2$$

*Different values $n, n'$ may give the same point $[x_{n,t}] = [x_{n',t}]$. The periodic points $[x_{n,t}] \in \mathbb{T}^2$ on the torus are in one to one correspondence with $n \in \mathcal{C}_t$. The number of periodic points with period $t$ is*

(3.3) $$\mathcal{N}_t \stackrel{\text{def}}{=} \sharp\left(\mathcal{C}_t\right) \ = \left|\det\left(\mathcal{T}_{0,(t)}\right)\right| = \left|\det\left(M_0^t - I\right)\right|$$
$$= \quad e^{\lambda_0 t} - 2 + e^{-\lambda_0 t} \qquad \left(\simeq e^{\lambda_0 t} \text{ for } t \gg 1\right)$$

*Proof.* — Using Eq.(2.10), the matrix of $\left(D\mathcal{T}_t\right)_x$ is

$$\left(D\mathcal{T}_t\right)_x = \left(DM^t\right)_x - Id = Q_{M^t x} \begin{pmatrix} \exp\left(\Lambda_{x,t}\right) & 0 \\ 0 & \exp\left(-\Lambda_{x,t}\right) \end{pmatrix} Q_x^{-1} - Id$$

with $\Lambda_{x,t} = \sum_{t'=0}^{t-1} \lambda_{M^{t'} x}$ , so $\lambda_{min} t \leqslant \Lambda_{x,t} \leqslant \lambda_{max} t$. We have supposed that $\lambda_{min} > 0$. Then

$$\det\left(\left(D\mathcal{T}_t\right)_x\right) = \det\left(\begin{pmatrix} \exp\left(\Lambda_{x,t}\right) & 0 \\ 0 & \exp\left(-\Lambda_{x,t}\right) \end{pmatrix} - Q_{M^t x}^{-1} Q_x\right)$$

The matrix $Q_{x'}^{-1} Q_x$ goes to identity (uniformly in $x, x' \in \mathbb{T}^2$) when the perturbation vanishes. Therefore, we can write $Q_{x'}^{-1} Q_x = Id + \varepsilon A_{x,x'}$, where $A_{x,x'}$ has matrix elements bounded by 1 in absolute value, and $\varepsilon$ goes to zero when the perturbation vanishes. One computes

$$\det\left(\left(D\mathcal{T}_t\right)_x\right) = 2\left(1 - \cosh\Lambda_{x,t}\right) + \varepsilon A_{11}\left(e^{-\Lambda_{x,t}} - 1\right)$$
$$+ \varepsilon A_{22}\left(e^{\Lambda_{x,t}} - 1\right) + \varepsilon^2 \det\left(A\right)$$

and deduces

$$|\det((D\mathcal{T}_t)_x)| > 2(1-\varepsilon)(\cosh(\lambda_{min}t) - 1) - 3\varepsilon > 0$$

for small enough $\varepsilon$. We deduce that $\mathcal{T}_t$ is a diffeomorphism on $\mathbb{R}^2$.     □

*Remarks. —*

- Note that two periodic points $x_{n',t}$, $x_{n,t}$ may belong to the same periodic orbit of $M_{torus}$. Explicitely:

$$x_{n',t} = M(x_{n,t}) \Leftrightarrow n' = M_0(n)$$

  so periodic orbits of $M$ on $\mathbb{T}^2$ with period $t$, are in one to one correspondence with periodic orbits of $M_0$ acting on the finite set $\mathcal{C}_t \equiv \mathbb{Z}^2/\Lambda_t$.
- Here is the number of periodic points with period $t$, for the example Eq. (2.11), see figure 3.1:

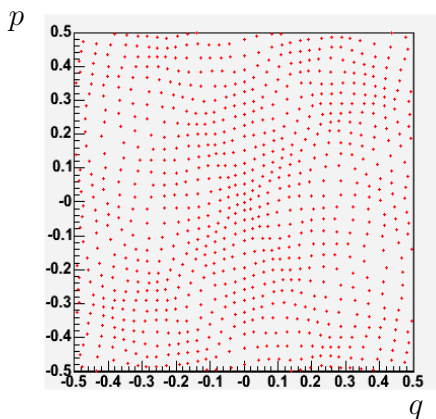| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | ... |
|-----|---|---|----|----|-----|-----|-----|------|------|-------|-------|-----|
| $\mathcal{N}_t$ | 1 | 5 | 16 | 45 | 121 | 320 | 841 | 2205 | 5776 | 15125 | 39601 | ... |



Figure 3.1. *Periodic points of map $M$, Eq. (2.12), with period $t = 7$. There are $\mathcal{N}_t = 841$ periodic points.*

## 3.2. Expression of the quantum map $\hat{M}_{torus}^t$, using periodic points

For a given $t \in \mathbb{Z}$, we consider

$$\left(\hat{\mathcal{P}}\hat{M}_{plane}^t\right) : \mathcal{H}_{plane} \to \mathcal{H}_{torus}$$

defined on a suitable dense domain (which contains the Schwartz space $\mathcal{S}(\mathbb{R})$), where $\hat{\mathcal{P}}$ is defined in Eq.(2.22), and $\hat{M}_{plane}$ is defined in Eq.(2.15). This operator $\left(\hat{\mathcal{P}}\hat{M}_{plane}^t\right)$ is important to look at, because if one wants any matrix element of the quantum map $\langle\tilde{\psi}_2|\hat{M}_{torus}^t|\tilde{\psi}_1\rangle$, with $|\tilde{\psi}_1\rangle, |\tilde{\psi}_2\rangle \in \mathcal{H}_{torus}$, then one just has to consider "lifted states on the plane" $|\psi_1\rangle$, $|\psi_2\rangle \in \mathcal{H}_{plane}$, such that $|\tilde{\psi}_i\rangle = \hat{\mathcal{P}}|\psi_i\rangle$, $i = 1, 2$, and then

$$(3.4) \qquad \langle\tilde{\psi}_2|\hat{M}_{torus}^t|\tilde{\psi}_1\rangle = \langle\psi_2|\hat{\mathcal{P}}\hat{M}_{plane}^t|\psi_1\rangle$$

We will now write $\hat{M}$ instead of $\hat{M}_{plane}$. One writes:

$$(3.5) \qquad \boxed{\left(\hat{\mathcal{P}}\hat{M}^t\right) = \sum_{n\in\mathbb{Z}^2} \hat{T}_{-n}\hat{M}^t = \sum_{n\in\mathbb{Z}^2} \hat{M}_{n,t}}$$

with

$$(3.6) \qquad \boxed{\hat{M}_{n,t} \stackrel{\text{def}}{=} \hat{T}_{-n}\hat{M}^t \qquad : \mathcal{H}_{plane} \to \mathcal{H}_{plane}}$$

where $n \in \mathbb{Z}^2$, and $t \in \mathbb{Z}$. The corresponding classical map is of course

$$(3.7) \qquad M_{n,t} : \begin{cases} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ x & \to & M_{n,t}(x) = M^t(x) - n \end{cases}$$

The map $M_{n,t}$ will be used many times in this paper. It is an hyperbolic map (translation of $M^t$), whose unique fixed point is the periodic point $x_{n,t}$, given by Eq.(3.2), because: $M_{n,t}(x_{n,t}) = M^t(x_{n,t}) - n = x_{n,t}$.

### 3.2.1. Periodicity of the decomposition

We decompose now the sum over $n \in \mathbb{Z}^2$ in Eq. (3.5), with respect to the lattice $\Lambda_t$ defined by Eq. (3.1). First, any $n' \in \mathbb{Z}^2$ can be decomposed in the unique way:

$$n' = n + \mathcal{T}_0(m), \qquad n \in \mathcal{C}_t, \quad m \in \mathbb{Z}^2$$

Observe that for $n, m \in \mathbb{Z}^2$ (we use Eq.(2.20)),

$$\begin{aligned} \hat{M}_{n+\mathcal{T}_0(m),t} &= \hat{T}_{-n-\mathcal{T}_0(m)}\hat{M}^t = \hat{T}_{-n-M_0(m)+m}\hat{M}^t \\ &= e^{i\mathcal{F}(n,m)/\hbar}\hat{T}_m\hat{T}_{-n}\hat{T}_{-M_0(m)}\hat{M}^t = e^{i\mathcal{F}(n,m)/\hbar}\hat{T}_m\hat{T}_{-n}\hat{M}^t\hat{T}_{-m} \\ &= e^{i\mathcal{F}(n,m)/\hbar}\hat{T}_m\hat{M}_{n,t}\hat{T}_{-m} \end{aligned}$$

The phase $\mathcal{F}$ comes from Eq.(2.18) and is given by:

$$\mathcal{F}(n,m) = \frac{1}{2}\left(n \wedge M_0(m) - m \wedge (n + M_0(m))\right)$$

But $2\mathcal{F}$ is an integer so $e^{i\mathcal{F}(n,m)/\hbar} = e^{i2\pi N\mathcal{F}(n,m)} = +1$ because we have supposed $N$ even. Therefore

(3.8)
$$\left(\hat{\mathcal{P}}\hat{M}^t\right) = \sum_{n\in\mathbb{Z}^2} \hat{M}_{n,t} = \sum_{m\in\mathbb{Z}^2} \hat{T}_m \hat{M}_{\mathcal{C}_t} \hat{T}_{-m}$$

with

$$\hat{M}_{\mathcal{C}_t} \stackrel{\text{def}}{=} \sum_{n\in\mathcal{C}_t} \hat{M}_{n,t}$$

From Proposition 3.1, this last expression of $\hat{M}_{\mathcal{C}_t}$ is a finite sum over periodic points, with $\mathcal{N}_t$ terms.

### 3.2.2. Trace of $\hat{M}_{torus}^t$

Remark that $\text{Tr}\left(\hat{M}_{torus}^t\right)$ is well defined because $\hat{M}_{torus}^t$ acts in $\mathcal{H}_{torus}$ which is a finite dimensional space. On the opposite $\hat{M}_{n,t} = \hat{T}_{-n}\hat{M}^t$ is a unitary operator in $\mathcal{H}_{plane} = L^2(\mathbb{R})$ and is not of Trace class. Nevertheless we will define a linear functional which can be thought as a trace.

PROPOSITION 3.2. — *For any $t \in \mathbb{Z}$, and any $n \in \mathbb{Z}^2$, the following integral is absolutely convergent:*

(3.9)
$$T\left(\hat{M}_{n,t}\right) \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \frac{dx}{2\pi\hbar} \langle x|\hat{M}_{n,t}|x\rangle$$

*where $|x\rangle$ is a standard coherent state at $x \in \mathbb{R}^2$ defined in Eq.(2.25). We have the relation*

(3.10)
$$\text{Tr}\left(\hat{M}_{torus}^t\right) = \sum_{n\in\mathcal{C}_t} T\left(\hat{M}_{n,t}\right)$$

Eq.(3.10) is an exact formula (not semi-classical). It is a sum over $\mathcal{N}_t$ terms. This formula does not use the assumption that $M$ is hyperbolic.

*Proof.* — A coherent state $|x\rangle$ belongs to $\mathcal{S}(\mathbb{R})$. Using closure relations Eq.(2.26) and Eq.(2.27), together with Eq.(3.4) and Eq.(3.8), we compute:

$$\mathrm{Tr}\left(\hat{M}^t_{torus}\right) = \int_{[0,1]^2} \frac{dx}{2\pi\hbar} \langle x|\hat{M}^t_{torus}|x\rangle_{torus} = \int_{[0,1]^2} \frac{dx}{2\pi\hbar} \langle x|\hat{\mathcal{P}}\hat{M}^t_{plane}|x\rangle$$

$$= \int_{[0,1]^2} \frac{dx}{2\pi\hbar} \sum_{n\in\mathcal{C}_t} \sum_{m\in\mathbb{Z}^2} \langle x|\hat{T}_m\hat{M}_{n,t}\hat{T}_{-m}|x\rangle$$

$$= \sum_{n\in\mathcal{C}_t} \int_{[0,1]^2} \sum_{m\in\mathbb{Z}^2} \frac{dx}{2\pi\hbar} \langle x-m|\hat{M}_{n,t}|x-m\rangle$$

The sums are absolutely convergent. In particular:

$$\int_{[0,1]^2} \sum_{m\in\mathbb{Z}^2} \frac{dx}{2\pi\hbar} \langle x-m|\hat{M}_{n,t}|x-m\rangle = \int_{\mathbb{R}^2} \frac{dx}{2\pi\hbar} \langle x|\hat{M}_{n,t}|x\rangle = \mathrm{T}\left(\hat{M}_{n,t}\right),$$

and we obtain $\mathrm{Tr}\left(\hat{M}^t_{torus}\right) = \sum_{n\in\mathcal{C}_t} \mathrm{T}\left(\hat{M}_{n,t}\right)$.                     $\square$

# 4. Semi-classical description of the dynamics

We do not have yet considered the semi-classical limit $\hbar \to 0$. In this section, we give the essential Theorem (Theorem 4.2 below) which shows that in the semi-classical limit, long time quantum dynamics can be expressed in terms of individual classical trajectories.

## 4.1. Neighborhood of a point in phase space and localized states

We first introduce the notion of a semi-classical neighborhood of a point in phase space. Let $x \in \mathbb{R}^2$ be a point in phase space, and $0 < \alpha < 1/2$. Let

$$(4.1) \qquad D_{x,\alpha} \overset{\text{def}}{=} \left\{ y \in \mathbb{R}^2 \ / \ |y - x| < \hbar^{1/2-\alpha} \right\}$$

be the disk of center $x$ and radius $\hbar^{1/2-\alpha}$, which shrinks to zero as $\hbar \to 0$. We define the **truncation operator**:

$$(4.2) \qquad \boxed{\hat{P}_{x,\alpha} \overset{\text{def}}{=} \int_{y\in D_{x,\alpha}} \frac{dy}{2\pi\hbar} |y\rangle\langle y| = Op_{AW}\left(\chi_{D_{x,\alpha}}\right)}$$

being the Anti-Wick quantization of the characteristic function of the disk $D_{x,\alpha}$ [51]. We will often drop the index $\alpha$, and write $\hat{P}_x \overset{\text{def}}{=} \hat{P}_{x,\alpha}$. In the case $x = 0$, we will drop the index $x$, and write: $\hat{P}_\alpha \overset{\text{def}}{=} \hat{P}_{x=0,\alpha}$. Notice that

in comparison with Eq.(2.26), the integral is restricted to the disk $D_{x,\alpha}$. The meaning of the operator $\hat{P}_x$ is that it truncates a quantum states in the vicinity of the point $x$. A quantum state is said to be localized at point $x$, if this truncation has no effect on it in the semi-classical limit. More precisely:

DEFINITION 4.1. — *A sequence of normalized quantum state $\psi_\hbar \in \mathcal{H}_{plane}$ (sequence which depends on $\hbar \to 0$) is said to be $\alpha-$**localized at point** $x \in \mathbb{R}^2$, if*

(4.3) $$\left\| \psi - \hat{P}_{x,\alpha} \psi \right\|_{L^2} = \mathcal{O}\left(\hbar^\infty\right)$$

Examples: a coherent state $|x\rangle$ is $\alpha-$localized at $x$, for any $0 < \alpha < 1/2$. For any *fixed* $n \in \mathbb{N}$, let $|n\rangle$ be the eigenstate of the harmonic oscillator: $\frac{1}{2}\left(\hat{p}^2 + \hat{q}^2\right)|n\rangle = \hbar\left(n + \frac{1}{2}\right)|n\rangle$, and let $|x,n\rangle = \hat{T}_x|n\rangle$. Then $|x,n\rangle$ is $\alpha-$localized at $x$, for any $0 < \alpha < 1/2$.

## 4.2. Semi-classical evolution in a neighborhood of a classical trajectory

Let $x \in \mathbb{R}^2$, and consider the classical trajectory $x(t) = M^t x \in \mathbb{R}^2$, for $t \in \mathbb{N}$. The following Theorem will be useful in order to compute matrix elements of the quantum evolution operator, of the form $\langle \psi_2 | \hat{M}^t | \psi_1 \rangle$, where $\psi_1$ is localized at $x(0)$, and $\psi_2$ is localized at $x(t)$. From Eq.(4.3), $\langle \psi_2 | \hat{M}^t | \psi_1 \rangle = \langle \psi_2 | \hat{P}_{x(t)} \hat{M}^t \hat{P}_{x(0)} | \psi_1 \rangle + \mathcal{O}\left(\hbar^\infty\right)$, so the computation involves the operator $\hat{P}_{x(t)} \hat{M}^t \hat{P}_{x(0)}$.

THEOREM 4.2. — *For any $C > 0$, any $t$, such that $1 \leqslant t < C \log\left(1/\hbar\right)$, any $0 < \alpha < 1/2$, any $K > 0$, there exists $C_K > 0$, such that for any $x = x(0) \in \mathbb{R}^2$, then in $L^2$ operator norm:*

(4.4) $$\left\| \hat{P}_{x(t)} \hat{M}^t \hat{P}_{x(0)} - \hat{P}_{x(t)} \hat{M} \hat{P}_{x(t-1)} \hat{M} \hat{P}_{x(t-2)} \ldots \hat{M} \hat{P}_{x(0)} \right\|_{L^2} \leqslant C_K \hbar^K$$

The proof is given in appendix A page 2564, but we give the main idea below.

*Remarks.* —

- To shorten we write that the error is $\mathcal{O}\left(\hbar^\infty\right)$ uniformly with respect to $x \in \mathbb{R}^2$.
- This relation means that in order to compute $\hat{P}_{x(t)} \hat{M}^t \hat{P}_{x(0)}$, we can introduce truncation operators all along the trajectory, without changing the result significantly in the semi-classical limit. In

other words the operator $\hat{P}_{x(t)}\hat{M}^t\hat{P}_{x(0)}$ depends only on the dynamics in the vicinity of the trajectory $x(0) \to x(t)$. This result will allow us to use normal forms in the next Section, which give a nice description of the dynamics in the vicinity of any trajectory.

- **Idea of the proof:** Let us explain here the main idea of the proof but at the level of classical dynamics. The proof follows the same idea. From definition Eq.(4.2), the operator $\hat{P}_{x(t)} = \int_{x \in D_t} \frac{dx}{2\pi\hbar}|x\rangle\langle x|$ truncates quantum states on the disk $D_t$ of small radius $\hbar^{1/2-\alpha}$, and center $x(t)$. The transcription of Eq.(4.4) in classical dynamics is

$$(4.5) \qquad D_t \cap M^t(D_0) = D_t \cap M\left(D_{t-1} \ldots M\left(D_1 \cap M\left(D_0\right)\right)\right)$$

To show this, let $G_1 = M(D_0) \setminus D_1$. Let $F_1$ such that $\mathbb{R}^2 = D_1 \cup F_1 \cup G_1$ is a disjoint union. So $F_1 \cap M(D_0) = \emptyset$. Then

$$(4.6) \quad \begin{aligned} D_t \cap M^t(D_0) &= D_t \cap M^{t-1}\left(\mathbb{R}^2 \cap M(D_0)\right) \\ &= D_t \cap M^{t-1}\left(\left(D_1 \cup F_1 \cup G_1\right) \cap M(D_0)\right) \\ &= \left(D_t \cap M^{t-1}\left(D_1 \cap M(D_0)\right)\right) \cup \left(D_t \cap M^{t-1}(G_1)\right) \end{aligned}$$

Now observe that $G_1$ is the set of points coming from $D_0$ but who leave the set $D_1$ in the unstable direction. Due to uniform hyperbolicity and the fact that we consider the dynamics on the cover $\mathbb{R}^2$, the set $G_1$ goes away from the trajectory $x(t)$ in the unstable direction. This gives: $D_t \cap M^{t-1}(G_1) = \emptyset$. Then Eq.(4.6) simplifies to $D_t \cap M^t(D_0) = D_t \cap M^{t-1}(D_1 \cap M(D_0))$. Repeating this argument, we get Eq. (4.5). (For illustration, see Figure A.2 page 2568).

- From the idea of the proof given above, it is clear that Eq.(4.4) holds because it concerns the phase space cover $\mathbb{R}^2$. Points who leave the trajectory in the unstable direction never come back. At the semi-classical level, there is no interference effects. The interference effects come when passing to the torus which is compact, and are due to the sum Eq.(3.5).

**Consequence for the trace.** There is a consequence of Theorem 4.2, which shows that the integral Eq.(3.9) up to a negligible error, can be restricted to a neighborhood of the fixed point $x_{n,t}$ of the map $M_{n,t}$, Eq.(3.7). This Lemma will be useful in order to obtain the semiclassical Trace formula.

LEMMA 4.3. — *For any $C > 0$, any $t$, such that $1 \leqslant t < C \log{(1/\hbar)}$, any $0 < \alpha < 1/2$:*

(4.7)
$$T\left(\hat{M}_{n,t}\right) = \int_{x \in D_{x_{n,t}}} \frac{dx}{2\pi\hbar} \langle x|\hat{M}_{n,t}|x\rangle + \mathcal{O}\left(\hbar^{\infty}\right)$$

$$= \mathrm{Tr}\left(\hat{M}_{n,t}\hat{P}_{x_{n,t}}\right) + \mathcal{O}\left(\hbar^{\infty}\right) = \mathrm{Tr}\left(\hat{P}_{x_{n,t}}\hat{M}_{n,t}\hat{P}_{x_{n,t}}\right) + \mathcal{O}\left(\hbar^{\infty}\right)$$

*The error is uniform with respect to the point $x_{n,t}$ (i.e. with respect to $t$ and $n \in \mathbb{Z}^2$).*

Note that $\hat{P}_x$ is a trace class operator, $\hat{M}_{n,t}$ is bounded, so $\left(\hat{M}_{n,t}\hat{P}_{x_{n,t}}\right)$ is also trace class. Although intuitive, our proof of Lemma 4.3 is quite long and is given in Appendix B.

## 5. Semi-classical non-stationary Normal Form

In the previous Section, we have explained that long time quantum dynamics can be expressed with the operator $\hat{P}_{x(t)}\hat{M}^t\hat{P}_{x(0)}$, where $x(t) = M^t x(0)$ is a given classical trajectory. Using truncation operators all along the trajectory, we have shown in Theorem 4.2, that the operator $\hat{P}_{x(t)}\hat{M}^t\hat{P}_{x(0)}$ depends in fact only of the vicinity of the trajectory $x(t')$, $t' \in [0,t]$. There appeared operators like $\hat{P}_{M(x)}\hat{M}\hat{P}_x$, with $x \in \mathbb{R}^2$. This suggests to use a local description of the dynamics along the trajectory, in terms of a Taylor expansion up to a given order $J$. Using convenient canonical coordinates, we can make this description in its simplest form, called a normal form expansion. This is Theorem 5.2 below. A normal form expression of the dynamics is particularly interesting for large time, because along a given trajectory we will obtain a product of normal forms operators which is quite easy to calculate (because they commute together). In particular, for a periodic orbit the computation of the trace will be explicit.

For classical hyperbolic map on the torus, David DeLatte in [19], has shown that there exists a global and uniform normal form expression, called non-stationary normal form, which is unique up to co-boundary terms. In this section we develop the semi-classical version of his result, and use it to control long time evolution. Semi-classical normal form for *individual* unstable trajectories is already a well known and very useful tool in semi-classical analysis, see [12, 13, 14], [59, 60],[54],[42, 41]. Our approach of semi-classical non stationary normal forms follows closely the exposition of J. Sjöstrand in [54], but will be adapted to the normal form approach of

David DeLatte [19], which is *uniform* over phase space and not individual for periodic orbits. Therefore it will give a satisfactory control of the normal form expressions for long (periodic) orbits.

In this Section we give the main result useful for our purpose, and in appendix D, we give the proofs and more details, in particular we present there the semi-classical non stationary normal form theory in terms of Hamiltonian flows. This could be useful in order to extend the present results to more general hyperbolic Hamiltonian flows.

## 5.1. Semi-classical non-stationary normal form

We give here the semi-classical version of a Theorem of David DeLatte [19] about non stationary normal forms.

THEOREM 5.1. — *Let $J \geqslant 2$ even, be the order of the normal form calculation. Let $0 < \alpha < 1/2$. The evolution operator $\hat{M}$ in the neighborhood of any point $x \in \mathbb{R}^2$ is well approximated by a normal form operator $\hat{N}_x$ :*

$$(5.1) \qquad \left\| \hat{P}_{M(x)} \hat{M} \hat{P}_x - \hat{P}_{M(x)} \left( \hat{\mathcal{T}}_{M(x)} \hat{N}_x \hat{\mathcal{T}}_x^{-1} \right) \hat{P}_x \right\|_{L^2} \leqslant C \hbar^A$$

*with $A = \left( \frac{1}{2} - \alpha' \right)(J+1) - 1$, any $\alpha < \alpha' < 1/2$, and $C$ independent of $x$. The operator*

$$(5.2) \qquad\qquad\qquad \hat{N}_x = \exp\left( -\frac{i}{\hbar} \hat{K}_x \right)$$

*is generated by a Hamiltonian with a total Weyl symbol $K_x(q,p)$ which is a normal form up to order $J$:*

$$(5.3) \qquad\qquad K_x(q,p) = \sum_{0 \leqslant l+j \leqslant J/2} \lambda_{l,j,(x)} \hbar^l (qp)^j$$

*$\lambda_{0,1,(x)} = \lambda_x$ is the local expansion rate already introduced in Eq.(2.7), and the meaning of the other $\lambda_{l,j,(x)}$ is discussed below. $\hat{\mathcal{T}}_x$ is a product of unitary operators:*

$$(5.4) \qquad\qquad \hat{\mathcal{T}}_x = \hat{T}_x \hat{Q}_x \hat{U}_{G_1,\mathbf{x}} \hat{U}_{G_2,\mathbf{x}} \ldots \hat{U}_{G_{n_{max}},\mathbf{x}}$$

*where $\hat{T}_x$ is the translation operator Eq.(2.17), $\hat{Q}_x$ is the Weyl quantization of the linear symplectic map $Q_x$, Eq.(2.9). The next operators give non linear corrections: the operator $\hat{U}_{G_n,\mathbf{x}} = \exp\left( -\frac{i}{\hbar} \hat{G}_{n,x} \right)$ is generated by $\hat{G}_{n,x}$ whose Weyl symbol is equal to*

$$G_{n,x}(q,p) = g_{l,a,b,(x)} \hbar^l q^a p^b, \qquad \text{with } 3 \leqslant 2l + a + b \leqslant J$$

in a neighborhood of the origin (the index $n$ enumerates all the indices $(l, a, b)$ in the range $3 \leqslant 2l + a + b \leqslant J$, and with increasing order of $(2l + a + b)$). The functions $g_{l,a,b,(x)}$ and $\lambda_{l,j,(x)}$ for $(l, j) \neq (0, 0)$, are continuous with respect to $x \in \mathbb{T}^2$. The function $\lambda_{0,0,(x)}$ is continuous with respect to $x \in \mathbb{R}^2$ (but not periodic).

The proof is given in appendix D.

*Remarks.* —

- Eq.(5.1) is interesting if the error vanishes in the semi-classical limit $\hbar \to 0$, so if $(J + 1) \left(\frac{1}{2} - \alpha\right) - 1 > 0 \Leftrightarrow J \geqslant \frac{1+2\alpha}{1-2\alpha}$, for example if $J = 2$ and $0 < \alpha < 1/6$.
- If $J = 2$, the normal form description is just at the level of linear approximation. It is the quantum version of Eq.(2.10) and relies on uniform hyperbolicity. In the proof, it will be clear that the next order terms are obtained iteratively from this linear approximation, as usual in normal form calculations.
- Eq.(5.1) has a classical version, with maps $N_x$, $\mathcal{T}_x$ instead of operators $\hat{N}_x$, $\hat{\mathcal{T}}_x$ respectively, and with similar Taylor expansions, but without the semiclassical terms $\hbar^l$, $l \geqslant 1$. The classical normal form is expressed in Figure 5.1, and corresponds to the theorem 1.1, p. 238 given in [19] by David DeLatte (the formal version). In [19][20], David DeLatte proved a more stronger result: if $M$ is analytic, then normal form series of $K_x$ and $\mathcal{T}_x$ are convergent in a neighborhood of $(q, p) = (0, 0)$, for $J \to \infty$. Thanks to truncation operators, we will not need this result.
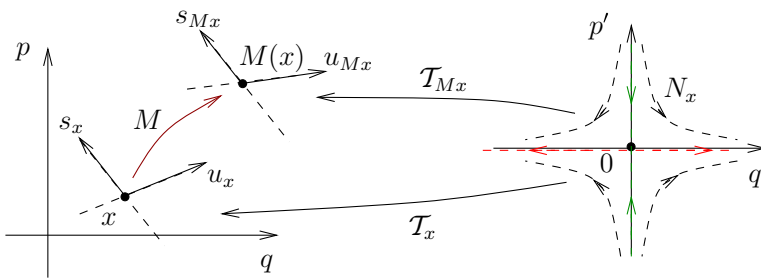


Figure 5.1. This picture traduces the conjugation relation Eq.(5.1) of the non stationary normal form. Here, in a neighborhood of a point $x$, the classical map $M \equiv \mathcal{T}_{M(x)} N_x \mathcal{T}_x^{-1}$ is conjugated to a map $N_x$ which is a normal form up to order $J$, and has 0 as hyperbolic fixed point.

- The main non trivial result in Theorem 5.1 is the continuity and periodicity of $N_x$ and $\mathcal{T}_x$ with respect to $x$. This is a global constraint over the torus, and relies on the uniform hyperbolicity of the dynamics. This continuity is already given in Theorem 2.1 for the expansion rate $\lambda_x$. In the proof, this appears in Lemma D.2.

- $\lambda_{0,0,(x)}$ is called the Action of the path $x \rightarrow M(x)$. It is given by the integral $\lambda_{0,0,(x)} = -\int \frac{1}{2}(pdq - qdp) - Hdt$ along the trajectory, as explained in Eq.(D.20) page 2590. $\lambda_{0,0,(x)}$ is a constant term in the function Eq.(5.3) and does not appear in the classical version of the Theorem, but is fundamental to explain the interference effects in quantum mechanics. For a geometric interpretation of $\lambda_{0,0}(x)$ in terms of parallel transport on a Complex line bundle over the phase space (called the prequantum bundle), see [25] and references therein.

- We will use later on, the fact that for $(j,l) \neq (0,0)$, $\lambda_{j,l,(x)}$ is a continuous function of $x \in \mathbb{T}^2$, and is therefore bounded:

(5.5)
$$\lambda_{j,l,min} \overset{\text{def}}{=} \min_{x \in \mathbb{T}^2}\left(\lambda_{j,l,(x)}\right) \leqslant \lambda_{j,l,(x)} \leqslant \lambda_{j,l,max} \overset{\text{def}}{=} \max_{x \in \mathbb{T}^2}\left(\lambda_{j,l,(x)}\right)$$

- The function $\lambda_{1,0,(x)} = \lambda_x$ is equal to the expansion rate introduced in Eq.(2.7), and is called the **Lyapounov cocycle.** The function $\lambda_{2,0,(x)}$ of $x$, is called the **Anosov cocycle** and is an obstruction to $C^2$ regularity of the unstable/stable foliation, see [40], or [37] p. 289. These two cocycles are the first terms of the series $\lambda_{j,l}$ of (semi-classical) cocycles.

## 5.2. Semi-classical normal form for a long orbit

We will now combine Eq.(4.4) and Eq.(5.1) along a given trajectory $x(t) = M^t x$ starting from $x \in \mathbb{R}^2$. Let us first define

$$\hat{N}_{x,t} \overset{\text{def}}{=} \hat{N}_{M^{t-1}(x)}\hat{N}_{M^{t-2}(x)} \ldots \hat{N}_x$$

to be the product of Normal forms $\hat{N}_x = \exp\left(-\frac{i}{\hbar}\hat{K}_x\right)$ along the trajectory. Quantum normal forms Hamiltonian $\hat{K}_x$ at different point $x$ commute together (this is proved in Eq.(5.23) page 2557). So the product $\hat{N}_{x,t}$ can be written

(5.6)
$$\boxed{\hat{N}_{x,t} = \exp\left(-\frac{i}{\hbar}t\hat{K}_{x,t}\right)}$$

where $\hat{K}_{x,t} = Op_{Weyl}(K_{x,t})$ has total Weyl symbol:

$$K_{x,t}(q,p) \overset{\text{def}}{=} \frac{1}{t} \sum_{s=0}^{t-1} K_{M^s x}(q,p)$$

which is just a time average along the trajectory. Using Eq.(5.3) we can write

(5.7)
$$\boxed{K_{x,t}(q,p) = \sum_{0 \leqslant l+j \leqslant J/2} \mu_{l,j,(x),t} \hbar^l (qp)^j}$$

with coefficients

(5.8)
$$\boxed{\mu_{j,l,(x),t} \overset{\text{def}}{=} \frac{1}{t} \sum_{s=0}^{t-1} \lambda_{j,l,(M^s x)}}$$

From Eq.(5.5), we deduce that $\mu_{j,l,(x),t}$ is bounded uniformly with respect to $x \in \mathbb{T}^2$ and $t \in \mathbb{Z}$ (i.e. for any orbit):

(5.9)
$$\lambda_{j,l,min} \leqslant \mu_{j,l,(x),t} \leqslant \lambda_{j,l,max}$$

We can now state:

THEOREM 5.2. — *For any $J \geqslant 2$, any $C > 0$, any $0 < \alpha < 1/2$, there exists $C_1 > 0$, such that for any $0 < t \leqslant C \log(1/\hbar)$, and any initial point $x \in \mathbb{R}^2$, any $1/2 > \alpha' > \alpha$,*
(5.10)
$$\left\| \hat{P}_{M^t(x)} \hat{M}^t \hat{P}_x - \hat{P}_{M^t(x)} \left( \hat{\mathcal{T}}_{M^t(x)} \hat{N}_{x,t} \hat{\mathcal{T}}_x^{-1} \right) \hat{P}_x \right\|_{L^2} \leqslant t\, C_1\, \hbar^{(J+1)\left(\frac{1}{2}-\alpha'\right)-1}$$

*As a corollary, for a periodic orbit, i.e. any fixed point $x_{n,t}$ of $M_{torus}^t$ given by Eq.(3.2):*

(5.11) $\quad \left| T\left(\hat{M}_{n,t}\right) - e^{iA_n/\hbar} \mathrm{Tr}\left(\hat{P}_\alpha \hat{N}_{(x_{n,t}),t} \hat{P}_\alpha\right) \right| \leqslant t C_1 \hbar^{(J+1)\left(\frac{1}{2}-\alpha'\right)-1-\alpha'}$

*where $\hat{M}_{n,t}$ is defined in Eq.(3.6), $T\left(\hat{M}_{n,t}\right)$ is defined in Eq.(3.9), and*

(5.12)
$$A_{n,t} \overset{\text{def}}{=} \frac{1}{2} n \wedge x_{n,t}$$

Remark. —

- The constant $A_{n,t}$ and $\mu_{0,0,(x_{n,t}),t} = -\frac{1}{t} \int \frac{1}{2}(pdq - qdp) - Hdt$ defined in Eq.(5.8), contribute to the total **action of the periodic orbit** defined by:

(5.13)
$$\mathcal{A}_{n,t} \overset{\text{def}}{=} A_{n,t} - t\mu_{0,0,(x_{n,t}),t}$$

*Proof.* — The proof of Eq.(5.10) combines Theorem 4.2 and Theorem 5.1 in three steps. First from Eq.(5.1), we deduce that

$$\left\| \hat{P}_{x(t)}\hat{M}\hat{P}_{x(t-1)}\hat{M}\hat{P}_{x(t-2)}\ldots\hat{M}\hat{P}_{x(0)} - \hat{P}_{x(t)}\left(\hat{\mathcal{T}}_{x(t)}\hat{N}_{x(t-1)}\hat{\mathcal{T}}_{x(t-1)}^{-1}\right)\right.$$
$$\left.\hat{P}_{x(t-1)}\left(\hat{\mathcal{T}}_{x(t-1)}\hat{N}_{x(t-2)}\hat{\mathcal{T}}_{x(t-2)}^{-1}\right)\hat{P}_{x(t-2)}\ldots\hat{P}_{x(0)}\right\|_{L^2} = Ct\hbar^A$$

We can now apply Theorem 4.2 to the sequence of hyperbolic maps $\left(\mathcal{T}_{M(x)}N_x\mathcal{T}_x^{-1}\right)$ in order to take off the intermediate operators $\hat{P}_{x(s)}$. We obtain:

$$\left\| \hat{P}_{x(t)}\left(\hat{\mathcal{T}}_{M^t(x)}\hat{N}_{x,t}\hat{\mathcal{T}}_x^{-1}\right)\hat{P}_{x(0)} - \hat{P}_{x(t)}\left(\hat{\mathcal{T}}_{x(t)}\hat{N}_{x(t-1)}\hat{\mathcal{T}}_{x(t-1)}^{-1}\right)\right.$$
$$\left.\hat{P}_{x(t-1)}\left(\hat{\mathcal{T}}_{x(t-1)}\hat{N}_{x(t-2)}\hat{\mathcal{T}}_{x(t-2)}^{-1}\right)\hat{P}_{x(t-2)}\ldots\hat{P}_{x(0)}\right\|_{L^2} = \mathcal{O}\left(\hbar^\infty\right)$$

Combining the last two equations with Eq.(4.4), we obtain finally Eq.(5.10). Now we will prove Eq.(5.11). First Eq.(5.10) for $x = x_{n,t}$ gives

$$\left\| \hat{P}_{M^t(x_{n,t})}\hat{M}^t\hat{P}_{x_{n,t}} - \hat{P}_{M^t(x_{n,t})}\left(\hat{\mathcal{T}}_{M^t(x_{n,t})}\hat{N}_{x_{n,t},t}\hat{\mathcal{T}}_{x_{n,t}}^{-1}\right)\hat{P}_{x_{n,t}}\right\|_{L^2}$$
$$\leqslant t\,C_1\,\hbar^{(J+1)\left(\frac{1}{2}-\alpha'\right)-1}$$

But $M^t x_{n,t} = x_{n,t} + n$, so $\hat{P}_{M^t(x_{n,t})} = \hat{T}_n\hat{P}_{x_{n,t}}\hat{T}_{-n}$. From Eq.(5.4), and $\hat{Q}_{M^t x_{n,t}} = \hat{Q}_{x_{nt}}$ (due to periodicity), we have

$$\hat{\mathcal{T}}_{M^t(x_{n,t})} = \hat{T}_{M(x_{n,t})}\hat{T}_{-x_{n,t}}\hat{\mathcal{T}}_{x_{n,t}} = e^{iA_{n,t}/\hbar}\hat{T}_n\hat{\mathcal{T}}_{x_{n,t}},$$

with $A_{n,t} \overset{\text{def}}{=} \frac{1}{2}n \wedge x_{n,t}$. The last equality comes from Eq.(2.18). We obtain:

$$\left\| \hat{P}_{x_{n,t}}\hat{M}_{n,t}\hat{P}_{x_{n,t}} - e^{iA_{n,t}/\hbar}\hat{P}_{x_{n,t}}\left(\hat{\mathcal{T}}_{x_{n,t}}\hat{N}_{x_{n,t},t}\hat{\mathcal{T}}_{x_{n,t}}^{-1}\right)\hat{P}_{x_{n,t}}\right\|_{L^2}$$
$$\leqslant t\,C_1\,\hbar^{(J+1)\left(\frac{1}{2}-\alpha'\right)-1}$$

Lemma B.10 page 2576 allows us to pass from operator norm to Trace norm estimates. It gives:

(5.14)
$$\left| \text{Tr}\left(\hat{P}_{x_{n,t}}\hat{M}_{n,t}\hat{P}_{x_{n,t}}\right) - e^{iA_{n,t}/\hbar}\text{Tr}\left(\hat{P}_{x_{n,t}}\left(\hat{\mathcal{T}}_{x_{n,t}}\hat{N}_{x_{n,t},t}\hat{\mathcal{T}}_{x_{n,t}}^{-1}\right)\hat{P}_{x_{n,t}}\right)\right|$$
$$\leqslant t\,C_1\,\hbar^{(J+1)\left(\frac{1}{2}-\alpha'\right)-1-\alpha'}$$

We want now to take off the operator $\hat{\mathcal{T}}_{x_{n,t}}$. Remind that $\mathcal{T}_x$ is a smooth canonical transformation which send point 0 to $x$. If $0 < \beta < \alpha$, we have in operator norm $\hat{P}_{x,\beta}\hat{\mathcal{T}}_x\hat{P}_{0,\alpha} = \hat{P}_{x,\beta}\hat{\mathcal{T}}_x + \mathcal{O}\left(\hbar^\infty\right)$, $\hat{P}_{0,\alpha}\hat{\mathcal{T}}_x\hat{P}_{x,\beta} = \hat{\mathcal{T}}_x\hat{P}_{0,\beta} +$

$\mathcal{O}(\hbar^\infty)$, and $\hat{P}_{x,\beta}\hat{P}_{0,\alpha} = \hat{P}_{x,\beta} + \mathcal{O}(\hbar^\infty)$. So, with $\beta > \gamma > \alpha$,

$$\mathrm{Tr}\left(\hat{P}_{x_{n,t},\alpha}\left(\hat{\mathcal{T}}_{x_{n,t}}\hat{N}_{x_{n,t},t}\hat{\mathcal{T}}_{x_{n,t}}^{-1}\right)\hat{P}_{x_{n,t},\alpha}\right)$$

$$= \mathrm{Tr}\left(\hat{P}_{x_{n,t},\alpha}\left(\hat{\mathcal{T}}_{x_{n,t}}\hat{P}_{0,\gamma}\hat{N}_{x_{n,t},t}\hat{P}_{0,\gamma}\hat{\mathcal{T}}_{x_{n,t}}^{-1}\right)\hat{P}_{x_{n,t},\alpha}\right) + \mathcal{O}(\hbar^\infty)$$

$$(5.15) \qquad = \mathrm{Tr}\left(\hat{P}_{x_{n,t},\beta}\left(\hat{\mathcal{T}}_{x_{n,t}}\hat{P}_{0,\gamma}\hat{N}_{x_{n,t},t}\hat{P}_{0,\gamma}\hat{\mathcal{T}}_{x_{n,t}}^{-1}\right)\hat{P}_{x_{n,t},\beta}\right) + \mathcal{O}(\hbar^\infty)$$

$$= \mathrm{Tr}\left(\hat{\mathcal{T}}_{x_{n,t}}\hat{P}_{0,\gamma}\hat{N}_{x_{n,t},t}\hat{P}_{0,\gamma}\hat{\mathcal{T}}_{x_{n,t}}^{-1}\right) + \mathcal{O}(\hbar^\infty)$$

$$= \mathrm{Tr}\left(\hat{P}_{0,\gamma}\hat{N}_{x_{n,t},t}\hat{P}_{0,\gamma}\right) + \mathcal{O}(\hbar^\infty)$$

For the second line, we have used the property that the trace does not depend on the choice of $\beta$, up to $\mathcal{O}(\hbar^\infty)$. Finally, Eq.(5.14,5.15,4.7) together give Eq.(5.11). $\qquad\qquad\square$

## 5.3. Post-Normalization

### 5.3.1. Post Normalization corrections

The Weyl symbol $K_{x,t}(q,p)$ given in Eq.(5.7) is a function of the product $(qp)$ only. Let us write it $K_{x,t}(qp)$. The quantized operator we have to consider in Eq.(5.6) is $\hat{K}_{x,t} = Op_{Weyl}(K_{x,t}(qp))$. First remark that, in order to compute the trace of the propagator, or its matrix elements, it is easier to deal with a function of the single operator $Op_{Weyl}(qp)$, and second, $Op_{Weyl}(K_{x,t}(qp)) \neq K_{x,t}(Op_{Weyl}(qp))$ in general (see e.g. Eq.(5.21) below). So we have to find a function $\tilde{K}_{x,t}$ of a single variable, such that

$$(5.16) \qquad \boxed{\hat{K}_{x,t} = Op_{Weyl}(K_{x,t}(qp)) = \tilde{K}_{x,t}(Op_{Weyl}(qp))}$$

PROPOSITION 5.3. — If $K_{x,t}(qp) = \sum_{0 \leqslant l+j \leqslant J/2} \mu_{l,j,x,t}\hbar^l (qp)^j$ is expressed as a power series in $(qp)$ and $\hbar$, (as we get in Eq.(5.7)), then $\tilde{K}_{x,t}$ is given by:

$$(5.17) \qquad \tilde{K}_{x,t}(qp) = \sum_{l,j \geqslant 0} \tilde{\mu}_{l,j,x,t}\hbar^l (qp)^j$$

where the coefficients $\tilde{\mu}_{l,j,x,t}$ are given explicitly from $\mu_{l,j,x,t}$ by:

$$(5.18) \qquad \tilde{\mu}_{l,j,x,t} \overset{\mathrm{def}}{=} \sum_{0 \leqslant m \leqslant [l/2]} E_{j',m}\mu_{l',j',x,t}, \qquad l' = l - 2m, \; j' = j + 2m$$

where $E_{j',m}$ is a numerical factor determined by recursive formula:

$$E_{j+1,m} \quad = \quad E_{j,m} - \frac{j^2}{4} E_{(j-1),(m-1)}, \text{ if } j, m \geqslant 1$$

(5.19) $\quad\quad E_{j,0} \;= 1, \quad E_{0,m} = 0, \; E_{1,m} = 0, \text{ for } m \geqslant 1$

*Remarks.* —

- In particular, at the principal and sub-principal level (i.e. order $l = 0, 1$ in $\hbar^l$), there is no change: i.e. $\tilde{\mu}_{l,j,x,t} = \mu_{l,j,x,t}$, if $l = 0$ or $l = 1$.
- In [31], paragraph 6, Alfonso Garcia-Saz gives such expressions as a special case of a more general problem. See also appendix I in [10].

*Proof.* — We use the Moyal start product $\sharp$ defined in Eq.(D.3), which is equivalent to the product of operators. In other words, we want to express $(qp)^j$ in terms of product of monomial terms $(qp)^{\sharp j} \overset{\text{def}}{=} (qp) \sharp (qp) \sharp \ldots \sharp (qp)$. We just apply Moyal product formula Eq.(D.4), with operators $A = qp$, $B = (qp)^j$, and observe that $(A\mathcal{D}^n B) = 0$, for $n = 1$ and $n \geqslant 3$, and $(A\mathcal{D}^2 B) = -2j^2 (qp)^{j-1}$. This gives

$$(5.20) \quad\quad (qp) \sharp (qp)^j = (qp)^{j+1} + \hbar^2 \frac{j^2}{4} (qp)^{j-1}$$

Or equivalently
(5.21)
$$Op_{Weyl} (qp)^{j+1} = (Op_{Weyl} (qp)) \left( Op_{Weyl} (qp)^j \right) - \hbar^2 \frac{j^2}{4} \left( Op_{Weyl} (qp)^{j-1} \right)$$

We deduce that

$$(5.22) \quad\quad (qp)^j = \sum_{m=0}^{[j/2]} E_{j,m} \hbar^{2m} (qp)^{\sharp(j-2m)}$$

This last formula says that each resonant term of $Op_{Weyl} (K_{x,t})$ can be written

$$\mu_{l,j,(x),t} \hbar^l Op_{Weyl} \left( (qp)^j \right) = \mu_{l,j,(x),t} \sum_{m=0}^{[j/2]} E_{j,m} \hbar^{l+2m} (Op_{Weyl} (qp))^{j-2m}$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

PROPOSITION 5.4. — $(qp)^n$ and $(qp)^m$ commute with respect to the star product:

$$[(qp)^n , (qp)^m]_\sharp = (qp)^n \sharp (qp)^m - (qp)^m \sharp (qp)^n = 0$$

therefore if $F, G \in C^\infty (\mathbb{R})$,

(5.23)             $[Op_{Weyl} (F (qp)) , Op_{Weyl} (G (qp))] = 0$

Proof. — By induction on the power: suppose that $[(qp)^n , (qp)^m]_\sharp = 0$, for $n, m \leqslant N$. Using Eq.(5.20), we compute

$$
\begin{aligned}
\left[(qp)^{N+1} , (qp)^n\right]_\sharp &= (qp)^{N+1} \sharp (qp)^n - (qp)^n \sharp (qp)^{N+1} \\
&= (qp) \sharp (qp)^N \sharp (qp)^n - \hbar^2 \frac{N^2}{4} (qp)^{N-1} \sharp (qp)^n \\
&\quad - (qp)^n \sharp (qp) \sharp (qp)^N + \hbar^2 \frac{N^2}{4} (qp)^n \sharp (qp)^{N-1} \\
&= 0
\end{aligned}
$$

from associativity of the star product. There is a more direct proof, using the post-normalization transformation $F \to \tilde{F}$ described above:

$$
[Op_{Weyl} (F (qp)) , Op_{Weyl} (G (qp))] = \left[\tilde{F} (Op_{Weyl} (qp)) , \tilde{G} (Op_{Weyl} (qp))\right]
$$
$$
= 0.
$$

□

### 5.3.2. Semi-classical Trace formula for a periodic orbit

Semi-classical expansions for the trace of hyperbolic normal forms is explicitly done in [43]. It can be used to give $\mathrm{Tr} \left(\hat{P}_\alpha \hat{N}_{x_n,t} \hat{P}_\alpha\right)$ in terms of the semi-classical post-normalized cocycles $\tilde{\mu}_{l,j,x_n,t}$ defined in Eq.(5.18). We do this in appendix E, Proposition E.1, page 2592. We can deduce the following proposition:

PROPOSITION 5.5. — Let $J \geqslant 2$. For any $C > 0$, any $0 < \alpha < 1/2$, there exists $C_1 > 0$ such that, for any time $0 < t \leqslant C \log (1/\hbar)$, any $n \in \mathcal{C}_t$, one has a semi-classical expression:

(5.24)             $\left| \mathrm{Tr} \left(\hat{P}_\alpha \hat{N}_{x_n,t,t} \hat{P}_\alpha\right) - T_{semi,J} (x_{n,t}) \right| \leqslant C_1 \hbar^{[J/2]}$

where $x_{n,t}$ is the fixed point of the map $M_{n,t}$, Eq.(3.7), and

$$
T_{semi,J} (x_{n,t}) \stackrel{\text{def}}{=} \exp \left(-it \frac{\mu_{0,0,x_n,t}}{\hbar}\right) \exp \left(-it \mu_{1,0,x_n,t}\right) \frac{1}{2 \sinh \left(\frac{\mu_{0,1,x_n,t} t}{2}\right)} \mathcal{E}_{x_n,t}
$$

with a semi-classical expansion

$$
\mathcal{E}_{x_n,t} = 1 + \hbar E_1 + \hbar^2 E_2 + \ldots + \hbar^{[J/2]-1} E_{([J/2]-1)}
$$

where $E_s$ depends on $t$ and $\tilde{\mu}_{l,j,x_n,t}$ , with $(l+j) \leqslant s+1$. We have the bound

$$|E_s| \leqslant t^s E_{max,s}$$

where $E_{max,s}$ does not depend on $t$ nor on $n$. With Eq.(5.11) we deduce that

(5.25) $$\boxed{\left| T\left(\hat{M}_{n,t}\right) - e^{iA_n/\hbar} T_{semi,J}\left(x_{n,t}\right)\right| \leqslant C_1 \hbar^{[J/2]}}$$

*Remark.* — Explicit expressions of $E_s$, $s \geqslant 1$ are quite complicated for large $s$, and are given in Eq.(E.9) page 2593, and Eq.(E.17) page 2595. It appears in appendix E that with Weyl quantization then $\mu_{1,0,x_n,t} = 0$ (and $\mu_{l,j} = 0$ for $l$ odd). We will use this simplification below.

*Proof.* — Proposition E.1, page 2592, gives

$$\left| \mathrm{Tr}\left(\hat{P}_\alpha \hat{N}_{x_{n,t},t} \hat{P}_\alpha\right) - T_{semi,J}\left(x_{n,t}\right)\right| \leqslant C_1 B,$$

where $C_1$ does not depend on $t$ and $n$, and with

$$B = \max_{n,t}\left(\left(2\sinh\left(\frac{\tilde{\mu}_{0,1}t}{2}\right)\right)^{-1} E_{[J/2]}\hbar^{[J/2]}\right)$$

given by the maximum over periodic orbits of the next order term in the series of $T_{semi,J}\left(x_{n,t}\right)$. The uniform control of the semi-classical cocycles, over the periodic orbits, Eq.(2553) gives $\tilde{\mu}_{0,1} \geqslant \lambda_{0,1,min} = \lambda_{min}$, where $\lambda_{min} > 0$ is defined in Eq.(2.8). Thus $\left(2\sinh\left(\frac{\tilde{\mu}_{0,1}t}{2}\right)\right)^{-1} \leqslant C_2 e^{-\lambda_{min}t/2} \leqslant C_2\hbar^{\left(\frac{t\lambda_{min}}{2t_E}\right)}$, because we can write $e^{-\lambda_{min}t/2} = e^{-\lambda_0 t_E\left(\frac{t\lambda_{min}}{2t_E}\right)} = \hbar^{\left(\frac{t\lambda_{min}}{2t_E}\right)}$. The uniform control of the semi-classical cocycles together with the bound Eq.(E.4) also gives that $\left|E_{[J/2]}\right| \leqslant t^s E_{max,s}$ where $E_{max,s}$ does not depend on $n, t$. Now if $|t| \leqslant \mathcal{O}\left(\log\left(1/\hbar\right)\right)$ then $|t^s| < \hbar^{-\varepsilon}$ for any $\varepsilon > 0$, so at final $C_1 B \leqslant C_3\hbar^{\left(\frac{t\lambda_{min}}{2t_E}\right)}\hbar^{-\varepsilon}\hbar^{[J/2]} \leqslant C_4\hbar^{[J/2]}$, if we take $\varepsilon = \frac{1}{2}\left(\frac{\lambda_{min}}{2t_E}\right)$. This gives Eq.(5.24).

From Eq.(5.24) and Eq.(5.11), we deduce that

$$\left| \mathrm{T}\left(\hat{M}_{n,t}\right) - e^{iA_n/\hbar} T_{semi,J}\left(x_{n,t}\right)\right| < C_1\hbar^{[J/2]} + C_1'\, \hbar^{(J+1)\left(\frac{1}{2}-\alpha'\right)-1-\alpha'}.$$

We observe that for large $J$, this gives a bound $\mathcal{O}\left(\hbar^\infty\right)$. So we deduce that the actual bound is given by the next order term in the series of

$T_{semi,J}(x_{n,t})$, namely $B = \mathcal{O}\left(\hbar^{[J/2]}\right)$ already computed above: in more precise terms, this is because

$$\left|\mathrm{T}\left(\hat{M}_{n,t}\right) - e^{iA_n/\hbar}T_{semi,J\infty}(x_{n,t})\right| = \mathcal{O}\left(\hbar^\infty\right),$$

and $|T_{semi,J\infty}(x_{n,t}) - T_{semi,J}(x_{n,t})| = \mathcal{O}\left(\hbar^{[J/2]}\right)$. $\qquad\qquad\square$

## 6. Trace of the quantum map

### 6.1. Semi-classical trace formula

The following Theorem is one of the main result of this paper. It gives a semi-classical expression for the trace $\mathrm{Tr}\left(\hat{M}_{torus}^t\right)$, as a finite sum over the fixed points $x_{n,t}$ of the classical map $M_{torus}^t$.

THEOREM 6.1. — *For any $K > 0$ , any $C > 0$, any $J \geqslant 2\left(K + C\right)$, there exists $C_1 > 0$, such that for any time $|t| < Ct_E$, with $t_E = \frac{1}{\lambda_0}\log\frac{1}{\hbar}$, any admissible value of $\hbar$ (see Eq.(2.21)), the trace of $\hat{M}_{torus}^t$ is given by the **semi-classical formula**:*

$$(6.1) \qquad \left|\mathrm{Tr}\left(\hat{M}_{torus}^t\right) - T_{semi,t,J}\right| \leqslant C_1\hbar^K$$

$$(6.2) \qquad T_{semi,t,J} = \sum_{n\in\mathcal{C}_t}\exp\left(i\frac{\mathcal{A}_{n,t}}{\hbar}\right)\frac{1}{2\sinh\left(\frac{\mu_{0,1,x_n,t}t}{2}\right)}\mathcal{E}_{x_{n,t}}$$

*where each term of the finite sum is associated with a fixed point $x_{n,t}$ of $M_{torus}^t$ given in Eq.(3.2). $\mathcal{A}_{n,t} = A_n - \mu_{0,0,x_n,t}t$ is the action of the periodic orbit, defined in Eq.(5.13). $\mu_{l,j,x_n,t}$ and $\tilde{\mu}_{l,j,x_n,t}$ are the semi-classical normal form coefficients (cocycles) of the periodic orbit computed up to a given order $J \geqslant 2$ (i.e. with $2\left(j+l\right) \leqslant J$). $\mathcal{E}_{x_{n,t}}$ is a semi-classical series:*

$$\mathcal{E}_{x_{n,t}} = 1 + \sum_{1\leqslant s\leqslant[J/2]-1}\hbar^s E_s = 1 + \hbar E_1 + \hbar^2 E_2 + \ldots + \hbar^{[J/2]-1}E_{[J/2]-1}$$

*where $E_s$ depends on $t$ and $\tilde{\mu}_{l,j,x_n,t}$ (with $(l+j) \leqslant s+1$), and in particular is bounded by*

$$|E_s| \leqslant t^s E_{max,s}$$

*where $E_{max,s}$ does not depend on $t$ nor on $x_{n,t}$.*

*Proof.* — We use Eq.(3.10), which is a sum with $\mathcal{N}_t = e^{\lambda_0 t} - 2 + e^{-\lambda_0 t}$ terms (from Eq.(3.3)). We write $\mathcal{N}_t \leqslant e^{\lambda_0 t} = e^{-\lambda_0 t_E(-t/t_E)} = \hbar^{(-t/t_E)} \leqslant \hbar^{-C}$. Each term $\mathrm{T}\left(\hat{M}_{n,t}\right)$ is approximated by a semiclassical expression given in Eq.(5.25), with an uniform error $C_1 \hbar^{[J/2]}$. By a simple triangular inequality (i.e. we sum the magnitudes of the error bounds) we deduce that

$$\left|\mathrm{Tr}\left(\hat{M}^t_{torus}\right) - T_{semi,t,J}\right| \leqslant \mathcal{N}_t C_1 \hbar^{[J/2]} \leqslant C_1 \hbar^{[J/2]-C}$$

This is $\mathcal{O}\left(\hbar^K\right)$ if $J > 2\left(K + C\right)$ (for $J$ even). $\qquad \square$

*Remarks.* —

- If we look at the whole proof, the limitation to these logarithmic time (any multiple of the Ehrenfest time $t_E = \frac{1}{\lambda_0} \log \frac{1}{\hbar}$) is present many times. But the main limitation is due to exponential proliferation of periodic orbits in the hyperbolic system, $\mathcal{N}_t \simeq e^{\lambda_0 t}$.
- The bound on the errors could be improved at many places in the proof, so the condition $J > 2\left(K + C\right)$ is not sharp. But the main limitation in our proof is that we simply estimate the total error as the sum of the absolute values of the error from each periodic orbit. However its seems that all these errors compensate each other, as observed in the next Section. However, a rigorous analysis of these compensations is out of reach for the moment.
- Let us comment again on the estimation used in the proof. We have $\left(2 \sinh\left(\frac{\mu_{0,1,x_n,t} t}{2}\right)\right)^{-1} \simeq e^{-\lambda_0 t/2}$ for large $t$, so, if we write Eq.(6.2) as $T_{semi,t,J} = \sum_{n \in \mathcal{C}_t} T_{n,t}$, each term associated to an individual periodic orbit $|T_{n,t}| \leqslant e^{-\lambda_0 t/2}$ decreases, but $\mathcal{N}_t = \sharp\left(\mathcal{C}_t\right) \simeq e^{\lambda_0 t}$ increases faster. We get the bound $|T_{semi,t,J}| \leqslant e^{\lambda_0 t/2}$ which is greater than $\dim\left(\mathcal{H}\right) = N = 1/\left(2\pi\hbar\right)$ for $t \geqslant \frac{2}{\lambda_0} \log\left(1/\hbar\right) = 2t_E$. This shows that for $t \geqslant 2t_E$, there are necessarily some cancellations among the complex amplitudes involved in Eq.(6.2), so that $|T_{semi,t,J}| \leqslant \dim\left(\mathcal{H}\right) = N$ is always satisfied. These cancellations are due to the leading complex terms $\exp\left(-i\frac{\mathcal{A}_{n,t}}{\hbar}\right)$.
- In some specific examples, namely the linear map $M_0$, Eq.(2.2), with particular values of $2\pi\hbar$, we observed in [27] that at $t \simeq 2t_E$, all the actions $\exp\left(-i\frac{\mathcal{A}_{n,t}}{\hbar}\right) = 1$ are equal and add together. This implies that the upper bound $\mathrm{Tr}\left(\hat{M}^{t=2t_E}_{torus}\right) = \dim\left(\mathcal{H}\right) = N$ is reached, and therefore that $\hat{M}^{t \simeq 2t_E}_{torus} \propto \hat{Id}$ which implies revivals of quantum

states and existence of scarred eigenstates, i.e. non equidistributed invariant semiclassical measures.

• Let us give now a "philosophical" remark which shows again that $t = 2t_E$ seems to be a critical value of time. If we define the "complexity" of the trace formula to be equal to the number of periodic orbits, it grows like $e^{\lambda_0 t}$. This complexity is larger than the "complexity" of the linear quantum problem (equal to the size of the matrix $\hat{M}$: $N^2 \simeq e^{2\lambda_0 t_E}$), for $t \geqslant 2t_E$. Again this shows that for $t \geqslant 2t_E$, periodic orbits manifest themselves in the semiclassical trace formula through collective averaging effects (see Berry in [29], or [25] where this open problem is also discussed).

## 6.2. Numerical results and observation of self-averaging effects

We illustrate the semi-classical formula for the trace Eq.(6.1), with the example defined by Eq.(2.12). We have computed $\text{Tr}\left(\hat{M}^t\right)$ numerically for $t = 0 \to 11$ (this is easy because the Hilbert space is finite dimensional, and we expect the numerical result to have a good accuracy). We compare it with the semi-classical approximation $\text{T}_{semi,t,J}$, Eq.(6.2), where every normal forms has been computed numerically[5] up to a given order $J \geqslant 2$. We choose successively $N = 1/(2\pi\hbar) = 10$, $N = 1/(2\pi\hbar) = 100$ with $J = 2$ and $J = 4$. The observed error is denoted by $\text{Error}_J(t) \overset{\text{def}}{=} \left|\text{Tr}\left(\hat{M}^t\right) - T_{semi,t,J}\right|$. The practical limitation in this numerical calculation is the increasing number $\mathcal{N}_t \simeq e^{\lambda_0 t}$ of periodic orbits; already $\mathcal{N}_{t=11} = 39601$. In Theorem 6.1, we have estimated that

$$\text{Error}_J(t) < C_1 \varepsilon_J(t), \quad \text{with} \quad \varepsilon_J(t) = \hbar^K = \hbar^{\frac{J}{2}-C} = \hbar^{J/2 - t/t_E}$$

with $t_E = \frac{1}{\lambda_0}\log(1/\hbar)$.

On Figure 6.1, we plot three functions of $(t/t_E)$ on a logarithmic scale: $\left|\text{Tr}\left(\hat{M}^t\right)\right|$, the observed error $\text{Error}_J(t)$ and the upper bound of the error $\varepsilon_J(t)$. We also draw the line $\text{Tr}_{Max} = 1/(2\pi\hbar)$ (because obviously $\left|\text{Tr}\left(\hat{M}^t\right)\right| \leqslant \dim \mathcal{H}_{torus} = 1/(2\pi\hbar)$ ), and the line of the particular value $\varepsilon_{\hbar,J} = \hbar^{J/2}$ which is expected to be an upper bound of the error from heuristic arguments given below. The unexpected observed fact is that $\text{Error}_J(t) < \varepsilon_{\hbar,J}$, although $\text{Error}_J(t) < \varepsilon_J(t) = \varepsilon_{\hbar,J}\hbar^{-t/t_E}$ has only been proved.

---

[5] The algorithm to compute the normal forms is explained in Section E.
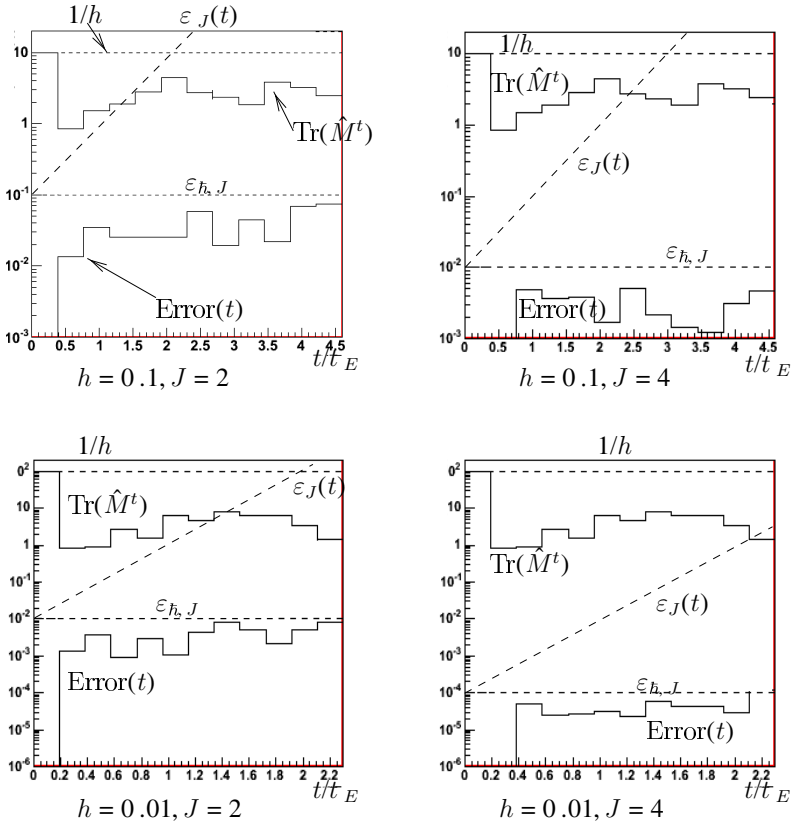
Figure 6.1. Numerical results for $\left| \text{Tr}\left( \hat{M}^t \right) \right|$ and its semiclassical approximation $T_{semi,t,J}$. We have plotted $\text{Error}_J(t) \overset{\text{def}}{=} \left| \text{Tr}\left( \hat{M}^t \right) - T_{semi,t,J} \right|$ in log. scale.

Remarks and observations:. —

- $\hat{M}$ is an unitary operator, so $\left| \text{Tr}\left( \hat{M}^t \right) \right| \leqslant \dim \mathcal{H}_{torus} = 1/(2\pi\hbar)$. However our error estimate gives $\varepsilon_J(t) = \hbar^{J/2 - t/t_E} \geqslant 1/(2\pi\hbar)$ for $t \geqslant \left( \frac{J}{2} + 1 \right) t_E$. So our estimates is not interesting for $t \geqslant \left( \frac{J}{2} + 1 \right) t_E$.
- According to Eq.(5.25), the error term $\text{Error}_J(t)$ is a sum of $\mathcal{N}_t \simeq e^{\lambda_0 t}$ complex numbers, each being bounded in magnitude by $B \simeq e^{-\lambda_0 t/2} \hbar^{J/2}$ (This appeared in the proof of Proposition 5.5). With

an heuristic point of view, if we think that these numbers behave as independent random variables, then the total error is estimated (by the central limit theorem) to be of order $\sqrt{\mathcal{N}_t}B \simeq \hbar^{J/2}$ as the numerical calculations shows. It would a nice result to explain such a behaviour.

# 7. Conclusion

The main result of this paper is the use of semiclassical non stationary normal form description of the hyperbolic dynamics first introduced by David DeLatte [19], and which gives some invariant semi-classical cocycles of the hyperbolic dynamics. With them, we obtain semi-classical expressions for long time of order $t \simeq C \log{(1/\hbar)}$, with any $C > 0$. These cocycles are a generalisation to all order in non linearities and in power of $\hbar$ of the Lyapounov cocycle which controls the linear instability of the dynamics. These cocycles allow us to have a control on the error term of the trace formula (or other semi-classical formula) for long time. We could ask the question if this Non-stationary Normal description presented in Section 5 is really necessary? the author believes that it is, because it gives Eq.(5.1) where $\mathcal{T}_x$ depends on the point $x$ and not on the trajectory $x(t) = M^t x$. Without this result, we could expect that the Normal Form construction would depend on the trajectory $x(t)$ as a whole, and could diverge with $t$ in an uncontrollable way. We would be then blocked.

There are many perspectives suggested by this work. Numerical calculations presented in Section 6.2 seem to show that our bounds are not sharp, and that the actual errors are much smaller. In particular, Gutzwiller trace formula at the leading order $J = 2$, i.e. without semi-classical normal forms corrections, could be correct for these "long time", with an error $\mathcal{O}(\hbar)$ uniform in time. To understand this surprising fact we should understand the way all the complex semi-classical contributions add and compensate together. We think that the Ruelle-Pollicott thermodynamical formalism of transfer operators in the context of prequantum dynamics, could help us in that direction. Some preliminary results in this direction are obtained in [25], in the case of a linear hyperbolic map, and we hope to be able to extend these results to non linear maps.

An other perspective is a generalization of this approach to a uniform hyperbolic flow, like the geodesic flow on a negative curvature manifold. We think that the approach of "semi-classical non stationary normal forms" developed in appendix D could be generalized for such models. A similar

but less obvious generalization could be done for maximal hyperbolic sets, situations which are met in general mixed chaotic systems, with homoclinic intersections of stable/unstable manifolds giving horseshoes [45].

## Appendix A. Proof of Theorem 4.2 page 2547 on semi-classical evolution in a neighborhood of a classical trajectory for long time

We first introduce some notations specific to this appendix.

### A.1. Some notations

Let $D \subset \mathbb{R}^2$ (a measurable set). We define:

$$\hat{P}_D \overset{\text{def}}{=} \int_{x \in D} \frac{d^2 x}{2\pi\hbar} |x\rangle\langle x|$$

where $|x\rangle$ is a coherent state defined by Eq. (2.25). One has the closure relation:

(A.1) $$\hat{P}_D + \hat{P}_{\mathbb{R}^2 \backslash D} = \hat{I}d$$

*Proof.* — $\hat{I}d = \int_{\mathbb{R}^2} \frac{d^2 x}{2\pi\hbar} |x\rangle\langle x| = \int_D \frac{d^2 x}{2\pi\hbar} |x\rangle\langle x| + \int_{\mathbb{R}^2 \backslash D} \frac{d^2 x}{2\pi\hbar} |x\rangle\langle x|.$  □

Let $0 < \beta < 1/2$ (The sharper results will correspond to $\beta$ close to 0). Let $D \subset \mathbb{R}^2$ a measurable set, or a sequence of sets $D_\hbar$ which depend on $\hbar$, with $\hbar \to 0$. In particular $D$ may be a point $D = \{x\}$, $x \in \mathbb{R}^2$.

DEFINITION A.1. — *Let's define the "**exterior domain**" $E(D)$, by:*

(A.2) $$\boxed{E(D) = \left\{ x \in \mathbb{R}^2 \text{ such that } \mathrm{dist}(x, D) > \hbar^{1/2 - \beta} \right\}}$$

*and its complementary, the "**interior**" $I(D)$*

$$\boxed{I(D) = \mathbb{R}^2 \backslash E(D)}$$

*Remarks: $I(D)$ is just the domain $D$ which has been thickened by $\hbar^{1/2-\beta}$. In particular $D \subset I(D)$. In Eq.(4.2), we have defined $\hat{P}_{x,\beta}$, which in the present notation is $\hat{P}_{I\{x\}}$.*

In this appendix, the classical dynamics is the map $M : \mathbb{R}^2 \to \mathbb{R}^2$ defined in Section 2, and the quantum map $\hat{M}$ has been defined in Eq.(2.15).

## A.2. Semi-classical evolution after finite time

The results of this appendix will rely on the following lemma which is a quite standard result on semi-classical evolution of wave packets after finite time (with respect to $\hbar \to 0$).

LEMMA A.2. — *For any $K \in \mathbb{N}$, and $\hbar$ small enough, there exists $C > 0$, such that for any $x \in \mathbb{R}^2$, and any domain $D \subset \mathbb{R}^2$ such that $D \subset E(Mx)$,*

$$\text{(A.3)} \qquad \left| \hat{P}_D \hat{M} |x\rangle \right|_{L^2} < C\hbar^K$$

*where $|x\rangle$ is a coherent state defined in Eq.(2.25).*

*Remarks.* —

- In all this appendix, we will use this result, and write: $\varepsilon = C\hbar^K$. At the end, $K$ will be chosen large enough. In common notations, we can write that $\left| \hat{P}_D \hat{M} |x\rangle \right|_{L^2} = \mathcal{O}(\hbar^\infty)$ uniformly over $x \in \mathbb{R}^2$ and $D \subset E(Mx)$. Lemma A.2 means that the evolved state $\hat{M}|x\rangle$ is "localized" at point $Mx$ at order $\mathcal{O}(\hbar^\infty)$. In our case, the uniformity in $x$ is due to the fact that the dynamics is defined on a torus (compact space).
- Lemma A.2 follows directly from more precise results obtained by A. Joye and G. Hagedorn in ref. [36] (theorem 3.2), or M. Combescure and D. Robert [16], theorem 3.1, or A. Iantchenko [41], Lemma 5.

COROLLARY A.3. — *Suppose that the domain $D \subset \mathbb{R}^2$ has finite measure $S(D) \stackrel{\text{def}}{=} \int_{x \in D} dx$. Then for any domain $D' \subset E(M(D))$*

$$\text{(A.4)} \qquad \left\| \hat{P}_{D'} \hat{M} \hat{P}_D \right\|_{L^2} < \left( \frac{S(D)}{2\pi\hbar} \right)^{1/2} \varepsilon$$

(We don't think that the dependence on $D$ of the right hand side is sharp).

*Proof.* — Let $\psi \in L^2(\mathbb{R})$ normalized, then

$$
\begin{aligned}
\left\| \hat{P}_{D'} \hat{M} \hat{P}_D \psi \right\|_{L^2} &= \left\| \hat{P}_{D'} \hat{M} \int_{x \in D} \frac{d^2 x}{2\pi\hbar} |x\rangle\langle x|\psi\rangle \right\|_{L^2} \\
&\leqslant \int_D \frac{d^2 x}{2\pi\hbar} |\langle x|\psi\rangle| \left\| \hat{P}_{D'} \hat{M} |x\rangle \right\|_{L^2} \\
&\leqslant \varepsilon \int_D \frac{d^2 x}{2\pi\hbar} |\langle x|\psi\rangle| \leqslant \left( \frac{S(D)}{2\pi\hbar} \right)^{1/2} \varepsilon
\end{aligned}
$$

In the last line we have used Eq.(A.3) because $D' \subset E(M(D)) \subset E(Mx)$ if $x \in D$. We have also used Cauchy-Schwarz inequality. $\qquad \square$

## A.3.  Long time Semi-classical evolution

Let $D \subset \mathbb{R}^2$ be a set at time $t = 0$, with finite measure. Let

$$I_0 (D) \overset{\text{def}}{=} D$$

And for any $t \geqslant 1$,

$$E_t (D) \overset{\text{def}}{=} E (M I_{t-1}), \qquad I_t (D) \overset{\text{def}}{=} \mathbb{R}^2 \backslash E_t (D)$$

$I_t (D)$ is just obtained from $D$, by evolution and thickening at each step, see figure A.1.
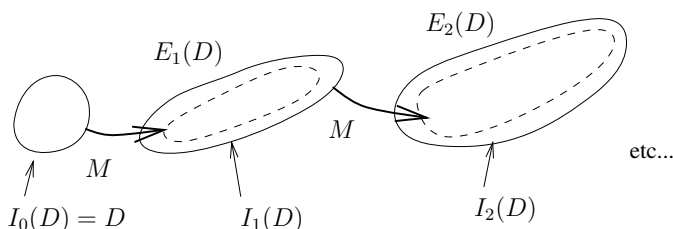


Figure A.1. Description of the sets $E_t (D)$ and $I_t (D)$, constructed inductively from $D$, by evolution and thickening.

LEMMA A.4. — For any time $t \geqslant 1$:

$$(A.5) \qquad \left\| \hat{P}_{E_t(D)} \hat{M}^t \hat{P}_D \right\|_{L^2} < \varepsilon \left( \sum_{s=0}^{t-1} \left( \frac{S (I_s (D))}{2\pi\hbar} \right)^{1/2} \right)$$

Proof. — From Eq.(A.1), we get $\hat{\text{Id}} = \hat{P}_{E_t(D)} + \hat{P}_{I_t(D)}$, and $E_t (D) = E_1 (I_{t-1} (D))$. Eq.(A.4) gives for any $t$:

$$\left\| \hat{P}_{E_1(D)} \hat{M} \hat{P}_D \right\|_{L^2} < \left( \frac{S (D)}{2\pi\hbar} \right)^{1/2} \varepsilon$$

where $\varepsilon$ does not depend on $t \in \mathbb{R}$, and $D$. By induction on time $t$ we get:

$$\left\| \hat{P}_{E_t(D)} \hat{M}^t \hat{P}_D \right\|_{L^2} = \left\| \hat{P}_{E_t(D)} \hat{M} \left( \hat{P}_{E_{t-1}(D)} + \hat{P}_{I_{t-1}(D)} \right) \hat{M}^{t-1} \hat{P}_D \right\|_{L^2}$$

$$\leqslant \left\| \hat{P}_{E_t(D)} \hat{M} \left( \hat{P}_{E_{t-1}(D)} \hat{M}^{t-1} \hat{P}_D \right) \right\|_{L^2}$$

$$+ \left\| \left( \hat{P}_{E_t(D)} \hat{M} \hat{P}_{I_{t-1}(D)} \right) \hat{M}^{t-1} \hat{P}_D \right\|_{L^2}$$

$$\leqslant \varepsilon \left( \sum_{s=0}^{t-2} \left( \frac{S (I_s (D))}{2\pi\hbar} \right)^{1/2} \right) + \varepsilon \left( \frac{S (I_{t-1} (D))}{2\pi\hbar} \right)^{1/2}$$

where we have used: $|A + B|_{L^2} \leqslant |A|_{L^2} + |B|_{L^2}$, $|AB|_{L^2} \leqslant |A|_{L^2} |B|_{L^2}$, and $\left|\hat{P}_D\right|_{L^2} \leqslant 1$, $\left|\hat{M}\right|_{L^2} = 1$. $\hfill \square$

### A.3.1. A particular application

We consider now a particular application of the previous results, which will be useful for hyperbolic dynamics. Let $t \geqslant 1$, and suppose that $D_0, D_1, \ldots, D_t$ is a sequence of sets. For any $s \in [1, t-1]$, let

$$G_s \overset{\text{def}}{=} I_1 (D_{s-1}) \backslash D_s$$

which corresponds to points coming from $D_{s-1}$ but not belonging to $D_s$.

LEMMA A.5. — *Suppose that for every* $s \in [1, t-1]$,

$$I_{t-s} (G_s) \cap D_t = \emptyset$$

*which means that "points which leave the domain $D_s$ do not come back to the domain $D_t$" (despite the thickening procedure), see Figure A.2. Then*

$$\left\| \hat{P}_{D_t} \hat{M}^t \hat{P}_{D_0} - \hat{P}_{D_t} \hat{M} \hat{P}_{D_{t-1}} \ldots \hat{M} \hat{P}_{D_1} \hat{M} \hat{P}_{D_0} \right\|_{L^2}$$

$$\leqslant \frac{\varepsilon}{\sqrt{2\pi\hbar}} \sum_{s=1}^{t-1} \left( (S(D_s))^{1/2} + \sum_{k=0}^{t-s-1} (S(I_k (G_s)))^{1/2} \right)$$

*Proof.* — For any $s \in [1, t-1]$, let

$$F_s \overset{\text{def}}{=} E_1 (D_{s-1}) \backslash D_s, \qquad G_s \overset{\text{def}}{=} I_1 (D_{s-1}) \backslash D_s,$$

So that $\mathbb{R}^2 = D_s \cup F_s \cup G_s$ as a disjoint union. The hypothesis $I_{t-s} (G_s) \cap D_t = \emptyset$ means that $D_t \subset E_{t-s} (G_s)$. Then Eq.(A.5) gives,

$$\left| \hat{P}_{D_t} \hat{M}^{t-s} \hat{P}_{G_s} \right|_{L^2} < \varepsilon \left( \sum_{k=0}^{t-s-1} \left( \frac{S (I_k (G_s))}{2\pi\hbar} \right)^{1/2} \right)$$

Also, $F_s \subset E_1 (D_{s-1})$, so Eq.(A.4) gives:

$$\left| \hat{P}_{F_s} \hat{M} \hat{P}_{D_{s-1}} \right|_{L^2} < \left( \frac{S (D_s)}{2\pi\hbar} \right)^{1/2} \varepsilon$$

Now we proceed by induction on $s$. We use $\hat{\text{Id}} = \hat{P}_{D_1} + \hat{P}_{F_1} + \hat{P}_{G_1}$ (from the disjoint union $\mathbb{R}^2 = D_s \cup F_s \cup G_s$), and write

$$\hat{P}_{D_t} \hat{M}^t \hat{P}_{D_0} = \hat{P}_{D_t} \hat{M}^{t-1} \left( \hat{P}_{D_1} + \hat{P}_{F_1} + \hat{P}_{G_1} \right) \hat{M} \hat{P}_{D_0}$$
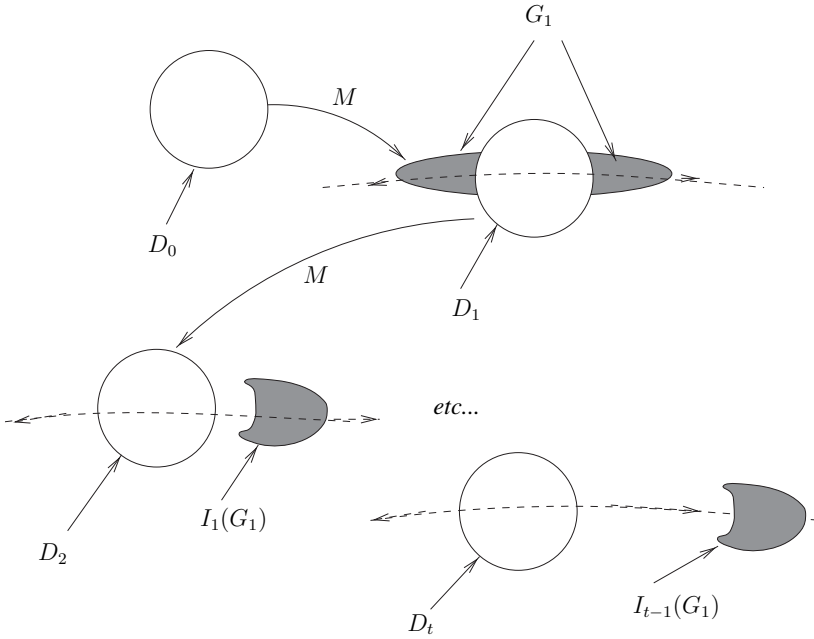
Figure A.2. Description of the sets $I_{t-s}(G_s)$ used in Lemma A.5.

so

$$\left\| \hat{P}_{D_t} \hat{M}^t \hat{P}_{D_0} - \hat{P}_{D_t} \hat{M}^{t-1} \hat{P}_{D_1} \hat{M} \hat{P}_{D_0} \right\|_{L^2}$$

$$\leqslant \left\| \hat{P}_{D_t} \hat{M}^{t-1} \left( \hat{P}_{F_1} \hat{M} \hat{P}_{D_0} \right) \right\|_{L^2} + \left\| \left( \hat{P}_{D_t} \hat{M}^{t-1} \hat{P}_{G_1} \right) \hat{M} \hat{P}_{D_0} \right\|_{L^2}$$

$$\leqslant \varepsilon \left( \frac{\mathrm{S}(D_1)}{2\pi\hbar} \right)^{1/2} + \varepsilon \left( \sum_{k=0}^{t-2} \left( \frac{\mathrm{S}(I_k(G_1))}{2\pi\hbar} \right)^{1/2} \right)$$

Similarly,

$$\left\| \hat{P}_{D_t} \hat{M}^{t-1} \hat{P}_{D_1} \hat{M} \hat{P}_{D_0} - \hat{P}_{D_t} \hat{M}^{t-2} \hat{P}_{D_2} \hat{M} \hat{P}_{D_1} \hat{M} \hat{P}_{D_0} \right\|_{L^2}$$

$$\leqslant \varepsilon \left( \frac{\mathrm{S}(D_2)}{2\pi\hbar} \right)^{1/2} + \varepsilon \left( \sum_{k=0}^{t-3} \left( \frac{\mathrm{S}(I_k(G_2))}{2\pi\hbar} \right)^{1/2} \right)$$

and therefore after few similar steps

$$\left\| \hat{P}_{D_t} \hat{M}^t \hat{P}_{D_0} - \hat{P}_{D_t} \hat{M} \hat{P}_{D_{t-1}} \ldots \hat{P}_{D_1} \hat{M} \hat{P}_{D_0} \right\|_{L^2}$$

$$\leqslant \varepsilon \sum_{s=1}^{t-1} \left( \frac{\mathrm{S}\,(D_s)}{2\pi\hbar} \right)^{1/2} + \varepsilon \left( \sum_{s=1}^{t-1} \sum_{k=0}^{t-s-1} \left( \frac{\mathrm{S}\,(I_k\,(G_s))}{2\pi\hbar} \right)^{1/2} \right)$$

$\square$

## A.4. Application to hyperbolic dynamics

### A.4.1. Proof of theorem 4.2

The preceding results were rather general. We will apply them to the hyperbolic dynamics considered in this paper. In order to prove theorem 4.2, we consider a trajectory $x\,(s) = M^s\,(x\,(0))$, $s = 1 \ldots t$, and disks $D_s$ of radius $\hbar^{1/2-\alpha}$ at each point $x\,(s)$. Using definitions introduced with Lemma A.5 page 2567, we remark that the hypothesis $I_{t-s}\,(G_s) \cap D_t = \emptyset$ is satisfied, because if $\beta < \alpha$, and due to hyperbolic instability of the trajectory $x\,(s)$, the dynamics sends the domain $G_s$ away from the domain $D_t$, in spite of the thickening (see figure A.2). We have $\mathrm{S}\,(D_s) \simeq \mathcal{O}\left(\hbar^{1/2-\alpha}\right)^2$ but the surface $\mathrm{S}\,(I_k\,(G_s))$ grows exponentially fast with $k$ (in the unstable direction) because of the thickening at each time step. A rough estimate gives:

$$\mathrm{S}\,(I_k\,(G)) = \mathcal{O}\left( \left(e^{\lambda_{max} k} \hbar^{1/2-\alpha}\right) \left(\hbar^{1/2-\alpha}\right) \right)$$

Suppose that $0 < t < C \log\,(1/\hbar) \Leftrightarrow e^{\lambda_{max} t} < \hbar^{-C\lambda_{max}}$. Then Lemma A.5, and $\varepsilon = C\hbar^K$, gives

$$\left\| \hat{P}_{D_t} \hat{M}^t \hat{P}_{D_0} - \hat{P}_{D_t} \hat{M} \hat{P}_{D_{t-1}} \ldots \hat{M} \hat{P}_{D_1} \hat{M} \hat{P}_{D_0} \right\|_{L^2}$$

$$\leqslant \frac{C\hbar^K}{\sqrt{2\pi\hbar}} \hbar^{1/2-\alpha} \sum_{s=1}^{t-1} \left( 1 + \sum_{k=0}^{t-s-1} e^{\lambda_{max} k/2} \right)$$

$$\leqslant \left( \frac{C}{e^{\lambda_{max}/2} - 1} \right) \hbar^K \hbar^{-\alpha} \hbar^{-C\lambda_{max}/2} t$$

For any $K' \geqslant 1$, the bound is $\mathcal{O}\left(\hbar^{K'}\right)$ if $K$ is chosen large enough. So we get Eq.(4.4).

### A.4.2. An other consequence

LEMMA A.6. — *Let $0 < \alpha < 1/2$. For any $C > 0$, any $t$, such that $|t| < C \log(1/\hbar)$, for any $n \in \mathbb{Z}^2$, any $x \in \mathbb{R}^2$, such that $|x - x_{n,t}| > \hbar^{1/2-\alpha}$ (which means that $x$ is at some distance of the hyperbolic fixed point $x_{n,t}$ of the hyperbolic map $M_{n,t}$, Eq.(3.7)) then*

$$(A.6) \qquad\qquad \langle x|\hat{M}_{n,t}|x\rangle = \mathcal{O}(\hbar^\infty)$$

*(uniformly other $x \in \mathbb{R}^2, n, t$), where $\hat{M}_{n,t} = \hat{T}_{-n}\hat{M}^t$ has been defined in Eq.(3.6). A light generalization is*

$$(A.7) \qquad\qquad \langle x'|\hat{M}_{n,t}|x\rangle = \mathcal{O}(\hbar^\infty)$$

*if $|x - x_{n,t}| > \hbar^{1/2-\alpha}$ and $|x - x'| < \hbar^{1/2-\alpha'}$, with $0 < \alpha' < \alpha$. This implies:*

$$(A.8) \qquad \mathrm{Tr}\left(\hat{P}_{x_{n,t},\alpha}\hat{M}_{n,t}\hat{P}_{x_{n,t},\alpha}\right) = \mathrm{Tr}\left(\hat{M}_{n,t}\hat{P}_{x_{n,t},\alpha}\right) + \mathcal{O}(\hbar^\infty)$$

*where $\hat{P}_{x_{n,t},\alpha}$ is defined in Eq.(4.2).*

Proof. — Let $0 < \gamma < \alpha'$, and let $D'_x$ be the disk of radius $\hbar^{1/2-\gamma}$ with center $x$. With the hypothesis $|x - x_{n,t}| > \hbar^{1/2-\alpha}$, and $|x - x'| < \hbar^{1/2-\alpha'}$, one checks that for $0 < \beta < \gamma$, then $D'_{x'} \subset E_t(D'_x)$. Then from Lemma A.4, and similar estimates as in Section A.4.1, one gets $\left|\hat{P}_{D'_{x'}}\hat{M}_{n,t}\hat{P}_{D'_x}\right| = \mathcal{O}(\hbar^\infty)$. This implies that $\langle x'|\hat{M}_{n,t}|x\rangle = \mathcal{O}(\hbar^\infty)$. To show Eq.(A.8), write $\mathrm{Tr}\left(\hat{M}_{n,t}\hat{P}_{x_{n,t},\alpha}\right) = \mathrm{Tr}\left(\hat{P}_{x_{n,t},\alpha}\hat{M}_{n,t}\hat{P}_{x_{n,t},\alpha}\right) + \mathrm{Tr}\left(\left(1 - \hat{P}_{x_{n,t},\alpha}\right)\hat{M}_{n,t}\hat{P}_{x_{n,t},\alpha}\right)$. But

$$\mathcal{I} \stackrel{\text{def}}{=} \mathrm{Tr}\left(\left(1 - \hat{P}_{x_{n,t},\alpha}\right)\hat{M}_{n,t}\hat{P}_{x_{n,t},\alpha}\right)$$
$$= \int_{|x|<\hbar^{1/2-\alpha}}\int_{|x'|>\hbar^{1/2-\alpha}}\frac{dx\,dx'}{(2\pi\hbar)^2}\langle x|x'\rangle\langle x'|\hat{M}_{n,t}|x\rangle$$

Using the exponential decrease of $|\langle x|x'\rangle| = C\exp\left(-|x - x'|^2/\hbar\right)$, and $\left|\langle x'|\hat{M}_{n,t}|x\rangle\right| \leqslant 1$, one first obtains:

$$\mathcal{I} = \int_{\hbar^{1/2-\alpha_1}<|x|<\hbar^{1/2-\alpha}}\int_{|x'-x|<\hbar^{1/2-\alpha'}}\frac{dx\,dx'}{(2\pi\hbar)^2}\langle x|x'\rangle\langle x'|\hat{M}_{n,t}|x\rangle + \mathcal{O}(\hbar^\infty)$$

with $0 < \alpha_1, \alpha' < \alpha$, and using Eq.(A.7), one finally obtains $\mathcal{I} = \mathcal{O}(\hbar^\infty)$, and deduces Eq.(A.8). $\qquad\qquad\square$

# Appendix B. Proof of Lemma 4.3 page 2548 for the trace of an individual orbit for long time

## B.1. General results to control localization of wave packets

We first give some results which will allow us to control the escape towards infinity in $\mathbb{R}^2$ of the phase-space distribution of wave packets after long time, under a hyperbolic dynamics. The usual semiclassical Egorov Theorem which describes the propagation of quantum observables does not work after a fixed constant times the Ehrenfest time ($\frac{1}{2}t_E$ in our case, see Section 1.2). The aim of the Section is to develop a weaker description which will be able to reach $t = C \log(1/\hbar)$, with any $C > 0$. Remind that the classical phase space is $x = (q, p) \in \mathbb{R}^2$. The quantum Hilbert space is $\mathcal{H}_{plane} = L^2(\mathbb{R})$ with Planck constant $\hbar$.

Let $m$ be an order function which possibly decreases at infinity ([21] p. 81, [49] p. 12, [23] p. 52).

DEFINITION B.1. — A **weight function** is a semi-classical symbol $W :$ $\mathbb{R}^2 \to \mathbb{R}^{+,*}$, $W \in S(m)$, such that $|W|_\infty = \max_x(W(x)) = 1$, and $W$ is elliptic.

Remark: $W$ elliptic means $W \geqslant \frac{1}{C} m$, with $C > 0$, then $W^{-1} \in S(m^{-1})$, see [21] p.100, [49] p. 13.

Example: $W(x) = m(x) = 1/\langle x \rangle^k$ with $\langle x \rangle = \sqrt{1 + q^2 + p^2}$, $k \geqslant 1$, or $W(x) = m(x) = \exp(-\langle x \rangle)$.

We write $\hat{W} = \mathrm{Op}_{Weyl}(W)$. We check that these operators have dense domain on $L^2(\mathbb{R})$.

DEFINITION B.2. — If $W$ is a weight function, a sequence of states $\psi_\hbar$, with $\hbar \to 0$, is **W-localized** if $\|\psi_\hbar\| = 1$ and if there exists $C > 0$ (independent of $\hbar$) such that

$$\left\| \hat{W}^{-1} \psi_\hbar \right\|_{L^2} \leqslant C$$

The intuitive idea of this definition, is that a W-localized quantum state has a distribution in phase space which is bounded by the function $W(x)$. This definition is very similar with weight functions used to treat tunnelling effect.

Remind that the dynamics considered in this paper is the map $M : \mathbb{R}^2 \to \mathbb{R}^2$, defined in Eq.(2.3) as a product $M = M_1 M_0$, with $M_0 \in \mathrm{SL}(2, \mathbb{R})$ linear hyperbolic and $M_1$ bounded. We also defined $\hat{M}$ in Eq.(2.15).

PROPOSITION B.3. — *Suppose that $W_0 \in S(m_0)$, $W_1 \in S(m_1)$ are weight functions such that*

$$W_1(x) \geqslant W_0\left(M^{-1}(x)\right), \qquad \forall x \in \mathbb{R}^2$$

*and $m_1(x) \geqslant m_0\left(M_0^{-1}(x)\right), \forall x$. Suppose that $\psi_0$ is $W_0$-localized. Then*

$$\psi_1 = \hat{M}\psi_0$$

*is $W_1$-localized.*

Proof. — Write

$$\left\|\hat{W}_1^{-1}\psi_1\right\| = \left\|\hat{W}_1^{-1}\hat{M}\psi_0\right\| = \left\|\hat{W}_1^{-1}\hat{M}\hat{W}_0\hat{W}_0^{-1}\psi_0\right\|$$

From Egorov theorem ([21] p.125., [49] page 138, [23] p.139)

$$\hat{M}\hat{W}_0\hat{M}^{-1} = \hat{W}' + \hbar\hat{R}$$

where $\hat{W}' = Op(W')$ and $W'$ has Weyl symbol $W'(x) = W_0\left(M^{-1}(x)\right)$, and belongs to $S(m')$, with order function $m' = m \circ M_0^{-1}$. The remainder $\hat{R}$ has Weyl symbol $R \in S(m')$.

Then

$$\left\|\hat{W}_1^{-1}\hat{M}_1\hat{W}_0\hat{W}_0^{-1}\psi_0\right\| = \left\|\hat{W}_1^{-1}\left(\hat{W}' + \hbar\hat{R}\right)\hat{M}\hat{W}_0^{-1}\psi_0\right\|$$

$$\leqslant \left\|\hat{W}_1^{-1}\left(\hat{W}' + \hbar\hat{R}\right)\right\| \left\|\hat{M}\right\| \left\|\hat{W}_0^{-1}\psi_0\right\|$$

By hypothesis, $W_1 \geqslant W' \Leftrightarrow \left|W_1^{-1}W'\right|_\infty \leqslant 1$, and $W_1^{-1}(W' + \hbar R) \in S\left(m_1^{-1}m'\right) = S(1)$. By Calderon-Vaillancourt Theorem ([21] p.85, [49] page 43, [23] p.69), $\left\|\hat{W}_1^{-1}\left(\hat{W}' + \hbar\hat{R}\right)\right\| \leqslant 1 + \mathcal{O}(\hbar)$. Also $\left\|\hat{M}\right\| = 1$, and $\left\|\hat{W}_0^{-1}\psi_0\right\| \leqslant C$. We get:

$$\left\|\hat{W}_1^{-1}\psi_1\right\| \leqslant C.(1 + \mathcal{O}(\hbar))$$

$$\square$$

PROPOSITION B.4. — *Suppose that $W_0$, $W_1$ are weight functions, and that $\psi_0$ is $W_0$-localized, and $\psi_1$ is $W_1$-localized. Then there exists $C > 0$ and $\hbar_0 > 0$ such that for any $\hbar < \hbar_0$:*

$$|\langle\psi_0|\psi_1\rangle| \leqslant C.|W_0W_1|_\infty$$

Proof. — We suppose $\left\|\hat{W}_0^{-1}\psi_0\right\| \leqslant C_0$ and $\left\|\hat{W}_1^{-1}\psi_1\right\| \leqslant C_1$. Then

$$|\langle\psi_0|\psi_1\rangle| = \left|\langle\psi_0|\hat{W}_0^{-1}\hat{W}_0\hat{W}_1\hat{W}_1^{-1}|\psi_1\rangle\right|$$

$$\leqslant \left\|\hat{W}_0^{-1}\psi_0\right\| \left\|\hat{W}_1^{-1}\psi_1\right\| \left\|\hat{W}_0\hat{W}_1\right\|$$

From Calderon-Vaillancourt Theorem, $\left\|\hat{W}_0\hat{W}_1\right\| \leqslant |W_0W_1|_\infty + \mathcal{O}(\hbar)$. We get:

$$|\langle\psi_0|\psi_1\rangle| \leqslant C_0C_1 |W_0W_1|_\infty + \mathcal{O}(\hbar)$$

But $|W_0W_1|_\infty \leqslant 1$, so

$$|\langle\psi_0|\psi_1\rangle| \leqslant C.|W_0W_1|_\infty$$

with $C > 0$.                                                                                                               $\square$


## B.2. Application

The previous paragraph will allow us to obtain the following Lemma:

LEMMA B.5. — *There exists $R > 0$, such that for any $C > 0$, $0 < \alpha < 1/2$, there exists $\hbar_0$, $C_1$, such that for any $t \leqslant C/\hbar$, $\hbar < \hbar_0$, $|x_0| > R$, one has*

$$\left|\langle x_0|\hat{M}^t|x_0\rangle\right| \leqslant C_1 e^{-\frac{(|x_0|-R)}{\hbar^{1/2-\alpha}}}$$

*where $|x_0\rangle$ is a coherent state at position $x_0 = (q_0, p_0) \in \mathbb{R}^2$.*

This last results means that the auto-correlation function $\left|\langle x_0|\hat{M}^t|x_0\rangle\right|$ decreases exponentially fast as $x_0$ goes to infinity. This is due to the hyperbolicity of the classical map $M$ on $\mathbb{R}^2$. The proof is given below. We deduce:

COROLLARY B.6. — *There exists $R > 0$, $C > 0$, such that for any $1 \leqslant t \leqslant C/\hbar$, $\hbar < \hbar_0$, one has*

$$\int_{|x_0|>R} \langle x_0|\hat{M}^t|x_0\rangle \frac{dx_0}{2\pi\hbar} = \mathcal{O}(\hbar^\infty)$$

(B.1)    $$\int_{|x-x_{n,t}|>R} \langle x|\hat{M}_{n,t}|x\rangle \frac{dx}{2\pi\hbar} = \mathcal{O}(\hbar^\infty)$$

The second equation is a easy generalisation of the first one, where we considered the map Eq.(3.6). This last result improves the result of convergence Eq.(3.9), because it shows that the integral is semiclassically negligible outside a disk of *fixed* radius $R$.


### B.2.1. Proof of Lemma B.5

In all the paragraph, let us denote by $x = (q, p) \in \mathbb{R}^2$ the unstable/stable coordinates in which the matrix $M_0$ , Eq.(2.2), is diagonal. $M_0 : (q, p) \rightarrow \left(e^{\lambda_0}q, e^{-\lambda_0}p\right)$.

**A first result on the classical map $M$:** Remind that the perturbation $M_1$ is bounded: $|M_1(x) - x| \leqslant C$, for any $x \in \mathbb{R}^2$. Without further assumption on $M_1$ (we don't assume here that $M = M_1 M_0$ is hyperbolic), we have the following simple result Lemma on the map $M = M_1 M_0$:

LEMMA B.7. — *For any $\varepsilon$, such that $0 < \varepsilon < \lambda_0$, let $\lambda = \lambda_0 - \varepsilon > 0$, $R = \frac{Ce^{-\lambda_0}}{(1 - e^{-\varepsilon})} > 0$. For any point $x = (q, p)$, let $x' = (q', p') = M(x)$. If $q > R$, then $q' > e^\lambda q$.*

This result guaranties that under the map $M$, points escape exponentially fast towards infinity if $q$ is large enough.

*Proof.* — We have $x' = M_1 M_0(x)$, so $\left| q' - e^{\lambda_0} q \right| \leqslant C$, so $q' \geqslant e^{\lambda_0} q - C$. And $e^{\lambda_0} q - C > e^\lambda q \Leftrightarrow q > R$. $\square$

**Choice of Weight functions $W_t$:** In order to apply later on Proposition B.3, let $x_0 = (q_0, p_0) \in \mathbb{R}^2$, with $q_0 > R$. Let $0 < \alpha < 1/2$ (by rescaling, we use semiclassical calculus at the scale $\hbar^{1/2 - \alpha}$, [21] p.82, [23] p.52). For any $t \in \mathbb{N}$, let us choose a weight function $W_t(q)$ (independent of $p$ and increasing in $q$), defined by $W_t(q) = 1$, for $q \geqslant q_0 e^{\lambda t}$, $W_t(q) = \exp\left( \left( q e^{-\lambda t} - q_0 \right) / \hbar^{1/2 - \alpha} \right)$ for $e^{\lambda t} R < q < q_0 e^{\lambda t}$, and $W_t(q) = W_{-\infty} \overset{\text{def}}{=} e^{-(q_0 - R)/\hbar^{1/2 - \alpha}}$ for $q < e^{\lambda t} R$. We smooth $W_t(q)$ in the vicinity of $R$ and $q_0$ and let it be still increasing but $C^\infty$. For any $t$, the function $W_t$ is a weight function according to definition B.1, but at the semiclassical scale $\tilde{x} = x / \left( \hbar^{1/2 - \alpha} \right)$. From Lemma B.7, we check that it satisfies:

LEMMA B.8. — *For any $x = (q, p) \in \mathbb{R}^2$, any $t \geqslant 1$, then*

$$W_t(x) \geqslant W_{t-1}\left( M^{-1}(x) \right)$$

LEMMA B.9. — *Let $\psi_0 = |x_0\rangle$ be a coherent state at position $x_0 \in \mathbb{R}^2$, with $q_0 > R$. Then $\psi_0$ is $W_0$-localized (according to definition B.2), and for any $0 \leqslant t \leqslant C/\hbar$, $\psi_t \overset{\text{def}}{=} \hat{M}^t \psi_0$ is $W_t$-localized.*

*Proof.* — $\psi_0$ is $W_0$-localized because as explained in Section 2.2.3, a Gaussian wave packet $\psi_0(q)$ is exponentially localized in $q$, around $q_0$. Then with Lemma B.8 and Lemma B.3, we deduce iteratively that $\psi_t$ is $W_t$-localized for any $t$. We have to be careful, and observe that in the proof of Lemma B.3, there appears $\left\| \hat{W}_t^{-1} \psi_t \right\| \leqslant C_{t-1} \cdot (1 + \mathcal{O}(\hbar))$. Under iterations, this gives $C_0 (1 + \mathcal{O}(\hbar))^t$, which is still $\mathcal{O}(1)$ if $t \leqslant C/\hbar$. $\square$

**Last step of the proof:** Now let us consider another weight function $W'(q)$ (again independent of $p$) defined by $W'(q) = 1$ for $q < q_0$, $W'(q) = \exp\left( -(q - q_0) / \hbar^{1/2 - \alpha} \right)$ if $q > q_0$. We modify $W'$ in the vicinity of $q_0$, so

that it is $C^\infty$. As above, it is clear that the coherent state $\psi_0 = |x_0\rangle$ is $W'$-localized. Then we apply Proposition B.4 to deduce that $|\langle\psi_0|\psi_t\rangle| \leqslant C.|W'W_t|_\infty$. Observe that $|W'W_t|_\infty = W_{-\infty} = e^{-(q_0-R)/\hbar^{1/2-\alpha}}$ so this gives

$$(\text{B.2}) \qquad \left|\langle x_0|\hat{M}^t|x_0\rangle\right| \leqslant C e^{-(q_0-R)/\hbar^{1/2-\alpha}}$$

The same analysis can be done for $q_0 < R$, and similarly for $|p_0| > R$ (in that last case, this is the stable direction, we have to work in the past, writing $\langle x_0|\hat{M}^t|x_0\rangle = \langle\psi_{-t}|\psi_0\rangle$). Finally we deduce Lemma B.5.

## B.3. A more refined estimate

Corollary B.6 is not precise enough to give Lemma 4.3 we are looking for. Indeed, instead of a disk of finite radius $R$, we have to reduce the integral Eq.(3.9) to a disk of smaller radius $\hbar^{1/2-\alpha}$. Let $0 < \alpha < 1/2$, and $t < C\log(1/\hbar)$, with $C > 0$. Decompose Eq.(3.9) in three parts:

$$T\left(\hat{M}_{n,t}\right) = \int_{|x-x_{n,t}|<\hbar^{1/2-\alpha}} \langle x|\hat{M}_{n,t}|x\rangle \frac{dx}{2\pi\hbar}$$
$$+ \int_{\hbar^{1/2-\alpha}<|x-x_{n,t}|<R} \langle x|\hat{M}_{n,t}|x\rangle \frac{dx}{2\pi\hbar}$$
$$+ \int_{R<|x-x_{n,t}|} \langle x|\hat{M}_{n,t}|x\rangle \frac{dx}{2\pi\hbar}$$

From Eq.(B.1) and Eq.(A.6), the last two integrals are $\mathcal{O}(\hbar^\infty)$. One obtains

$$T\left(\hat{M}_{n,t}\right) = \int_{x\in D_{x_{n,t}}} \frac{dx}{2\pi\hbar} \langle x|\hat{M}_{n,t}|x\rangle + \mathcal{O}(\hbar^\infty) = \text{Tr}\left(\hat{M}_{n,t}\hat{P}_{x_{n,t}}\right) + \mathcal{O}(\hbar^\infty)$$

where $\hat{P}_{x_{n,t}}$ is defined in Eq.(4.2). Eq.(A.8) gives

$$\text{Tr}\left(\hat{M}_{n,t}\hat{P}_{x_{n,t}}\right) = \text{Tr}\left(\hat{P}_{x_{n,t}}\hat{M}_{n,t}\hat{P}_{x_{n,t}}\right) + \mathcal{O}(\hbar^\infty).$$

Finally Lemma 4.3 is proved.

### B.3.1. Estimates of Trace norm in terms of Operator norm

We consider operators $\hat{P}_{x,\alpha}$ defined in Eq.(4.2), with $x = 0$.

LEMMA B.10. — *Suppose that $\hat{A}$ is bounded:* $\left\| \hat{A} \right\|_{L^2} = \mathcal{O}(1)$, *and* $\left\| \hat{P}_\alpha \hat{A} \right\|_{L^2} = \mathcal{O}\left( \hbar^K \right)$ *with* $K \geqslant 0$. *Let* $\alpha' > \alpha$. *Then in Trace norm:*

$$\left\| \hat{P}_{\alpha'} \hat{A} \right\|_1 = \mathcal{O}\left( \hbar^{K-\alpha'} \right), \qquad \left\| \hat{P}_{\alpha'} \hat{A} \hat{P}_{\alpha'} \right\|_1 = \mathcal{O}\left( \hbar^{K-\alpha'} \right)$$

*Similarly if* $\left\| \hat{B} \right\|_{L^2} = \mathcal{O}(1)$, *and* $\left\| \hat{P}_\alpha \hat{B} \hat{P}_\alpha \right\|_{L^2} = \mathcal{O}\left( \hbar^K \right)$ *then*

(B.3) $$\left\| \hat{P}_{\alpha'} \hat{B} \hat{P}_{\alpha'} \right\|_1 = \mathcal{O}\left( \hbar^{K-\alpha'} \right)$$

*Proof.* — We first give basic estimates:

(B.4) $$\left\| \hat{P}_\alpha \right\|_1 = \int_{|x| < \hbar^{1/2-\alpha}} \frac{1}{2\pi\hbar} = \mathcal{O}\left( \hbar^{-2\alpha} \right)$$

If $\alpha' > \alpha$,

$$\left\| \hat{P}_\alpha \left( 1 - \hat{P}_{\alpha'} \right) \right\|_{L^2} = \mathcal{O}\left( \hbar^\infty \right)$$

and

$$\left\| \hat{P}_\alpha \right\|_{L^2} < 1$$

One has $\left\| \hat{P}_{\alpha'} \hat{P}_\alpha \hat{A} \right\|_1 \leqslant \left\| \hat{P}_{\alpha'} \right\|_1 \left\| \hat{P}_\alpha \hat{A}_1 \right\|_{L^2} = \mathcal{O}\left( \hbar^{K-\alpha'} \right)$ (see [30], prop 5.4 page 62). Write $1 = 1 - \hat{P}_\alpha + \hat{P}_\alpha$, and

$$\left\| \hat{P}_{\alpha'} \hat{A} \right\|_1 \leqslant \left\| \hat{P}_{\alpha'} \hat{P}_\alpha \hat{A} \right\|_1 + \left\| \hat{P}_{\alpha'} \left( 1 - \hat{P}_\alpha \right) \hat{A} \right\|_1$$

where

$$\begin{aligned} \left\| \hat{P}_{\alpha'} \left( 1 - \hat{P}_\alpha \right) \hat{A} \right\|_1 &\leqslant \left\| \hat{P}_{\alpha'} \left( 1 - \hat{P}_\alpha \right) \right\|_1 \left\| \hat{A} \right\|_{L^2} \\ &\leqslant \left\| \hat{P}_{\alpha'} \left( 1 - \hat{P}_\alpha \right) \right\|_{L^2} \left\| \hat{A} \right\|_{L^2} \\ &= \mathcal{O}\left( \hbar^\infty \right) \end{aligned}$$

Therefore $\left\| \hat{P}_{\alpha'} \hat{A} \right\|_1 = \mathcal{O}\left( \hbar^{K-\alpha'} \right)$ and $\left\| \hat{P}_{\alpha'} \hat{A} \hat{P}_{\alpha'} \right\|_1 \leqslant \left\| \hat{P}_{\alpha'} \hat{A} \right\|_1 \left\| \hat{P}_{\alpha'} \right\|_{L^2} = \mathcal{O}\left( \hbar^{K-\alpha'} \right)$. For the second estimate, let $\hat{A} = \hat{B}\hat{P}_\alpha$. This gives $\left\| \hat{P}_{\alpha'} \hat{B} \hat{P}_\alpha \right\|_1 = \left\| \hat{P}_{\alpha'} \hat{A} \right\|_1 = \mathcal{O}\left( \hbar^{K-\alpha'} \right)$. As above, one can show that

$$\left\| \hat{P}_{\alpha'} \hat{B} \hat{P}_{\alpha'} \right\|_1 \leqslant \left\| \hat{P}_{\alpha'} \hat{B} \hat{P}_\alpha \right\|_1 + \mathcal{O}\left( \hbar^\infty \right).$$

$\square$

## Appendix C.  Useful results for finite time evolution

In this appendix we collect well known results on semiclassical finite time evolution. We state them in the context of the present paper. They are used in other parts of the paper.

### C.1.  Truncation of Taylor series of the Hamiltonian

THEOREM C.1. — *Let $H_1(x,t)$ and $H_2(x,t)$ be two symbols on $x = (q,p) \in \mathbb{R}^2$, such that 0 is a fixed point (i.e. $(DH_1)_{x=0} = 0$), and suppose that they have identical Taylor series in $(x,\hbar)$ at the origin $(0,0,0)$, up to order $J$:*

$$H_1(x,t) = H_2(x,t) + \mathcal{O}\left((x,\hbar)^{J+1}\right)$$

*(i.e. Taylor terms $\hbar^l q^a p^b$ are identical if $l + a + b \leqslant J$).*
*Let $\hat{U}_1 = \exp\left(-i\hat{H}_1/\hbar\right)$ and $\hat{U}_2 = \exp\left(-i\hat{H}_2/\hbar\right)$, with $\hat{H}_j = Op_{Weyl}(H_j)$, $j = 1, 2$. Let $0 < \alpha < 1/2$, and $\hat{P}_\alpha$ be defined by Eq.(4.2). Then*

(C.1)
$$\left\|\left(\hat{U}_1 - \hat{U}_2\right)\hat{P}_\alpha\right\|_{L^2} \leqslant C\hbar^A$$

*with $A = \left(\frac{1}{2} - \alpha'\right)(J+1) - 1$, for any $\alpha' > \alpha$.*

This result is interesting in the semiclassical limit if $A > 0 \Leftrightarrow J > \frac{1+2\alpha}{1-2\alpha}$, for example if $J = 2$ and $0 < \alpha < 1/6$. This theorem will be used page 2591 for the proof of Theorem 5.1.

### C.2.  Proof of Theorem C.1

We will use the Duhamel formula [44](which is not semiclassical):

PROPOSITION C.2. — *Suppose that $\hat{H}(t)$ is a self-adjoint operator in a Hilbert space $\mathcal{H}$, and $\psi(t) \in \mathcal{H}$ is solution of*

$$i\hbar\frac{d\psi}{dt} = \hat{H}(t)\psi + \xi(t)$$

*where $\xi(t) \in \mathcal{H}$, $\|\xi(t)\| \leqslant \mu(t)$, and $\|\psi(0)\| = 1$. Suppose that $\varphi(t)$ is solution of*

$$i\hbar\frac{d\varphi}{dt} = \hat{H}(t)\varphi$$

*with $\varphi(0) = \psi(0)$. Then*

(C.2)
$$\|\psi(t) - \varphi(0)\| \leqslant \frac{1}{\hbar}\int_0^t \mu(s)\,ds$$

We can deduce a "semiclassical version" of the Duhamel formula:

COROLLARY C.3. — *If* $H_1, H_2 \in S(1)$ *are real bounded semiclassical symbols (cf [21], p.81), and if* $K = H_1 - H_2 \in S^{-k}(1)$ *(i.e.* $K \in \hbar^k S(1)$*), and* $\hat{U}_1 = \exp\left(-i\hat{H}_1/\hbar\right)$, $\hat{U}_2 = \exp\left(-i\hat{H}_2/\hbar\right)$, *with* $\hat{H}_j = Op_{Weyl}(H_j)$, $j = 1, 2$, *then*

$$\left\|\hat{U}_1 - \hat{U}_2\right\|_{L^2} \leqslant \mathcal{O}\left(\hbar^{k-1}\right)$$

*Proof.* — $K \in S^{-k}(1)$, thus $\left\|\hat{K}\right\|_{L^2} = \mathcal{O}\left(\hbar^k\right)$, from Calderon-Vaillancourt Theorem. Let $\psi \in \mathcal{H}$, be normalized: $\|\psi\| = 1$, and $\psi_1(t)$, $\psi_2(t)$, solutions of $i\hbar\frac{d\psi_j}{dt} = \hat{H}_j \psi_j$, $j = 1, 2$, with $\psi_1(0) = \psi_2(0) = \psi$. This gives $i\hbar d\psi_2/dt = \left(\hat{H}_1 - \hat{K}\right)\psi_2 = \hat{H}_1 \psi_2 + \xi(t)$, with $\|\xi(t)\| \leqslant \left\|\hat{K}\right\|_{L^2} = \mathcal{O}\left(\hbar^k\right)$. From Eq.(C.2), we deduce $\left\|\hat{U}_2\psi - \hat{U}_1\psi\right\|_{L^2} \leqslant \frac{1}{\hbar}\mathcal{O}\left(\hbar^k\right)$.                    □

**Micro-Localization Lemma:**

LEMMA C.4. — *Let* $0 < \alpha < \alpha' < 1/2$, *and* $H(x,t)$ *a real symbol, with fixed point 0:* $DH_{/x=0} = 0$. *Let* $\hat{H}(t) = Op_{Weyl}(H(t))$, *and* $\hat{H}'(t) = \hat{H}(t)\hat{P}_{\alpha'}$ *be the truncated Hamiltonian, where* $\hat{P}_{\alpha'}$ *is defined in Eq.(4.2). Let* $\hat{U}(t)$ *and* $\hat{U}'(t)$ *be the evolution operators defined by* $i\hbar\frac{d\hat{U}(t)}{dt} = \hat{H}(t)\hat{U}(t)$, $i\hbar\frac{d\hat{U}'(t)}{dt} = \hat{H}'(t)\hat{U}'(t)$, *and* $\hat{U}(0) = \hat{U}'(0) = \hat{Id}$. *Then for any finite time* $t$,

$$\left\|\left(\hat{U}(t) - \hat{U}'(t)\right)\hat{P}_\alpha\right\|_{L^2} = \mathcal{O}\left(\hbar^\infty\right)$$

*Remark.* — Remind that $\hat{P}_\alpha$ truncates on a disk of radius $\hbar^{1/2-\alpha}$, and notice that $\hbar^{1/2} \ll \hbar^{1/2-\alpha} \ll \hbar^{1/2-\alpha'}$ in the semiclassical limit. Proposition C.4 means that the evolution of a quantum states in the vicinity of a fixed point (or more generally of a trajectory) after finite time, depends only of $H(x)$ in the vicinity of this fixed point. The proof of Lemma C.4 relies on Egorov Theorem, and can be found for example in [41], Lemma 3, Lemma 4.

**Proof of Theorem C.1:**

*Proof.* — We use notations of Lemma C.4 and Corollary C.3, and combine them. Let $H'_j = H_j\chi'$, $j = 1, 2$, where $\chi'(x)$ truncates at radius $r' = \hbar^{1/2-\alpha'}$ (i.e. $\chi'(x) = 1$, if $|x| < r'$, $\chi'(x) = 0$, if $|x| > 2r'$). Similarly, let $\chi(x)$ which truncates at radius $r = \hbar^{1/2-\alpha}$. Let $\hat{U}_j = \exp\left(-i\hat{H}_j/\hbar\right)$, $\hat{U}'_j = \exp\left(-i\hat{H}'_j/\hbar\right)$. By hypothesis, we have $K = (H'_2 - H'_1) \in S^{-k}_{\alpha'}(1)$, with

$\hbar^k = \hbar^{(1/2-\alpha')J}$ i.e. $k = (1/2 - \alpha') J$. From Corollary C.3, $\left\|\hat{U}_1' - \hat{U}_2'\right\|_{L^2} = \mathcal{O}\left(\hbar^{k-1}\right)$. Let $\hat{\chi} = Op_{Weyl}(\chi)$. One has

$$\left\|\left(\hat{U}_1 - \hat{U}_2\right)\hat{\chi}\right\|_{L^2} = \left\|\left(\hat{U}_1 - \hat{U}_1' + \hat{U}_1' - \hat{U}_2' + \hat{U}_2' - \hat{U}_2\right)\hat{\chi}\right\|$$
$$\leqslant \left\|\left(\hat{U}_1 - \hat{U}_1'\right)\hat{\chi}\right\| + \left\|\left(\hat{U}_1' - \hat{U}_2'\right)\hat{\chi}\right\| + \left\|\left(\hat{U}_2' - \hat{U}_2\right)\hat{\chi}\right\|$$
$$\leqslant \mathcal{O}\left(\hbar^\infty\right) + \mathcal{O}\left(\hbar^{k-1}\right) + \mathcal{O}\left(\hbar^\infty\right)$$

$\square$

# Appendix D. Proof of Theorem 5.1 on Semiclassical Non Stationary Normal Forms

In this appendix we develop the Theory of Semi-classical Non-Stationary Normal form for hyperbolic map on the torus, which extends the results of David DeLatte [19] from the classical to the semi-classical case.

## D.1. Description of the dynamics:

Let us write $x = (q, p) \in \mathbb{R}^2$. Remind that the classical map $M = M_1 M_0 :$ $\mathbb{R}^2 \to \mathbb{R}^2$ was defined in Eq.(2.3) as the result of a first flow $M_0$ generated by the quadratic Hamiltonian $H_0$ for time interval $t \in [0, 1]$, followed by the "perturbation" $M_1$ generated by the Hamiltonian $H_1$ on time interval $t \in [1, 2]$. $H_1(x)$ is periodic with respect to $\mathbb{Z}^2 \subset \mathbb{R}^2$. So the dynamics is defined by a *time dependant* Hamiltonian function written $H(x, t)$ with period 2 in $t$.

For any $t, t' \in \mathbb{R}$, we denote $M_{t,t'} : \mathbb{R}^2 \to \mathbb{R}^2$ the flow generated by the Hamiltonian $H(x, t)$, in the time interval $[t, t']$. In particular $M = M_{0,2}$.

Using the Weyl quantization procedure, we have defined the corresponding quantum Hamiltonian $\hat{H}(t) = Op_{Weyl}(H(x, t))$, in Section 2.2. Conversely $H(x, t)$ is called the total semiclassical symbol of $\hat{H}(t)$. The unitary evolution operator $\hat{M}_{t,t'}$ is defined by $\frac{d\hat{M}_{t,t'}}{dt'} = -\frac{i}{\hbar}\hat{H}(t')\hat{M}_{t,t'}$, and $\hat{M}_{t,t} = \hat{Id}$.

**Stable and unstable tangent vectors for intermediate time $t$:** In Section 2.1.3, we explained that for every point $x \in \mathbb{T}^2$ , there is a basis of tangent vectors $(u_x, s_x) \in T_x\mathbb{T}^2$, continuous with respect to $x \in \mathbb{T}^2$, and tangent to the unstable/stable foliation. We can define this basis for

intermediate time $t \in [0,2]$ (and then for any time $t \in \mathbb{R}$ by requiring periodicity), giving vectors $u_{x,t}, s_{x,t}$ as follows: the direction of $u_{x,t}$ (respect. $s_{x,t}$) is the direction $(DM_{0,t})_{x'} u_{x'}$ with $x = M_{0,t}(x')$ (respect. for $s_{x'}$), and we fix the choice of $u_{x,t}$ (respect. $s_{x,t}$) by requiring that

(D.1) $$u_{x,t} \wedge s_{x,t} = 1$$

i.e. they form a symplectic basis, and we also impose that $\|u_{x,t}\| = \|s_{x,t}\|$.

Let us define the lattices $\Gamma_0 = \mathbb{Z}^2$, $\Gamma_t = M_{0,t}(\Gamma_0)$ for $t \in [0,1]$, and $\Gamma_t = \Gamma_0$ for $t \in [1,2]$. Let us define $\mathbb{T}_t \overset{\text{def}}{=} \mathbb{R}^2/\Gamma_t$. If we consider $t \in S^1 = \mathbb{R}/(2\mathbb{Z})$, then the phase space and time containing points $(x,t)$ forms a non trivial bundle of tori $\mathcal{B} \to S^1$ over $S^1$, whose fiber over $t \in S^1$ is the torus $\mathbb{T}_t^2 = \mathbb{R}^2/\Gamma_t$. $u_{x,t}$ and $s_{x,t}$ are continuous in $x \in \mathbb{R}^2$ and periodic with respect to $\Gamma_t$, and also continuous and periodic in $t \in \mathbb{R}$. In other words they are continuous functions of $(x,t) \in \mathcal{B}$. In this section periodicity in $x$ and $t$ will always refer to the lattice $\Gamma_t$ and to $S^1 = \mathbb{R}/(2\mathbb{Z})$ respectively.

Define the expansion rate $\lambda(x,t,t')$ by

$$(DM_{t,t'})_x (u_{x,t}) = e^{\lambda(x,t,t')} u_{M_{t,t'}(x),t'}$$

and the local expansion rate $\eta(x,t)$ by

(D.2) $$\eta(x,t) = \left(\frac{d\lambda}{dt'}\right)_{t'=t}.$$

$\eta(x,t)$ is a piece-wise continuous function and periodic. In particular

$$\lambda_x = \int_0^2 \eta(x(t),t)\, dt = \lambda(x,0,2)$$

gives the expansion rate already defined in Eq.(2.7).

## D.2. Preliminaries on Time-dependant canonical transformations

In order to simplify the description of the dynamics, and obtain the non-stationary normal form, we will perform time dependant canonical transformations at the semi-classical level ([49], [23], [21]). In this paragraph, we prepare some results and fix the notations.

### D.2.1. Moyal formula

We remind that if $A, B$ are two symbols and $\hat{A} = Op_{Weyl}(A)$ and $\hat{B} = Op_{Weyl}(B)$, then the Moyal product $\sharp$ between symbols correspond to product of operators and is defined by ([49] p. 41)

$$(D.3) \qquad A\sharp B = \sigma_{Weyl}\left(\hat{A}\hat{B}\right)$$

Where $\sigma_{Weyl}$ gives the symbol of an operator. The Moyal formula gives the symbol of a product of operators:

$$(D.4) \qquad A\sharp B = \sum_{n\geqslant 0} (-i\hbar)^n \frac{(-1)^n}{2^n n!} (A\mathcal{D}^n B) = AB - i\hbar\frac{1}{2}\{A, B\} + \dots$$

where $\mathcal{D} = \overleftarrow{\frac{\partial}{\partial p}}\overrightarrow{\frac{\partial}{\partial q}} - \overleftarrow{\frac{\partial}{\partial q}}\overrightarrow{\frac{\partial}{\partial p}}$, and $\{A, B\} = (A\mathcal{D}B) = \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p}$ is the Poisson Bracket. As a consequence the symbol of the commutator is given by

$$(D.5) \qquad \mathfrak{X}_A(B) \stackrel{\text{def}}{=} \sigma_{Weyl}\left(\frac{-1}{i\hbar}\left[\hat{A}, \hat{B}\right]\right)$$

$$= \frac{i}{\hbar}(A\sharp B - B\sharp A)$$

$$= \sum_{n\geqslant 0} (\hbar)^{2n} \frac{(-1)^n}{2^{2n}(2n+1)!} (A\mathcal{D}^{2n+1}B)$$

$$= \{A, B\} - \frac{\hbar^2}{24}(A\mathcal{D}^3 B) + \dots$$

The first term, which is the classical Poisson Bracket will also be written:

$$\mathfrak{X}_{class,A}(B) \stackrel{\text{def}}{=} \{A, B\}$$

These formula exist in a more general framework of star-product, and all what follows can be applied to any star-product, in particular to any choice of quantization procedure.

### D.2.2. Time dependant semiclassical canonical transformation

Let $G_t(x)$ be a semiclassical symbol which depends on $t$, $\hat{G}_t = Op_{Weyl}(G_t(x))$ and $\hat{U}_t = \exp\left(-i\hat{G}_t/\hbar\right)$ the unitary transformation in $L^2(\mathbb{R})$ generated by $\hat{G}_t$ for any fixed value of $t$.

PROPOSITION D.1. — *For any $(t, t')$, the transformed evolution opera-tor:*

(D.6) $$\hat{M}'_{t,t'} = \hat{U}_{t'}^{-1} \hat{M}_{t,t'} \hat{U}_t$$

*is generated by a transformed Hamiltonian symbol $H'$, (i.e. $d\hat{M}'_{t,t'}/dt' = -\frac{i}{\hbar}\hat{H}'(t')\hat{M}'_{t,t'}$, and $\hat{M}'_{t,t} = \hat{I}d$, with $\hat{H}' = Op_{Weyl}(H')$), given by*

(D.7) $$H' = \exp(-\mathfrak{X}_{G_t}) H + R$$

*with:*

$$R = \sum_{k \geqslant 0} \frac{(-1)^k}{(k+1)!} \mathfrak{X}_{G_t}^k \left(\frac{\partial G_t}{\partial t}\right) = \left(\frac{\partial G_t}{\partial t}\right) - \frac{1}{2}\mathfrak{X}_G \left(\frac{\partial G_t}{\partial t}\right) + \dots$$

Notice that the term $R$ is due to the time dependence of the canonical transformation, and that $\exp(-\mathfrak{X}_{class,G_t}) H = H \circ U_t^{-1}$ is the function $H$ after the classical canonical transformation generated by $G_t$.

*Proof.* — For the proof, we consider the enlarged phase space $(t, \tau, q, p) \in \mathbb{R}^4$ with the symplectic two form $\omega = dq \wedge dp + dt \wedge d\tau$, where $t$ is considered here as a dynamical variable, and $\tau$ is its conjugated variable. This is a well known and useful trick both in classical and quantum mechanics which allows to map a time dependant dynamics onto a time independent dynamics, so that we can use well known tools. With $\hat{\tau} \overset{\text{def}}{=} -i\hbar\partial/\partial t$, and $\hat{\mathbf{H}} \overset{\text{def}}{=} \hat{\tau} + \hat{H}$, the time-dependant Schrödinger equation $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$ can be written

(D.8) $$\hat{\mathbf{H}}\psi = \left(\hat{\tau} + \hat{H}\right)\psi = 0$$

i.e. as the time-independent Schrödinger equation for $\psi(q, t)$. We will use bold fonts when dealing with the enlarged phase space or enlarged Hilbert space $L^2(\mathbb{R}^2)$. Let $\hat{\mathbf{G}} = Op_{Weyl}(\mathbf{G})$, where $\mathbf{G}(q, p, t, \tau) = G_t(q, p)$ is independent of $\tau$, let $\hat{\mathbf{U}} = \exp(-i\hat{\mathbf{G}}/\hbar)$, and consider the transformed Hamiltonian

(D.9) $$\hat{\mathbf{H}}' \overset{\text{def}}{=} \hat{\mathbf{U}}^{-1}\hat{\mathbf{H}}\hat{\mathbf{U}}$$

whose total symbol is:

$$\mathbf{H}' \overset{\text{def}}{=} \exp(-\mathfrak{X}_{\mathbf{G}})\mathbf{H} = \mathbf{H} - \mathfrak{X}_{\mathbf{G}}(\mathbf{H}) + \frac{1}{2!}\mathfrak{X}_{\mathbf{G}}^2(\mathbf{H}) + \dots$$

where $\mathfrak{X}_{\mathbf{G}}(\mathbf{H}) = \left(\sum_{k \geqslant 0} \frac{(-1)^k \hbar^{2k}}{2^{2k}(2k+1)!} \mathbf{G}\mathbf{D}^{2k+1}\mathbf{H}\right)$, with

$$\mathbf{D} = \left(\frac{\overleftarrow{\partial}}{\partial\tau}\frac{\overrightarrow{\partial}}{\partial t} - \frac{\overleftarrow{\partial}}{\partial t}\frac{\overrightarrow{\partial}}{\partial\tau}\right) + \left(\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial p}\right) = \mathcal{D}_t + \mathcal{D}_q.$$

Because $\mathbf{G}$ does not depend on $\tau$, we have $\mathbf{G}\,\mathbf{D}\,\mathbf{H} = \left(-\frac{\partial \mathbf{G}}{\partial t}\right) + \mathbf{G}\mathcal{D}_q\mathbf{H} = \left(-\frac{\partial G_t}{\partial t}\right) + G_t\mathcal{D}_q H$ and for $k \geqslant 2$, $\mathbf{G}\,\mathbf{D}^k\,\mathbf{H} = \mathbf{G}\left(\mathcal{D}_t + \mathcal{D}_q\right)^k \mathbf{H} = G_t\mathcal{D}_q^k H$. Then $\mathfrak{X}_{\mathbf{G}}\left(\mathbf{H}\right) = \left(-\frac{\partial G_t}{\partial t}\right) + \mathfrak{X}_{G_t}\left(H\right)$, and $\mathfrak{X}_{\mathbf{G}}^k\left(\mathbf{H}\right) = -\mathfrak{X}_{G_t}^{k-1}\left(\frac{\partial G_t}{\partial t}\right) + \mathfrak{X}_{G_t}^k\left(H\right)$, for $k \geqslant 1$. Therefore we get

$$\mathbf{H}' = \exp\left(-\mathfrak{X}_{\mathbf{G}}\right)\mathbf{H} = \tau + H'$$

with

$$H' = \exp\left(-\mathfrak{X}_{G_t}\right)H + R$$

with $R = -\sum_{k \geqslant 1} \frac{(-1)^k}{k!}\mathfrak{X}_{G_t}^{k-1}\left(\frac{\partial G_t}{\partial t}\right)$.

Now, for any $u \in \mathbb{R}$, let $\hat{\mathbf{M}}_u = \exp\left(-iu\hat{\mathbf{H}}/\hbar\right)$ and $\hat{\mathbf{M}}_u' = \exp\left(-iu\hat{\mathbf{H}}'/\hbar\right)$. From Eq.(D.9), we have the conjugation relation $\hat{\mathbf{M}}_u' = \hat{\mathbf{U}}^{-1}\hat{\mathbf{M}}_u\hat{\mathbf{U}}$. But $\left(\hat{\mathbf{U}}\psi\right)(q,t) = \left(\hat{U}_t\psi\right)(q,t)$ and $\left(\hat{\mathbf{M}}_u\psi\right)(q,t) = \left(\hat{M}_{t-u,t}\psi\right)(q,t-u)$. So the conjugation relation can be written: $\hat{M}_{t-u,t}' = \hat{U}_t^{-1}\hat{M}_{t-u,t}\hat{U}_{t-u}$, or $\hat{M}_{t,t'}' = \hat{U}_{t'}^{-1}\hat{M}_{t,t'}\hat{U}_t$, for any $t, t'$. $\qquad\square$

### D.2.3. Translations

If $x = (q,p) \in \mathbb{R}^2$, and $x' = (q',p') \in \mathbb{R}^2$, we write $x \wedge x' = qp' - pq'$. Let $A(t) = (a(t), b(t)) \in \mathbb{R}^2$, $t \in \mathbb{R}$, be any path, and as a special case of Proposition D.1, suppose that $G_t$ is a linear function of $x = (q,p)$:

$$(D.10) \qquad G_t(q,p) = a(t)p - b(t)q = A(t) \wedge x$$

We get:

$$(D.11) \qquad H'(x,t) = \frac{dA}{dt} \wedge \left(x - \frac{1}{2}A\right) + H(x - A(t), t)$$

*Proof.* — We have $\left(\frac{\partial G_t}{\partial t}\right) = \frac{da}{dt}p - \frac{db}{dt}q = \frac{dA}{dt} \wedge x$, and since $G_t$ is linear, $\mathfrak{X}_G = \mathfrak{X}_{class,G} = \{G_t, .\}$. Then $\mathfrak{X}_{G_t}\left(\frac{\partial G_t}{\partial t}\right) = \{G_t, \frac{\partial G_t}{\partial t}\} = \frac{da}{dt}b - a\frac{db}{dt} = \frac{dA}{dt} \wedge A$, so $\mathbf{R} = \frac{dA}{dt} \wedge \left(x - \frac{1}{2}A\right)$. Also we have $\exp\left(-\mathfrak{X}_{G_t}\right)H = \exp\left(-\mathfrak{X}_{class,G_t}\right)H = H(q - a(t), p - b(t))$, i.e. the translated function, as transformed by the classical canonical transformation. $\qquad\square$

### D.2.4. Symplectic linear transformations

Suppose that $G_t(q,p)$ is quadratic in $q, p$ variables, with time-dependant coefficients. Then except from the quadratic terms who may change, the transformed Hamiltonian $H'$ given in Proposition D.1 is just the classical transform of $H$:

$$(D.12) \qquad H' = \exp\left(-\mathfrak{X}_{class,G_t}\right)H + \text{quadratic terms}$$

Proof. — Again $\mathfrak{X}_{G_t} = \mathfrak{X}_{class,G_t} = \{G_t, .\}$. Also, for any quadratic homogeneous functions $A, B$, then $\{A, B\}$ is also quadratic. We deduce that $\mathfrak{X}_{class.G_t}^k \left( \frac{\partial G_t}{\partial t} \right)$ and $R$ are quadratic homogeneous function of $(q, p)$. Then Eq.(D.7) gives Eq.(D.12). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

### D.2.5. Time-dependant transformations of a special kind

When dealing with next order terms, we will only need symbols $G_t$ of the simple factorized form:

$$G_t(x) = g(t)\,\tilde{G}(x)$$

Then

(D.13) $$H' = \left( \frac{\partial G_t}{\partial t} \right) + \exp\left(-\mathfrak{X}_{G_t}\right) H$$

since $\mathfrak{X}_{G_t} \left( \frac{\partial G_t}{\partial t} \right) = g(t)\,g'(t)\,\mathfrak{X}_{\tilde{G}}\left(\tilde{G}\right) = 0$.

## D.3. Pre-Normalization

Consider a given point $x_0 = (q_0, p_0) \in \mathbb{R}^2$ at time $t = 0$, and $x(t) = (q(t), p(t)) = M_{0,t}x_0$ the trajectory passing through it. See figure D.1. We will sometimes use the condensed notation $\mathbf{x} = (x(t), t) \in \mathbb{R}^3$.
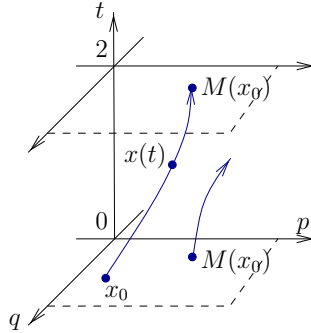


Figure D.1. Trajectory $x(t)$.

During Pre-normalization, we will perform successively two canonical transformations (a translation and a linear symplectic transformation) in order to simplify the Taylor expansion of $H(q, p, t)$ along the trajectory

$x(t)$, up to degree two in $(q, p)$. Higher degrees will be treated in a second stage called the normalization process. We will use the notation $x' = (q', p')$ for the new coordinates and $H'_{\mathbf{x}}(q', p')$ for the transformed Hamiltonian, which depends on time through $\mathbf{x} = (x(t), t)$.

Notice that if $F(\mathbf{x}) = F(x, t)$ is a periodic function with respect to $t$, then it fulfills the obvious relation:

$$(D.14) \qquad F(x(t), t)_{t=2} = F((Mx)(t), t)_{t=0}$$

### D.3.1. Translation

We decide to map the trajectory $x(t) = (q(t), p(t))$ onto the origin $x' = (q', p') = 0$, using a time-dependant translation, generated by Eq.(D.10), with $A(t) = -x(t)$. From Eq.(D.11), the value of the transformed Hamiltonian $H'$ at the origin is then

$$H'_{\mathbf{x}}(x' = 0) = -\mathcal{A}_{\mathbf{x}}$$

with

$$(D.15) \qquad \mathcal{A}_{\mathbf{x}} = \frac{1}{2}\frac{dx}{dt} \wedge x - H(x) = \frac{1}{2}\left(p\frac{dq}{dt} - q\frac{dp}{dt}\right) - H(x).$$

We recognize the time derivative of the classical action along the trajectory $x(t)$ (see [25] for a geometric interpretation). From Eq.(D.11) and using classical Hamilton equations for $x(t)$, one obtains that the derivative of $H'_{\mathbf{x}}$ at the origin is zero: $(\partial_{x'}H')(x' = 0) = 0$, as expected because the origin is now a fixed point. Higher derivatives are equal to derivatives of $H$ at $x(t)$:

$$\left(\partial_{x'}^k H'\right)_{x'=0} = \left(\partial_x^k H\right)_{x=x(t)}, \qquad \forall k \geqslant 2$$

### D.3.2. Linear Symplectic transformation

Similarly to Eq.(2536), let $Q_{\mathbf{x}} \in SL(2, \mathbb{R})$ be the symplectic matrix which transforms the canonical basis of $\mathbb{R}^2$ to the basis $(u_{\mathbf{x}}, s_{\mathbf{x}})$ at point $x(t)$ and time $t$ defined above. We use this canonical transformation in order to simplify the quadratic terms of the Hamiltonian. This transformation is generated by a quadratic function $G$, and from Eq.(D.12), the transformed Hamiltonian $H'$ differs from $H$ only by different quadratic terms. From Eq.(2.7) and Eq.(D.2), the new quadratic terms are

$$(H'_{\mathbf{x}}(q', p'))_{degree\,2} = \eta_{\mathbf{x}}q'p'$$

### D.3.3. Result of the pre-normalization process

Up to now, we have chosen an initial point $x_0 \in \mathbb{R}^2$, and have performed successively a translation and a linear symplectic transform:

$$\mathcal{T}_{pre,\mathbf{x}} \overset{\text{def}}{=} T_{x(t)} Q_{\mathbf{x}}$$

so that the trajectory $x(t)$ is mapped onto the origin, and the new Hamiltonian $H'_{\mathbf{x}}(q',p')$ has a Taylor expansion in variables $x' = (q',p')$ at the origin of the form:

$$(H'_{\mathbf{x}})_{deg \leqslant 2}(q',p') = -\mathcal{A}_{\mathbf{x}} + \eta_{\mathbf{x}} q' p'$$

$$(H'_{\mathbf{x}})_{deg \geqslant 3}(q',p') = (H \circ \mathcal{T}_{pre,\mathbf{x}})(q',p')$$

Terms of degree lower or equal to two are now in normal form, which means that the stable and unstable manifold are tangential to the principal axis, see figure D.2.
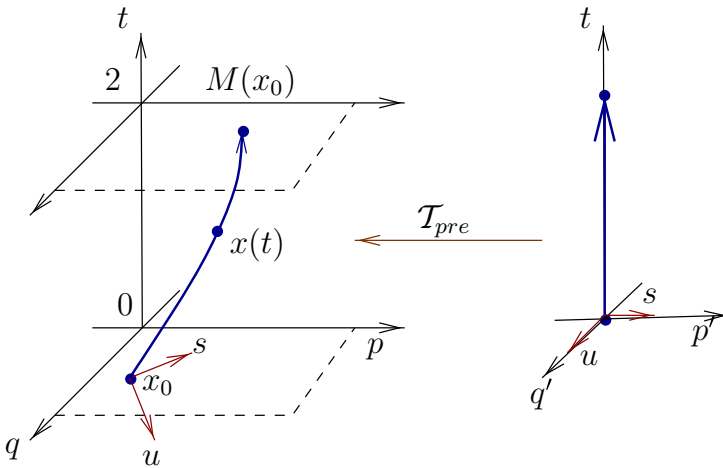


Figure D.2. The Pre-normalization transformation $\mathcal{T}^{-1}_{pre,\mathbf{x}}$ translates the trajectory to the origin, and transforms the unstable/stable directions onto the principal axis $(q',p')$.

Eq.(D.6) with $t = 0$, $t' = 2$, gives

$$\hat{M} = \hat{\mathcal{T}}_{pre,Mx_0} \hat{M}'_{x_0} \hat{\mathcal{T}}^{-1}_{pre,x_0}$$

where $\mathcal{T}_{pre,x_0} = \mathcal{T}_{pre,(x_0,t=0)}$. This relation gives Theorem 5.1 page 2550 at the linear order $J = 2$ (we will explain below how to transform the time

dependent term $\eta_{\mathbf{x}} q' p'$ into the time independent $\lambda_{0,1,x_0} q' p'$, when treating resonant terms).

Notice that the Hamiltonian $H'_{\mathbf{x}}(q', p')$ in pre-normal form we have obtained, is continuous with respect to $\mathbf{x} = (x, t)$, and periodic in $t$, as explained in Eq.(D.14). The action $\mathcal{A}_{\mathbf{x}}$ is not periodic in $x$ with respect to the lattice $\Gamma_t$, but the terms of higher degrees are.

## D.4. Normalization

We go on with the normalization process, starting now from the Hamiltonian $H_{\mathbf{x}}(q', p')$ given by the pre-normalization above, and which has a Taylor series in $x' = (q', p')$ at the origin of the form:

$$(\text{D.16}) \qquad H_{\mathbf{x}}(q', p') = -\mathcal{A}_{\mathbf{x}} + \eta_{\mathbf{x}} q' p' + \sum_{l \geqslant 0} \sum_{\alpha + \beta \geqslant 0} v_{\alpha, \beta, l, \mathbf{x}} \hbar^l q'^{\alpha} p'^{\beta}$$

In this expansion, every term is a periodic and continuous function of $\mathbf{x} = (x, t) \in \mathbb{R}^3$, (except for $\mathcal{A}_{\mathbf{x}}$ which is not periodic in $x$), and $v_{\alpha, \beta, l, \mathbf{x}} = 0$, if $l = 0$ and $\alpha + \beta \leqslant 2$. We will transform Eq.(D.16) into a normal form, by successive iterations. At the first stage, we have no "quantum terms", i.e. $v_{\alpha, \beta, l, \mathbf{x}} = 0$ if $l \geqslant 1$. But during the normalization process such terms will possibly appear. We will check that due to the special form of the Moyal formula Eq.(D.5), only terms with even value of $l$ will appear.

**Notations:** To simplify the notations, we will write $\eta(t) \stackrel{\text{def}}{=} \eta_{(x(t), t)}$, $v_{\alpha, \beta, l}(t) \stackrel{\text{def}}{=} v_{\alpha, \beta, l, (x(t), t)}$ etc.. in order to show the time dependence along the trajectory $x(t)$, and keep in mind that they also depend continuously and periodically on $x$ and $t$. Let

$$(\text{D.17}) \qquad n \stackrel{\text{def}}{=} \alpha + \beta + 2l \in \mathbb{N}$$

be the degree of the monomial $\hbar^l q'^{\alpha} p'^{\beta}$. We will use the notation $\mathcal{O}(n) \stackrel{\text{def}}{=} \mathcal{O}((\hbar, q', p')^n)$ which means higher degree terms with respect to this graduation.

### D.4.1. Homological equation

Suppose that we want to suppress the monomial term $v_{\alpha, \beta, l}(t) \hbar^l q'^{\alpha} p'^{\beta}$ with degree $n = \alpha + \beta + 2l$, from the Hamiltonian Eq.(D.16). For that purpose, we consider a time dependant canonical transformation generated by the symbol $G_{\mathbf{x}}(t, q', p') = g_{\mathbf{x}}(t) \hbar^l q'^{\alpha} p'^{\beta} \chi(q', p')$, where $g_{\mathbf{x}}(t)$ is a still

unknown function, but supposed to be continuous and periodic with $\mathbf{x} = (q, p, t)$. The function $\chi(q', p')$ is a $C_0^\infty$ cut-off function[6] with compact support such that $\chi(q', p') = 1$ for $(q', p') \in D_1 = \left\{ (q', p') \,/\, |q'|^2 + |p'|^2 < 1 \right\}$. From Eq.(D.13), the new Hamiltonian will be

$$H' = \left( \frac{\partial G}{\partial t} \right) + \exp\left(-\mathfrak{X}_G\right) H$$

Consider $(q', p') \in D_1$. We have

$$
\begin{aligned}
\mathfrak{X}_G(H)(q', p') &= \mathfrak{X}_G\left(\eta(t)\, q'p'\right) + \mathcal{O}(n+1) \\
&= \eta(t)\left\{\mathbf{G}, q'p'\right\} + \mathcal{O}(n+1) \\
&= \eta(t)(\beta - \alpha)\, g(t)\, \hbar^l q'^\alpha p'^\beta + \mathcal{O}(n+1)
\end{aligned}
$$

So $H'$ contains the monomial

$$K(q', p') = \left( v_{\alpha, \beta, l}(t) + \frac{dg}{dt} + \eta(t)(\alpha - \beta)\, g(t) \right) \hbar^l q'^\alpha p'^\beta$$

and differs from $H$ by this term and terms of higher orders only. We wish to suppress this term, by solving $K = 0$. This gives

$$(D.18) \qquad g_\mathbf{x}(t) = \left( C(x_0) - \int_0^t v_\mathbf{x}(s)\, e^{(\alpha - \beta)\int_0^s \eta_\mathbf{x} ds'}\, ds \right) e^{-(\alpha - \beta)\int_0^t \eta_\mathbf{x} ds}$$

where $C(x_0)$ is a constant which depends only on $x_0 = M_{-t}(x) \in \mathbb{R}^2$ (i.e. the point of the trajectory at $t = 0$). The continuity in $\mathbf{x}$ is obvious, but we have to impose the periodicity relation Eq.(D.14), which writes:

$$g_{x_0(2)}(2) = g_{(Mx_0)}(0)$$

This gives a relation for the function $C(x_0)$, called the **homological equation** (this is usual in normal form calculations (see [3] chap 5, [45] p.100):

$$(D.19) \qquad \boxed{C(Mx_0) = (C(x_0) - P(x_0))\, e^{-(\alpha - \beta)\lambda_{x_0}}}$$

with

$$\lambda_{x_0} = \int_0^2 \eta_{\mathbf{x}(t)}\, dt,$$

which is the expanding rate along a piece of trajectory, and with

$$P(x_0) = \int_0^2 v_\mathbf{x}(s)\, e^{(\alpha - \beta)\int_0^s \eta_\mathbf{x} ds'}\, ds.$$

We have to check now that the above homological equation has a solution $C(x_0)$ continuous with respect to $x_0 \in \mathbb{R}^2$. We have to consider two cases.

---

[6] The cut-off function $\chi$ is necessary otherwise the canonical transformation generated by $G$ is not defined other all $(q', p') \in \mathbb{R}^2$. Think for example of $G = C\, p'q'^2$.

The **"non resonant case"** $\alpha \neq \beta$: If $\alpha > \beta$, we can express $C(x_0)$ from the past image $M^{-t}x_0$ and iterate:

$$
\begin{aligned}
C(x_0) &= \left(C\left(M^{-1}x_0\right) - P\left(M^{-1}x_0\right)\right) e^{-(\alpha-\beta)\lambda_{M^{-1}x_0}} \\
&= \left(C\left(M^{-2}x_0\right) - P\left(M^{-2}x_0\right)\right) e^{-(\alpha-\beta)\left(\lambda_{M^{-2}x_0} + \lambda_{M^{-1}x_0}\right)} \\
&\quad - P\left(M^{-1}x_0\right) e^{-(\alpha-\beta)\lambda_{M^{-1}x_0}}
\end{aligned}
$$

etc...

Similarly, if $\alpha < \beta$, we use images of $x_0$ in the future:

$$
\begin{aligned}
C(x_0) &= P(x_0) + C(Mx_0) e^{-(\beta-\alpha)\lambda_{x_0}} \\
&= P(x_0) + e^{-(\beta-\alpha)\lambda_{x_0}} \left(P(Mx_0) + C\left(M^2x_0\right) e^{-(\beta-\alpha)\lambda_{Mx_0}}\right)
\end{aligned}
$$

etc... The following Lemma shows that these iterations converge nicely.

LEMMA D.2. — *(David DeLatte [19]). For $\alpha > \beta$, the solution of the homological equation Eq.(D.19) is*

$$
C(x_0) = -\sum_{t=1}^{\infty} P\left(M^{-t}x_0\right) e^{-(\alpha-\beta)\Lambda_{-t}(x_0)}
$$

*which is a continuous function of $x_0$, and with*

$$
\Lambda_{-t}(x_0) = \sum_{-t \leqslant k \leqslant -1} \lambda_{M^k x_0}, \qquad \text{for } t \geqslant 1
$$

*is the expanding rate along a piece of trajectory of length $t$.*

*For $\alpha < \beta$, the solution*

$$
C(x_0) = \sum_{t=0}^{+\infty} P\left(M^t x_0\right) e^{-(\beta-\alpha)\Lambda_t(x_0)}
$$

*is continuous in $x_0$, with*

$$
\begin{aligned}
\Lambda_t(x_0) &= \sum_{0 \leqslant k \leqslant t-1} \lambda_{M^k x_0}, \qquad \text{for } t \geqslant 1 \\
&= 0, \qquad \text{for } t = 0
\end{aligned}
$$

*Proof.* — From uniform hyperbolicity assumption, there exist $\lambda_{min} > 0$, $t_{min} > 0$ such that $\forall x_0 \in \mathbb{T}^2, t > t_{min}$   $\Lambda_t(x_0) > |t|\lambda_{min}$. $P(x)$ is a continuous function of $x \in \mathbb{T}^2$, so bounded by $|P(x)| < P_{max}$. Now if $C^{(N)}(x) = -\sum_{t=1}^{N}(\ldots)$ is the partial sum of $C(x)$ as given above, with $N \in \mathbb{N}$, then $\forall x \in \mathbb{T}^2, \left|C^{(N+1)}(x) - C^{(N)}(x)\right| < e^{-\lambda_{min}N}P_{max}$ . This implies that $C^{(N)}(x)$ converges uniformly for $N \to \infty$, and that $C(x_0)$ is continuous in $x_0$.    $\square$

**The "resonant case"** $\alpha = \beta$**:** In that case we cannot solve the Homological equation, so we keep the monomial term of $H$. We can however make a small simplification of the Hamiltonian. Let us suppose that $H$ has a resonant term $\eta_{l,j,\mathbf{x}}\hbar^l (q'p')^j$, where $\eta_{l,j,\mathbf{x}}$ is time dependant because of $\mathbf{x} = (x(t), t)$. Let us define:

$$\lambda_{l,j,(x_0)} \overset{\text{def}}{=} \int_0^2 \eta_{l,j,\mathbf{x}} dt$$

We decompose the resonant term into $\eta_{l,j,\mathbf{x}} = \tilde{\eta}_{l,j,\mathbf{x}} + \frac{1}{2}\lambda_{l,j,(x_0)}$. From Eq.(D.18), the first time-dependent term $\tilde{\eta}_{l,j,\mathbf{x}}\hbar^l (q'p')^j$ can be eliminated after a time dependant canonical transformation given by $g_{\mathbf{x}}(t) = -\int_0^t \tilde{\eta}_{l,j,\mathbf{x}} dt$. There remains a time-independent term $\frac{1}{2}\lambda_{l,j,(x_0)}\hbar^l (q'p')^j$. In particular the resonant term $(l,j) = (0,0)$ is the action term Eq.(D.15), and gives

$$\text{(D.20)} \quad \lambda_{0,0,(x)} = -\int_0^2 \mathcal{A}_{\mathbf{x}} dt = -\int_0^2 \left( \frac{1}{2}\left( p\frac{dq}{dt} - q\frac{dp}{dt} \right) - H(x) \right) dt$$

### D.4.2. Result of the normalization process, and final proof of Theorem 5.1

By successive iterations of the normalization process described above, we can discard every non resonant term of the Hamiltonian Eq.(2587), up to a given degree. We have performed a sequence of time-dependant canonical transformations in a given order indexed by $n = 1, 2, \ldots n_{max}$, and $\mathbf{x}$:

$$\hat{\mathcal{T}}_{\mathbf{x}} = T_{x(t)} Q_{\mathbf{x}} \hat{U}_{G_1,\mathbf{x}} \hat{U}_{G_2,\mathbf{x}} \ldots \hat{U}_{G_{n_{max}},\mathbf{x}}$$

In this product, each term $\hat{U}_{G_n,\mathbf{x}}$ is generated by a bounded symbol $G_{n,\mathbf{x}}(q', p', t)$ with compact support, equal to

$$G_{n,\mathbf{x}}(t, q', p') = g_{l,a,b,\mathbf{x}}(t)\, \hbar^l q'^\alpha p'^\beta$$

inside the disk $D_1$ of radius 1. The resulting time-dependant Hamiltonian $H_{\mathbf{x}}(q', p')$ is a total symbol which grows quadratically at infinity and has a Taylor expansion in $x' = (q', p')$ at the origin of the form:

$$H_{\mathbf{x}}(q', p') = \frac{1}{2} \sum_{2(l+j) \leqslant J} \lambda_{l,j,(x)}\hbar^l (q'p')^j + \mathcal{O}(J+1)$$

i.e. the Taylor terms of degree less or equal to $J$ are in normal form, with time-independent coefficients $\lambda_{l,j,(x)}$. We can apply Eq.(D.6), with time $t = 0$, $t' = 2$, and deduce that

$$\hat{M} = \hat{\mathcal{T}}_{M(x_0)} \hat{M}_{J,x_0} \hat{\mathcal{T}}_{x_0}^{-1}$$

where $\hat{M}_{J,x_0}$ is generated by $\hat{H}_{x_0(t)} = Op_{Weyl}\left(H_{x_0(t)}\right)$ on time two, i.e. $\hat{M}_{J,x_0} = \hat{M}_{J,x_0,t=2}$, solution of $d\hat{M}_{J,x_0,t}/dt = -\left(i/\hbar\right)\hat{H}_{x_0(t)}\hat{M}_{J,x_0,t}$, and $\hat{M}_{J,x_0,t=0} = \hat{I}d$. Also $\hat{\mathcal{T}}_x$ means $\hat{\mathcal{T}}_{\mathbf{x}=(x,t=0)}$, and involves

$$g_{l,a,b,(x)} \overset{\text{def}}{=} g_{l,a,b,\mathbf{x}}\,(t=0)\,.$$

In order to simplify the notations, we rescale time by a factor $1/2$, so that $t \in [0,1]$, and correspondingly we multiply $H_{x_0(t)}$ by a factor two. We obtain finally:

(D.21) $$H_x\left(q',p'\right) = K_x\left(q',p'\right) + \mathcal{O}\left(J+1\right)$$

$$K_x\left(q',p'\right) = \sum_{l+j\leqslant J/2} \lambda_{l,j,(x)}\hbar^l\left(q'p'\right)^j$$

The functions $\lambda_{l,j,(x)}$, and $g_{l,a,b,(x)}$ are continuous with respect to $x \in \mathbb{R}^2$. They are also periodic in $x$ with respect to the lattice $\mathbb{Z}^2$, except for the action $\lambda_{0,0,(x)}$ . To deduce Eq.(2550) of Theorem 5.1, we still have to compare $\hat{M}$ with the unitary map generated by the normal form part $K_x$ only, dropping out the terms of degree higher than $J$, and estimate the error. This is exactly what gives Theorem C.1, page 2577. Thus, we have finally proved Theorem 5.1.

## Appendix E. Trace of a semi-classical hyperbolic normal form

In this appendix, we give explicit semi-classical expressions for the trace and matrix elements of a hyperbolic normal form. They are already obtained in [43], but we adapt them to the present context. Consider the Hilbert space $\mathcal{H}_{plane} = L^2\left(\mathbb{R}\right)$, and the quadratic normal form Hamiltonian:

$$\hat{K}_0 = Op_{Weyl}\left(qp\right) = \frac{1}{2}\left(\hat{p}\hat{q} + \hat{q}\hat{p}\right) = \hbar\hat{I}$$

with

$$\hat{I} \equiv -i\frac{1}{2}\left(\frac{\partial}{\partial q}q + q\frac{\partial}{\partial q}\right)$$

Let

(E.1) $$\hat{K} \overset{\text{def}}{=} \sum_{l,j\geqslant 0} \tilde{\mu}_{l,j}\hbar^l\hat{K}_0^j$$

where $\tilde{\mu}_{l,j} \in \mathbb{R}$, but $\tilde{\mu}_{0,1} > 0$. In this paper, the operator $\hat{K}$ appears in Eq.(5.16) and Eq.(5.17), as a result of the (post) normalization procedure

along a classical trajectory. Remind that $\tilde{\mu}_{l,j} = \mu_{l,j}$, if $l \leqslant 1$. The operator $\hat{K}$ generates a unitary propagator operator after time $t$:

(E.2) $$\hat{N}(t) \overset{\text{def}}{=} \exp\left(-it\hat{K}/\hbar\right)$$

The truncation operators $\hat{P}_\alpha$ has been defined in Eq.(2546).

PROPOSITION E.1. — *For any $0 < \alpha < 1/2$, one has a semi-classical expression:*

(E.3) $$\text{Tr}\left(\hat{P}_\alpha \hat{N}(t) \hat{P}_\alpha\right) = T_{semi} + \mathcal{O}(\hbar^\infty)$$

$$T_{semi} \overset{\text{def}}{=} \exp\left(-it\frac{\mu_{0,0}}{\hbar}\right) \exp\left(-it\mu_{1,0}\right) \frac{1}{2\sinh\left(\frac{\mu_{0,1}t}{2}\right)} \mathcal{E}$$

*with a semi-classical series*

$$\mathcal{E} = 1 + \sum_{s \geqslant 1} \hbar^s E_s = 1 + \hbar E_1 + \hbar^2 E_2 + \dots$$

*where $E_s$ depends on $t$ and $\tilde{\mu}_{l,j}$ , with $(l+j) \leqslant s+1$, and are given explicitly in the proof. We have the control:*

(E.4) $$|E_s| \leqslant t^s E_{max,s}$$

*where $E_{max,s}$ does not depend on $t$.*

We prove this Proposition below. The idea and techniques of the proof are taken from [43] and related work cited in this paper. In this Proposition, the time $t$ is fixed with respect to $\hbar$, but the result remains valid for $|t| = \mathcal{O}(\log(1/\hbar))$, because the semiclassical expansion remains unchanged.

## E.1. A first useful formal asymptotic formula

Define $\mu(X) \overset{\text{def}}{=} \sum_{l,j \geqslant 0} \tilde{\mu}_{l,j} \hbar^l X^j$. From Eq.(E.1) and Eq.(E.2), one can write $\hat{N}_t = \exp\left(-\frac{i}{\hbar} t\mu\left(\hbar\hat{I}\right)\right)$. Define:

$$r(X) \overset{\text{def}}{=} \mu(X) - (\mu_{0,0} + \hbar\mu_{1,0} + \mu_{0,1}X) = \sum_{l+j \geqslant 2} \tilde{\mu}_{l,j} \hbar^l X^j$$

This decomposition is natural because for a given $I \in \mathbb{R}$, the expression:

(E.5) $$S = -\frac{it}{\hbar} r(\hbar I) = -it \sum_{l+j \geqslant 2} \tilde{\mu}_{l,j} \hbar^{l+j-1} I^j = \mathcal{O}(\hbar)$$

is an asymptotic series in $\hbar$:

(E.6)
$$S = \sum_{s \geqslant 1} \hbar^s S_s$$

with

$$S_s = \left( -it \sum_{j=0}^{s+1} \tilde{\mu}_{(l=s-j+1),j} I^j \right), \qquad s \geqslant 1$$

The first terms are explicitely:

$$S_1 = -it \left( \tilde{\mu}_{2,0} + \tilde{\mu}_{1,1} I + \tilde{\mu}_{0,2} I^2 \right)$$
$$S_2 = -it \left( \tilde{\mu}_{3,0} + \tilde{\mu}_{2,1} I + \tilde{\mu}_{1,2} I^2 + \tilde{\mu}_{0,3} I^3 \right),$$

Then

(E.7)
$$\exp\left( -\frac{i}{\hbar} t \mu \left( \hbar I \right) \right) = \exp\left( -\frac{i}{\hbar} t \mu_{0,0} \right) \exp\left( -it\mu_{1,0} I \right) \exp\left( -t\mu_{0,1} \right) \mathcal{E}\left( I \right)$$

with the series in $\hbar$ and $I$:

(E.8)
$$\mathcal{E}\left( I \right) \overset{\text{def}}{=} \exp\left( S \right) = 1 + \sum_{s \geqslant 1} \hbar^s E_s \left( I \right)$$

where $E_s\left( I \right)$ is a polynomial of degree $2s$ in $I$:

$$E_s\left( I \right) = \sum_{j=0}^{2s} E_{s,j} I^j$$

The first two terms $E_s\left( I \right)$ are:

(E.9)
$$E_1\left( I \right) = S_1 = -it \left( \tilde{\mu}_{2,0} + \tilde{\mu}_{1,1} I + \tilde{\mu}_{0,2} I^2 \right)$$

(E.10)
$$E_2\left( I \right) = S_2 + \frac{1}{2} S_1^2 = -\left( it\tilde{\mu}_{3,0} + \frac{1}{2} t^2 \tilde{\mu}_{2,0}^2 \right) - I \left( it\tilde{\mu}_{2,1} + t^2 \tilde{\mu}_{2,0} \tilde{\mu}_{1,1} \right)$$

(E.11)
$$- I^2 \left( it\tilde{\mu}_{1,2} + \frac{1}{2} t^2 \tilde{\mu}_{1,1}^2 + t^2 \tilde{\mu}_{2,0} \tilde{\mu}_{0,2} \right) - I^3 \left( it\tilde{\mu}_{0,3} + t^2 \tilde{\mu}_{1,1} \tilde{\mu}_{0,2} \right)$$

(E.12)
$$- I^4 \left( \frac{1}{2} t^2 \tilde{\mu}_{0,2}^2 \right).$$

(E.13)  etc..

However, in this paper, due to the special choice of Weyl quantization, we have $\tilde{\mu}_{l,j} = 0$ if $l$ is odd. This simplifies:

$$E_1\left(I\right) = -it\left(\tilde{\mu}_{2,0} + \tilde{\mu}_{0,2}I^2\right)$$

$$E_2\left(I\right) = -\left(\tfrac{1}{2}t^2\tilde{\mu}_{2,0}^2\right) - I\left(it\tilde{\mu}_{2,1}\right) - I^2\left(t^2\tilde{\mu}_{2,0}\tilde{\mu}_{0,2}\right) - I^3\left(it\tilde{\mu}_{0,3}\right) - I^4\left(\tfrac{1}{2}t^2\tilde{\mu}_{0,2}^2\right).$$

We get:

$$\hat{N}\left(t\right) = \exp\left(-it\frac{\tilde{\mu}_{0,0}}{\hbar}\right)\exp\left(-it\tilde{\mu}_{1,0}\right)\exp\left(-\frac{it}{\hbar}r\left(\hbar\hat{I}\right)\right)\hat{N}_0\left(t\right)$$

with

(E.14)          $$\hat{N}_0\left(t\right) \overset{\text{def}}{=} \exp\left(-it\frac{\tilde{\mu}_{0,1}}{\hbar}\hat{K}_0\right) = \exp\left(-i\mu\hat{I}\right)$$

and

$$\mu \overset{\text{def}}{=} \tilde{\mu}_{0,1}t$$

We have for any $k \in \mathbb{N}$,

$$\left(i\frac{\partial}{\partial\mu}\right)^k\hat{N}_0\left(t\right) = \left(\hat{I}\right)^k\hat{N}_0\left(t\right)$$

so for any formal series $\mathcal{E}\left(I\right)$ in $I$, this gives:

$$\mathcal{E}\left(i\frac{\partial}{\partial\mu}\right)\hat{N}_0\left(t\right) = \mathcal{E}\left(\hat{I}\right)\hat{N}_0\left(t\right)$$

It means that we may substitute $\hat{I} \Leftrightarrow \left(i\frac{\partial}{\partial\mu}\right)$ in the formal series. We deduce the formal result

(E.15)          $$\boxed{\hat{N}\left(t\right) = \exp\left(-it\frac{\tilde{\mu}_{0,0}}{\hbar}\right)\exp\left(-it\tilde{\mu}_{1,0}\right)\mathcal{E}\left(i\frac{\partial}{\partial\mu}\right)\hat{N}_0\left(t\right)}$$

   *Remark.* —   In the right hand side of Eq.(E.15), only $\hat{N}_0\left(t\right)$ is an operator.

## E.2. Semi-classical expression of the Trace

   We first compute the Trace of Eq.(E.14) in the linear case. From Eq.(3.9), the functional $\mathrm{T}\left(\hat{N}_0\right)$ is well defined. It's value is

(E.16)          $$T_\mu \overset{\text{def}}{=} \mathrm{T}\left(\hat{N}_0\left(t\right)\right) = \frac{1}{2\sinh\left(\frac{\mu}{2}\right)}, \qquad \mu = \lambda_{0,1}t$$

Remark: it can also be written $T_\mu = 1/\sqrt{\left|\det\left(N_0\left(t\right) - I\right)\right|}$ where $N_0\left(t\right) \in SL\left(2,\mathbb{R}\right)$ is the classical map.

*Proof.* — From its Schwartz kernel $\langle x'|\hat{N}_0(t)|x\rangle = e^{\mu/2}\delta(x' - e^{\mu}x)$, and a suitable regularization, one has $\mathrm{T}\left(\hat{N}_0(t)\right) = e^{\mu/2}\int dx\delta(x - e^{\mu}x) = e^{\mu/2}\frac{1}{|1-e^{\mu}|} = \frac{1}{2\sinh\left(\frac{\mu}{2}\right)}.$                    $\square$

In Lemma 4.3 page 2548, we showed that the value $T_\mu$ comes microlocally from the origin:

$$\mathrm{Tr}\left(\hat{P}_\alpha\hat{N}_0(t)\hat{P}_\alpha\right) = T_\mu + \mathcal{O}(\hbar^\infty)$$

From Eq.(E.8), $\mathcal{E}\left(i\frac{\partial}{\partial\mu}\right)T_\mu = 1 + \sum_{s,j}\hbar^s E_{s,j}\left(i\frac{\partial}{\partial\mu}\right)^j T_\mu = T_\mu\mathcal{E}$ with

$$\mathcal{E} = 1 + \sum_s \hbar^s E_s, \qquad E_s = \sum_{j=0}^{2s} E_{s,j}I^{(j)}$$

and

$$I^{(j)} \overset{\text{def}}{=} \frac{(i)^j}{T_\mu}\frac{d^j T_\mu}{d\mu^j}$$

In other words, we have substituted $I^j$ by $I^{(j)}$ in the series Eq.(E.8). Finally, we deduce from Eq.(E.15) that:

$$\mathrm{Tr}\left(\hat{P}_\alpha\hat{N}(t)\hat{P}_\alpha\right) = \exp\left(-it\frac{\tilde{\mu}_{0,0}}{\hbar}\right)\exp(-it\tilde{\mu}_{1,0})\,\mathcal{E}\left(i\frac{\partial}{\partial\mu}\right)T_\mu + \mathcal{O}(\hbar^\infty)$$

$$= \exp\left(-it\frac{\tilde{\mu}_{0,0}}{\hbar}\right)\exp(-it\tilde{\mu}_{1,0})\,\mathcal{E}\,T_\mu + \mathcal{O}(\hbar^\infty)$$

We have obtained Eq.(E.3). We can calculate the first terms $I^{(j)}$. With $s \overset{\text{def}}{=} \sinh\left(\frac{\mu}{2}\right)$ and $c \overset{\text{def}}{=} \cosh\left(\frac{\mu}{2}\right)$, we obtain:

(E.17)              $I^{(1)} = -i\frac{c}{2s}, \quad I^{(2)} = -\frac{1}{4s^2}\left(2c^2 - s^2\right)$

$$I^{(3)} = -\frac{i}{8s^3}\left(-6c^3 + 5cs^2\right), \qquad I^{(4)} = \frac{1}{16s^4}\left(24c^4 - 28s^2c^2 + 5s^4\right).$$

**Expressions for large values of $t$:** If $t \gg 1$, then $\mu = \lambda_{0,1}t \gg 1$, and

$$T_\mu = e^{-\mu/2}\left(1 - e^{-\mu}\right)^{-1} = \sum_{k\geqslant 0}\exp\left(-\mu\left(k + \frac{1}{2}\right)\right)$$

thus

$$I^{(j)} = (i)^j \frac{1}{T_\mu} \partial_\mu^j T_\mu = (-i)^j \frac{1}{T_\mu} \sum_{k \geqslant 0} \left( k + \frac{1}{2} \right)^j \exp\left( -\mu \left( k + \frac{1}{2} \right) \right)$$

$$= \left( \frac{-i}{2} \right)^j + \sum_{k \geqslant 1} \left( \left( k + \frac{1}{2} \right)^j - \left( k - \frac{1}{2} \right)^j \right) e^{-k\mu}$$

$$= \left( \frac{-i}{2} \right)^j + \mathcal{O}\left( e^{-\tilde{\mu}_{0,1} t} \right)$$

and we deduce in particular that $I^{(j)}$ is uniformly bounded for $t \in [1, +\infty[$. From Eq.(E.8), we deduce that

$$|E_s| \leqslant t^s E_{max,s}$$

where $E_{max,s}$ does not depend on $t$.

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