ANNALES

DE

L'INSTITUT FOURIER

Claus MAZANTI SORENSEN

A generalization of level-raising congruences for algebraic modular forms

Tome 56, n° 6 (2006), p. 1735-1766.

<http://aif.cedram.org/item?id=AIF_2006__56_6_1735_0>

Ann. Inst. Fourier, Grenoble
56, 6 (2006) 1735-1766

A GENERALIZATION OF LEVEL-RAISING CONGRUENCES FOR ALGEBRAIC MODULAR FORMS

by Claus MAZANTI SORENSEN

ABSTRACT. — In this paper, we extend the results of Ribet and Taylor on level-raising for algebraic modular forms on the multiplicative group of a definite quaternion algebra over a totally real field F. We do this for automorphic representations of an arbitrary reductive group G over F, which is compact at infinity. In the special case where G is an inner form of $\mathrm{GSp}(4)$ over $\mathbb{Q}$, we use this to produce congruences between Saito-Kurokawa forms and forms with a generic local component.

RÉSUMÉ. — Dans cet article, on étend des résultats d'augmentation du niveau de Ribet et Taylor, au cas de formes modulaires algébriques pour une algèbre à division sur un corps totalement réel F. On travaille avec des représentations automorphes d'un groupe réductif G sur F, compact à l'infini. Dans le cas particulier où G est une forme intérieure de $\mathrm{GSp}(4)$ sur $\mathbb{Q}$, on utilise ces résultats pour construire des congruences entre des formes de Saito-Kurokawa et des formes avec des composantes locales génériques.

1. Introduction

In this paper, we will prove a generalization of the following theorem of Ribet [13]:

THEOREM 1.1. — *Let $f \in S_2(\Gamma_0(N))$ be an eigenform, and let $\lambda|\ell$ be a finite place of $\overline{\mathbb{Q}}$ such that $\ell \geqslant 5$ and f is not congruent to an Eisenstein series modulo λ. If $q \nmid N\ell$ is a prime number such that $\ell \nmid 1+q$ and the following condition is satisfied,*

$$a_q(f)^2 \equiv (1+q)^2 \pmod{\lambda},$$

then there exists a q-new eigenform $\tilde{f} \in S_2(\Gamma_0(Nq))$ congruent to f modulo λ.

Keywords: Level-raising, algebraic modular forms.
Math. classification: 11F33, 11F70.

Two eigenforms f and $\tilde{f}$ are said to be congruent modulo λ if their Hecke eigenvalues are congruent for almost all primes, that is, if $a_p(f) \equiv a_p(\tilde{f}) \pmod{\lambda}$ for almost all p. The proof of this theorem can be reduced, via the Jacquet-Langlands correspondence, to the corresponding statement for $D^\times$ where D is a definite quaternion algebra over $\mathbb{Q}$.

Our goal in this paper is to prove that an automorphic form of Saito-Kurokawa type is congruent to an automorphic form which is *not* of Saito-Kurokawa type. Since functorialty is not yet available, we are considering an inner form G of $\mathrm{PGSp}(4)/\mathbb{Q}$ such that $G(\mathbb{R})$ is compact. By a form on $G \simeq \mathrm{SO}(5)$ of Saito-Kurokawa type we mean a theta lift from $\widetilde{\mathrm{SL}}(2)$. We achieve this goal as a result of Theorem 9.3 below.

We apply some of the ideas and methods of [18] and [19]. The level-raising part of Taylor's proof carries over to a much more general setup. Namely, the following: We let F denote a totally real number field with adeles $\mathbb{A} = F_\infty \times \mathbb{A}^\infty$, and let G be a connected reductive F-group such that $G^1_\infty = G_\infty \cap G(\mathbb{A})^1$ is compact and G^{der} is simple and simply connected. When $F \neq \mathbb{Q}$, this just means that G_∞ is compact. However, when $F = \mathbb{Q}$ and Z_G is split, it suffices that G^{der}_∞ is compact. There are plenty of such groups. In fact, any split simple F-group not of type A_n ($n \geqslant 2$), D_{2n+1} or E_6 has infinitely many inner forms which are compact at infinity. Throughout, we fix a Haar measure $\mu = \otimes \mu_v$ on $G(\mathbb{A}^\infty)$. It is convenient to state our results using the following notion of congruence. As K varies over the compact open subgroups of $G(\mathbb{A}^\infty)$, the centers $Z(\mathcal{H}_{K,\mathbb{Z}})$ of the Hecke algebras form an inverse system. To an automorphic representation π of $G(\mathbb{A})$ we associate the character

$$\eta_\pi : \varprojlim Z(\mathcal{H}_{K,\mathbb{Z}}) \to \mathbb{C}$$

such that $\eta_\pi = \eta_{\pi^K} \circ \mathrm{pr}_K$ for every compact open subgroup K such that $\pi^K \neq 0$. If λ is a finite place of $\overline{\mathbb{Q}}$, we say that $\tilde{\pi}$ and π are congruent modulo λ if their characters are. We write $\tilde{\pi} \equiv \pi \pmod{\lambda}$. A similar notion makes sense locally, and then $\tilde{\pi} \equiv \pi \pmod{\lambda}$ if and only if $\tilde{\pi}_v \equiv \pi_v \pmod{\lambda}$ for all finite v. Moreover, when both $\tilde{\pi}_v$ and π_v are unramified, $\tilde{\pi}_v \equiv \pi_v \pmod{\lambda}$ simply means the Satake parameters are congruent. Before we can state the main theorem, we need the following definition.

Definition 1.2. — *Let π be an automorphic representation of $G(\mathbb{A})$ such that $\pi_\infty = \mathbf{1}$. We say that π is abelian modulo λ, a finite place of $\overline{\mathbb{Q}}$, if there exists an automorphic character χ of $G(\mathbb{A})$ with $\chi_\infty = \mathbf{1}$ such that $\pi \equiv \chi \pmod{\lambda}$.*

This is the analogue of the notion Eisenstein modulo λ in [6, p. 1269]. Since G^{der} is anisotropic in our setup, there are no cusps and we prefer the terminology abelian modulo λ. The following theorem is in some sense the main result of this paper.

THEOREM 1.3. — *Let $\pi = \otimes\pi_v$ be an automorphic representation of $G(\mathbb{A})$ such that $\pi_\infty = \mathbf{1}$, and let $\lambda|\ell$ be a finite place of $\overline{\mathbb{Q}}$ such that π is non-abelian modulo λ. Suppose w is a finite place of F where π_w is unramified and*

$$\pi_w \equiv \mathbf{1} \pmod{\lambda}.$$

Let $K_w \subset G_w$ be a hyperspecial subgroup and let $J_w = K_w \cap K'_w$ be a parahoric subgroup, where K'_w is another maximal compact subgroup. Suppose $\ell \nmid [K'_w : J_w]$. Then there exists an automorphic representation $\tilde{\pi} = \otimes\tilde{\pi}_v$ of $G(\mathbb{A})$ with $\tilde{\pi}_\infty = \mathbf{1}$,

- $\tilde{\pi}_w^{J_w} \neq \tilde{\pi}_w^{K_w} + \tilde{\pi}_w^{K'_w}$,
- $\tilde{\pi} \equiv \pi \pmod{\lambda}$.

This theorem has no content unless $\pi_w^{J_w} = \pi_w^{K_w} + \pi_w^{K'_w}$. There is a more precise version later in this paper. If G_w^{der} has rank one, J_w is an Iwahori subgroup and one can conclude that $\tilde{\pi}_w^{K_w} = 0$ but $\tilde{\pi}_w^{J_w} \neq 0$. This was first proved by Bellaiche in his thesis [1], using the ideas of Clozel [6]. By a theorem of Serre, [16], the eigensystem of a modular form mod ℓ comes from an algebraic modular form mod ℓ on $D^\times$, where $D/\mathbb{Q}$ now is the quaternion algebra with ramification locus $\{\infty, \ell\}$. Combining this result with the Jacquet-Langlands correspondence yields the result of Ribet after stripping powers of ℓ from the level.

There is another proof of Ribet's theorem relying on the so-called Ihara lemma. It states that for $q \nmid N\ell$, the degeneracy maps $X_0(Nq) \rightrightarrows X_0(N)$ induce an injection

$$H^1(X_0(N), \mathbb{Z}_\ell)^{\oplus 2} \to H^1(X_0(Nq), \mathbb{Z}_\ell)$$

with torsion-free cokernel. The proof of this lemma reduces to the congruence subgroup property of the group $\mathrm{SL}_2(\mathbb{Z}[1/q])$. In our case we are looking at functions on a finite set, and the analogue of the Ihara lemma can be proved by imitating the combinatorial argument of Taylor [18, p. 274] in the diagonal weight 2 case. See section 5.3 below.

We mention a few applications of our main theorem. First, let $E/\mathbb{Q}$ be an imaginary quadratic extension and let $G^* = \mathrm{U}(2,1)$ be the quasi-split unitary $\mathbb{Q}$-group in 3 variables split over E. Let $G = \mathrm{U}(3)$ be an inner form of G^* such that G_∞ is compact. For primes q inert in E, the semisimple

rank of $G(\mathbb{Q}_q)$ is one and we recover the result of Clozel [6]. In the split case we obtain the following as a corollary:

THEOREM 1.4. — *Let $\pi = \otimes\pi_p$ be an automorphic representation of $G(\mathbb{A})$ with $\pi_\infty = \mathbf{1}$, and let $\lambda|\ell$ be a finite place of $\overline{\mathbb{Q}}$ such that π is non-abelian modulo λ. Suppose $q \neq \ell$ is a prime, split in E, such that π_q is unramified and $\ell \nmid 1 + q + q^2$. If moreover, for $\mathfrak{q}|q$,*

$$\mathfrak{t}_{\pi,\mathfrak{q}} \equiv \begin{pmatrix} q & & \\ & 1 & \\ & & q^{-1} \end{pmatrix} \pmod{\lambda},$$

then there exists an automorphic representation $\tilde{\pi} = \otimes\tilde{\pi}_p$ of $G(\mathbb{A})$ with $\tilde{\pi}_\infty = \mathbf{1}$ such that

- *$\tilde{\pi}_q$ is generic and $\tilde{\pi}_q^{J_q} \neq 0$, where J_q is any maximal proper parahoric,*
- *$\tilde{\pi} \equiv \pi \pmod{\lambda}$.*

We cannot prove by our methods that $\tilde{\pi}_q$ is ramified. On the other hand, Bellaiche has a result in his thesis in the split case, [1, p. 218], proving that $\tilde{\pi}_q$ is ramified under the additional assumption that π occurs with multiplicity one (and discarding finitely many primes ℓ). We classify the Iwahori-spherical representations of GL(3) and compute the dimensions of their parahoric fixed spaces. This allows us to conclude that $\tilde{\pi}_q$ is either a full unramified principal series or induced from a Steinberg representation.

It seems very likely that our method and corollary can be extended to allow $\pi_\infty \neq \mathbf{1}$, but we have chosen not to do it here for the sake of brevity. In that case it would follow that if π is endoscopic abelian (that is, nearly equivalent to a weak transfer of a character of an endoscopic group), then it is congruent to a $\tilde{\pi}$ which is not endoscopic abelian. This is true even for $\mathrm{U}(n)$, for all $n \geqslant 2$. For $n = 3$ this phenomenon has been applied to the Bloch-Kato conjecture for certain Hecke characters of E by Bellaiche [1].

In our second application, we let G be an inner form of $\mathrm{Gsp}(4)$ such that $G^{\mathrm{der}}(\mathbb{R})$ is compact. Concretely, $G = \mathrm{GSpin}(f)$ for some definite quadratic form f in 5 variables over $\mathbb{Q}$. In this situation, our main theorem yields the following:

THEOREM 1.5. — *Let $\pi = \otimes\pi_p$ be an automorphic representation of $G(\mathbb{A})$ with $\pi_\infty = \mathbf{1}$, and let $\lambda|\ell$ be a finite place of $\overline{\mathbb{Q}}$ such that π is non-abelian modulo λ. Suppose $q \neq \ell$ is a prime such that π_q is unramified. If the Hecke matrix satisfies*

$$\mathfrak{t}_{\pi_q \otimes |\nu|^{-3/2}} \equiv \begin{pmatrix} 1 & & \\ & q & \\ & & q^2 \\ & & & q^3 \end{pmatrix} \pmod{\lambda},$$

then there exists an automorphic representation $\tilde{\pi} = \otimes\tilde{\pi}_p$ *of* $G(\mathbb{A})$ *with* $\tilde{\pi}_\infty = \mathbf{1}$ *such that*

- $\tilde{\pi}_q$ *is generic and* $\tilde{\pi}_q^{J_q} \neq 0$*, where* J_q *is the Klingen parahoric,*
- $\tilde{\pi} \equiv \pi \pmod{\lambda}$.

By the Klingen parahoric, we mean the inverse image of the standard Klingen parabolic in $\mathrm{Gsp}(4, \mathbb{F}_q)$. Briefly, the proof relies on the computations of Ralf Schmidt [15]. If $m(\pi) = 1$, Bellaiche's methods seem to apply and one can probably show that $\tilde{\pi}_q$ is induced from a twisted Steinberg representation on the standard Klingen-Levi. It is known that Saito-Kurokawa lifts (that is, theta lifts from $\widetilde{\mathrm{SL}}(2)$) are locally non-generic. Therefore, if π is of Saito-Kurokawa type, it is congruent to a $\tilde{\pi}$ which is not of Saito-Kurokawa type. Our interest in it stems from our desire to apply it to the Bloch-Kato conjecture for the motives attached to classical modular forms, and we plan to study this in a sequel paper. In particular, we hope to establish a mod ℓ analogue of a result of Skinner and Urban [17], which is valid for *all* (not necessarily ordinary) modular forms of classical weight at least 4.

This work forms part of my doctoral dissertation at the California Institute of Technology. I would like to acknowledge the impact of the ideas of Ribet, Taylor, Clozel and Bellaiche.

2. The abstract setup and Taylor's lemma

2.1. The abstract setup

In this section, we fix a subring $\mathcal{O} \subset \mathbb{C}$ and denote by $L \subset \mathbb{C}$ its field of fractions. Let H be a commutative $\mathbb{C}$-algebra. We do not require H to be of finite dimension. However, we assume H comes equipped with an involution $\phi \mapsto \phi^\vee$. Here, by involution we mean a $\mathbb{C}$-linear automorphism of order two. Moreover, we fix an $\mathcal{O}$-order $H_\mathcal{O} \subset H$ preserved by $\vee$ (by an $\mathcal{O}$-order we mean an $\mathcal{O}$-subalgebra which is the $\mathcal{O}$-span of a $\mathbb{C}$-basis for H). Then we look at a triple $(V, \langle -, - \rangle_V, V_\mathcal{O})$ consisting of the following data:

- V is a finite-dimensional $\mathbb{C}$-space with an action $r_V : H \to \mathrm{End}_\mathbb{C}(V)$,
- $\langle -, - \rangle_V$ is a non-degenerate, symmetric, $\mathbb{C}$-bilinear form $V \times V \to \mathbb{C}$,
- $V_\mathcal{O} \subset V$ is an $\mathcal{O}$-lattice (that is, the $\mathcal{O}$-span of a $\mathbb{C}$-basis).

We impose the following compatibility conditions between these data:

- $r_V(\phi^\vee)$ is the adjoint of $r_V(\phi)$ with respect to $\langle -, - \rangle_V$,
- $V_\mathcal{O} \subset V$ is preserved by the order $H_\mathcal{O} \subset H$,

- $V_{\mathcal{O}}/(V_{\mathcal{O}} \cap V_{\mathcal{O}}^{\vee})$ and $V_{\mathcal{O}}^{\vee}/(V_{\mathcal{O}} \cap V_{\mathcal{O}}^{\vee})$ are torsion $\mathcal{O}$-modules.

Here $V_{\mathcal{O}}^{\vee} = \{v \in V : \langle v, V_{\mathcal{O}} \rangle_V \subset \mathcal{O}\}$ is the dual lattice of $V_{\mathcal{O}}$ in V. Choose nonzero annihilators A_V and B_V in $\mathcal{O}$ of the torsion modules above, that is, such that

$$A_V \langle V_{\mathcal{O}}, V_{\mathcal{O}} \rangle_V \subset \mathcal{O} \quad \text{and} \quad \langle v, V_{\mathcal{O}} \rangle_V \subset \mathcal{O} \Rightarrow B_V v \in V_{\mathcal{O}}.$$

Now let $(U, \langle -, - \rangle_U, U_{\mathcal{O}})$ be another such triple and choose annihilators A_U and B_U for it as above. Suppose we are given a map $\delta : U \to V$, which is H-linear, and in addition has the following properties:

- $U = \ker(\delta) \oplus \ker(\delta)^{\perp}$,
- $V = \mathrm{im}(\delta) \oplus \mathrm{im}(\delta)^{\perp}$,
- $\delta(U_{\mathcal{O}}) \subset V_{\mathcal{O}} \cap \delta(U)$, and the quotient is killed by $C \in \mathcal{O} - \{0\}$.

We consider its adjoint map $\delta^{\vee} : V \to U$ defined in the obvious way. Let $V^{\mathrm{old}} = \mathrm{im}(\delta)$ and $V^{\mathrm{new}} = \mathrm{im}(\delta)^{\perp}$. These are H-stable subspaces of V, and by assumption we have an orthogonal decomposition $V = V^{\mathrm{old}} \oplus V^{\mathrm{new}}$.

Definition 2.1. — *Let* $V_{\mathcal{O}}^{\mathrm{old}} = V_{\mathcal{O}} \cap V^{\mathrm{old}}$ *and* $V_{\mathcal{O}}^{\mathrm{new}} = V_{\mathcal{O}} \cap V^{\mathrm{new}}$.

These $H_{\mathcal{O}}$-stable submodules of $V_{\mathcal{O}}$ span V^{old} and V^{new} respectively. They are orthogonal, but their sum is not always all of $V_{\mathcal{O}}$. Note that $\delta(U_{\mathcal{O}}) \subset V_{\mathcal{O}}^{\mathrm{old}}$ and $CV_{\mathcal{O}}^{\mathrm{old}} \subset \delta(U_{\mathcal{O}})$ by assumption. Now we look at the quotients of $\mathbb{T}_{\mathcal{O}}$, the image of $H_{\mathcal{O}}$ in $\mathrm{End}_{\mathcal{O}}(V_{\mathcal{O}})$, cut out by these submodules:

$$\mathbb{T}_{\mathcal{O}}^{\mathrm{old}} \subset \mathrm{End}_{\mathcal{O}}(V_{\mathcal{O}}^{\mathrm{old}}) \quad \text{and} \quad \mathbb{T}_{\mathcal{O}}^{\mathrm{new}} \subset \mathrm{End}_{\mathcal{O}}(V_{\mathcal{O}}^{\mathrm{new}})$$

denote the images of $H_{\mathcal{O}}$. Clearly we have natural surjective maps $\mathbb{T}_{\mathcal{O}} \twoheadrightarrow \mathbb{T}_{\mathcal{O}}^{\mathrm{old}}$ and $\mathbb{T}_{\mathcal{O}} \twoheadrightarrow \mathbb{T}_{\mathcal{O}}^{\mathrm{new}}$ given by restriction, and $\mathbb{T}_{\mathcal{O}}$ acts faithfully on $V_{\mathcal{O}}$.

2.2. An extension of Taylor's lemma

Note that $\mathbb{T}_{\mathcal{O}}$ acts naturally on $U'_{\mathcal{O}} = U_{\mathcal{O}} \cap \ker(\delta)^{\perp}$. Moreover, one can easily check that the action factors through $\mathbb{T}_{\mathcal{O}}^{\mathrm{old}}$. By a congruence module we mean a $\mathbb{T}_{\mathcal{O}}$-module, such that the action factors through both $\mathbb{T}_{\mathcal{O}}^{\mathrm{old}}$ and $\mathbb{T}_{\mathcal{O}}^{\mathrm{new}}$. The following lemma was stated for $\mathcal{O} = \mathbb{Z}$, trivial annihilators, and injective δ in [19, p. 331].

Lemma 2.2. — $U'_{\mathcal{O}}/(U'_{\mathcal{O}} \cap E^{-1}\delta^{\vee}\delta(U_{\mathcal{O}}))$ *is a congruence module for* $E = A_U B_V C^2$.

Proof. — Suppose $\phi \in H_{\mathcal{O}}$ acts trivially on $V_{\mathcal{O}}^{\mathrm{new}}$. We must show that $E\phi$ maps $U'_{\mathcal{O}}$ into $\delta^{\vee}\delta(U_{\mathcal{O}})$. Note first that $\phi^{\vee}$ also acts trivially on $V_{\mathcal{O}}^{\mathrm{new}}$,

so it maps $V_{\mathcal{O}}$ into $V_{\mathcal{O}}^{\mathrm{old}}$. Now let $u = \delta^\vee(v) \in U_{\mathcal{O}}$ for some $v \in V^{\mathrm{old}}$. Note that

$$A_U C\langle v, V_{\mathcal{O}}^{\mathrm{old}}\rangle_V \subset A_U\langle v, \delta(U_{\mathcal{O}})\rangle_V \subset A_U\langle u, U_{\mathcal{O}}\rangle_U \subset \mathcal{O},$$

so $A_U C\langle \phi v, V_{\mathcal{O}}\rangle_V \subset \mathcal{O}$, and hence $A_U B_V C(\phi v) \in V_{\mathcal{O}}^{\mathrm{old}}$. We deduce that

$$A_U B_V C^2(\phi v) \in \delta(U_{\mathcal{O}}),$$

and we get the result by applying $\delta^\vee$ to this: $E(\phi u) \in \delta^\vee\delta(U_{\mathcal{O}})$. □

As in [19, p. 331], we have the following useful corollary:

COROLLARY 2.3. — *Let $\mathcal{O} = \mathcal{O}_L$ be the ring of integers of a number field $L \subset \mathbb{C}$. Suppose $u \in U_{\mathcal{O}} - \{0\}$ is an eigenvector for $H_{\mathcal{O}}$, with character $\eta\colon H_{\mathcal{O}} \to \mathcal{O}$. Assume:*

- *$\mathcal{E}(Lu \cap (U_{\mathcal{O}} + \ker\delta)) \subset \mathcal{O}u$, for some nonzero ideal $\mathcal{E} \subset \mathcal{O}$,*
- *$\delta^\vee\delta(u) \in mU_{\mathcal{O}}$, for some nonzero $m \in \mathcal{O}$.*

Then η induces a homomorphism $\mathbb{T}_{\mathcal{O}}^{\mathrm{new}} \to \mathcal{O}/(\mathcal{O} \cap mE^{-1}\mathcal{E}^{-1})$, where $E = A_U B_V C^2$.

We remark that $m = 0 \Rightarrow u \in \ker\delta$. If we factor the fractional ideal $\mathcal{O} \cap mE^{-1}\mathcal{E}^{-1}$ into prime powers and project further, we get the following: For every (nonzero) prime ideal $\lambda \subset \mathcal{O}$ there is a homomorphism

$$\mathbb{T}_{\mathcal{O}}^{\mathrm{new}} \to \mathcal{O}/\lambda^n$$

induced by η, where n is a non-negative integer satisfying the inequality

$$n \geqslant v_\lambda(m) - v_\lambda(E) - v_\lambda(\mathcal{E}).$$

Here we should think of $v_\lambda(m)$ as the main term, and the other two as controllable error terms. In our applications we want to show that the right-hand-side is positive.

3. Compactness at infinity

Let F be a totally real number field, and let ∞ be the set of archimedean places. We denote the ring of adeles by $\mathbb{A} = \mathbb{A}_F = F_\infty \times \mathbb{A}^\infty$. We consider a connected reductive F-group G, and let $A = A_G$ denote the F-split component of its center $Z = Z_G$. Each F-rational character $\chi \in X^*(G)_F$ gives a continuous homomorphism $G(\mathbb{A}) \to \mathbb{R}_+^*$ by composing with the idele norm, and we define

$$G(\mathbb{A})^1 = \{g \in G(\mathbb{A})\colon |\chi(g)| = 1, \forall \chi \in X^*(G)_F\}.$$

It is known to be unimodular. By the product formula, $G(F)$ is a discrete subgroup of $G(\mathbb{A})^1$, and the quotient $G(F)\backslash G(\mathbb{A})^1$ has finite volume. Moreover, this quotient is compact if and only if G^{ad} is anisotropic. Later, we are naturally led to studying groups for which $G_\infty^1 = G_\infty \cap G(\mathbb{A})^1$ is compact.

PROPOSITION 3.1. — *G_∞^1 is compact if and only if one of the following holds:*

- *G_∞ is compact,*
- *$F = \mathbb{Q}$, $\mathrm{rk}_{\mathbb{Q}} Z = \mathrm{rk}_{\mathbb{R}} Z$, and G_∞^{der} is compact.*

Proof. — Suppose first that G_∞^1 is compact. We may assume that $A \neq 1$ (otherwise $G_\infty = G_\infty^1$ is compact). Choosing a basis for $X^*(A)$, we see that (with $r = \dim A$)

$$A_\infty^1 \simeq \left\{x \in F_\infty^* \colon \prod_{v\in\infty} |x_v|_v = 1\right\}^r.$$

Therefore $\{x \in F_\infty^* \colon \prod_{v|\infty} |x_v|_v = 1\}$ is compact, and we conclude that F has a unique infinite place. That is, $F = \mathbb{Q}$. If $\mathrm{rk}_{\mathbb{Q}} Z < \mathrm{rk}_{\mathbb{R}} Z$, the $\mathbb{Q}$-anisotropic component A' is not $\mathbb{R}$-anisotropic. The converse is clear. □

4. Hecke algebras and algebraic modular forms

4.1. Hecke algebras

From now on we fix a totally real number field F, and a connected reductive F-group G, not a torus, such that G_∞^1 is compact. We consider the locally profinite group of finite adeles $G(\mathbb{A}^\infty)$, and choose a Haar measure $\mu = \otimes\mu_v$ on it once and for all. We consider the vector space of all locally constant compactly supported $\mathbb{C}$-valued functions

$$\mathcal{H} = \mathcal{H}(G(\mathbb{A}^\infty)) = C_c^\infty(G(\mathbb{A}^\infty), \mathbb{C}).$$

This becomes an associative $\mathbb{C}$-algebra, without neutral element, under μ-convolution. There is a canonical involution on $\mathcal{H}$ defined by $\phi^\vee(x) = \phi(x^{-1})$. This is an anti-automorphism. If $K \subset G(\mathbb{A}^\infty)$ is a compact open subgroup,

$$e_K = \mu(K)^{-1}\chi_K \in \mathcal{H}$$

is an idempotent. This is a neutral element in the subalgebra of K-biinvariant compactly supported functions:

$$\mathcal{H}_K = \mathcal{H}(G(\mathbb{A}^\infty), K) = C_c(G(\mathbb{A}^\infty)//K, \mathbb{C}) = e_K \star \mathcal{H} \star e_K.$$

Clearly $\vee$ preserves $\mathcal{H}_K$. In addition, there is a canonical $\mathbb{Z}$-order $\mathcal{H}_{K,\mathbb{Z}} \subset \mathcal{H}_K$ consisting of all $\mu(K)^{-1}\mathbb{Z}$-valued functions. As a ring, $\mathcal{H}_{K,\mathbb{Z}}$ is isomorphic to $C_c(G(\mathbb{A}^\infty)//K, \mathbb{Z})$ endowed with the K-normalized convolution. If R is a commutative ring, with neutral element, we then define

$$\mathcal{H}_{K,R} = R \otimes_{\mathbb{Z}} \mathcal{H}_{K,\mathbb{Z}}.$$

The algebras $\mathcal{H}_K$ are not always commutative. However, by a result of Bernstein [2], $\mathcal{H}_K$ is a finite module over its center $Z(\mathcal{H}_K)$. Now, suppose $J \subset K$ is a (proper) compact open subgroup. Then obviously $\mathcal{H}_K \subset \mathcal{H}_J$. However, $\mathcal{H}_K$ is not a subring since $e_K \neq e_J$. There is a natural retraction $\mathcal{H}_J \twoheadrightarrow \mathcal{H}_K$ defined by $\phi \mapsto e_K \star \phi \star e_K$. It does map $e_J \mapsto e_K$, but does not preserve $\star$ unless we restrict it to the centralizer $Z_{\mathcal{H}_J}(e_K)$. Clearly, $Z_{\mathcal{H}_J}(\mathcal{H}_K)$ maps to the center $Z(\mathcal{H}_K)$. In particular,

$$Z(\mathcal{H}_J) \to Z(\mathcal{H}_K), \quad \phi \mapsto \phi \star e_K = e_K \star \phi,$$

gives a canonical homomorphism of algebras. It maps $Z(\mathcal{H}_{J,\mathbb{Z}})$ into $Z(\mathcal{H}_{K,\mathbb{Z}})$.

4.2. Algebraic modular forms

Note that $G(F) \subset G(\mathbb{A}^\infty)$ is a discrete subgroup. We consider the Hilbert space of L^2-functions on the quotient, $L^2(G(F)\backslash G(\mathbb{A}^\infty))$. There is a unitary representation r of $G(\mathbb{A}^\infty)$ on this space given by right translations. We consider the smooth vectors,

$$\mathcal{A} = L^2(G(F)\backslash G(\mathbb{A}^\infty))^\infty = C^\infty(G(F)\backslash G(\mathbb{A}^\infty), \mathbb{C}),$$

consisting of locally constant functions. This is an admissible representation:

$$\mathcal{A} = \bigcup \mathcal{A}_K, \quad \text{where} \quad \mathcal{A}_K \simeq C(G(F)\backslash G(\mathbb{A}^\infty)/K, \mathbb{C}),$$

and K runs over all compact open subgroups of $G(\mathbb{A}^\infty)$. Therefore, the Hecke algebra $\mathcal{H}$ acts on $\mathcal{A}$ in the usual way. We have the following compatibility between this action and the inner product:

$$(r(\bar{\phi})f, g) = (f, r(\phi^\vee)g).$$

For a compact open subgroup $K \subset G(\mathbb{A}^\infty)$, the space of K-invariants

$$\mathcal{A}_K \simeq C(G(F)\backslash G(\mathbb{A}^\infty)/K, \mathbb{C}) = r(e_K)\mathcal{A}$$

is finite-dimensional. Indeed the double coset space $X_K = G(F)\backslash G(\mathbb{A}^\infty)/K$ is finite. Functions in $\mathcal{A}_K$ are examples of algebraic modular forms. Clearly,

$\mathcal{H}_K$ acts on $\mathcal{A}_K$, and the order $\mathcal{H}_{K,\mathbb{Z}}$ preserves the lattice of $\mathbb{Z}$-valued functions:

$$\mathcal{A}_{K,\mathbb{Z}} = C(G(F)\backslash G(\mathbb{A}^\infty)/K, \mathbb{Z}) \subset \mathcal{A}_K.$$

For a commutative ring R we let $\mathcal{A}_{K,R} = R \otimes_{\mathbb{Z}} \mathcal{A}_{K,\mathbb{Z}}$. The R-algebra $\mathcal{H}_{K,R}$ acts on this module, and we let $\mathbb{T}_{K,R}$ denote the image of the center $Z(\mathcal{H}_{K,R})$ in $\mathrm{End}_R \mathcal{A}_{K,R}$. Hence $\mathbb{T}_{K,R}$ is a commutative R-algebra. Now, suppose $J \subset K$ is a (proper) compact open subgroup. Then $\mathcal{A}_K \subset \mathcal{A}_J$, and the canonical homomorphism $Z(\mathcal{H}_{J,R}) \to Z(\mathcal{H}_{K,R})$ descends to the restriction map $\mathbb{T}_{J,R} \to \mathbb{T}_{K,R}$.

4.3. Pairings

We define a pairing on $\mathcal{A}_K$ as follows. Here $(-,-)$ denotes the Petersson inner product.

Definition 4.1. — *For $f, g \in \mathcal{A}_K$, we define a symmetric bilinear form by*

$$\langle f, g\rangle_K = \mu(K)^{-1}(f, \bar{g}) = \sum_{x \in X_K} f(x)g(x)|G(F) \cap {}^xK|^{-1},$$

where we use the notation ${}^xK = xKx^{-1}$.

The factors $|G(F)\cap {}^xK|^{-1}$ are missing in [18] and [19]. If K is sufficiently small, for example if $K = \prod_{v<\infty} K_v$ and some K_v is torsion-free (this is the case if K_v is a sufficiently deep principal congruence subgroup), then indeed $G(F) \cap {}^xK = 1$. For $\phi \in \mathcal{H}_K$ and $f, g \in \mathcal{A}_K$ we have the compatibility relation

$$\langle r(\phi)f, g\rangle_K = \langle f, r(\phi^\vee)g\rangle_K.$$

Next we have to show the quotient $\mathcal{A}_{K,\mathcal{O}}/\mathcal{A}^\vee_{K,\mathcal{O}}$ is torsion and find a good annihilator A_K. The fact that it is torsion is immediate: It is killed by the positive integer

$$\prod_{x \in X_K} |G(F) \cap {}^xK|.$$

This is 1 if K is sufficiently small in the sense above.

Lemma 4.2. — *Let $K = \prod_{v<\infty} K_v \subset G(\mathbb{A}^\infty)$ be a decomposable compact open subgroup, and let ℓ be a prime number. Suppose $\ell \nmid |K_v|$ for some $v < \infty$. Then there exists a positive integer A_K, not divisible by ℓ, such that*

$$A_K\langle \mathcal{A}_{K,\mathcal{O}}, \mathcal{A}_{K,\mathcal{O}}\rangle_K \subset \mathcal{O}.$$

Proof. — Choose some torsion-free subgroup $\widetilde{K}_v \subset K_v$ and let $\widetilde{K} = \widetilde{K}_v K^v$. Then

$$\langle \mathcal{A}_{\widetilde{K},\mathcal{O}}, \mathcal{A}_{\widetilde{K},\mathcal{O}} \rangle_{\widetilde{K}} \subset \mathcal{O}$$

as we have observed above. Therefore, for $f, g \in \mathcal{A}_{K,\mathcal{O}} \subset \mathcal{A}_{\widetilde{K},\mathcal{O}}$, we have

$$[K_v \colon \widetilde{K}_v] \langle f, g \rangle_K = \langle f, g \rangle_{\widetilde{K}} \in \mathcal{O}.$$

We then take $A_K = [K_v \colon \widetilde{K}_v]$. This is not divisible by ℓ. □

Note that $\ell \nmid |K_v|$ if K_v is torsion-free and $v \nmid \ell$. For large ℓ this is automatic:

LEMMA 4.3. — *Suppose there exists an F-embedding $G \hookrightarrow \mathrm{GL}(n)$. Let $K = \prod_{v<\infty} K_v$ be arbitrary and let $\ell > [F \colon \mathbb{Q}]n + 1$ be a prime number. Then $\ell \nmid |K_v|$ holds for infinitely many places v.*

Proof. — K_v embeds into a conjugate of $\mathrm{GL}(n, \mathcal{O}_v)$. Therefore $|K_v|$ divides

$$|\mathrm{GL}(n, \mathcal{O}_v)| = p^\infty \prod_{i=1}^{n} (q^i - 1).$$

Assume ℓ divides $|K_v|$ for almost all v. Then p has order at most $[F \colon \mathbb{Q}]n$ in $(\mathbb{Z}/\ell)^*$ for almost all primes p. Now, $(\mathbb{Z}/\ell)^*$ is cyclic of order $\ell - 1$, so by Dirichlet's theorem on primes in arithmetic progressions we conclude that $\ell \leqslant [F \colon \mathbb{Q}]n + 1$. □

5. Parahoric level structure and the concrete setup

5.1. Parahoric subgroups

From now on we assume for simplicity that G^{der} is simple (that is, it has no nontrivial connected normal subgroups). Moreover, we fix a compact open subgroup

$$K = \prod_{v<\infty} K_v \subset G(\mathbb{A}^\infty).$$

It is known that $K_v \subset G_v$ is a hyperspecial maximal compact subgroup for almost all places v, that is, $K_v = \underline{G}(\mathcal{O}_{F_v})$ for a smooth affine group scheme $\underline{G}$ of finite type over $\mathcal{O}_{F_v}$ with generic fiber G. Such exist precisely when G_v is unramified. Let us look at a fixed finite place w of F where K_w is hyperspecial. Then write $K = K_w K^w$, where

$$K^w = \prod_{v \neq w} K_v \subset G(\mathbb{A}^{\infty,w}).$$

Let $\mathcal{B}_w$ denote the reduced Bruhat-Tits building of G_w (that is, the building of G_w^{ad}). We have assumed G^{der} is simple, so $\mathcal{B}_w$ is a simplicial complex. Let $x \in \mathcal{B}_w$ be the vertex fixed by K_w, and let (x, x') be an edge in the building. Then consider the maximal compact subgroup $K'_w \subset G_w$ fixing the vertex x', and the parahoric subgroup $J_w = K_w \cap K'_w$ associated with the edge (x, x'). Let $K' = K'_w K^w$ and $J = J_w K^w$ be the corresponding subgroups of $G(\mathbb{A}^\infty)$.

LEMMA 5.1. — $\langle K_w, K'_w \rangle = G_w^0 := \left\{ g \in G_w \colon |\chi(g)| = 1, \forall \chi \in X^*(G)_{F_w} \right\}$.

Proof. — This follows from Bruhat-Tits theory. □

Note that $G_w^{\mathrm{der}} \subset G_w^0 \subset G_w^1 = G_w \cap G(\mathbb{A})^1$.

5.2. The concrete setup

Now we want to apply our general results in the following setup: Let $L \subset \mathbb{C}$ be a number field, and let $\mathcal{O} = \mathcal{O}_L$ be its ring of integers. We let $H = Z(\mathcal{H}_J)$. This is a commutative $\mathbb{C}$-algebra, and it comes with the involution defined by $\phi^\vee(x) = \phi(x^{-1})$. $V = \mathcal{A}_J$ is a finite-dimensional $\mathbb{C}$-space on which $Z(\mathcal{H}_J)$ acts. The order $Z(\mathcal{H}_{J,\mathcal{O}})$ preserves the lattice $V_{\mathcal{O}} = \mathcal{A}_{J,\mathcal{O}}$. The space V comes with the bilinear form $\langle -, - \rangle_J$. The compatibility conditions between these data are satisfied. Let $U = \mathcal{A}_K \oplus \mathcal{A}_{K'}$. Then $Z(\mathcal{H}_J)$ acts on this space via the natural maps to $Z(\mathcal{H}_K)$ and $Z(\mathcal{H}_{K'})$. The lattice $U_{\mathcal{O}} = \mathcal{A}_{K,\mathcal{O}} \oplus \mathcal{A}_{K',\mathcal{O}}$ is preserved by $Z(\mathcal{H}_{J,\mathcal{O}})$. The bilinear form on U is given by the sum $\langle -, - \rangle_K \oplus \langle -, - \rangle_{K'}$. The degeneracy map δ is given by

$$\delta \colon \mathcal{A}_K \oplus \mathcal{A}_{K'} \overset{\mathrm{sum}}{\to} \mathcal{A}_J,$$

which is clearly $Z(\mathcal{H}_J)$-linear. Obviously, $\ker(\delta)$ consists of all pairs $(f, -f)$, where

$$f \in \mathcal{A}_K \cap \mathcal{A}_{K'} = \{ G_w^0 K^w - \text{invariant functions } f \in \mathcal{A} \}.$$

The decompositions $U = \ker(\delta) \oplus \ker(\delta)^\perp$ and $V = \mathrm{im}(\delta) \oplus \mathrm{im}(\delta)^\perp$ are immediate because of the relation between the pairings and the inner product.

5.3. The combinatorial Ihara lemma

The proof of the following lemma is a straightforward generalization of [18, p. 274]:

LEMMA 5.2. — $\mathcal{A}_{J,\mathcal{O}} \cap \delta(\mathcal{A}_K \oplus \mathcal{A}_{K'}) = \delta(\mathcal{A}_{K,\mathcal{O}} \oplus \mathcal{A}_{K',\mathcal{O}})$.

Proof. — Let us first set up some machinery for the proof. We define an equivalence relation on X_J by saying that $x, y \in X_J$ are equivalent ($x \sim y$) if and only if

$$\exists \text{ chain } x = x_0, \dots, x_d = y \text{ such that}$$
$$\forall i\colon \pi(x_i) = \pi(x_{i+1}) \text{ or } \pi'(x_i) = \pi'(x_{i+1}).$$

This gives a partition of X_J into equivalence classes X_J^j. For each j, we fix a representative $y^j \in X_J^j$. Correspondingly, we have a radius function $d\colon X_J \to \mathbb{Z}_{\geqslant 0}$ defined as follows: Given $x \in X_J$, there is a unique j such that $x \sim y^j$. Then $d(x)$ is the minimal length of a chain connecting x to y^j. Now, suppose $g = \delta(f, f') \in \mathcal{A}_{J,\mathcal{O}}$ for some $f \in \mathcal{A}_K$ and $f' \in \mathcal{A}_{K'}$. We want to show $g \in \delta(\mathcal{A}_{K,\mathcal{O}} \oplus \mathcal{A}_{K',\mathcal{O}})$.

CLAIM. — *We may assume that $f(\pi(y^j)) = 0$ for all j.*

To see this, note that $X_K = \sqcup\pi(X_J^j)$ and $X_{K'} = \sqcup\pi'(X_J^j)$. We then define $\tilde{f} \in \mathcal{A}_K$ such that $\tilde{f}|\pi(X_J^j) \equiv f(\pi(y^j))$, and $\tilde{f}' \in \mathcal{A}_{K'}$ such that $\tilde{f}'|\pi'(X_J^j) \equiv f(\pi(y^j))$. Then

$$g = \delta(f - \tilde{f}, f' + \tilde{f}')$$

and $(f - \tilde{f})(\pi(y^j)) = 0$ for all j. This proves the claim, so from now on assume that $f(\pi(y^j)) = 0$ for all j. By induction on $m \geqslant 0$, we now prove the following:

$$\forall x \in X_J \text{ with } d(x) = m\colon f(\pi(x)) \in \mathcal{O} \text{ and } f'(\pi'(x)) \in \mathcal{O}.$$

This is sufficient, for then $f \in \mathcal{A}_{K,\mathcal{O}}$ and $f' \in \mathcal{A}_{K',\mathcal{O}}$. Note that $f(\pi(x)) \in \mathcal{O}$ if and only if $f'(\pi'(x)) \in \mathcal{O}$. The start $m = 0$ is essentially just our assumption, so assume the statement is true for $m - 1 \geqslant 0$ and consider $x \in X_J$ with $d(x) = m$. Let

$$x = x_0, x_1, \dots, x_m = y^j$$

be a chain of minimal length. Then $x' = x_1 \in X_J$ has $d(x') = m - 1$, so by induction $f(\pi(x')) \in \mathcal{O}$ and $f'(\pi'(x')) \in \mathcal{O}$. However, $\pi(x) = \pi(x')$ or $\pi'(x) = \pi'(x')$. In either case we get the statement for x. □

6. Applying the abstract theory

6.1. Computing $\delta^\vee\delta$

To apply the abstract theory it is necessary to compute $\delta^\vee\delta$ explicitly.

LEMMA 6.1. — *The endomorphism $\delta^\vee\delta$ is given by the 2×2 matrix*

$$\delta^\vee\delta = \begin{pmatrix} [K\colon J] & [K\colon J]e_K \\ [K'\colon J]e_{K'} & [K'\colon J] \end{pmatrix}.$$

Proof. — $\delta^\vee\delta$ is an endomorphism of $\mathcal{A}_K \oplus \mathcal{A}_{K'}$, and we write is as

$$\delta^\vee\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $b\colon \mathcal{A}_{K'} \to \mathcal{A}_K$ and so on. Using the definition it is not hard to see that

$$\langle af, g\rangle_K = \langle f, g\rangle_J = [K\colon J]\langle f, g\rangle_K$$

for all $f, g \in \mathcal{A}_K$. In particular, $a = [K\colon J]$. In the same way one computes b, c and d. □

6.2. The main lemma

In our situation, Corollary 2.3 gives the following crucial lemma.

LEMMA 6.2. — *Let $f \in \mathcal{A}_{K,\mathcal{O}}$ be an eigenform for $Z(\mathcal{H}_{K,\mathcal{O}})$ with character $\eta_f\colon \mathbb{T}_{K,\mathcal{O}} \to \mathcal{O}$. Assume f is not G_w^0-invariant modulo $\lambda|\ell$, where ℓ satisfies the following:*

There exists at least two places v such that $\ell \nmid |K_v|$.

Then the reduction of η_f modulo λ^n factors through $\mathbb{T}_{J,\mathcal{O}}^{\mathrm{new}}$ when

$$v_\lambda\big(\eta_f(e_{K,K'}) - [K\colon J][K'\colon J]\big) - v_\lambda([K'\colon J]\big) \geqslant n,$$

where we introduce the notation $e_{K,K'} = [K\colon J][K'\colon J](e_K \star e_{K'} \star e_K) \in Z(\mathcal{H}_{K,\mathbb{Z}})$.

Proof. — To produce an eigenvector in $U_\mathcal{O} = \mathcal{A}_{K,\mathcal{O}} \oplus \mathcal{A}_{K',\mathcal{O}}$, we take

$$\vec{f} = [K'\colon J](f, -r(e_{K'})f) \in \mathcal{A}_{K,\mathcal{O}} \oplus \mathcal{A}_{K',\mathcal{O}}.$$

The factor $[K'\colon J]$ is included since $r(e_{K'})f$ does not necessarily take values in $\mathcal{O}$. Clearly, $\vec{f}$ is an eigenvector for $Z(\mathcal{H}_{J,\mathcal{O}})$, and its character is the composite

$$\eta_{\vec{f}}\colon Z(\mathcal{H}_{J,\mathcal{O}}) \to Z(\mathcal{H}_{K,\mathcal{O}}) \xrightarrow{\eta_f} \mathcal{O}.$$

Using the explicit formula for $\delta^\vee\delta$ in Lemma 6.1 above, it follows that

$$\delta^\vee\delta(\vec{f}) = \big(\eta_f(e_{K,K'}) - [K\colon J][K'\colon J]\big)(-f, 0).$$

Now, since $(-f, 0) \in U_\mathcal{O}$, in Corollary 2.3 we can take

$$m = \eta_f(e_{K,K'}) - [K\colon J][K'\colon J] \in \mathcal{O}$$

as long as this is nonzero. However, note that $\vec{f}$ must belong to the kernel of δ if $m = 0$. Hence f must be invariant under the group G_w^0 (say, on the right). Now, let

$$\mathcal{F} = \{x \in L \colon xf \in \mathcal{A}_{K,\mathcal{O}} + \mathcal{A}_K \cap \mathcal{A}_{K'}\}.$$

This is clearly an $\mathcal{O}$-submodule of L containing $\mathcal{O}$. Obviously, $\mathcal{F} = L$ if $f \in \mathcal{A}_K \cap \mathcal{A}_{K'}$. However, f is not G_w^0-invariant, so $\mathcal{F}$ is a fractional ideal. To see this note that

$$A_K \langle f, g \rangle_K \mathcal{F} \subset \mathcal{O}$$

for every $g \in \mathcal{A}_{K,\mathcal{O}} \cap (\mathcal{A}_K \cap \mathcal{A}_{K'})^\perp$. These g span $(\mathcal{A}_K \cap \mathcal{A}_{K'})^\perp$ so f must belong to $\mathcal{A}_K \cap \mathcal{A}_{K'}$ if $\langle f, g \rangle_K = 0$ for all such g. Now, the nonzero ideal $\widetilde{\mathcal{E}} = \mathcal{F}^{-1}$ satisfies:

$$\widetilde{\mathcal{E}}\big(Lf \cap (\mathcal{A}_{K,\mathcal{O}} + \mathcal{A}_K \cap \mathcal{A}_{K'})\big) \subset \mathcal{O}f.$$

Therefore, $\mathcal{E} = [K' \colon J]\widetilde{\mathcal{E}}$ satisfies the primitivity condition in Corollary 2.3:

$$\mathcal{E}\big(L\vec{f} \cap (\mathcal{A}_{K,\mathcal{O}} \oplus \mathcal{A}_{K',\mathcal{O}} + \ker \delta)\big) \subset \mathcal{O}\vec{f}.$$

Suppose $\lambda \subset \mathcal{O}$ is a maximal ideal such that $v_\lambda(\widetilde{\mathcal{E}}) \neq 0$. Then $\lambda^{-1} \subset \mathcal{F}$. It follows that $f \in \lambda(\mathcal{A}_{K,\mathcal{O}} + \mathcal{A}_K \cap \mathcal{A}_{K'})$, and hence the reduction $\bar{f} \in \mathcal{A}_{K,\mathbb{F}_\lambda}$ is G_w^0-invariant. Since $\ell \nmid |K_v|$ holds for at least one $v \neq w$, by assumption, we can find A_K and $A_{K'}$ indivisible by ℓ according to Lemma 4.2. Note also that we can take $C = 1$ by Lemma 5.2. □

7. Semisimplicity

7.1. Automorphic representations and the decomposition of $\mathcal{A}_K$

Henceforth assume G^{der} is simple and simply connected. There is an admissible representation of $G(\mathbb{A}^\infty)$ on the space

$$\mathcal{A} = C^\infty(G(F)\backslash G(\mathbb{A}^\infty), \mathbb{C})$$

given by right translations. For a compact open subgroup $K \subset G(\mathbb{A}^\infty)$, we look at the $\mathcal{H}_K$-module of K-invariants $\mathcal{A}_K$. Recall that an automorphic representation of $G(\mathbb{A})$ is an irreducible representation π of $G(\mathbb{A})$ (on some Hilbert space) such that

$$\mathrm{Hom}_{G(\mathbb{A})^1}(\pi, L^2(G(F)\backslash G(\mathbb{A})^1)) \neq 0.$$

We let $m(\pi)$ denote the dimension of this space. We then have an isomorphism,

$$\mathcal{A}_K \simeq \bigoplus_{\pi \in \Pi_{\mathrm{unit}}(G(\mathbb{A}))\colon \pi_\infty = \mathbf{1}} m(\pi)\pi^K,$$

On the right we have a finite direct sum over the automorphic representations π of $G(\mathbb{A})$ such that $\pi_\infty = \mathbf{1}$ and $\pi^K \neq 0$. These π are automatically unitary.

7.2. Semisimplicity in characteristic zero

It is known that each π^K is a simple module over $\mathcal{H}_K$, and hence $\mathcal{A}_K$ is semisimple. Moreover, by Schur's lemma, the center $Z(\mathcal{H}_K)$ acts on π^K by a $\mathbb{C}$-algebra homomorphism $\eta_{\pi^K}\colon Z(\mathcal{H}_K) \to \mathbb{C}$. For a character $\eta\colon Z(\mathcal{H}_K) \to \mathbb{C}$, we denote by $\mathcal{A}_K(\eta)$ the η-isotypic component. That is, the eigenspace

$$\mathcal{A}_K(\eta) = \big\{f \in \mathcal{A}_K\colon r(\phi)f = \eta(\phi)f, \forall \phi \in Z(\mathcal{H}_K)\big\}.$$

Then there is a direct sum decomposition $\mathcal{A}_K = \oplus_\eta \mathcal{A}_K(\eta)$. Clearly, $\mathcal{A}_K(\eta) \neq 0$ if and only if $\eta = \eta_{\pi^K}$ for some π. The image $\mathbb{T}_K \subset \operatorname{End}\mathcal{A}_K$ of the center $Z(\mathcal{H}_K)$ is a commutative semisimple $\mathbb{C}$-algebra, that is, a direct product of copies of $\mathbb{C}$.

LEMMA 7.1. — *The eigenspace $\mathcal{A}_K(\eta)$ is nonzero if and only if η factors through $\mathbb{T}_K$.*

Proof. — Obviously, η factors if $\mathcal{A}_K(\eta) \neq 0$. Conversely, suppose η factors and look at its kernel $\mathfrak{m} = \ker(\eta) \subset \mathbb{T}_K$. This is a maximal ideal. Since $\mathbb{T}_K$ acts faithfully on $\mathcal{A}_K$, which is finite-dimensional, $\mathfrak{m}$ belongs to the support of $\mathcal{A}_K$. By the theory of associated primes, $\mathfrak{m}$ contains a prime ideal of the form $\operatorname{Ann}_{\mathbb{T}_K}(f)$ with $f \in \mathcal{A}_K$. All primes are maximal in $\mathbb{T}_K$, so in fact $\mathfrak{m} = \operatorname{Ann}_{\mathbb{T}_K}(f)$. Clearly $\mathfrak{m}$ contains $T - \eta(T)$ for every $T \in \mathbb{T}_K$, so $f \in \mathcal{A}_K(\eta)$, and f must be nonzero as $\mathfrak{m} \neq \mathbb{T}_K$. □

Now, consider the $\mathcal{H}_{K,\mathbb{Q}}$-module $\mathcal{A}_{K,\mathbb{Q}}$, and the image $\mathbb{T}_{K,\mathbb{Q}}$ of the center $Z(\mathcal{H}_{K,\mathbb{Q}})$ in the endomorphism algebra $\operatorname{End}_{\mathbb{Q}}\mathcal{A}_{K,\mathbb{Q}}$. $\mathbb{T}_{K,\mathbb{Q}}$ can be viewed as a subring of $\mathbb{T}_K \simeq \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{T}_{K,\mathbb{Q}}$. We deduce that $\mathbb{T}_{K,\mathbb{Q}}$ is a reduced commutative finite-dimensional $\mathbb{Q}$-algebra, that is, a product of number fields by Nakayama's lemma:

$$\mathbb{T}_{K,\mathbb{Q}} \simeq L_1 \times \cdots \times L_t.$$

Visibly, $\mathbb{T}_{K,\mathbb{Q}}$ is a semisimple $\mathbb{Q}$-algebra. The L_i occurring in $\mathbb{T}_{K,\mathbb{Q}}$ are totally real or CM.

7.3. Semisimplicity in positive characteristic

Now let R be a field of characteristic $p > 0$. We are interested in when $\mathcal{A}_{K,R}$ is a semisimple module over $Z(\mathcal{H}_{K,R})$. As we have seen, this means that $\mathbb{T}_{K,R}$ is a semisimple R-algebra. We have $\mathbb{T}_{K,R} \simeq R \otimes_{\mathbb{F}_p} \mathbb{T}_{K,\mathbb{F}_p}$, so equivalently, when is $\mathbb{T}_{K,\mathbb{F}_p}$ semisimple? There is always a surjective homomorphism

$$\xi\colon \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{T}_{K,\mathbb{Z}} \twoheadrightarrow \mathbb{T}_{K,\mathbb{F}_p}.$$

Indeed the image of $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{T}_{K,\mathbb{Z}}$ in $\mathrm{End}_{\mathbb{F}_p} \mathcal{A}_{K,\mathbb{F}_p}$ equals the image of $\mathbb{F}_p \otimes_{\mathbb{Z}} Z(\mathcal{H}_{K,\mathbb{Z}})$. However, the natural map from this last algebra to $Z(\mathcal{H}_{K,\mathbb{F}_p})$ is surjective. Let

$$\widetilde{\mathbb{T}}_{K,\mathbb{Z}} = \bigl\{T \in \mathbb{T}_{K,\mathbb{Q}}\colon T(\mathcal{A}_{K,\mathbb{Z}}) \subset \mathcal{A}_{K,\mathbb{Z}}\bigr\}.$$

This is a finite free $\mathbb{Z}$-module containing $\mathbb{T}_{K,\mathbb{Z}}$ as a subgroup of finite index.

LEMMA 7.2. — *The kernel* $\ker(\xi)$ *is nilpotent. It is trivial if and only if* $p \nmid [\widetilde{\mathbb{T}}_{K,\mathbb{Z}}\colon \mathbb{T}_{K,\mathbb{Z}}]$.

Proof. — It is enough to show that every element in $\ker(\xi)$ is nilpotent. Under the identification $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{T}_{K,\mathbb{Z}} \simeq \mathbb{T}_{K,\mathbb{Z}}/p\mathbb{T}_{K,\mathbb{Z}}$, the kernel $\ker(\xi)$ corresponds to the ideal

$$(\mathbb{T}_{K,\mathbb{Z}} \cap p\widetilde{\mathbb{T}}_{K,\mathbb{Z}})/p\mathbb{T}_{K,\mathbb{Z}}.$$

Let $T \in \mathbb{T}_{K,\mathbb{Z}} \cap p\widetilde{\mathbb{T}}_{K,\mathbb{Z}}$. Obviously, $\widetilde{\mathbb{T}}_{K,\mathbb{Z}}$ is integral over $\mathbb{Z}$, so there is an equation

$$(p^{-1}T)^n + a_{n-1}(p^{-1}T)^{n-1} + \cdots + a_1(p^{-1}T) + a_0 = 0$$

for certain $a_i \in \mathbb{Z}$. Multiplying by p^n we see that $T^n \in p\mathbb{T}_{K,\mathbb{Z}}$. For the last assertion, note that $\ker(\xi) = 0$ if and only if $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{T}_{K,\mathbb{Z}} \to \mathbb{F}_p \otimes_{\mathbb{Z}} \widetilde{\mathbb{T}}_{K,\mathbb{Z}}$ is injective. □

In particular, $\ker(\xi)$ is contained in the Jacobson radical. We let $\overline{\mathbb{T}}_{K,\mathbb{Z}}$ denote the integral closure of $\mathbb{Z}$ in $\mathbb{T}_{K,\mathbb{Q}}$. It contains $\widetilde{\mathbb{T}}_{K,\mathbb{Z}}$ as a subgroup of finite index.

LEMMA 7.3. — $p \nmid \Delta_K := [\overline{\mathbb{T}}_{K,\mathbb{Z}}\colon \widetilde{\mathbb{T}}_{K,\mathbb{Z}}] \cdot \prod_i \Delta_{L_i/\mathbb{Q}} \Rightarrow \mathbb{T}_{K,\mathbb{F}_p}$ *is semisimple.*

Proof. — Note first that $\mathbb{F}_p \otimes_{\mathbb{Z}} \widetilde{\mathbb{T}}_{K,\mathbb{Z}} \simeq \mathbb{F}_p \otimes_{\mathbb{Z}} \overline{\mathbb{T}}_{K,\mathbb{Z}}$ since $p \nmid [\overline{\mathbb{T}}_{K,\mathbb{Z}}\colon \widetilde{\mathbb{T}}_{K,\mathbb{Z}}]$. Now,

$$\mathbb{F}_p \otimes_{\mathbb{Z}} \overline{\mathbb{T}}_{K,\mathbb{Z}} \simeq \prod_i \mathcal{O}_{L_i}/p\mathcal{O}_{L_i} \simeq \prod_i \prod_{\mathfrak{p}|p} \mathcal{O}_{L_i}/\mathfrak{p},$$

since p is unramified in every L_i occurring in $\mathbb{T}_{K,\mathbb{Q}}$. There is an embedding,

$$\mathbb{T}_{K,\mathbb{F}_p} \simeq \mathbb{T}_{K,\mathbb{Z}}/(\mathbb{T}_{K,\mathbb{Z}} \cap p\widetilde{\mathbb{T}}_{K,\mathbb{Z}}) \hookrightarrow \widetilde{\mathbb{T}}_{K,\mathbb{Z}}/p\widetilde{\mathbb{T}}_{K,\mathbb{Z}} \simeq \mathbb{F}_p \otimes_{\mathbb{Z}} \widetilde{\mathbb{T}}_{K,\mathbb{Z}},$$

and it follows that $\mathbb{T}_{K,\mathbb{F}_p}$ is semisimple. □

The converse holds at least for $p \nmid [\widetilde{\mathbb{T}}_{K,\mathbb{Z}} : \mathbb{T}_{K,\mathbb{Z}}]$ (that is, when ξ is injective).

7.4. The simple modules

Let R be a perfect field of characteristic $p \geqslant 0$. Up to isomorphism, the simple $Z(\mathcal{H}_{K,R})$-modules are given by an extension R'/R with an action given by a surjective R-algebra homomorphism $\eta : Z(\mathcal{H}_{K,R}) \twoheadrightarrow R'$. If η is a submodule of $\mathcal{A}_{K,R}$, the extension R'/R is finite and η factors through $\mathbb{T}_{K,R}$. If $p \nmid \Delta_K$, there exists a finite extension L/R such that we have a direct sum decomposition

$$\mathcal{A}_{K,L} = \bigoplus_{\eta} \mathcal{A}_{K,L}(\eta),$$

This is still true when $p|\Delta_K$, provided $\mathcal{A}_{K,L}(\eta)$ denotes the generalized eigenspace:

$$\mathcal{A}_{K,L}(\eta) = \big\{ f \in \mathcal{A}_{K,L} : \forall \phi \in Z(\mathcal{H}_{K,L}), (r(\phi) - \eta(\phi))^n f = 0 \text{ for some } n \geqslant 1 \big\}.$$

Observe the following:

LEMMA 7.4. — *Let R be a field, and choose a finite extension L/R as above. Then let L'/L be an arbitrary extension. Suppose $\eta' : Z(\mathcal{H}_{K,L'}) \to L'$ occurs in $\mathcal{A}_{K,L'}$. Then $\eta' = 1 \otimes \eta$ for some character $\eta : Z(\mathcal{H}_{K,L}) \to L$ occurring in $\mathcal{A}_{K,L}$. Moreover,*

$$\mathcal{A}_{K,L'}(1 \otimes \eta) \simeq L' \otimes_L \mathcal{A}_{K,L}(\eta),$$

so η and $\eta' = 1 \otimes \eta$ occur with the same multiplicity.

Proof. — Both $\mathcal{A}_{K,L}$ and $\mathcal{A}_{K,L'} \simeq L' \otimes_L \mathcal{A}_{K,L}$ have decompositions into direct sums of generalized eigenspaces. Under this isomorphism, $L' \otimes_L \mathcal{A}_{K,L}(\eta) \hookrightarrow \mathcal{A}_{K,L'}(1 \otimes \eta)$. Therefore, every η' occurring in $\mathcal{A}_{K,L'}$ must come from an η, and the above injection must be an isomorphism. □

Let us apply these results to $R = \mathbb{Q}$. We conclude that there exists a number field $L/\mathbb{Q}$ such that $\mathcal{A}_{K,L}$ is a direct sum of eigenspaces for characters $Z(\mathcal{H}_{K,L}) \to L$. Furthermore, if $\eta : Z(\mathcal{H}_K) \to \mathbb{C}$ is a character such that $\mathcal{A}_K(\eta) \neq 0$, then η restricts to a $\mathbb{Q}$-algebra homomorphism $Z(\mathcal{H}_{K,\mathbb{Q}}) \to L$ occurring in $\mathcal{A}_{K,L}$. In addition, since $Z(\mathcal{H}_{K,\mathbb{Z}})$ preserves $\mathcal{A}_{K,\mathcal{O}_L}$, η even restricts to a ring homomorphism $Z(\mathcal{H}_{K,\mathbb{Z}}) \to \mathcal{O}_L$ occurring in $\mathcal{A}_{K,\mathcal{O}_L}$.

8. End of the proof

8.1. Invariance modulo λ

The following is a more refined version of the notion abelian modulo λ.

DEFINITION 8.1. — *Let π be an automorphic representation of $G(\mathbb{A})$ such that $\pi_\infty = \mathbf{1}$. We say that π is abelian modulo λ relative to K if $\pi^K \neq 0$ and there exists an automorphic character χ of $G(\mathbb{A})$, trivial on $G_\infty K$, such that $\eta_{\pi^K}(\phi) \equiv \eta_\chi(\phi) \pmod{\lambda}$, $\forall \phi \in Z(\mathcal{H}_{K,\mathbb{Z}})$.*

If this does *not* hold, we can find eigenforms in π^K to which our main lemma applies:

LEMMA 8.2. — *Let π be an automorphic representation of $G(\mathbb{A})$ such that $\pi_\infty = \mathbf{1}$. If π is non-abelian modulo λ relative to K, then the eigenspace $\mathcal{A}^\circ_{K,\mathbb{F}}(\bar{\eta})$ contains no nonzero G_w^{der}-invariant functions, where w is a place such that K_w is hyperspecial.*

Proof. — Choose a number field $L/\mathbb{Q}$ such that $\mathcal{A}_{K,L}$ is a direct sum of eigenspaces and let $\mathcal{O} = \mathcal{O}_L$. Denote by

$$\eta = \eta_{\pi^K} : Z(\mathcal{H}_{K,\mathbb{Z}}) \to \mathcal{O}$$

the character giving the action on π^K. As we have observed above it occurs in $\mathcal{A}_{K,\mathcal{O}}$, that is, there exists an eigenform $0 \neq f \in \mathcal{A}_{K,\mathcal{O}}$ with $\eta_f = \eta$. We consider a finite place $\lambda|\ell$ of $\overline{\mathbb{Q}}$, and a finite place w of F such that G_w is unramified. Let $\bar{f} = 1 \otimes f \in \mathcal{A}_{K,\mathbb{F}}$ be the reduction modulo λ, where $\mathbb{F} = \mathcal{O}/\lambda \cap \mathcal{O}$ is a finite extension of $\mathbb{F}_\ell$. By scaling f, we can assume that $\bar{f} \neq 0$. Let us assume $\bar{f}$ is G_w^{der}-invariant. Now, G^{der} is simple, simply connected and G_w^{der} is noncompact. By the strong approximation theorem, $\bar{f}$ is in fact $G^{\mathrm{der}}(\mathbb{A}^\infty)$-invariant. There is a short exact sequence

$$1 \to G^{\mathrm{der}}(\mathbb{A}^\infty) \to G(\mathbb{A}^\infty) \xrightarrow{\nu} G^{\mathrm{ab}}(\mathbb{A}^\infty) \to 1.$$

It follows that $\bar{f}$ lives on $G^{\mathrm{ab}}(\mathbb{A}^\infty)$. More precisely, there exists a unique function $\tilde{f} : G^{\mathrm{ab}}(\mathbb{A}^\infty) \to \mathbb{F}$ such that $\bar{f} = \tilde{f} \circ \nu$. It fits into the diagram

$$\begin{array}{ccc} X_K = G(F)\backslash G(\mathbb{A}^\infty)/K & \xrightarrow{\bar{f}} & \mathbb{F} \\ \downarrow{\scriptstyle \nu} & \nearrow{\scriptstyle \tilde{f}} & \\ Y_K = \nu(G(F))\backslash G^{\mathrm{ab}}(\mathbb{A}^\infty)/\nu(K) & & \end{array}$$

If R is a ring we denote by $\mathcal{A}^{\mathrm{ab}}_{K,R}$ the module of R-valued functions on Y_K. Pulling back via ν, identifies $\mathcal{A}^{\mathrm{ab}}_{K,R}$ with an $\mathcal{H}_{K,R}$-submodule of $\mathcal{A}_{K,R}$. Then

$0 \neq \tilde{f} \in \mathcal{A}_{K,\mathbb{F}}^{\mathrm{ab},\circ}(\bar{\eta})$. By the Deligne-Serre lifting lemma (that is, Lemme 6.11 in their paper [7, p. 522]) we can lift $\bar{\eta}$ to characteristic zero: There exists an eigenform $0 \neq f' \in \mathcal{A}_{K,L_\lambda}^{\mathrm{ab}}$ such that its character $\eta' \colon Z(\mathcal{H}_{K,\mathbb{Z}}) \to \mathcal{O}_\lambda$ reduces to $\bar{\eta}$ modulo $\lambda \cap \mathcal{O}$. From the results of the previous section we see that in fact η' maps into $\mathcal{O}$, and it occurs in $\mathcal{A}_{K,L}^{\mathrm{ab}}$ (and therefore in $\mathcal{A}_K^{\mathrm{ab}}$). However, $\mathcal{A}_K^{\mathrm{ab}}$ is just the space of $\mathbb{C}$-valued functions on the finite abelian group Y_K, so the characters form a basis. We conclude that there exists a character χ such that $\eta(\phi) \equiv \eta_\chi(\phi) \pmod{\lambda}$ for all $\phi \in Z(\mathcal{H}_{K,\mathbb{Z}})$. □

8.2. Proof of the main theorem

We can now prove the more precise version of Theorem 1.3 alluded to in the introduction.

THEOREM 8.3. — *Let $K = \prod_{v<\infty} K_v \subset G(\mathbb{A}^\infty)$ be a compact open subgroup. Let $\lambda|\ell$ be a finite place of $\overline{\mathbb{Q}}$ such that there exists at least two finite places v where $\ell \nmid |K_v|$ (this is automatic if $\ell > [F\colon \mathbb{Q}]n+1$). Let $\pi = \otimes \pi_v$ be an automorphic representation of $G(\mathbb{A})$ such that $\pi_\infty = \mathbf{1}$ and $\pi^K \neq 0$. Assume π is non-abelian modulo λ relative to K. Let w be a finite place of F such that K_w is hyperspecial, and let $J_w = K_w \cap K'_w$ be a parahoric subgroup, where $K'_w \neq K_w$ is maximal compact. Let $J = J_w K^w$ and $K' = K'_w K^w$. Suppose $\ell \nmid [K'\colon J]$ and*

$$\eta_{\pi^K}(e_{K,K'}) \equiv \eta_{\mathbf{1}}(e_{K,K'}) \pmod{\lambda},$$

where

$$e_{K,K'} = [K\colon J][K'\colon J](e_K \star e_{K'} \star e_K) \in Z(\mathcal{H}_{K,\mathbb{Z}}).$$

Then there exists an automorphic representation $\tilde{\pi} = \otimes \tilde{\pi}_v$ of $G(\mathbb{A})$ such that $\tilde{\pi}_\infty = \mathbf{1}$ and $\tilde{\pi}^{K^w} \neq 0$ satisfying the following:

- *$\tilde{\pi}_w^{J_w} \neq \tilde{\pi}_w^{K_w} + \tilde{\pi}_w^{K'_w}$, and*
- *$\eta_{\tilde{\pi}^J}(\phi) \equiv \eta_{\pi^K}(e_K \star \phi) \pmod{\lambda}$, for all $\phi \in Z(\mathcal{H}_{J,\mathbb{Z}})$.*

Proof. — The reduction $\bar{\eta}_{\pi^K}$ modulo $\lambda \cap \mathcal{O}$ factors through $\mathbb{T}_{J,\mathbb{Z}}^{\mathrm{new}}$ by the main lemma (Lemma 6.2). That is, there exists a character $\eta' \colon Z(\mathcal{H}_{J,\mathbb{Z}}) \to \mathbb{F}$ factoring through $\mathbb{T}_{J,\mathbb{Z}}^{\mathrm{new}}$ such that $\eta'(\phi) = \eta_{\pi^K}(e_K \star \phi) \pmod{\lambda}$ for all $\phi \in Z(\mathcal{H}_{J,\mathbb{Z}})$. As above, there is a surjective homomorphism with nilpotent kernel

$$\mathbb{F}_\ell \otimes_{\mathbb{Z}} \mathbb{T}_{J,\mathbb{Z}}^{\mathrm{new}} \twoheadrightarrow \mathbb{T}_{J,\mathbb{F}_\ell}^{\mathrm{new}}.$$

Thus η' gives rise to a character $\mathbb{T}_{J,\mathbb{F}}^{\mathrm{new}} \to \mathbb{F}$, also denoted by η'. By a standard argument (used above in section 7.2), there is an eigenform $f' \in$

$\mathcal{A}_{J,\mathbb{F}}^{\mathrm{new}}$ with character η'. Now we apply the Deligne-Serre lifting lemma, [7, p. 522], to the finite free module $\mathcal{A}_{J,\mathcal{O}_\lambda}^{\mathrm{new}}$ where $\mathcal{O}_\lambda$ is the completion of $\mathcal{O}$ at $\lambda \cap \mathcal{O}$. It gives the existence of a character $\tilde{\eta}\colon \mathbb{T}_{J,\mathbb{Z}}^{\mathrm{new}} \to \widetilde{\mathcal{O}}_\lambda$ occurring in $\mathcal{A}_{J,\widetilde{\mathcal{O}}_\lambda}^{\mathrm{new}}$ and reducing to η', where $\widetilde{\mathcal{O}}_\lambda$ is the ring of integers in a finite extension of L_λ. Since $\mathbb{T}_{J,\mathbb{Z}}^{\mathrm{new}}$ preserves the lattice $\mathcal{A}_{J,\mathbb{Z}}^{\mathrm{new}}$, the values $\tilde{\eta}(\phi)$ all lie in the ring of integers of some number field, $\mathcal{O}_{\widetilde{L}}$. We deduce that there exists a character $\tilde{\eta}\colon Z(\mathcal{H}_{J,\mathbb{Z}}) \to \mathcal{O}_{\widetilde{L}}$, occurring in $\mathcal{A}_J^{\mathrm{new}}$, such that

$$\tilde{\eta}(\phi) \equiv \eta_{\pi^K}(e_K \star \phi) \pmod{\lambda}$$

for all $\phi \in Z(\mathcal{H}_{J,\mathbb{Z}})$. From the decomposition of $\mathcal{A}_J$ in terms of automorphic representations, it follows that the newspace $\mathcal{A}_J^{\mathrm{new}}$ has the following description:

$$\mathcal{A}_J^{\mathrm{new}} \simeq \bigoplus_{\pi \in \Pi_{\mathrm{unit}}(G(\mathbb{A}))\colon\, \pi_\infty = \mathbf{1}} m(\pi)\big(\pi^J/(\pi^K + \pi^{K'})\big),$$

as $Z(\mathcal{H}_J)$-modules. The center $Z(\mathcal{H}_J)$ acts on the quotient $\pi^J/(\pi^K + \pi^{K'})$ by the character η_{π^J}. We conclude that there exists an automorphic representation $\tilde{\pi}$ of $G(\mathbb{A})$ with $\tilde{\pi}_\infty = \mathbf{1}$ and $\tilde{\pi}^J \neq \tilde{\pi}^K + \tilde{\pi}^{K'}$, such that $\eta_{\tilde{\pi}^J} = \tilde{\eta}$. In particular,

$$\eta_{\tilde{\pi}^J}(\phi) \equiv \eta_{\pi^K}(e_K \star \phi) \pmod{\lambda}$$

for all $\phi \in Z(\mathcal{H}_{J,\mathbb{Z}})$. This finishes the proof □

9. Applications

9.1. The rank one situation

When the F_w-rank of G_w^{der} is one, the condition $\tilde{\pi}_w^{J_w} \neq \tilde{\pi}_w^{K_w} + \tilde{\pi}_w^{K'_w}$ forces $\tilde{\pi}_w$ to be ramified:

COROLLARY 9.1. — *With notation as above, let w be a finite place of F such that K_w is hyperspecial and the F_w-rank of G_w^{der} is one. Let $I_w = K_w \cap K'_w$ be an Iwahori subgroup, where K'_w is maximal compact. Let $I = I_w K^w$ and $K' = K'_w K^w$. Suppose $\ell \nmid [K'\colon I]$ and*

$$\eta_{\pi^K}(e_{K,K'}) \equiv \eta_{\mathbf{1}}(e_{K,K'}) \pmod{\lambda},$$

with $e_{K,K'}$ as in Theorem 8.3. Then there exists an automorphic representation $\tilde{\pi} = \otimes \tilde{\pi}_v$ of $G(\mathbb{A})$ such that $\tilde{\pi}_\infty = \mathbf{1}$ and $\tilde{\pi}^{K^w} \neq 0$ satisfying the following properties:

- *$\tilde{\pi}_w^{I_w} \neq 0$ and $\tilde{\pi}_w^{K_w} = 0$,*

- $\eta_{\tilde{\pi}^I}(\phi) \equiv \eta_{\pi^K}(e_K \star \phi) \pmod{\lambda}$, *for all* $\phi \in Z(\mathcal{H}_{I,\mathbb{Z}})$.

This corollary is a slight generalization of Bellaiche's [1, Theorem 1.4.6, p. 215]: It gives results modulo arbitrary $\lambda|\ell$, the level-raising condition is weaker, and we get information about the action of the center of the Iwahori-Hecke algebra on $\tilde{\pi}_w^{I_w}$. Bellaiche's proof is different. He uses results of Lazarus and Vigneras from modular representation theory, such as the computation of the composition series of universal modules. With his stronger level-raising condition, $\eta_{\pi^K}(\phi) \equiv \eta_{\mathbf{1}}(\phi)$ for all $\phi \in \mathcal{H}_{K_w}$, one can conclude that $\tilde{\pi}_w$ is the actual Steinberg representation of G_w, see [1, p. 221].

9.2. U(3) - the split case

In this subsection, we let $E/\mathbb{Q}$ denote an imaginary quadratic extension of $\mathbb{Q}$, even though much of what we have to say is true for CM extensions. We consider the quasi-split unitary $\mathbb{Q}$-group in 3 variables, $G^* = \mathrm{U}(2,1)$, split over E. We let $G = \mathrm{U}(3)$ be an arbitrary inner form of G^* such that G_∞ is compact. Such exist since E is imaginary. The rank is odd, so we may even assume G is quasi-split at all finite primes, but we do not need that here. Now, we will focus on primes q split in E. First, we make some remarks on the parahoric subgroups of $\mathrm{GL}_3(E_\mathfrak{q}) \simeq \mathrm{GL}_3(\mathbb{Q}_q)$. There is the hyperspecial maximal compact subgroup $K = \mathrm{GL}_3(\mathbb{Z}_q)$, and the Iwahori subgroup

$$I = \{g \in K \colon g \equiv \left(\begin{smallmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{smallmatrix}\right) \pmod{q}\}.$$

There is only one $\mathrm{GL}_3(\mathbb{Q}_q)$-conjugacy class of maximal proper parahorics. We take

$$J = \{g \in K \colon g \equiv \left(\begin{smallmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{smallmatrix}\right) \pmod{q}\} = K \cap \mu^{-1} K \mu,$$

where

$$\mu = \left(\begin{smallmatrix} q & & \\ & q & \\ & & 1 \end{smallmatrix}\right),$$

as a representative. The following is a slightly stronger version of Theorem 1.4.

THEOREM 9.2. — *Let* $\pi = \otimes \pi_p$ *be an automorphic representation of* $G(\mathbb{A})$ *with* $\pi_\infty = \mathbf{1}$. *Let* $\lambda|\ell$ *be a finite place of* $\overline{\mathbb{Q}}$ *such that* π *is non-abelian modulo* λ. *Choose a compact open subgroup* $K = \prod K_p \subset G(\mathbb{A}^\infty)$ *such that* $\pi^K \neq 0$. *If* $\ell \leqslant 3$, *or we are in the situation where* $E = \mathbb{Q}(\sqrt{-7})$ *and* $\ell = 7$, *assume* $\ell \nmid |K_p|$ *for at least two primes* p. *Let* $q \neq \ell$ *be a prime,*

split in E, such that K_q is hyperspecial. If $\ell \nmid 1+q+q^2$, and the following is satisfied

$$\mathfrak{t}_{\pi,\mathfrak{q}} \equiv \begin{pmatrix} q & & \\ & 1 & \\ & & q^{-1} \end{pmatrix} \pmod{\lambda},$$

where $\mathfrak{q}|q$, then there exists an automorphic representation $\tilde{\pi} = \otimes \tilde{\pi}_p$ of $G(\mathbb{A})$ with $\tilde{\pi}_\infty = \mathbf{1}$ and $\tilde{\pi}^{K^q} \neq 0$ satisfying the following conditions,

- *$\tilde{\pi}_q$ is either an irreducible unramified principal series or induced from a Steinberg representation. In particular $\tilde{\pi}_q$ is generic, not L^2, and $\tilde{\pi}_q^{J_q} \neq 0$,*
- *$\eta_{\tilde{\pi}^J}(\phi) \equiv \eta_{\pi^K}(e_K \star \phi) \pmod{\lambda}$, for all $\phi \in Z(\mathcal{H}_{J,\mathbb{Z}})$, where $J = J_q K^q$.*

Proof. — We first need to classify all the Iwahori-spherical representations of $\mathrm{GL}_3(\mathbb{Q}_q)$. It is a theorem of Borel and Casselman that these are precisely the constituents of unramified principal series. Using the theory developed by Bernstein and Zelevinsky, nicely summarized in [9], we obtain the following table: $\nu = |\cdot|$ is the absolute value,

		constituent of	representation	unitary	tempered	L^2	generic
I		$\chi_1 \times \chi_2 \times \chi_3$	$\chi_1 \times \chi_2 \times \chi_3$	below	$\lvert\chi_i\rvert = 1$		•
II	a	$\chi_1\nu^{1/2} \times \chi_1\nu^{-1/2} \times \chi_2$	$\chi_1 \operatorname{St}_{\mathrm{GL}(2)} \times \chi_2$	$\lvert\chi_i\rvert = 1$	$\lvert\chi_i\rvert = 1$		•
	b	$\chi_1\chi_2^{-1} \neq \nu^{\pm 3/2}$	$\chi_1 \mathbf{1}_{\mathrm{GL}(2)} \times \chi_2$	$\lvert\chi_i\rvert = 1$			
III	a	$\chi\nu \times \chi \times \chi\nu^{-1}$	$\chi \operatorname{St}_{\mathrm{GL}(3)}$	$\lvert\chi\rvert = 1$	$\lvert\chi\rvert = 1$	$\lvert\chi\rvert = 1$	•
	b		χV_P				
	c		χV_Q				
	d		$\chi \mathbf{1}_{\mathrm{GL}(3)}$	$\lvert\chi\rvert = 1$			

Table A: Iwahori-spherical representations of GL(3)

The irreducible representation $\chi_1 \times \chi_2 \times \chi_2$ in group I is unitary if and only if either all the χ_i are unitary, or, $\chi_1\chi_2^{-1} = \nu^\alpha$ with $0 < \alpha < 1$ and χ_3 unitary (after a permutation). In the table, P and Q denote the parabolics of $G = \mathrm{GL}_3(\mathbb{Q}_q)$ of type $(2,1)$ and $(1,2)$ respectively. Moreover, $V_P = C^\infty(P\backslash G)/\mathbb{C}$ and V_Q is defined similarly. They are not unitary, and therefore irrelevant for the theory of automorphic forms. Next, we list the dimensions of their parahoric fixed spaces:

		representation	remarks	K	J	I
I		$\chi_1 \times \chi_2 \times \chi_3$		1	3	6
II	a	$\chi_1 \operatorname{St}_{\mathrm{GL}(2)} \times \chi_2$		0	1	3
	b	$\chi_1 \mathbf{1}_{\mathrm{GL}(2)} \times \chi_2$		1	2	3
III	a	$\chi \operatorname{St}_{\mathrm{GL}(3)}$		0	0	1
	b	χV_P	not unitary	0	1	2
	c	χV_Q	not unitary	0	1	2
	d	$\chi \mathbf{1}_{\mathrm{GL}(3)}$	irrelevant	1	1	1

Table B: Dimensions of the parahoric fixed spaces

To compute these dimensions, we use the following observation: If P is parabolic and J is parahoric, a choice of representatives $g \in P\backslash G/J$ determines an isomorphism

$$\operatorname{Ind}_P^G(\tau)^J \simeq \bigoplus_{g \in P\backslash G/J} \tau^{P \cap gJg^{-1}},$$

for every representation τ of a Levi factor M_P. In particular, if $P = B$ is the Borel subgroup and τ is an unramified character, the dimension of $\operatorname{Ind}_B^G(\tau)^J$ equals the number of double cosets $|B\backslash G/J|$. With this information, the proof proceeds as follows: Our main theorem gives us an automorphic representation $\tilde{\pi}$ congruent to π (modulo λ) such that $\tilde{\pi}_q^{J_q} \neq \tilde{\pi}_q^{K_q} + \tilde{\pi}_q^{K_q'}$. Since $\tilde{\pi}_q$ must be unitary, we see from table B that it is of type I or IIa. Then, from table A, we derive that $\tilde{\pi}_q$ is generic and not L^2. Finally, note that there is a bijection $K/J \simeq \mathrm{GL}_3(\mathbb{F}_q)/\overline{P}$, so $[K\colon J] = 1 + q + q^2$. □

Remark. — This corollary has no content unless π_q is induced from the determinant (type IIb), that is, unramified and non-generic (and not 1-dimensional), which is the case for the endoscopic lifts from $\mathrm{U}(2) \times \mathrm{U}(1)$ considered in [1, p. 250]. In fact, the results we get for $\mathrm{U}(n)$ indicate that an endoscopic abelian lift π is congruent to a $\tilde{\pi}$ which is not endoscopic abelian. In his thesis [1, p. 218], Bellaiche also has a result in the split case. Apparently, if you only allow ℓ outside a finite set and π occurs with multiplicity 1, then you can obtain a $\tilde{\pi}$ with $\tilde{\pi}_q$ ramified. Hence, from our analysis, $\tilde{\pi}_q$ is induced from Steinberg. It looks like the preceding corollary is related to the $n = 3$ case of conjecture 5.3 in [19, p. 35], providing an analogue of Ihara's lemma, and to the work of Mann [11]. We also note that automorphic representations of unitary groups with a generic component at a split prime, come up naturally in the proof of the local Langlands correspondence for $\mathrm{GL}(n)$ [8].

9.3. Gsp(4)

In this subsection we view Gsp(4) as an algebraic $\mathbb{Q}$-subgroup of GL(4) by realizing it with respect to the standard skew-diagonal symplectic form. With this choice, the set of upper triangular matrices form a Borel subgroup $B = TU$. There are two maximal parabolic subgroups containing B, namely the Siegel parabolic

$$P = M_P \ltimes N_P = \left\{ \begin{pmatrix} g & \\ & \nu^\tau g^{-1} \end{pmatrix} \begin{pmatrix} 1 & & r & s \\ & 1 & t & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\},$$

where ${}^\tau g$ denotes the skew-transpose, and the Klingen parabolic

$$Q = M_Q \ltimes N_Q = \left\{ \begin{pmatrix} \nu & & & \\ & g & & \\ & & \nu^{-1}\det g & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & c & \\ & 1 & & -c \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r & s \\ & 1 & & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}.$$

We consider an inner form G of Gsp(4) such that $G^{\mathrm{der}}(\mathbb{R})$ is compact. Concretely we have $G = \mathrm{GSpin}(f)$, where f is some definite quadratic form in 5 variables over $\mathbb{Q}$. Now, let us first describe the parahoric subgroups of $\mathrm{Gsp}_4(\mathbb{Q}_q)$. There is the hyperspecial maximal compact subgroup $K_q = \mathrm{Gsp}_4(\mathbb{Z}_q)$, and the Iwahori subgroup I_q consisting of elements in K_q with upper triangular reduction mod q. Similarly, P and Q define (non-conjugate) parahoric subgroups J'_q and J_q called the Siegel parahoric and the Klingen parahoric respectively. One can easily check that we have the identity,

$$J'_q = K_q \cap hK_q h^{-1}, \text{ where } h = \begin{pmatrix} & & 1 & \\ & & & 1 \\ q & & & \\ & q & & \end{pmatrix}.$$

However, $J_q = K_q \cap K'_q$, where K'_q is the non-special maximal compact subgroup containing I_q. It is called the paramodular group. Since P and Q are not associated parabolics, the classification of the Iwahori-spherical representations of $\mathrm{Gsp}_4(\mathbb{Q}_q)$ is much more complicated than for $\mathrm{GL}_3(\mathbb{Q}_q)$. Fortunately, this has been done by Ralf Schmidt. The tables we need are Table 1 and Table 3 in the forthcoming paper [15]. With the permission of Ralf Schmidt, we have reproduced the information we need in Appendix B as Table C an Table D. We use the notation from this appendix below. If π has a Galois representation $\rho_{\pi,\lambda}$ (for example, if it transfers to a cuspidal representation Π of Gsp(4) with Π_∞ in the discrete series, see [10], [20]), then $\rho_{\pi,\lambda}(\mathrm{Fr}_p)$ and $\mathbf{t}_{\pi_p \otimes |\nu|^{-3/2}}$ have the same eigenvalues. In this case, π is abelian modulo λ if some twist of $\bar{\rho}_{\pi,\lambda}$ has the form $\mathbf{1} \oplus \bar{\omega}_\ell \oplus \bar{\omega}_\ell^2 \oplus \bar{\omega}_\ell^3$. We obtain the following strengthening of Theorem 1.5.

THEOREM 9.3. — *Let $\pi = \otimes \pi_p$ be an automorphic representation of $G(\mathbb{A})$ with $\pi_\infty = \mathbf{1}$. Let $\lambda | \ell$ be a finite place of $\overline{\mathbb{Q}}$ such that π is non-abelian modulo λ. Choose a compact open subgroup $K = \prod K_p$ such that*

$\pi^K \neq 0$. If $\ell \leqslant 5$ assume $\ell \nmid |K_p|$ for at least two primes p. Let $q \neq \ell$ be a prime such that K_q is hyperspecial. Suppose

$$\mathfrak{t}_{\pi_q \otimes |\nu|^{-3/2}} \equiv \begin{pmatrix} 1 & & \\ & q & \\ & & q^2 \\ & & & q^3 \end{pmatrix} \pmod{\lambda}.$$

Then there exists an automorphic representation $\tilde{\pi} = \otimes \tilde{\pi}_p$ of $G(\mathbb{A})$ with $\tilde{\pi}_\infty = \mathbf{1}$ and $\tilde{\pi}^{K^q} \neq 0$ satisfying the following conditions,

- *$\tilde{\pi}_q$ is generic and Klingen-spherical,*
- *$\eta_{\tilde{\pi}^J}(\phi) \equiv \eta_{\pi^K}(e_K \star \phi) \pmod{\lambda}$, for all $\phi \in Z(\mathcal{H}_{J,\mathbb{Z}})$, where $J = J_q K^q$.*

Moreover, if in addition $q^4 \neq 1 \pmod{\ell}$, $\tilde{\pi}_q$ must be of type I, IIa or IIIa.

Proof. — We apply the main theorem (Theorem 8.3) to the Klingen parahoric J_q. An easy computation shows that $[K_q' \colon J_q] = q$. We get an automorphic representation $\tilde{\pi}$, congruent to π modulo λ, such that the component at q satisfies the identity:

$$\tilde{\pi}_q^{J_q} \neq \tilde{\pi}_q^{K_q} + \tilde{\pi}_q^{K_q'}.$$

In particular, $\tilde{\pi}_q^{J_q} \neq 0$. We must have that $\tilde{\pi}_q^{K_q} \cap \tilde{\pi}_q^{K_q'} = 0$, for otherwise $\dim \tilde{\pi}_q = 1$ and therefore $\tilde{\pi}$ is one-dimensional by the strong approximation theorem. However, π is assumed to be non-abelian modulo λ. Thus, equivalently we have

$$\dim \tilde{\pi}_q^{J_q} > \dim \tilde{\pi}_q^{K_q} + \dim \tilde{\pi}_q^{K_q'}.$$

From Schmidt's tables, [15, p. 16], (that is, Table D in Appendix B), we deduce that this inequality is satisfied precisely when $\tilde{\pi}_q$ is of type I, IIa, IIIa, IVb, IVc, Va or VIa. However, those representations of type IVb and IVc are not unitary and can therefore be ruled out immediately. We are then left with the possible types I, IIa, IIIa, Va and VIa. Then, from the tables [15, p. 9], (Table C in Appendix B), we read off that $\tilde{\pi}_q$ is generic. Indeed all the representations of type Xa are generic, for X arbitrary.

Now, let us show that the types Va and VIa can also be ruled out if we assume $q^4 \neq 1 \pmod{\ell}$. Suppose first that $\tilde{\pi}_q$ is of type Va, that is, the unique subrepresentation of some $|\cdot| \xi_0 \times \xi_0 \rtimes |\cdot|^{-1/2} \sigma$ where ξ_0 has order two, see [15, p. 7] for an explanation of the notation. By the main theorem, the center of the Klingen-Hecke algebra $Z(\mathcal{H}_{J_q,\mathbb{Z}})$ acts on $\tilde{\pi}_q^{J_q}$ by a character $\eta_{\tilde{\pi}_q^{J_q}}$ satisfying the congruence

$$\eta_{\tilde{\pi}_q^{J_q}}(\phi) \equiv \eta_{\pi_q^{K_q}}(e_{K_q} \star \phi) \equiv \eta_{\mathbf{1}}(e_{K_q} \star \phi) \pmod{\lambda},$$

for all $\phi \in Z(\mathcal{H}_{J_q,\mathbb{Z}})$. We get immediately that the analogous statement is also true for the center of the Iwahori-Hecke algebra $Z(\mathcal{H}_{I_q,\mathbb{Z}})$. This, however, acts by a character on the Iwahori-fixed vectors in the principal series $|\cdot|\xi_0 \times \xi_0 \rtimes |\cdot|^{-1/2}\sigma$ (for it has an unramified Langlands quotient, so is generated by any nonzero K_q-fixed vector). Hence, $Z(\mathcal{H}_{I_q,\mathbb{Z}})$ acts on every constituent of this principal series by the same character $\eta_{\tilde{\pi}_q^{I_q}}$. In particular, the action of the spherical Hecke algebra $\mathcal{H}_{K_q,\mathbb{Z}} \simeq Z(\mathcal{H}_{I_q,\mathbb{Z}})$ on the K_q-fixed vectors of the unramified quotient (type Vd) is given by a character congruent to $\eta_{\mathbf{1}}$. In terms of their Satake parameters we therefore must have (modulo the action of the Weyl group):

$$\begin{pmatrix} q^{-1/2}\sigma(q) & & & \\ & q^{-1/2}\xi_0\sigma(q) & & \\ & & q^{1/2}\xi_0\sigma(q) & \\ & & & q^{1/2}\sigma(q) \end{pmatrix} \equiv \begin{pmatrix} q^{-3/2} & & & \\ & q^{-1/2} & & \\ & & q^{1/2} & \\ & & & q^{3/2} \end{pmatrix} \pmod{\lambda}.$$

Since $\xi_0(q) = -1$ we conclude that $q \equiv -1$ or $q^2 \equiv -1$ modulo ℓ. Secondly, assume $\tilde{\pi}_q$ is of type VIa, that is, the unique irreducible subrepresentation of some $|\cdot| \times \mathbf{1} \rtimes |\cdot|^{-1/2}\sigma$. Then, by the argument above, we conclude that the unramified quotient of this principal series must be congruent to $\mathbf{1}$. That is, in terms of their Satake parameters:

$$\begin{pmatrix} q^{-1/2}\sigma(q) & & & \\ & q^{-1/2}\sigma(q) & & \\ & & q^{1/2}\sigma(q) & \\ & & & q^{1/2}\sigma(q) \end{pmatrix} \equiv \begin{pmatrix} q^{-3/2} & & & \\ & q^{-1/2} & & \\ & & q^{1/2} & \\ & & & q^{3/2} \end{pmatrix} \pmod{\lambda}.$$

It follows that $q^2 \equiv 1$. The types I, IIa and IIIa cannot be excluded, even if π has trivial central character. □

Remark. — There exists q with $q^4 \neq 1 \pmod{\ell}$ precisely when $\ell \geqslant 7$. In this case $\tilde{\pi}_q$ is an unramified principal series (type I) or induced from a twisted Steinberg representation $\chi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \chi'$ or $\chi \rtimes \chi'\,\mathrm{St}_{\mathrm{GL}(2)}$ (type IIa and IIIa respectively). If one can show that $\tilde{\pi}_q$ is para-ramified, meaning that $\tilde{\pi}_q$ has no nonzero K_q'-fixed vectors, one can conclude that it is of type IIIa and therefore induced from a twisted Steinberg representation on the Klingen-Levi. It seems possible to prove this if $m(\pi) = 1$, using the methods of [1] and [6]. We hope to return to this point in another paper. The result above only gives non-trivial congruences if π_q is non-generic. If π is of Saito-Kurokawa type (that is, a theta-lift from the $\widetilde{\mathrm{SL}}(2)$), it is locally non-generic, and we get a $\tilde{\pi}$ congruent to π which is not of Saito-Kurokawa type. If we know $\tilde{\pi}_q$ is of type IIIa, we can apply this strategy to the Bloch-Kato conjecture for the motives attached to classical modular forms of weight (at least) 4, using the methods of [1]. We should note that

if we choose to work with the Siegel-parahoric J'_q, we can only conclude that $\tilde{\pi}_q$ is generic *or* a Saito-Kurokawa lift.

Appendix A. Congruent representations

The compact open subgroups $K \subset G(\mathbb{A}^\infty)$ form a directed set by opposite inclusion, that is $K \preccurlyeq J \Leftrightarrow K \supset J$. Let R be a commutative ring. As K varies over the compact open subgroups, the centers $Z(\mathcal{H}_{K,R})$ form an inverse system of R-algebras with respect to the canonical maps $Z(\mathcal{H}_{K,R}) \leftarrow Z(\mathcal{H}_{J,R})$ when $K \supset J$. Let

$$\mathcal{Z}_{G(\mathbb{A}^\infty),R} = \varprojlim Z(\mathcal{H}_{K,R}).$$

In this limit, it is enough to let K run through a neighborhood basis at the identity. Thus $\mathcal{Z}_{G(\mathbb{A}^\infty),R}$ is a commutative R-algebra, and it comes with projections $(K \supset J)$

$$\begin{array}{ccc} & \mathcal{Z}_{G(\mathbb{A}^\infty),R} & \\ {\scriptstyle \mathrm{pr}_K}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_J} \\ Z(\mathcal{H}_{K,R}) & \xleftarrow[e_K \star \phi \leftarrow \phi]{} & Z(\mathcal{H}_{J,R}) \end{array}$$

All we have said makes sense for any locally profinite group, so in particular we have local analogues $\mathcal{Z}_{G_v,R}$ for each finite place v. If $\mu = \otimes \mu_v$, it follows that

$$\mathcal{Z}_{G(\mathbb{A}^\infty),R} \simeq \bigotimes_{v<\infty} \mathcal{Z}_{G_v,R},$$

a restricted tensor product. Indeed the decomposable groups $K = \prod K_v$ form a cofinal system. It remains to determine the algebras $\mathcal{Z}_{G_v,R}$. By [5, p. 14], there exists a neighborhood basis at 1 consisting of compact open subgroups $K_v \subset G_v$ with Iwahori factorization with respect to a fixed minimal parabolic. If G_v is unramified, for such a K_v the canonical map $Z(\mathcal{H}_{K_v,R}) \to \mathcal{H}^{\mathrm{sph}}_{v,R}$ to the spherical Hecke algebra at v is an isomorphism [3], [4]. This is a well-known result due to Bernstein when K_v is an actual Iwahori subgroup. Therefore,

$$G_v \text{ unramified } \Longrightarrow \mathcal{Z}_{G_v,R} \simeq \mathcal{H}^{\mathrm{sph}}_{v,R}.$$

The reason for introducing these objects is the following: Let $\pi = \otimes \pi_v$ be an irreducible admissible representation of $G(\mathbb{A})$. Then there exists a unique character

$$\eta_\pi \colon \mathcal{Z}_{G(\mathbb{A}^\infty),\mathbb{Z}} \to \mathbb{C},$$

such that $\eta_\pi = \eta_{\pi^K} \circ \mathrm{pr}_K$ for every K such that $\pi^K \neq 0$. Uniqueness is clear, and the existence reduces to showing that $\eta_{\pi^J}(\phi) = \eta_{\pi^K}(e_K \star \phi)$ for $K \supset J$ when $\pi^K \neq 0$. Similarly, we have characters η_{π_v} locally, and $\eta_\pi = \otimes \eta_{\pi_v}$ under the isomorphism above. If π is automorphic and $\pi_\infty = \mathbf{1}$, the character η_π maps into the ring of integers of some number field. Our work suggests the following definition:

DEFINITION A.1. — *Let π and $\tilde{\pi}$ be automorphic representations of $G(\mathbb{A})$, both trivial at infinity, and let λ be a finite place of $\overline{\mathbb{Q}}$. Then we say that π and $\tilde{\pi}$ are congruent modulo λ, and we write $\tilde{\pi} \equiv \pi \pmod{\lambda}$, if for all $\phi \in \mathcal{Z}_{G(\mathbb{A}^\infty),\mathbb{Z}}$ we have*

$$\eta_{\tilde{\pi}}(\phi) \equiv \eta_\pi(\phi) \pmod{\lambda}.$$

Analogously, it makes sense to say the local components $\tilde{\pi}_v$ and π_v are congruent. Then $\tilde{\pi} \equiv \pi \pmod{\lambda}$ if and only $\tilde{\pi}_v \equiv \pi_v \pmod{\lambda}$ for all $v < \infty$. This is the kind of local-global compatibility aimed for in Parson's thesis [12]. Parson has another definition of being congruent. We do not know how the two definitions are related. Note also that if $\tilde{\pi}_v$ and π_v are both unramified, then $\tilde{\pi}_v \equiv \pi_v \pmod{\lambda}$ means that the Satake parameters are congruent as it should. With these definitions, our results translate into those stated in the introduction.

Appendix B. Iwahori-spherical representations of Gsp(4)

In this appendix we reproduce parts of Table 1 and Table 3 in [15]. We are grateful to Ralf Schmidt for his permission to do so. We stress that the tables in [15] contain more information than what is listed here (such as Atkin-Lehner eigenvalues and signs of ϵ-factors). Below, we employ the notation of [14]. Thus ν denotes the normalized absolute value of a non-archimedean local field. If χ_1, χ_2 and σ are unramified characters, we recall that $\chi_1 \times \chi_2 \rtimes \sigma$ denotes the principal series of Gsp(4) obtained from

$$T \ni \mathrm{diag}(x, y, zy^{-1}, zx^{-1}) \mapsto \chi_1(x)\chi_2(y)\sigma(z) \in \mathbb{C}^*$$

by normalized induction. Similarly, if π is a representation of GL(2), we denote by $\pi \rtimes \sigma$ and $\sigma \rtimes \pi$ the representations of Gsp(4) induced from $\mathrm{diag}(X, z^\tau X^{-1}) \mapsto \pi(X)\sigma(z)$ and $\mathrm{diag}(z, X, z^{-1}\det X) \mapsto \sigma(z)\pi(X)$ respectively. By $L((-))$ we mean the unique irreducible quotient (the Langlands quotient) when it exists. The representations $\tau(S, \nu^{-1/2}\sigma)$ and $\tau(T, \nu^{-1/2}\sigma)$ are the constituents of $\mathbf{1} \rtimes \sigma\,\mathrm{St}_{\mathrm{GL}(2)}$. They are occasionally called limit of discrete series. ξ_0 is the non-trivial unramified quadratic character.

		constituent of	representation	tempered	L^2	generic
I		$\chi_1 \times \chi_2 \rtimes \sigma$	$\chi_1 \times \chi_2 \rtimes \sigma$	$\lvert\chi_i\rvert = \lvert\sigma\rvert = 1$		•
II	a	$\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$	$\chi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\lvert\chi\rvert = \lvert\sigma\rvert = 1$		•
	b	$\chi^2 \notin \{\nu^{\pm 1}, \nu^{\pm 3}\}$	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$			
III	a	$\chi \times \nu \rtimes \nu^{-1/2}\sigma$,	$\chi \rtimes \sigma\,\mathrm{St}_{\mathrm{GL}(2)}$	$\lvert\chi\rvert = \lvert\sigma\rvert = 1$		•
	b	$\chi \notin \{\mathbf{1}, \nu^{\pm 2}\}$	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GL}(2)}$			
IV	a	$\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$	$\sigma\,\mathrm{St}_{\mathrm{Gsp}(4)}$	•	•	•
	b		$L((\nu^2, \nu^{-1}\sigma\,\mathrm{St}_{\mathrm{GL}(2)}))$			
	c		$L((\nu^{3/2}\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma))$			
	d		$\sigma\mathbf{1}_{\mathrm{Gsp}(4)}$			
V	a	$\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$	$\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$	•	•	•
	b	$\xi_0^2 = \mathbf{1}$, $\xi_0 \neq \mathbf{1}$	$L((\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$			
	c		$L((\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{-1/2}\sigma))$			
	d		$L((\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma))$			
VI	a	$\nu \times \mathbf{1} \rtimes \nu^{-1/2}\sigma$	$\tau(S, \nu^{-1/2}\sigma)$	•		•
	b		$\tau(T, \nu^{-1/2}\sigma)$	•		
	c		$L((\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$			
	d		$L((\nu, \mathbf{1} \rtimes \nu^{-1/2}\sigma))$			

Table C: Iwahori-spherical representations of Gsp(4)

In the following table, our notation is different from [15]. Recall that in our setup K is hyperspecial, K' is paramodular, J is the Klingen parahoric, J' the Siegel parahoric and I is the Iwahori subgroup of Gsp(4).

		representation	remarks	K	K'	J	J'	I
I		$\chi_1 \times \chi_2 \rtimes \sigma$		1	2	4	4	8
II	a	$\chi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$		0	1	2	1	4
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$		1	1	2	3	4
III	a	$\chi \rtimes \sigma\,\mathrm{St}_{\mathrm{GL}(2)}$		0	0	1	2	4
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GL}(2)}$		1	2	3	2	4
IV	a	$\sigma\,\mathrm{St}_{\mathrm{Gsp}(4)}$		0	0	0	0	1
	b	$L((\nu^2, \nu^{-1}\sigma\,\mathrm{St}_{\mathrm{GL}(2)}))$	not unitary	0	0	1	2	3
	c	$L((\nu^{3/2}\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma))$	not unitary	0	1	2	1	3
	d	$\sigma\mathbf{1}_{\mathrm{Gsp}(4)}$	irrelevant	1	1	1	1	1
V	a	$\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$		0	0	1	0	2
	b	$L((\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$		0	1	1	1	2
	c	$L((\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{-1/2}\sigma))$		0	1	1	1	2
	d	$L((\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma))$		1	0	1	2	2
VI	a	$\tau(S, \nu^{-1/2}\sigma)$		0	0	1	1	3
	b	$\tau(T, \nu^{-1/2}\sigma)$		0	0	0	1	1
	c	$L((\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$		0	1	1	0	1
	d	$L((\nu, \mathbf{1} \rtimes \nu^{-1/2}\sigma))$		1	1	2	2	3

Table D: Dimensions of the parahoric fixed spaces

BIBLIOGRAPHY

[1] J. Bellaiche, "Congruences endoscopiques et représentations galoisiennes", thesis, Université Paris XI - Orsay, 2002, http://www.math.columbia.edu/~jbellaic/.

[2] J. N. Bernstein, "Le «centre» de Bernstein", in *Representations of reductive groups over a local field*, Travaux en Cours, Hermann, Paris, 1984, Edited by P. Deligne, p. 1-32.

[3] C. Bushnell, "Personal communication", November 2004.

[4] C. J. Bushnell, "Representations of reductive p-adic groups: Localization of Hecke algebras and applications", *J. London Math. Soc. (2)* **63** (2001), no. 2, p. 364-386.

[5] W. Casselman, "Introduction to the theory of admissible representations of p-adic reductive groups", preprint, 1995, http://www.math.wisc.edu/~ram/p-adic-book.ps.

[6] L. Clozel, "On Ribet's level-raising theorem for U(3)", *Amer. J. Math.* **122** (2000), no. 6, p. 1265-1287.

[7] P. Deligne & J.-P. Serre, "Formes modulaires de poids 1", *Ann. Sci. École Norm. Sup. (4)* **7** (1974), p. 507-530.

[8] M. Harris & R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[9] S. S. Kudla, "The local Langlands correspondence: The non-Archimedean case", in *Motives (Seattle, WA, 1991)*, Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, p. 365-391.

[10] G. Laumon, "Sur la cohomologie à supports compacts des variétés de Shimura pour $\mathrm{GSp}(4)_{\mathbf{Q}}$", *Compositio Math.* **105** (1997), no. 3, p. 267-359.

[11] W. Mann, "Local level-raising for GL_n", thesis, Harvard, 2001.

[12] J. Parson, "Level-raising congruences in the representation theory of reductive groups over local fields", thesis, Princeton, 2003.

[13] K. A. Ribet, "Congruence relations between modular forms", in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)* (Warsaw), PWN, 1984, p. 503-514.

[14] P. Sally & M. Tadić, "Induced representations and classifications for $\mathrm{GSp}(2, F)$ and $\mathrm{Sp}(2, F)$", *Mém. Soc. Math. France (N.S.)* (1993), no. 52, p. 75-133.

[15] R. Schmidt, "Iwahori-spherical representations of GSp(4) and Siegel modular forms of degree 2 with square-free level", *J. Math. Soc. Japan* **57** (2005), no. 1, p. 259-293.

[16] J.-P. Serre, "Two letters on quaternions and modular forms (mod p)", *Israel J. Math.* **95** (1996), p. 281-299, With introduction, appendix and references by R. Livné.

[17] C. Skinner & E. Urban, "Sur les déformations p-adiques de certaines représentations automorphes", preprint, May 2004.

[18] R. Taylor, "On Galois representations associated to Hilbert modular forms", *Invent. Math.* **98** (1989), no. 2, p. 265-280.

[19] ———, "Representations of Galois groups associated to Hilbert modular forms", in *Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988)*, Perspect. Math., vol. 11, Academic Press, Boston, MA, 1990, p. 323-336.

[20] R. Weissauer, "Four dimensional Galois representations", preprint, http://www.mathi.uni-heidelberg.de/~weissaue/papers.html.

Manuscrit reçu le 15 septembre 2005,
accepté le 21 novembre 2005.

Claus MAZANTI SORENSEN
California Institute of Technology
Department of Mathematics
253-37 Caltech,
Pasadena, CA 91125 (USA)
claus@caltech.edu