ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 50, nº 2 (2000), p. 519-535

<http://www.numdam.org/item?id=AIF_2000__50_2_519_0>

Ann. Inst. Fourier, Grenoble
50, 2 (spécial Cinquantenaire) (2000), 519–535

INVERTIBLE COHOMOLOGICAL FIELD THEORIES AND WEIL–PETERSSON VOLUMES

by Yuri I. MANIN and Peter ZOGRAF

0. Introduction and summary.

The aim of this paper is to record some progress in understanding intersection numbers on moduli spaces of stable pointed curves and their generating functions. Continuing the study started in [KoM], [KoMK] and pursued further in [KaMZ], [KabKi], we work with Cohomological Field Theories (CohFT), which is the same as cyclic algebras over the modular operad $H_*M_{g,n} = H_*(\overline{M}_{g,n+1}, K)$ in the sense of [GK] (here K is a field of characteristic zero). Such algebras form a category with symmetric tensor product and identical object. Isomorphism classes of (slightly rigidified) invertible objects form K–points of an infinite–dimensional abelian algebraic group which can be called *the Picard group* of the respective category. This group can be identified with the product of K^* and a vector space L consisting of certain families of elements in all $H^*(\overline{M}_{g,n}, K)^{S_n}$. The space L is naturally graded by (co)dimension.

(A) The first problem is the calculation of L. To our knowledge, it is unsolved, and we want to stress its importance. We know two independent elements κ_a and μ_a in each odd codimension $a \geqslant 1$, and one element κ_a in each even codimension $a \geqslant 2$. There are only two divisorial classes in L, namely, κ_1 and μ_1. As it follows from the results of [KaMZ], the genus zero restriction of L is generated by the κ-classes, and in genus one there is just one additional class μ_1 (see [KabKi]).

Keywords : Moduli spaces – Cohomological field theories – Weil–Petersson volumes.
Math classification: 14H10 – 14N10 – 58D29.

(B) The second problem is a description of a formal function on L (or $K^* \times L$), the potential of the respective invertible CohFT.

It turns out that evaluating this function *on the subspace generated by κ_a only*, we get essentially E. Witten's generating function for τ–intersection numbers (total free energy of two dimensional gravity) but written in a different coordinate system: the respective nonlinear coordinate change is given by Schur polynomials. The proof uses the explicit formula for higher Weil–Petersson volumes of arbitrary genus derived in [KaMZ]. This completes the calculation of the total generating function for these volumes: the genus zero component was calculated in [KaMZ], and the results of [IZu] about the genus expansion of the total free energy now allow us to express all higher genus contributions through the genus zero one. This is a version of the argument given in [Zo], see also Theorem 4.1 below. It also leads to a complete proof of Itzykson's conjecture about asymptotics of the classical Weil–Petersson volumes of $\overline{M}_{g,n}$, see [Zo], formula (7):

$$\frac{\langle \kappa_1^{3g-3+n} \rangle}{n!(3g-3+n)!} = C^n n^{-1+\frac{5}{2}(g-1)}(B_g + O(1/n)), \quad n \to \infty.$$

For genus zero it was proved in [KaMZ]. The constant C does not depend on g. It was mentioned in [KaMZ] that it is expressible via the first zero of the Bessel function J_0. We supply here a detailed proof and extend it to all genera.

Looking at the identification of generating functions from the other side, we realize that the complete potential as a function on the moduli space of invertible CohFT's is a natural infinite dimensional extension of the total free energy. It would be interesting to derive differential equations for this function extending the Virasoro constraints and higher KdV equations.

Finally, we discuss several related constructions and analogies:

(C) The cohomology space of any projective algebraic smooth manifold V carries a canonical structure of CohFT (quantum cohomology). The identity object of the category CohFT is the quantum cohomology of a point. It is tempting to interpret any other invertible CohFT, T, as generalized quantum cohomology of a point, and to define *the generalized quantum cohomology of* V as the cyclic modular algebra $H^*(V) \otimes T$. The potential of this theory restricted to the κ–subspace can be derived from the potential of V with gravitational descendants. Of course, it is an *ad hoc* prescription which must be replaced by a more geometric construction.

This sheds new light on the mathematical nature of the *large phase space* (see [W]): according to the comments (B), it must be a part of a *very* *large phase space* which is a suitably interpreted moduli space of generalized quantum cohomology.

(D) The structure of the operad $H^*(\overline{M}_{g,n}, K)$ restricted to the $n = 2$ part of it gives rise to an interesting bialgebra. A part of its primitive elements can be obtained from L. It would be interesting to describe all of them. The action of this bialgebra on $H^*(V)$ can be derived from a complexified version of the path groupoid of V. This raises several problems of which the most interesting is probably the construction of the quantum motivic fundamental group, upon which this groupoid might act.

1. The space L.

By definition, $L \subset \prod_{g,n} H^*(\overline{M}_{g,n}, K)$ is the linear space formed by all families $l = \{l_{g,n} \in H^*(\overline{M}_{g,n}, K)\}$ satisfying the following conditions:

(i) $l_{g,n} \in H^{ev}(\overline{M}_{g,n}, K)^{S_n}$.

(ii) For every boundary morphism

$$b : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n} \tag{1}$$

with $g = g_1 + g_2, n_1 + n_2 = n$, we have

$$b^*(l_{g,n}) = l_{g_1,n_1+1} \otimes 1 + 1 \otimes l_{g_2,n_2+1}. \tag{2}$$

(iii) For any boundary morphism

$$b' : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n} \tag{3}$$

we have

$$b'^*(l_{g,n}) = l_{g-1,n+2}. \tag{4}$$

Recall that the boundary morphisms b (resp. b') glue together a pair of marked points situated on different connected components (resp. on the same one) of the appropriate universal curves.

PROPOSITION 1.1. — (a) *L is graded by codimension in the following sense: put $L^{(a)} = L \cap \prod_{g,n} H^{2a}(\overline{M}_{g,n}, K)$, then $L = \prod_{a\geqslant 1} L^{(a)}$.*

(b) *Let (H, h) be a one–dimensional vector space over K endowed with a metric h and an even basic vector Δ_0 such that $h(\Delta_0, \Delta_0) = 1$. For any structure of the invertible CohFT on (H, h) given by the maps*

$I_{g,n} : H^{\otimes n} \to H^*(\overline{M}_{g,n}, K)$ put $c_{g,n} = I_{g,n}(\Delta_0^{\otimes n})$. Let P be the group of the isomorphism classes of such theories with respect to the tensor multiplication (isomorphisms should identify the basic vectors). Then the following map $K^* \times L \to P$ is a group isomorphism:

$$(t, \{l_{g,n}\}) \mapsto \{c_{g,n} = t^{2g-2+n} \exp(l_{g,n})\}. \tag{5}$$

The proof is straightforward: cf. [KoMK], sec. 3.1, where most of it is explained for genus zero. Notice that the statement on the grading means that if one puts $l_{g,n} = \sum_a l_{g,n}^{(a)}$ with $l_{g,n}^{(a)} \in H^{2a}(\overline{M}_{g,n}, K)$, then $l \in L$ is equivalent to $l^{(a)} \in L$ for all a, where $l^{(a)} = \{l_{g,n}^{(a)}\}$.

2. Classes κ_a and μ_a.

Put

$$\kappa_a = \left\{ \kappa_{g,n;\,a} = \pi_* \left(c_1 \left(\omega_{g,n} \left(\sum_{i=1}^n x_i \right) \right)^{a+1} \right) \in H^{2a}(\overline{M}_{g,n}, K)^{S_n} \right\} \tag{6}$$

where $\omega_{g,n}$ is the relative dualizing sheaf of the universal curve $\pi = \pi_{g,n} : C_{g,n} \to \overline{M}_{g,n}$, and x_i are the structure sections of π.

Furthermore, with the same notation put

$$\mu_a = \{ \mu_{g,n;\,a} = \mathrm{ch}_a(\pi_* \omega_{g,n}) \}. \tag{7}$$

Classes $\mu_{g,n;a}$ vanish for all even a: see [Mu].

Both κ_a and μ_a belong to $L^{(a)}$: for κ this was noticed, e.g., in [AC], and for μ in [Mu]. These elements are linearly independent for odd a because, for example, $\mu_{g,n;\,a}$ is lifted from $\overline{M}_{g,0}$ for $g \geqslant 2$, or from $\overline{M}_{1,1}$, whereas $\kappa_{g,n;\,a}$ is not.

Remark 2.1. — The classes $\kappa_{g,n;\,a}$ are denoted $\omega_{g,n}(a)$ in [KaMZ]. We use this opportunity to clarify one apparent notational ambiguity in that paper. Namely, the initial definition of $\omega_{g,n}(a)$ given by (0.1) in [KaMZ] coincides with our (6). On the other hand, specializing the formula (2.4) of [KaMZ] for $\omega_{g,n}(a_1, \ldots, a_p)$ to the case $p = 1$ we obtain a formally different expression

$$\omega_{g,n}(a) = \pi_*(c_1(x_{n+1}^* \omega_{g,n+1})^{a+1}),$$

where $C_{g,n}$ is canonically identified with $\overline{M}_{g,n+1}$ and $x_{n+1} : \overline{M}_{g,n+1} \to C_{g,n+1}$ is the respective section. Actually, the two definitions coincide because there exists a canonical isomorphism on $C_{g,n}$

$$\omega_{g,n}\left(\sum_{i=1}^{n} x_i\right) \simeq x_{n+1}^{*}\omega_{g,n+1}.$$

3. Three generating functions.

Witten's *total free energy* of two dimensional gravity is the formal series

(8)

$$F(t_0,t_1,\ldots) = \sum_{g=0}^{\infty} F_g(t_0,t_1,\ldots) = \sum_{g=0}^{\infty} \sum_{\sum(i-1)l_i=3g-3} \langle \tau_0^{l_0}\tau_1^{l_1}\ldots\rangle \prod_{i=0}^{\infty} \frac{t_i^{l_i}}{l_i!}.$$

Here $\langle \tau_0^{l_0}\tau_1^{l_1}\ldots\rangle$ is the intersection number defined as follows. Consider a partition $d_1+\ldots+d_n = 3g-3+n$ such that l_0 of the summands d_i are equal to 0, l_1 are equal to 1, etc, and put

$$\langle \tau_0^{l_0}\tau_1^{l_1}\ldots\rangle = \langle \tau_{d_1}\cdots\tau_{d_n}\rangle = \int_{\overline{M}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n} \tag{9}$$

where $\psi_i = \psi_{g,n;\,i} = c_1(x_i^*\omega_{g,n})$ and $n=\sum l_i$.

The generating function for *higher Weil–Petersson volumes* was introduced (in a slightly different form) in [KaMZ]:

$$\begin{aligned} K(x,s_1,\ldots) &= \sum_{g=0}^{\infty} K_g(x,s_1,\ldots) \\ &= \sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\sum_{|\mathbf{m}|=3g-3+n} \langle \kappa_1^{m_1}\kappa_2^{m_2}\ldots\rangle \frac{x^n}{n!}\prod_{a=1}^{\infty}\frac{s_a^{m_a}}{m_a!}. \end{aligned} \tag{10}$$

Here and below $\mathbf{m} = (m_1,m_2,\ldots m_{3g-3+n})$ with $|\mathbf{m}| = \sum_{a=1}^{3g-3+m} am_a$, and

$$\langle \kappa_1^{m_1}\kappa_2^{m_2}\ldots\rangle = \int_{\overline{M}_{g,n}} \kappa_{g,n;\,1}^{m_1}\kappa_{g,n;\,2}^{m_2}\cdots\kappa_{g,n;\,3g-3+n}^{m_{3g-3+n}} \tag{11}$$

(all integrals over the moduli spaces of unstable curves are assumed to be zero).

Actually, (10) is a specialization of the general notion of the the *potential* of a CohFT, $(H,h;\{I_{g,n}\})$, which is a formal series in coordinates of H and is defined as $\langle \exp(\gamma) \rangle$ where γ is the generic even element of H and the functional $\langle * \rangle$ as above is the integration over fundamental classes. Considering variable invertible CohFT's, we will get the potential as a formal function on $H \times K^* \times L$ which is a series in $(x, t, s_a, r_b, \ldots)$. Here x stands for the coordinate on H dual to Δ_0, t for the coordinate on K^* as in (5), s_a (resp. r_b) are coordinates on L dual to κ_a (resp. μ_a), and dots stand for the unexplored part of L (if there is any). More precisely, potential of the individual theory (5) is

$$\Phi(x,t) = \sum_g \Phi_g(x,t) = \sum_g \langle \exp(x\Delta_0) \rangle_g$$
$$= \sum_{g,n} \frac{\langle I_{g,n}(x^n \Delta_0^{\otimes n}) \rangle}{n!} = \sum_{g,n} \frac{x^n}{n!} \int_{\overline{M}_{g,n}} t^{2g-2+n} \exp(l_{g,n}) .$$

Now make $l_{g,n}$ generic, that is, put

$$l_{g,n} = \sum_{a,b,\ldots} (s_a \kappa_{g,n;\,a} + r_b \mu_{g,n;\,b} + \ldots).$$

Collecting the terms of the right dimension, we finally get the series

$$\Phi(x,t,s_1,s_2,\ldots,r_1,r_2,\ldots) = \sum_g \Phi_g(x,t,s_1,s_2,\ldots,r_1,r_2,\ldots) \tag{12}$$

$$= \sum_{g,n} \frac{x^n}{n!} t^{2g-2+n} \sum \langle \kappa_1^{m_1} \kappa_2^{m_2} \ldots \mu_1^{p_1} \mu_2^{p_2} \ldots \rangle \prod_{a=1}^{\infty} \frac{s_a^{m_a}}{m_a!} \prod_{b=1}^{\infty} \frac{r_b^{p_b}}{p_b!} \cdots$$

where the inner summation is taken over $\mathbf{m}, \mathbf{p}, \ldots$ with $|\mathbf{m}| + |\mathbf{p}| + \ldots = 3g - 3 + n$.

Putting here $t = 1, r_b = \ldots = 0$, we obtain (10). The following Theorem 4.1 shows, that making an invertible change of variables in (8) restricted to $t_1 = 0$ we again can get (10). Thus (12) can be considered as a natural infinite dimensional extension of (8).

Some information about the coefficients of (12) with non–vanishing p_b is obtained in [FP]. If L is not generated by the κ– and μ–classes, these coefficients provide the natural generalization of Hodge integrals of [FP].

4. Relation between generating functions.

The main result of this section is:

THEOREM 4.1. — *For every $g = 0, 1, \dots$ we have*

$$K_g(x, s_1, \dots) = F_g(t_0, t_1, \dots)|_{t_0=x, t_1=0, t_{j+1}=p_j(s_1,\dots,s_j)}.$$

Here p_j are the Schur polynomials defined by

$$1 - \exp\left(-\sum_{i=1}^{\infty} \lambda^i s_i\right) = \sum_{j=1}^{\infty} \lambda^j p_j(s_1, \dots, s_j).$$

Proof. — The statement of the theorem is a convenient reformulation of formula (2.17) of [KaMZ]:

$$\frac{\langle \kappa_1^{m_1} \dots \kappa_{3g-3+n}^{m_{3g-3+n}} \rangle}{m_1! \dots m_{3g-3+n}!}$$

$$(13) \qquad = \sum_{k=1}^{\|\mathbf{m}\|} \frac{(-1)^{\|\mathbf{m}\|-k}}{k!} \sum_{\mathbf{m}=\mathbf{m}^{(1)}+\dots+\mathbf{m}^{(k)}} \frac{\langle \tau_0^n \tau_{|\mathbf{m}^{(1)}|+1} \cdots \tau_{|\mathbf{m}^{(k)}|+1} \rangle}{\mathbf{m}^{(1)}! \dots \mathbf{m}^{(k)}!},$$

where $\|\mathbf{m}\| = \sum m_i$, $\mathbf{m}^{(i)} = (m_1^{(i)}, \dots, m_{3g-3+n}^{(i)}) \neq 0$, $\mathbf{m}^{(i)}! = m_1^{(i)}! \dots m_{3g-3+n}^{(i)}!$. To show this, fix $\mathbf{m} = (m_1, \dots, m_{3g-3+n})$ and compare the coefficients at the monomial $x^n s_1^{m_1} \dots s_{3g-3+n}^{m_{3g-3+n}}$ in the series K_g and $F_g(x, 0, p_1, p_2, \dots)$. The contribution from the term

$$(14) \qquad \langle \tau_0^{l_0} \tau_1^{l_1} \dots \rangle \prod \frac{t_j^{l_j}}{l_j!}$$

to this coefficient is nontrivial if and only if $l_0 = n$, $l_1 = 0$ and there exists a partition $\mathbf{m} = \mathbf{m}^{(1)} + \dots + \mathbf{m}^{(k)}$ with $k = l_2 + l_3 + \dots$ such that for any $j = 2, 3, \dots$ there are exactly l_j sequences $\mathbf{m}^{(i)} = (m_1^{(i)}, m_2^{(i)}, \dots)$ with $|\mathbf{m}^{(i)}| = j - 1$. Then, obviously, $\langle \tau_0^{l_0} \tau_1^{l_1} \dots \rangle = \langle \tau_0^n \tau_{|\mathbf{m}^{(1)}|+1} \cdots \tau_{|\mathbf{m}^{(k)}|+1} \rangle$. Let us subdivide the set of $k = l_2 + l_3 + \dots$ sequences $\mathbf{m}^{(i)}$ into subsets consisting of equal sequences and denote the cardinalities of these subsets by $l_2^{(1)}, l_2^{(2)}, \dots, l_3^{(1)}, l_3^{(2)}, \dots, l_j^{(1)}, l_j^{(2)}, \dots$, so that $\sum_a l_j^{(a)} = l_j$. Since the polynomials p_j are explicitly given by

$$p_j(s_1, \dots, s_j) = -\sum_{|\mathbf{m}|=j} \prod_{i=1}^{j} (-1)^{m_i} \frac{s_i^{m_i}}{m_i!},$$

we can rewrite the contribution from (14) corresponding to the partition $\mathbf{m} = \mathbf{m}^{(1)} + \ldots + \mathbf{m}^{(k)}$ as

$$\langle \tau_0^n \tau_{|\mathbf{m}^{(1)}|+1} \cdots \tau_{|\mathbf{m}^{(k)}|+1} \rangle \frac{x^n}{\prod l_j!} \prod_{j\geqslant 2} \frac{l_j!}{\prod_a l_j^{(a)}!} \prod_{i=1}^{k} \left(- \prod_{b=1}^{3g-3+n} (-1)^{m_b^{(i)}} \frac{s_b^{m_b^{(i)}}}{m_b^{(i)}!} \right)$$

(15)

$$= (-1)^{\|\mathbf{m}\|-k} \frac{1}{\prod_{j,a} l_j^{(a)}!} \frac{\langle \tau_0^n \tau_{|\mathbf{m}^{(1)}|+1} \cdots \tau_{|\mathbf{m}^{(k)}|+1} \rangle}{\mathbf{m}^{(1)}! \ldots \mathbf{m}^{(k)}!} \frac{x^n s_1^{m_1} \ldots s_{3g-3+n}^{m_{3g-3+n}}}{n!}.$$

Since there are exactly $k!/\prod_{j,a} l_j^{(a)}!$ partitions of $\mathbf{m} = (m_1, \ldots, m_{3g-3+n})$ that are equivalent to the above partition $\mathbf{m} = \mathbf{m}^{(1)} + \ldots + \mathbf{m}^{(k)}$ under permutation of indices $1, \ldots, k$, the coefficient in (15) coincides with the corresponding term in (13), which completes the proof.

COROLLARIES 4.2. — It follows from Theorem 4.1 that all known properties of the function F can be automatically translated to K. First, F satisfies the Virasoro constraints (or, equvalently, $\partial^2 F/\partial t_0^2$ is a solution to the KdV hierarchy). The same is true for K up to a change of variables. However, one should take into account that in order to get K, we first restrict F to the $t_1 = 0$ hyperplane so that the derivatives in the t_1-direction must be first eliminated.

Second, we can express each K_g with $g \geqslant 1$ in terms of K_0 in the same way as it was done in [IZu] for F. Let us remind the genus expansion of F obtained in [IZu] (cf. also [EYY]). Put $u_0 = F_0''$ where a prime denotes the derivative with respect to t_0, and define a sequence of formal series $I_1, I_2, \ldots$ by

$$I_1 = 1 - \frac{1}{u_0'}, \quad I_{k+1} = \frac{I_k'}{u_0'}, \quad k = 1, 2, \ldots$$

Then $F_1 = \frac{1}{24} \log u_0'$, and for $g \geqslant 2$

$$(16) \qquad F_g = \sum_{\sum (i-1)m_i = 3g-3} \langle \tau_2^{m_2} \tau_3^{m_3} \ldots \tau_{3g-2}^{m_{3g-2}} \rangle \, u_0'^{\,2g-2+\|\mathbf{m}\|} \prod_{k=2}^{3g-2} \frac{I_k^{m_k}}{m_k!}.$$

Substituting here $t_0 = x, t_1 = 0$ and $t_{i+1} = p_i(s_1, \ldots, s_i), i \geqslant 1$ we get the identical formula for K_g.

5. Topological interpretation of the genus expansion.

Differentiating n times the both sides of (16) with respect to t_0 and using the definition of I_k, one easily gets that

$$\frac{\partial^n F_g}{\partial t_0^n} = \sum_{|\mathbf{m}|=3g-3+n} a_{\mathbf{m}}^{g,n} \left(\frac{\partial^3 F_0}{\partial t_0^3}\right)^{1-g-\|\mathbf{m}\|} \prod_{i=1}^{3g-3+n} \left(\frac{\partial^{i+3} F_0}{\partial t_0^{i+3}}\right)^{m_i}$$

with certain constants $a_{\mathbf{m}}^{g,n}$ depending on g, n and $\mathbf{m} = (m_1, \ldots, m_{3g-3+n})$. These constants are linear combinations of the intersection numbers $\langle \tau_2^{l_2} \tau_3^{l_3} \ldots \rangle$, and the corresponding $p\,(3g-3+n) \times p\,(3g-3)$ matrix (where $p\,(k)$ is the partition number of k) can be written quite explicitly (cf. [EYY], Theorem 1 and Appendix D). Below we give a topological interpretation of the numbers $a_{\mathbf{m}}^{g,n}$; details will appear elsewhere [GoOrZo].

Denote by $t_i \in H_{2i}(BU, \mathbb{Q})$ the class associated with the hyperplane section line bundle on $\mathbb{C}P^i$; the classes t_i form the canonical multiplicative basis of the homology ring $H_*(BU, \mathbb{Q})$. Let us treat the classes $\frac{1}{i!}\,\kappa_{g,n;\,i} \in H^{2i}(\overline{M}_{g,n}, \mathbb{Q})$ as the components of the Chern character of a (virtual) vector bundle on $\overline{M}_{g,n}$, and denote by $\alpha_{g,n;\,k}$ the corresponding roots. Put $Q(\alpha) = 1 + \sum_{i=1}^{\infty} \alpha^i t_i$ and consider the homology class in $H_{6g-6+2n}(BU, \mathbb{Q})$ defined by

$$[\overline{M}_{g,n}] = \int_{\overline{M}_{g,n}} \prod_k Q(\alpha_{g,n;\,k}) = \sum_{|\mathbf{m}|=3g-3+n} b_{\mathbf{m}}^{g,n}\, t_1^{m_1} \ldots t_{3g-3+n}^{m_{3g-3+n}}.$$

Similar to $K(x, s_1, \ldots)$, we introduce the formal series

$$B(x, t_1, \ldots) = \sum_{g,n} \frac{x^n}{n!} \sum_{|\mathbf{m}|=3g-3+n} b_{\mathbf{m}}^{g,n} \prod_{i=1}^{3g-3+n} t_i^{m_i}.$$

THEOREM 5.1. — (a) *The classes $[\overline{M}_{0,n}]$ form a multiplicative basis of the homology ring $H_*(BU, \mathbb{Q})$ and for any g, n we have the decomposition*

$$[\overline{M}_{g,n}] = \sum_{|\mathbf{m}|=3g-3+n} a_{\mathbf{m}}^{g,n} \prod_{i=1}^{3g-3+n} [\overline{M}_{0,n}]^{m_i}$$

in $H_*(BU, \mathbb{Q})$ with the same constants $a_{\mathbf{m}}^{g,n}$ as defined above.

(b) *The series $K(x, s_1, \ldots)$ and $B(x, t_1, \ldots)$ are related via the formula*

$$K(x, s_1, \ldots) = B(x, t_1, \ldots)\,\big|_{t_i = q_i(s_1, \ldots, s_i)},$$

where q_i are the ordinary Schur polynomials defined by

$$\exp\left(\sum_{i=1}^{\infty}\lambda^i s_i\right) = 1 + \sum_{j=1}^{\infty}\lambda^j q_j(s_1,\dots,s_j).$$

The proof essentially follows from Theorem 4.1 and formula (16).

Remark 5.2. — If we formally put $Q(\psi) = \sum_{i=0}^{\infty}\psi^i t_i$, then for Witten's total free energy we have the formula

$$F(t_0,t_1,\dots) = \sum_{g,n}\frac{1}{n!}\int_{\overline{M}_{g,n}}\prod_{i=1}^{n} Q(\psi_{g,n;i}).$$

This is reminiscent of J. Morava's approach to Witten's two dimensional gravity [Mo], although we do not make use of complex cobordisms.

6. Asymptotics for the volumes.

We start with defining some constants. Consider the classical Bessel function

$$J_0(z) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{z^2}{4}\right)^m .$$

Denote by j_0 its first positive zero and put

$$x_0 = -\frac{1}{2}\, j_0 J_0'(j_0), \qquad y_0 = -\frac{1}{4}\, j_0^2, \qquad A = -j_0^{-1} J_0'(j_0). \tag{17}$$

Constants x_0 and A are positive. Put also $V_{g,n} = \langle \kappa_1^{3g-3+n}\rangle$. Recall that the Kähler form ω_{WP} of the Weil–Petersson metric on $M_{g,n}$ extends as a closed current to $\overline{M}_{g,n}$ and $[\omega_{WP}] = 2\pi^2\kappa_1$ in the real cohomology (see [Wo]). This means that, up to a normalization, $V_{g,n}$ are the classical Weil–Petersson volumes of moduli spaces. In the unstable range we put by definition $V_{g,n} = 0$ for $2g+n \leqslant 2$ and $V_{0,3} = 1$.

THEOREM 6.1. — *We have the following asymptotical expansions valid for any fixed g and $n \to \infty$:*

$$\frac{V_{g,n}}{n!(n+3g-3)!} = (n+1)^{\frac{5g-7}{2}}\, x_0^{-n}\left(B_g + \sum_{k=1}^{\infty}\frac{B_{g,k}}{(n+1)^k}\right) \tag{18}$$

where

$$B_0 = \frac{1}{A^{\frac{1}{2}}\Gamma\left(-\frac{1}{2}\right) x_0^{\frac{1}{2}}}, \quad B_1 = \frac{1}{48}, \tag{19}$$

and for $g \geqslant 2$

$$B_g = \frac{A^{\frac{g-1}{2}}}{2^{2g-2}\,(3g-3)!\,\Gamma\left(\frac{5g-5}{2}\right) x_0^{\frac{5g-5}{2}}} \langle \tau_2^{3g-3} \rangle. \tag{20}$$

Remark 6.2. — By definition,

$$\langle \tau_2^{3g-3} \rangle = \int_{\overline{M}_{g,3g-3}} \psi_1^2 \dots \psi_{3g-3}^2.$$

These numbers can be consecutively calculated using a simple recursive formula established in [IZu], sec. 6. We reproduce it here for a reader's convenience. Put $b_0 = -1$, $b_1 = \frac{1}{24}$, $b_g = \frac{(5g-3)(5g-5)}{2^g\,(3g-3)!} \langle \tau_2^{3g-3} \rangle$ for $g \geqslant 2$. Then we have

$$b_{g+1} = \frac{25g^2 - 1}{24} b_g + \frac{1}{2} \sum_{m=1}^{g} b_{g+1-m} b_m \, .$$

Equivalently, the formal series $\chi(t) = \sum_{g=0}^{\infty} b_g \, t^{-\frac{5g-1}{2}}$ satisfies the first Painlevé equation

$$\frac{1}{3}\chi'' + \chi^2 - t = 0.$$

The constants $B_{g,k}$ in the asymptotical expansion (18) are also calculable in terms of known quantities: see below.

Proof 6.3 of Theorem 6.1. — The proof breaks into two logically distinct but tightly interwoven parts.

In the first part we calculate the smallest singularity of the function

$$\varphi_g(x) = \sum_{n=0}^{\infty} \frac{V_{g,n}}{n!\,(n+3g-3)!}\, x^n. \tag{21}$$

and establish its behavior near this singularity. It turns out that the convergence radius of (21) is x_0 (see (17)), and that near this point the function (or its derivative for $g = 0, 1$) can be represented as a convergent Laurent series in $(x_0 - x)^{1/2}$ with the leading term

$$\varphi_0''(x) \sim -\frac{1}{A^{\frac{1}{2}}}(x_0 - x)^{\frac{1}{2}}, \tag{22}$$

$$\varphi_1'(x) \sim \frac{1}{48\,(x_0 - x)} \tag{23}$$

and for $g \geqslant 2$

$$\varphi_g(x) \sim \frac{A^{\frac{g-1}{2}}}{2^{2g-2}(3g-3)!}\,\langle \tau_2^{3g-3}\rangle\,(x_0 - x)^{-\frac{5g-5}{2}}\,. \tag{24}$$

The right hand side of (18) can now be obtained in the following way. Expand each monomial $a_\nu(x_0 - x)^\nu$ into the Taylor series near $x = 0$ and take the sum of the standard asymptotic expansions of their coefficients in n. The leading singularities (22)–(24) produce the leading terms in (18).

The justification of this procedure is the second part of the proof. Instead of proceeding directly, we will show that we can apply to our situation Theorem 6.1 of [O], Chapter 4, which is a version of the stationary phase method.

Now we will explain the precise meaning of (22)–(24) and prove the statements above.

The genus zero generating function $y(x) := \varphi_0''(x)$ near zero *as a formal series* is obtained by inverting the Bessel function $x(y) := -\sqrt{y}\,J_0'(2\sqrt{y})$. A simple direct proof of this fact based upon an earlier result by P. Zograf is given in the introduction of [KaMZ]. The derivative of $x(y)$ in y is $J_0(2\sqrt{y})$. All roots of this function are real and simple (see e.g. [O]). Clearly, the first root is y_0 (see (17)). On the half–line $(-\infty, y_0)$ the function $x(y)$ strictly increases from $-\infty$ to x_0. It follows that these half–lines are identified by a well defined *set–theoretic* inverse function $y(x)$. Since $x(y)$ is holomorphic (in fact, entire) and $y(x)$ is a series in x with positive coefficients, it converges to the set–theoretic inverse in the interval $(-x_0, x_0)$ and in the open disk $|x| < x_0$. Near $y = y_0$ we have $x(y) = x_0 - A(y_0 - y)^2 + O((y_0 - y)^3)$ (see (17)). Projecting the complex graph of $x(y)$ in $\mathbb{C}^2$ to the y–axis we see that, over a sufficiently small disk around $y = y_0$, the function $(x_0 - x)^{\frac{1}{2}}$ lifted to this graph is a local parameter at (x_0, y_0). Hence $y(x)$ can be expanded into Taylor series in $(x_0 - x)^{\frac{1}{2}}$ starting with

$$y = y_0 - \frac{1}{A^{1/2}}\,(x_0 - x)^{\frac{1}{2}} + O((x_0 - x)). \tag{25}$$

This explains (and proves) (22).

Notice that x_0 is the closest to zero singularity only on the branch of $y(x)$ described above and analytically continued to the cut complex plane $\mathbb{C} \setminus [x_0, \infty]$. The other branches contain a sequence of real positive ramification points tending to zero. In fact, the critical values of $J_0(x)$ are real and tend to zero.

We now pass to the cases $g \geqslant 1$. Formula (10) shows that $\varphi_g(x)$ is the specialization of $K_g(x,s)$ at $s_1 = 1, s_2 = s_3 = \ldots = 0$. Therefore, in view of Theorem 4, it is the specialization of $F_g(t_0, t_1, \ldots)$ at $t_0 = x$, $t_1 = 0$, $t_k = \frac{(-1)^k}{(k-1)!}$ for $k \geqslant 1$. In particular, u_0 gets replaced by $y(x)$ and prime means now the derivation in x so that $\varphi_1(x) = \frac{1}{24} \log y'(x)$ which together with (22) establishes (23).

Finally, to treat the case $g \geqslant 2$ denote by $f_k(x)$ the specialization of I_k corresponding to the above values of t_i. So, we have

$$f_2(x) = \frac{y''(x)}{y'(x)^3}, \quad f_{k+1} = \frac{f_k'(x)}{y'(x)} \tag{26}$$

and

$$\varphi_g(x) = \sum_{\sum (i-1) l_i = 3g-3} \langle \tau_2^{l_2} \tau_3^{l_3} \ldots \tau_{3g-2}^{l_{3g-2}} \rangle \, y'(x)^{2g-2+\sum l_i} \prod_{k=2}^{3g-2} \frac{f_k(x)^{l_k}}{l_k!} . \tag{27}$$

Since $y'(x)\, x'(y) = 1$ on the graph of our function, the r. h. s. of (27) can be rewritten in the following way:

$$\sum_{\sum (i-1) l_i = 3g-3} (-1)^{\sum l_i} \langle \tau_2^{l_2} \tau_3^{l_3} \ldots \tau_{3g-2}^{l_{3g-2}} \rangle \frac{1}{x'(y)^{2g-2+\sum l_i}} \prod_{k=2}^{3g-2} \frac{x^{(k)}(y)^{l_k}}{l_k!} \Bigg|_{y=y(x)} \tag{28}$$

Each term of (28) as a function of y is meromorphic on the whole y–plane. Hence we can substitute into its Laurent series at x_0 the Taylor series of $y(x)$ at x_0 and get the Laurent series for $\varphi_g(x)$.

In particular, an easy induction starting with (25) shows that all $f_k(x)$ are finite at x_0. On the other hand, $y'(x)$ replacing u_0' in (16) starts with $\frac{1}{2A^{\frac{1}{2}}} (x_0 - x)^{-\frac{1}{2}}$. Therefore the leading singularity in (27) is furnished by the unique term for which the sum $\sum l_i$ is maximal, that is,

$$(l_2, \ldots, l_{3g-2}) = (3g-3, 0, \ldots, 0).$$

To deduce (24) from here, it remains to notice that $f_2(x_0) = 2A$.

We now pass to the second part of the proof. Let $q(x)$ denote $y(x)$ or one of the summands in (27). We want to get the asymptotic expansion (for $n \to \infty$) of the integral taken over a small circle around zero

$$\frac{1}{2\pi i} \int x^{-(n+1)} \, q(x) dx. \tag{29}$$

According to the previous discussion, this contour can be continuously deformed in the definition domain of our branch of $y(x)$. The natural choice is to take Γ to be the inverse image of the circle $|y| = y_0$ under the map $x \mapsto y(x)$. We owe to Don Zagier the crucial remark that Γ is a closed curve without self–intersections, $x \mapsto x(y)$ is a homeomorphism of Γ, and the circle $|x| = x_0$ lies strictly inside Γ except for one point $x = x_0$ where this circle touches Γ. In other words, $|x(y_0 e^{i\phi})| > x_0$ unless $\phi \in 2\pi i\mathbb{Z}$. (The easiest way to convince oneself in this is to look at the computer generated graph of $|x(y_0 e^{i\phi})| - x_0$ produced for us by Don Zagier. One can also supply a straightforward analytic proof which we omit.) Denote by Γ_+ the part of Γ in the upper half–plane run over clockwise and by Γ_- the lower part run over counterclockwise so that

$$(30) \qquad \frac{1}{2\pi i}\int x^{-(n+1)}\, q(x)dx = \frac{1}{2\pi i}\left(\int_{\Gamma_+} - \int_{\Gamma_-}\right) x^{-(n+1)}\, q(x)dx.$$

Now we have to check that the conditions (I)–(V) stated in [O], IV.6.1 are satisfied so that we can apply Theorem 6.1 of loc. cit. to our $\int_{\Gamma_\pm}$. Except for (III), everything is already checked. In particular, the crucial condition (V) is precisely the fact that Γ includes the circle $|x| = x_0$. A part of the condition (III) requires the order of $q(x)$ at x_0 (Olver's $\lambda - 1$) to be greater than -1. Since in our situation this order can be $\frac{5}{2}\,(1-g)$ (for the leading term when $g \geqslant 2$) we have to start with representing $q(x)$ as $q_\infty(x) + q_0(x)$ where q_∞ is a linear combination of negative powers of $(x_0 - x)^{\frac{1}{2}}$ and the order of $q_0(x)$ is greater than -1. Then the first summand is treated directly whereas for the second one we have the stationary phase expansion.

This completes the proof of Theorem 6.1. We will now complement it by an explicit prescription for the calculation of all the coefficients $B_{g,k}$. To adapt Olver's formula IV.(6.19) to our situation, notice that his μ is 1, and his $\lambda - 1$ is now the order of $q_0(x)$ at $x = x_0$ which depends on g and on the summand in (27) that we are treating. Define the coefficients a_k by

$$(31) \qquad x_0 e^v q_0(x_0 e^v) = \sum_{s=0}^{\infty} a_k v^{k+\lambda-1}$$

near $v = 0$ (this is Olver's formula IV.(6.09)). Then the asymptotic expansion of (29) for even g reads

$$(32) \quad \frac{1}{2\pi i}\int_{\Gamma_\pm} x^{-(n+1)} q_0(x)dx = \pm\frac{1}{2\pi}\, x_0^{-(n+1)} \sum_{k=0}^{\infty} \Gamma(k+\lambda)\, \frac{a_k}{(n+1)^{k+\lambda}}$$

from which arbitrary number of coefficients $B_{g,k}$ can be calculated.

7. The path groupoid in quantum cohomology.

The first term of any operad is a monoid. Since H_*M is the operad of coalgebras (with comultiplication induced by the diagonal), we get a bialgebra $B_* = \oplus_{g\geqslant 1} H_*(\overline{M}_{g,2})$ and the dual bialgebra $B^* = \prod_{g\geqslant 1} H^*(\overline{M}_{g,2})$. More precisely, denote here by b the family of boundary morphisms

$$b_{g_1,g_2} : \overline{M}_{g_1,2} \times \overline{M}_{g_1,2} \to \overline{M}_{g_1+g_2,2}$$

glueing the second point of the first curve to the first point of the second curve. It induces a multiplication on the homology coalgebra and a comultiplication on the cohomology algebra

$$b_* : B_* \otimes B_* \to B_*, \quad b^* : B^* \to B^* \otimes B^*$$

making each space a bialgebra. Moreover, renumbering the structure sections $x_1 \leftrightarrow x_2$ we get an involution s of $\overline{M}_{g,2}$ inducing an involution of B_*, resp. B^*. The latter is an automorphism of the comultiplication on homology, resp. multiplication on cohomology, but an antiautomorphism of the remaining two structures, because from the definition one sees that

$$b \circ \sigma_{1,2} = s \circ b \circ (s \times s)$$

where $\sigma_{1,2}$ is the permutation of factors.

PROPOSITION 7.1.— *Let V be a smooth projective manifold. Then the family of Gromov–Witten correspondences $I_{g,2} \in A_*(V^2 \times \overline{M}_{g,2})$ defines an action of the algebra B_* and a coaction of the coalgebra B^* on $H^*(V)$.*

This is a part of the general statement that $H^*(V)$ is a cyclic algebra over the modular operad H_*M.

The geometry of this (co)action (explained e.g. in [KoM]) shows that morally it reflects the properties of the complexified groupoid of paths on V: a stable map of a curve with two marked points to V should be considered as a complex path from the first marked point to the second one. This agrees with the composition by glueing the endpoint of one path to the starting point of another, and the smoothing of the resulting singularity plays the role of homotopy. Notice that there may well exist non–constant stable maps of genus zero with only two marked points whereas there are no stable curves with this property. For this reason, one may hope that B_* (and possibly B^*) can be extended by the genus zero component, perhaps by taking an appropriate (co)homology of the respective Artin stack $\mathcal{M}_{0,2}$ which might be close to the (co)homology of BG_m.

As we mentioned in the Introduction, it would be highly desirable to introduce quantum fundamental group of V, with appropriate action of our path groupoid. In any case, we are interested in the primitive elements h of B^* satisfying $b^*(h) = h \otimes 1 + 1 \otimes h$. Clearly, restricting L to its $n = 2$ part we get a supply of such elements. Their full classification may be easier than that of L.

Acknowledgements. — We are grateful to Boris Dubrovin for very stimulating discussions and Don Zagier for his help with the proof of Theorem 6.1. PZ thanks MPIM (Bonn) for hospitality and support during 1998-99 academic year.

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Manuscrit reçu le 17 mai 1999.

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